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# Advanced properties of spectral functions in frequency and time domains for diffraction by a wedge-shaped region

J.M.L. BERNARD

## 1) Introduction

The scattering by objects with singular geometries is a particularly delicate problem ([1]-[19],[22]-[29]), and the Sommerfeld-Maliuzhinets representation of field, when it is associated to Maliuzhinets inversion [3]-[4] and single face representation [7]-[8] methods, is a powerful tool to investigate the acoustic and electromagnetic waves diffraction by a complex wedge-shaped region, in particular based on remarkable properties of spectral functions in this representation. In first step of this chapter, we review some general expressions and properties in scalar case for these problems : general properties in complex plane, spectral representation of Green function, single face representation of spectral function and consequences, are analysed for scatterers with imperfectly reflecting surfaces that can extend to infinity. In a second step, we consider the general solution for a wedge with face impedances of arbitrary signs (passive or active case), then the solution for the diffraction of a skew incident plane wave by a passive anisotropic impedance wedge of any angle in vectorial case [9] (2D 1/2 electromagnetism problems), and detail efficient expansions of special functions used for them and their properties. In a third step, an explicitly causal representation of field in time domain is developed within a large domain of validity, including the case of a dispersive wedge with multimode boundary conditions.

## 2) Basic properties of spectral function in Sommerfeld-Maliuzhinets representation

### 2.1) Sommerfeld Maliuzhinets representation for scattering by a wedge-shaped sector

Let us consider the case of diffraction in free space of a scalar incident plane wave,

$$u^i(\rho, \varphi) = U^i(\omega) e^{ik\rho \cos(\varphi - \varphi_0)}, \quad (1)$$

by a scatterer (finite or infinite) enclosed in a wedge-shaped region, defined in cylindrical coordinates  $(\rho, \varphi)$  as the domain outside the free space angular sector  $-\Phi_r \leq \varphi \leq \Phi_l$  with origin  $O$ , while  $z$  axis is defined as along the edge of the sector. The angle of illumination  $\varphi_0$  is considered with  $-\Phi_r \leq \varphi_0 \leq \Phi_l$ , and the plane wave field  $u^i$  verifies the Helmholtz equation  $(\Delta + k^2)u^i(\rho, \varphi) = 0$  in whole space. An implicit harmonic dependence on time  $e^{i\omega t}$  is understood and henceforth suppressed with  $|\arg(i\omega)| \leq \frac{\pi}{2}$ , and  $k = \frac{\omega}{c}$  denotes the wave number of the exterior medium with  $|\arg(ik)| < \frac{\pi}{2}$ . Generally,

$|\arg(ik)| < \frac{\pi}{2}$  is most often associated to some losses in free space, while  $|\arg(ik)| = \frac{\pi}{2}$  is considered as a limit case.

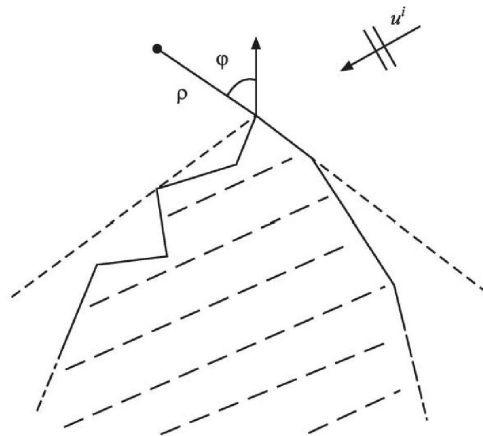


figure 1 : Geometry for a wedge-shaped region

The term  $U^i(\omega)$  is considered as an analytical function, regular for  $\text{Im}\omega \leq 0$  and  $O(\omega^{-b})$  with  $b > 1$  as  $|\text{Re}\omega| \rightarrow \infty$ . The scattered field  $u_{sc}$  can be then assumed with the same behaviour, as analytical for  $\text{Im}\omega \leq 0$  and  $O(\omega^{-v})$  with  $v > 1$  as  $|\text{Re}\omega| \rightarrow \infty$ . The characteristics of the scatterer are supposed to be independent of  $z$  coordinate and the total field in the free space region,  $u = u_{sc} + u^i$ , satisfies the Helmholtz equation [1][2],

$$(\Delta + k^2)u(\rho, \varphi) = 0. \quad (2)$$

in the free space, as  $|\arg(ik)| < \frac{\pi}{2}$  (note :  $|\arg(ik)| = \frac{\pi}{2}$  is considered in the sense of the limit). It is analytic with respect to  $\rho$ ,  $\varphi$  and  $\varphi_0$ , except possibly at the origin, and it exists a constant  $s_0$  such that  $\int_0^\infty |u(\rho, \varphi)e^{-s_0\rho}|d\rho < \infty$ . The problem can be considered in acoustic, but also in electromagnetism when we consider  $u$  as the  $z$ -component of the electric or the magnetic field parallel to the edge of the wedge.

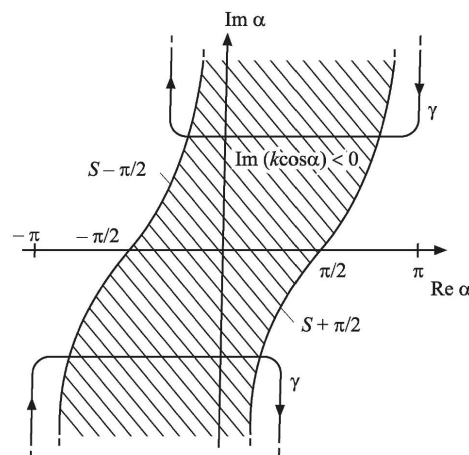


figure 2 : Complex path of integration

We can then search the total field  $u$  for  $-\Phi_r \leq \varphi \leq \Phi_l$  as a Sommerfeld-Maliuzhinets integral [3],[8],

$$u(\rho, \varphi) = \frac{1}{2\pi i} \int_{\gamma} f(\alpha + \varphi, \varphi_0) e^{ik\rho \cos\alpha} d\alpha, \quad (3)$$

which satisfies the Helmholtz equation. In this representation,  $f$  is an analytic function and the path  $\gamma$  consists of two branches: one, named  $\gamma_+$ , going from  $(i\infty + \arg(ik) + (a_1 + \frac{\pi}{2}))$  to  $(i\infty + \arg(ik) - (a_2 + \frac{\pi}{2}))$  with  $0 < a_{1,2} < \pi$ , as  $\text{Im}\alpha \geq d > 0$ , above all the singularities of the integrand, and the other, named  $\gamma_-$ , obtained by inversion of  $\gamma_+$  with respect to  $\alpha = 0$ .

We can develop the expression, considering a deformation of  $\gamma$  to the steepest descent path  $SDP$  for large  $k\rho$ , composed of two paths  $SDP_{\pm}$  centered on the stationary phase points  $\alpha = \pm \pi$ . Assuming that, between  $\gamma$  and  $SDP$ ,  $f(\alpha + \varphi, \varphi_0) e^{ik\rho \cos\alpha}$  is meromorphic and vanishes at infinity, we can write for  $|\varphi| < \Phi$ ,

$$u(\rho, \varphi) = u_i + \sum u_r^{\pm} + \sum u_{tr}^{\pm} + \sum u_s^{\pm} + \frac{e^{-ik\rho}}{2\pi i} \sum_{\pm} \int_{SDP_{\pm}} f(\alpha + \varphi, \varphi_0) e^{ik\rho(\cos\alpha+1)} d\alpha, \quad (4)$$

where

- the terms  $u_i$ ,  $u_r^{\pm}$  and  $u_{tr}^{\pm}$ , for incident, reflected and transmitted (if any) Geometrical Optics plane waves, correspond to real poles of  $f(\alpha + \varphi)$  captured during deformation of  $\gamma$  to  $SDP$ . i.e. the respective real poles  $\alpha$ , in the band  $|\text{Re}\alpha| < \pi$ , verifying in particular,

$$\begin{aligned} \text{for } u_i : \alpha + \varphi &= \varphi_0, \text{ i.e. } \alpha = \varphi_0 - \varphi, \\ \text{for } u_r^{\pm} : \alpha + \varphi &= -\varphi_0 + 2\Phi_{\pm}, \text{ i.e. } \alpha = (\Phi_{\pm} - \varphi) + (\Phi_{\pm} - \varphi_0), \end{aligned} \quad (5)$$

for  $-\Phi_r \leq \varphi \leq \Phi_l$ . The first term gives the incident field in the illuminated zone, while the second term expresses the reflection by the semi-infinite plane parts at  $\varphi = \Phi_{\pm}$  (if any) of the scatterer enclosed by face  $\pm$ .

- the terms  $u_s^{\pm}$  correspond to complex poles of  $f(\alpha + \varphi)$ , if any, captured during deformation of  $\gamma$  to  $SDP$ , for terms of complex plane guided waves excited by the edge (note: these terms are different from creeping waves guided by faces if they are curved).

- the last term, named  $u_d$ , is principally radiated cylindrically when the observation point  $M$  at  $(\rho, \varphi)$  is far, i.e.  $k\rho = k|OM| \rightarrow \infty$ . Approximating  $f(\alpha + \varphi)$  on  $SDP_{\pm}$  in vicinity of  $\alpha = \pm \pi$ , we obtain

$$u_d(M) \sim \frac{-e^{-i\pi/4 - ik\rho}}{\sqrt{2\pi k\rho}} F(\varphi, \varphi_0), \quad (6)$$

where

$$F(\varphi, \varphi_0) = f(\pi + \varphi, \varphi_0) - f(-\pi + \varphi, \varphi_0), \quad (7)$$

is the diffraction (or far field) coefficient attached to the wedge-shaped region relatively to the origin.

Remark : Let us notice that we can add a constant or any  $2\pi$ -periodic function to  $f$  without changing  $F$  (and  $u(\rho, \varphi)$ ). Some conditions at infinity on  $f$  and on its singularities are generally added to enforce its unique determination.

Remark : Considering the Green's surface integral expression of the scattered field [1]-[2] and the behaviour of incident field at infinity as  $|\arg(ik)| < \frac{\pi}{2}$ , we note that, for  $\Phi_r = \Phi_l = \Phi > \pi/2$ , and the conditions  $\frac{\pi}{2} < \Phi \mp \varphi_0 < \frac{3\pi}{2}$ ,  $\frac{\pi}{2} < \Phi \mp \varphi < \frac{3\pi}{2}$  satisfied simultaneously for convergence of integrals, we have,

$$\begin{aligned} F(\varphi, \varphi_0) &= \frac{-1}{2} \sum_{\pm} \int_{L_{0,\infty}^{\pm}} (u \nabla_M(e^{ik\hat{\tau}.OM}) - \nabla_M(u)e^{ik\hat{\tau}.OM}) \cdot \hat{n}^{\pm} ds_M, \\ 0 &= \frac{-1}{2} \sum_{\pm} \int_{L_{0,\infty}^{\pm}} (u^i \nabla_M(e^{ik\hat{\tau}.OM}) - \nabla_M(u^i)e^{ik\hat{\tau}.OM}) \cdot \hat{n}^{\pm} ds_M, \end{aligned} \quad (8)$$

where  $L_{0,\infty}^+ \cup L_{0,\infty}^-$  encloses both faces of the physical surface of the scatterer, within the wedge-shaped region  $|\varphi| < \Phi$ ,  $\hat{\tau}$  refers to the direction of observation of angle  $\varphi$ , and  $\hat{n}^{\pm}$  is the outward normal to the face  $L_{0,\infty}^{\pm}$ , on each smooth segment composing each face.

## 2.2) Basic properties on total field $u$ and spectral function $f$

Some elementary properties can be assumed to hold :

- (a') the only incoming plane wave, from the free space sector  $-\Phi_r \leq \varphi \leq \Phi_l$ , is the incident field;
- (b') as  $\rho \rightarrow 0$ , the limit of the field  $u$  is finite and does not depend on  $\varphi$  with  $u(0, \varphi) \equiv u(0) < u_0$ , while the derivatives  $\partial_{\rho}u$  and  $\partial_{\varphi}u/\rho$  are  $O(\rho^{c_0-1}(\ln\rho)^{d_0})$ , where  $u_0$ ,  $d_0$  and  $c_0 > 0$  can be chosen as three constants independent of the pulsation  $\omega$  (note :  $\partial_{\rho}u$  and  $\partial_{\varphi}u/\rho$  are then locally summable with respect to  $\rho$  in the vicinity of the origin). This property applies for an origin taken at any point out of or upon the scatterer;
- (c') the field, except possibly its geometrical optics part when  $\text{Im}(k) \neq 0$ , does not grow at infinity. In addition, some bounds on the far field are assumed, and for large  $k\rho$ , as  $\omega$  or  $\rho$  vary, with  $|\arg(ik)| \leq \frac{\pi}{2}$ , the field is in general  $O(1 + |e^{ik\rho\cos(\varphi-\varphi_0)}|)$  when the scattering object satisfies passive boundary conditions, and more particularly, is  $O(e^{ik\rho\cos(\varphi-\varphi_0)})$  for a scatterer with passive impedance surface [3],[7]-[8].

These properties lead us to assume some elementary conditions on  $f$  [3]-[4],[8]-[9] :

- (a)  $(f(\alpha) - u^i(O)/(\alpha - \varphi_0))$  is regular at points with  $\text{Re}(\alpha)$  belonging to the free space angular sector with origin  $O$ ,  $-\Phi_r \leq \text{Re}(\alpha) \leq \Phi_l$ , which ensures (a').
- (b) there exists some constants  $g^{\pm}$ , some analytic function  $h$ , and some Maliuzhinets contour  $\gamma$  such that  $|f(\alpha + \varphi) \mp f(-\alpha + \varphi) - g^{\pm}| < |h(\alpha)|$  on and inside the loop formed by the upper branch  $\gamma_+$  of  $\gamma$ , when  $-\Phi_r \leq \varphi \leq \Phi_l$ , the function  $h$  being summable on  $\gamma_+$ , regular on and within it. Since  $\gamma$  is odd, we can add a constant to  $f$  without changing  $u$ , which implies that we can choose, from now, to define  $f$  with

$g^- = 0$ . We then note that we can write, for  $-\Phi_r \leq \varphi \leq \Phi_l$ ,

$$\begin{aligned} f(i|\ln\rho| + \varphi) - f(-i|\ln\rho| + \varphi) &= -iu(0) + O\left(\frac{\rho\partial u(\rho, \varphi)}{\partial\rho}\right), \\ f(i|\ln\rho| + \varphi) + f(-i|\ln\rho| + \varphi) &= O\left(\frac{\partial u(\rho, \varphi)}{\partial\varphi}\right), \\ \text{with } \frac{\rho\partial u}{\partial\rho} &= O(e^{c_1\ln\rho}(\ln\rho)^{d_1}), \frac{\partial u(\rho, \varphi)}{\partial\varphi} = O(e^{c_2\ln\rho}(\ln\rho)^{d_2}), \end{aligned} \quad (9)$$

as  $\rho \rightarrow 0$ , with  $u(0) = ig^+$ ,  $c_{1,2} \geq c_0 > 0$ , and that we have, for  $\alpha$  belonging to the semi-infinite domain bounded by the loop  $\gamma_+$ ,

$$\lim_{\text{Im}\alpha \rightarrow \infty} f(\alpha + \varphi) = -\lim_{\text{Im}\alpha \rightarrow \infty} f(-\alpha + \varphi) = -\frac{i}{2}u(0), \quad (10)$$

i.e., from definition of  $\gamma_+$ , for  $\arg(ik) - \frac{\pi}{2} \leq \text{Re}\alpha \leq \arg(ik) + \frac{\pi}{2}$ . This ensures (b').

- (c)  $f(\alpha + \varphi)$  has no singularity, except possibly those associated to incident, reflected, transmitted plane waves not vanishing at infinity, or bounded guided waves, in the zone with  $\text{Re}(ik\cos\alpha) \geq 0$  as  $|\text{Re}(\alpha)| \leq \pi$ ,  $-\Phi_r \leq \varphi \leq \Phi_l$ ,  $|\arg(ik)| \leq \frac{\pi}{2}$ . This ensures (c').

Remark : Letting  $\Phi_r = \Phi_l = \Phi$ , we note from (9)-(10) that, we have, as  $\arg(ik) - \frac{\pi}{2} - \Phi \leq \pm \text{Re}\alpha' \leq \arg(ik) + \frac{\pi}{2} + \Phi$ ,

$$f(\alpha') = \mp \frac{i}{2}u(0) + O(e^{\pm ic_{\pm}\alpha'}) \text{ with } c_{\pm} > 0, \text{ as } \pm \text{Im}\alpha' \rightarrow \infty. \quad (11)$$

Remark : Using (c') when  $|\arg(ik)| < \frac{\pi}{2}$ ,  $f(\alpha + \varphi)$  has no singularity in the region with  $\text{Re}(ik\cos\alpha) > 0$  as  $|\text{Re}(\alpha)| < \pi$  when  $\text{Re}(ik\cos(\varphi - \varphi_0)) < 0$ , i.e.  $\frac{\pi}{2} < |\varphi - \varphi_0| < \frac{3\pi}{2}$ .

## 2.3) Higher order expressions with Sommerfeld-Maliuzhinets integrals

### 2.3.1) Basic equalities for the differentiation of integral representation of $u$

Considering previous properties, in particular convergence at infinity of  $f(\alpha + \varphi)$  for large imaginary argument, we can obtain different reduced expressions for a differential operator applied to  $u$ , in particular, from the use of integration by parts. So, for positive or null integers  $n, m, k, p, l$ , and  $\delta = 0$  or  $1$ , we have the basic equality,

$$\begin{aligned}
& (ik\rho)^n \left(\frac{\partial}{ik\partial\rho}\right)^m (ik\rho)^k \left(\frac{\partial}{ik\rho\partial\varphi}\right)^p \left(\frac{\partial}{ik\partial\rho}\right)^l \left(\frac{\partial}{ik\rho\partial\varphi}\right)^\delta (u(\rho, \varphi)) = \\
& = \frac{1}{2\pi i} \int_\gamma e^{ik\rho\cos\alpha} \left(\frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right)\right)^n \left((\cos\alpha)^m \left(\frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right)\right)^{k-p}\right. \\
& \quad \left. \left[\frac{\partial^p f(\alpha + \varphi, \varphi_\circ)}{\partial\varphi^p} (\cos\alpha)^l (\sin\alpha)^\delta\right] d\alpha\right) \\
& = \frac{1}{2\pi i} \int_{\gamma_+} \left[ \left(\frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right)\right)^n \left((\cos\alpha)^m \left(\frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right)\right)^{k-p}\right) \right. \\
& \quad \left. [(\cos\alpha)^l (\sin\alpha)^\delta \left(\frac{\partial}{\partial\alpha}\right)^p f(\alpha + \varphi, \varphi_\circ)] \right. \\
& \quad \left. - \left(\frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right)\right)^n \left((\cos\alpha)^m \left(\frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right)\right)^{k-p}\right) \right. \\
& \quad \left. [(\cos\alpha)^l (-\sin\alpha)^\delta \left(\frac{\partial}{\partial\alpha}\right)^p f(-\alpha + \varphi, \varphi_\circ)] \right] e^{ik\rho\cos\alpha} d\alpha \quad (12)
\end{aligned}$$

where  $\left(\frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right)\right)^n \equiv \frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right) \frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right) \dots \frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right)$   $n$  times. By using (12), we note in particular that we have,

$$\begin{aligned}
& (\rho/ik)(\Delta + k^2)u = (ik\rho \left(\frac{\partial}{ik\partial\rho}\right)^2 + \frac{\partial}{ik\partial\rho} + \frac{\partial}{ik\rho\partial\varphi} \frac{\partial}{\partial\varphi} - ik\rho)u \\
& = \frac{1}{2\pi i} \int_\gamma \left[ \frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right) (f(\alpha + \varphi, \varphi_\circ)(1 - (\sin\alpha)^2)) + f(\alpha + \varphi, \varphi_\circ) \cos\alpha \right. \\
& \quad \left. + \sin\alpha \frac{\partial}{\partial\alpha} f(\alpha + \varphi, \varphi_\circ) - \frac{\partial}{\partial\alpha} \left(\frac{(\cdot)}{\sin\alpha}\right) f(\alpha + \varphi, \varphi_\circ) \right] e^{ik\rho\cos\alpha} d\alpha = 0. \quad (13)
\end{aligned}$$

in accordance with the Helmholtz equation verified by  $u$ , which implies the equality,

$$\left(\frac{\partial}{\rho\partial\rho} + \left(\frac{\partial}{\rho\partial\varphi}\right)^2\right)u = -(k^2 + \left(\frac{\partial}{\partial\rho}\right)^2)u.$$

In complement, considering cartesian coordinates  $(x^\pm, y^\pm)$  attached to each faces, with  $y^\pm$  axes respectively directed as the outward normal to the faces  $\varphi = \pm\Phi$ , we note that,

$$\begin{aligned}
& \frac{\partial u}{\partial y^\pm} = \mp \frac{\partial u}{\rho\partial\varphi}, \left(\frac{\partial}{\partial y^\pm}\right)^2 u = -(k^2 + \left(\frac{\partial}{\partial\rho}\right)^2)u \text{ for } \varphi = \pm\Phi, \\
& \left(\frac{\partial u}{ik\partial y^\pm}\right)^n|_{\varphi=\pm\Phi} = \frac{1}{2\pi i} \int_\gamma (\mp \sin\alpha)^n f(\alpha \pm \Phi, \varphi_\circ) e^{ik\rho\cos\alpha} d\alpha, \\
& \left(\frac{\partial u}{ik\partial x^\pm}\right)^n|_{\varphi=\pm\Phi} = \frac{1}{2\pi i} \int_\gamma (\mp \cos\alpha)^n f(\alpha \pm \Phi, \varphi_\circ) e^{ik\rho\cos\alpha} d\alpha. \quad (14)
\end{aligned}$$

Therefore, we can exploit these reductions for derivatives of fields, in conjunction with the Maliuzhinets inversion theorem, to determine functional equations on  $f$  from differential boundary equations.

Remark : Considering  $y^\pm = \mp \rho \sin(\varphi \mp \Phi)$  and  $x^\pm = \mp \rho \cos(\varphi \mp \Phi)$ , we note that,

$$\begin{aligned} \frac{\partial}{\partial y^\pm} &= \mp \cos(\varphi \mp \Phi) \frac{\partial}{\rho \partial \varphi} \mp \sin(\varphi \mp \Phi) \frac{\partial}{\partial \rho}, \\ \frac{\partial}{\partial x^\pm} &= \pm \sin(\varphi \mp \Phi) \frac{\partial}{\rho \partial \varphi} \mp \cos(\varphi \mp \Phi) \frac{\partial}{\partial \rho}. \end{aligned} \quad (15)$$

### 2.3.2) Basic example of determination of functional equations on $f$

For example of use of (12), we can consider a wedge with general high order boundary conditions on plane faces  $\varphi = \pm \Phi$  (taking  $\Phi_r = \Phi_l = \Phi$ ), that are combinations of higher order derivatives of fields [16]-[18], following,

$$\sum_{n,m,k,p,\delta} a_{n,m,k,p,\delta}^\pm \left( (ik\rho)^n \left( \frac{\partial}{ik\partial\rho} \right)^m (ik\rho)^k \left( \frac{\partial}{ik\rho\partial\varphi} \right)^p \left( \frac{\partial}{ik\rho\partial\varphi} \right)^\delta u(\rho, \varphi) \right) |_{\varphi=\pm\Phi} = 0, \quad (16)$$

where  $n, m, k, p$  and  $\delta$  are positive or null integers, with  $\delta \leq 1$ . The coefficients  $a_{n,m,k,p,\delta}^\pm$  are generally obtain from asymptotic considerations on losses, permittivities and permeabilites of materials, and geometry near the edge. Using an approach similar (slightly modified) to the one used by Maliuzhinets in [3] for its inversion theorem, we can exploit (12) and then derive functional equations on  $f(\alpha)$  in the complex plane.

For that, considering each sum in (16) as  $O(\rho^{-Q^\pm-d})$  in vicinity of the origin, with  $0 \leq d < 1$  and  $Q^\pm$  integer, we multiply (16) by  $\rho^{Q^\pm+1}$ , and using (12), apply a Laplace transform on  $\rho$ . Considering the behaviour of integrand in the loop  $\gamma_+$ , we can close the loop  $\gamma_+$  at infinity, and derive, from the residue theorem, the functional equations on  $f$ ,

$$\left( \frac{\partial}{\partial \alpha} \left( \frac{(\cdot)}{\sin \alpha} \right) \right)^{Q^\pm+1} \sum_{n,m,k,p,\delta}^{N,M,K,P,1} a_{n,m,k,p,\delta}^\pm (D_\alpha^\pm(f(\alpha \pm \Phi)) - D_{-\alpha}^\pm(f(-\alpha \pm \Phi))) = 0, \quad (17)$$

where  $n,m,k,p$  and  $\delta$  are positive or null integers, with  $\delta \leq 1$ ,

$$\begin{aligned} D_\alpha^\pm(f(\alpha \pm \Phi)) &= \\ &\left( \frac{\partial}{\partial \alpha} \left( \frac{(\cdot)}{\sin \alpha} \right) \right)^n [(\cos \alpha)^m \left( \frac{\partial}{\partial \alpha} \left( \frac{(\cdot)}{\sin \alpha} \right) \right)^{k-p} ((\sin \alpha)^\delta \left( \frac{\partial}{\partial \alpha} \right)^p f(\alpha \pm \Phi, \varphi_\circ))] \end{aligned}$$

which implies, after integration,

$$\begin{aligned} &\sum_{n,m,k,p,\delta}^{N,M,K,P} a_{n,m,k,p,\delta}^\pm (D_\alpha^\pm(f(\alpha \pm \Phi)) - D_{-\alpha}^\pm(f(-\alpha \pm \Phi))) \\ &= \sum_{q=1}^{Q^\pm+1} b_q^\pm \sin \alpha (\cos \alpha)^{q-1}, \end{aligned} \quad (18)$$

Considering  $u$  finite at the origin, we can use (11) in (18), and comparing each side for  $\text{Im} \alpha \rightarrow \infty$ , we obtain  $Q^\pm + 1 = \max_{a_{n,m,k,p,\delta}^\pm \neq 0} (m + p - k - n + \delta)$ .



This type of (difference) functional equations, encountered for wedges with plane faces in [4],[16]-[18], can also be derived asymptotically from conditions on curved faces. By analytical continuation, these equations permit a determination everywhere from the one in the band  $|\operatorname{Re}\alpha| \leq \Phi$ . Furthermore, the constants  $b_q^\pm$  for  $1 \leq q < Q^\pm + 1$  are attached to additional conditions on  $u$  at the edge, that we can relate to conditions at infinity on  $f$  satisfying (18), and, in complement, to reciprocity principle (if it applies) on  $u$  [18].

Remark : if we use that  $f(\alpha + \Phi) + f(-\alpha + \Phi) \rightarrow 0$  for  $|\operatorname{Im}\alpha| \rightarrow \infty$  on and inside the loops  $\gamma_\pm$  of  $\gamma$ , we notice that  $b_{M+1}^\pm = 0$  if  $N = K = P = 0$ .

