Fredholm backstepping for critical operators and application to rapid stabilization for the linearized water waves

Ludovick Gagnon, Amaury Hayat, Shengquan Xiang, Christophe Zhang

FREDHOLM BACKSTEPPING FOR CRITICAL OPERATORS AND APPLICATION TO RAPID STABILIZATION FOR THE LINEARIZED WATER WAVES

LUDOVICK GAGNON, AMAURY HAYAT, SHENGQUAN XIANG, AND CHRISTOPHE ZHANG

Abstract. Fredholm-type backstepping transformation, introduced by Coron and Lü, has become a powerful tool for rapid stabilization with fast development over the last decade. Its strength lies in its systematic approach, allowing to deduce rapid stabilization from approximate controllability. But limitations with the current approach exist for operators of the form $|D_x|^\alpha$ for $\alpha \in (1, 3/2]$. We present here a new compactness/duality method which hinges on Fredholm's alternative to overcome the $\alpha = 3/2$ threshold. More precisely, the compactness/duality method allows to prove the existence of a Riesz basis for the backstepping transformation for skew-adjoint operator verifying $\alpha > 1$, a key step in the construction of the Fredholm backstepping transformation, where the usual methods only work for $\alpha > 3/2$. The illustration of this new method is shown on the rapid stabilization of the linearized capillary-gravity water wave equation exhibiting an operator of critical order $\alpha = 3/2$.

Keywords: water waves, compactness/duality method, Fredholm transformation, backstepping, rapid stabilization.
2010 MSC: 35S50, 76B15, 93B05.

Contents

Université de Lorraine, CNRS, Inria équipe SPHINX, F-54000 Nancy, France. E-mail: ludovick.gagnon@inria.fr.

CERMICS, École des Ponts ParisTech, 6 - 8, Avenue Blaise Pascal, Cité Descartes—Champs sur Marne, 77455 Marne la Vallée, France. E-mail: amaury.hayat@enpc.fr.

Bâtiment des Mathématiques, EPFL, Station 8, CH-1015 Lausanne, Switzerland. E-mail: shengquan.xiang@epfl.ch.

Université de Lorraine, CNRS, Inria équipe SPHINX, F-54000 Nancy, France. E-mail: Christophe.zhang@inria.fr.

1. Introduction

Since its introduction by Coron and Lü for the rapid stabilization of the Korteweg-de Vries equation [16] and Kuramoto-Sivashinsky equation [17], the Fredholm transformation has been applied successfully in the past decade for the rapid stabilization of a large class of equations. It consists in finding an operator-isomorphism pair (K, T) that maps a system of the form

$$\partial_t u = Au + BKu \tag{1.1}$$

to the rapidly exponentially stable system

$$\partial_t v = (A - \lambda)v \tag{1.2}$$

where A is the generator of a strong semigroup B is an unbounded operator and λ is a positive number arbitrarily large. Compared with the original Volterra transformation introduced by Krstić and Balogh [5], the Fredholm transformation possesses the advantage of presenting a systematic approach to prove the rapid stabilization from spectral properties of the spatial operator A and from suitable controllability assumptions. However, the classical approach for the Fredholm transformation fails to deal with operators of the form $|D_x|^\alpha$ for $\alpha \in (1, 3/2]$. Indeed, one key step in proving the existence of the Fredholm transformation T is to prove that the family $T\varphi_n$ is a Riesz basis of the state space, where

$$\{(\varphi_n, \lambda_n)\}_n \text{ are the eigenmodes of the spatial operator } A.$$

The usual way to tackle this problem is to prove that the family $T\varphi_n$ is quadratically close to the eigenfunction basis φ_n (see Definition A.1 (3) for a precise statement), but the lack of growth of the high frequency eigenvalues λ_n like n^α prevents the use of such criteria, leaving the Fredholm alternative for operators behaving as $|D_x|^\alpha$ for $\alpha \in (1, 3/2]$ an open question since then.

1.1. **The compactness/duality method.**

In this paper we present a new method to answer this question. This method is based on a new compactness/duality approach to prove that the family $T\varphi_n$ is indeed a Riesz basis in sharp spaces for the whole range $\alpha > 1$. The challenging part in proving that the family $T\varphi_n$ is a Riesz basis is the coercivity estimate (see the left-hand side estimate of (A.1)). We proceed by a contradiction argument to prove this inequality. Using the expression of $T\varphi_n$, we are able to prove that T can be decomposed in an invertible part and a compact part. Then, the desired uniform inequality can be deduced from the ω-independent property. A further inspection on the duality between the ω-independence of $T\varphi_n$ in H^r and the density of $T^*\overline{\varphi_n}$ in $(H^r)^* \simeq H^{-r}$ finally leads to the required property.

A second important step of our method is to deal with the so-called $TB = B$ uniqueness condition, introduced in [11] for finite-dimensional systems, used implicitly in [16, 17] and introduced explicitly in [12] for the first time in PDEs to deal with the nonlocal term arising from the distributed controls. In the original approach, proposed by [16] and used since then, this condition is solved thanks to the quadratically close property mentioned above. With this new method, we are able to sidestep this limit thanks to a fine decomposition of the $TB = B$ condition, allowing to define the transformation T along with the feedback law K.

Beyond the $\alpha = 3/2$ threshold, this new method leads to sharp Riesz basis properties for a large class of skew-adjoint operators including Fourier multiplier based operators as long as the high frequency scales as n^α for $\alpha > 1$.

We apply this new strategy to prove the rapid stabilization of the linearized capillary-gravity water waves equation, exhibiting a spatial operator behaving like $|D_x|^{3/2}$. This is an example corresponding to the critical case $\alpha = 3/2$ which remained out of reach until now.

1.2. **The capillary-gravity water waves equation.**

We introduce the linearized capillary-gravity water waves equation (following [3, 4, 28, 29]) relevant for modelling the motion and stability of perfect fluids where the surface tension and capillarity cannot be neglected; for instance for small characteristic scales or when waves are breaking and at certain waves frequencies ([29, 40]). Thus, consider the 2-D capillary-gravity water waves for an homogeneous, inviscid, incompressible, irrotational fluid over a flat bottom on which an external pressure is applied. The volume of the fluid is described by

$$\Omega(t) = \{(x, y) \in \mathbb{T} \times \mathbb{R} \,|\, -h \leq y \leq \eta(x, t)\},$$

where $y = -h$ is the bottom of the fluid, $y = \eta(x, t)$ is the deformation from the rest $y = 0$ of the free surface and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. The evolution of the velocity field U of the fluid and of the free surface are governed by 2-D free surface Euler equation,

$$\begin{cases} \partial_t U + (U.\nabla)U = -\nabla p - g e_2, & (x, y) \in \Omega(t), \\ \text{div } U = 0, \ \text{rot } U = 0, & (x, y) \in \Omega(t), \\ U.n = 0, & (x, y) \in \mathbb{T} \times \{-h\}, \end{cases}$$

satisfying the boundary conditions on the free surface $y = \eta(t, x)$,

$$\begin{cases} \partial_t \eta = \sqrt{1 + |\nabla\eta|^2} U.n, & (x, y) \in \mathbb{T} \times \{\eta(t, x)\}, \\ p = p_{atm} + P_{ext} - \sigma\kappa(\eta), & (x, y) \in \mathbb{T} \times \{\eta(t, x)\}, \end{cases}$$

where p is the pressure, g the gravitational constant, $n := \frac{1}{\sqrt{1+|\nabla\eta|^2}}(-\nabla\eta, 1)^t$ the outward normal vector to the surface η, $e_2 = (0, 1)^t$ the unit vector, $\sigma > 0$ is the surface tension coefficient and

$$\kappa(\eta) = \partial_x \left(\frac{\partial_x \eta}{\sqrt{1 + |\partial_x \eta|^2}} \right) = \frac{\partial_x^2 \eta}{(1 + (\partial_x \eta)^2)^{3/2}},$$

is the mean curvature of the surface. The first part is the Euler equation on U describing incompressible and irrotational fluids with an impermeable bottom respectively. The second part is the boundary conditions on the free surface: the kinematic equation on the surface for η asserting that particles on the surface remains on the surface along time, and the pressure at the surface, including the surface tensison and the localized external pressure $P_{ext}(t, x, \eta(t, x))$. The incompressible and irrotational assumption implies that the velocity field is represented by a velocity potential $\Phi : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ such that $U = \nabla_{x,y}\Phi$. The 2-D free surface Euler equation implies that the velocity potential satisfies

$$\begin{cases} \partial_t \Phi + \frac{1}{2}|\nabla\phi|^2 + gy = -(p - p_{atm}), & (x, y) \in \Omega(t), \\ \Delta\Phi = 0, & (x, y) \in \Omega(t), \\ \partial_n \Phi = 0, & (x, y) \in (0, 2\pi) \times \{-h\}, \\ \partial_t \eta = \sqrt{1 + |\nabla\eta|^2} \partial_n \Phi, & (x, y) \in (0, 2\pi) \times \{\eta(t, x)\}. \end{cases}$$

It was first noted by Zakharov [40] that the preceding equation on the velocity potential Φ is an Hamiltonian system, where $\psi := \Phi|_{y=\eta}$ and η are (generalized) canonical variables. Moreover,

Φ is completely determined through the Laplace equation and the knowledge of $\psi := \Phi|_{y=\eta}$ and η. This leads to study the Dirichlet-to-Neumann map

$$G[\eta, h] : \psi \mapsto \sqrt{1+|\nabla\eta|^2}\partial_n\Phi|_{y=\eta} = \partial_y\Phi(x,\eta,t) - \partial_x\eta(x,t)\partial_x\Phi(x,\eta,t).$$

We refer for instance to [3, 4, 28, 29] and the references therein for the properties of the Dirichlet-to-Neumann map as well as its application to the well-posedness of the Cauchy problem of the gravity and capillary-gravity water waves.

Using the Dirichlet-to-Neumann map, one may reformulate the capillary-gravity water waves as,

$$\begin{cases} \partial_t\eta - G[\eta,h]\psi = 0, \\ \partial_t\psi + g\eta + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{(G[\eta,h]\psi + \nabla\eta.\nabla\psi)^2}{2(1+|\nabla\eta|^2)} = \sigma\kappa(\eta) - P_{ext}. \end{cases} \tag{1.3}$$

The Dirichlet-to-Neumann operator is nonlinear with respect to the surface elevation. We therefore consider the linearization around $(\eta,\psi) = (0,0)$, yielding (fixing $\sigma = 1$),

$$\begin{cases} \partial_t\eta - G[0,h]\psi = 0, \\ \partial_t\psi + g\eta - \partial_x^2\eta = -P_{ext}, \end{cases} \tag{1.4}$$

where $G[0,h] = |D_x|\tanh(h|D_x|)$, defined as a Fourier multiplier on periodic functions. Set $\mathscr{L}$ the operator

$$\mathscr{L} := -i\left((g-\partial_x^2)G[0,h]\right)^{1/2}, \tag{1.5}$$

and let $u = \psi + \mathscr{L}G[0,h]^{-1}\eta$, we end up with,

$$\partial_t u = \mathscr{L}u + P_{ext}.$$

To be more precise, we consider the external pressure (the control) to be of the distributed control form $P_{ext} = B_1(x)w_1(t) + B_2(x)w_2(t)$. Notice that $\mathscr{L}$ has double eigenvalues (see Section 1.8), which means according to [19] two distributed controls are required to control/stabilize the system instead of one. Hence, for ease of notations we consider

$$\partial_t u = \mathscr{L}u + Bw(t), \tag{1.6}$$

where B is a two dimensional control operator $B : w \in \mathbb{C}^2 \to w_1B_1 + w_2B_2$.

1.3. **Statement of the main results.**

We consider in this paper the rapid stabilisation of (1.6), that is to seek, for any $\lambda > 0$, a two-dimensional control feedback law $w(t) = Ku(t,\cdot)$ such that the solution of (1.6) satisfies

$$\|u(t)\| \lesssim e^{-\lambda t}\|u_0\|, \quad \forall t \in (0,+\infty).$$

To state our main results, we first introduce condition (2.1) for the exact controllability of (1.6) given below in Section 2 (see Proposition 2.1 and its proof in Appendix B). Using the backstepping method with a Fredholm transformation, we are able to prove (see Theorem 2.2 and Corollary 2.3 for a precise statement)

THEOREM 1.1. *Let $B \in (H^{-3/4})^2$ satisfying Assumption 1 concerning controllability. Then, for any $\lambda > 0$, there exists a bounded linear operator $K \in \mathcal{L}(H^{3/4};\mathbb{C}^2)$ and an operator T being an isomorphism from $H^r(\mathbb{T})$ to itself for any $r \in (-1,1)$ and maps the closed-loop system*

$$\partial_t u = \mathscr{L}u + BK(u), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}, \tag{1.7}$$

to the system

$$\partial_t v = \mathscr{L} v - \lambda v, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T},$$

where $\mathscr{L}$ is the linearized water-wave operator given in (1.5). Consequently, the closed-loop system (1.7) is exponentially stable in H^r for any $r \in (-1,1)$.

The proof of Theorem 1.1 relies on our compactness/duality method, allowing to prove the existence of a Fredholm operator for the backstepping method in the critical case $\alpha = 3/2$. Our method in fact extends to the more general cases (see Theorem 2.4 for a precise statement).

THEOREM 1.2 (General skew-adjoint operators)**.** *Let $\alpha > 1$. Let $B \in (H^{-3/4})^2$ satisfying Assumption 1 concerning controllability. Let $h(s)$ a real valued-function satisfying*

- $|n_1 - n_2| n_1^{\alpha-1} \lesssim |h(n_1) - h(n_2)|$ *for any* $(n_1, n_2) \in \mathbb{N}^*$.
- $s^\alpha \lesssim |h(s)| \lesssim s^\alpha$ *for any* $s \in [1, +\infty)$.

Then, for any $\lambda > 0$, there exists a bounded linear operator $K \in \mathcal{L}(H^{3/4}; \mathbb{C}^2)$ and an operator T such that T is an isomorphism from $H^r(\mathbb{T})$ to itself for any $r \in (1/2 - \alpha, \alpha - 1/2)$ and maps the system

$$\partial_t u = i\, h(|D_x|)u + BK(u), \ (t,x) \in \mathbb{R}_+ \times \mathbb{T}, \tag{1.8}$$

to the system

$$\partial_t v = i\, h(|D_x|)v - \lambda v, \ (t,x) \in \mathbb{R}_+ \times \mathbb{T}.$$

Consequently, the closed-loop system (1.8) is exponentially stable in H^r for $r \in (1/2 - \alpha, \alpha - 1/2)$ with decay rate λ.

In the next subsection, we introduce formally the Fredholm backstepping method for PDEs as well as the main steps of our proof.

1.4. **The Fredholm-type backstepping method.** Before stating our main results, we introduce formally the backstepping method for PDEs. The backstepping method to consists to prove the existence of a feedback law $w = Ku$ and an invertible operator T mapping the solution u of the equation to be stabilized,

$$\begin{cases} u_t(t) = Au(t) + Bw(t), \\ u(0) = u_0, \end{cases} \tag{1.9}$$

onto the solution z of an exponentially stable equation (thanks to the natural dissipation of A and the strong damping effect of $-\lambda I$),

$$\begin{cases} v_t(t) = (A - \lambda I)v(t), \\ v(0) = v_0, \end{cases} \tag{1.10}$$

where $u(t)$ and $v(t) = Tu(t)$ belong to a Hilbert space H, A is the generator of a semigroup over the state space, B is an unbounded operator satisfying some admissibility condition (see for instance [36] for a definition) and the control $w(t) = Ku(t)$ is of feedback form to achieve stabilisation.
The main challenge is to find an operator-isomorphism pair (T, K) such that this mapping can be achieved. This problem is equivalent to find (T, K) solving,

$$T(A + BK) = (A - \lambda I)T, \tag{1.11}$$

shown by taking formally the time derivative of $z = Tu$ and using (1.9) and (1.10).

