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# FORMULATION AND WELL-POSEDNESS OF UNILATERAL MULTIBODY DYNAMICS

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The classical theory of rigid bodies systems undergoing perfect bilateral constraints has received firm mathematical foundations for a long time. We extend it to the case where there are perfect unilateral constraints also. The formulation follows the line initiated by Schatzman (1978) and Moreau (1983). We give a parametrization-free formulation and try to identify the most general form of impact constitutive equation which is compatible with well-posedness. Then, well-posedness is proved under the assumption of analyticity of the data, since it is known, thanks to Bressan (1960) and Schatzman (1978), that uniqueness does not hold in the  $C^\infty$  setting.

## 1. Background: multibody systems with perfect bilateral constraints

Consider a finite collection of rigid bodies undergoing perfect bilateral constraints. It is classically associated with a manifold  $Q$  (called the configuration manifold of the system) of finite dimension  $d$  (called the number of degrees-of-freedom of the system).

A motion of the system is a curve on  $Q$ , that is a mapping  $q(t)$  from a real interval  $I$  into  $Q$ . When a motion is smooth enough, there exists a time-derivative  $\dot{q}(t)$  in tangent space  $T_{q(t)}Q$  also called a (generalized) velocity of the system. We shall also denote it by  $(q(t), \dot{q}(t))$ . An arbitrary element  $v$  (also denoted by  $(q, v)$ ,  $q$  being the basepoint of  $v$ ) of the tangent bundle  $TQ$  is called a (virtual) state of the system.

Given the mass distribution in each of the solids, one classically obtains the kinetic energy  $K$  which appears to define a positive definite quadratic form on each tangent space  $T_qQ$  of the configuration manifold, endowing it with a Riemannian structure. The induced scalar

product and norm will be denoted by  $(\cdot, \cdot)_q$  and  $\|\cdot\|_q$ , whereas the duality product between tangent space  $T_qQ$  and cotangent space  $T_q^*Q$  will be denoted by  $\langle \cdot, \cdot \rangle_q$ . With this notation, we have:

$$K(q, v) = \frac{1}{2} \|v\|_q^2.$$

The modelling of forces makes use of duality. Internal and external forces acting on the system in the configuration  $q$  are represented by a linear form  $f \in T_q^*Q$ . Given an arbitrary virtual velocity  $(q, v)$  of the system, the real number  $\langle f, v \rangle_q$  is the virtual power of internal and external forces for the virtual velocity  $(q, v)$ . The cotangent vector  $f$  is usually allowed to depend on the current state  $(q, v)$  and also on time  $t$ . Hence, we are given a mapping  $f : TQ \times \mathbb{R}^+ \rightarrow T^*Q$  satisfying:

$$\forall (q, v) \in TQ, \quad \forall t \in \mathbb{R}^+, \quad \Pi_Q^*(f(q, v; t)) = q, \quad (1)$$

( $\Pi_Q^*$  is the natural projection on  $T^*Q$ ), called the force mapping.

Since such a structure appears systematically in the modelling of the dynamics of rigid bodies systems with perfect bilateral constraints, we are led to make the following definition.

**Definition 1** *A simple discrete mechanical system is a pair  $(Q, f)$  where:*

- *$Q$  is a finite-dimensional Riemannian manifold called the configuration manifold.*
- *$f : TQ \times \mathbb{R}^+ \rightarrow T^*Q$  is a mapping satisfying (1), called the force mapping.*

Now, consider an arbitrary smooth motion  $q(t)$  of the system. The power of inertial forces is by definition the time derivative of the kinetic energy:

$$\frac{d}{dt}K(q, \dot{q}) = \frac{d}{dt} \frac{1}{2} \|\dot{q}(t)\|_{q(t)}^2 = \left( \frac{D}{dt} \dot{q}(t), \dot{q}(t) \right)_{q(t)} = \left\langle \flat \frac{D}{dt} \dot{q}(t), \dot{q}(t) \right\rangle_{q(t)},$$

where  $D/dt$  denotes the covariant derivative along the curve  $q(t)$  associated with the Levi-Civita connection, and  $\flat$  the isomorphism from  $TQ$  onto  $T^*Q$  canonically associated with the Riemannian metric. For any virtual velocity  $(q, v)$  of the system, the real number  $\langle \flat D\dot{q}/dt, v \rangle_q$  is naturally called the virtual power of inertial forces in the virtual velocity  $(q, v)$ . Given any local parametrization of the system (local chart), it admits the well-known representation:

$$\flat \frac{D}{dt} \dot{q} = \left( \frac{d}{dt} \frac{\partial}{\partial \dot{q}^i} K(q, \dot{q}) - \frac{\partial}{\partial q^i} K(q, \dot{q}) \right) dq^i.$$