### 3) Spectral function attached to radiation of a single face and properties

#### 3.1) Basic integral expression of radiation of one face

We now apply Green's theorems [1],[2], when applied to Helmholtz equation, to derive important properties on representation of spectral functions attached to the diffraction of a 2D polygonal scatterer. To simplify the notation without losing generality, we take from now  $\Phi_r = \Phi_l = \Phi$ . From the properties (b'), (c') on  $u$ , the scattered field in free space  $u_{sc} = u - u^i$  for  $|\varphi| < \Phi$  can be written as the sum  $u_+ + u_-$  of the radiations of equivalent surface currents carried by the faces  $\varphi = +\Phi$  and  $-\Phi$ , following,

$$\begin{aligned} u_\pm(\rho, \varphi) &= \frac{-i}{4} \int_{0^+}^{\infty} (u(\rho', \varphi') \frac{\partial H_0^{(2)}(kR)}{\partial n} - \frac{\partial u(\rho', \varphi')}{\partial n} H_0^{(2)}(kR))|_{\varphi'=\pm\Phi} d\rho' \\ &= \frac{-i}{4} \int_{0^+}^{\infty} (u(\rho', \varphi') \nabla H_0^{(2)}(kR) - \nabla(u(\rho', \varphi')) H_0^{(2)}(kR))|_{\varphi'=\pm\Phi} \cdot \hat{n} d\rho', \end{aligned} \quad (19)$$

with  $R = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi')}$ ,  $\partial(\cdot)/\partial n = \hat{n} \nabla(\cdot) = \mp \partial(\cdot)/\rho' \partial \varphi'$ ,  $\hat{n}$  the outward normal to the face  $\varphi' = \pm\Phi$ ,  $|\varphi| < \Phi$ ,  $|\arg(ik)| < \frac{\pi}{2}$ , and where  $H_0^{(2)}$  is the Hankel function of second kind [20].

We will show that it is possible, after expressing the spectral function corresponding to  $u_\pm$  with  $f$ , to express  $f$  for the total field from the radiation of a single face in far field. This expression will then have the remarkable property, from Green's theorem, to have a path of integration that can be deformed without changing the result. For this, we begin by using a Sommerfeld-Maliuzhinets representation of  $H_0^{(2)}(kR)$ .

Remark : The possibility to deform integration paths without change of result is well-known in Cauchy theorem for complex plane, and in Green's theorem for 2D/3D spaces.

#### 3.2) Spectral function associated to $H_0^{(2)}(kR)$

As shown in [7],[8],[13] the spectral function associated to Sommerfeld-Maliuzhinets representation of  $H_0^{(2)}(kR)$ , is given by,

$$f_{H_0^{(2)}}(\alpha) = \frac{-1}{2\pi} \int_{\mathcal{S}} e^{ik\rho'\cos(\alpha'-\varphi')} (\tan(\frac{1}{2}(\alpha - \alpha')) - g_o(\alpha')) d\alpha', \quad (20)$$

where  $\mathcal{S}$  is the path from  $-i\infty - \arg(ik)$  to  $i\infty + \arg(ik)$  with  $\text{Im}(k\sin\alpha) = 0$ , as  $R = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi - \varphi')}$ ,  $|\varphi'| > \frac{\pi}{2}$ ,  $|\varphi| < |\varphi'|$ . The terms  $g_o$ , normally unnecessary because  $\gamma$  is odd, can be chosen as  $g_o(\alpha) = -\tan(\frac{\alpha}{2})$  in order to ameliorate the convergence of the integral. The expression (20) is a determination for  $|\varphi'| > \frac{\pi}{2}$ , regular as  $\alpha \in ]\mathcal{S} - \pi, \mathcal{S} + \pi[$  (i.e. the domain in complex plane limited by  $\mathcal{S} - \pi$  and  $\mathcal{S} + \pi$ ), and verifying (after capture of poles of  $\tan(\frac{1}{2}(\alpha - \alpha'))$  as  $\alpha$  varies),

$$f_{H_0^{(2)}}(\pi + \alpha') - f_{H_0^{(2)}}(-\pi + \alpha') = -2ie^{ik\rho'\cos(\alpha'-\varphi')}, \quad (21)$$

which is a functional equation that can be considered, by argument of analiticity, in complement of (20) for a determination of  $f_{H_0^{(2)}}$  everywhere as  $\alpha$  varies.

The path  $\mathcal{S}$  can be also deformed continuously, as long as the integrand remains bounded in (20), without changing  $f_{H_0^{(2)}}$ , and this expression can be then continued for  $|\varphi'| \leq \frac{\pi}{2}$  from shift of  $\mathcal{S}$  and summation of residues due to poles of  $\tan(\frac{1}{2}(\alpha - \alpha'))$  captured by  $\mathcal{S}$ .

### 3.3) Expression of spectral function associated to $u_{\pm}$ and properties

Therefore, we can write  $f_{H_0^{(2)}}$  corresponding to  $H_0^{(2)}(kR)$  for arbitrary  $\varphi'$ . Letting  $\varphi' = \epsilon\Phi$ , with  $\epsilon \equiv +$  or  $-$ , and  $\Phi > 0$ , a real positive constant, we can write the spectral function  $f_{H_0^{(2)}}^{\epsilon} \equiv f_{H_0^{(2)}}^{\epsilon}$ , associated to  $H_0^{(2)}(kR_{\epsilon})$  with

$$R_{\epsilon} = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi - \epsilon\Phi)}, \quad (22)$$

following,

$$f_{H_0^{(2)}}^{\epsilon}(\alpha) = \frac{-1}{2\pi} \int_{\mathcal{S}_{\epsilon}} e^{ik\rho'\cos(\alpha'-\epsilon\Phi)} (\tan(\frac{1}{2}(\alpha - \alpha')) - g_o(\alpha')) d\alpha', \quad (23)$$

where  $\mathcal{S}_{\epsilon} = \mathcal{S} + a_{\epsilon}$ , with  $a_{\epsilon}$  a constant satisfying  $\Phi - \frac{3\pi}{2} < \epsilon a_{\epsilon} < \Phi - \frac{\pi}{2}$ . This gives us a regular value of  $f_{H_0^{(2)}}^{\epsilon}(\alpha)$  between  $\mathcal{S} + \epsilon(\Phi - \frac{5\pi}{2})$  and  $\mathcal{S} + \epsilon(\Phi + \frac{\pi}{2})$ , bounded at infinity in this domain if  $\rho' \neq 0$ , verifying everywhere from (21),

$$f_{H_0^{(2)}}^{\epsilon}(\pi + \alpha') - f_{H_0^{(2)}}^{\epsilon}(-\pi + \alpha') = -2ie^{ik\rho'\cos(\alpha'-\epsilon\Phi)}, \quad (24)$$

which allows an explicit Sommerfeld Maliuzhinets integral representation for  $H_0^{(2)}(kR_{\epsilon})$  as  $|\varphi| < \Phi$ .

Using the expression (23) for  $H_0^{(2)}(kR_\epsilon)$  and  $\partial_n H_0^{(2)}(kR_\epsilon)$  in (19), we then obtain a Sommerfeld-Maliuzhinets representation of  $u_\pm$  with  $f$  [7],[8]:

$$u_\pm(\rho, \varphi) = \frac{1}{2\pi i} \int_\gamma f_\pm(\alpha + \varphi) e^{ik\rho \cos\alpha} d\alpha, \quad (25)$$

where

$$\begin{aligned} f_\epsilon(\alpha) &= \frac{1}{4\pi i} \int_{\mathcal{S}_\epsilon} \epsilon f(\epsilon\pi + \alpha') (\tan(\frac{1}{2}(\alpha - \alpha')) - g_\circ(\alpha')) d\alpha', \\ f_\epsilon(\alpha + \varphi) - f_\epsilon(-\alpha + \varphi) &= \frac{1}{2\pi i} \int_{\mathcal{S}_\epsilon} \frac{\epsilon f(\epsilon\pi + \alpha') \sin\alpha}{\cos\alpha + \cos(\varphi - \alpha')} d\alpha', \end{aligned} \quad (26)$$

for  $\alpha$  between  $\mathcal{S}_\epsilon - \pi$  and  $\mathcal{S}_\epsilon + \pi$ , provided  $\frac{\pi}{2} < \Phi - \epsilon a_\epsilon < \frac{3\pi}{2}$ ,  $\frac{\pi}{2} < \Phi - \epsilon\varphi_\circ < \frac{3\pi}{2}$  with  $\epsilon \equiv +$  or  $-$ .

Since  $f$  is an analytic function,  $f_\epsilon$  can be analytically continued in the whole complex plane. We note in particular, that, taking account of the poles of  $\tan(\frac{1}{2}(\alpha - \alpha'))$  which can be captured by  $\mathcal{S}_\epsilon$  as  $\alpha$  varies,  $f_\pm$  satisfies

$$f_\pm(\pi + \alpha) - f_\pm(-\pi + \alpha) = \pm f(\pm\pi + \alpha). \quad (27)$$

Concerning the dependence on  $\varphi_\circ$  (or  $\Phi$ ), the expression (26) has been determined for  $\frac{\pi}{2} < \Phi - \epsilon\varphi_\circ < \frac{3\pi}{2}$ , but we can consider  $f_\epsilon(\alpha)$  outside this domain by analytical continuation on  $\varphi_\circ$  (or  $\Phi$ ), which corresponds to take account of the contribution of any singularity that would go through  $\mathcal{S}_\epsilon$  as  $\varphi_\circ$  (or  $\Phi$ ) goes into these regions.

Remark : As  $\sum_{\pm} f(\epsilon\pi + \epsilon a_\epsilon \pm (\alpha' - \epsilon a_\epsilon))$  exponentially vanishes for large  $\alpha'$  on  $\mathcal{S}_\epsilon$  (see property (b) considered on  $f$  and (11)), we note that  $f_\epsilon(\alpha + \varphi) - f_\epsilon(-\alpha + \varphi)$  is bounded for large  $\alpha$  on  $\gamma$ , even if  $f_\epsilon(\alpha + \varphi)$  is not when  $f(\pm i\infty) \neq 0$ . This property is in accordance with the expression  $u_\pm$  given by (19), which is finite as  $\rho \rightarrow 0$ , while its normal derivative is not when  $u$  is non null at the origin.

#### 4) Far field radiation of one face and single face expression of spectral function $f$

##### 4.1) The spectral function $f$ derived from fields on a semi straight line and properties

The steepest descent path method [1]-[2] can be applied to (25) to find the far field radiation of the face  $\varphi = \epsilon\Phi$ , also denoted  $\varphi = \pm\Phi$ . From the regularity of  $f_\epsilon$ , we can deform  $\gamma$  to stationary phase points  $\alpha = +\pi$  and  $-\pi$ , when  $\frac{\pi}{2} < \Phi - \epsilon\varphi_\circ < \frac{3\pi}{2}$  and  $\frac{\pi}{2} < \Phi - \epsilon\varphi < \frac{3\pi}{2}$ , and thus, out of the reflected and shadowed regions

$2\Phi - \epsilon(\varphi + \varphi_0) < \pi$  and  $|\varphi - \varphi_0| > \pi$ . In this case, the equation (25) then gives us,

$$u_{\pm}(\rho, \varphi) = \frac{-e^{-i\pi/4}}{\sqrt{2\pi k\rho}} e^{-ik\rho} (f_{\pm}(\pi + \varphi) - f_{\pm}(-\pi + \varphi) + O(1/(k\rho))), \quad (28)$$

which, from (27), can be written,

$$u_{\pm}(\rho, \varphi) = \frac{-e^{-i\pi/4}}{\sqrt{2\pi k\rho}} e^{-ik\rho} (\pm f(\pm\pi + \varphi) + O(1/(k\rho))). \quad (29)$$

At this step, let us notice that this result is a generalization of Michaeli's result [19] obtained for Dirichlet boundary conditions. In other respects, we can consider the far field derived from (19), using formulas [20],

$$\begin{aligned} H_0^{(2)}(kR) &= \sqrt{2/(\pi kR)} e^{-ikR+i\pi/4} (1 + O(1/(kR))), \\ \partial_n H_0^{(2)}(kR) &= -i\sqrt{2/(\pi kR)} e^{-ikR+i\pi/4} \partial_n(kR) (1 + O(1/(kR))), \end{aligned} \quad (30)$$

for large  $R$ , with  $\partial_n(kR) = \mp \partial(kR)/\rho' \partial\varphi'|_{\varphi=\pm\Phi} = (\pm k\rho \sin(\varphi - \varphi')/R)|_{\varphi=\pm\Phi}$ ,  $R = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi')}$ . Taking into account that

$$R = \rho - \rho' \cos(\varphi \mp \Phi) + \frac{(\rho' \sin(\varphi \mp \Phi))^2}{R + \rho - \rho' \cos(\varphi \mp \Phi)}, \quad (31)$$

when  $R + \rho - \rho' \cos(\varphi \mp \Phi) \neq 0$ , and considering the properties (b)-(c) on  $u$ , we then obtain the expression for the far field from (19) which, compared with (29), gives us :

$$\begin{aligned} f(\pm\pi + \varphi) &= \frac{1}{2} \int_{0+}^{\infty} (iku(\rho', \pm\Phi) \sin(\varphi \mp \Phi) \\ &\quad \pm \frac{\partial u}{\partial n}(\rho', \pm\Phi)) e^{ik\rho' \cos(\varphi \mp \Phi)} d\rho', \end{aligned} \quad (32)$$

as  $\frac{\pi}{2} < \Phi \mp \varphi_0 < \frac{3\pi}{2}$  and  $\frac{\pi}{2} < \Phi \mp \varphi < \frac{3\pi}{2}$  with  $|\arg(ik)| < \frac{\pi}{2}$ , i.e. as  $e^{ik\rho' \cos(\varphi \mp \Phi)}$  and  $e^{ik\rho' \cos(\varphi_0 \mp \Phi)}$  vanish at infinity. We can also consider the field along the line  $\varphi = \pm\Phi_e^{\pm}$  with  $0 < \Phi_e^{\pm} \leq \Phi$ , letting  $\pm\pi + \varphi = \alpha \pm \Phi_e^{\pm}$ . In this case, we can write, when  $\frac{\pi}{2} < \Phi_e^{\pm} \mp \varphi_0 < \frac{3\pi}{2}$  and  $\frac{\pi}{2} < \mp\alpha + \pi < \frac{3\pi}{2}$ ,

$$\begin{aligned} f(\alpha \pm \Phi_e^{\pm}) &= \frac{1}{2} \int_{0+}^{\infty} (-iku(\rho', \pm\Phi_e^{\pm}) \sin\alpha \pm \frac{\partial u}{\partial n}(\rho', \pm\Phi_e^{\pm})) e^{-ik\rho' \cos\alpha} d\rho' \\ &= \frac{1}{2} \int_{0+}^{\infty} (-iku(\rho', \pm\Phi_e^{\pm}) \sin\alpha - \frac{\partial u}{\rho \partial \varphi}(\rho', \pm\Phi_e^{\pm})) e^{-ik\rho' \cos\alpha} d\rho', \end{aligned} \quad (33)$$

which is also valid, by analytic continuation, as  $\text{Re}(ik(\cos\alpha - \cos(\Phi_e^{\pm} \mp \varphi_0))) > 0$ ,  $|\text{Re}\alpha| < \pi$ . This expression implies that,

$$\begin{aligned} f(ix \pm \Phi_e^{\pm}) - f(-ix \pm \Phi_e^{\pm}) &= -iu(0, \pm\Phi_e^{\pm}) + O\left(\frac{\partial u(\rho, \pm\Phi_e^{\pm})}{\partial \ln \rho} \Big|_{|\ln \rho|=x}\right), \\ f(ix \pm \Phi_e^{\pm}) + f(-ix \pm \Phi_e^{\pm}) &= O\left(\frac{\partial u(\rho, \varphi)}{\partial \varphi} \Big|_{|\ln \rho|=x, \varphi=\pm\Phi_e^{\pm}}\right), \end{aligned} \quad (34)$$

as  $\rho \rightarrow 0$  and  $x = |\ln \rho| \rightarrow \infty$ , which is in accordance with the property (b) of the spectral function  $f$  attached to the field  $u$ .

Thus, the definition of the field and its normal derivatives on one single face (a semi-line) leads us to a (explicit) definition of a unique analytic function  $f$  attached to the total radiated field (uniqueness and existence), with  $f(ix \pm \Phi_e^\pm) + f(-ix \pm \Phi_e^\pm)$  vanishing at infinity as  $\frac{\pi}{2} < \Phi_e^\pm \mp \varphi_o < \frac{3\pi}{2}$ ,  $0 < \Phi_e^\pm \leq \Phi$ , in accordance with condition (b).

Remark : From (33), valid as  $\text{Re}(ik(\cos\alpha - \cos(\Phi_e^\pm \mp \varphi_o))) > 0$ , in the band  $|\text{Re}\alpha| < \pi$ , for  $0 < \Phi_e^\pm \leq \Phi$ , we can write in this band, as  $\text{Re}(ik\cos\alpha) > \text{Re}(ik)$ ,  $|\varphi| \leq \Phi$ ,

$$f(\alpha + \varphi) = \frac{1}{2} \int_{0^+}^{\infty} (-iku(\rho, \varphi)\sin\alpha - \frac{\partial u}{\rho \partial \varphi}(\rho, \varphi))e^{-ik\rho\cos\alpha} d\rho. \quad (35)$$

Remark : It is possible to recover (33) by another way, as already noticed in [7]. For this, we can use the inversion theorem of Maliuzhinets [3] (taking care about time convention) on  $u(\rho, \pm \Phi)$  and  $\partial u / \partial n(\rho, \pm \Phi)$  to obtain,

$$\begin{aligned} f(\alpha + \varphi) - f(-\alpha + \varphi) &= -ik \int_{0^+}^{\infty} u(\rho, \varphi)\sin\alpha e^{-ik\rho\cos\alpha} d\rho, \\ f(\alpha + \varphi) + f(-\alpha + \varphi) &= - \int_{0^+}^{\infty} \frac{\partial u}{\rho \partial \varphi}(\rho, \varphi) e^{-ik\rho\cos\alpha} d\rho, \end{aligned} \quad (36)$$

for  $\text{Re}(ik\rho\cos\alpha) > \text{Re}(ik\rho)$  as  $|\text{Re}\alpha| < \pi$ . Elementary combinations and analyticity then permit us to recover (33). This way is not used in this section because it does not allow to simply relate (33) to Green's theorem, which is a key point to derive the remarkable property of deformation of the integration path given in next section.

## 4.2) The spectral function $f$ derived from fields on a piecewise smooth semi line

### 4.2.1) Single face expressions of $f$ along $L_{0,\infty}^\pm$ with the use of tangent angles

Using the properties of Green's theorem [1], it is worth noticing [7]-[8] that the contour of integration along  $\varphi = \pm \Phi$  in (19) can be deformed into any path  $L_{0,\infty}^\pm$  without changing the field  $u_\pm(M)$ , provided that  $L_{0,\infty}^\pm$  is a piecewise smooth semi line, and that, during the deformation, the integral remains definite,  $M$  is not captured by the path during its deformation, and no source passes through the path. Afterwards, we can write (32) in a more general form, with integration along the new general path  $L_{0,\infty}^\pm$ , a piecewise smooth semi line from the origin to infinity, following,

$$\begin{aligned} f(\pm \pi + \varphi) &= \frac{1}{2} \int_{L_{0,\infty}^\pm} (iku(\rho', \varphi')\sin(\varphi - \varphi'_t) \\ &\pm \frac{\partial u}{\partial n}(\rho', \varphi'))e^{ik\rho'\cos(\varphi - \varphi')} dl'(\rho', \varphi'), \end{aligned} \quad (37)$$

where  $dl'(\rho', \varphi')$  is the element of length,  $\frac{\partial}{\partial n}$  is the outward normal derivative, and  $\varphi'_t$  is the tangent angle on each smooth segment composing  $L_{0,\infty}^\pm$ .

If we divide the piecewise smooth semi line  $L_{0,\infty}^\pm$  (deriving from a deformation of the faces  $\varphi = \pm \Phi$  enclosing the scatterer, described above) into  $L_{0,\Delta^\pm}^\pm$  (i.e.  $0 < l' < \Delta^\pm$ ) and  $L_{\Delta^\pm,\infty}^\pm$  (i.e.  $l' > \Delta^\pm$ ), we have

$$\begin{aligned} f(\pm \pi + \varphi) &= \frac{1}{2} \int_{L_{0,\Delta^\pm}^\pm} (iku(\rho', \varphi') \sin(\varphi - \varphi'_t) \\ &\pm \frac{\partial u}{\partial n}(\rho', \varphi')) e^{ik\rho' \cos(\varphi - \varphi')} dl'(\rho', \varphi') + f_{L_{\Delta^\pm,\infty}^\pm}(\pm \pi + \varphi), \end{aligned} \quad (38)$$

where  $f_{L_{\Delta^\pm,\infty}^\pm}(\alpha) = e^{-ik\rho_{\Delta^\pm} \cos(\alpha - \varphi_{\Delta^\pm})} f_e^\pm(\alpha)$ ,  $f_e^\pm(\alpha)$  is the spectral function related to the Sommerfeld-Maliuzhinets representation of the field in coordinates with origin at points  $(\rho_{\Delta^\pm}, \varphi_{\Delta^\pm})$  with  $l' = \Delta^\pm$ . We can then write, by analytic continuation,

$$\begin{aligned} f(\alpha) &= \frac{1}{2} \int_{L_{0,\Delta^\pm}^\pm} (-iku(\rho', \varphi') \sin(\alpha - \varphi'_t) \\ &\pm \frac{\partial u}{\partial n}(\rho', \varphi')) e^{-ik\rho' \cos(\alpha - \varphi')} dl'(\rho', \varphi') + f_{L_{\Delta^\pm,\infty}^\pm}(\alpha). \end{aligned} \quad (39)$$

#### 4.2.2) Single face expressions of $f$ along $L_{0,\infty}^\pm$ with the use of gradient

Using the expression of normal derivative with gradient, we derive another form of the expression of the spectral function associated to total field, in presence of a scatterer enclosed by a surface with piecewise smooth faces  $L_{0,\infty}^\pm$ , ended by semi-infinite planes with tangent angles  $\pm \Phi^\pm$  (note : in the case of a closed scatterer, we can take  $\Phi^\pm = \pi$ ). Therefore, the spectral function  $f$  satisfies the single-face expressions, as  $\frac{\pi}{2} < \Phi^\pm \mp \varphi_o < \frac{3\pi}{2}$  and  $\frac{\pi}{2} < \Phi^\pm \mp \varphi < \frac{3\pi}{2}$  [8], following,

$$\begin{aligned} f(\pm \pi + \varphi) &= \frac{1}{2} \int_{L_{0,\infty}^\pm} \mp (u(M) \nabla_M (e^{ik\hat{\tau} \cdot \overline{OM}}) \\ &- \nabla_M (u(M)) e^{ik\hat{\tau} \cdot \overline{OM}}) \cdot \hat{n}^\pm ds(M), \end{aligned} \quad (40)$$

where  $\hat{\tau}(\varphi) = \cos\varphi \hat{x} + \sin\varphi \hat{y}$  refers to the direction of observation of angle  $\varphi$ , and  $\hat{n}^\pm ds = \mp (\hat{z} \wedge d(\overline{OM}))$  on each smooth segment composing  $L_{0,\infty}^\pm$ . This expression is in agreement with (37) and the expression (8) of  $F(\varphi, \varphi_o) = f(\pi + \varphi) - f(-\pi + \varphi)$ . As for (39), we then derive,

$$\begin{aligned} f(\alpha) &= \frac{1}{2} \int_{L_{0,\Delta^\pm}^\pm} \mp (u(M) \nabla_M (e^{-ik\hat{\tau}_e \cdot \overline{OM}}) \\ &- \nabla_M (u(M)) e^{-ik\hat{\tau}_e \cdot \overline{OM}}) \cdot \hat{n}^\pm ds(M) + f_{L_{\Delta^\pm,\infty}^\pm}(\alpha), \end{aligned} \quad (41)$$

where  $\widehat{\tau}_e$  refers to the complex angle  $\alpha$ , with  $\widehat{\tau}_e = \cos\alpha \widehat{x} + \sin\alpha \widehat{y}$  verifying  $\widehat{\tau}_e \cdot \widehat{\tau}_e = 1$  (note : we use  $\widehat{\tau}_e = -\widehat{\tau}(\mp \pi + \alpha)$ , with scalar product of complex vectors defined by analytic continuation (no hermitian product and no conjugate)).

Remark : When  $L_{0,\infty}^\pm$  has a succession of singularities at points  $M_n$ , we can consider  $L_{0,\infty}^\pm$  as the addition of curved line segments without  $M_n$ , i.e. the integral as the addition of improper (or limit) integrals on open paths, in the sense that the ends of paths are not strictly considered at  $M_n$  but as  $M \rightarrow M_n$ .

## 5) Expression of $f$ from the far field function $F$ and its consequences

### 5.1) Integral expression of $f$ relatively to $F$

Considering (25), we can write for the spectral function  $f$  attached to the total field,

$$f(\alpha) = f_+(\alpha) + f_-(\alpha) + f_i(\alpha) - A_F, \quad (42)$$

where  $f_i(\alpha) = \frac{\text{Res}f|_{\alpha=\varphi_0}}{2} \cot(\frac{1}{2}(\alpha - \varphi_0))$ , the  $f_\pm$  verify (26) and (27), while,

$$A_F = \frac{1}{2} \sum_{\pm} \lim_{\text{Im}\alpha \rightarrow \pm\infty} (f_+(\alpha) + f_-(\alpha)),$$

so that  $f(i\infty) = -f(-i\infty)$  as specified in the condition (b). The constant  $A_F$  does not change the field since  $\gamma$  is odd, and is only here to define uniquely  $f$  with  $f(i\infty) = -f(-i\infty)$ .

When both conditions  $\frac{\pi}{2} < \Phi \mp \varphi_0 < \frac{3\pi}{2}$  are satisfied (permitted with  $\pi > \Phi > \pi/2$ ), we then obtain, from (26) and (27), for  $\alpha \in ]\mathcal{S} - \pi, \mathcal{S} + \pi[$ ,

$$\begin{aligned} f(\alpha, \varphi_0) &= \frac{1}{4\pi i} \int_{\mathcal{S}} F(\alpha', \varphi_0) (\tan(\frac{1}{2}(\alpha - \alpha')) - g_0(\alpha')) d\alpha' \\ &+ \frac{\text{Res}f|_{\alpha=\varphi_0}}{2} \cot(\frac{1}{2}(\alpha - \varphi_0)) - A_F, \end{aligned} \quad (43)$$

with

$$\begin{aligned} F(\alpha', \varphi_0) &= f(\pi + \alpha', \varphi_0) - f(-\pi + \alpha', \varphi_0), \\ A_F &= \frac{-1}{4\pi i} \int_{\mathcal{S}} F(\alpha', \varphi_0) g_0(\alpha') d\alpha', \end{aligned} \quad (44)$$

where the far field function (also known as the diffraction coefficient at the edge in (7)) vanishes at infinity on  $\mathcal{S}$ , and  $f(i\infty) = -f(-i\infty)$  is verified.

As first consequence, we conclude to the existence of  $f$  from  $F$  as  $\Phi > \pi/2$ , defined from (43) and its analytical continuation with (44), which implies the definition of representation (3) for the field  $u$ .