The backstepping method was first introduced in finite dimension as a chain of integrators for feedback laws [27], but was then extended to PDEs by Krstić and Balogh [5] for Volterra transformations of the second kind,

$$z(t,x) = Tu = u(t,x) - \int_0^x k(x,y)u(t,y)dy.$$

The abstract equation (1.11) then transfers to the existence of a solution to a non-standard PDE on the kernel k. In addition to leaving the classical framework of Cauchy theory, the kernels resulting from the Volterra transformation present boundary conditions on the diagonal $k(x,x)$ that proves to be very difficult to handle. Despite these difficulties, several methods have been developed to solve the PDE on the kernel of the Volterra transformation (successive approximations [27], explicit representations [27] or method of characteristics [15]) leading to a rich literature, the invertibility of the Volterra transformation being guaranteed.

More recently, the Fredholm-type transformation as follows

$$z(t,\cdot) = Tu(t,\cdot) = (Id + \mathcal{T}_{comp})u(t,\cdot),$$

was introduced by Coron and Lü for the rapid stabilization of the Korteweg-de Vries equation [16] and Kuramoto-Sivashinsky equation [17] by means of the backstepping method. In this transformation, T is a Fredholm operator with an invertible and a compact part. Although much more technical, the Fredholm transformation provides a systematic approach to the backstepping method based on the spectral properties of the operator A and the controllability properties to prove the rapid stabilization for a large class of equations.

Let us elaborate on the techniques involved with the backstepping method with the Fredholm transformation. A first crucial step is to consider the so-called uniqueness condition $TB = B$ to change the abstract equation (1.11) into a system of two equations

$$TA + BK = (A - \lambda I)T, \tag{1.12}$$

$$TB = B. \tag{1.13}$$

The uniqueness condition $TB = B$ was first introduced in [11] to prove the existence of (T,K) solving (1.12)-(1.13) in finite dimension, used implicitly in [16, 17] and finally stated explicitly in [12] to remove nonlocal terms involved in (1.11) for distributed controls. The proof of existence of an invertible transformation T over the state space and a feedback law stabilizing (1.9) is then divided in the following steps:

Step 1: Let $\{\lambda_n + \lambda\}_n \cap \sigma(A) = \emptyset$. Notice that (1.12) is equivalent to

$$T\varphi_n = K_n(A - (\lambda_n + \lambda)I)^{-1}B,$$

where $K_n = K\varphi_n$. We prove that

$$\tilde{T}\varphi_n = (A - (\lambda_n + \lambda)I)^{-1}B,$$

is a Riesz basis family of H.

Step 2: Let $B = \sum_n b_n\varphi_n$. Use the Riesz basis properties to solve the $TB = B$ uniqueness condition in a suitable sense

$$TB = \sum_n b_n T\varphi_n = \sum_n b_n K_n \tilde{T}\varphi_n = B.$$

Step 3: Show from $TB = B$ that $(K_n)_n$ is uniformly bounded. Then, using operator equality and Kato's perturbation theory, prove that $T : H \to H$ is continuous and invertible.

Step 4: Thanks to the operator equality, the operator $A+BK$ generates a semigroup on H (and indeed it also generates semigroups in H^β with a certain range of β). Conclude on the rapid stabilization using the operator equality.

Aside of the seemingly different approach of hyperbolic systems [13, 14, 15, 41, 42], the proof Step 1 and 2 relied heavily in the literature on the quadratically close criterion. Roughly speaking it amounts to show after some computations that

$$\sum_{n\in\mathbb{N}}\sum_{p\in\mathbb{N}\setminus\{n\}}\frac{1}{|\lambda_n-\lambda_p+\lambda|^2}<+\infty$$

which holds if the eigenvalues λ_n of the operator A scales as n^α with $\alpha>3/2$ but fails as soon as $\alpha\leq 3/2$.

We introduce in this article the compactness/duality approach to overcome the limitations of the quadratically close criterion coupled with ω-independence/density properties. Indeed this criterion corresponds to the rather strong Hilbert-Schmidt criterion for compactness. In fact the compactness part can be proved in a more general way, and we establish ω-independence thanks to a new duality observation, thus overcoming the apparent limit of $\alpha>3/2$ (see Remark 4.9). This allows us to prove Step 1 for operators with eigenvalues scaling as n^α (for instance $|D_x|^\alpha$) with $\alpha\in(1,3/2]$.

Let us also stress that the $TB=B$ uniqueness condition is more difficult to handle than it seems since B is an unbounded operator. Indeed, if B were to be bounded, then from

$$\sum_n b_n K_n\tilde{T}\varphi_n=B,$$

one would be tempted to deduce that the sequence b_nK_n belongs to ℓ^2. Moreover, the controllability assumption implies that the sequence of $|b_n|$ is bounded from below (it is impossible from the assumption that B is bounded but let us assume it is for the sake of the argument) and therefore one would conclude that $K_n\in\ell^2$. But then, with the expression of T, it is not difficult to prove that in this case that the transformation T would be compact and therefore not invertible.

The proof of the decomposition of $TB=B$ for B unbounded and admissible still follow the same idea, with the slight modification that TB is seen as a singular and bounded part. Then, one adjusts the behaviour of K_n by hand by letting $K_n\sim c+k_n$ where c is a constant. If the Riesz basis is quadratically close to the eigenfunctions, then one obtains,

$$\sum_n b_nk_n\tilde{T}\varphi_n=\sum_n b_n(\varphi_n-\tilde{T}\varphi_n)$$

and the right-hand side is bounded in H using the quadratically close argument and the boundedness of the sequence b_n in ℓ^∞ (roughly speaking provided by the admissibility). Without the quadratically close property this direct argument fails. However, we prove here that even if the Riesz basis is not quadratically close to the eigenfunctions, we are still able to reach the same final conclusion by a close inspection of the left-hand side.

1.5. **Related works on the backstepping with a Fredholm transform.** There are essentially two type of systems in the literature for which the rapid stabilisation was achieved through the backstepping method with a Fredholm transformation : either the operator A is of first order ($\alpha=1$) or of second order or higher ($\alpha\geq 2$). We have so far excluded from our discussion the case $\alpha=1$ as it seems to be a very specific case with techniques on its own. Indeed, the rapid stabilisation for hyperbolic systems was established in [14, 15] through direct methods or by identifying the isomorphism applied to the eigenbasis leading to the Riesz basis [13, 41, 42]. The

other results found in the literature were concerned with operators such that $\alpha \geq 2$, and in these case the Riesz basis properties was proved through the quadratically close criterion, thanks to the sufficient growth of the eigenvalues. Following the steps described in the previous section, the rapid stabilisation was obtained for the linearized bilinear Schrödinger equation [12], the KdV equation [16], the Kuramoto-Sivashinksi equation [17], a degenerate parabolic operator [20] and finally the heat equation for which the backstepping is proved in sharp spaces [19]. The variety of the PDEs for which this methodology can be applied tends to show that there exists an abstract theory for operators of order $\alpha > 1$. Theorem 1.2 demonstrates this fact for skew-adjoint operators for $\alpha > 1$. This abstract setting could allow to lift some difficult questions raised when trying to apply the backstepping with the Volterra transformation. One such difficult is seen for instance for degenerate parabolic equations ([20]), where the Fredholm transformation lead to the study of well-known spectral properties of the Sturm-Liouville equation, whereas the PDE on the kernel of the Volterra transformation amounts to describe the propagation of bicharacteristics from a boundary satisfying a degenerate equation, a notoriously difficult problem.

Finally, we shall remark that there are many other useful stabilization techniques in the literature that may also apply to similar systems, for instance the damping stabilization of waves [1, 7, 26], the multiplier methods [25], Riccati theory [6, 9, 30], Gramian method [37], equivalence between observability and stabilizability [35], quantitative finite time stabilization [38, 39], various Lyapunov approaches [18, 22, 23, 24] and among others.

1.6. **Related works on controllability and stabilization of the water waves system.**

A considerable amount of literature exists on the control of fluids. However, few works address the controllability and stabilization of the water waves. Recently, this subject has drawn more and more attention. The controllability properties of these systems was first investigated in [31] using the moment method, and the controllability assumption was sharpened to quasi-linear systems in [3], where the control is localized in the domain. A recent work [43] investigated the water waves in 3-D, highlighting the need for the geometric control condition for the controllability (the 2-D case satisfying automatically the geometric control condition as the control problem reduces to a 1-D equation).

Concerning stabilization, despite fruitful stabilisation results were obtained in the literature for fluids, only few result seems to exist regarding the water waves. We may refer to the asymptotic stabilization and exponential stabilisation results of water waves systems [1, 2] that are based on external "damping" forces and "observability" of the closed-loop systems. Alternatively, stabilizability properties of linearized water waves systems with controls acting on the boundary have been recently studied in [32, 33]. Our contribution, as a direct consequence of the Fredholm backstepping transformation we obtain a rapid stabilisation result, that is exponential stability with arbitrarily decay rate.

1.7. **Outline of the paper.** The paper is divided in the following way. First, the main results are stated precisely in Section 2 as well as their possible extensions, and the strategy of the proof is presented in Section 3. Section 4 begins with the statement of technical estimates used throughout the article as well as the proof of the Riesz basis (Step 1 of Section 1.4) using the compactness/duality argument. In Section 5 we prove that the uniqueness condition $TB = B$ holds in $H^{-1/2-\varepsilon}, \varepsilon \in (0, 1/2)$ (Step 2) and define properly the feedback law K, the isomorphism T (Step 3) as well as the operator equality (1.12). The well-posedness and rapid stabilisation of the closed-loop system (2.2) is proved (Step 4) in Section 6 and in turn prove that the sharp spaces for which the backstepping transform is establish coincide with the sharp space for the well-posedness of the closed-loop system. Finally, Appendix A recalls the basic definitions for

Riesz basis, Appendix B proves the exact controllability of (1.6), Appendix C is devoted to some basic estimates concerning the linearized water waves opeartor, Appendix D and E are dedicated to technical proofs on the Fredholm transform for the backstepping method already existing in the literature and Appendix F extends the proof of Theorem 2.2 to Theorem 2.4 for the general cases.

1.8. **Notations and spaces.** We begin by recalling that the eigenfunctions of the operator $\mathscr{L}$ on the torus coincide with the classical Fourier series $\{e^{inx}\}_{n\in\mathbb{Z}}$ in $\mathbb{T}$. Notice that for any $n\in\mathbb{N}^*$

$$\mathscr{L}e^{\pm inx} = -i\left((g+n^2)n\tanh(hn)\right)^{1/2}e^{\pm inx},$$

thus

$$\mathscr{L}\sin(nx) = \lambda_n\sin(nx) \quad\text{and}\quad \mathscr{L}\cos(nx) = \lambda_n\cos(nx), \tag{1.14}$$

$$\text{with } \lambda_n := -i\left(n(g+n^2)\tanh(hn)\right)^{1/2}. \tag{1.15}$$

Since every nonzero eigenvalue has multiplicity two, any given function can be separated by odd part and even part which corresponds to the orthonormal basis

$$\varphi_n^1 = (\sqrt{\pi}^{-1}\sin(nx)),\ \varphi_n^2 = (\sqrt{\pi}^{-1}\cos(nx)),\quad \text{for } n\in\mathbb{N}^*, \varphi_0^2 = \sqrt{2\pi}^{-2}. \tag{1.16}$$

In this logic, we decompose the space $H^r(\mathbb{T};\mathbb{C})$ as $H_1^r\oplus H_2^r$, for any $r\in\mathbb{R}$, where

$$H_1^r := \{a\in H^r(\mathbb{T};\mathbb{C})\mid a=\sum_{n\in\mathbb{N}^*}a_n\varphi_n^1\},\ H_2^r := \{a\in H^r(\mathbb{T};\mathbb{C})\mid a=\sum_{n\in\mathbb{N}}a_n\varphi_n^2\}.$$

Since we are working on $\mathbb{T}$, the inner product $\langle\cdot,\cdot\rangle_{H_i^m}$ is well-defined and given by, for any $f=\sum_n f_n\varphi_n^i$ and $g=\sum_n g_n\varphi_n^i$ belong to H_i^m,

$$\langle f,g\rangle_{H_1^m} = \sum_{n\in\mathbb{N}^*}(n^m f_n)(n^m\overline{g_n}),\ \ \langle f,g\rangle_{H_2^m} = f_0g_0+\sum_{n\in\mathbb{N}^*}(n^m f_n)(n^m\overline{g_n}).$$

We finally recall some properties of $G[0,h]$ and $\mathscr{L}$: for $s>3/2$, we have $G[0,h]:H^s(\mathbb{T};\mathbb{K})\to H^{s-1}(\mathbb{T};\mathbb{K})$ and for $s\in\mathbb{R}$, we have $\mathscr{L}:H^s(\mathbb{T};\mathbb{K})\to H^{s-3/2}(\mathbb{T};\mathbb{K})$, for $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$.

2. Main results

We first introduce the assumption leading to the exact controllability of 1.6.

Assumption 1. *Let the operator $B=(B_1,B_2)$ be such that $B_1\in H_1^{-3/4}$ and $B_2\in H_2^{-3/4}$. Then, assume the following condition,*

$$b_0\neq 0 \ \text{ and } \ c_1<|b_n^i|<c_2, \quad \text{for } i\in\{1,2\},\ n\in\mathbb{N}^*. \tag{2.1}$$

Thanks to the condition (2.1), we have the following controllability result for (1.6).

PROPOSITION 2.1. *Let $T>0$ and assume that (2.1) holds. For any $(u_0,u_f)\in(L^2)^2$ there exists a control $v\in L^2(0,T)$ such that the unique solution of (1.6) with initial state u_0 satisfies $u(T)=u_f$.*

This is proved using the moments method and using Haraux's refined version of the Ingham's inequality. As the proof is similar to the one given by Reid [31] for a similar water waves system, we postpone it to the Appendix B.

Our main result is the following

THEOREM 2.2. *Let $B = (B_1, B_2) \in H_1^{-3/4} \times H_2^{-3/4}$ such that (2.1) holds. Then, for any $\lambda > 0$, there exists an explicit bounded linear operator $K \in \mathcal{L}(H_1^{3/4} \times H_2^{3/4}; \mathbb{C}^2)$ and an isomorphism T from $H^r(\mathbb{T})$ to itself for any $r \in (-1, 1)$ that maps the system*

$$\partial_t u = \mathscr{L} u + BK(u), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}, \tag{2.2}$$

to the system

$$\partial_t v = \mathscr{L} v - \lambda v, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}, \tag{2.3}$$

where $\mathscr{L}$ is the linearized water-wave operator given in (1.5).

A direct consequence of this theorem is the existence of an explicit control law for the rapid exponential stabilization of the system (1.6).

COROLLARY 2.3 (Exponential stability)**.** *For any $\lambda > 0$, there exists an explicit feedback functional $K \in \mathcal{L}(H_1^{3/4} \times H_2^{3/4}; \mathbb{C}^2)$ such that for any $r \in (-1, 1)$, for any initial state $u(t)|_{t=0} = u_0 \in H^r$, the closed-loop system (2.2) has a unique solution $u \in C^0([0, +\infty); H^r(\mathbb{T}; \mathbb{C}))$. In addition, this unique solution decays exponentially with rate λ*

$$\|u(t, \cdot)\|_{H^r} \lesssim e^{-\lambda t} \|u_0\|_{H^r}, \qquad \forall t \in (0, +\infty).$$

We shall remark here that the bound $r = 1$ is sharp in the sense that for $r \geq 1$ the unbounded operator $A + BK$ does not anymore generate a strongly continuous semigroup in H^r. We also underline that, while the isomorphism T depends on the regularity of the state space H^r, the feedback law is, surprisingly, independent of r. This independence was already noticed in [19].