The fundamental principle of classical dynamics asserts that the virtual power of inertial forces should equal the virtual power of external and internal forces, for any virtual velocity. We obtain immediately the equation of motion:

$$\forall t, \quad \flat \frac{D}{dt} \dot{q}(t) = f(q(t), \dot{q}(t); t).$$

Next, we are given an initial state  $(q_0, v_0) \in TQ$ . Then, the evolution problem associated with the dynamics of rigid bodies system with perfect bilateral constraints is the Cauchy problem:

**Problem I.** Find  $T > 0$  and  $q \in C^2([0, T[; Q)$  such that:

- $(q(0), \dot{q}(0)) = (q_0, v_0)$ ,
- $\forall t \in [0, T[, \quad \flat \frac{D}{dt} \dot{q}(t) = f(q(t), \dot{q}(t), t)$ .

Well-posedness is ensured by the following.

**Theorem 2 (Cauchy)** *The Riemannian configuration manifold is assumed to be of class  $C^2$  and the mapping  $f : TQ \times \mathbb{R}^+ \rightarrow T^*Q$  is of class  $C^1$ . Then, there exists a unique maximal solution for problem I.*

More precisely, theorem 2 states that there exists  $T_m > 0$  ( $T_m \in \mathbb{R}^+ \cup \{+\infty\}$ ) and  $q_m \in C^2([0, T_m[, Q)$  being a solution of problem I such that any other solution of problem I is a restriction of  $q_m$ . Of course, we expect that  $T_m = +\infty$ , in which case the dynamics is said to be *eternal*. This situation cannot be taken for granted, in general. In the usual cases encountered in mechanics, eternal dynamics is ensured by the following general sufficient condition.

**Theorem 3** *The configuration manifold  $Q$  is assumed to be a complete Riemannian manifold. The effort mapping  $f$  is supposed to admit the following estimate:*

$$\|f(q, v; t)\|_q^* \leq l(t) \left( 1 + d(q, q_0) + \|v\|_q \right),$$

for all  $(q, v) \in TQ$  and almost all  $t \in [t_0, +\infty[$ , where  $d(\cdot, \cdot)$  is the Riemannian distance and  $l(t)$ , a (necessarily nonnegative) function of  $L^1_{loc}(\mathbb{R}; \mathbb{R})$ .

Then, the dynamics is eternal:  $T_m = +\infty$ .

## 2. Formulation of unilateral multibody dynamics

The consideration of elementary examples shows that the dynamics of rigid bodies systems can lead to predictions of the motion in which

some bodies of the system *overlap* in the real world. Of course, this cannot be allowed. Hence, very often, one has to add the statement of non-penetration conditions to a simple discrete mechanical system. This is a simple occurrence of a unilateral constraint. Here, we shall recall briefly the general formulation of the resulting evolution problem. This is due essentially to Schatzman (1978) and Moreau (1983).

Consider a simple discrete mechanical system according to definition 1. A *unilateral constraint* is a restriction on the admissible motions of the system which is expressed by means of a finite number  $n$  of smooth real-valued functions  $\varphi_i$  on the configuration manifold  $Q$ , so that the set of all admissible configurations  $A$  is given by:

$$A = \{q \in Q ; \forall i \in \{1, 2, \dots, n\}, \varphi_i(q) \leq 0\}. \quad (2)$$

The set of all active constraints in the admissible configuration  $q \in A$  is defined by:

$$J(q) = \{i \in \{1, 2, \dots, n\} ; \varphi_i(q) = 0\}.$$

The following hypothesis is usual in this framework. The functions  $\varphi_i$  are assumed to be *functionally independent* in the sense that, for all  $q \in A$ , the  $d\varphi_i(q)$  ( $i \in J(q)$ ) are linearly independent in  $T^*Q$ .

The cone of admissible right velocities in the configuration  $q$  is defined by:

$$V(q) = \{v \in T_qQ ; \forall i \in J(q), \langle d\varphi_i(q), v \rangle_q \leq 0\},$$

and we denote by  $N^*(q)$  its polar cone for the duality  $(T_qQ, T_q^*Q)$ :

$$N^*(q) = \left\{ \sum_{i=1}^n \lambda_i d\varphi_i(q) ; \forall i \in J(q), \lambda_i \geq 0, \quad \forall i \notin J(q), \lambda_i = 0 \right\},$$

the polar cone of  $V(q)$  for the euclidean structure of  $T_qQ$  being  $N(q) = \#(N^*(q))$  ( $\# = \flat^{-1}$ ).