The expression (43) can be also generalized by analytical continuation on  $\alpha$  and  $\varphi_o$ . For that, we begin to write (43) in the form,

$$f(\alpha, \varphi_o) = \frac{-1}{4\pi i} \left( \int_{\mathcal{S}+\pi} - \int_{\mathcal{S}-\pi} \right) f(\alpha', \varphi_o) \left( \cot\left(\frac{1}{2}(\alpha - \alpha')\right) + \cot\left(\frac{\alpha'}{2}\right) \right) d\alpha' + \frac{\text{Res}f|_{\alpha=\varphi_o}}{2} \cot\left(\frac{1}{2}(\alpha - \varphi_o)\right) - A_F, \quad (45)$$

We can assume that the nearest singularities of  $f(\alpha, \varphi_o)$  on real axis are the incident pole at  $\alpha = \varphi_o$  and the geometrical optics reflection poles at  $\alpha = 2\Phi_{\pm} - \varphi_o = \pm 2\Phi^{\pm} - \varphi_o$  (if infinite faces are present), where the constant  $\Phi^{\pm} \geq \Phi$  refers to the angle of semi-infinite plane composing each physical face, while the singularities apart of real axis, which are independent of  $\varphi_o$ , are with  $|\text{Im}\alpha| > \delta > 0$ . We can then use Cauchy theorem, and deform  $\mathcal{S}$  to a new path  $\mathcal{S}_0^{\pm}$ , with capture of poles on real axis [13]. This leads us to write for  $\Phi \geq \pm \varphi_o > \pi/2$ ,

$$f(\alpha, \varphi_o) = - \left( \int_{\mathcal{S}_0^{\pm}+\pi} - \int_{\mathcal{S}_0^{\pm}-\pi} \right) f(\alpha', \varphi_o) \left( \cot\left(\frac{\alpha - \alpha'}{2}\right) + \cot\left(\frac{\alpha'}{2}\right) \right) \frac{d\alpha'}{4\pi i} - A_F = \frac{1}{4\pi i} \int_{\mathcal{S}_0^{\pm}} F(\alpha', \varphi_o) \left( \tan\left(\frac{\alpha - \alpha'}{2}\right) - g_o(\alpha') \right) d\alpha' - A_F, \quad (46)$$

as  $\alpha \in ]\mathcal{S}_0^{\pm} - \pi, \mathcal{S}_0^{\pm} + \pi[$ . For  $|\arg(ik)| = \pi/2 - 0^+$ , let us note that we can consider that  $\mathcal{S}_0^{\pm} \equiv \mathcal{S} \pm i\delta \text{sign}(\arg(ik))$ , and that (46) remains valid by continuation for complex  $\varphi_o$  in the domain  $\Omega_{\delta, \Phi}^{\pm}$  defined by  $\Phi \geq \pm \text{Re}(\varphi_o) > \pi/2$  with  $|\text{Im}(\varphi_o)| < \delta$ .

Remark : Both conditions  $\frac{\pi}{2} < \Phi \mp \varphi_o < \frac{3\pi}{2}$ , considered to apply (43), are satisfied only if  $2\Phi > \pi$ . With these conditions, all poles of  $f(\alpha) - \frac{\text{Res}f|_{\alpha=\varphi_o}}{2} \cot\left(\frac{1}{2}(\alpha - \varphi_o)\right)$  are outside the band  $]\mathcal{S} - \pi, \mathcal{S} + \pi[$ , in particular the GO reflection poles at  $\alpha_{\pm} = -\varphi_o \pm 2\Phi^{\pm}$  with  $\Phi^{\pm} \geq \Phi$ , since the conditions  $\frac{\pi}{2} < \Phi \mp \varphi_o < \frac{3\pi}{2}$  implies  $2\Phi \mp \varphi_o > \frac{\pi}{2} + \Phi > \pi$  and thus  $2\Phi^{\pm} \mp \varphi_o > \pi$ .

## 5.2) Equations on $f$ from equations on $F$ and consequences

The form (46) has important consequence to deduce new analytical properties of  $f(\alpha, \beta)$  on incident variable  $\beta$ . In particular, let  $F(\alpha', \beta)$  satisfy analytical functional equations relatively to variable  $\beta$ ,

$$\mathcal{G}_{\beta}^{\pm}(F(\alpha', \beta)) = 0, \text{ as } \alpha' \in \mathcal{S}_0^{\pm}, \beta \in \Omega_{\delta, \Phi}^{\pm}, \quad (47)$$

where  $\mathcal{G}_{\beta}^{\pm}$  is an analytical operator, linear relatively to  $F$ , we directly deduce from (46),

$$\mathcal{G}_{\beta}^{\pm}(f(\alpha, \beta)) = 0, \quad (48)$$

for  $\beta \in \Omega_{\delta, \Phi}^{\pm}$  as  $\alpha \in ]\mathcal{S}_0^{\pm} - \pi, \mathcal{S}_0^{\pm} + \pi[$ , then everywhere by analytical continuation.



It is worth noticing that when reciprocity principle applies, we have  $F(\alpha', \varphi_o) = F(\varphi_o, \alpha')$ , and functional equations for  $F$  on variable  $\varphi_o$  derive from those on  $\alpha'$ . Thus, (48) is a remarkable way, while  $f(\alpha', \varphi_o) \neq f(\varphi_o, \alpha')$ , to exploit reciprocity and obtain equations on second variable of  $f$ .

The property (48) was first noted and illustrated in [13] in the particular case where  $\mathcal{G}_\beta^\pm(F(\alpha', \beta)) = 0$  are difference functional equations derived from boundary conditions on faces of a wedge, when reciprocity applies and  $F(\alpha', \varphi_o) = F(\varphi_o, \alpha')$ . In this type of problems, we can generally write, from boundary conditions on the field for each face of a wedge  $\varphi = \pm \Phi$ , functional difference equations on  $f$  [4],[16]-[18], following,

$$\mathcal{A}_\alpha^\pm f(\alpha \pm \Phi, \beta) - \mathcal{A}_{-\alpha}^\pm f(-\alpha \pm \Phi, \beta) = \sin \alpha \sum_{q=1}^{Q^\pm} a_q^\pm (\cos \alpha)^{q-1}, \quad (49)$$

with  $\mathcal{A}_\alpha = \mathcal{A}_{\alpha+2\pi}$ , and thus deduce,

$$\mathcal{A}_{\pi+\alpha}^\pm F(\alpha \pm \Phi, \beta) + \mathcal{A}_{\pi-\alpha}^\pm F(-\alpha \pm \Phi, \beta) = 0, \quad (50)$$

which gives us, from  $F(\alpha, \alpha') = F(\alpha', \alpha)$  when reciprocity applies,

$$\mathcal{A}_{\pi+\beta}^\pm F(\alpha, \beta \pm \Phi) + \mathcal{A}_{\pi-\beta}^\pm F(\alpha, -\beta \pm \Phi) = 0. \quad (51)$$

This equation permits, from the application of (48), to obtain functional equations relatively to the second variable on  $f$ , following,

$$\mathcal{A}_{\pi+\beta}^\pm f(\alpha, \beta \pm \Phi) + \mathcal{A}_{\pi-\beta}^\pm f(\alpha, -\beta \pm \Phi) = 0, \quad (52)$$

valid, by analytical continuation, for any  $\alpha$  and  $\beta$ .

### 5.3) Expression of $f$ from $F$ with shifted origin

Considering the expression of  $u_d$  in (6), we can express, relatively to new coordinates  $(\rho_1, \varphi_1)$  when the origin is shifted at the point  $O_1 \equiv (\rho_s, \varphi_s)$ , the far field function  $F_1$ , as follows,

$$\begin{aligned} F_1(\varphi_1, \varphi_o) &= - \lim_{\rho_1 \rightarrow \infty} (\sqrt{2\pi k \rho_1} e^{i\pi/4 + ik\rho_1} u_d(M)) \\ &= \lim_{\rho_1 \rightarrow \infty} (\sqrt{(\rho_1/\rho)} e^{ik(\rho_1-\rho)} F(\varphi, \varphi_o)) \\ &= e^{-ik\rho_s \cos(\varphi_1-\varphi_s)} F(\varphi_1, \varphi_o), \end{aligned} \quad (53)$$

since  $\varphi \sim \varphi_1$  and  $\rho = |OM| \sim \rho_1 + \frac{OO_1 \cdot O_1 M}{|O_1 M|}$  as  $\rho_1 \rightarrow \infty$  [1]. Thus, we can use (46) for the spectral function  $f_1$ , attached to Sommerfeld Maliuzhinets representation for coordinates  $(\rho_1, \varphi_1)$  and origin  $O_1$ , with its far field function  $F_1$ , and we can write,

$$f_1(\alpha, \varphi_0) = \frac{1}{4\pi i} \int_{\mathcal{S}_0^\pm} F_1(\alpha', \varphi_0) \left( \tan\left(\frac{1}{2}(\alpha - \alpha')\right) - g_0(\alpha') \right) d\alpha' - A_{F_1}$$

$$\text{with } F_1(\alpha', \varphi_0) = e^{-ik\rho_s \cos(\alpha' - \varphi_s)} F(\alpha', \varphi_0), \quad (54)$$

as  $\alpha \in ]\mathcal{S}_0^\pm - \pi, \mathcal{S}_0^\pm + \pi[$ ,  $\Phi \geq \pm \varphi_0 > \pi/2$ ,  $|\varphi_s| < \pi/2$ , where we note that we can let,

$$-\cos(\alpha' - \varphi_s) = \cos(\alpha' - \varphi'_s) \text{ with } \pi/2 < |\varphi'_s| \leq \pi. \quad (55)$$

This expression of  $f_1$  can be analytically continued for  $|\varphi_s| > \pi/2$  by shift of  $\mathcal{S}_0^\pm$ , as long as the integrand continues to vanish at infinity, taking account of singularities captured during it.

Remark : We note that,

$$e^{-ik\rho_s \cos(\alpha' - \varphi_s)} F(\alpha', \varphi_0) =$$

$$= e^{ik\rho_s \cos(\pi + \alpha' - \varphi_s)} f(\pi + \alpha', \varphi_0) - e^{ik\rho_s \cos(-\pi + \alpha' - \varphi_s)} f(-\pi + \alpha', \varphi_0), \quad (56)$$

but the reader will notice that we have generally,

$$f_1(\alpha', \varphi_0) \neq e^{ik\rho_s \cos(\alpha' - \varphi_s)} f(\alpha', \varphi_0). \quad (57)$$

Indeed, in the Sommerfeld Maliuzhinets integral representation, the spectral function has to be  $O(\alpha^{b'} (\cos \alpha)^{a'})$  at infinity on and inside the loops  $\gamma_\pm$  of  $\gamma$ , where  $a'$  and  $b'$  are two constants, and we then generally need (54) to express  $f_1$  from  $f$ .

Remark : The expression (1) of the incident field  $u^i$  for a shift of origin can be used with,

$$e^{iku_0 \cdot (OM)} = e^{iku_0 \cdot (OO_1 + O_1M)} = (e^{ik\rho_s \cos(\varphi_0 - \varphi_s)}) e^{ik\rho_1 \cos(\varphi_1 - \varphi_0)}. \quad (58)$$

6) The spectral function  $f_0(\alpha)$  for the diffraction by a wedge, with passive or active impedance faces, and an illumination normal to the edge

6.1) Definition of  $f_0(\alpha)$  in general case, with passive or active faces

The boundary conditions, for the 2D diffraction by a constant impedance wedge with straight faces  $\varphi = \pm \Phi$ , are in general,

$$\left( \frac{\partial}{\partial n} - ik \sin \theta^\pm \right) u(\rho, \varphi)|_{\varphi = \pm \Phi} = 0, \text{ i.e. } \left( \frac{\mp \partial}{ik\rho \partial \varphi} - \sin \theta^\pm \right) u(\rho, \varphi)|_{\varphi = \pm \Phi} = 0,$$

where  $\sin \theta^\pm$  are the relative impedances of the faces  $\varphi = \pm \Phi$ , with  $|\text{Re} \theta^\pm| \leq \pi/2$ . These conditions are a particular case of (16), and the spectral function  $f_0$  attached to the total field  $u$  then satisfies,

$$\begin{aligned} (\sin \alpha + \sin \theta^+) f_0(\alpha + \Phi) - (-\sin \alpha + \sin \theta^+) f_0(-\alpha + \Phi) &= 0, \\ (\sin \alpha - \sin \theta^-) f_0(\alpha - \Phi) - (-\sin \alpha - \sin \theta^-) f_0(-\alpha - \Phi) &= 0, \end{aligned} \quad (59)$$

with  $f_0(i\infty) = -f_0(-i\infty)$ . In whole generally, we consider  $s^\pm = \text{sign}(\text{Re}(\sin\theta^\pm))$ , with  $\text{Re}(\sin\theta^\pm) \neq 0$ , that can be positive (passive impedance face) or negative (active impedance face). Letting  $\mu = \pi/2\Phi$  and  $\varphi_\circ$  the angle of illumination, we have,

$$\begin{aligned} f_0(\alpha) &= \chi_0(\alpha) - (\chi_0(i\infty) + \chi_0(-i\infty))/2, \\ \chi_0(\alpha) &= \Psi(\alpha, \sin\theta^+, \sin\theta^-)P_n(\sin(\mu\alpha))\sigma(\alpha)/\Psi(\varphi_\circ, \sin\theta^+, \sin\theta^-), \\ \sigma(\alpha) &= \mu\cos(\mu\varphi_\circ)/(\sin(\mu\alpha) - \sin(\mu\varphi_\circ)), \\ \Psi(\alpha, \sin\theta^+, \sin\theta^-) &= \Psi^+(\alpha, \sin\theta^+)\Psi^-(\alpha, \sin\theta^-). \end{aligned} \quad (60)$$

where  $P_n$  is a polynomial of order  $n$ , and  $f_0(i\infty) = -f_0(-i\infty)$ . The function  $P_n(\sin(\mu\alpha))\sigma(\alpha)$ , that has the pole of incidence  $\alpha = \varphi_\circ$  as  $\text{Re}\alpha < \Phi$ , verifies,

$$P_n(\sin(\mu(\alpha \pm \Phi)))\sigma(\alpha \pm \Phi) = P_n(\sin(\mu(-\alpha \pm \Phi)))\sigma(-\alpha \pm \Phi), \quad (61)$$

while  $\Psi(\alpha)$ , without any zero or pole as  $\text{Re}\alpha < \Phi$ , is the solution of,

$$\begin{aligned} (\sin\alpha + \sin\theta^+)\Psi(\alpha + \Phi) - (-\sin\alpha + \sin\theta^+)\Psi(-\alpha + \Phi) &= 0, \\ (\sin\alpha - \sin\theta^-)\Psi(\alpha - \Phi) - (-\sin\alpha - \sin\theta^-)\Psi(-\alpha - \Phi) &= 0, \end{aligned} \quad (62)$$

so that  $f_0$  verifies (59). The polynomial  $P_n$  will be defined so that  $f_0$  satisfies the conditions (a)-(b)-(c) on the spectral function for a scatterer in a wedge-shaped sector illuminated by a unit plane wave.

## 6.2) General exact derivation of $\Psi(\alpha) = \Psi^+(\alpha)\Psi^-(\alpha)$

### 6.2.1) An efficient expression of $\Psi(\alpha) = \Psi^+(\alpha)\Psi^-(\alpha)$ with $\Gamma$ function

The function  $\Psi(\alpha)$  can be derived from the development of (200)-(201) of appendix C. From (62), we have,

$$\begin{aligned} \partial_\alpha \ln \frac{\Psi(\alpha \pm \Phi)}{\Psi(-\alpha \pm \Phi)} &= \pm \partial_\alpha \ln \frac{(\sin\alpha - \sin\theta^\pm)}{(\sin\alpha + \sin\theta^\pm)}, \\ \frac{\Psi'}{\Psi}(\alpha \pm \Phi) + \frac{\Psi'}{\Psi}(-\alpha \pm \Phi) &= S^\pm(\alpha), \\ S^\pm(\alpha) &= \pm s^\pm \partial_\alpha \ln \frac{(\sin\alpha - s^\pm \sin\theta^\pm)}{(\sin\alpha + s^\pm \sin\theta^\pm)}, \end{aligned} \quad (63)$$

where  $s^\pm = \text{sign}(\text{Re}(\sin\theta^\pm))$ ,  $|\text{Re}\theta^\pm| \leq \pi/2$ . We can then consider  $s(\alpha) = \partial \ln(\Psi(\alpha))$  in appendix C as the solution of,

$$\begin{aligned} s(\alpha \pm \Phi) + s(-\alpha \pm \Phi) &= S^\pm(\alpha), \\ S^\pm(\alpha) &= \frac{\pm 2\sin\theta^\pm \cos\alpha}{(\sin\alpha + s^\pm \sin\theta^\pm)(\sin\alpha - s^\pm \sin\theta^\pm)} \\ &= \frac{\pm s^\pm \cos\alpha}{(\sin\alpha - s^\pm \sin\theta^\pm)} - \frac{\pm s^\pm \cos\alpha}{(\sin\alpha + s^\pm \sin\theta^\pm)}. \end{aligned} \quad (64)$$

Using residue theorem, we have,

$$\begin{aligned} r_{\pm}(\omega) &= \int_{-i\infty}^{+i\infty} S^{\pm}(\alpha') e^{i\omega\alpha'} d\alpha' = \pm (-2\pi i) s^{\pm} \frac{e^{i\omega s^{\pm}\theta^{\pm}} + e^{i\omega(-s^{\pm}\theta^{\pm}+\pi)}}{1 + e^{i\omega\pi}}, \\ R_{\pm}(i\nu) &= \frac{-i}{2\sqrt{2\pi}} r_{\pm}(i\nu) = \mp \frac{s^{\pm}\sqrt{2\pi}}{2} \frac{e^{-\nu s^{\pm}\theta^{\pm}} + e^{-\nu(-s^{\pm}\theta^{\pm}+\pi)}}{1 + e^{-\nu\pi}}, \end{aligned} \quad (65)$$

where  $0 < s^{\pm}\text{Re}\theta^{\pm} \leq \pi/2$ , and we derive from (200)-(201),

$$\begin{aligned} \Psi(\alpha) &= \Psi^{+}(\alpha, \sin\theta^{+})\Psi^{-}(\alpha, \sin\theta^{-}), \\ \frac{\Psi^{\pm'}}{\Psi^{\pm}}(\alpha) &= i s^{\pm} \int_0^{i\infty} \frac{e^{i\omega s^{\pm}\theta^{\pm}} + e^{i\omega(-s^{\pm}\theta^{\pm}+\pi)}}{(1 + e^{i\omega\pi})\sin(2\omega\Phi)} \sin(\omega(\alpha \pm \Phi)) d\omega \\ &= -s^{\pm} \int_0^{\infty} \frac{e^{-\nu s^{\pm}\theta^{\pm}} + e^{-\nu(-s^{\pm}\theta^{\pm}+\pi)}}{(1 + e^{-\nu\pi})} \frac{\sinh(\nu(\alpha \pm \Phi))}{\sinh(2\nu\Phi)} d\nu \\ &= -s^{\pm} \int_0^{\infty} \frac{\cosh(\nu(\pi/2 - s^{\pm}\theta^{\pm}))}{\cosh(\nu\pi/2)} \frac{\sinh(\nu(\alpha \pm \Phi))}{\sinh(2\nu\Phi)} d\nu, \end{aligned} \quad (66)$$

as  $|\text{Re}\alpha| \leq \Phi$ , where we have noted to simplify,

$$\Psi^{\pm}(\alpha) \equiv \Psi^{\pm}(\alpha, \sin\theta^{\pm}). \quad (67)$$

After integration, we then obtain,

$$\begin{aligned} \Psi^{\pm}(\alpha) &= B^{\pm} \exp \left( s^{\pm} \int_0^{\infty} \frac{e^{-\nu s^{\pm}\theta^{\pm}} + e^{-\nu(\pi-s^{\pm}\theta^{\pm})}}{(1 + e^{-\nu\pi})} \right. \\ &\quad \left. \times \frac{(1 - \cosh(\nu(\alpha \pm \Phi)))}{\nu \sinh(2\nu\Phi)} d\nu \right), \end{aligned} \quad (68)$$

as  $|\text{Re}\alpha| \leq \Phi$ , where  $B^{\pm} = \Psi^{\pm}(\mp \Phi)$  is a free constant, that we could take equal to unity without changing the expression of  $f_0$ . Considering that we have [20],

$$\begin{aligned} (1 + e^{i\omega\pi})^{-1} &= \sum_{m=0}^N (-e^{i\omega\pi})^m + \frac{(-e^{i\omega\pi})^{N+1}}{1 + e^{i\omega\pi}}, \\ \int_0^{\infty} e^{-\mu x} \frac{\sinh(\beta x)}{\sinh(bx)} dx &= \frac{1}{2b} (\partial \ln(\Gamma)(\frac{1}{2} + \frac{\mu + \beta}{2b}) - \partial \ln(\Gamma)(\frac{1}{2} + \frac{\mu - \beta}{2b})) \\ &= \partial_{\beta} (\ln(\Gamma(\frac{1}{2} + \frac{(\mu + \beta)}{2b})\Gamma(\frac{1}{2} + \frac{(\mu - \beta)}{2b}))), \\ \int_0^{\infty} e^{-\mu x} \frac{1 - \cosh(\beta x)}{x \sinh(bx)} dx &= -\ln \left( \frac{\Gamma(\frac{1}{2} + \frac{(\mu + \beta)}{2b})\Gamma(\frac{1}{2} + \frac{(\mu - \beta)}{2b})}{\Gamma(\frac{1}{2} + \frac{\mu}{2b})\Gamma(\frac{1}{2} + \frac{\mu}{2b})} \right), \end{aligned} \quad (69)$$

as  $|\operatorname{Re}(\mu \pm \beta)| < |\operatorname{Re}(b)|$ , leads us to expand (68), as  $|\operatorname{Re}\alpha| \leq \Phi + (N + 1)\pi$ , following,

$$\begin{aligned} \Psi^\pm(\alpha) &= B_N^\pm \left( \prod_{m=0}^N \left[ \Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}(\alpha \pm \Phi + s^\pm \theta^\pm + m\pi)\right) \right. \right. \\ &\Gamma\left(\frac{1}{2} - \frac{1}{4\Phi}(\alpha \pm \Phi - (s^\pm \theta^\pm + m\pi))\right) \Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}(\alpha \pm \Phi + \pi - s^\pm \theta^\pm + m\pi)\right) \\ &\Gamma\left(\frac{1}{2} - \frac{1}{4\Phi}(\alpha \pm \Phi - (\pi - s^\pm \theta^\pm + m\pi))\right) \left. \right]^{s^\pm (-1)^{m+1}} \times \\ &\exp\left(s^\pm \int_0^\infty (-e^{-\nu\pi})^{N+1} \frac{e^{-\nu s^\pm \theta^\pm} + e^{-\nu(\pi - s^\pm \theta^\pm)}}{(1 + e^{-\nu\pi})} \frac{(1 - \cosh(\nu(\alpha \pm \Phi)))}{\nu \sinh(2\nu\Phi)} d\nu\right), \quad (70) \end{aligned}$$

where  $N$  is an arbitrary positive integer,  $s^\pm = \operatorname{sign}(\operatorname{Re}(\sin\theta^\pm))$ ,  $\operatorname{Re}(\sin\theta^\pm) \neq 0$ . The  $B_N^\pm$  are constants depending of  $N$ , while  $\Psi^\pm(\alpha)/\Psi^\pm(\varphi_0)$  does not depends of it.

Since  $e^{-\nu(N+1)\pi}$  rapidly decreases for large  $\nu$  or  $N$ , this expression has the advantage to give an efficient calculation for  $N \sim 1$ , and all analytic properties from those of  $\Gamma$  function when we consider  $N \rightarrow \infty$ . It is then particularly simple to note that  $\Psi(\alpha)$  is a meromorphic function of the complex plane.

### 6.2.2) An efficient expression of the Maliuzhinets function $\Psi_\Phi$ with $\Gamma$

We note, from (68) and the use of  $2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b)$ , that we have,

$$\begin{aligned} \frac{\Psi(\alpha)}{\Psi(\varphi')} &= \prod_{\pm} \exp\left(s^\pm \int_0^\infty \sum_{\epsilon'=-1,1} \left( \frac{\cosh(\nu(\varphi' \pm \Phi + \epsilon'(s^\pm \theta_1^\pm - \frac{\pi}{2}))) - 1}{2\nu \sinh(2\nu\Phi) \cosh(\frac{\nu\pi}{2})} \right. \right. \\ &\left. \left. + \frac{1 - \cosh(\nu(\alpha \pm \Phi + \epsilon'(s^\pm \theta_1^\pm - \frac{\pi}{2})))}{2\nu \sinh(2\nu\Phi) \cosh(\frac{\nu\pi}{2})} \right) d\nu\right), \quad (71) \end{aligned}$$

for  $|\operatorname{Re}\alpha| < \Phi$ . We can then write  $\Psi$ , for  $|\operatorname{Re}\alpha + \epsilon'(s^\pm \theta_1^\pm - \frac{\pi}{2})| < \Phi + (N + 1)\pi$  as  $\epsilon' = -1$  and  $+1$ ,

$$\begin{aligned} \Psi(\alpha) &= C_N \prod_{\epsilon'=-1,1} \left( \prod_{\pm} \right. \\ &\prod_{m \geq 0} \left[ \Gamma\left(\frac{1}{2} + \frac{(\alpha \pm \Phi + \epsilon'(s^\pm \theta_1^\pm - \frac{\pi}{2})) + \frac{\pi}{2} + l\pi}{4\Phi}\right) \right. \\ &\Gamma\left(\frac{1}{2} + \frac{-(\alpha \pm \Phi + \epsilon'(s^\pm \theta_1^\pm - \frac{\pi}{2})) + \frac{\pi}{2} + l\pi}{4\Phi}\right) \left. \right]^{s^\pm (-1)^{m+1}} \\ &\times \exp\left(s^\pm \int_0^\infty \frac{1 - \cosh(\nu(\alpha \pm \Phi + \epsilon'(s^\pm \theta_1^\pm - \frac{\pi}{2})))}{2\nu (-e^{-\nu\pi})^{N+1} \sinh(2\nu\Phi) \cosh(\frac{\nu\pi}{2})} d\nu\right) \left. \right) \\ &= C_N \prod_{\epsilon'=-1,1} \left( \prod_{\pm} (\Psi_{\Phi,N}(\alpha \pm \Phi + \epsilon'(s^\pm \theta_1^\pm - \frac{\pi}{2})))^{s^\pm} \right), \quad (72) \end{aligned}$$

where  $C_N$  is a constant, and  $\Psi_{\Phi,N}(\alpha)$  is given, for  $|\operatorname{Re}\alpha| < 2\Phi + (N+1)\pi + \frac{\pi}{2}$ , by,

$$\begin{aligned} \Psi_{\Phi,N}(\alpha) &= \left( \prod_{m \geq 0}^N \right. \\ & \left. \left[ \Gamma\left(\frac{1}{2} + \frac{\alpha + \frac{\pi}{2} + l\pi}{4\Phi}\right) \Gamma\left(\frac{1}{2} + \frac{-\alpha + \frac{\pi}{2} + l\pi}{4\Phi}\right) \right]^{(-1)^{m+1}} \right) \\ & \times \exp\left( \int_0^\infty (-e^{-\nu\pi})^{N+1} e^{-\nu\frac{\pi}{2}} \frac{1 - \cosh(\nu\alpha)}{\nu \sinh(2\nu\Phi)(1 + e^{-\nu\pi})} d\nu \right), \end{aligned} \quad (73)$$

As for the expansion of  $\Psi^\pm(\alpha)$  with  $\Gamma$  function,  $\Psi_{\Phi,N}(\alpha)/\Psi_{\Phi,N}(0)$  is independent of  $N$ . Comparing this expression with the one given for the Maliuzhinets function  $\Psi_\Phi$  given in [29], we remark that

$$\Psi_{\Phi,N=-1}(\alpha) = \Psi_\Phi(\alpha), \quad \Psi_{\Phi,N=-1}(0) = 1, \quad (74)$$

and,

$$\Psi_{\Phi,N}(\alpha)/\Psi_{\Phi,N}(0) = \Psi_\Phi(\alpha), \quad (75)$$

which represents a new efficient expansion of  $\Psi_\Phi(\alpha)$  with  $\Gamma$  function.