As stated in the introduction, the system (1.6) is all the more interesting as it represents the critical case $\alpha = 3/2$ where the usual method fails. To prove Theorem 2.2, we have to overcome this difficulty by introducing a new method. This allows us to free ourselves from the bound $\alpha = 3/2$ and, in fact, the method presented in this paper is more general: it can be extended at no cost for a large class of systems satisfying $\alpha > 1$. More precisely, we have the following theorem

THEOREM 2.4 (General skew-adjoint operators)**.** *Let $\alpha > 1$. Let $B = (B_1, B_2) \in H_1^{-3/4} \times H_2^{-3/4}$ such that (2.1) holds. Let $h(s)$ a real valued function satisfying*

- *$|n_1 - n_2| n_1^{\alpha - 1} \lesssim |h(n_1) - h(n_2)|$ for any $(n_1, n_2) \in \mathbb{N}^*$.*
- *$s^\alpha \lesssim |h(s)| \lesssim s^\alpha$ for any $s \in [1, +\infty)$.*

For any $\lambda > 0$, there exists an explicit bounded linear operator $K \in \mathcal{L}(H_1^{3/4} \times H_2^{3/4}; \mathbb{C}^2)$ and an isomorphism T from $H^r(\mathbb{T})$ to itself for $r \in (1/2 - \alpha, \alpha - 1/2)$ that maps the system

$$\partial_t u = i\, h(|D_x|) u + BK(u), \ (t, x) \in \mathbb{R}_+ \times \mathbb{T}, \tag{2.4}$$

to the system

$$\partial_t v = i\, h(|D_x|) v - \lambda v, \ (t, x) \in \mathbb{R}_+ \times \mathbb{T}.$$

Consequently, the closed-loop system (2.4) is exponentially stable in H^r for $r \in (1/2 - \alpha, \alpha - 1/2)$ with decay rate λ.

A way to adapt the proof of Theorem 2.2 to this case is given in Appendix F. The only significant difference is in the derivation of the regularity of the stabilizing feedback (see step (4) in Section 3 below) for which one need a finer decomposition and an iteration to reach the desired regularity.

2.1. **Extension of the main results.** The results of this article extends readily to more general cases or to other boundary conditions with minor modifications of the proof. We describe below such extensions.

2.1.1. *Rapid stabilisation in H^s.* The fundamental assumption guiding the appropriate Sobolev spaces in which the rapid stabilisation holds is the controllability Assumption 1 since the coefficients of the feedback law K, which enters in the decomposition of the isomorphism T, are obtained through the $TB = B$ uniqueness condition. The controllability Assumption 1 such that $b_n^i \sim 1$ implies the controllability of the linearized system in L^2 and leads to an isomorphism T from H^r for any $r \in (-1, 1)$. If we replace this assumption by the condition that guarantees the controllability in H^s space for some $s \in \mathbb{R}$, namely $b_n^i \sim n^{-s}$, then we can adapt slightly the proof of existence of the Riesz basis and construct the isomorphism T in $H^{s+r}(\mathbb{T})$ to itself for any $r \in (-1, 1)$ and a bounded linear operator $K : H^{s+3/4} \to \mathbb{R}^2$ such that the closed-loop system is exponentially stable in H^{s+r} for any $r \in (-1, 1)$.

2.1.2. *Conservation of mass.* In this paper we have not taken into account the "conservation of mass" condition

$$\int_{\mathbb{T}} u(x)dx = \langle u(t), \varphi_0^2 \rangle = 0,$$

which concerns the even space H_2^r. In fact by choosing B such that $\langle B_2, \varphi_0^2 \rangle = 0$, the backstepping method can be applied. Indeed, by following the same steps as in our proof, one can build an isomorphism that maps the even part of the water-waves system, with mass conservation, to the target system

$$\partial_t v = \mathscr{L} v - \lambda v,$$

with mass conservation, which is of course still exponentially stable. The reason for this is that the projection on φ_0^2 commutes with the isomorphism T and the closed-loop $A + BK$ obtained in the general case.

Thus, the same proof also leads to rapid stabilization in cases where there is a "conservation of mass" condition.

2.1.3. *Water waves in bounded domain.* We have investigated the linearized water waves system in a periodic domain. In fact we can also study the same system in a bounded domain with Neumann boundary conditions, the controllability of which was obtained by Reid [31]. In this framework, since all eigenvalues are simple, we are able to establish controllability, and rapid stabilization by backstepping, using only one control term.

3. Strategy and outline

In this Section we briefly comment on the strategy to prove Theorem 2.2 which is the task of the next two section, while the proof of Corollary concerning the well-posedness and stability of the closed-loop system will be discussed later on in Section 6. Given the decomposition along odd and even functions, showing Theorem 2.2 amounts to proving the following proposition.

PROPOSITION 3.1. *Let $i \in \{1, 2\}$. Let $B_i \in H_i^{-3/4}$ satisfying (2.1). For any $\lambda > 0$, there exists a bounded linear operator $K_i \in \mathcal{L}(H_i^{3/4}; \mathbb{C})$ and an isomorphism T_i from L^2 to itself (which is also an isomorphism from H_i^r to itself for any $r \in (-1, 1)$) which maps the system*

$$\partial_t u = \mathscr{L} u + B_i K_i(u), \quad u \in H_i^r \tag{3.1}$$

to the system

$$\partial_t v = \mathscr{L} v - \lambda v, \quad v \in H^r_i, \tag{3.2}$$

where $\mathscr{L}$ is the linearized water-wave operator given in (1.5).

If Proposition 3.1 holds, then the operator-isomorphism pair (K, T) of Theorem 2.2 is simply $T = T_1 \oplus T_2$ and $K = (K_1, K_2)$. In the following we will focus on the odd space $i = 1$, the even space $i = 2$ can be treated in the same way.

Notations: for reader's convenience we will also drop for now on the index 1 and denote again H^r, K, T, φ_n instead of H^r_1, K_1, T_1, φ^1_n.

Before proving Proposition 3.1, let us make some formal observations to get an intuition of the problem. To map systems (3.1) onto (3.2), what we would like to do is to obtain (formally) the following operator equality

$$T(A + BK) = (A - \lambda)T, \tag{3.3}$$

where $A := \mathscr{L}$. As noted in [12, 17, 19], a good approach to this aim is to add a condition on TB, and to require instead the two following operator equalities

$$\begin{aligned} TA + BK &= (A - \lambda)T, \\ TB &= B, \end{aligned} \tag{3.4}$$

in a certain sense to be specified. Note that (3.4) implies formally (3.3) and requiring (3.4) instead of (3.3) allows to deal with operator equations that are linear with (T, K). It also usually ensures the uniqueness of solution (see for instance [19]). In finite dimension, (3.3) corresponds to an equivalent formulation of the pole-shifting theorem [13, Section 2]. Applying the first operator equality on the orthonormal basis $(\varphi_n)_{n\in\mathbb{N}^*}$ gives

$$\lambda_n(T\varphi_n) + BK(\varphi_n) = (A - \lambda)(T\varphi_n), \tag{3.5}$$

where we used the fact that φ_n is an eigenvector of A. Observe that (3.5) is a (nonlocal) differential equation on $(T\varphi_n)$. Projecting now on a vector φ_p and recalling that $b_p = \langle B, \varphi_p\rangle$, this becomes

$$\begin{aligned} (\lambda_n + \lambda)\langle (T\varphi_n), \varphi_p\rangle + \langle B, \varphi_p\rangle K(\varphi_n) &= \langle A(T\varphi_n), \varphi_p\rangle, \\ &= \langle (T\varphi_n), A^*\varphi_p\rangle = \lambda_p \langle (T\varphi_n), \varphi_p\rangle. \end{aligned} \tag{3.6}$$

This gives the following formal expression

$$T\varphi_n = \sum_{p\in\mathbb{N}^*} \langle (T\varphi_n), \varphi_p\rangle \varphi_p = (-K(\varphi_n)) \sum_{p\in\mathbb{N}^*} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda}. \tag{3.7}$$

These formal calculations lead us to introduce the following notations that will be used all along the proof. We define

- The families

$$q_n := \sum_{p\in\mathbb{N}^*} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda}, \quad K_n = K(\varphi_n), \quad n \in \mathbb{N}^*. \tag{3.8}$$

- The operator

$$S : n^{-r}\varphi_n \mapsto n^{-r} q_n. \tag{3.9}$$

Note that S is completely defined as an operator on H^r since $(n^{-r}\varphi_n)_{n\in\mathbb{N}^*}$ is an orthonormal basis of H^r for any $r \in \mathbb{R}$.

- The operator

$$\tau : \varphi_n \mapsto b_n \varphi_n. \tag{3.10}$$

Note that since b_n are uniformly bounded by above and below, this is an isomophism from L^2 to L^2, and in fact also from H^r to H^r for any $r \in \mathbb{R}$.

- The operator T defined on H^r

$$T : n^{-r}\varphi_n \mapsto (-K_n)n^{-r}\tau q_n. \tag{3.11}$$

Note that this expression of T corresponds exactly to the expression (3.7) obtained from the formal calculations.

In the following we will show that, for a good choice of $(K_n)_{n\in\mathbb{N}^*}$, the operator T thus defined is an isomorphism from H^r to itself for $r \in (-1, 1)$, and T and K satisfy (3.4) in a sense to be specified.

We are going to show successively the following steps:

(1) Show that S is a Fredholm operator from $H^r \to H^r$ for any $r \in (-1, 1)$.

(2) Show that $(q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis for L^2 using a duality argument and the fact that S is Fredholm.

(3) Further show that $(n^{-r}q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis for H^r for any $r \in (-1, 1)$ by showing it is ω-independent using a duality argument between the density of $(n^{-r}q_n)_{n\in\mathbb{N}^*}$ in H^r and the ω-independence of $(n^r\overline{q_n})_{n\in\mathbb{N}^*}$ in H^{-r}.

(4) Provide an explicit candidate of $(K_n)_{n\in\mathbb{N}}$ which satisfies $TB = B$ in $H^{-3/4}$ sense. Show that $(|K_n|)_{n\in\mathbb{N}}$ is bounded from above and that $b_nK_n = -(\lambda + k_n)$ for any $n \in \mathbb{N}^*$, where $(k_n n^{\varepsilon})_{n\in\mathbb{N}^*} \in l^\infty$ for any $\varepsilon \in [0, 1/2)$.

(5) Show that T is bounded from H^r in itself for $r \in (-1, 1)$ and the first operator equality (3.4) holds in $\mathcal{L}(H^{3/4}; H^{-3/4})$.

(6) Show that T is a Fredholm operator from $H^{-3/4}$ to $H^{-3/4}$.

(7) Show that T is an isomorphism from $H^{-3/4}$ to $H^{-3/4}$ using a Fredholm argument and spectral theory in $H^{-3/4}$.

(8) Show that T is an isomorphism from L^2 to L^2 and in fact an isomorphism from H^r to itself for any $r \in (-1, 1)$.

Let us briefly discuss step 6 to 8 as, at first sight, it seems odd to prove the invertibility in $H^{-3/4}$ and not in the classical L^2 space for instance. The main motivation is to avoid working in the space $D(A + BK) := \{f \in L^2 : (A + BK)f \in L^2\}$ before proving the invertibility of T. Indeed, in the preesent setting, the space $D(A + BK)$ does not have nice properties shared by the Sobolev spaces such as the density of C^∞ functions. This comes from the fact that B is not regular enough, and therefore one is not able to conclude that $\varphi_n \in D(A + BK)$ for any $n \in \mathbb{N}^*$. Hence, it is easier to first prove the invertibility in weaker but classical Sobolev spaces (step 6

and 7) before deducing the invertibility in the required spaces (step 8). In turn, the invertibility of T in H^r allows to construct an equivalent norm to H^r, which allows to prove that $D(A+BK)$ is an Hilbert space, a non-trivial task to prove without the invertibility of T. We underline that if our setting is close to the linearized bilinear Schrödinger equation, the fact that the control is real-valued in [12] allows to decouple the real and imaginary part of the solution to deal directly with the space $D(A+BK)$, which is not the case here.

We start by introducing some technical Lemmas in Subsection 4.1. Then we prove Proposition 3.1, following the outline above : we prove steps (1)-(3) in Section 4 and steps (4)-(8) in Section 5. Finally, we prove the well-posedness of the closed-loop system obtained and Corollary 2.3 in Section 6.

4. Compactness/duality method for Riesz basis

Following the outline described in Section 3, this section is devoted to the proofs of Steps (1)-(3). These steps form a first important part of the proof of our main theorem: they revolve around Riesz basis properties for some important families of functions derived from the back-stepping operator equalities (3.4). As we have mentioned in the introduction, we introduce here a new method based on compactness and duality, namely, we prove in a general way that the transformations involved in our backstepping method are Fredholm operators.

4.1. Some basic estimates.

In this section we introduce some technical Lemmas that will be used in the following. For readers' convenience we put part of proofs in Appendix C.

The first lemma is a direct consequence of the existence of $c, C > 0$ such that

$$cn^{3/2} \leq |\lambda_n| \leq Cn^{3/2}, \ \forall n \in \mathbb{N}^*.$$

LEMMA 4.1. *Let $s \in \mathbb{R}$. Let ρ from the resolvent set of the operator $\mathscr{L}$. We know that*

$$\mathscr{L} : H^{s+3/4} \to H^{s-3/4} \text{ is continuous,}$$
$$(\mathscr{L} - \rho)^{-1} : H^{s-3/4} \to H^{s+3/4} \text{ is continuous.}$$

We turn to,

LEMMA 4.2. *For any $s < 1/2$, there exists $C > 0$ such that*

$$\sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{n^s}{|\lambda_n - \lambda_p|} \leq C(p^{-1/2+s} \log(p) + p^{-3/2}), \ \forall p \in \mathbb{N}^*.$$

Let us now show Lemma 4.2

Proof of Lemma 4.2. Let $s < 1/2$, we have

$$\sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{n^s}{|\lambda_n - \lambda_p|} = I_1 + I_2 + I_3,$$

where

$$I_1 = \sum_{n \in \mathbb{N}^*, \, n \leq p/2} \frac{n^s}{|\lambda_n - \lambda_p|},$$
$$I_2 = \sum_{n \in \mathbb{N}^* \setminus \{p\}, \, p/2 < n < 2p} \frac{n^s}{|\lambda_n - \lambda_p|},$$
$$I_3 = \sum_{n=2p}^{+\infty} \frac{n^s}{|\lambda_n - \lambda_p|}.$$

We will show that all these three terms can be bounded by $C(p^{-1/2+s}\log(p)+p^{-3/2})$ where C is a constant independent of p. For this, we introduce the following basic estimates (the proofs may be found in Appendix C).

LEMMA 4.3. *There exists $c>0$ such that for any $(n,m)\in\mathbb{N}^*$*

$$|\lambda_n-\lambda_m|\geq c|n-m|^{3/2},$$
$$|\lambda_n-\lambda_m|\geq c|n-m|n^{1/2}.$$

LEMMA 4.4. *For any $s\neq -1$, there exists $C>0$ such that for any $p\in\mathbb{N}^*$*

$$\sum_{n=1}^{p} n^s\leq C(1+p^{1+s}).$$

LEMMA 4.5. *For any $s\in\mathbb{R}$ and $\varepsilon_1>0$, there exists $C>0$ and $\varepsilon>0$ such that $\varepsilon\in(0,\varepsilon_1)$ and*

$$\sum_{n=1}^{p} n^s\log(n)\leq C(1+p^{1+s+\varepsilon}).$$

Let us look at I_1, since there exists $C>0$ independent of p and n satisfying $n\leq p/2$, such that $|\lambda_p-\lambda_n|\geq C^{-1}p^{3/2}$, using Lemma 4.3

$$\sum_{n\in\mathbb{N}^*,n\leq p/2}\frac{n^s}{|\lambda_n-\lambda_p|}\lesssim p^{-3/2}\sum_{n\in\mathbb{N}^*,n\leq p/2} n^s\lesssim p^{-3/2}+p^{-1/2+s}\log(p), \tag{4.1}$$

where in the rightmost inequality we used Lemma 4.4 if $s\neq -1$, applies and $p^{1+s}\leq p^{1+s}\log(p)$ for p large enough, and if $s=-1$ then we simply used that $\left(\sum_{n=0}^{p}1/n\right)=O(\log(p))$.