The equation of motion has to be corrected with some reaction efforts  $R$ :

$$\flat \frac{D\dot{q}^+}{dt} = f(q, \dot{q}^+; t) + R. \quad (3)$$

We require the unilateral constraint to be perfect, that is, the reaction effort  $R$  has to take values in the normal cone  $N^*(q)$ . It is well known that the dynamics of such a system involves some time-discontinuities of velocity called *impacts*. Therefore, we cannot expect that the equation of motion (3) should be satisfied in a classical sense, but rather in a distributional sense. Actually, it is usual to require that  $R$  should be a vector-valued mesasure rather than a general distribution. Hence, we

define the class of motions  $MMA(I; Q)$  (motions with measure acceleration) to be the set of all absolutely continuous motions  $q(t)$  from a real interval  $I$  to  $Q$  admitting a right velocity  $\dot{q}^+(t)$  at every instant  $t$  of  $I$  and such that the function  $\dot{q}^+(t)$  has locally bounded variation over  $I$ . Bounded variation is classically defined only for functions taking values in a normed vector space. However, for any absolutely continuous curve  $q(t)$  on a Riemannian manifold, parallel translation along  $q(t)$  classically provides intrinsic identification of the tangent spaces at different points of the curve and so, the definitions can easily be carried over to this case (for a precise mathematical setting, see Ballard (2000)). Any motion  $q \in MMA(I; Q)$  admits a left and right velocity,  $\dot{q}^-$  and  $\dot{q}^+$ , in the classical sense at any instant. Moreover, any motion  $q \in MMA(I; Q)$  is intrinsically associated with the covariant Stieltjes measure  $\mathfrak{D}\dot{q}^+$  of its right velocity  $\dot{q}^+$ . The equation of motion takes the form:

$$\mathfrak{bD}\dot{q}^+ = f(q, \dot{q}^+; t) dt + R,$$

where  $dt$  denotes the Lebesgue measure. It remains to express the condition that  $R$  should take values in  $N^*(q)$  when  $R$  is a measure.

**Convention.** We shall write:

$$R \in -N^*(q(t))$$

to mean: there exist  $n$  nonpositive real measures  $\lambda_i$  such that:

$$R = \sum_{i=1}^n \lambda_i d\varphi_i(q(t)) \quad \text{and} \\ \forall i \in \{1, 2, \dots, n\}, \quad \text{Supp } \lambda_i \subset \{t; \varphi_i(q(t)) = 0\}. \quad (4)$$

With this convention, the final form of the equation of motion is:

$$R = \mathfrak{bD}\dot{q}^+ - f(q(t), \dot{q}^+(t); t) dt \in -N^*(q(t)) \quad (5)$$

It is classical that this equation of motion has to be completed with some constitutive impact equation describing the outcome of any impact occurrence. We shall assume an impact constitutive equation of general form:

$$\forall t, \quad \dot{q}^+(t) = \mathcal{F}(q(t), \dot{q}^-(t)). \quad (6)$$

To ensure compatibility with the equation of motion (5), the mapping  $\mathcal{F}$  should satisfy:

$$\forall q \in A, \quad \forall v^- \in -V(q), \quad \begin{aligned} \mathcal{F}(q, v^-) &\in V(q), \\ \mathcal{F}(q, v^-) - v^- &\in -N(q). \end{aligned} \quad (7)$$

Moreover, we add the assumption that the kinetic energy of the system cannot increase during an impact:

$$\forall q \in A, \quad \forall v^- \in -V(q), \quad \|\mathcal{F}(q, v^-)\|_q \leq \|v^-\|_q. \quad (8)$$

Now, we formulate the evolution problem associated with the dynamics of rigid bodies systems with perfect bilateral and unilateral constraints. The initial condition is assumed to be compatible with the realization of the constraint:  $v_0 \in V(q_0)$ .

**Problem II.** Find  $T > 0$  and  $q \in MMA([0, T[; Q)$  such that:

- $(q(0), \dot{q}^+(0)) = (q_0, v_0)$ ,
- $\forall t \in [0, T[, \quad q(t) \in A$ ,
- $R \stackrel{\text{def}}{=}} \text{bD}\dot{q}^+ - f(q(t), \dot{q}^+(t); t) dt \in -N^*(q(t))$ ,
- $\forall t \in ]0, T[, \quad \dot{q}^+(t) = \mathcal{F}(q(t), \dot{q}^-(t))$ .