Remark : We also note that,

$$(\Psi_\Phi(\alpha))^2 = \frac{\Psi^\pm(\alpha \mp \Phi, \sin\theta^\pm = 1)}{\Psi^\pm(\mp \Phi, \sin\theta^\pm = 1)}. \quad (76)$$

Remark : For  $|\operatorname{Im}\alpha| \rightarrow \infty$ , the principal contribution, in the integral expressions (68), comes from the vicinity of  $\nu = 0$ . We then note that,

$$\begin{aligned} \int_0^\infty \nu^{2m} \frac{\sinh(\beta\nu)}{\sinh(b\nu)} d\nu &= \partial_{\beta^{1+2m}} \left( \ln\left(\frac{\pi}{\cos(\frac{\pi}{2b}\beta)}\right) \right) \\ \partial_\alpha(\ln(\Psi^\pm(\alpha))) &= s^\pm \partial_\alpha \left( \ln\left(\cos\left(\frac{\pi(\alpha \pm \Phi)}{4\Phi}\right)\right) \right) + O(1/\alpha^2) \end{aligned} \quad (77)$$

and thus,

$$\Psi^\pm(\alpha) \sim C_\pm \left( \cos\left(\frac{\pi}{4\Phi}(\alpha \pm \Phi)\right) \right)^{s^\pm} \text{ with } C_\pm \text{ a constant.} \quad (78)$$

A complete asymptotics for  $\Psi^\pm$  will be given in a further section.

### 6.2.3) Additional expressions for $\Psi^\pm$ and $\Psi_\Phi$

Other expressions of  $\Psi^\pm$  can be derived from, this time, the use of (202) of appendix C. Considering  $s(\alpha) = \partial_\alpha \ln \Psi^\pm(\alpha, \sin\theta^\pm)$ ,  $S^\pm(\alpha) = \pm s^\pm \partial_\alpha \ln \frac{(\sin\alpha - s^\pm \sin\theta^\pm)}{(\sin\alpha + s^\pm \sin\theta^\pm)}$  and  $S^\mp(\alpha) = 0$

with  $s^\pm = \text{sign}(\text{Re}(\sin\theta^\pm))$ , we have, from (202), as  $|\text{Re}\alpha| < \Phi$  with  $\text{Re}(\sin\theta^\pm) \neq 0$ ,

$$\begin{aligned}
S^\pm(\alpha) &= \pm s^\pm \partial_\alpha (\ln(\tan(\frac{\alpha - s^\pm \theta^\pm}{2})) - \ln(\tan(\frac{\alpha + s^\pm \theta^\pm}{2}))) \\
&= \mp s^\pm (\frac{1}{\cos(\alpha + (s^\pm \theta^\pm - \frac{\pi}{2}))} + \frac{1}{\cos(\alpha - (s^\pm \theta^\pm - \frac{\pi}{2}))}), \\
\partial_\alpha \ln \Psi^\pm(\alpha, \sin\theta^\pm) &= \frac{-i}{8\Phi} \int_{-i\infty}^{+i\infty} \pm S^+(\alpha') \tan(\frac{\pi}{4\Phi}(\alpha \pm \Phi - \alpha')) d\alpha' \\
&= \frac{is^\pm}{8\Phi} \int_{-i\infty}^{+i\infty} \sum_{\epsilon=-1,1} \frac{1}{\cos(\alpha' + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2}))} \tan(\frac{\pi}{4\Phi}(\alpha \pm \Phi - \alpha')) d\alpha', \quad (79)
\end{aligned}$$

where, by definition,  $0 < \text{Re}(s^\pm \theta^\pm) \leq \pi/2$ . After integration, we can choose to add a constant integral, and write as  $|\text{Re}\alpha| < 2\Phi$ ,

$$\begin{aligned}
\ln \Psi^\pm(\alpha \mp \Phi, \sin\theta^\pm) &= \frac{-is^\pm}{2\pi} \int_{-i\infty}^{+i\infty} \sum_{\epsilon=-1,1} \frac{\ln(\frac{\cos(\frac{\pi}{4\Phi}(\alpha - \alpha'))}{\cos(\frac{\pi}{4\Phi}\alpha')})}{\cos(\alpha' + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2}))} d\alpha' + \ln B^\pm \\
&= \frac{-is^\pm}{2\pi} \ln(\cos(\frac{\pi}{4\Phi}\alpha)) \int_{-i\infty}^{+i\infty} \sum_{\epsilon=-1,1} \frac{1}{\cos(\alpha' + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2}))} d\alpha' + \ln B^\pm \\
&+ \frac{-is^\pm}{2\pi} \int_{-i\infty}^{+i\infty} \sum_{\epsilon=-1,1} \frac{1}{\cos(\alpha' + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2}))} \ln(\frac{\cos(\frac{\pi}{4\Phi}(\alpha - \alpha'))}{\cos(\frac{\pi}{4\Phi}\alpha)\cos(\frac{\pi}{4\Phi}\alpha')}) d\alpha', \quad (80)
\end{aligned}$$

where we notice that  $B^\pm = \Psi^\pm(\mp \Phi, \sin\theta^\pm)$ . In a similar manner, but, this times, when  $|\text{Re}(\alpha + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2}))| < 2\Phi$  for  $\epsilon = -1$  and  $\epsilon = 1$ , we can write, after integration and addition of another constant integral,

$$\begin{aligned}
\ln \Psi^\pm(\alpha \mp \Phi, \sin\theta^\pm) &= \frac{-is^\pm}{2\pi} \int_{-i\infty}^{+i\infty} \sum_{\epsilon=-1,1} \frac{\ln(\frac{\cos(\frac{\pi}{4\Phi}(\alpha - \alpha'))}{\cos(\frac{\pi}{4\Phi}\alpha' + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2}))})}{\cos(\alpha' + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2}))} d\alpha' + \ln C^\pm \\
&= \frac{-is^\pm}{2\pi} \int_{-i\infty}^{+i\infty} \sum_{\epsilon=-1,1} \frac{\ln(\frac{\cos(\frac{\pi}{4\Phi}(\alpha + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2}) - \alpha'))}{\cos(\frac{\pi}{4\Phi}\alpha')})}{\cos\alpha'} d\alpha' + \ln C^\pm \\
&= s^\pm (\sum_{\epsilon=-1,1} \ln(\cos(\frac{\pi}{4\Phi}\alpha + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2})))) (\frac{-i}{2\pi} \int_{-i\infty}^{+i\infty} \frac{d\alpha'}{\cos\alpha'}) + \ln C^\pm \\
&+ \frac{-is^\pm}{2\pi} \int_{-i\infty}^{+i\infty} \sum_{\pm} \frac{\ln(\frac{\cos(\frac{\pi}{4\Phi}(\alpha + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2}) - \alpha'))}{\cos(\frac{\pi}{4\Phi}(\alpha + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2})))\cos(\frac{\pi}{4\Phi}\alpha')})}{\cos\alpha'} d\alpha', \quad (81)
\end{aligned}$$

where we have used a shift of the integration path, permitted from  $0 < \text{Re}(s^\pm \theta^\pm) \leq \pi/2$ . Considering that,

$$\int_{-i\infty}^{+i\infty} \frac{-i}{\cos(\alpha' + \epsilon(s^\pm \theta^\pm - \frac{\pi}{2}))} d\alpha' = \int_{-i\infty}^{+i\infty} \frac{-i}{\cos\alpha'} d\alpha' = \pi. \quad (82)$$

we obtain, from (80), as  $|\operatorname{Re}\alpha| < 2\Phi$ ,

$$\begin{aligned} \Psi^\pm(\alpha \mp \Phi, \sin\theta^\pm) &= B^\pm \exp\left(s^\pm \ln\left(\cos\left(\frac{\pi}{4\Phi}\alpha\right)\right)\right) \\ &+ \frac{-is^\pm}{2\pi} \sum_{\epsilon=-1,1} \int_{-i\infty}^{+i\infty} \frac{\ln\left(1 + \tan\left(\frac{\pi}{4\Phi}\alpha'\right)\tan\left(\frac{\pi}{4\Phi}\alpha\right)\right)d\alpha'}{\cos\left(\alpha' + \epsilon\left(s^\pm\theta^\pm - \frac{\pi}{2}\right)\right)} \\ &= B^\pm \left(\cos\left(\frac{\pi}{4\Phi}\alpha\right)\right)^{s^\pm} \prod_{\epsilon=-1,1} \exp\left(s^\pm \int_{-\infty}^{+\infty} \frac{\ln\left(1 + i\tanh\left(\frac{\pi}{4\Phi}\nu\right)\tan\left(\frac{\pi}{4\Phi}\alpha\right)\right)}{2\pi\cos\left(i\nu + \epsilon\left(s^\pm\theta^\pm - \frac{\pi}{2}\right)\right)}d\nu\right), \end{aligned} \quad (83)$$

while, from (81), when  $|\operatorname{Re}(\alpha + \epsilon(s^\pm\theta^\pm - \frac{\pi}{2}))| < 2\Phi$  as  $\epsilon = -1$  and  $\epsilon = 1$ ,

$$\begin{aligned} \Psi^\pm(\alpha \mp \Phi, \sin\theta^\pm) &= C^\pm \prod_{\epsilon=-1,1} \left(\cos\left(\frac{\pi}{4\Phi}\left(\alpha + \epsilon\left(s^\pm\theta^\pm - \frac{\pi}{2}\right)\right)\right)\right)^{\frac{s^\pm}{2}} \\ &\exp\left(s^\pm \int_{-\infty}^{+\infty} \frac{\ln\left(1 + i\tanh\left(\frac{\pi}{4\Phi}\nu\right)\tan\left(\frac{\pi}{4\Phi}\left(\alpha + \epsilon\left(s^\pm\theta^\pm - \frac{\pi}{2}\right)\right)\right)}{2\pi\cosh\nu}d\nu\right), \end{aligned} \quad (84)$$

The terms  $B^\pm$  and  $C^\pm$  are free constants that we can take arbitrary without changing the spectral function, while we can let  $\tanh(\frac{\pi}{4\Phi}\nu)$  for  $-\tanh(\frac{\pi}{4\Phi}\nu)$  in (83) and (84) without changing the expressions, so that  $\Psi^\pm(\alpha \mp \Phi, \sin\theta^\pm) = \Psi^\pm(-\alpha \mp \Phi, \sin\theta^\pm)$ .

Considering the definition of  $\Psi^\pm$  with function  $\Psi_\Phi$  (as used in (72)-(74)) following,

$$\Psi^\pm(\alpha \mp \Phi) = C^\pm \prod_{\epsilon'=-1,1} \left(\Psi_\Phi\left(\alpha + \epsilon'\left(s^\pm\theta_1^\pm - \frac{\pi}{2}\right)\right)^{s^\pm}\right) \text{ with } \Psi_\Phi(0) = 1, \quad (85)$$

we compare it with (84), and obtain, in a original manner, another expression of the Maliuzhinets function  $\Psi_\Phi$  for  $|\operatorname{Re}(\alpha)| < 2\Phi$ ,

$$\Psi_\Phi(\alpha) = \left(\cos\left(\frac{\pi}{4\Phi}\alpha\right)\right)^{\frac{1}{2}} \exp\left(\int_{-\infty}^{+\infty} \frac{\ln\left(1 - i\tanh\left(\frac{\pi}{4\Phi}\nu\right)\tan\left(\frac{\pi}{4\Phi}\alpha\right)\right)}{2\pi\cosh\nu}d\nu\right), \quad (86)$$

This expression, previously quoted in [12], is useful to give the asymptotics of  $\Psi_\Phi$ , in particular its first order, as  $|\operatorname{Im}\alpha| \rightarrow \infty$ .

Remark : Let us note that, for  $\operatorname{Im}(w) \rightarrow \pm\infty$  with  $|\operatorname{Re}(w)| < \pi/2$ , we can write,

$$\begin{aligned} &\prod_{\epsilon=-1,1} \exp\left(\int_{-\infty}^{+\infty} \frac{\ln\left(1 + i\tanh\left(\frac{\pi}{4\Phi}\nu\right)\tan\left(\frac{\pi}{4\Phi}\alpha\right)\right)}{2\pi\cos\left(i\nu + \epsilon w\right)}d\nu\right) \\ &\sim \prod_{\epsilon=-1,1} \exp\left(\frac{1}{2}\ln\left(1 \mp i\epsilon\tan\left(\frac{\pi}{4\Phi}\alpha\right)\right)\int_{-\infty}^{+\infty} \frac{d\nu'}{\pi\cos\left(i\nu'\right)}\right) \\ &\sim \exp\left(\ln\left(\frac{1}{\cos\left(\frac{\pi}{4\Phi}\alpha\right)}\right)\right) = \left(\cos\left(\frac{\pi}{4\Phi}\alpha\right)\right)^{-1}, \end{aligned} \quad (87)$$

and thus, we have, from (83),

$$\Psi^\pm(\alpha \mp \Phi, \sin\theta^\pm) \sim B^\pm \text{ as } \operatorname{Im}(\theta^\pm) \rightarrow \infty. \quad (88)$$



### 6.3) Complete asymptotic expressions of $\frac{\Psi^\pm}{\Psi^\pm}$ and $\Psi^\pm$

#### 6.3.1) Asymptotic expressions

For a complete asymptotic expansions of  $\frac{\Psi^\pm}{\Psi^\pm}$  and  $\Psi^\pm$  for large  $\text{Im}\alpha$ , we can transform (66) and (68) expressions, using residue theorem. Considering  $\frac{\Psi^\pm}{\Psi^\pm}$ , for  $\text{Im}\alpha \neq 0$  with  $\epsilon = \text{sign}(\text{Im}\alpha)$  and  $|\text{Re}\alpha| < \Phi$ , we develop (66), following,

$$\begin{aligned}
\frac{\Psi^{\pm'}}{\Psi^\pm}(\alpha) &= i s^\pm \int_0^{i\infty} \frac{e^{i\omega s^\pm \theta^\pm} + e^{i\omega(-s^\pm \theta^\pm + \pi)}}{(1 + e^{i\omega\pi}) \sin(2\omega\Phi)} \sin(\omega(\alpha \pm \Phi)) d\omega \\
&= \frac{\epsilon s^\pm}{4} \int_{-i\infty}^{i\infty} \frac{\cos(\omega(\pi/2 - s^\pm \theta^\pm))}{\cos(\omega\pi/2) \sin(2\omega\Phi)} (e^{i\omega\epsilon(\alpha \pm \Phi)} - e^{-i\omega\epsilon(\alpha \pm \Phi)}) d\omega \\
&= -\pi i \left( \frac{\epsilon s^\pm}{4\Phi} \right) + \frac{\epsilon s^\pm}{4} \int_{0^+ - i\infty}^{0^+ + i\infty} \frac{\cos(\omega(\pi/2 - s^\pm \theta^\pm))}{\cos(\omega\pi/2) \sin(2\omega\Phi)} e^{i\omega\epsilon(\alpha \pm \Phi)} d\omega \\
&\quad - \frac{\epsilon s^\pm}{4} \int_{-0^+ - i\infty}^{-0^+ + i\infty} \frac{\cos(\omega(\pi/2 - s^\pm \theta^\pm))}{\cos(\omega\pi/2) \sin(2\omega\Phi)} e^{-i\omega\epsilon(\alpha \pm \Phi)} d\omega \\
&= -\pi i \left( \frac{\epsilon s^\pm}{4\Phi} \right) + 2 \times \frac{\epsilon s^\pm}{4} \int_{0^+ - i\infty}^{0^+ + i\infty} \frac{\cos(\omega(\pi/2 - s^\pm \theta^\pm))}{\cos(\omega\pi/2) \sin(2\omega\Phi)} e^{i\omega\epsilon(\alpha \pm \Phi)} d\omega, \quad (89)
\end{aligned}$$

We can then shift the integration path to  $\text{Re}\alpha = \delta > 0$ , while considering residues (first and second order ones) due to the capture of poles between  $0^+$  and the real positive constant  $\delta$ , and we obtain,

$$\begin{aligned}
\frac{\Psi^{\pm'}}{\Psi^\pm}(\alpha) &= -i\epsilon s^\pm \left( \frac{\pi}{4\Phi} \right) + \frac{\epsilon s^\pm}{2} (-2\pi i) \\
&\times \left[ \left( \sum_{m \geq 1, \frac{m\pi}{2\Phi} \neq 2p+1}^{\frac{m\pi}{2\Phi} < \delta} (-1)^m \frac{1}{2\Phi} \frac{\cos(\frac{m\pi(\pi/2 - s^\pm \theta^\pm)})}{\cos(\frac{m\pi(\pi/2)}{2\Phi})} e^{i\frac{m\pi}{2\Phi}\epsilon(\alpha \pm \Phi)} \right) \right. \\
&+ \left( \sum_{p \geq 0, 2p+1 \neq \frac{m\pi}{2\Phi}}^{2p+1 < \delta} (-1)^{p+1} \frac{2}{\pi} \frac{\cos((2p+1)(\pi/2 - s^\pm \theta^\pm))}{\sin(2(2p+1)\Phi)} e^{i(2p+1)\epsilon(\alpha \pm \Phi)} \right) \\
&+ \left( \sum_{p \geq 0, 2p+1 = \frac{m\pi}{2\Phi}}^{2p+1 < \delta} \frac{(-1)^{m+p+1} 2}{2\Phi} \frac{1}{\pi} (-\pi/2 - s^\pm \theta^\pm) \sin\left(\frac{m\pi(\pi/2 - s^\pm \theta^\pm)}{2\Phi}\right) \right. \\
&\left. + i\epsilon(\alpha \pm \Phi) \cos\left(\frac{m\pi(\pi/2 - s^\pm \theta^\pm)}{2\Phi}\right) \right) e^{i\frac{m\pi}{2\Phi}\epsilon(\alpha \pm \Phi)} \left. \right] \\
&+ \frac{\epsilon s^\pm}{2} e^{i\delta\epsilon(\alpha \pm \Phi)} \int_{\delta - i\infty}^{\delta + i\infty} \frac{\cos(\omega(\pi/2 - s^\pm \theta^\pm))}{\cos(\omega\pi/2) \sin(2\omega\Phi)} e^{i(\omega - \delta)\epsilon(\alpha \pm \Phi)} d\omega, \quad (90)
\end{aligned}$$

where we take care that  $\frac{\delta 2\Phi}{\pi}$  and  $\frac{\delta - 1}{2}$  are not integers. After integration of (90), or using directly (68) by a similar analysis, we obtain  $\Psi^\pm$ , for  $\text{Im}\alpha \neq 0$  with  $\epsilon = \text{sign}(\text{Im}\alpha)$  and

$|\operatorname{Re}\alpha| < \Phi$ ,

$$\begin{aligned}
\Psi^\pm(\alpha) &= B^\pm e^{-i\epsilon s^\pm \frac{\pi}{4\Phi}(\alpha \pm \Phi)} \\
&\times \exp \left[ \frac{s^\pm}{2} \int_{0^+ - i\infty}^{0^+ + i\infty} \frac{\cos(\omega(\pi/2 - s^\pm \theta^\pm))}{\cos(\omega\pi/2)\sin(2\omega\Phi)} \frac{e^{i\epsilon\omega(\alpha \pm \Phi)} - 1}{i\omega} d\omega \right] \\
&= B^\pm e^{D_\pm} e^{-i\epsilon s^\pm \frac{\pi}{4\Phi}(\alpha \pm \Phi)} \\
&\times \exp \left[ -\pi s^\pm \left( \left( \sum_{m \geq 1, \frac{m\pi}{2\Phi} \neq 2p+1}^{\frac{m\pi}{2\Phi} < \delta} \frac{(-1)^m \frac{1}{2\Phi} \cos(\frac{m\pi(\pi/2 - s^\pm \theta^\pm)}{2\Phi})}{\frac{m\pi}{2\Phi} \cos(\frac{m\pi(\pi/2)}{2\Phi})} e^{i\epsilon \frac{m\pi}{2\Phi}(\alpha \pm \Phi)} \right) \right. \right. \\
&+ \left( \sum_{p \geq 0, 2p+1 \neq \frac{m\pi}{2\Phi}}^{2p+1 < \delta} \frac{(-1)^{p+1} \frac{2}{\pi} \cos((2p+1)(\pi/2 - s^\pm \theta^\pm))}{(2p+1)\sin(2(2p+1)\Phi)} e^{i\epsilon(2p+1)(\alpha \pm \Phi)} \right) \\
&+ \left( \sum_{p \geq 0, 2p+1 = \frac{m\pi}{2\Phi}}^{2p+1 < \delta} \frac{(-1)^{m+p+1} \frac{2}{\pi} \frac{2\Phi}{m\pi} \left( -(\frac{\pi}{2} - s^\pm \theta^\pm) \sin(\frac{m\pi(\frac{\pi}{2} - s^\pm \theta^\pm)}{2\Phi}) \right)}{2\Phi} \right. \\
&\left. \left. + i\epsilon(\alpha \pm \Phi) \cos(\frac{m\pi(\frac{\pi}{2} - s^\pm \theta^\pm)}{2\Phi}) e^{i\epsilon \frac{m\pi}{2\Phi}(\alpha \pm \Phi)} \right) \right) \\
&\left. + \frac{s^\pm e^{i\epsilon\delta(\alpha \pm \Phi)}}{2\delta} \int_{\delta - i\infty}^{\delta + i\infty} \frac{\delta \cos(\omega(\pi/2 - s^\pm \theta^\pm))}{i\omega \cos(\omega\pi/2)\sin(2\omega\Phi)} e^{i\epsilon(\omega - \delta)(\alpha \pm \Phi)} d\omega \right], \tag{91}
\end{aligned}$$

where we note that  $B^\pm = \Psi^\pm(\mp \Phi, \sin\theta^\pm)$ , while,

$$\begin{aligned}
D_\pm &= \frac{-s^\pm}{2i} \int_{0^+ - i\infty}^{0^+ + i\infty} \frac{\cos(\omega(\pi/2 - s^\pm \theta^\pm))}{\cos(\omega\pi/2)\sin(2\omega\Phi)} \frac{1}{\omega} d\omega \\
&= \frac{-s^\pm}{2i} \int_{-i\infty}^{+i\infty} \partial_\omega \left( \left( \frac{\cos(\omega\pi/2) \frac{\sin(2\omega\Phi)}{\omega}}{\cos(\omega(\pi/2 - s^\pm \theta^\pm))} \right)^{-1} \right) \frac{1}{\omega} d\omega. \tag{92}
\end{aligned}$$

The last integral terms in (90) and (91) being  $O(1)$ , and considering that

$$\begin{aligned}
\left| \exp(z) - \sum_{m=0}^M \frac{z^m}{m!} \right| &= \left| \int_0^z \frac{t^M}{M!} e^t dt \right| \leq \frac{|z|^{M+1} \max(1, |e^z|)}{(M+1)!}, \\
\exp(z_1 + z_2 + \dots) &= \exp(z_1) \times \exp(z_2) \times \dots \tag{93}
\end{aligned}$$

we have obtained, thanks to (90) and (91), the complete asymptotic expansions for the functions  $\frac{\Psi^\pm}{\Psi^\pm}(\alpha)$  and  $\Psi^\pm(\alpha)$  as  $|\operatorname{Im}\alpha| \rightarrow \infty$ .

Remark : Second order residue in (90) and (91) concerns the case  $2p+1 = \frac{m\pi}{2\Phi}$ . It implies that we have  $\frac{\pi}{2\Phi} = \frac{2p_0+1}{m_0}$ ,  $2p+1 = (2q+1)(2p_0+1)$ , and  $m = (2q+1)m_0$ , with  $q = 0, 1, 2, 3, \dots$ , when  $2p_0+1$  and  $m_0$  are mutually prime.

### 6.3.2) Some complements on the determination of $D_\pm$

For the calculus of  $D_{\pm}$ , we can also write,

$$D_{\pm} = \frac{is^{\pm}}{2} \int_{-i\infty}^{+i\infty} \frac{\left(\frac{\cos(\omega(\pi/2 - s^{\pm}\theta^{\pm}))}{\cos(\omega\pi/2)} - 1\right) \frac{1}{\omega}}{\sin(2\omega\Phi)} d\omega + \frac{is^{\pm}}{2} \int_{0^{+} - i\infty}^{0^{+} + i\infty} \frac{1}{\sin(2\omega\Phi)} \frac{1}{\omega} d\omega. \quad (94)$$

We can then use [20], the method of integration by parts, and shifts of integration path, to reduce the first and second integrals. We note first that we have,

$$\begin{aligned} \left(\frac{\cos(\omega a)}{\cos(\omega\pi/2)} - 1\right) \frac{1}{\omega} &= \frac{1}{\pi i} \sum_{\epsilon'=\pm 1, -1} \left( \int_{-i\infty}^{+i\infty} \frac{(\cos(\omega(\alpha + \epsilon' a)) - 1)}{2\omega \cos(\alpha)} d\alpha \right) \\ &= \frac{1}{\pi i} \sum_{\epsilon'=\pm 1, -1} \left( \int_{-i\infty}^{+i\infty} \frac{(\cos(\omega\alpha) - 1)}{2\omega \cos(\alpha + \epsilon' a)} d\alpha \right) \\ &= \frac{1}{\pi i} \sum_{\epsilon'=\pm 1, -1} \int_0^{i\infty} \sin(\omega\alpha) \ln\left(\frac{1}{i} \tan\left(\frac{\alpha + \epsilon' a + \frac{\pi}{2}}{2}\right)\right) d\alpha, \end{aligned}$$

for  $|\operatorname{Re}\omega| < 1$ ,  $|\operatorname{Re}a| < \pi/2$ . (95)

while we remark that,

$$\begin{aligned} \frac{i}{2} \int_{-i\infty+0^{+}}^{+i\infty+0^{+}} \frac{1}{\omega \sin(2\omega\Phi)} d\omega &= \frac{i}{2} \int_{-i\infty+0^{+}}^{+i\infty+0^{+}} \frac{1}{\omega \sin(\omega\pi/2)} d\omega \\ &= \frac{i}{2} \int_{-i\infty}^{+i\infty} \frac{1}{(1+\omega)\cos(\omega\pi/2)} d\omega = i \int_0^{+i\infty} \frac{1}{(1-\omega^2)\cos(\omega\pi/2)} d\omega \\ &= - \int_0^{\infty} \frac{1}{(1+\nu^2)\cosh(\nu\pi/2)} d\nu = -\ln(2), \end{aligned} \quad (96)$$

We can then write,

$$\begin{aligned} D_{\pm} + s^{\pm} \ln(2) &= \frac{i}{2} \frac{2s^{\pm}}{\pi i} \sum_{\epsilon'=\pm 1, -1} \int_0^{i\infty} \ln\left(\frac{1}{i} \tan\left(\frac{\alpha + \epsilon'(\pi/2 - s^{\pm}\theta^{\pm}) + \frac{\pi}{2}}{2}\right)\right) \\ &\times \int_0^{i\infty} \frac{\sin(\omega\alpha)}{\sin(2\omega\Phi)} d\omega d\alpha \\ &= \frac{s^{\pm}}{\pi} \sum_{\epsilon'=\pm 1, -1} \int_0^{i\infty} \ln\left(\frac{1}{i} \tan\left(\frac{\alpha + \epsilon'(\pi/2 - s^{\pm}\theta^{\pm}) + \frac{\pi}{2}}{2}\right)\right) \frac{\pi i}{4\Phi} \tan\left(\frac{\pi}{4\Phi} \alpha\right) d\alpha \\ &= \frac{-1}{i} \frac{s^{\pm}}{2\pi} \sum_{\epsilon'=\pm 1, -1} \int_{-i\infty}^{+i\infty} \frac{\ln(\cos(\frac{\pi}{4\Phi} \alpha))}{\cos(\alpha + \epsilon'(\pi/2 - s^{\pm}\theta^{\pm}))} d\alpha. \end{aligned} \quad (97)$$

With this reduction of  $D_{\pm}$ , it is worth noticing that the first order of this asymptotics is in perfect agreement with the one given by (83).