Then we turn to I_2, using Lemma 4.3 we have

$$\sum_{n\in\mathbb{N}^*\setminus\{p\},p/2<n<2p}\frac{n^s}{|\lambda_n-\lambda_p|}\lesssim p^s\sum_{n\in\mathbb{N}^*\setminus\{p\},p/2<n<2p}\frac{1}{|\lambda_n-\lambda_p|}\lesssim p^s\sum_{n\in\mathbb{N}^*\setminus\{p\},p/2<n<2p}\frac{1}{|n-p|p^{1/2}}.$$

Notice that

$$\sum_{n\in\mathbb{N}^*\setminus\{p\},p/2<n<2p}\frac{1}{|n-p|}\leq\sum_{k\leq p/2}\frac{1}{k}+\sum_{k=1}^{p}\frac{1}{k}\lesssim\log(p),$$

hence

$$\sum_{n\in\mathbb{N}^*\setminus\{p\},p/2<n<2p}\frac{n^s}{|\lambda_n-\lambda_p|}\lesssim p^{-1/2+s}\log(p), \tag{4.2}$$

which gives the bound on I_2.

Finally we look at I_3, since $n>2p$, there exists a constant $C>0$ independent of p such that $|\lambda_n-\lambda_p|\geq C^{-1}n^{3/2}$ from Lemma 4.3, thus

$$I_3=\sum_{n=2p}^{+\infty}\frac{n^s}{|\lambda_n-\lambda_p|}\lesssim\sum_{n=2p}^{+\infty}n^{s-3/2}\lesssim\int_{2p}^{+\infty}x^{s-3/2}dx\lesssim p^{-1/2+s} \tag{4.3}$$

where we used that $s<1/2$ thus $s-3/2<-1$. Combining (4.1), (4.2) and (4.3) we deduce that

$$I_1+I_2+I_3\lesssim p^{-1/2+s}\log(p)+p^{-3/2}.$$

This ends the proof of Lemma 4.2. □

4.2. **Step (1): a general Fredholm operator.** In this Section we show the following Proposition

PROPOSITION 4.6. *For any $r \in (-1,1)$, there exists a compact operator S_c from H^r into itself such that the operator S defined by (3.9) satisfies on H^r*

$$S = \frac{1}{\lambda} Id + S_c. \tag{4.4}$$

In particular, S is a Fredholm operator (of index 0) from H^r into itself.

First we show that for any $a \in H^r$, denoting $a_n = (a, n^{-r}\varphi_n)_{H^r}$ such that $a = \sum_{n\in\mathbb{N}^*} a_n n^{-r}\varphi_n$, we have

$$Sa = \frac{a}{\lambda} + \left(\sum_{n\in\mathbb{N}^*} a_n n^{-r} \left(\sum_{p\in\mathbb{N}^*\setminus\{n\}} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda} \right) \right). \tag{4.5}$$

Indeed, from the definition of (3.9),

$$\begin{aligned} Sa &= \sum_{n\in\mathbb{N}^*} a_n n^{-r} S(\varphi_n) = \sum_{n\in\mathbb{N}^*} a_n n^{-r} q_n \\ &= \sum_{n\in\mathbb{N}^*} a_n n^{-r} \left(\sum_{p\in\mathbb{N}^*} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda} \right) \\ &= \frac{1}{\lambda} \left(\sum_{n\in\mathbb{N}^*} a_n n^{-r}\varphi_n \right) + \sum_{n\in\mathbb{N}^*} a_n n^{-r} \left(\sum_{p\in\mathbb{N}^*\setminus\{n\}} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda} \right), \end{aligned}$$

which, given the definition of a, is exactly (4.5). Now we show the following

LEMMA 4.7. *For any $r \in (-1,1)$ there exists $\varepsilon = \varepsilon(r) > 0$ such that the operator S_c defined by*

$$S_c : \sum_{n\in\mathbb{N}^*} a_n n^{-r}\varphi_n \mapsto \sum_{n\in\mathbb{N}^*} a_n n^{-r} \left(\sum_{p\in\mathbb{N}^*\setminus\{n\}} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda} \right)$$

is continuous from H^r to $H^{r+\varepsilon}$. In particular this operator is compact from H^r in itself.

The proof of this lemma is based on a careful estimation allowed by the Lemma 4.2, and we give its proof below. Proposition 4.6 then follows from (4.5) and Lemma 4.7.

REMARK 4.8. *Note, as a corollary of Lemma 4.7 and the expression of S given by (4.5), that for any $r \in (-1,1)$, there exists $C > 0$ such that for any $(a_n)_{n\in\mathbb{N}^*} \in l^2$ one has*

$$\left\| \sum_{n\in\mathbb{N}^*} a_n n^{-r} q_n \right\|_{H^r}^2 \leq C \sum_{n\in\mathbb{N}^*} |a_n|^2. \tag{4.6}$$

which means that S is a bounded operator from H^r into itself. In fact, Proposition 4.6 is stronger since it shows that S is even a Fredholm operator.

REMARK 4.9. *As we have mentioned in the introduction, previous works on the backstepping method use the quadratically close criterion to prove that q_n is a Riesz basis. In our case, one would then seek to prove that*

$$\sum_{n\in\mathbb{N}^*} \| n^{-r} q_n - \frac{1}{\lambda} n^{-r}\varphi_n \|_{H^r}^2 < +\infty. \tag{4.7}$$

In terms of S_c, this amounts to

$$\sum_{n\in\mathbb{N}^*} \|S_c n^{-r}\varphi_n\|^2_{H^r} < +\infty \tag{4.8}$$

which is the Hilbert-Schmidt compactness criterion for S_c. However, in our case, (4.7)–(4.8) does not hold. Our new compactness/duality method illustrates that the relevant property is not (4.7), which is particular case of compactness thanks to the Hilbert-Schmidt criterion, but rather the compactness of S_c in itself, which we prove here in a more general way. This, together with the duality argument presented in Steps (2) and (3) below, can also lead to Riesz basis properties thanks to the Fredholm alternative.

Interestingly, this illustrates that there is a link between the growth of the eigenvalues and the class of compact operators that appear in the Fredholm decomposition of S.

Proof of Lemma 4.7. We start by considering two different cases: let $r \in (-1, 0]$, what we need to show is that there exists $C > 0$ and $\varepsilon > 0$ such that for any $(a_n)_{n\in\mathbb{N}^*} \in l^2$,

$$\left\| \sum_{n\in\mathbb{N}^*} a_n n^{-r} \left(\sum_{p\in\mathbb{N}^*\setminus\{n\}} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda} \right) \right\|_{H^{r+\varepsilon}} \leq C \left\| \sum_{n\in\mathbb{N}^*} a_n n^{-r}\varphi_n \right\|_{H^r}.$$

Notice that the following inequality holds in the H^{-1} space

$$\sum_{n\in\mathbb{N}^*} a_n n^{-r} \left(\sum_{p\in\mathbb{N}^*\setminus\{n\}} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda} \right) = \sum_{p\in\mathbb{N}^*} \varphi_p \left(\sum_{n\in\mathbb{N}^*\setminus\{p\}} \frac{a_n n^{-r}}{\lambda_n - \lambda_p + \lambda} \right),$$

it suffices to show that

$$\sum_{p\in\mathbb{N}^*} p^{2r+2\varepsilon} \left| \sum_{n\in\mathbb{N}^*\setminus\{p\}} \frac{a_n n^{-r}}{\lambda_n - \lambda_p + \lambda} \right|^2 \leq C \sum_{n\in\mathbb{N}^*} |a_n|^2. \tag{4.9}$$

Let us look at the left-hand side and use Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\sum_{p\in\mathbb{N}^*} p^{2r+2\varepsilon} \left| \sum_{n\in\mathbb{N}^*\setminus\{p\}} \frac{a_n n^{-r}}{\lambda_n - \lambda_p + \lambda} \right|^2 \\ &\leq \sum_{p\in\mathbb{N}^*} p^{2r+2\varepsilon} \left(\sum_{n\in\mathbb{N}^*\setminus\{p\}} \frac{|a_n|^2 n^{-2r-1/2+2\varepsilon}}{|\lambda_n - \lambda_p|} \right) \left(\sum_{n\in\mathbb{N}^*\setminus\{p\}} \frac{n^{1/2-2\varepsilon}}{|\lambda_n - \lambda_p|} \right). \end{aligned}$$

Then, using Lemma 4.2, Fubini theorem (since all terms are nonnegative), and again Lemma 4.2 (since $2r < 1/2$) we get

$$
\begin{aligned}
&\sum_{p\in\mathbb{N}^*} p^{2r+2\varepsilon}\left|\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{a_n n^{-r}}{\lambda_n-\lambda_p+\lambda}\right|^2\\
&\lesssim \sum_{p\in\mathbb{N}^*} p^{2r+2\varepsilon}\left(\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{|a_n|^2 n^{-2r-1/2+2\varepsilon}}{|\lambda_n-\lambda_p|}\right)\left(p^{-2\varepsilon}\log(p)+p^{-3/2}\right)\\
&\lesssim \sum_{n\in\mathbb{N}^*}|a_n|^2 n^{-2r-1/2+2\varepsilon}\sum_{p\in\mathbb{N}^*\setminus\{n\}}\frac{p^{2r+2\varepsilon}\left(p^{-2\varepsilon}\log(p)+p^{-3/2}\right)}{|\lambda_n-\lambda_p|}\\
&\lesssim \sum_{n\in\mathbb{N}^*}|a_n|^2 n^{-2r-1/2+2\varepsilon}\sum_{p\in\mathbb{N}^*\setminus\{n\}}\frac{p^{2r+2\varepsilon}\left(p^{-3/2\varepsilon}+p^{-3/2}\right)}{|\lambda_n-\lambda_p|}\\
&\lesssim \sum_{n\in\mathbb{N}^*}|a_n|^2 n^{-2r-1/2+2\varepsilon}\left(n^{-1/2+2r+\varepsilon/2}\log(n)+n^{-2+2r+2\varepsilon}\log(n)+n^{-3/2}\right)\\
&\lesssim \sum_{n\in\mathbb{N}^*}|a_n|^2(n^{-1+3\varepsilon}+n^{-5/2+5\varepsilon}+n^{-2-2r+2\varepsilon}).
\end{aligned}
\tag{4.10}
$$

Note that the limiting term is the last one, and since $r\in(-1,0)$ we can choose ε depending only on r such that $r+1-\varepsilon\geq 0$ and such that we have $(n^{-1+3\varepsilon}+n^{-3/2+5\varepsilon}+n^{-2-2r+2\varepsilon})\leq 3$ (choose for instance $\varepsilon=\min(r+1,3/10)$). This means that

$$
\sum_{p\in\mathbb{N}^*} p^{2r+2\varepsilon}\left|\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{a_n n^{-r}}{\lambda_n-\lambda_p+\lambda}\right|^2\lesssim\sum_{n\in\mathbb{N}^*}|a_n|^2,
$$

and this ends the proof of Lemma 4.7 in the case $r\in(-1,0]$. Note that this could also work for $r\in(0,1/4)$. However, for $r\geq 1/4$ Lemma 4.2 cannot be used to get the fifth line of (4.10), which explains the disjonction of cases.

Let us now assume that $r\in(0,1)$. Just like in the previous case, it suffices to show (4.9). Let us apply again Cauchy-Schwarz inequality on the left-hand side of (4.9), but slightly differently as before. We have

$$
\sum_{p\in\mathbb{N}^*} p^{2r+2\varepsilon}\left|\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{a_n n^{-r}}{\lambda_n-\lambda_p+\lambda}\right|^2\leq\sum_{p\in\mathbb{N}^*} p^{2r+2\varepsilon}\left(\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{|a_n|^2}{|\lambda_n-\lambda_p|}\right)\left(\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{n^{-2r}}{|\lambda_n-\lambda_p|}\right).
$$

Using again Lemma 4.2 since $-2r<1/2$, then Fubini theorem, and then again Lemma 4.2 by choosing $\varepsilon>0$ such that $\varepsilon<1/3$ and $2r-3/2+2\varepsilon<1/2$ (which always exists since $2r<2$),

we have

$$\begin{aligned}
&\sum_{p\in\mathbb{N}^*} p^{2r+2\varepsilon}\left|\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{a_n n^{-r}}{\lambda_n-\lambda_p+\lambda}\right|^2\\
&\lesssim \sum_{p\in\mathbb{N}^*} p^{2r+2\varepsilon}\left(\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{|a_n|^2}{|\lambda_n-\lambda_p|}\right)(p^{-1/2-2r}\log(p)+p^{-3/2})\\
&\lesssim \sum_{p\in\mathbb{N}^*}\left(\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{|a_n|^2}{|\lambda_n-\lambda_p|}\right)(p^{-1/2+3\varepsilon}+p^{-3/2+2r+2\varepsilon})\\
&\lesssim \sum_{n\in\mathbb{N}^*}|a_n|^2\left(\sum_{p\in\mathbb{N}^*\setminus\{n\}}\frac{p^{-1/2+3\varepsilon}+p^{-3/2+2r+2\varepsilon}}{|\lambda_n-\lambda_p|}\right)\\
&\lesssim \sum_{n\in\mathbb{N}^*}|a_n|^2\left(n^{-1+3\varepsilon}\log(n)+n^{-3/2}+n^{-2+2r+2\varepsilon}\log(n)\right)\\
&\lesssim \sum_{n\in\mathbb{N}^*}|a_n|^2\left(n^{-1+4\varepsilon}+n^{-3/2}+n^{-2+2r+3\varepsilon}\right)
\end{aligned}$$

Then by choosing ε such that $\varepsilon<1/4$ and $2r-3/2+3\varepsilon<1/2$ we have $-2+2r+3\varepsilon<0$ and $-1+4\varepsilon<0$, hence there exists a constant $C>0$ (depending only on r) such that

$$\sum_{p\in\mathbb{N}^*} p^{2r+2\varepsilon}\left|\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{a_n n^{-r}}{\lambda_n-\lambda_p+\lambda}\right|^2\leq C\sum_{n\in\mathbb{N}^*}|a_n|^2.$$

This ends the proof of Lemma 4.7. □

4.3. **Step (2): a Riesz basis for L^2.** In this section we prove the existence of a Riesz basis.

PROPOSITION 4.10. *The family $(q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis of L^2*

Showing this amounts to showing that S is an isomorphism from L^2 to L^2. Since we know from Proposition 4.6 that S is a Fredholm operator (of index 0), it suffices to show that $ker(S^*)=0$. But we know that

$$\dim(coker(S))=\dim(ker(S^*))=\dim(ker(S))<+\infty, \tag{4.11}$$

where S^* denotes the adjoint of S. Hence it suffices to show that $ker(S)=\{0\}$. This is equivalent to say that for any $(a_n)_{n\in\mathbb{N}^*}\in l^2$, such that

$$\sum_{n\in\mathbb{N}^*} a_n q_n=0, \tag{4.12}$$

one has $a_n=0$, for all $n\in\mathbb{N}^*$. In other words this is equivalent to show that $(q_n)_{n\in\mathbb{N}^*}$ is ω-independent in L^2 (see Definition A.1). Notice that

$$q_n=\sum_p\frac{\varphi_p}{\lambda_n-\lambda_p+\lambda}\quad\text{and}\quad\overline{q_n}=\sum_p\frac{\varphi_p}{\lambda_p-\lambda_n+\lambda}.$$

Thus we have the following Lemma:

LEMMA 4.11. *The sequences $(q_n)_{n\in\mathbb{N}^*}$ and $(\overline{q_n})_{n\in\mathbb{N}^*}$ satisfy the following:*

(i) *$(q_n)_{n\in\mathbb{N}^*}$ is either ω-independent in L^2 or L^2-dense.*

(ii) $(\overline{q_n})_{n\in\mathbb{N}^*}$ *is either* ω*-independent in* L^2 *or* L^2*-dense.*
(iii) $(q_n)_{n\in\mathbb{N}^*}$ *is* ω*-independent in* $L^2 \iff (\overline{q_n})_{n\in\mathbb{N}^*}$ *is* ω*-independent in* L^2*.*
(iv) $(q_n)_{n\in\mathbb{N}^*}$ *is* L^2*-dense* $\iff (\overline{q_n})_{n\in\mathbb{N}^*}$ *is* L^2*-dense.*
(v) $(q_n)_{n\in\mathbb{N}^*}$ *is* L^2*-dense* $\iff (\overline{q_n})_{n\in\mathbb{N}^*}$ *is* ω*-independent in* L^2*.*
(vi) $(\overline{q_n})_{n\in\mathbb{N}^*}$ *is* L^2*-dense* $\iff (q_n)_{n\in\mathbb{N}^*}$ *is* ω*-independent in* L^2*.*

Consequently, we know that $\{q_n\}_{n\in\mathbb{N}^*}$ *(resp.* $\{\overline{q_n}\}_{n\in\mathbb{N}^*}$*) is both* ω*-independent in* L^2 *and* L^2 *dense.*

Proof of Proposition 4.10. The proof of Proposition 4.10 is equivalent to prove (4.12) implies $a_n = 0, \forall n \in \mathbb{N}^*$. But since from Lemma 4.11, $(q_n)_{n\in\mathbb{N}^*}$ is ω-independent in L^2, we conclude directly that $a_n = 0, \forall n \in \mathbb{N}^*$, hence the proof. □

Hence, it remains to prove Lemma 4.11.