The equation of motion is understood in the sense of convention (4), and the impact constitutive equation is supposed to fulfill requirements (7) and (8).

### 3. Well-posedness of unilateral multibody dynamics

To study existence and uniqueness of solution for problem II, we need to state some regularity hypotheses on the data. It was pointed out by Bressan (1960) and Schatzman (1978), through a striking example, that under the strong hypotheses that the data are all of class  $C^\infty$  there may exist non-unique solutions for problem II. It was Percivale (1985) and Schatzman (1998) who were the first to notice that this trouble could be overcome by assuming that the data are *analytic*. But their analysis was limited to one-degree-of-freedom systems. The general case is treated in Ballard (2000), Ballard (2001) and is now briefly recalled.

From now on, we assume the following

**Regularity hypothesis.** The Riemannian configuration manifold  $Q$ , the functions  $\varphi_i$  and the mapping  $f : TQ \times \mathbb{R}^+ \rightarrow T^*Q$  are *analytic*.

**Proposition 4** *Let  $q_0 \in A$  and  $v_0 \in V(q_0)$ . Then, there exist  $T_a > 0$ , an analytic curve  $q_a : [0, T_a[ \rightarrow Q$  and  $n$  analytic functions  $\lambda_{ai} : [t_0, T_a[ \rightarrow \mathbb{R}$  such that:*

- $(q_a(0), \dot{q}_a^+(0)) = (q_0, v_0)$ ,
- $\forall t \in [0, T_a[$ ,  $\frac{D}{dt} \dot{q}_a(t) = f(q_a(t), \dot{q}_a(t); t) + \sum_{i=1}^n \lambda_{ai}(t) d\varphi_i(q_a(t))$ ,
- $\forall t \in [0, T_a[$ ,  $\forall i = 1, 2, \dots, n$ ,  $\lambda_{ai}(t) \leq 0$ ,  $\varphi_i(q_a(t)) \leq 0$ ,  $\lambda_{ai}(t) \varphi_i(q_a(t)) = 0$ .

Moreover, the solution of this evolution problem is unique in the sense that any other analytic solution  $(T, q, \lambda_1, \dots, \lambda_n)$  is either a restriction or an analytic extension of  $(T_a, q_a, \lambda_{a1}, \dots, \lambda_{an})$ .

An early proof of proposition 4 can be found in Lötstedt (1982).

**Corollary 5** *There exists an analytic solution  $(T_a, q_a)$  for problem II.*

Naturally, the analytic solution furnished by corollary 5 will cease to exist at the first instant of impact. This is the reason why we have considered the wider class *MMA* which contains motions which are not differentiable in the classical sense. Considering motions in *MMA* will allow us to extend the solution beyond the first instant of impact. But, we must ensure that admitting the wider class of solutions *MMA* will not introduce parasitic solutions. This is the aim of the following theorem.

**Theorem 6** *Let  $(T_a, q_a)$  be the solution for problem II furnished by corollary 5, and  $(T, q)$  be an arbitrary solution for problem II. Then, there exists a real number  $T_0$  ( $0 < T_0 \leq \min\{T_a, T\}$ ) such that:*

$$q|_{[0, T_0[} \equiv q_a|_{[0, T_0[}.$$

*In other terms, there is local uniqueness for problem II.*

Local uniqueness is the difficult part in the study of well-posedness of problem II.

**Corollary 7** *There exists a unique maximal solution for problem II.*

It was noticed above that the analytical solution for problem II furnished by corollary 5 fails to exist at the first instant of impact. To overcome this fact, we have proved that local uniqueness still holds in the wider class of motion *MMA* which allows impacts. However, the maximal solution for problem II may stop to exist at finite time for unphysical reasons. In other terms, we still do not know if the class *MMA* is wide enough. Actually, it is wide enough as shown by the following theorem which should be brought aside theorem 3.



**Theorem 8** *The configuration manifold  $Q$  is assumed to be a complete Riemannian manifold and the mapping  $f$  is supposed to admit the following estimate:*

$$\|f(q, v; t)\|_q^* \leq l(t) \left( 1 + d(q, q_0) + \|v\|_q \right),$$

for all  $(q, v) \in TQ$  and almost all  $t \in [0, +\infty[$ , where  $d(\cdot, \cdot)$  is the Riemannian distance and  $l(t)$ , a (necessarily nonnegative) function of  $L^1_{loc}(\mathbb{R}; \mathbb{R})$ .

Then, the dynamics is eternal, that is, the maximal solution for problem II is defined on  $[0, +\infty[$ .

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