#### 6.4) Links between $\Psi^{\pm}$ for passive and active impedances, and positions of poles

##### 6.4.1) The link between $\Psi^{\pm}$ for passive and active impedances

For arbitrary sign of  $\text{Re}(\sin\theta^\pm)$ ,  $\text{Re}(\sin\theta^\pm) > 0$  (passive) or  $\text{Re}(\sin\theta^\pm) < 0$  (active), we note that the functions  $\Psi^\pm$  in the expression of  $\Psi(\alpha) = \Psi^+(\alpha)\Psi^-(\alpha)$  verify,

$$\begin{aligned} (\sin\alpha \pm \sin\theta^\pm)\Psi^\pm(\alpha \pm \Phi) - (-\sin\alpha \pm \sin\theta^\pm)\Psi^\pm(-\alpha \pm \Phi) &= 0, \\ \Psi^\pm(\alpha \mp \Phi) - \Psi^\pm(-\alpha \mp \Phi) &= 0, \end{aligned} \quad (98)$$

i.e., when  $\text{Re}(\sin\theta^\pm) \neq 0$ ,

$$\begin{aligned} \frac{\Psi^{\pm'}}{\Psi^\pm}(\alpha \pm \Phi) + \frac{\Psi^{\pm'}}{\Psi^\pm}(-\alpha \pm \Phi) &= \pm \partial_\alpha \ln \frac{(\sin\alpha - \sin\theta^\pm)}{(\sin\alpha + \sin\theta^\pm)}, \\ \frac{\Psi^{\pm'}}{\Psi^\pm}(\alpha \mp \Phi) + \frac{\Psi^{\pm'}}{\Psi^\pm}(-\alpha \mp \Phi) &= 0. \end{aligned} \quad (99)$$

Since we have,

$$\partial_\alpha \ln \frac{(\sin\alpha - \sin\theta^\pm)}{(\sin\alpha + \sin\theta^\pm)} = -\partial_\alpha \ln \frac{(\sin\alpha + \sin\theta^\pm)}{(\sin\alpha - \sin\theta^\pm)}, \quad (100)$$

and the function  $\Psi^\pm$  has to remain without any zero or pole in the band  $|\text{Re}\alpha| \leq \Phi$ , we can, letting  $\Psi^\pm(\alpha, \sin\theta^\pm) \equiv \Psi^\pm(\alpha)$  for impedance  $\sin\theta^\pm$ , derive  $\Psi^\pm$  with an active impedance face directly from  $\Psi^\pm$  with passive impedance face, following,

$$\Psi^\pm(\alpha, \sin\theta^\pm) = B_0/\Psi^\pm(\alpha, -\sin\theta^\pm) \text{ as } \text{Re}(\sin\theta^\pm) \neq 0, \quad (101)$$

with  $B_0$  a constant, as we can yet notice from general expression (70).

#### 6.4.2) Positions of poles and zeros of $\Psi^\pm(\alpha)$

Letting  $s^\pm = \text{sign}(\text{Re}(\sin\theta^\pm))$  and using (70), we directly derive zeros and poles of  $\Psi^\pm$  from poles of the function  $\Gamma(z)$ . The poles and zeros of  $\Psi^\pm(\alpha) \equiv \Psi^\pm(\alpha, \sin\theta^\pm)$ , respectively in passive and active cases, are linked, and we note that, when  $s^\pm = 1$  (resp.  $s^\pm = -1$ ),  $\Psi^\pm(\alpha, \sin\theta^\pm)$  has zeros (resp. poles) at

$$\begin{aligned} \alpha &= \pm (s^\pm\theta^\pm + (4p+1)\Phi + 2m\pi), \\ \text{and } \alpha &= \pm (\pi - s^\pm\theta^\pm + (4p+1)\Phi + 2m\pi), \end{aligned} \quad (102)$$

and poles (resp. zeros) at

$$\begin{aligned} \alpha &= \pm (\pi + s^\pm\theta^\pm + (4p+1)\Phi + 2m\pi), \\ \text{and } \alpha &= \pm (2\pi - s^\pm\theta^\pm + (4p+1)\Phi + 2m\pi), \end{aligned} \quad (103)$$

for  $m \geq 0, p \geq 0$ .

#### 6.5) Determination of $P_n(\sin(\mu\alpha))$ in general for passive or active faces

The polynomial  $P_n$  permits to complete the satisfaction of conditions (a)-(b)-(c) on  $f_0$ . Considering a unit incident plane wave, we have  $f_0(\alpha) \sim 1/(\alpha - \varphi_0)$  in vicinity of

incidence pole  $\alpha = \varphi_0$ . This implies  $P_n(\sin(\mu\varphi_0))\sigma(\alpha) \sim 1/(\alpha - \varphi_0)$ , and thus,

$$P_n(\sin(\mu\varphi_0)) = 1. \quad (104)$$

From unique definition of  $u$  at  $\rho = 0$ , we have to satisfy  $f_0(\alpha) = O(1)$  as  $|\text{Im}\alpha| \rightarrow \infty$ , and considering (91) (see also (83) or (78)), we can write,

$$f_0(\alpha) \sim C_0 \frac{\prod \cos(\frac{\mu}{2}(\alpha \pm \Phi))^{s^\pm}}{(\sin(\mu\alpha))^{1-n}} = O(1),$$

which implies  $(n-1)\mu + \sum_{\pm} \frac{s^\pm}{2}\mu = 0$ ,

i.e.  $n = \sum_{\pm} (1 - s^\pm)/2 = \text{the number of active faces.}$  (105)

For  $|\text{Re}\varphi| < \Phi$  and passive impedance wedge with  $\text{Re}(\sin\theta^\pm) > 0$ , the poles of  $\Psi(\alpha + \varphi)$  in the expression of  $f_0(\alpha + \varphi)$  are outside the strip  $|\text{Re}\alpha| < \pi$ , but it is no more the case for active impedance faces.

In this latter case, the  $n$  zeros of  $P_n(\sin(\mu\alpha))$  have to suppress, for each active face with  $s^\pm = -1$ , the non physical pole  $\alpha_s^\pm$  of  $\Psi^\pm(\alpha, \sin\theta^\pm)$  as  $|\text{Re}\alpha| < \pi + \Phi$  that can give poles of  $f_0(\alpha + \varphi)$  within the region  $|\text{Re}\alpha| < \pi$  with  $\text{Re}(ik\cos\alpha) > 0$  as  $|\text{Re}\varphi| < \Phi$ , which leads to cancel non physical contribution  $e^{ik\rho\cos(\alpha_s^\pm - \varphi)}$  that can diverge at infinity as  $|\text{Re}\varphi| < \Phi$  (note : for real  $k$ , notice that the non physical pole  $\alpha_s^\pm$  is the one with  $\pm \text{sign}(\arg(ik))\text{Im}\alpha_s^\pm > 0$ , among  $\alpha_1^\pm = \pm (s^\pm\theta^\pm + \Phi)$  and  $\alpha_2^\pm = \pm (\pi - s^\pm\theta^\pm + \Phi)$ ). We then have,

$$P_n(\sin(\mu\alpha)) = \prod_{\pm} \left( \frac{\sin(\mu\alpha) - \sin(\mu\alpha_s^\pm)}{\sin(\mu\varphi_0) - \sin(\mu\alpha_s^\pm)} \right)^{\frac{1-s^\pm}{2}},$$

$$s^\pm = \text{sign}(\text{Re}(\sin\theta^\pm)), \text{Re}(\sin\theta^\pm) \neq 0. \quad (106)$$

which completes the definition of  $f_0$  for arbitrary impedances (passive or active).

## 7) Analysis for a wedge at skew incidence (2D1/2) and associated special functions

We study in this section the special functions used for the electromagnetic diffraction of a skew incident plane wave by a passive anisotropic wedge of any angle, their expansions and their properties. The problem is then no more scalar, as in strict 2D problem, but more generally vectorial.

We present here our solution [9],[22] for balanced hybrid conditions (when the product of diagonal terms of impedance matrix is equal to unity), and study its relation with Fisanov function used in the solution of Lyalinov and Zhu in [23], and with Maliuzhinets function. On this occasion, we insist on the common links with the  $\Gamma$  special function [20], as in (91) for  $\Psi^\pm$ , which reveals remarkable analytic properties and permits efficient calculus.

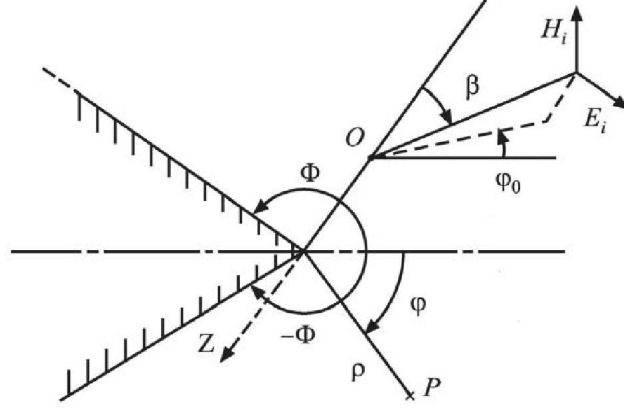


Figure 3 : Geometry at skew incidence

7.1) Our solution given in [9]

The plane wave incident on the wedge is vectorial. It is characterized by the  $z$ -components of the electric and magnetic fields along the edge, respectively  $E_z^i$  and  $H_z^i$ , given by

$$\begin{bmatrix} E_z^i(\rho, \varphi, z) \\ Z_o H_z^i(\rho, \varphi, z) \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} e^{ik[\rho \sin \beta \cos(\varphi - \varphi') - z \cos \beta]}. \quad (107)$$

In this expression,  $k$  ( $\text{Im}k \leq 0$ ) and  $Z_o$  denote respectively the exterior medium wave number and impedance, and  $\beta$  is the angle of the incident direction with the edge of the wedge.

The  $z$ -components of the field, in cylindrical coordinates  $(\rho, \varphi, z)$ , are searched in the form of Sommerfeld-Maliuzhinets integrals [9], following

$$\begin{bmatrix} E_z(\rho, \varphi, z) \\ Z_o H_z(\rho, \varphi, z) \end{bmatrix} = \frac{e^{-ikz \cos \beta}}{2\pi i} \int_{\gamma} \begin{bmatrix} f_1(\alpha + \varphi) \\ f_2(\alpha + \varphi) \end{bmatrix} e^{ik\rho \sin \beta \cos \alpha} d\alpha, \quad (108)$$

which satisfy the Helmholtz equation  $(\Delta + k^2)u = 0$  on each component. The contour  $\gamma$  is odd and composed of two symmetric loops as for 2D problems. Conditions at the edge and at infinity are assumed, and we consider an anisotropic constant impedance boundary condition  $(E - n^\pm(n^\pm E)) = (Z^\pm)(n^\pm \wedge H)$  on each face  $\varphi = \pm \Phi$ , where  $n^\pm$  is the unit vector along the outward-pointing normal to the face, and  $(Z^\pm)/Z_o$  a relative constant impedance tensor. This implies coupled functional equations

$$A_\alpha^\pm \begin{bmatrix} f_1(\alpha \pm \Phi) \\ f_2(\alpha \pm \Phi) \end{bmatrix} - A_{-\alpha}^\pm \begin{bmatrix} f_1(-\alpha \pm \Phi) \\ f_2(-\alpha \pm \Phi) \end{bmatrix} = \sin \alpha \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad (109)$$

with

$$A_{\alpha}^{\pm} = \begin{bmatrix} \cos\beta\cos\alpha & \sin\alpha \pm \eta_h^{\pm}\sin\beta \\ \sin\alpha \pm \sin\beta/\eta_e^{\pm} & -\cos\beta\cos\alpha \end{bmatrix}, \quad (110)$$

when  $(Z^{\pm})/Z_o$  is characterized by a class of diagonal relative impedance matrices with elements  $\eta_h^{\pm}, \eta_e^{\pm}$ .

Considering the oddness of  $\gamma$ , we note that we can choose to add some constants to spectral functions  $f_{1,2}$  without changing the field, and the constants  $c_{1,2}$  have no influence on it (see details in [9]). In these circumstances, we note that the previous sections on general properties of spectral functions, in Sommerfeld Maliuzhinets representation of 2D fields, can apply providing that we let,

$$f(\alpha) \equiv e^{-ikz\cos\beta} (f_{1,2}(\alpha) - \frac{f_{1,2}(i\infty) + f_{1,2}(-i\infty)}{2}), \quad (111)$$

in frequency domain, with  $u \equiv E_z(\rho, \varphi, z)$  or  $Z_o H_z(\rho, \varphi, z)$ ,  $\varphi_o \equiv \varphi'$ , and  $e^{ik\rho\sin\beta\cos\alpha}$  in place of  $e^{ik\rho\cos\alpha}$  in integral representation of  $u$ .

By employing an original factorization technique for coupled equations [9], we found a closed-form solution when the products of these elements are equal to unity, i.e.  $\eta_h^{\pm}\eta_e^{\pm} = 1$ . This case of anisotropy is important. It corresponds to a geometrical optics reflection coefficient which is independent of the polarization of the incident field.

We let  $\eta_h^{\pm} = 1/\eta_e^{\pm} = \sin\theta_1^{\pm}$  with  $0 < \text{Re}\theta_1^{\pm} \leq \pi/2$ ,  $\delta = \ln(\tan(\beta/2))$ , and obtain [9], [22],

$$\begin{bmatrix} f_1(\alpha) \\ f_2(\alpha) \end{bmatrix} = \frac{\Psi_{an}(\alpha)}{\Psi_{an}(\varphi')} \begin{bmatrix} \cos\Delta(\alpha) & \sin\Delta(\alpha) \\ -\sin\Delta(\alpha) & \cos\Delta(\alpha) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \sigma(\alpha), \quad (112)$$

where  $\Delta(\alpha) = v_{an}(\alpha) - v_{an}(\varphi')$ ,  $\sigma(\alpha) = \mu\cos(\mu\varphi')(\sin(\mu\alpha) - \sin(\mu\varphi'))^{-1}$ ,  $\mu = \pi/2\Phi$ , and the functions  $\Psi_{an}$  and  $v_{an}$  verify,

$$\begin{aligned} \frac{\Psi_{an}(\alpha \pm \Phi)}{\Psi_{an}(-\alpha \pm \Phi)} &= \left( \frac{\det A_{-\alpha}^{\pm}}{\det A_{\alpha}^{\pm}} \right)^{\frac{1}{2}}, \\ v_{an}(\alpha \pm \Phi) - v_{an}(-\alpha \pm \Phi) &= \arctan\left( \frac{n(\alpha)}{l^{\pm}(\alpha)} \right), \end{aligned} \quad (113)$$

with

$$\begin{aligned} \det(A_{\alpha}^{\pm}) &= -\sin^2\beta(\sin\alpha \pm \sin(\theta_1^{\pm} - i\delta)) \\ &\quad \times (\sin\alpha \pm \sin(\theta_1^{\pm} + i\delta)), \\ \det(A_{\alpha}^{\pm})\det(A_{-\alpha}^{\pm}) &= (n(\alpha))^2 + (l^{\pm}(\alpha))^2, \\ l^{\pm}(\alpha) &= \cos^2\beta \sin^2\alpha - \cos^2\alpha - ((\eta_h^{\pm})^2 - 1)\sin^2\beta, \\ n(\alpha) &= \cos\beta \sin(2\alpha), \end{aligned} \quad (114)$$

and  $\sin(\beta)\sin(\theta_1^{\pm} - \epsilon i\delta) = (\eta_h^{\pm} + \epsilon\cos\beta\sqrt{(\eta_h^{\pm})^2 - 1})$ . The explicit closed-form expressions of the two special functions  $\Psi_{an}(\alpha)$  and  $v_{an}(\alpha)$  is derived [9],[22], for

$|\operatorname{Re}\alpha| < \Phi + (N + 1)\pi$ , following,

$$\begin{aligned}
\Psi_{an}(\alpha) &= \prod_{\epsilon=-1,1} \prod_{\pm} \\
& \left( \prod_{l \geq 0}^N \left[ \Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}((\alpha \pm \Phi) + \theta_1^\pm - \epsilon i\delta + l\pi)\right)^{-\frac{1}{2}} \right. \right. \\
& \Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}(-(\alpha \pm \Phi) + \theta_1^\pm - \epsilon i\delta + l\pi)\right)^{-\frac{1}{2}} \\
& \Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}((\alpha \pm \Phi) + \pi - \theta_1^\pm + \epsilon i\delta + l\pi)\right)^{-\frac{1}{2}} \\
& \left. \Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}(-(\alpha \pm \Phi) + \pi - \theta_1^\pm + \epsilon i\delta + l\pi)\right)^{-\frac{1}{2}} \right]^{(-1)^l} \\
& \times \exp\left(\int_0^\infty (-1)^{N+1} \times \frac{e^{-\nu(\theta_1^\pm - \epsilon i\delta)} + e^{-\nu(\pi - \theta_1^\pm + \epsilon i\delta)}}{2 \times (1 + e^{-\nu\pi})} \right. \\
& \left. \times e^{-\nu(N+1)\pi} \frac{1 - \cosh(\nu(\alpha \pm \Phi))}{\nu \sinh(2\nu\Phi)} d\nu\right)
\end{aligned} \tag{115}$$

and

$$\begin{aligned}
v_{an}(\alpha) &= i \sum_{\pm} \sum_{\epsilon=-1,+1} \\
& \left( \sum_{l \geq 0}^N \ln\left(\left[\Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}(-(\alpha \pm \Phi) + \theta_1^\pm - \epsilon i\delta + l\pi)\right)^{\mp\frac{1}{2}} \right. \right. \\
& \Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}((\alpha \pm \Phi) + \theta_1^\pm - \epsilon i\delta + l\pi)\right)^{\mp\frac{1}{2}} \\
& \Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}((\alpha \pm \Phi) + \pi - \theta_1^\pm + \epsilon i\delta + l\pi)\right)^{\pm\frac{1}{2}} \\
& \left. \Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}(-(\alpha \pm \Phi) + \pi - \theta_1^\pm + \epsilon i\delta + l\pi)\right)^{\pm\frac{1}{2}} \right]^\epsilon \\
& + \epsilon \int_0^\infty \frac{e^{-\nu(\theta_1^\pm - \epsilon i\delta)} - e^{-\nu(\pi - \theta_1^\pm + \epsilon i\delta)}}{2 \times (1 - e^{-\nu\pi})} \\
& \left. \times e^{-\nu(N+1)\pi} \frac{\mp (\cosh(\nu(\alpha \pm \Phi)) - 1)}{\nu \sinh(2\nu\Phi)} d\nu\right),
\end{aligned} \tag{116}$$

for arbitrary  $N \geq -1$ , where we have used  $\frac{1}{1 \pm x} = \sum_{l \geq 0}^N (\mp x)^l + \frac{(\mp x)^{N+1}}{1 \pm x}$  as in [24]-[25]

(note:  $\prod_{l \geq 0}^{-1} = 1$  and  $\sum_{l \geq 0}^{-1} () = 0$ ). Since  $e^{-\nu(N+1)\pi}$  rapidly decreases for large  $\nu$  or  $N$ , these

expressions have the advantage to give an efficient calculation for  $N \sim 1$ , and all analytic properties from those of  $\Gamma$  function when we consider  $N \rightarrow \infty$ .

Besides, it is worth noticing that it is then particularly simple to realize that  $\Psi_{an}(\alpha)e^{\pm i v_{an}(\alpha)}$  is a meromorphic function of the complex plane, for which we can identify all complex zeros and poles. For the particular case of normal incidence (i.e. for  $\delta = 0$  when we have no more coupling between TM and TE waves), or for isotropic unit impedance ( $\theta_1^\pm = \pi/2$ ), we recover the expressions given in [4],[12] and [24]-[25].



Remark : Among all analytical properties of  $\Gamma$ , we can quote the remarkable identities,

$$\begin{aligned}\Gamma(z+1) &= z\Gamma(z), \quad -z\Gamma(-z)\Gamma(z) = \pi/\sin(\pi z), \quad \Gamma(1) = 1, \\ \Gamma(nz) &= (2\pi)^{\frac{1}{2}(1-n)} n^{nz-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right).\end{aligned}\quad (117)$$

7.2) The relation with the solution given by Lyalinov and Zhu [23]

We now exhibit the relations of the special functions  $\Psi_{an}$  and  $v_{an}$  of our solution with the Fisanov function  $\chi_{\Phi}(\alpha)$ , used by Lyalinov and Zhu in [23]. For that, we let  $N = -1$  in (115) and (116), and note that,

$$\begin{aligned}\frac{\Psi_{an}(\alpha)}{\Psi_{an}(\varphi')} &= \prod_{\epsilon=-1,1} \prod_{\pm} \\ &\exp\left(\int_0^{\infty} \frac{\cosh(\nu(\pi/2 - \theta_1^{\pm} + \epsilon i\delta))}{2 \times \cosh(\nu(\pi/2))}\right. \\ &\quad \times \frac{\cosh(\nu(\varphi' \pm \Phi)) - \cosh(\nu(\alpha \pm \Phi))}{\nu \sinh(2\nu\Phi)} d\nu) \\ &= \prod_{\epsilon=-1,1} \prod_{\pm} \\ &\exp\left(\int_0^{\infty} \frac{\sinh(\nu(\pi - \theta_1^{\pm} + \epsilon i\delta)) - \sinh(\nu(-\theta_1^{\pm} + \epsilon i\delta))}{2\sinh(\nu\pi)}\right. \\ &\quad \times \frac{\cosh(\nu(\varphi' \pm \Phi)) - \cosh(\nu(\alpha \pm \Phi))}{\nu \sinh(2\nu\Phi)} d\nu),\end{aligned}\quad (118)$$

and

$$\begin{aligned}v_{an}(\alpha) - v_{an}(\varphi') &= i \sum_{\pm} \sum_{\epsilon=-1,+1} \\ &\epsilon \int_0^{\infty} \frac{\sinh(\nu(\pi/2 - \theta_1^{\pm} + \epsilon i\delta))}{2 \sinh(\nu\pi/2)} \\ &\quad \times \frac{\mp (\cosh(\nu(\alpha \pm \Phi)) - \cosh(\nu(\varphi' \pm \Phi)))}{\nu \sinh(2\nu\Phi)} d\nu) \\ &= i \sum_{\pm} \sum_{\epsilon=-1,+1} \\ &\epsilon \int_0^{\infty} \frac{\sinh(\nu(\pi - \theta_1^{\pm} + \epsilon i\delta)) + \sinh(\nu(-\theta_1^{\pm} + \epsilon i\delta))}{2 \sinh(\nu\pi)} \\ &\quad \times \frac{\mp (\cosh(\nu(\alpha \pm \Phi)) - \cosh(\nu(\varphi' \pm \Phi)))}{\nu \sinh(2\nu\Phi)} d\nu).\end{aligned}\quad (119)$$

and then compare with the integral expression of  $\chi_\Phi(\alpha)$  given in [26],

$$\chi_\Phi(\alpha) = \exp\left(\frac{1}{2} \int_0^\infty \frac{1}{\nu \sinh(\nu\pi)} \left(\frac{\alpha}{2\Phi} - \frac{\sinh(\nu\alpha)}{\sinh(2\nu\Phi)}\right) d\nu\right). \quad (120)$$

Since, we have,

$$\begin{aligned} & 2(\sinh(\nu(\pi+x))\cosh(\nu y) - \sinh(\nu x)\cosh(\nu y)) = \\ & = (\sinh(\nu(y+\pi+x)) - \sinh(\nu(y-\pi-x)) - \\ & - (\sinh(\nu(y+x)) - \sinh(\nu(y-x))) \\ & = (\sinh(\nu(\pi+y+x)) - \sinh(\nu(y+x)) + \\ & + (\sinh(\nu(y-x)) - \sinh(\nu(-\pi+y-x))), \end{aligned} \quad (121)$$

we can express  $\frac{\Psi_{an}(\alpha)}{\Psi_{an}(\varphi')}$  as a combination of functions of the type  $\left(\frac{\chi_\Phi(\pi+z+\alpha)\chi_\Phi(z'+\alpha)}{\chi_\Phi(z+\alpha)\chi_\Phi(-\pi+z'+\alpha)}\right)^{\frac{1}{2}} \left(\frac{\chi_\Phi(\pi+z+\varphi')\chi_\Phi(z'+\varphi')}{\chi_\Phi(z+\varphi')\chi_\Phi(-\pi+z'+\varphi')}\right)^{-\frac{1}{2}}$ , and in a similar manner, since

$$\begin{aligned} & 2(\sinh(\nu(\pi+x))\cosh(\nu y) + \sinh(\nu x)\cosh(\nu y)) = \\ & = (\sinh(\nu(y+\pi+x)) - \sinh(\nu(y-\pi-x)) + \\ & + (\sinh(\nu(y+x)) - \sinh(\nu(y-x))) \\ & = (\sinh(\nu(\pi+y+x)) + (\sinh(\nu(y+x))) - \\ & - (\sinh(\nu(y-x)) + \sinh(\nu(-\pi+y-x))), \end{aligned} \quad (122)$$

we can express  $\cos(v_{an}(\alpha) - v_{an}(\varphi'))$  and  $\sin(v_{an}(\alpha) - v_{an}(\varphi'))$  as combinations of functions  $\left(\frac{\chi_\Phi(\pi+z+\alpha)\chi_\Phi(z+\alpha)}{\chi_\Phi(z'+\alpha)\chi_\Phi(-\pi+z'+\alpha)}\right)^{\pm\frac{1}{2}} \left(\frac{\chi_\Phi(\pi+z+\varphi')\chi_\Phi(z+\varphi')}{\chi_\Phi(z'+\varphi')\chi_\Phi(-\pi+z'+\varphi')}\right)^{\mp\frac{1}{2}}$ , which implies that  $\frac{\Psi_{an}(\alpha)\cos(v_{an}(\alpha)-v_{an}(\varphi'))}{\Psi_{an}(\varphi')}$  and  $\frac{\Psi_{an}(\alpha)\sin(v_{an}(\alpha)-v_{an}(\varphi'))}{\Psi_{an}(\varphi')}$  are meromorphic functions in  $\alpha$  which express as combinations of  $\frac{\chi_\Phi(\pi+z+\alpha)}{\chi_\Phi(-\pi+z'+\alpha)} \left(\frac{\chi_\Phi(\pi+z+\varphi')}{\chi_\Phi(-\pi+z'+\varphi')}\right)^{-1}$  and  $\frac{\chi_\Phi(z'+\alpha)}{\chi_\Phi(z+\alpha)} \left(\frac{\chi_\Phi(z'+\varphi')}{\chi_\Phi(z+\varphi')}\right)^{-1}$  terms. This result permits to recover the expressions obtained by Zhu and Lyalinov [23].