Proof of Lemma 4.11. The relations (iii) and (iv) are direct consequences of the conjugacy of q_n and $\overline{q_n}$. We only focus on the proof of (i) and (vi), as (ii) and (v) can be treated similarly.

The proof of (vi) is further separated by two parts:

$$(\overline{q_n})_{n\in\mathbb{N}^*} \text{ is not } L^2\text{-dense} \Longrightarrow (q_n)_{n\in\mathbb{N}^*} \text{ is not } \omega\text{-independent in } L^2,$$
$$(\overline{q_n})_{n\in\mathbb{N}^*} \text{ is not } L^2\text{-dense} \Longleftarrow (q_n)_{n\in\mathbb{N}^*} \text{ is not } \omega\text{-independent in } L^2.$$

On the one hand, suppose that $(\overline{q_n})$ is not L^2-dense, then there exists some *nontrivial* function $\overline{a} = \sum_{p\in\mathbb{N}^*} \overline{a_p}\varphi_p \in L^2$, namely $(\overline{a_p})_{p\in\mathbb{N}^*} \in l^2$, such that

$$\langle \overline{q_n}, \overline{a}\rangle_{L^2} = 0, \ \forall n \in \mathbb{N}^*,$$

which is equivalent to

$$\sum_{p\in\mathbb{N}^*} \frac{a_p}{\lambda_p - \lambda_n + \lambda} = 0, \ \forall n \in \mathbb{N}^*.$$

Thus

$$\sum_{p\in\mathbb{N}^*} a_p q_p = \sum_{p\in\mathbb{N}^*} a_p \sum_{m\in\mathbb{N}^*} \frac{\varphi_m}{\lambda_p - \lambda_m + \lambda} = \sum_{m\in\mathbb{N}^*} \varphi_m \sum_{p\in\mathbb{N}^*} \frac{a_p}{\lambda_p - \lambda_m + \lambda} = 0,$$

and consequently $(q_p)_{p\in\mathbb{N}^*}$ is not ω-independent in L^2.

On the other hand, suppose that $(q_n)_{n\in\mathbb{N}^*}$ is not ω-independent in L^2, then there exists some *nontrivial* sequence $(a_p)_{p\in\mathbb{N}^*} \in l^2$, namely $a = \sum_p a_p\varphi_p \in L^2$, such that

$$0 = \sum_{p\in\mathbb{N}^*} a_p q_p = \sum_{p\in\mathbb{N}^*} a_p \sum_{m\in\mathbb{N}^*} \frac{\varphi_m}{\lambda_p - \lambda_m + \lambda} = \sum_{m\in\mathbb{N}^*} \varphi_m \sum_{p\in\mathbb{N}^*} \frac{a_p}{\lambda_p - \lambda_m + \lambda}.$$

Since $(\varphi_m)_{m\in\mathbb{N}^*}$ is a basis of L^2, this implies that

$$\sum_{p} \frac{a_p}{\lambda_p - \lambda_n + \lambda} = 0, \ \forall n \in \mathbb{N}^*,$$

which is equivalent to

$$(\overline{q_n}, \overline{a})_{L^2} = 0, \ \forall n \in \mathbb{N}^*.$$

Hence the sequence $(\overline{q_n})_{n\in\mathbb{N}^*}$ is not L^2-dense. This ends the proof of the property (vi).

Finally, we turn to the property (i). The proof of this point is now classical (see for instance [19]) and we put it in Appendix D for readers' convenience. We however underline that the proof in Appendix D does not require that the family $(q_n)_{n\in\mathbb{N}^*}$ is quadratically close. □

4.4. **Step (3): extending the Riesz basis property to a sharp range of Sobolev spaces.** We are now going to use Proposition 4.10 as well as Proposition 4.6 to show the following

PROPOSITION 4.12. *For any $r \in (-1,1)$, the family $(n^{-r}q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis of H^r.*

Proof of Proposition 4.12. First of all note, given the definition of S in (3.9), that showing Proposition 4.12 is equivalent to showing that S is an isomorphism from H^r to itself. Since we know, from Proposition 4.6, that S is a Fredholm operator (of index 0) from H^r to itself, it is enough to show that $ker(S) = \{0\}$ (where now $ker(S)$ is now a subset of H^r). As previously, this is equivalent to show that $(n^{-r}q_n)_{n\in\mathbb{N}^*}$ is ω-independent in H^r in the sense of Definition A.1, *e.g.* for any $(a_n)_{n\in\mathbb{N}^*} \in l^2$, such that

$$\sum_{n\in\mathbb{N}^*} a_n n^{-r} q_n = 0, \tag{4.13}$$

one has $a_n = 0$, for all $n \in \mathbb{N}^*$.

The case $r = 0$ is treated in Proposition 4.10, so we now assume $r \neq 0$. We consider two different cases:

- **Case (1)** $r > 0$**.** Let $(a_n)_{n\in\mathbb{N}^*} \in l^2$, and assume that (4.13) holds, we can set $c_n = (a_n n^{-r})_{n\in\mathbb{N}^*} \in l^2$, and (4.13) becomes

$$\sum_{n\in\mathbb{N}^*} c_n q_n = 0. \tag{4.14}$$

  Since $(q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis from Proposition 4.10, we have that $(q_n)_{n\in\mathbb{N}^*}$ is ω-independent in L^2 (see in particular Lemma 4.11) and therefore (4.14) implies that

$$c_n = 0, \quad \forall n \in \mathbb{N}^*,$$

  which implies that $a_n = 0$ for all $n \in \mathbb{N}^*$. Thus $(n^{-r}q_n)_{n\in\mathbb{N}^*}$ is ω-independent, hence $ker(S) = \{0\}$ (in H^r) and this ends the proof.

  REMARK 4.13. *This is summarized by the following: if $r > 0$, $ker(S)$ seen as a subset of H^r is included in $ker(S)$ seen as a subset of L^2, which is $\{0\}$ from Proposition 4.10.*

- **Case (2)** $r < 0$. This is the more challenging case as $ker(S)$ seen as a subset of H^r is not anymore included in $ker(S)$ seen as a subset of L^2 (but rather the opposite holds). To show this, we proceed by contradiction and use a dual argument between ω-independence in H^r and density in H^{-r}.
  Let assume by contradiction that $(n^{-r}q_n)_{\omega\in\mathbb{N}^*}$ is not ω-independent in H^r. Then there exists some nontrivial $(a_n)_{n\in\mathbb{N}^*} \in l^2$ such that (4.13) holds. Projecting on $(m^{-r}\varphi_m)_{m\in\mathbb{N}^*}$, we have

$$\begin{aligned} 0 &= \left(\sum_{n\in\mathbb{N}^*} a_n n^{-r} \left(\sum_{p\in\mathbb{N}^*} \frac{p^{-r}p^r\varphi_p}{\lambda_n - \lambda_p + \lambda} \right), m^{-r}\varphi_m \right)_{H^r} \\ &= \sum_{n\in\mathbb{N}^*} a_n n^{-r} m^r \frac{1}{\lambda_n - \lambda_m + \lambda}, \quad \forall\, m \in \mathbb{N}^*. \end{aligned} \tag{4.15}$$

  Now, let us set $h = \sum_{n\in\mathbb{N}^*} a_n n^r \varphi_n$ which belongs to H^{-r} since $(a_n)_{n\in\mathbb{N}^*} \in l^2$. From assumption, a_n is not identically 0 thus $h \neq 0$ since $n^r\varphi_n$ is a basis of H^{-r}. Since $r < 0$,

$(-r) > 0$ and therefore, from Case (1), $(n^r q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis of H^{-r}. In particular, this implies that $(n^r q_n)_{n\in\mathbb{N}^*}$ is a dense family in H^{-r} and there exists $m_0 \in \mathbb{N}^*$ such that

$$\langle h, m_0^r q_{m_0}\rangle_{H^{-r}} \neq 0,$$

Expending the expression of h and q_{m_0}, using the fact that the λ_i are imaginary, this means

$$\sum_{n\in\mathbb{N}^*} a_n m_0^r \frac{n^{-r}}{\lambda_n - \lambda_{m_0} + \lambda} \neq 0,$$

which is in contradiction with (4.15). Thus $(a_n)_{n\in\mathbb{N}^*}$ is identically 0 and $(n^{-r}q_n)_{n\in\mathbb{N}^*}$ is ω-independant in H^r, which means that $ker(S)$ (seen as a subset of H^r) is reduced to $\{0\}$ and S is an isomorphism from H^r to H^r. This ends the proof of Proposition 4.12.

□

5. Construction of the feedback-isomorphism pair

In this Section we construct the feedback-isomorphism pair to end the proof of Proposition 3.1. In the previous Section, we have established some important properties, namely that specific families of functions form Riesz basis in appropriate spaces: accordingly, this allowed us to define a general isomorphism, given by S, using the first backstepping equality in (3.4). We now use the second one to propose a feedback law. Then, using the Riesz basis properties and the isomorphism S, we prove that the corresponding transformation T is an isomorphism in an array of spaces. This will end the proof of Proposition 3.1 and Theorem 2.2. It will then remain to prove that the closed-loop system given by the feedback is well-posed to have Corollary 2.3.

5.1. **Step (4): construction and basic properties of the stabilizing feedback.** Let us now provide a candidate for $(K_n)_{n\in\mathbb{N}^*}$. Recall that our goal is to have $(K_n)_{n\in\mathbb{N}^*}$ such that the TB=B condition holds in some sense. Expressing this condition on the basis φ_n it becomes

$$\sum_{n\in\mathbb{N}^*} (-K_n b_n)\tau q_n = \sum_{n\in\mathbb{N}^*} b_n \varphi_n. \tag{5.1}$$

Note that the right-hand side belongs to $H^{-3/4}$ and in fact given (2.1) it even belongs to $H^{-1/2-\varepsilon}$ for any $\varepsilon \in (0, 1/2)$. Since $(n^{1/2+\varepsilon}q_n)_{n\in\mathbb{N}^*}$ (and hence $(n^{1/2+\varepsilon}\tau q_n)_{n\in\mathbb{N}^*}$) is a Riesz basis of $H^{-1/2-\varepsilon}$ we have the following:

LEMMA 5.1. *There exists a unique sequence $(-K_n)_{n\in\mathbb{N}^*}$ such that for any $\varepsilon \in (0, 1/2)$ the condition (5.1) holds in $H^{-1/2-\varepsilon}$ and*

$$((-K_n b_n)n^{-1/2-\varepsilon})_{n\in\mathbb{N}^*} \in l^2.$$

Let us set

$$k_n := -(b_n K_n + \lambda), \tag{5.2}$$

The goal of this section will be to show the following Lemma

LEMMA 5.2. *The two following hold*

1. *The sequence $(K_n)_{n\in\mathbb{N}^*}$ defined by Lemma 5.1 is uniformly bounded.*
2. *For any $r \in (-1, 1)$, there exists $\varepsilon > 0$ such that the operator k defined by*

$$k : \varphi_n \mapsto k_n \tau q_n \tag{5.3}$$

is continuous from H^r to $H^{r+\varepsilon}$. In particular this operator is a compact operator from H^r to H^r.

From the definition of K_n given by Lemma 5.1 and the $TB = B$ condition given by (5.1), we have,

$$\sum_{n\in\mathbb{N}^*}(-K_n b_n)\tau q_n = \sum_{n\in\mathbb{N}^*} b_n\varphi_n. \tag{5.4}$$

Recall that this expression is well defined in $H^{-3/4}$ (and in fact in $H^{-1/2-\varepsilon}$ for any $\varepsilon\in(0,1/2)$). Using the definition of k_n and τq_n, we get that in $H^{-1/2-\varepsilon}$ sense

$$\sum_{n\in\mathbb{N}^*}\lambda\sum_{p\in\mathbb{N}^*}\frac{b_p\varphi_p}{\lambda_n-\lambda_p+\lambda}+\sum_{n\in\mathbb{N}^*}k_n\tau q_n=\sum_{n\in\mathbb{N}^*}b_n\varphi_n.$$

which gives

$$\sum_{n\in\mathbb{N}^*}\lambda\sum_{p\in\mathbb{N}^*\setminus\{n\}}\frac{b_p\varphi_p}{\lambda_n-\lambda_p+\lambda}+\sum_{n\in\mathbb{N}^*}b_n\varphi_n+\sum_{n\in\mathbb{N}^*}k_n\tau q_n=\sum_{n\in\mathbb{N}^*}b_n\varphi_n.$$

Hence,

$$\sum_{n\in\mathbb{N}^*}\lambda\sum_{p\in\mathbb{N}^*\setminus\{n\}}\frac{b_p\varphi_p}{\lambda_n-\lambda_p+\lambda}=-\sum_{n\in\mathbb{N}^*}k_n\tau q_n. \tag{5.5}$$

Let us remark that this equality a priori holds in $H^{-1/2-\varepsilon}$ for any $\varepsilon\in(0,1/2)$. However, we removed from the equation the most singular part and both terms are in fact more regular. Nevertheless, using a rough estimation looking at the sum of the modulus of the terms of the left-hand side we cannot conclude that this term is in L^2: indeed, we are here in a critical case where, for p large enough,

$$|b_p|^2\left(\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{1}{|\lambda_n-\lambda_p+\lambda|}\right)^2\geq\frac{1}{p},$$

and therefore the sum over $p\in\mathbb{N}^*$ cannot converge. However, we can show that it belongs to $H^{-\varepsilon}$ for any $\varepsilon>0$. Indeed, we have

$$\sum_{n\in\mathbb{N}^*}\lambda\sum_{p\in\mathbb{N}^*\setminus\{n\}}\frac{b_p\varphi_p}{\lambda_n-\lambda_p+\lambda}=\sum_{p\in\mathbb{N}^*}b_p\varphi_p\lambda\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{1}{\lambda_n-\lambda_p+\lambda}$$

this equality holds in $H^{-3/4}$ using Fubini theorem. Now we have

$$\begin{aligned}\left|\sum_{p\in\mathbb{N}^*}b_p\varphi_p\lambda\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{1}{\lambda_n-\lambda_p+\lambda}\right|^2_{H^{-\varepsilon}}&=\sum_{p\in\mathbb{N}^*}p^{-2\varepsilon}|b_p|^2\lambda^2\left|\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{1}{\lambda_n-\lambda_p+\lambda}\right|^2\\&\lesssim\sum_{p\in\mathbb{N}^*}p^{-2\varepsilon}\left(\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{1}{|\lambda_n-\lambda_p|}\right)^2,\end{aligned} \tag{5.6}$$

where we used the fact that b_p are bounded and that λ_n and λ_p are imaginary while λ is real. We use now Lemma 4.2 which gives that, together with (5.6),

$$\left|\sum_{p\in\mathbb{N}^*}b_p\varphi_p\lambda\sum_{n\in\mathbb{N}^*\setminus\{p\}}\frac{1}{\lambda_n-\lambda_p+\lambda}\right|^2_{H^{-\varepsilon}}\lesssim\sum_{p\in\mathbb{N}^*}p^{-2\varepsilon}\frac{\log^2(p)}{p} \tag{5.7}$$

and this converges for any $\varepsilon > 0$, hence the left-hand side of (5.5) belongs to $H^{-\varepsilon}$ for any $\varepsilon > 0$. Let us select $\varepsilon \in (0,1)$. Rewriting (5.5), we have

$$\sum_{n\in\mathbb{N}^*} \lambda \sum_{p\in\mathbb{N}^*\setminus\{n\}} \frac{b_p\varphi_p}{\lambda_n - \lambda_p + \lambda} = -\sum_{n\in\mathbb{N}^*} (k_n n^{-\varepsilon})(n^{\varepsilon}\tau q_n). \tag{5.8}$$