### 7.3) $\Psi_{an}$ in terms of $\Psi_\Phi$ functions and expressions of $\Psi_\Phi$

#### 7.3.1) $\Psi_\Phi(\alpha)$ from the expansion of $\Psi_{an}$ with $\Gamma$ function

From (115), the expression of  $\frac{\Psi_{an}(\alpha)}{\Psi_{an}(\varphi')}$  is given by,

$$\begin{aligned} \frac{\Psi_{an}(\alpha)}{\Psi_{an}(\varphi')} &= \prod_{\epsilon=-1,1} \left( \prod_{\pm} \right. \\ & \exp\left(\int_0^\infty \sum_{\epsilon'=-1,1} \left(\frac{\cosh(\nu(\varphi' \pm \Phi + \epsilon'(\theta_1^\pm - \epsilon i\delta - \frac{\pi}{2}))) - 1}{4\nu \sinh(2\nu\Phi)\cosh(\frac{\nu\pi}{2})}\right) - 1 \right. \\ & \left. \left. + \frac{1 - \cosh(\nu(\alpha \pm \Phi + \epsilon'(\theta_1^\pm - \epsilon i\delta - \frac{\pi}{2})))}{4\nu \sinh(2\nu\Phi)\cosh(\frac{\nu\pi}{2})}\right) d\nu\right), \end{aligned} \quad (123)$$

for  $|\operatorname{Re}\alpha| < \Phi$ . Considering that [20],

$$\begin{aligned}
\int_0^\infty e^{-\mu x} \frac{\sinh(\beta x)}{\sinh(bx)} dx &= \partial_\beta (\ln(\Gamma(\frac{1}{2} + \frac{(\mu + \beta)}{2b}) \Gamma(\frac{1}{2} + \frac{(\mu - \beta)}{2b}))), \\
\int_0^\infty e^{-\mu x} \frac{1 - \cosh(\beta x)}{x \sinh(bx)} dx &= -\ln\left(\frac{\Gamma(\frac{1}{2} + \frac{(\mu + \beta)}{2b}) \Gamma(\frac{1}{2} + \frac{(\mu - \beta)}{2b})}{\Gamma(\frac{1}{2} + \frac{\mu}{2b}) \Gamma(\frac{1}{2} + \frac{\mu}{2b})}\right), \\
\frac{(-e^{-\nu\pi})^{N+1}}{(1 + e^{-\nu\pi})} &= (-e^{-\nu\pi})^{N+1} + \frac{(-e^{-\nu\pi})^{N+2}}{(1 + e^{-\nu\pi})},
\end{aligned} \tag{124}$$

we can write, for  $|\operatorname{Re}\alpha + \epsilon'(\theta_1^\pm - \epsilon i\delta - \frac{\pi}{2})| < \Phi + (N + 1)\pi$  as  $\epsilon' = -1$  and  $1$ ,

$$\begin{aligned}
\Psi_{an}(\alpha) &= C_N \prod_{\epsilon=-1,1} \left( \prod_{\epsilon'=-1,1} \left( \prod_{\pm} \right. \right. \\
&\quad \left. \prod_{l \geq 0}^N \left[ \Gamma\left(\frac{1}{2} + \frac{(\alpha \pm \Phi + \epsilon'(\theta_1^\pm - \epsilon i\delta - \frac{\pi}{2})) + \frac{\pi}{2} + l\pi}{4\Phi}\right) \right]^{-\frac{1}{2}} \right. \\
&\quad \left. \Gamma\left(\frac{1}{2} + \frac{-(\alpha \pm \Phi + \epsilon'(\theta_1^\pm - \epsilon i\delta - \frac{\pi}{2})) + \frac{\pi}{2} + l\pi}{4\Phi}\right)^{-\frac{1}{2}} \right]^{(-1)^l} \\
&\quad \times \exp\left(\int_0^\infty \frac{1 - \cosh(\nu(\alpha \pm \Phi + \epsilon'(\theta_1^\pm - \epsilon i\delta - \frac{\pi}{2})))}{4\nu(-e^{-\nu\pi})^{N+1} \sinh(2\nu\Phi) \cosh(\frac{\nu\pi}{2})} d\nu\right) \\
&= C_N \prod_{\epsilon=-1,1} \left( \prod_{\epsilon'=-1,1} \left( \prod_{\pm} (\Psi_{\Phi,N}(\alpha \pm \Phi + \epsilon'(\theta_1^\pm - \epsilon i\delta - \frac{\pi}{2})))^{\frac{1}{2}} \right) \right),
\end{aligned} \tag{125}$$

where  $C_N$  is a constant, and  $\Psi_{\Phi,N}(\alpha)$  is given, for  $|\operatorname{Re}\alpha| < 2\Phi + (N + 1)\pi + \frac{\pi}{2}$ , by,

$$\begin{aligned}
\Psi_{\Phi,N}(\alpha) &= \left( \prod_{l \geq 0}^N \right. \\
&\quad \left[ \Gamma\left(\frac{1}{2} + \frac{\alpha + \frac{\pi}{2} + l\pi}{4\Phi}\right) \Gamma\left(\frac{1}{2} + \frac{-\alpha + \frac{\pi}{2} + l\pi}{4\Phi}\right) \right]^{(-1)^{l+1}} \\
&\quad \exp\left(\int_0^\infty (-e^{-\nu\pi})^{N+1} e^{-\nu\frac{\pi}{2}} \frac{1 - \cosh(\nu\alpha)}{\nu \sinh(2\nu\Phi)(1 + e^{-\nu\pi})} d\nu\right),
\end{aligned} \tag{126}$$

for  $N \geq -1$ , and  $\prod_{l \geq 0}^{-1}() = 1$ , which in perfect agreement with (73). Comparing this expression with the one given for  $\Psi_\Phi$  in [29], we conclude, as in (74), that we have,

$$\Psi_\Phi(\alpha) = \Psi_{\Phi,N}(\alpha) / \Psi_{\Phi,N}(0) \text{ with } \Psi_{\Phi,N=-1}(0) = 1, \tag{127}$$

where  $\Psi_\Phi$  is the Maliuzhinets function, first defined in [4] for the diffraction by an impedance wedge, and verifying  $\frac{\Psi_\Phi(\alpha+2\Phi)}{\Psi_\Phi(\alpha-2\Phi)} = \cot(\frac{1}{2}(\alpha + \frac{\pi}{2}))$ .

The expression (127) is an efficient expansion to calculate the function  $\Psi_\Phi(\alpha)$ , when  $N$  is finite. It also permits to recover all analytical properties if we let tend  $N$  to infinity, in particular from well known remarkable identities satisfied by  $\Gamma$  in the complex plane.

Besides, it is worth noticing that the constants,  $C_N$  and  $\Psi_{\Phi,N}(0)$ , have no influence on the solution, and thus, that we can take  $C_N = 1$ , and  $\Psi_\Phi$  or  $\Psi_{\Phi,N}$  indifferently, without changing the field.

7.3.2) Complete asymptotic expression of  $\Psi_\Phi(\alpha)$  for large  $\text{Im}\alpha$ 

For an asymptotic expression of  $\Psi_\Phi$  for large  $\text{Im}\alpha$ , we can transform the expression given by (127) as  $N = -1$ , using residue theorem. Letting  $\epsilon = \text{sign}(\text{Im}\alpha)$ , we can then write, as  $\text{Im}\alpha \neq 0$  and  $|\text{Re}\alpha| < 2\Phi + \frac{\pi}{2}$ ,

$$\begin{aligned}
\Psi_\Phi(\alpha) &= \exp\left[\frac{i}{4}\int_{-i\infty}^{i\infty}\frac{1-\cos(\omega\alpha)}{\omega\cos(\omega\pi/2)\sin(2\omega\Phi)}d\omega\right] \\
&= \exp\left[\frac{1}{8i}\int_{-i\infty}^{i\infty}\frac{(e^{i\epsilon\omega\alpha}-i\epsilon\omega\alpha-1)+(e^{-i\epsilon\omega\alpha}+i\epsilon\omega\alpha-1)}{\omega\cos(\omega\pi/2)\sin(2\omega\Phi)}d\omega\right] \\
&= \exp\left[\frac{(-\pi i)}{8i}\left(\frac{i\epsilon\alpha}{2\Phi}\right)+\frac{1}{8}\int_{0^+-i\infty}^{0^++i\infty}\frac{1}{i\omega}\frac{e^{i\epsilon\omega\alpha}-1}{\cos(\omega\pi/2)\sin(2\omega\Phi)}d\omega\right. \\
&\quad \left.+\frac{(\pi i)}{8i}\left(\frac{-i\epsilon\alpha}{2\Phi}\right)+\frac{1}{8}\int_{-0^+-i\infty}^{-0^++i\infty}\frac{1}{i\omega}\frac{e^{-i\epsilon\omega\alpha}-1}{\cos(\omega\pi/2)\sin(2\omega\Phi)}d\omega\right] \\
&= e^{-i\epsilon\frac{\pi}{8\Phi}\alpha}\times\exp\left[\frac{1}{8}\int_{0^+-i\infty}^{0^++i\infty}\frac{1}{i\omega}\frac{e^{i\epsilon\omega\alpha}-1}{\cos(\omega\pi/2)\sin(2\omega\Phi)}d\omega\right. \\
&\quad \left.+\frac{1}{8}\int_{-0^+-i\infty}^{-0^++i\infty}\frac{1}{i\omega}\frac{e^{-i\epsilon\omega\alpha}-1}{\cos(\omega\pi/2)\sin(2\omega\Phi)}d\omega\right] \\
&= e^{-i\epsilon\frac{\pi}{8\Phi}\alpha}\times\exp\left[\frac{1}{8}\int_{0^+-i\infty}^{0^++i\infty}\frac{1}{i\omega}\frac{e^{i\epsilon\omega\alpha}-1}{\cos(\omega\pi/2)\sin(2\omega\Phi)}d\omega\right. \\
&\quad \left.+\frac{1}{8}\int_{0^+-i\infty}^{0^++i\infty}\frac{1}{i\omega}\frac{e^{i\epsilon\omega\alpha}-1}{\cos(\omega\pi/2)\sin(2\omega\Phi)}d\omega\right] \\
&= e^{-i\epsilon\frac{\pi}{8\Phi}\alpha}\exp\left[\frac{1}{4}\int_{0^+-i\infty}^{0^++i\infty}\frac{1}{\cos(\omega\pi/2)\sin(2\omega\Phi)}\frac{-1}{i\omega}d\omega\right] \\
&\quad \times\exp\left[\frac{1}{4}\int_{0^+-i\infty}^{0^++i\infty}\frac{1}{\cos(\omega\pi/2)\sin(2\omega\Phi)}\frac{e^{i\epsilon\omega\alpha}}{i\omega}d\omega\right]. \tag{128}
\end{aligned}$$

In the last exponential term, we can then shift the path to  $\text{Re}\alpha = \delta > 0$ , taking care that  $\frac{\delta 2\Phi}{\pi}$  and  $\frac{\delta-1}{2}$  are not integers, while adding residues terms (first and second order), due to the capture of poles, between  $0^+$  and the constant  $\delta$ , during the shifting. We then derive, as  $|\text{Re}\alpha| < 2\Phi + \frac{\pi}{2}$  and  $\text{Im}\alpha \neq 0$ ,

$$\begin{aligned}
\Psi_\Phi(\alpha) &= e^B e^{-i\epsilon\frac{\pi}{8\Phi}\alpha} \\
&\quad \times\exp\left[-\frac{\pi}{2}\left(\left(\sum_{m\geq 1, \frac{m\pi}{2\Phi}\neq 2p+1}^{\frac{m\pi}{2\Phi}<\delta}\frac{(-1)^m\frac{1}{2\Phi}}{\frac{m\pi}{2\Phi}\cos(\frac{m\pi(\pi/2)}{2\Phi})}e^{i\epsilon\frac{m\pi}{2\Phi}\alpha}\right)\right.\right. \\
&\quad \left.+\left(\sum_{p\geq 0, 2p+1\neq\frac{m\pi}{2\Phi}}^{2p+1<\delta}\frac{(-1)^{p+1}\frac{2}{\pi}}{(2p+1)\sin(2(2p+1)\Phi)}e^{i\epsilon(2p+1)\alpha}\right)\right. \\
&\quad \left.+\left(\sum_{p\geq 0, 2p+1=\frac{m\pi}{2\Phi}}^{2p+1<\delta}\frac{(-1)^{m+p+1}\frac{2}{2\Phi}}{\frac{2}{\pi}\frac{2\Phi}{m\pi}}i\epsilon\alpha e^{i\frac{m\pi}{2\Phi}\epsilon\alpha}\right)\right) \\
&\quad \left.+\frac{e^{i\epsilon\delta\alpha}}{4\delta}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\delta}{i\omega\cos(\omega\pi/2)\sin(2\omega\Phi)}e^{i\epsilon(\omega-\delta)\alpha}d\omega\right], \tag{129}
\end{aligned}$$

where,

$$B = \frac{i}{4} \int_{0+i\infty}^{0^++i\infty} \frac{1}{\cos(\omega\pi/2)\sin(2\omega\Phi)} \frac{1}{\omega} d\omega = \frac{i}{4} \int_{-i\infty}^{+i\infty} \frac{\partial_\omega \frac{\omega}{\cos(\omega\pi/2)\sin(2\omega\Phi)}}{\omega} d\omega. \quad (130)$$

The last integral term in (129) being  $O(1)$ , we have obtained a complete asymptotic expansion for the function  $\Psi_\Phi(\alpha)$  as  $|\text{Im}\alpha| \rightarrow \infty$ .

### 7.3.3) Some complement on the determination of $B$

For the calculus of  $B$ , we can also write,

$$B = \frac{i}{4} \int_{-i\infty}^{+i\infty} \frac{(\frac{1}{\cos(\omega\pi/2)} - 1)}{\sin(2\omega\Phi)\omega} d\omega + \frac{i}{4} \int_{0^++i\infty}^{+i\infty} \frac{1}{\sin(2\omega\Phi)} \frac{1}{\omega} d\omega. \quad (131)$$

We then use (95)-(96) and the method of integration by parts to simplify the first and second integrals. From (95)-(96), we note first that,

$$\begin{aligned} \left(\frac{1}{\cos(\frac{\pi}{2}\omega)} - 1\right) \frac{1}{\omega} &= \frac{2}{\pi i} \int_0^{i\infty} \frac{(\cos(\omega\alpha) - 1)}{\omega \cos(\alpha)} d\alpha \\ &= \frac{2}{\pi i} \int_0^{i\infty} \sin(\omega\alpha) \ln\left(\frac{1}{i} \tan\left(\frac{\alpha + \frac{\pi}{2}}{2}\right)\right) d\alpha, |\text{Re}\omega| < 1, \\ \frac{i}{2} \int_{-i\infty+0^+}^{+i\infty+0^+} \frac{1}{\omega \sin(2\omega\Phi)} d\omega &= -\ln(2), \end{aligned} \quad (132)$$

and we can then write,

$$\begin{aligned} B + \frac{\ln(2)}{2} &= \frac{i}{2} \frac{2}{\pi i} \int_0^{i\infty} \ln\left(\frac{1}{i} \tan\left(\frac{\alpha + \frac{\pi}{2}}{2}\right)\right) \int_0^{i\infty} \frac{\sin\omega\alpha}{\sin(2\omega\Phi)} d\omega d\alpha \\ &= \frac{1}{\pi} \int_0^{i\infty} \ln\left(\frac{1}{i} \tan\left(\frac{\alpha + \frac{\pi}{2}}{2}\right)\right) \frac{\pi i}{4\Phi} \tan\left(\frac{\pi}{4\Phi}\alpha\right) d\alpha \\ &= \frac{-1}{i} \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} \frac{\ln(\cos(\frac{\pi}{4\Phi}\alpha))}{\cos(\alpha)} d\alpha = \frac{-1}{2\pi} \int_{-\infty}^{+\infty} \frac{\ln(\cosh(\frac{\pi}{4\Phi}\nu))}{\cosh(\nu)} d\nu. \end{aligned} \quad (133)$$

With this reduction of  $B$ , it is worth noticing that the first order of this asymptotic is in perfect agreement with the one given by (86).

## 7.4) Expressions for the Fisanov function $\chi_\Phi$

### 7.4.1) An efficient expansion of the Fisanov function $\chi_\Phi$ with $\Gamma$ function

The Fisanov function  $\chi_\Phi$ , used in [23] and [26], can be developed in an original manner from the special function  $\Gamma$ . We write the second order derivative of (120), for  $|\text{Re}(\alpha)| < \pi + 2\Phi$ , following,

$$\begin{aligned}
\frac{\partial^2}{\partial \alpha^2} \ln(\chi_\Phi(\alpha)) &= -\frac{1}{2} \int_0^\infty \frac{\nu}{\sinh(\nu\pi)} \frac{\sinh(\nu\alpha)}{\sinh(2\nu\Phi)} d\nu \\
&= -\frac{1}{2} \int_0^\infty \left( 2e^{-\nu\pi} \sum_{l \geq 0}^N e^{-2l\pi\nu} + \frac{e^{-\nu(N+1)2\pi}}{\sinh(\nu\pi)} \right) \frac{\nu \sinh(\nu\alpha)}{\sinh(2\nu\Phi)} d\nu.
\end{aligned} \tag{134}$$

Considering then that we have [20],

$$\begin{aligned}
\int_0^\infty e^{-\mu\nu} \frac{\nu \sinh(\nu\alpha)}{\sinh(2\nu\Phi)} d\nu &= \frac{-1}{4\Phi} \partial_\mu \left( (\partial \ln \Gamma) \left( \frac{1}{2} + \frac{(\mu + \alpha)}{4\Phi} \right) - \right. \\
&\quad \left. - (\partial \ln \Gamma) \left( \frac{1}{2} + \frac{(\mu - \alpha)}{4\Phi} \right) \right) \\
&= -\frac{\partial^2}{\partial \alpha^2} \left( \ln \left( \Gamma \left( \frac{1}{2} + \frac{(\mu + \alpha)}{4\Phi} \right) \right) - \ln \left( \Gamma \left( \frac{1}{2} + \frac{(\mu - \alpha)}{4\Phi} \right) \right) \right),
\end{aligned} \tag{135}$$

we can write, for  $|\operatorname{Re}(\alpha)| < (N + 1)2\pi + \pi + 2\Phi$ ,

$$\begin{aligned}
\ln(\chi_\Phi(\alpha)) &= (A_N + B_N \alpha) + \\
&\quad \sum_{l \geq 0}^N \ln \left( \frac{\Gamma \left( \frac{1}{2} + \frac{1}{4\Phi} ((2l + 1)\pi + \alpha) \right)}{\Gamma \left( \frac{1}{2} + \frac{1}{4\Phi} ((2l + 1)\pi - \alpha) \right)} \right) \\
&\quad + \int_0^\infty \frac{e^{-\nu(N+1)2\pi}}{2\sinh(\nu\pi)} \frac{(\alpha - \frac{\sinh(\nu\alpha)}{\nu})}{\sinh(2\nu\Phi)} d\nu,
\end{aligned} \tag{136}$$

Thus, since  $\chi_\Phi(0) = 1$ , we obtain

$$\chi_\Phi(\alpha) = e^{B_N \alpha} \chi_{\Phi, N}(\alpha), \tag{137}$$

with,

$$\begin{aligned}
\chi_{\Phi, N}(\alpha) &= \prod_{l \geq 0}^N \left( \frac{\Gamma \left( \frac{1}{2} + \frac{1}{4\Phi} ((2l + 1)\pi + \alpha) \right)}{\Gamma \left( \frac{1}{2} + \frac{1}{4\Phi} ((2l + 1)\pi - \alpha) \right)} \right) \\
&\quad \times \exp \left( \int_0^\infty e^{-\nu(N+1)2\pi} \frac{(\nu\alpha - \sinh(\nu\alpha))}{2\nu \sinh(\nu\pi) \sinh(2\nu\Phi)} d\nu \right),
\end{aligned} \tag{138}$$

for  $|\operatorname{Re}(\alpha)| < (N + 1)2\pi + \pi + 2\Phi$ ,  $N \geq -1$ ,  $\prod_{l \geq 0}^{-1} = 1$ . Considering the derivative of the integral term in the expression of  $\chi_{\Phi, N}$ , null at  $\alpha = 0$ , the constant  $B_N$  is given by,

$$B_N = \frac{1}{2} \int_0^\infty \frac{\left( \frac{1}{2\Phi} - \frac{\nu}{\sinh(2\nu\Phi)} \right)}{\nu \sinh(\nu\pi)} d\nu - 2 \sum_{l \geq 0}^N (\partial \ln \Gamma) \left( \frac{1}{2} + \frac{1}{4\Phi} ((2l + 1)\pi) \right), \tag{139}$$

but also, from  $\chi_\Phi(\pi) = 2^{\frac{1}{2} - \frac{\pi}{4\Phi}}$  [26], following,

$$B_N = \frac{1}{\pi} \ln \left( \frac{2^{\frac{1}{2} - \frac{\pi}{4\Phi}}}{\chi_{\Phi, N}(\pi)} \right).$$

The expressions (137) and (138) are efficient tools to calculate the functions  $\chi_\Phi$  and

$\chi_{\Phi,N}$  when  $N$  is finite, permitting to recover all the analytical properties from remarkable identities of  $\Gamma$  when we let tend  $N$  to infinity.

It is important to notice that, considering the wedge problem with anisotropic impedance condition, the constant  $B_N$  has no influence on the solution, and thus, that we can choose  $\chi_{\Phi,N}$  in place of  $\chi_{\Phi}$ , at  $N$  arbitrarily fixed, without changing the final expression of the field.

Remark :  $\chi_{\Phi,N}$  and  $\chi_{\Phi}$  are meromorphic functions with the same poles and zeros, and we have  $\chi_{\Phi,N}(\alpha + 2\Phi) = (e^{-B_N 4\Phi} \cos(\frac{\alpha}{2})) \chi_{\Phi,N}(\alpha - 2\Phi)$ .

#### 7.4.2) About an asymptotic expansion of the function $\chi_{\Phi,N}$

Let  $N = -1$  in (138). Using a manner similar to the one use for  $\Psi_{\Phi}$ , we can develop the complete asymptotics for  $\ln\chi_{\Phi,N}$  and  $\chi_{\Phi,N}$ . For that, we let  $\epsilon = \text{sign}(\text{Im}(\alpha))$ , use residue theorem and write,

$$\begin{aligned} \ln\chi_{\Phi,N}(\alpha) &= \frac{i\epsilon}{2} \int_{-i\infty}^{+i\infty} \frac{(\omega\epsilon\alpha - \sin(\omega\epsilon\alpha))}{2\omega\sin(\omega\pi)\sin(2\omega\Phi)} d\omega \\ &= \frac{-\pi i\epsilon\alpha^2}{16\pi\Phi} - \frac{\epsilon}{4} \int_{-i\infty+0^+}^{+i\infty+0^+} \frac{e^{i\epsilon\omega\alpha} - i\epsilon\omega\alpha - 1}{2\omega\sin(\omega\pi)\sin(2\omega\Phi)} d\omega \\ &\quad + \frac{\epsilon}{4} \int_{-i\infty-0^+}^{+i\infty-0^+} \frac{e^{-i\epsilon\omega\alpha} + i\epsilon\omega\alpha - 1}{2\omega\sin(\omega\pi)\sin(2\omega\Phi)} d\omega \\ &= \frac{-i\epsilon\alpha^2}{16\Phi} + \epsilon B'_0 + B'_1\alpha - \frac{\epsilon}{4} \int_{-i\infty+0^+}^{+i\infty+0^+} \frac{e^{i\epsilon\omega\alpha}}{\omega\sin(\omega\pi)\sin(2\omega\Phi)} d\omega, \end{aligned} \quad (140)$$

where,

$$\begin{aligned} B'_0 &= \int_{-i\infty+0^+}^{+i\infty+0^+} \frac{d\omega/4}{\omega\sin(\omega\pi)\sin(2\omega\Phi)} = \int_{-i\infty}^{+i\infty} \partial_{\omega^3} \left( \frac{-\omega^2/8}{\sin(\omega\pi)\sin(2\omega\Phi)} \right) \ln(\omega) d\omega \\ &= -\frac{i\pi}{2} \int_0^{+i\infty} \partial_{\omega^3} \left( \frac{\omega^2/4}{\sin(\omega\pi)\sin(2\omega\Phi)} \right) d\omega = \frac{i\pi}{2} \partial_{\omega^2} \left( \frac{\omega^2/4}{\sin(\omega\pi)\sin(2\omega\Phi)} \right) \Big|_{\omega=0} \\ &= \frac{i\pi}{2} \frac{1}{4} \frac{1}{\pi 2\Phi} \left( \frac{(\pi)^2}{3} + \frac{(2\Phi)^2}{3} \right) = \frac{i}{12} \left( \frac{(\pi)^2}{4\Phi} + \Phi \right), \\ B'_1 &= \int_{-i\infty+0^+}^{+i\infty+0^+} \frac{i/4}{\sin(\omega\pi)\sin(2\omega\Phi)} d\omega = \int_0^{+\infty} \partial_{\nu} \left( \frac{\nu^2/2}{\sinh(\nu\pi)\sinh(2\nu\Phi)} \right) \frac{1}{\nu} d\nu. \end{aligned} \quad (141)$$

Shifting the path of integration to  $\text{Re}\omega = \delta > 0$  in the last integral term of (140), and using residue theorem (first and second order) for poles captured during it, we obtain the complete asymptotic expansion of  $\ln\chi_{\Phi,N}(\alpha)$  and thus of  $\chi_{\Phi,N}(\alpha)$ .

Remark : We note that, using (96), we can also simplify  $B'_1$  following,

$$\begin{aligned} B'_1 &= \frac{i}{4} \int_{-i\infty}^{+i\infty} \frac{\left(\frac{\omega}{\sin(2\omega\Phi)} - \frac{1}{2\Phi}\right)}{\omega \sin(\omega\pi)} d\omega + \frac{i}{8\Phi} \int_{-i\infty+0^+}^{+i\infty+0^+} \frac{1}{\omega \sin(\omega\pi)} d\omega \\ &= \frac{i}{4} \int_{-i\infty}^{+i\infty} \frac{\left(\frac{\omega}{\sin(2\omega\Phi)} - \frac{1}{2\Phi}\right)}{\omega \sin(\omega\pi)} d\omega - \frac{\ln(2)}{4\Phi}. \end{aligned} \quad (142)$$

## 8) Explicitly causal expression in time domain for a dispersive wedge-shaped region

### 8.1) Basic elements on causality and integral expressions

Let us consider the total field  $u(\rho, \varphi; i\omega)$  in presence of a 2D scatterer, enclosed in the region outside the free space sector  $|\varphi| < \Phi$ , when it is illuminated by a plane wave,

$$u^i(\rho, \varphi; i\omega) = U^i(i\omega) e^{ik\rho \cos(\varphi - \varphi_0)} \text{ with } ik = \frac{i\omega}{c}, \quad (143)$$

We assume that, in the domain  $|\arg(i\omega)| \leq \pi/2$ ,  $U^i(i\omega)$  and  $ik$  are holomorphic functions of  $i\omega$ , and  $|\arg(ik)| \leq \pi/2$ . The function  $U^i(i\omega)$  is  $O(\omega^{-b})$  as  $|\operatorname{Re}\omega| \rightarrow \infty$ ,  $b > 1$ , while  $u(\rho, \varphi; i\omega)$  satisfies the conditions (a'), (b') and (c'). The time domain response  $\bar{u}^i(\rho, \varphi; t)$  (or  $\bar{u}(t)$ ) is then given by,

$$\bar{u}^i(\rho, \varphi; t) = \lim_{A \rightarrow \infty} \frac{1}{2\pi i} \int_{-iA+\sigma}^{+iA+\sigma} u^i(\rho, \varphi; s) e^{st} ds, \quad \sigma \geq 0. \quad (144)$$

Closing the contour of integration at infinity, we note that

$$\bar{u}^i(\rho, \varphi; t) = 0 \text{ for } t < \mathcal{T}(\rho, \varphi) = \frac{-\rho \cos(\varphi - \varphi_0)}{c_\infty}, \quad (145)$$

where  $c_\infty = \lim_{|\omega| \rightarrow \infty} c > 0$ , and we can write,

$$u^i(\rho, \varphi; s) = \int_{\mathcal{T}(\rho, \varphi)}^{\infty} \bar{u}^i(\rho, \varphi; t) e^{-st} dt. \quad (146)$$

The time domain field  $\bar{u}(\rho, \varphi; t)$  is real. It is here assumed to be bounded, and, in presence of the wedge-shaped region illuminated by  $\bar{u}^i(\rho, \varphi; t)$ , it is causal with respect to the front of the incident wave, so that we have,

$$\bar{u}(\rho, \varphi; t) = 0 \text{ as } t < \mathcal{T}(\rho, \varphi). \quad (147)$$

Noticing that  $\mathcal{T}(\rho, \varphi) > \mathcal{T}_0(\rho) = -\rho/c_\infty$ , we can add that we have,

$$\bar{u}(\rho, \varphi; t) = \bar{u}^i(\rho, \varphi; t) = 0 \text{ as } t < \mathcal{T}_0(\rho) = -\rho/c_\infty. \quad (148)$$

We then have,



$$u(\rho, \varphi; s) = \int_{-\rho/c_\infty}^{\infty} \bar{u}(t) e^{-st} dt, \quad (149)$$

which implies,

$$\begin{aligned} u(\rho, \varphi; s^*) &= (u(\rho, \varphi; s))^*, \\ u(\rho, \varphi; s) &= O(e^{s\rho/c}) \text{ for } \operatorname{Re}(s) > 0, \end{aligned}$$

and thus no singularity of  $u(\rho, \varphi; s)$  as  $\operatorname{Re}(s) > 0$ . Therefore, we can write,

$$\begin{aligned} \bar{u}(t) &= \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{+i\infty+\sigma} u(\rho, \varphi; s) e^{st} ds, \quad \sigma \geq 0, \\ \bar{u}(t) &= \frac{1}{2\pi} \left( \int_0^\infty u(\rho, \varphi; i\omega) e^{i\omega t} d\omega + \int_{-\infty}^0 u(\rho, \varphi; i\omega) e^{i\omega t} d\omega \right) \\ &= \frac{1}{\pi} \operatorname{Re} \left( \int_0^\infty u(\rho, \varphi; i\omega) e^{i\omega t} d\omega \right). \end{aligned} \quad (150)$$

Rather to express causality from analyticity of  $u(\rho, \varphi; s)$  for complex  $s = i\omega$ , we now aim to reduce  $\bar{u}(t)$  to an explicitly causal expression depending on position, i.e. an integral on a 'finite continuous' path  $\mathcal{H}$  whose ends depend on  $(\rho, \varphi)$  and  $t$ .

For that, we will transform the expressions in (150), from the integral representation of  $u$  with spectral function  $f(\alpha + \varphi, i\omega)$ . After analysing the properties of  $f(\alpha + \varphi, i\omega)$ , in particular those concerning the positions of their singularities in  $\alpha$  for real  $\omega$ , we will define and use some transform of it, that presents some intricate causality properties. Dispersive multilayered wedges with higher order boundary conditions can be then considered. To simplify, we assume in what follows that, in the free space,

$$c = \frac{\omega}{k} \text{ is a real positive constant independent of } \omega. \quad (151)$$

Remark : the problem when  $c = \frac{\omega}{k}$  depends on  $\omega$ , i.e. when the free space is a homogeneous dispersive medium, is studied in [13]-[14] for the diffraction by a passive impedance wedge, and we note in general that, from (148),

$$\int_{-i\infty+\sigma}^{+i\infty+\sigma} u(\rho', \varphi; s) e^{-s\rho/c+s\tau} ds = 0 \text{ as } -\frac{\rho}{c_\infty} + \tau < -\frac{\rho'}{c_\infty}. \quad (152)$$

## 8.2) Spectral function $f(\alpha + \varphi; i\omega)$ and elementary properties

We consider that the spectral function  $f(\alpha + \varphi; i\omega)$ , in Sommerfeld-Maliuzhinets representation (3) of field  $u$ , has poles of Geometrical Optics on real axis whose positions are independent of  $\omega$ , in particular those related to incident and reflected angles, while its complex singularities, whose positions depend analytically on  $i\omega$ , generally derive from singularities of reflection coefficients  $R^\pm(\beta; i\omega)$  for both faces.

For now, we assume that each face is composed of passive materials (and metamaterials), characterized by complex relative permittivity  $\epsilon_r$  and permeability  $\mu_r$  with  $|\arg(i\epsilon_r)| < \pi/2$  and  $|\arg(i\mu_r)| < \pi/2$ , which cover non penetrable surfaces verifying passive impedance boundary conditions. As  $\epsilon_r - 1$  and  $\mu_r - 1$  tend to zero for  $\omega \rightarrow \infty$  from causality principle [21], the contribution of materials vanishes at high frequencies while the one of impedance surface can continue to be present.

The function  $f(\alpha'; i\omega)$  is meromorphic in the domain  $|\operatorname{Re}(\alpha')| \leq \pi + \Phi$ , with real angle poles (independent of  $\omega$ ), and possible complex angle poles (depending analytically on  $i\omega$ ), while, at fixed  $\omega$ ,  $f(\alpha'; i\omega)$  is bounded at infinity in this domain. Concerning the trajectories  $\alpha' = \alpha_s(i\omega)$  for complex poles, we consider the following properties :

Each trajectory described by complex poles of  $f(\alpha'; i\omega)$ ,  $\alpha' = \alpha_s(i\omega)$ , numbered  $s \in N$ , when  $\omega$  varies from  $\pm 0^+$  to  $\pm A$ , can cross the domain with  $|\operatorname{Re}(\alpha')| < \pi + \Phi - \epsilon_A$ , with  $\epsilon_A > 0$ , but these trajectories have no end points in any closed subdomain of it (with  $\epsilon_A \rightarrow 0^+$  as  $A \rightarrow \infty$ ); in other words, we suppose that the end points are outside the band  $|\operatorname{Re}(\alpha')| < \pi + \Phi - \epsilon_A$ , or at infinity; Moreover, we take precaution that we have no cusp ( $\partial_\omega(\alpha_s(i\omega)) \neq 0$ ). (153)

Besides, using the single face expression (33) of  $f$ , we note that,

$$f(\alpha^* + \varphi; (i\omega)^*) = (f(\alpha + \varphi; i\omega))^*, \quad (154)$$

and thus, we can add that  $\alpha_s$  verifies,

$$\alpha_s((i\omega)^*) = (\alpha_s(i\omega))^*. \quad (155)$$

The general assumption given in (153) has no use for a wedge with passive (dispersive) impedance boundaries, since, in this case, we have no pole depending on  $\omega$  in the band  $|\operatorname{Re}(\alpha)| \leq \pi + \Phi$  [13]-[14], but must be considered when the faces are composed of a multilayer (see appendix A) and  $f(\alpha; i\omega)$  has possible active poles depending on  $\omega$  in the band  $|\operatorname{Re}(\alpha)| \leq \pi + \Phi$ .

### 8.3) Elementary transform $\mathcal{F}_A^\pm$ of $f$ and its analytical continuation $\mathcal{F}_{a,A}^\pm$

#### 8.3.1) Definitions of domains $\Omega^\pm$ and of transforms $\mathcal{F}_A^\pm$ , for real $\pm \omega > 0$

The spectral function  $f(\alpha + \varphi; i\omega)$  verifies the property (b), and thus we have,

$f(\alpha'; i\omega)$  is bounded and regular in  $\{\alpha' = \alpha + \varphi, \alpha \in \Omega, |\varphi| \leq \Phi\}$  where  $\Omega$  is the region of the strip  $|\operatorname{Re}(\alpha)| \leq \pi$  with  $\operatorname{Re}(i\omega \cos \alpha) > 0$ , i.e. the region where the function  $e^{i\omega \rho \cos \alpha / c}$  diverges as  $\rho \rightarrow \infty$ . (156)

For real  $\pm \omega > 0$ , the domain  $\Omega$  is denoted  $\Omega^\pm$ . It is composed of the subdomains  $\Omega_{\epsilon=1}^\pm$  and  $\Omega_{\epsilon=-1}^\pm$ . Each  $\Omega_\epsilon^\pm$  is bounded by the loops  $\mathcal{L}_\epsilon^\pm$  defined following,

$$\begin{aligned}\mathcal{L}_\epsilon^+ &\equiv ]\epsilon(i\infty + \pi^+), \epsilon(\pi + i0^+)] \cup [\epsilon(\pi + i0^+), \epsilon i0^+] \cup [\epsilon i0^+, \epsilon i\infty[, \\ \mathcal{L}_\epsilon^- &\equiv [\epsilon(i\infty), \epsilon i0^+ [ \cup [\epsilon i0^+, \epsilon(i0^+ - \pi)] \cup [\epsilon(i0^+ - \pi), \epsilon(-\pi + i\infty)[, \end{aligned} \quad (157)$$

where we notice that  $\mathcal{L}_\epsilon^+ \equiv (\mathcal{L}_{-\epsilon}^-)^*$ , and thus,  $\Omega_\epsilon^\pm$  is the conjugate of  $\Omega_{-\epsilon}^\mp$ . In this domain of regularity,  $f$  is from (33),  $O(\omega^{-v})$  as  $|\operatorname{Re}\omega| \rightarrow \infty$  for any finite  $\operatorname{Im}\omega \leq 0$ , with  $v > 1$ . We then define, as  $\alpha \in \Omega^\pm$ , for any  $\tau$  real,

$$\begin{aligned}\mathcal{F}_A^+(\alpha + \varphi, \tau) &= \frac{1}{2\pi} \int_0^{+A} f(\alpha + \varphi; i\omega) e^{i\omega\tau} d\omega, \\ \mathcal{F}_A^-(\alpha + \varphi, \tau) &= \frac{1}{2\pi} \int_{-A}^0 f(\alpha + \varphi; i\omega) e^{i\omega\tau} d\omega, \end{aligned} \quad (158)$$

with  $A$  a positive real constant. This expression also permits to consider,  $\tau \equiv \tau_e(\alpha) = \rho \cos\alpha / c + t$ , with  $\alpha$  belonging to the frontier  $\mathcal{L}^\pm \equiv \mathcal{L}_{\epsilon=1}^\pm \cup \mathcal{L}_{\epsilon=-1}^\pm$ , where  $\cos\alpha$  is real, and to write,

$$\begin{aligned}\mathcal{F}_A^+(\alpha + \varphi, \tau_e(\alpha)) &= \frac{1}{2\pi} \int_0^{+A} f(\alpha + \varphi; i\omega) e^{i\omega\tau_e(\alpha)} d\omega, \\ \mathcal{F}_A^-(\alpha + \varphi, \tau_e(\alpha)) &= \frac{1}{2\pi} \int_{-A}^0 f(\alpha + \varphi; i\omega) e^{i\omega\tau_e(\alpha)} d\omega, \end{aligned} \quad (159)$$

We note that, so defined, the functions  $\mathcal{F}_A^\pm$  verify,

$$\begin{aligned}(\mathcal{F}_A^\pm(\alpha + \varphi, \tau))^* &= \mathcal{F}_A^\mp(\alpha^* + \varphi, \tau), \\ (\mathcal{F}_A^\pm(\alpha + \varphi, \tau_e(\alpha)))^* &= \mathcal{F}_A^\mp(\alpha^* + \varphi, \tau_e(\alpha^*)). \end{aligned} \quad (160)$$

### 8.3.2) Definition of analytical continuation $\mathcal{F}_{a,A}^\pm$ of $\mathcal{F}_A^\pm$ in the domain with poles of $f$

We can then consider the analytical continuation  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau_e(\alpha))$  (resp.  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau)$ ) of  $\mathcal{F}_A^\pm(\alpha + \varphi, \tau_e(\alpha))$  (resp.  $\mathcal{F}_A^\pm(\alpha + \varphi, \tau)$ ) in the domain with poles of  $f$ , complementary to  $\Omega^\pm$  in the band  $|\operatorname{Re}(\alpha)| \leq \pi$ , which corresponds to the part of the domain  $|\operatorname{Re}(\alpha)| \leq \pi$  where  $e^{i\omega\tau_e(\alpha)} = O(1)$  as  $\pm \omega > 0$ . The frontiers, from which  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau_e(\alpha))$  (resp.  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau)$ ) can be continued, are the lines  $\{\alpha = ix \mp 0^+ \operatorname{sign}(x), x \in R\}$  and the real axis, where  $\mathcal{F}_A^\pm(\alpha + \varphi, \tau_e(\alpha))$  (resp.  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau)$ ) is known.

Because of (153), this continuation can be done without any singularity and cut in it, and on its boundary, as  $|\varphi| < \Phi - \epsilon_A$ , even when  $A \rightarrow \infty$ . The function  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau_e(\alpha))$  (resp.  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau)$ ) will be then regular in the domain  $\Omega^{\pm*} \equiv \Omega^\mp$  of  $\alpha$ , where  $e^{i\omega\tau_e(\alpha)} = O(1)$ , bounded by  $\mathcal{L}^{\pm*} \equiv \mathcal{L}^\mp$ , in complement of the loop  $\mathcal{L}^\pm \equiv \mathcal{L}_{\epsilon=1}^\pm \cup \mathcal{L}_{\epsilon=-1}^\pm$  on which  $\mathcal{F}_A^\pm(\alpha + \varphi, \tau_e(\alpha))$  (resp.  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau)$ ) was initially definite.

Remark : As mentioned earlier, for the particular case of the diffraction by a passive impedance wedge,  $f(\alpha + \varphi; i\omega)$  is regular as  $|\text{Re}(\alpha + \varphi)| \leq \pi + \Phi$  except at real Geometrical Optics poles, and thus in  $\Omega^{\pm*}$ , and we can directly consider the integral expression of  $\mathcal{F}_A^\pm$  for  $\mathcal{F}_{a,A}^\pm$  in it, but it is no more the case for a multilayered wedge.

### 8.3.3) An example of continuation for an integral with a continuous line of poles

Let us consider the basic example of the exponential integral  $E_1$  defined with,

$$E_1(z) = e^{-z} \int_{-\infty}^0 \frac{e^s}{z-s} ds, \text{ as } |\arg(z)| < \pi,$$

$$E_1(z) = -\gamma - \ln z - \sum_{n \geq 1} \frac{(-z)^n}{nn!}. \quad (161)$$

The trajectory of poles in the integral is a straight semi-line from 0 to  $-\infty$ , which gives us the cut of  $E_1(z)$ , the same than for  $\ln z$ . An analytical continuation of  $E_1(z)$  through this cut changes its determination.

By definition, we continue an analytical function by successive use of Taylor expansions, whose domains of convergence are limited by the radius of convergence (determined by circle with first singularity encountered), which amounts to say that we increase continuously the domain of holomorphy, and thus deform in our case the cut, to avoid  $z$  to cross it. The new cut is a curved semi-line  $L_{-\infty,0}$ , with same ends points 0 and  $-\infty$ , from which the continuation can be named  $E_{1,L_{-\infty,0}}(z)$ .

Now, an remarkable result, due to the use of Cauchy theorem, is that this continuation can directly be done from the integral representation (161) by 'continuous' deformation of the integration path, and thus, by a change of the initial path to  $L_{-\infty,0}$ , which gives us,

$$E_{1,L_{-\infty,0}}(z) = e^{-z} \int_{L_{-\infty,0}} \frac{e^s}{z-s} ds, \text{ as } z \notin L_{-\infty,0},$$

$$E_{1,L_{-\infty,0}}(z) = -\gamma - \ln_L z - \sum_{n \geq 1} \frac{(-z)^n}{nn!} \text{ with } \ln_L \neq \ln, \quad (162)$$

This kind of results will be used to take account of complex poles of  $f$  depending of  $\omega$ , for the derivation of the continuation  $\mathcal{F}_{a,A}^\pm$  of  $\mathcal{F}_A^\pm$ , provided (153) is satisfied.

### 8.3.4) Expression of $\mathcal{F}_{a,A}^\pm$ in vicinity of real poles of $f$ independent of $\omega$

On the path  $\mathcal{L}^\pm = \mathcal{L}_{\epsilon=1}^\pm \cup \mathcal{L}_{\epsilon=-1}^\pm$  of the strip  $|\text{Re}(\alpha)| \leq \pi$ , we have real Geometrical Optics singularities of  $f(\alpha + \varphi; i\omega)$  at  $\alpha + \varphi = \alpha_r$ , that have the particularity to be independent of  $\omega$  (for incident and reflected GO waves), as  $|\varphi| < \Phi$ . To take account of

them for the continuation on  $\mathcal{L}^\pm$ , we can write,

$$\begin{aligned} \mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau_e(\alpha)) &= \frac{\pm 1}{2\pi} \int_0^{\pm A} (f(\alpha + \varphi; i\omega) e^{i\omega\tau_e(\alpha)} \\ &\quad - \sum_r \frac{f_r(i\omega) e^{i\omega\tau_e(\alpha)}}{\alpha + \varphi - \alpha_r}) d\omega + \frac{1}{2\pi} \sum_{\alpha_r} g_r(\alpha + \varphi, \tau_e(\alpha)), \end{aligned} \quad (163)$$

where

$$f_r(i\omega) = \frac{1}{[\partial_\alpha(1/f)](\alpha_r, i\omega)}, \quad g_r^\pm(\alpha + \varphi, \tau_e(\alpha)) = \frac{\pm \int_0^{\pm A} f_r(i\omega) e^{i\omega\tau_e(\alpha)} d\omega}{\alpha + \varphi - \alpha_r}. \quad (164)$$

with  $|\operatorname{Re}(\alpha)| < \pi + \Phi$ . This expression can also be considered for  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau)$ .

#### 8.3.4) Analytical continuation $\mathcal{F}_{a,A}^\pm$ of $\mathcal{F}_A^\pm$ for complex poles of $f$ depending of $\omega$

In the case of a multilayered wedge, we have to consider the presence of poles  $\alpha_s(\omega)$  of  $f(\alpha; i\omega)$  crossing the domain  $|\operatorname{Re}(\alpha)| < \pi + \Phi$  and verifying (153). By definition,  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau_e(\alpha))$  is the regular continuation of  $\mathcal{F}_A^\pm(\alpha + \varphi, \tau_e(\alpha))$  in  $\Omega^{\pm*}$ , bounded by  $\mathcal{L}^{\pm*}$ , where  $e^{i\omega\tau_e(\alpha)} = O(1)$  as  $\pm\omega > 0$ , with no branch point and cut in  $\Omega^{\pm*}$  and on  $\mathcal{L}^{\pm*}$  as  $|\varphi| < \Phi - \epsilon_A$  (note :  $\epsilon_A \rightarrow 0^+$  as  $A \rightarrow \infty$ ). For that, we write,

$$\begin{aligned} \mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau_e(\alpha)) &= \frac{\pm 1}{2\pi} \int_0^{\pm A} (f(\alpha + \varphi; i\omega) e^{i\omega\tau_e(\alpha)} \\ &\quad - \sum_s \frac{f_s(i\omega) e^{i\omega\tau_e(\alpha)}}{\alpha + \varphi - \alpha_s(i\omega)}) d\omega + \frac{1}{2\pi} \sum_s g_s^\pm(\alpha + \varphi, \tau_e(\alpha)), \end{aligned} \quad (165)$$

where we have,

$$\begin{aligned} f_s(-i\omega^*) &= (f_s(i\omega))^*, \quad \alpha_s(-i\omega^*) = (\alpha_s(i\omega))^*, \\ f_s(i\omega) &= \frac{1}{[\partial_\alpha(1/f)](\alpha_s(i\omega); i\omega)}, \\ g_s^\pm(\alpha + \varphi, \tau_e(\alpha)) &= \pm \int_{\pm 0^+}^{\pm A} \frac{f_s(i\omega) e^{i\omega\tau_e(\alpha)}}{\alpha + \varphi - \alpha_s(i\omega)} d\omega, \\ &= \pm \int_{\mathcal{T}_A^\pm} \frac{f_s(i\omega) e^{i\omega\tau_e(\alpha)}}{\alpha + \varphi - \alpha_s(i\omega)} \frac{1}{i\partial_{i\omega}(\alpha_s(i\omega))} d\alpha_s(i\omega), \\ &= \pm \int_{\alpha_{s,0^\pm}}^{\alpha_{s,\pm A}} \frac{f_s(i\omega) e^{\tau_e(\alpha)(\alpha_s^{(-1)}(\alpha_s))}}{\alpha + \varphi - \alpha_s} (-i\partial_v(\alpha_s^{(-1)}(v))|_{v=\alpha_s}) d\alpha_s, \end{aligned} \quad (166)$$

with  $\mathcal{T}_A^\pm = \{\alpha_s(i\omega), \omega \in ]0^\pm, \pm A]\}$ ,  $i\omega = \alpha_s^{(-1)}(\alpha_s(i\omega))$ ,  $\alpha_{s,\pm A} = \alpha_s(\pm iA)$  and  $\alpha_{s,0^\pm} = \alpha_s(i0^\pm)$ . From (153),  $\partial_\omega(\alpha_s(i\omega)) \neq 0$  (no cusp), and the endpoints value  $\alpha_{s,0^\pm}$  and  $\alpha_{s,\pm A}$  can be considered outside the band  $|\operatorname{Re}(\alpha)| < \pi + \Phi - \epsilon_A$ .

In this hypothesis,  $g_s$  can be analytically continued as in (162), from  $\alpha$  on the imaginary axis to  $\alpha$  in  $\Omega^{\pm*}$ , by continuously deforming  $\mathcal{T}_A^{\pm}$  as  $\alpha + \varphi$  approaches the trajectory of  $\alpha_s$ , in order to avoid to cross it.

Thereafter, taking account of the continuation with (163) and (165), we have in general,

$$\begin{aligned} \mathcal{F}_{a,A \rightarrow \infty}^{\pm}(\alpha + \varphi, \tau_e(\alpha)) \text{ is regular in } \Omega^{\pm*} \text{ and on } \mathcal{L}^{\pm} \cup \mathcal{L}^{\pm*} \\ \text{for } \tau_e(\alpha) = \rho \cos \alpha / c + t \text{ with arbitrary real } t, |\varphi| < \Phi. \end{aligned} \quad (167)$$

These results can also be considered for  $\mathcal{F}_{a,A}^{\pm}(\alpha + \varphi, \tau)$ , as  $\tau$  is real.

Remark : Considering (160) and the definition of  $\mathcal{F}_{a,A}^{\pm}$ , we note that,

$$(\mathcal{F}_{a,A}^{\pm}(\alpha + \varphi, \tau_e(\alpha)))^* = \mathcal{F}_{a,A}^{\mp}(\alpha^* + \varphi, \tau_e(\alpha^*)). \quad (168)$$

#### 8.4) Fourier transform of $f$ and properties related to causality

After noting the regularity of  $f(\alpha' + \varphi; i\omega)$  on imaginary axis for  $|\text{Im}\alpha'| > 0$ , we define on it,

$$\mathcal{F}_A(\alpha' + \varphi, \tau) = \frac{1}{2\pi} \int_{-A-i0^+}^{+A-i0^+} f(\alpha' + \varphi, i\omega) e^{i\omega\tau} d\omega, \quad (169)$$

for any  $\tau$  real. Considering the single face expression (33), which is valid for  $\text{Re}(ik(\cos\alpha - \cos(\Phi_e^{\pm} \mp \varphi_o))) > 0$ ,  $0 < \Phi_e^{\pm} < \Phi$ , and thus, the expression (35) for  $f(\alpha + \varphi)$  as  $\text{Re}(ik\cos\alpha) > \text{Re}(ik)$ ,  $|\varphi| \leq \Phi$ , we can, for  $\text{Re}\alpha' = 0$  as  $\alpha' \neq 0$ , interchange the order of the integration (permitted even for  $A \rightarrow \infty$ ) and write,

$$\begin{aligned} \mathcal{F}_A(\alpha' + \varphi, \tau) = \frac{1}{4\pi} \int_{0^+}^{\infty} \int_{-A}^{+A} (-iku(\rho, \varphi) \sin\alpha' \\ - \frac{\partial u}{\rho \partial \varphi}(\rho, \varphi)) e^{i\omega t'} d\omega d\rho, \end{aligned} \quad (170)$$

as  $t' = -\rho' \cos\alpha' / c + \tau$ , where  $u$  satisfies (149) and the conditions (a'), (b') and (c').

Since  $\bar{u}(t)$  is causal with respect to the front of the incident plane wave, the contour of integration in  $\omega$  can be closed at infinity for  $\text{Im}(\omega) < 0$  when  $t' < -\rho'/c$ , and we can write,

$$\mathcal{F}_{A \rightarrow \infty}(\alpha + \varphi, \tau) \rightarrow 0 \text{ for } \text{Re}\alpha = 0, |\varphi| \leq \Phi, \alpha \neq 0 \text{ at any fixed } \tau < 0. \quad (171)$$

We then consider the analytical continuation  $\mathcal{F}_{a,A}(\alpha + \varphi, \tau)$  of  $\mathcal{F}_A(\alpha + \varphi, \tau)$ , for any  $\tau$  real, with no cut and no singularities as  $|\text{Re}(\alpha + \varphi)| < \pi + \Phi - \epsilon_A$ , except the poles of Geometrical Optics on real axis. For that, we exploit that, in our construction of  $\mathcal{F}_{a,A}^{\pm}$  with (163) and (165),  $\tau_e(\alpha)$  remains an independent function that we are free to take as a constant, and note the regularity of  $\mathcal{F}_{a,A}^{\pm}(\alpha + \varphi, \tau)$  and  $\mathcal{F}_A^{\mp}(\alpha + \varphi, \tau)$  in  $\Omega^{\pm*} \equiv \Omega^{\mp}$ .

Therefore, we can write,

$$\begin{aligned}\mathcal{F}_{a,A}(\alpha + \varphi, \tau) &= \mathcal{F}_{a,A}^+(\alpha + \varphi, \tau) + \mathcal{F}_A^-(\alpha + \varphi, \tau) \text{ in } \Omega^{+*} \equiv \Omega^-, \\ \mathcal{F}_{a,A}(\alpha + \varphi, \tau) &= \mathcal{F}_A^+(\alpha + \varphi, \tau) + \mathcal{F}_{a,A}^-(\alpha + \varphi, \tau) \text{ in } \Omega^{-*} \equiv \Omega^+, \end{aligned} \quad (172)$$

with  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau) = \mathcal{F}_A^\pm(\alpha + \varphi, \tau)$  in  $\Omega^{\mp*} \equiv \Omega^\pm$  and on their frontiers  $\mathcal{L}^{\mp*} \equiv \mathcal{L}^\pm$ . The regularity of  $\mathcal{F}_{a,A}$  in  $\alpha$  allows us to extend (171), and we can now assume a vanishing property for  $\mathcal{F}_{a,A \rightarrow \infty}$ , as follows,

$$\mathcal{F}_{a,A \rightarrow \infty}(\alpha + \varphi, \tau) \rightarrow 0 \text{ at any fixed } \tau < 0 \text{ as } |\operatorname{Re}(\alpha + \varphi)| < \pi + \Phi. \quad (173)$$

The factor  $\tau_e(\alpha)$  being real on  $\mathcal{L}^\pm \cup \mathcal{L}^{\pm*}$ , we deduce that, for  $\alpha \in \mathcal{L}^\pm \cup \mathcal{L}^{\pm*}$ ,

$$\mathcal{F}_{a,A}(\alpha + \varphi, \tau_e(\alpha)), \text{ and thus } \mathcal{F}_{a,A}^+(\alpha + \varphi, \tau_e(\alpha)) + \mathcal{F}_{a,A}^-(\alpha + \varphi, \tau_e(\alpha)), \quad (174)$$

verifies the causality property,

$$\mathcal{F}_{a,A \rightarrow \infty}(\alpha + \varphi, \tau_e(\alpha)) \rightarrow 0 \text{ for } \tau_e(\alpha) < 0, \text{ as } \alpha \in \mathcal{L}^\pm \cup \mathcal{L}^{\pm*}, |\varphi| < \Phi. \quad (175)$$

Remark : Considering the problem when  $c = \frac{\omega}{k}$  depends on  $\omega$ , we note in general from (152) that,

$$\int_{-\infty - i0^+}^{+\infty - i0^+} u(\rho', \varphi; i\omega) e^{-s\rho' \cos\alpha'/c + s\tau} d\omega = 0 \text{ as } \tau < 0 \text{ when } \operatorname{Re}\alpha' = 0, \quad (176)$$

and thus, that (171) and (173) apply in whole generality, when the free space is an homogeneous dispersive medium.