From Proposition 4.12 and the fact that τ is an isomorphism from H^r to H^r for any $r \in (-1,1)$, $(n^{\varepsilon}\tau q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis of $H^{-\varepsilon}$ and therefore there exists a unique $(k_n n^{-\varepsilon})_{n\in\mathbb{N}^*} \in l^2$ such that (5.8) holds. In particular this shows that $(k_n)_{n\in\mathbb{N}^*}$ defined in (5.2) satisfies

$$(k_n n^{-\varepsilon})_{n\in\mathbb{N}^*} \in l^2 \text{ for any } \varepsilon \in (0, 3/4]. \tag{5.9}$$

By definition, $K_n b_n = -(\lambda + k_n)$, this together with (5.9) and the uniform boundedness of $(b_n)_{n\in\mathbb{N}^*}$ given by (2.1) implies that

$$(K_n n^{-\varepsilon})_{n\in\mathbb{N}^*} \in l^\infty \text{ for any } \varepsilon \in (0, 3/4]. \tag{5.10}$$

In fact, we can even show more than that: looking at (5.4) again and expliciting τq_n, we have

$$\sum_{n\in\mathbb{N}^*} (-K_n b_n)\left(\frac{b_n\varphi_n}{\lambda} + \sum_{p\in\mathbb{N}^*\setminus\{n\}} \frac{b_p\varphi_p}{\lambda_n - \lambda_p + \lambda}\right) = \sum_{n\in\mathbb{N}^*} b_n\varphi_n,$$

which gives, by decomposing K_n in λ and b_n in the first term only

$$\sum_{n\in\mathbb{N}^*} \frac{k_n b_n}{\lambda}\varphi_n + \left(\sum_{n\in\mathbb{N}^*} (-K_n b_n) \sum_{p\in\mathbb{N}^*\setminus\{n\}} \frac{b_p\varphi_p}{\lambda_n - \lambda_p + \lambda}\right) = 0,$$

therefore, using again Fubini theorem in H^{-1} and identifying the coefficients along the basis $(\varphi_n)_{n\in\mathbb{N}^*}$,

$$\frac{k_n b_n}{\lambda} = \sum_{m\in\mathbb{N}^*\setminus\{n\}} K_m b_m \frac{b_n}{\lambda_m - \lambda_n + \lambda}. \tag{5.11}$$

hence, using (5.10), the boundedness of $(b_m)_{m\in\mathbb{N}^*}$, and Lemma 4.2 we have, for any $s > 0$,

$$\left|\frac{k_n}{\lambda}\right| \lesssim \sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{m^s}{|\lambda_m - \lambda_n|} \lesssim \frac{\log(n)}{n^{1/2-s}}.$$

This implies that

$$(k_n n^{\varepsilon})_{n\in\mathbb{N}^*} \in l^\infty \quad \text{for any } \varepsilon \in [0, 1/2), \tag{5.12}$$

and, in particular $k_n \to 0$ and, from (5.2), $(K_n)_{n\in\mathbb{N}^*} \in l^\infty$. This ends the proof of the first point of Lemma 5.2.

Let us now study the operator k given by (5.3). Let $r \in (-1,1)$, set $\varepsilon \in (0, 1-r)$ such that $\varepsilon < 1/4$. Let $f \in H^r$ and denote $f_n = \langle f, n^{-r}\varphi_n\rangle_{H^r}$ such that $f = \sum_{n\in\mathbb{N}^*} f_n n^{-r}\varphi_n$,

$$k(f) = \sum_{n\in\mathbb{N}^*} f_n k_n n^{-r}\tau q_n = \sum_{n\in\mathbb{N}^*} (f_n k_n n^{\varepsilon})(n^{-(r+\varepsilon)}\tau q_n),$$

the last equality holds a priori in H^r since k_n is bounded. However, we know from (5.12) that $(k_n n^{\varepsilon})_{n\in\mathbb{N}^*}$ is uniformly bounded and thus $(f_n k_n n^{\varepsilon})_{n\in\mathbb{N}^*} \in l^2$. In addition, $r + \varepsilon < 1$ and

we know from Proposition 4.12 that $(n^{-(r+\varepsilon)}q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis of $H^{r+\varepsilon}$ therefore so is $(n^{-(r+\varepsilon)}\tau q_n)_{n\in\mathbb{N}^*}$. Thus $k(f)\in H^{r+\varepsilon}$ and

$$\|k(f)\|_{H^{r+\varepsilon}} \lesssim \|f_n k_n n^{\varepsilon}\|_{l^2} \lesssim \|f\|_{H^r}.$$

This ends the proof of Lemma 5.2.

5.2. **Step (5): boundedness of the corresponding backstepping transformation.** Let us now look at the boundedness of T and show that it satisfies the operator equality (3.4) in some sense. We show the following

LEMMA 5.3. *The operator T given by (3.11) is a bounded operator from H^r to H^r for any $r\in(-1,1)$. Moreover and we have the following operator equality*

$$TA+BK=(A-\lambda)T \quad in \quad \mathcal{L}(H^{3/4+s};H^{-3/4+s}),\ \forall s\in\left(-\frac{1}{4},\frac{1}{4}\right). \tag{5.13}$$

Proof of Lemma 5.3. We start by proving that T is a bounded operator using the previous section. Let $r\in(-1,1)$. We introduce the operator τ_K

$$\tau_K : n^{-r}\tau q_n \to (-K_n)n^{-r}\tau q_n, \tag{5.14}$$

which is well defined since $(n^{-r}\tau q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis of H^r (recall that $(n^{-r}\tau q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis of H^r and τ is an isomorphism from H^r to H^r). From the definition of T given in (3.11) we have

$$T=\tau_K\circ\tau\circ S. \tag{5.15}$$

Since both τ and S are bounded from H^r to itself is suffices to show that τ_K is bounded to show that T is bounded from H^r to itself. Since we showed in the previous section that $(K_n)_{n\in\mathbb{N}^*}$ is uniformly bounded from above (see Lemma 5.2), and $(n^{-r}\tau q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis of H^r, we have that for any $f=\sum_{n\in\mathbb{N}^*} f_n n^{-r}\tau q_n\in H^r$,

$$\|\tau_K f\|_{H^r} = \|\sum_{n\in\mathbb{N}^*}(-K_n)f_n n^{-r}\tau q_n\|_{H^r} \lesssim \|(f_nK_n)_n\|_{l^2} \lesssim \|(f_n)_n\|_{l^2} \lesssim \|f\|_{H^r}$$

Thus τ_K is bounded from H^r in itself and so is T.

Let us now prove the operator equality (5.13). Observe first that all terms make sense: indeed, $K: n^{-r}\varphi_n\mapsto n^{-r}K_n$ is a bounded operator from $H^{3/4+s}$ to $\mathbb{C}$ for any $s\in(-1/4,1/4)$. On the other hand $B\in H^{-3/4}$ and in fact in $H^{-3/4+s}$ for any $s\in(-1/4,1/4)$ since $(b_n)_{n\in\mathbb{N}^*}$ is uniformly bounded. So B can be formally seen as an operator from $\mathbb{C}$ to $H^{-3/4+s}$ for any $s\in(-1/4,1/4)$. Thus, BK is a bounded operator from $H^{3/4+s}$ to $H^{-3/4+s}$ for any $s\in(-1/4,1/4)$. Similarly one can check that both AT and TA are bounded operators from $H^{3/4+s}$ to $H^{-3/4+s}$ when $s\in(-1/4,1/4)$ (since T is a bounded operator in H^r for any $r\in(-1,1)$). To show (5.13) it suffices to check that it holds against $n^{-3/4-s}\varphi_n$ for any $n\in\mathbb{N}^*$ and $s\in(-1/4,1/4)$. Since the operators are linear it suffices to check that it holds against φ_n for any $n\in\mathbb{N}^*$. From the

definition of T (see (3.11)) we have (in $H^{-3/4+s}$ space)

$$\begin{aligned}
&[(TA+BK)-(A-\lambda)T]\,\varphi_n = \lambda_n T\varphi_n + BK_n - (A-\lambda)T\varphi_n \\
&= \lambda_n(-K_n)\left(\sum_{p\in\mathbb{N}^*}\frac{b_p\varphi_p}{\lambda_n-\lambda_p+\lambda}\right) + BK_n - (-K_n)\left(\sum_{p\in\mathbb{N}^*}\frac{b_p(\lambda_p-\lambda)\varphi_p}{\lambda_n-\lambda_p+\lambda}\right). \\
&= (-K_n)\left(\sum_{p\in\mathbb{N}^*} b_p\varphi_p\right) + BK_n = 0.
\end{aligned}$$

This ends the proof of Lemma 5.3 □

5.3. **Step (6): Fredholm operator property in $H^{-3/4}$.** We now prove the following Lemma

LEMMA 5.4. *T defined by (3.11) with K_n defined by Lemma 5.1 is a Fredholm operator (of index 0) from $H^{-3/4}$ to $H^{-3/4}$.*

Proof of Lemma 5.4. In order to show this[1] it suffices to show that $T_c = T - Id$ is a compact operator in $H^{-3/4}$. Let $f \in H^{-3/4}$, and denote $\left(f, n^{3/4}\varphi_n\right)_{H^{-3/4}} = f_n$ such that we have

$$f = \sum_{n\in\mathbb{N}^*} f_n(n^{3/4}\varphi_n),$$

where $(f_n)_{n\in\mathbb{N}^*} \in l^2$. We have, using the fact that T is bounded from $H^{-3/4}$ to $H^{-3/4}$,

$$\begin{aligned}
T_c f &= Tf - f \\
&= \sum_{n\in\mathbb{N}^*} f_n(-K_n)n^{3/4}\tau q_n - \sum_{n\in\mathbb{N}^*} f_n n^{3/4}\varphi_n \\
&= \left(\sum_{n\in\mathbb{N}}\frac{f_n}{b_n}n^{3/4}\lambda\tau q_n\right) - \left(\sum_{n\in\mathbb{N}^*} f_n n^{3/4}\varphi_n\right) + \left(\sum_{n\in\mathbb{N}}\frac{f_n}{b_n}n^{3/4}k_n\tau q_n\right).
\end{aligned} \tag{5.16}$$

Now observe that, as previously,

$$\begin{aligned}
&\left(\sum_{n\in\mathbb{N}^*}\frac{f_n}{b_n}n^{3/4}\lambda\tau q_n\right) - \left(\sum_{n\in\mathbb{N}^*} f_n n^{3/4}\varphi_n\right) \\
&= \left(\sum_{n\in\mathbb{N}^*}\frac{f_n}{b_n}n^{3/4}\lambda\sum_{p\in\mathbb{N}^*\setminus\{n\}}\frac{b_p\varphi_p}{\lambda_n-\lambda_p+\lambda}\right) + \left(\sum_{n\in\mathbb{N}^*} f_n n^{3/4}\lambda\frac{\varphi_n}{\lambda}\right) - \left(\sum_{n\in\mathbb{N}^*} f_n n^{3/4}\varphi_n\right) \\
&= \lambda\left(\sum_{n\in\mathbb{N}^*}\frac{f_n}{b_n}n^{3/4}\sum_{p\in\mathbb{N}^*\setminus\{n\}}\frac{b_p\varphi_p}{\lambda_n-\lambda_p+\lambda}\right).
\end{aligned}$$

Together with (5.16) and the definition of the operator k (5.3), this gives

$$T_c f = \lambda\left(\sum_{n\in\mathbb{N}^*}\frac{f_n}{b_n}n^{3/4}\sum_{p\in\mathbb{N}^*\setminus\{n\}}\frac{b_p\varphi_p}{\lambda_n-\lambda_p+\lambda}\right) + k(\tau^{-1}f).$$

[1] One could wonder: why showing this in $H^{-3/4}$ while this seems to hold in L^2 as well? This will become clearer in the next section: it is easier to show first that T is an isomorphism in $H^{-3/4}$ rather than in L^2 and then deduce that T is an isomorphism in L^2.

From the definition S_c given by Lemma 4.7 and τ given in (3.10), this can be expressed as

$$T_c f = \lambda \tau \circ S_c \circ \tau^{-1} f + k \circ \tau^{-1} f.$$

We know from Lemma 5.2 that k is a compact operator from $H^{-3/4}$ to $H^{-3/4}$ and τ^{-1} is an isomorphism from $H^{-3/4}$ to $H^{-3/4}$ thus $k \circ \tau^{-1}$ is a compact operator from $H^{-3/4}$ to $H^{-3/4}$. Similarly S_c is a compact operator from $H^{-3/4}$ to $H^{-3/4}$ from Lemma 4.7, and therefore $\tau \circ S_c \circ \tau^{-1}$ is a compact operator from $H^{-3/4}$ to $H^{-3/4}$. Hence, T_c is a compact operator from $H^{-3/4}$ to $H^{-3/4}$. This ends the proof of Lemma 5.4. □

5.4. **Step (7): invertibility in $H^{-3/4}$.** We show the following Lemma

LEMMA 5.5. *T defined by (3.11) with K_n given by Lemma 5.1 defines an isomorphism from $H^{-3/4}$ to $H^{-3/4}$.*

Proof of Lemma 5.5. We already now that T is a bounded operator from $H^{-3/4}$ to $H^{-3/4}$ and that T is also a Fredholm operator of index 0. Therefore, using the fact that the adjoint of a Fredholm operator is still a Fredholm operator from Schauder's theorem, and that $dim(ker(T)) = dim(coker(T)) = dim(ker(T^*)) < +\infty$, it suffices to show that

$$ker(T^*) = \{0\} \text{ in } H^{-3/4}, \tag{5.17}$$

where T^* is the adjoint of T (taken as an operator from $H^{-3/4}$ for $H^{-3/4}$). From that point the method is inspired by what is done for the Schrödinger equation or the heat equation in [12, 19]: the proof is composed of three main steps:

1. There exists $\rho \in \mathbb{C}$ such that both $A + BK + \lambda Id + \rho Id$ and $A + \rho Id$ are invertible operator from $H^{3/4}$ to $H^{-3/4}$.
2. For such a ρ, if $ker(T^*) \neq \{0\}$, then $(A + \rho Id)^{-1}$ has an eigenvector h which belongs to $ker(T^*)$.
3. No eigenvector of $(A + \rho Id)^{-1}$ belong to $ker(T^*)$.

From *Step 2* and *Step 3* we deduce that $ker(T^*) = \{0\}$. Given that this is very similar to what is done in [12, 19], the rigorous proof is postponed to the Appendix E. In addition, we also provide another direct method to prove the first step instead of the method using the perturbation theory of operators that is introduced in [12]. □

5.5. **Step (8): invertibility on a range of Sobolev spaces.** Now that we know from Lemma 5.5 that T is an isomorphism from $H^{-3/4}$ to $H^{-3/4}$, we are going to show the following proposition

PROPOSITION 5.6. *For any $r \in (-1, 1)$, the operator T given by (3.11) is an isomorphism from H^r to H^r. In particular T is an isomorphism from L^2 to L^2.*

Proof of Proposition 5.6. We first show that for any $n \in \mathbb{N}^*$, $K_n \neq 0$. Let $m \in \mathbb{N}^*$. Since T is an isomorphism from $H^{-3/4}$ to $H^{-3/4}$ from Lemma 5.5 there exists $h \in H^{-3/4}$ such that

$$Th = m^{3/4} \tau q_m \tag{5.18}$$

As $h \in H^{-3/4}$ there exists $(h_n)_{n \in \mathbb{N}^*} \in l^2$ such that

$$h = \sum_{n \in \mathbb{N}^*} h_n n^{3/4} \varphi_n,$$

hence

$$Th = \sum_{n \in \mathbb{N}^*} (-K_n) h_n n^{3/4} \tau q_n.$$

As $(n^{3/4}q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis of $H^{-3/4}$ and $\tau : \ H^{-3/4} \to H^{-3/4}$ is an isomorphism, then $(n^{3/4}\tau q_n)_{n\in\mathbb{N}^*}$ is also a Riesz basis of $H^{-3/4}$ which means, together with (5.18), that

$$h_m(-K_m) = 1,$$

hence $K_m \neq 0$.