### 8.5) Explicitly causal expression of the field in time domain with $\mathcal{F}_{a,A \rightarrow \infty}$

From the expression (3) of  $u$ , and the domain of regularity of  $\mathcal{F}_{a,A}^\pm(\alpha + \varphi, \tau_e(\alpha))$ , the field in time domain is given by,

$$\begin{aligned}\bar{u}(t) &= \frac{1}{2\pi} \left( \int_0^\infty u(\rho, \varphi; i\omega) e^{i\omega t} d\omega + \int_{-\infty}^0 u(\rho, \varphi; i\omega) e^{i\omega t} d\omega \right) \\ &= \lim_{A \rightarrow \infty} \frac{1}{2\pi i} \left( \int_{\mathcal{L}^+} \mathcal{F}_{a,A}^+(\alpha + \varphi, \tau_e(\alpha)) d\alpha + \int_{\mathcal{L}^-} \mathcal{F}_{a,A}^-(\alpha + \varphi, \tau_e(\alpha)) d\alpha \right), \end{aligned} \quad (177)$$

with  $\tau_e(\alpha) = \rho \cos\alpha/c + t$ , where the double loops  $\mathcal{L}^\pm$  are oriented, as  $\gamma$  in (3) when we take  $\pm k > 0$ ,  $a_{1,2} = 0$  and  $d = 0^+$ .

Considering the analyticity of  $\mathcal{F}_{a,A}^+(\alpha + \varphi, \tau_e(\alpha))$  and  $\mathcal{F}_{a,A}^-(\alpha + \varphi, \tau_e(\alpha))$ , we can deform in (177) both paths  $\mathcal{L}^+$  and  $\mathcal{L}^-$  (oriented) to the common path  $\mathcal{D} \equiv \mathcal{D}_+ \cup \mathcal{D}_-$  independent of  $\omega$ , with the loops  $\mathcal{D}_\pm$  (oriented as loops  $\gamma_\pm$  when we take  $a_1 = \pi - \arg(ik) - \frac{\pi}{2}$ ,  $a_2 = \pi + \arg(ik) - \frac{\pi}{2}$ , and  $d = 0^+$  in (3)) verifying,

$$\begin{aligned} \mathcal{D}_{\pm} \equiv & ] \pm (i\infty + \pi), \pm (\pi + i0^+) ] \cup [ \pm (\pi + i0^+), \pm (-\pi + i0^+) ] \\ & \cup [ \pm (-\pi + i0^+), \pm (-\pi + i\infty) ], \end{aligned} \quad (178)$$

and we can write, from (174),

$$\bar{u}(t) = \frac{1}{2\pi i} \int_{\mathcal{D}} \mathcal{F}_{a,A \rightarrow \infty}(\alpha + \varphi, \tau_e(\alpha)) d\alpha, \quad (179)$$

with  $\tau_e(\alpha)$  real on  $\mathcal{D}$ .

We then use the causality property (175) on  $\mathcal{F}_{a,A \rightarrow \infty}$  and obtain an expression for  $\bar{u}(t)$ , explicitly causal, on account of a finite path depending of time and position, following,

$$\bar{u}(t) = \frac{1}{2\pi i} \int_{\mathcal{H}(\rho,t)} \mathcal{F}_{a,A \rightarrow \infty}(\alpha + \varphi, \tau_e(\alpha)) d\alpha, \quad (180)$$

where  $\mathcal{H} = \mathcal{H}_+ \cup \mathcal{H}_-$  is the finite subset of  $\mathcal{D}$  with  $\tau_e(\alpha) = \rho \cos \alpha / c + t > 0$ , and  $\mathcal{F}_{a,A}$  is given by (174) on  $\mathcal{H}$ . Therefore, letting,

$$iv(\rho, t) = \begin{cases} i \ln(ct/\rho + \sqrt{(ct/\rho)^2 - 1}), & ct \geq \rho \\ -\arccos(ct/\rho), & -\rho < ct < \rho \end{cases}, \quad (181)$$

the finite path  $\mathcal{H}_+$  (oriented as  $\mathcal{D}_+$ ) can be described,

$$\begin{aligned} \text{as } |ct| < \rho : & [\pi + iv(\rho, t) + i0^+, -\pi - iv(\rho, t) + i0^+], \\ \text{as } ct \geq \rho : & ]\pi + iv(\rho, t), \pi + i0^+ ] \cup [\pi + i0^+, -\pi + i0^+ ] \\ & \cup [-\pi + i0^+, iv(\rho, t) - \pi], \end{aligned} \quad (182)$$

above all the real singularities of the integrand as  $|\operatorname{Re}(\alpha)| < \pi$  (due to Geometrical Optics part of the field), while  $\mathcal{H}_-$  is its inverse below the real axis.

Remark : Considering (168) for (180), we note that,

$$\bar{u}(t) = \operatorname{Re} \left( \frac{1}{\pi i} \int_{\mathcal{H}(\rho,t)} \mathcal{F}_{a,A \rightarrow \infty}^+(\alpha + \varphi, \tau_e(\alpha)) d\alpha \right). \quad (183)$$

Appendix A : Analysis of the reflection coefficient attached to a multilayered face of a wedge

Let us consider the reflection coefficient attached to each face  $\varphi = \pm \Phi$ , semi-planes composed of a passive surface with relative impedance  $\sin \theta_{imp}^{\pm}$  covered by a passive material of total depth  $d$  with relative permittivity  $\epsilon_r$  and permittivity  $\mu_r$  varying in depth



and frequency, following,

$$R^\pm(\beta) = \prod_{j=1}^{N_e} \frac{\cos\beta - \sin\theta_n^\pm}{\cos\beta + \sin\theta_n^\pm}, \quad (184)$$

where  $\beta$  is the angle with the outward normal at each face, and the complex constants  $\sin\theta_n^\pm$  have real parts of arbitrary signs [16]-[18] with  $|\operatorname{Re}(\theta_n^\pm)| \leq \pi/2$ ,  $|\arg(i\omega)| < \pi/2$ .

Let us consider to simplify the case with  $|\arg(i\omega)| = |\arg(ik)| = \pi/2$ , i.e.  $\omega$  and  $k$  real.

When the permittivity and the permeability of materials have positive real part, which is always the case at very large frequency, we can generally assume that the  $\sin\theta_n^\pm$  with  $\operatorname{Re}(\sin\theta_n^\pm) < 0$  (active modes) verify  $\arg(ik)\operatorname{Im}(\sin\theta_n^\pm) < 0$  while the  $\sin\theta_n^\pm$  with  $\operatorname{Re}(\sin\theta_n^\pm) > 0$  (passive modes) have  $\operatorname{Im}(\sin\theta_n^\pm)$  of arbitrary sign. In this case, the first poles of  $f(\alpha; i\omega)$  from imaginary axis are, for each mode, at  $\alpha = \pm(\pi + \theta_n^\pm + \Phi)$ .

Besides, on limited domains of frequencies, permittivity and permeability can have negative real part (if metamaterials are present), and the term  $\arg(ik)\operatorname{Im}(\sin\theta_n^\pm)$  for  $\sin\theta_n^\pm$  with  $\operatorname{Re}\sin\theta_n^\pm < 0$  (active modes) can have arbitrary sign. In this case, the first poles of  $f(\alpha; i\omega)$  from imaginary axis are then, for each mode, at  $\alpha = \pm(\pi + \theta_n^\pm + \Phi)$  when  $\operatorname{Re}(\sin\theta_n^\pm) > 0$  (passive modes), while they are at  $\alpha = \pm(-\theta_n^\pm + \Phi)$  (if  $\arg(ik)\operatorname{Im}(\sin\theta_n^\pm) > 0$ ) or at  $\alpha = \pm(\pi + \theta_n^\pm + \Phi)$  (if  $\arg(ik)\operatorname{Im}(\sin\theta_n^\pm) < 0$ ) when  $\operatorname{Re}(\sin\theta_n^\pm) < 0$  (active modes).

Remark : to analyse the expression of poles of  $f$  given here, the reader can refer to appendix B, where we give the solution for a wedge with arbitrary passive or active impedance faces, and we describe the suppression of non physical poles for  $f$  when active impedances are present.

Remark : We can generally assume that, as  $\omega$  tends to infinity,

$$\epsilon_r \rightarrow 1, \mu_r \rightarrow 1, \text{ and thus, } R^\pm(\beta) \rightarrow e^{-2i\omega d \cos\beta/c} \frac{\cos\beta - \sin\theta_{imp}^\pm}{\cos\beta + \sin\theta_{imp}^\pm}, \quad (185)$$

with  $\operatorname{Re}(\theta_{imp}^\pm) > 0$ , while we have, for the modes that are active with  $\operatorname{Re}(\theta_n^\pm) < 0$ ,

$$\begin{aligned} \operatorname{Re}(\theta_n^\pm) &\rightarrow 0 \text{ as } |\operatorname{Re}(\omega)| \rightarrow \infty, \\ |\operatorname{Im}(\theta_n^\pm)| &\rightarrow \infty \text{ as } \omega \rightarrow 0. \end{aligned} \quad (186)$$

Remark : let us note that, in free space, we can write  $k = \omega\sqrt{\epsilon_0\mu_0}$  ( $= \omega/c$ ) for the wave number, while for modelisation of passive metamaterials, we have to write  $k = \omega\sqrt{\epsilon}\sqrt{\mu}$  ( $= \omega\sqrt{\epsilon_r}\sqrt{\mu_r}/c$ ) to ensure passivity and  $\operatorname{Re}(ik) > 0$ .

Appendix B : Some miscellaneous properties of spectral functions for an impedance wedge with passive impedance faces

B.1) The spectral function  $f_0(\alpha)$  for a wedge with straight passive faces

B.1) The definition of  $f_0(\alpha)$  in passive impedance case

Let us consider the spectral function  $f_0(\alpha)$  associated to the diffraction of an unit plane wave of incidence angle  $\varphi_0$  by a wedge with straight faces of relative impedances  $\sin\theta^\pm$  with positive real part (passive impedances) [4]. It is given by,

$$\begin{aligned} f_0(\alpha) &= \Psi(\alpha)\sigma(\alpha)/\Psi(\varphi_0) \\ \sigma(\alpha) &= \mu\cos(\mu\varphi_0)/(\sin(\mu\alpha) - \sin(\mu\varphi_0)), \end{aligned} \quad (187)$$

where the function  $\Psi(\alpha)$ , without any zero or pole as  $\text{Re}\alpha < \Phi$ , is the solution of

$$\begin{aligned} (\sin\alpha + \sin\theta^+)\Psi(\alpha + \Phi) - (-\sin\alpha + \sin\theta^+)\Psi(-\alpha + \Phi) &= 0, \\ (\sin\alpha - \sin\theta^-)\Psi(\alpha - \Phi) - (-\sin\alpha - \sin\theta^-)\Psi(-\alpha - \Phi) &= 0, \end{aligned} \quad (188)$$

i.e., when  $\text{Re}(\sin\theta^\pm) \neq 0$ ,

$$\begin{aligned} \partial_\alpha \ln \frac{\Psi(\alpha \pm \Phi)}{\Psi(-\alpha \pm \Phi)} &= \pm \partial_\alpha \ln \frac{(\sin\alpha - \sin\theta^\pm)}{(\sin\alpha + \sin\theta^\pm)}, \\ \frac{\Psi'}{\Psi}(\alpha \pm \Phi) + \frac{\Psi'}{\Psi}(-\alpha \pm \Phi) &= S^\pm(\alpha), \\ S^\pm(\alpha) &= \pm \partial_\alpha \ln \frac{(\sin\alpha - \sin\theta^\pm)}{(\sin\alpha + \sin\theta^\pm)}, \end{aligned} \quad (189)$$

(note: the case  $\text{Re}(\sin\theta^\pm) = 0$  can be considered by taking  $\text{Re}(\sin\theta^\pm) \rightarrow 0^+$ ). The expression of  $\Psi$  can be expressed from (113)-(115), considering  $\Psi$  in place of  $\Psi_{an}$  with  $\delta = 0$ . In passive case with  $0 < \text{Re}(\sin\theta^\pm) \leq \pi/2$ , we can then write, as  $|\text{Re}\alpha| \leq \Phi + (N+1)\pi$ ,

$$\begin{aligned} \Psi(\alpha) &= B'_N \prod_{\pm} \left( \prod_{m=0}^N \left[ \Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}(\alpha \pm \Phi + \theta^\pm + m\pi)\right) \right. \right. \\ &\quad \Gamma\left(\frac{1}{2} - \frac{1}{4\Phi}(\alpha \pm \Phi - (\theta^\pm + m\pi))\right) \Gamma\left(\frac{1}{2} + \frac{1}{4\Phi}(\alpha \pm \Phi + \pi - \theta^\pm + m\pi)\right) \\ &\quad \left. \left. \Gamma\left(\frac{1}{2} - \frac{1}{4\Phi}(\alpha \pm \Phi - (\pi - \theta^\pm + m\pi))\right) \right]^{(-1)^{m+1}} \right) \\ &\quad \times \exp\left(\int_0^\infty (-e^{-\nu\pi})^{N+1} \frac{e^{-\nu\theta^\pm} + e^{-\nu(\pi-\theta^\pm)}}{(1+e^{-\nu\pi})} \frac{(1 - \cosh(\nu(\alpha \pm \Phi)))}{\nu \sinh(2\nu\Phi)} d\nu\right), \end{aligned} \quad (190)$$

where  $B'_N$  is an arbitrary constant. This expression is suitable for numerical calculus (with  $N = 1$  or  $2$ ) or to derive the analytical properties of  $\Psi(\alpha)$  (with  $N \rightarrow \infty$ ) from those of  $\Gamma$ . We note that the function  $\Psi(\alpha)$  in passive case, i.e. when  $0 < \text{Re}(\sin\theta^\pm) \leq \pi/2$ , has zeros at,

$$\begin{aligned} \alpha &= \pm (\theta^\pm + (4p+1)\Phi + 2m\pi) \\ \text{and } \alpha &= \pm (-\theta^\pm + (4p+1)\Phi + (2m+1)\pi) \end{aligned}$$

and has poles at,

$$\alpha = \pm (\theta^\pm + (4p + 1)\Phi + (2m + 1)\pi)$$

$$\text{and } \alpha = \pm (2\pi - \theta^\pm + (4p + 1)\Phi + 2m\pi)$$

for  $m \geq 0, p \geq 0$ , and that we have [12],

$$\Psi(\alpha) = A \cos\left(\frac{\pi\alpha}{2\Phi}\right) (1 + O(\alpha^\nu e^{-\mu'|\text{Im}\alpha|})), \quad (191)$$

for large  $|\text{Im}\alpha|$ , with constants  $A, \nu$ , and  $\mu' > 0$ .

B.2) Some miscellaneous results concerning integrals as  $\Phi = \pi/2$  and  $\text{Re}(\sin\theta^\pm) \geq 0$

We note that, for  $\Phi = \pi/2$ , we have,

$$\begin{aligned} & \frac{(-\sin\alpha + \sin\theta^-)}{\Psi(\alpha + \frac{\pi}{2})} - \frac{(\sin\alpha + \sin\theta^-)}{\Psi(-\alpha + \frac{\pi}{2})} \\ &= (\sin\theta^- - \sin\theta^+) \left( \frac{1}{\Psi(\alpha + \frac{\pi}{2})} - \frac{1}{\Psi(-\alpha + \frac{\pi}{2})} \right), \\ & \frac{(\sin\alpha + \sin\theta^-)}{\Psi(\alpha - \frac{\pi}{2})} - \frac{(-\sin\alpha + \sin\theta^-)}{\Psi(-\alpha - \frac{\pi}{2})} = 0. \end{aligned} \quad (192)$$

Using that  $\Psi(-\alpha)\Psi(\alpha) = A_0^2(\cos\alpha + \sin\theta^+)(\cos\alpha + \sin\theta^-)$ , and  $\Psi(\alpha)/\cos\alpha \rightarrow A_0$  as  $\alpha \rightarrow i\infty$ , we can let  $s(\alpha) = \frac{A_0(\cos\alpha + \sin\theta^-)}{\Psi(\alpha)}$  and derive, from (202) of appendix C,

$$\begin{aligned} & \frac{(\cos\alpha + \sin\theta^-)}{\Psi(\alpha)} - \frac{1}{A_0} = \\ &= \frac{-i(\sin\theta^- - \sin\theta^+)}{4\pi} \int_{-i\infty}^{+i\infty} H(\alpha') \tan\left(\frac{1}{2}(\alpha + \frac{\pi}{2} - \alpha')\right) d\alpha', \\ & \frac{i\Psi(-\alpha)/A_0^2}{(\cos\alpha + \sin\theta^+)} - \frac{i}{A_0} = \\ &= \frac{(\sin\theta^- - \sin\theta^+)}{4\pi} \int_{-i\infty}^{+i\infty} H(\alpha') \tan\left(\frac{1}{2}(\alpha + \frac{\pi}{2} - \alpha')\right) d\alpha', \end{aligned} \quad (193)$$

for  $\frac{-3\pi}{2} < \text{Re}(\alpha) < \frac{\pi}{2}$ , where  $H(\alpha') = 2\sin\alpha' / (\Psi(\alpha' + \frac{\pi}{2})(\sin\alpha' + \sin\theta^+))$  and

$$H(\alpha) = \frac{1}{\Psi(\alpha + \frac{\pi}{2})} - \frac{1}{\Psi(-\alpha + \frac{\pi}{2})} = \frac{\sin\alpha/\sin\theta^+}{\Psi(\alpha + \frac{\pi}{2})} + \frac{\sin\alpha/\sin\theta^+}{\Psi(-\alpha + \frac{\pi}{2})}. \quad (194)$$

From oddness of  $H(\alpha)$ ,  $\tan(\frac{1}{2}(\alpha + \frac{\pi}{2} - \alpha'))$  can be replaced by  $\frac{-\sin\alpha'}{(\cos(\alpha + \frac{\pi}{2}) + \cos\alpha')}$ , and we note that, after deformation of integration path, we have,

$$\begin{aligned} & \int_{-i\infty}^{+i\infty} \frac{1}{\Psi(\varphi')} \frac{\cos\varphi'}{\cos\alpha + \sin\varphi'} d\varphi' = -2\pi i \frac{\frac{\sin\alpha + \sin\theta^-}{\Psi(\alpha - \frac{\pi}{2})} - \frac{1}{A_0}}{(\sin\theta^+ - \sin\theta^-)}, \\ & \int_{-i\infty}^{+i\infty} \frac{\cos\varphi'}{\Psi(\varphi')} \frac{\cos\varphi'}{\cos\alpha + \sin\varphi'} d\varphi' = 2\pi i \sin\theta^+ \frac{\frac{\sin\alpha + \sin\theta^-}{\Psi(\alpha - \frac{\pi}{2})} - \frac{1}{A_0}}{(\sin\theta^+ - \sin\theta^-)} + \frac{i\pi}{A_0}, \end{aligned} \quad (195)$$

as  $|\operatorname{Re}\alpha| < \frac{\pi}{2}$ . Similar expressions with  $-\sin\varphi'$  in place of  $\sin\varphi'$  can be obtained from  $\Psi(-\alpha) \rightarrow \Psi(\alpha)$  when  $\theta^\pm \rightarrow \theta^\mp$  (or by continuation with capture of pole).

Appendix C : General analysis of the solution of  $s(\alpha \pm \Phi) - \varepsilon s(-\alpha \pm \Phi) = S^\pm(\alpha)$

We study here the solution of the functional equations for  $s$ ,

$$s(\alpha \pm \Phi) - \varepsilon s(-\alpha \pm \Phi) = S^\pm(\alpha), \quad (196)$$

when  $s$  is regular as  $|\operatorname{Re}\alpha| \leq \Phi$ , with  $\varepsilon$  being  $+1$  or  $-1$ . We consider the case where the functions  $S^\pm$  are analytic on the imaginary axis and  $O(\exp(-a|\operatorname{Im}\alpha|))$ ,  $a > 0$ , as  $|\operatorname{Im}\alpha| \rightarrow \infty$ , and thus  $s(i\infty) - \varepsilon s(-i\infty) = 0$ . The solution  $s$ , regular as  $|\operatorname{Re}\alpha| \leq \Phi$  (even at infinity), can be written following,

$$s(\alpha) = \frac{s(i\infty) + s(-i\infty)}{2} + \frac{s(i\infty) - s(-i\infty)}{2} \operatorname{sign}(\operatorname{Im}\alpha) + s_o(\alpha), \quad (197)$$

where  $s_o$  is absolutely integrable on any line  $\operatorname{Re}\alpha = \alpha_o$  as  $|\alpha_o| \leq \Phi$ . This type of equation has been solved in particular in the works of Maliuzhinets, Tuzhilin, Bernard for the analytical determination of the solutions of wedge diffraction problems.

We consider the case with  $s(i\infty) - s(-i\infty) = 0$ , i.e. with arbitrary  $s(i\infty) + s(-i\infty)$  when  $\varepsilon = 1$ , or with  $s(\pm i\infty) = 0$  when  $\varepsilon = -1$ . By use of Fourier transform  $\mathcal{F}(\omega)$  of  $s_o(\alpha)$ , we then obtain a system of two equations :

$$\mathcal{F}(\omega)e^{\mp i\omega\Phi} - \varepsilon\mathcal{F}(-\omega)e^{\pm i\omega\Phi} = \mathcal{H}^\pm(\omega). \quad (198)$$

Multiplying them by  $\exp(\mp i\omega\Phi)$  and making the difference, the function  $\mathcal{F}$  satisfies,

$$2i\mathcal{F}(\omega)\sin(2\omega\Phi) = \mathcal{H}^-(\omega)e^{i\omega\Phi} - \mathcal{H}^+(\omega)e^{-i\omega\Phi}. \quad (199)$$

We notice that the analytic function obtained by dividing the right member of the previous equality with  $2i\sin(2\omega\Phi)$  does verify (198), and then, that the inverse Fourier transform (integral taken in the sense of principal value) of this expression satisfies (196) for  $|\operatorname{Re}\alpha| < \Phi$ . So, we have the developed form of  $s$ , for  $|\operatorname{Re}\alpha| < \Phi$ ,

$$\begin{aligned} s(\alpha) - \frac{s(i\infty) + s(-i\infty)}{2} &= \\ &= \frac{i}{\sqrt{2\pi}} \text{v.p.} \int_{-i\infty}^{+i\infty} \left( \frac{R_+(\omega)e^{-i\omega\Phi}}{i\sin(2\omega\Phi)} - \frac{R_-(\omega)e^{i\omega\Phi}}{i\sin(2\omega\Phi)} \right) e^{-i\omega\alpha} d\omega, \\ &= \frac{i}{\sqrt{2\pi}} \sum_{\pm} \int_0^{i\infty} \frac{\pm R_{\pm}(\omega)(e^{-i\omega(\alpha\pm\Phi)} + \varepsilon e^{i\omega(\alpha\pm\Phi)})}{i\sin(2\omega\Phi)} d\omega, \\ &= \frac{1}{\sqrt{2\pi}} \sum_{\pm} \int_0^{\infty} \frac{\pm R_{\pm}(i\nu)(e^{\nu(\alpha\pm\Phi)} + \varepsilon e^{-\nu(\alpha\pm\Phi)})}{\sinh(2\nu\Phi)} d\nu, \end{aligned} \quad (200)$$

with

$$\begin{aligned}
 R_{\pm}(\omega) &= \frac{-i}{2\sqrt{2\pi}} \int_{-i\infty}^{+i\infty} S^{\pm}(\alpha') e^{i\omega\alpha'} d\alpha', \\
 R_{\pm}(i\nu) &= \frac{-i}{2\sqrt{2\pi}} \int_{-i\infty}^{+i\infty} S^{\pm}(\alpha') e^{-\nu\alpha'} d\alpha', \\
 R_{\pm}(\omega) + \varepsilon R_{\pm}(-\omega) &= 0,
 \end{aligned} \tag{201}$$

where the quantity  $s(i\infty) + s(-i\infty)$  is arbitrary when  $\varepsilon = 1$  and equal to zero when  $\varepsilon = -1$ . The term v.p. means that we take the principal value of the integral.

For the case where the function  $s$  is  $\partial v_{an}$  and  $\partial \ln(\Psi_{an})$ , we have  $\varepsilon = -1$  and  $s(\pm i\infty) = 0$ , while  $R_{\pm}$  can be evaluated explicitly by the method of residues. So, we obtain a simple integral expression of  $v_{an}$  and  $\Psi_{an}$ , corresponding to (115) and (116).

In other respects, we notice that, since the functions  $S^{\pm}$  are absolutely integrable, it is possible to change the order of integration, so that we obtain, for  $|\operatorname{Re}\alpha| < \Phi$ ,

$$\begin{aligned}
 s(\alpha) &= \frac{s(i\infty) + s(-i\infty)}{2} + \frac{-i}{8\Phi} \int_{-i\infty}^{+i\infty} d\alpha' (S^+(\alpha') \tan(\frac{\pi}{4\Phi}(\alpha + \Phi - \alpha')) \\
 &\quad - S^-(\alpha') \tan(\frac{\pi}{4\Phi}(\alpha - \Phi - \alpha'))).
 \end{aligned} \tag{202}$$

Considering  $S^{\pm} = O(\exp(-a|\operatorname{Im}\alpha|))$  and  $\varepsilon = -1$ , and developing the  $\tan(\cdot)$  terms, we have,

$$s(\alpha) = \frac{\pm 1}{8\Phi} \int_{-i\infty}^{+i\infty} d\alpha' (S^+(\alpha') - S^-(\alpha')) + O(e^{-c|\operatorname{Im}\alpha|}), \tag{203}$$

as  $\operatorname{Im}\alpha \rightarrow \pm\infty$ , with  $c > 0$  being some constants independent of  $\alpha$ . Then, letting  $s = \partial v_{an}$ , we derive  $v_{an}(\alpha) = A_1 + O(e^{-c|\operatorname{Im}\alpha|})$ , while, letting  $s = \partial \ln(\Psi_{an})$ , we obtain  $\Psi_{an} = A_2 \exp(\mu|\operatorname{Im}\alpha|)(1 + O(e^{-c|\operatorname{Im}\alpha|}))$  with  $\mu = \pi/2\Phi$ ,  $A_1$  and  $A_2$  being some constants. This behaviour, deduced from an expression valid for  $|\operatorname{Re}\alpha| < \Phi$ , remains valid in any band  $|\operatorname{Re}\alpha| < \text{constant}$ , from the use of functional equations or the deformation of the integration path.

Remark : We notice, as Maliuzhinets, that

$$\text{v.p.} \int_{-i\infty}^{+i\infty} \equiv \frac{1}{2} \left( \int_{-i\infty-0^+}^{+i\infty-0^+} + \int_{-i\infty+0^+}^{+i\infty+0^+} \right), \tag{204}$$

Remark : The expression (202) can be continued analytically for  $|\operatorname{Re}\alpha| \geq \Phi$  by considering the residue due to the poles crossing the path of integration, so that we verify easily that the continuation of this expression satisfies the initial functional equations. This latter expression continues to satisfy the functional equation when the analytic function  $S^{\pm}$ , regular on the imaginary axis, is only assumed to be absolutely integrable.

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