Now, we are going to show that $(K_n)_{n\in\mathbb{N}^*}$ is bounded by below. Note that from (5.2)

$$|K_n b_n| = |\lambda + k_n|,$$

where $(k_n n^{\varepsilon})_{n\in\mathbb{N}^*} \in l^{\infty}$, which means that $k_n \to 0$ when $n \to +\infty$. Hence there exists $n_0 \in \mathbb{N}^*$ such that for any $n \geq n_0$

$$|K_n b_n| \geq \frac{\lambda}{2} > 0. \tag{5.19}$$

As K_n and b_n do not vanish, $\min_{n\leq n_0}(|K_n b_n|) > 0$. This together with (5.19) means that $|K_n b_n|$ is uniformly bounded by below. Since b_n is uniformly bounded we deduce that there exists a constant $c > 0$ independent of n such that

$$|K_n| \geq c > 0, \ \ \forall\, n \in \mathbb{N}^*.$$

We can now conclude: since $(K_n)_{n\in\mathbb{N}^*}$ is uniformly bounded by below and by above (see Lemma 5.2), the operator

$$\tau_K : n^{-r}\tau q_n \mapsto (-K_n)n^{-r}\tau q_n$$

is an isomorphism from H^r to H^r, for any $r \in (-1, 1)$. We used here that $(n^{-r}\tau q_n)_{n\in\mathbb{N}^*}$ is a Riesz basis of H^r. Using the same fact, the operator

$$n^{-r}\varphi_n \mapsto n^{-r}\tau q_n$$

is also an isomorphism from H^r to H^r, hence composing using τ_K

$$T : n^{-r}\varphi_n \mapsto (-K_n)n^{-r}\tau q_n$$

is an isomorphism from H^r to H^r. This ends the proof of Proposition 5.6

REMARK 5.7. *This last argument simply means that $T = \tau_K \circ \tau \circ S$, each of these three operator being an isomorphism from H^r to H^r.*

□

6. Well-posedness and stability of the closed-loop system

The well-posedness of the closed-loop system is based on semigroup theory. We start by showing the well-posedness of the closed-loop system in L^2. The closed-loop system has the form

$$\begin{cases} \partial_t u = \mathscr{L}u + BKu, & (t,x) \in (0,T)\times\mathbb{T}, \\ u|_{t=0} = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{6.1}$$

with

$$B = \sum_{n\in\mathbb{N}^*} b_n\varphi_n \in H^{-1/2-\varepsilon},$$

$$K : \varphi_n \to K_n, \text{ thus } H^{1/2+\varepsilon} \to \mathbb{C}.$$

Recall that in order to simplify the notations we have the operator

$$\mathscr{L} + BK = A + BK.$$

Clearly, $A+BK$ can be defined on $H^{1/2+\varepsilon}$, for example

$$A+BK : H^{3/4} \to H^{-3/4}.$$

However, it does not map regular enough Sobolev spaces to the "desired" L^2 space:

$$A+BK : H^{3/2} \to H^{-1/2-\varepsilon}.$$

Therefore, we shall define the unbounded operator $A+BK$ as follows:

$$A+BK : D(A+BK) \subset L^2 \to L^2,$$
$$D(A+BK) := \{f \in L^2 : (A+BK)f \in L^2\},$$

where the space $D(A+BK)$ is endowed with the usual norm

$$\|y\|_{D(A+BK)} := \|y\|_{L^2} + \|(A+BK)y\|_{L^2},\ \forall y \in D(A+BK).$$

Keep in mind that $H^{3/2} \cap D(A+BK) = \{0\}$ since B does not belong to L^2. In particular φ_n does not belong to $D(A+BK)$. Inspired from the definition we shall compare $D(A+BK)$ with H^1.

REMARK 6.1 ($D(A+BK)$ is a subset of $H^{1-\varepsilon}$)**.** *Remark that $D(A+BK) \subset H^{1-\varepsilon}$, for any $\varepsilon > 0$. Indeed, by the definition of $D(A+BK)$ we know that every f belonging to $D(A+BK)$ satisfies $Af \in H^{-3/2}$, thus $BKf \in H^{-3/2}$. This implies that Kf is well-defined and belongs to $\mathbb{C}$. Therefore, BKf also belong to the space $H^{-1/2-\varepsilon}$, we know from the definition of $D(A+BK)$ that Af belongs to $H^{-1/2-\varepsilon}$. Hence, $f = A^{-1}(Af) \in H^{1-\varepsilon}$.*

In the following we shall prove that $A+BK$ generates a strong C^0-semigroup on L^2, namely the system (6.1) is well-posed on L^2. Before that, we define an equivalent norm on L^2 as

$$\|y\|_{L^2_d} := \|Ty\|_{L^2},\ \forall y \in L^2. \tag{6.2}$$

PROPOSITION 6.2. *The unbounded operator $A+BK$ is the infinitesimal generator of a strong C^0-semigroup on $(L^2, \|\cdot\|_{L^2_d})$.*

Proof of Proposition 6.2. This proof is inspired by [12] for the well-posedness of the closed-loop Schrödinger equations and [13] for the well-posed of closed-loop water tank systems. Remark that we unfortunately cannot adapt the proof from [19] concerning well-posedness of the heat equations for the lack of smoothing effects.

Step 1. Extend the operator equality on $D(A+BK)$.

For any function f chosen from $D(A+BK)$ we aim to show that

$$(TA+BK)f = ((A-\lambda)T)\, f \ \text{ in } L^2. \tag{6.3}$$

Recall that this holds for $f \in H^{3/4+s}$ in $H^{-3/4+s}$ for any $s \in (-1/4, 1/4)$ but *a priori* we do not have any information in L^2. Using the $TB = B$ condition we know that for any $f \in D(A+BK) \subset H^{3/4}$ the following holds in $H^{-3/4}$

$$(TA+BK)f = (TA+TBK)f = T(A+BK)f.$$

However $T(A+BK)f \in L^2$ from the definition of $D(A+BK)$. Hence the previous equality also holds in L^2. This means that $(TA+BK)f \in L^2$. Hence, since we know from Lemma 5.3 that the operator equality (6.3) holds in $H^{-3/4}$ sense, it also holds in L^2. As a direct consequence we know that $ATf \in L^2$, thus $Tf \in H^{3/2}$. This implies that every $f \in D(A+BK)$ also belongs to

$T^{-1}(H^{3/2})$.

Step 2. On the characterization of $D(A+BK)$.

In the preceding step we have shown that $D(A+BK)$ is a subspace of $T^{-1}(H^{3/2})$. Actually, we have the following lemma which shows that they are coincident:

LEMMA 6.3. *The following holds $D(A+BK) = T^{-1}(H^{3/2})$.*

It remains to prove that $T^{-1}(H^{3/2}) \subset D(A+BK)$. Suppose that $\tilde{f} \in T^{-1}(H^{3/2}) \subset H^{1-\varepsilon}$. The operator equality (6.3) holds in $H^{-3/4}$ sense. We also know from the choice of $\tilde{f}$ that $(A-\lambda)T\tilde{f} \in L^2$, which means that (6.3) holds in L^2 sense, namely

$$T(A+BK)\tilde{f} \in L^2 \Longrightarrow (A+BK)\tilde{f} \in L^2 \Longrightarrow \tilde{f} \in D(A+BK).$$

This concludes to proof of the Lemma.

REMARK 6.4. *Looking at the target system (2.3), it is relatively easy to show that this sytem is well-posed in L^2 and that the operator $(A-\lambda)$ generates a semigroup on L^2. One can observe that $H^{3/2}$ is exactly the domain of the infinitesimal generator of this semigroup. Thus, what Lemma 6.3 means is that $D(A+BK)$ defined as above is indeed the right space to consider since it corresponds exactly the inverse image of the domain of the infinitesimal semigroup generator of the target system.*

Step 3. Show that $D(A+BK)$ is dense in L^2.

Thanks to Lemma 6.3, we are able to show that $D(A+BK)$ is dense in L^2. Indeed, as $H^{3/2}$ is dense in L^2, for any function $k \in L^2$ and any $\varepsilon > 0$ we can select some function $h \in H^{3/2}$ such that $\|h - Tk\|_{L^2} \leq \varepsilon$. By the fact that T is an isomorphism on L^2, the function $T^{-1}h \in T^{-1}(H^{3/2}) = D(A+BK)$ and satisfies $\|T^{-1}h - k\|_{L^2} \leq C\varepsilon$.

Step 4. Further characterization on $D(A+BK)$, the operator T and the operator equality.

We show the following Lemma

LEMMA 6.5. *$D(A+BK)$ is a Hilbert space. Moreover,*

$$T : D(A+BK) = T^{-1}(H^{3/2}) \to H^{3/2}$$

is an isomorphism, and

$$TA + BK = (A-\lambda)T \in \mathcal{L}(D(A+BK); L^2).$$

Proof of Lemma 6.5. We first show that $D(A+BK)$ is complete and hence is a Hilbert space. Given $(f_n)_{n\in\mathbb{N}^*}$ a Cauchy sequence in $D(A+BK)$, it holds

$$\|(A+BK)(f_n - f_m)\|_{L^2} + \|f_n - f_m\|_{L^2} \xrightarrow{n,m\to+\infty} 0.$$

Since $(f_n)_{n\in\mathbb{N}^*}$ is a sequence of elements of $D(A+BK)$, according to the operator equality (6.3),

$$\|(A-\lambda)(Tf_n - Tf_m)\|_{L^2} = \|T(A+BK)(f_n - f_m)\|_{L^2} \to 0,$$

thus

$$\|(A(Tf_n - Tf_m)\|_{L^2} \to 0 \quad \text{which implies} \quad \|Tf_n - Tf_m\|_{H^{3/2}} \to 0$$

Since $H^{3/2}$ is complete, there exists some $Tf \in H^{3/2}$ such that Tf_n tends to Tf in $H^{3/2}$. As a direct consequence we immediately obtain f_n tends to f in L^2. Notice that $Tf \in H^{3/2}$ hence $f = T^{-1}(Tf)$ belongs to $T^{-1}(H^{3/2}) = D(A+BK)$ from Lemma 6.3, and so we have the operator equality (6.3)

$$T(A+BK)f = (A-\lambda)Tf \text{ in } L^2.$$

Moreover,

$$\|T(A+BK)f_n - T(A+BK)f\|_{L^2} = \|(A-\lambda)T(f_n-f)\|_{L^2} \to 0,$$

where the last line comes from the fact that $Tf_n \to Tf$ in $H^{3/2}$. Thus $(f_n)_{n\in\mathbb{N}^*}$ converges to f in $D(A+BK)$ endowed with the $\|\cdot\|_{D(A+BK)}$-norm.

Next, we turn to the second part of Lemma 6.5 and we show that T is an isomorphism. For any $y \in D(A+BK) = T^{-1}(H^{3/2})$, we know that, since the operator equality holds in L^2,

$$\begin{aligned}\|y\|_{D(A+BK)} = \|(A+BK)y\|_{L^2} + \|y\|_{L^2} &\sim \|T(A+BK)y\|_{L^2} + \|y\|_{L^2}\\ &= \|(A-\lambda)Ty\|_{L^2} + \|y\|_{L^2}\\ &\sim \|Ty\|_{H^{3/2}},\end{aligned}$$

which implies that T is an isomorphism.

Finally, we immediately get from the definition of $D(A+BK)$ and the fact that it is a Hilbert space that

$$TA + BK, (A-\lambda)T \in \mathcal{L}(D(A+BK); L^2).$$

□

Step 5. Prove that the operator, $A+BK : D(A+BK) \to L^2$, is closed.

The standard argument leads to the closeness of $A+BK : D(A+BK) \to L^2$: suppose that a sequence $f_n \in D(A+BK)$ satisfies

$$(f_n, (A+BK)f_n) \to (f,g) \text{ in } L^2 \times L^2,$$

for some $(f,g) \in L^2 \times L^2$, then we must have $f \in D(A+BK)$ and $(A+BK)f = g$. The proof is divided into 2 steps:

- Since f_n tends to f in L^2, we know that Af_n tends to Af in $H^{-3/2}$. Thus from the assumption that
$$(A+BK)f_n \to g \text{ in } L^2$$
we obtain
$$BKf_n \to g - Af \text{ in } H^{-3/2},$$
which implies that Kf_n converges to some constant $c \in \mathbb{C}$, in particular,
$$BKf_n \to cB \text{ in } H^{-3/4}.$$
Now the assumption further yields
$$Af_n \to g - cB \text{ in } H^{-3/4},$$
hence f_n tends to $A^{-1}(g-cB)$ in $H^{3/4}$ sense, which combined with the assumption that f_n tends to f in L^2 imply that $f = A^{-1}(g-cB)$ in $H^{3/4}$ and
$$f_n \to f \text{ in } H^{3/4}.$$
- By the convergence of f_n in $H^{3/4}$ space, we get
$$(A+BK)f_n = Af_n + BKf_n \to Af + BKf \text{ in } H^{-3/4}.$$
This, to be combined with the assumption, yield
$$g = (A+BK)f \text{ in } H^{-3/4} \text{ and in fact in } L^2 \text{ since } g \in L^2,$$
in particular, it also means that $f \in D(A+BK)$.

Therefore, $\mathscr{L}_C : D(A+BK) \to L^2$ is closed.

Step 6. Dissipativity of $(A+BK)$ *and* $(A+BK)^*$ *in* $(L^2, \|\cdot\|_{L^2_d})$.

Let $f \in D(A+BK) \subset H^{3/4}$. Thanks to *Step 1*, we know that $Tf \in H^{3/2}$, $Af \in H^{-1/4}$ and $TBKf \in H^{-3/4}$ and the operator equality (6.3) holds in L^2. Thus from the definition of $\|\cdot\|_{L^d}$ defined in (6.2) and denoting $\langle\cdot,\cdot\rangle_{L^d}$ the associated scalar product,

$$\begin{aligned}\operatorname{Re}\langle (A+BK)f, f\rangle_{L^2_d} &= \operatorname{Re}\langle T(A+BK)f, Tf\rangle_{L^2}\\ &= \operatorname{Re}\langle (A-\lambda)Tf, Tf\rangle_{L^2} = \langle -\lambda Tf, Tf\rangle_{L^2} \le 0.\end{aligned}$$

We have used here the fact that $Re(\langle Af, f\rangle_{L^2}) = 0$ since A is skew-adjoint.

Similarly, we can check the dissipativity of $(A+BK)^*$, which is defined as follows:

$$\begin{aligned}(A+BK)^* : D((A+BK)^*) \subset L^2_d \to L^2_d,\\ D((A+BK)^*) := \{g \in L^2_d : (A+BK)^* g \in L^2_d\},\end{aligned}$$

such that for any $f \in D(A+BK) = T^{-1}(H^{3/2})$ and any $g \in D((A+BK)^*)$ there is

$$\langle (A+BK)f, g\rangle_{L^2_d} = \langle f, (A+BK)^* g\rangle_{L^2_d}.$$

The preceding formula is equivalent to

$$\langle Tf, (-A-\lambda)Tg\rangle_{H^{3/2},H^{-3/2}} = \langle T(A+BK)f, Tg\rangle_{L^2} = \langle Tf, T(A+BK)^* g\rangle_{L^2},$$

again using that A is skew-adjoint. Thus

$$T(A+BK)^* g = (-A-\lambda)Tg \text{ in } L^2, \quad \text{for any } g \in D((A+BK)^*),$$

which further yields $Tg \in H^{3/2}$ and thus $D((A+BK)^*) \subset H^{1-\varepsilon}$. Therefore, for any $g \in D((A+BK)^*)$ we have that

$$\begin{aligned}\operatorname{Re}\langle (A+BK)^* g, g\rangle_{L^2_d} &= \operatorname{Re}\langle T(A+BK)^* g, Tg\rangle_{L^2}\\ &= \operatorname{Re}\langle (-A-\lambda)Tg, Tg\rangle_{L^2} = \langle -\lambda Tg, Tg\rangle_{L^2} \le 0.\end{aligned}$$

Step 7. Lumer–Philipps theorem.

Combine Step 3–5 and using Lumer-Philipps theorem we conclude that the unbounded operator $(A+BK)$ with domain $D(A+BK)$ is the infinitesimal generator of a strong semigroup in L^2.

□

Proof of Corollary 2.3. This Corollary is devoted to the exponential stability of the closed-loop system (2.2). The stability analysis in H^r space is based on semigroup theory on dissipative operators. It turns out that $r = -1/2$ is the threshold between "regular" semigroups and "singular" semigroups: as we shall see in the following, if $r \in (-1, -1/2)$, the operator $(A+BK)$ is defined on $D_r(A+BK) = H^{r+3/2}$; while, if $r \in [-1/2, 1)$, the domain of definition of $(A+BK)$ becomes $D_r(A+BK) = T^{-1}(H^{r+3/2})$ which is dense in H^r but does not include regular functions like φ_n (this is the reason why we call this case "singular"). It is also noteworthy that $r = 1$ is the threshold to have a strong semigroup generated by $(A+BK)$, and that $r = -1$ is the limit to have the operator equality (5.13).

Step 1: $A+BK : D(A+BK) \subset L^2 \to L^2$ generator an exponential stable C^0-semigroup in L^2. For any $u_0 \in D(A+BK)$, according to Proposition 6.2 the function $u(t)$ defined as

$$u(t) := e^{(A+BK)t}u_0 \in C^1\left([0,+\infty); L^2\right) \cap C^0\left([0,+\infty); D(A+BK)\right)$$

is the solution of

$$u_t = Au + BKu \ \text{ in } L^2.$$

Therefore,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle u,u\rangle_{L^2_d} &= \mathrm{Re}\,\langle u_t,u\rangle_{L^2_d} = \mathrm{Re}\,\langle Tu_t,Tu\rangle_{L^2} = \mathrm{Re}\,\langle (TA+BK)u,Tu\rangle_{L^2}\\ &= \mathrm{Re}\,\langle (A-\lambda)Tu,Tu\rangle_{L^2} = -\lambda\langle Tu,Tu\rangle_{L^2},\end{aligned}$$

which yields

$$\|u(t)\|_{L^2} \le \|T^{-1}\|\|u(t)\|_{L^2_d} \le \|T^{-1}\|e^{-\lambda t}\|u_0\|_{L^2_d} \le \|T\|\|T^{-1}\|e^{-\lambda t}\|u_0\|_{L^2}.$$

Finally, thanks to the fact that $D(A+BK)$ is dense in L^2, for any $u_0 \in L^2$ the unique solution

$$u(t) := e^{(A+BK)t}u_0 \in C^0\left([0,+\infty); L^2\right)$$

still satisfies (6) and decays exponentially.

Step 2: $A+BK : D_r(A+BK) \subset H^r \to H^r$ generates an exponential stable C^0-semigroup in H^r for $r \in [-1/2, 1)$.

For $r \in [-1/2,1)$ we can adapt the same proof given by Proposition 6.2 concerning the case $r=0$ to show that $A+BK$ is the infinitesimal generator of a strong C^0-semigroup on $(H^r, \|\cdot\|_{H^r_d})$ to get the well-posedness and the exponential stability of the closed-loop system (2.2): $A+BK$ is defined on

$$\begin{aligned}D_r(A+BK) &:= \{f \in H^r : (A+BK)f \in H^r\},\\ \|f\|_{D_r(A+BK)} &:= \|(A+BK)f\|_{H^r} + \|f\|_{H^r},\ \forall f \in D_r(A+BK),\end{aligned}$$

satisfying

$$\begin{gathered}D_r(A+BK) \text{ is a Hilbert space endowed with the } D_r(A+BK)\text{-norm},\\ D_r(A+BK) = T^{-1}(H^{r+3/2}) \text{ is dense in } H^r,\\ T : D_r(A+BK) \to H^r \text{ is an isomorphism},\\ TA+BK = (A-\lambda)T \in \mathcal{L}(D_r(A+BK); H^r),\\ A+BK : D_r(A+BK) \to H^r_d \text{ is closed and dissipative (so is } (A+BK)^*).\end{gathered}$$

Remark that $r=1$ is the threshold at and above which the operator $A+BK$ does not generate a strong C^0-semigroup on H^r anymore. Indeed even $D_r(A+BK)$ is not well-defined in this circumstance: suppose that f belongs to $D(A+BK)$. Then f is obviously an element from H^r with $r \ge 1$, thus $Af \in H^{r-3/2} \subset H^{-1/2}$. Thus $BKf \in H^{-1/2}$, which is in contradiction with the fact that $B \notin H^{-1/2}$.

Step 3: $A+BK : D(A+BK) = H^{r+3/2} \to H^r$ generates an exponential stable C^0-semigroup in H^r for $r \in (-1,-1/2)$.

In fact for any $r \in (-1,-1/2)$ the operator $A+BK$ also generates a strongly continuous semigroup on $H^{-3/4+s}$, where the domain of the unbounded operator $A+BK$ given by $D(A+BK)$ as follows

$$A+BK : D(A+BK) \subset H^{3/2+r} \to H^r,$$

$$D(A+BK) := \{f \in H^r : (A+BK)f \in H^r\},$$

satisfies $D(A+BK) = H^{3/2+r}$ and $\|\cdot\|_{D(A+BK)} = \|\cdot\|_{H^{3/2+r}}$. Indeed, the situation is much easier here since T is an isomorphism from $H^{3/2+r}$ in itself (*resp.* from H^r in itself) and BK belongs to $\mathcal{L}(H^{3/2+r}; H^r)$, thus $D(A+BK) = H^{3/2+r}$ holds obviously, and moreover the operator equality (5.13) implies that $D(A+BK)$ is exactly $H^{3/2+r}$ endowed with the same norm. Next, it is standard to show that $A+BK \in \mathcal{L}(H^{3/2+r}; H^r)$ is a closed operator. To conclude to the existence of a semigroup on H^r the only thing left to show is that $(A+BK)$ and $(A+BK)^*$ are dissipative operators in $(H^{-3/4+s}, \|\cdot\|_{H_d^{-3/4+s}})$ where

$$\|h\|_{H_d^{-3/4+s}} := \|Th\|_{H^{-3/4+s}},\ \forall\, h \in H^{-3/4+s}.$$

This is straightforward as the operator equality (5.13) holds in $\mathcal{L}(H^{3/4+s}, H^{-3/4+s})$ and can directly be used, for any $f \in H^{3/4+s}$ there is

$$\begin{aligned}\mathrm{Re}(\langle (A+BK)f, f\rangle_{H_d^{-3/4+s}}) &= \mathrm{Re}(\langle T(A+BK)f, Tf\rangle_{H^{-3/4+s}})\\ &= \mathrm{Re}(\langle (A-\lambda)Tf, Tf\rangle_{H_d^{-3/4+s}}) = -\lambda\|f\|_{H_d^{-3/4}}.\end{aligned}$$

This implies the well-posedness and stability of the closed-loop system (2.2) in H^r with $r \in (-1,-1/2)$: for any initial state $u_0 \in H^r$ the closed-loop system has a unique solution $u(t)$ that decays exponentially:

$$\begin{aligned}u(t) &:= e^{(A+BK)t}u_0 \in C^0\left([0,+\infty); H^r\right),\\ \|u(t)\|_{H^r} &\lesssim e^{-\lambda t}\|u_0\|_{H^r},\ \forall t \in (0,+\infty).\end{aligned}$$

REMARK 6.6. *Besides, the stability in H^r for $r \in (-1,1)$ can be also understood in the following sense, weaker than a semigroup stability: for $r \in (-1,1)$ there exists some $r_0 \in (-1,-1/2)$ such that $r \in (r_0,1)$. Let $u_0 \in H^r$. Since $(A+BK)$ generates a strongly continuous semigroup on H^{r_0} there exists a unique solution of the closed-loop system (2.2) $u \in C^0([0,T], H^{r_0})$ associated to the initial condition $u_0 \in H^r \subset H^{r_0}$. We know that in the H^{r_0} space*

$$u(t) = e^{(A+BK)t}u_0 = T^{-1}e^{(A-\lambda)t}(Tu_0) \in C^0([0,T], H^{r_0}).$$

Meanwhile, keep in mind that the right-hand side term in the preceding equation also belongs to $C^0([0,T], H^r)$ (since T and T^{-1} are isomorphisms from H^r to itself) and decays exponentially in H^r, we know that $u(t)$ is exponentially stable in H^r.

□

7. Conclusion

We have presented a compactness/duality method to overcome the limitations of the classical Fredholm backstepping method. This allows to prove the rapid stabilisation of the linearized capillary-gravity water waves system (1.7). More precisely, this compactness/duality method allows to construct a Riesz basis for skew-adjoint operators behaving like $|D_x|^\alpha$ for $\alpha > 1$, that is beyond the $\alpha > 3/2$ threshold imposed by the typical quadratically close criterion. We were moreover able to prove that the uniqueness condition $TB = B$ can also be handled without the quadratically close criterion, using fine estimations. The rapid stabilisation was proved in the spaces H^r, $r \in (-1,1)$. These bounds are sharp in the sense that the operator $A+BK$ cannot generate a strong semigroup for $r = 1$, while $r = -1$ is the limit for the operator equality (5.13) to hold. Moreover, the feedback law is shown to be independent of r. Finally, we are able to prove the existence of the isomorphism T in regular spaces $H^s, s \in \mathbb{R}$ – as long as the control

operator B is regular enough and satisfies the equivalent of the controllability Assumption 1 – a crucial step in proving the local rapid stabilisation of the nonlinear system (1.3) using the regularity-consuming nonlinear estimates of [4, 28].

Acknowledgements. Amaury Hayat and Shengquan Xiang would like to warmly thank Mathematisches Forschungsinstitut Oberwolfach for their hospitality, their kindness, and the inspiration this place and this working environment provided. The authors would also like to thank the PEPS 2022 program of CNRS. Ludovick Gagnon was partially supported by the French Grant ANR ODISSE (ANR-19-CE48-0004-01) and the French Grant ANR TRECOS (ANR-20-CE40-0009). Amaury Hayat was financially supported by Ecole des Ponts Paristech. Shengquan Xiang was financially supported by the Chair of Partial Differential Equations at EPFL.

Appendix A. Riesz basis in Hilbert spaces

We recall here some definitions about vector families in Hilbert spaces (see for instance [8, 10, 12, 19]).

DEFINITION A.1 (Vector family)**.** *Let X be a Hilbert space. A family of vectors $\{\xi_n\}_{n\in\mathcal{I}}$, where $\mathcal{I}=\mathbb{Z}$, $\mathbb{N}$, or $\mathbb{N}^*$ is said to be*

(1) ***Dense** in X, if $\overline{span\{\xi_i; i\in\mathcal{I}\}}=X$.*
(2) ***ω-independent** in X, if*

$$\sum_{k\in\mathcal{I}} c_k\xi_k = 0 \text{ in } X \text{ with } \{c_n\}_{n\in\mathcal{I}}\in\ell^2(\mathcal{I}) \Longrightarrow c_n=0,\ \forall n\in\mathcal{I}.$$

(3) ***Quadratically close** to a family of vector $\{e_n\}_{n\in\mathcal{I}}$, if*

$$\sum_{k\in\mathcal{I}} \|\xi_k - e_k\|_X^2 < +\infty.$$

(4) ***Riesz basis** of X, if it is the image of an isomorphism (on X) of some orthonormal basis.*

An equivalent definition of Riesz basis can also be stated as follows

DEFINITION A.2 (Riesz basis)**.** *A family of vectors $\{\xi_n\}_{n\in\mathcal{I}}$, where $\mathcal{I}=\mathbb{Z}$, $\mathbb{N}$, or $\mathbb{N}^*$ of X is a called a Riesz basis of X, if it is dense in X and if there exist $C_1, C_2>0$ such that for any $\{a_n\}_{n\in\mathcal{I}}\in\ell^2(\mathcal{I})$ we have*

$$C_1\sum_{k\in\mathcal{I}}|a_k|^2 \le \|\sum_{k\in\mathcal{I}} a_k\xi_k\|_X^2 \le C_2\sum_{k\in\mathcal{I}}|a_k|^2. \tag{A.1}$$

The following lemma has been heavily used in the literature as a criteria for Riesz basis.

LEMMA A.3. *Let $\{\xi_n\}_{n\in\mathcal{I}}$ be quadratically close to an orthonormal basis $\{e_n\}_{n\in\mathcal{I}}$. Suppose that $\{\xi_n\}_{n\in\mathcal{I}}$ is either dense in X or ω-independent in X, then $\{\xi_n\}_{n\in\mathcal{I}}$ is a Riesz basis of X.*

Finally in this paper we also make use of the following Lemma

LEMMA A.4. *Let X, Y be Hilbert spaces. Let $T: X\to Y$ be an isomorphism. Suppose that $\{\xi_n\}_{n\in\mathcal{I}}$ is a Riesz basis of X, then with $\zeta_n := T\xi_n$, the family $\{\zeta_n\}_{n\in\mathcal{I}}$ is a Riesz basis of Y.*

Appendix B. Controllability and Proof of Proposition 2.1

Proof. We first study the controllability of the system (1.6). From the time-reversibility of (1.6), it is sufficient to prove the null-controllability, that it is always possible to drive the solution from any initial data to $u(T) = 0$, the proof of which relies on the moment method. Indeed, let $u_0 \in L^2$ and write the solution of (1.6) with initial condition u_0 by the Duhamel formula and the existence of a strongly continuous semigroup $(e^{\mathscr{L}t})_{t\geq 0}$ generated by $\mathscr{L}$ on L^2,

$$u(t) = e^{\mathscr{L}t}u_0 + \int_0^t e^{\mathscr{L}(t-s)}Bv(s)ds.$$

We also decompose the state into odd and even parts, and only study the odd part for the ease of notations. Decompose (the odd function) u_0 under the eigenbasis and considering $t = T$,

$$u_0 = \sum_{n\in\mathbb{N}^*} u_0^n \sin(nx) \quad \text{and } B = \sum_{n\in\mathbb{N}^*} b_n \sin(nx)$$

the null-controllability is equivalent to

$$\begin{aligned} 0 &= \sum_{n\in\mathbb{Z}^*} e^{\lambda_n t}u_0^n \sin(nx) + \int_0^T e^{\mathscr{L}(T-s)} \sum_{n\in\mathbb{Z}^*} b_n \sin(nx)v(s)ds \\ &= \sum_{n\in\mathbb{Z}^*} e^{\lambda_n t}u_0^n e^{inx} + \int_0^T \sum_{n\in\mathbb{Z}^*} e^{\lambda_n(T-s)}b_n \sin(nx)v(s)ds. \end{aligned}$$

This implies in particular that $b_n \neq 0$ for any $n \in \mathbb{N}^*$ and is equivalent to,

$$-\frac{u_0^n}{b_n} = \int_0^T e^{-\lambda_n s}v(s)ds. \tag{B.1}$$

We see here the reason why we required $0 < c \leq |b_n|$ in the controllability condition (2.1). Thanks to this controllability assumption (2.1), we have

$$\frac{u_0^n}{b_n} \in \ell^2(\mathbb{N}^*;\mathbb{C}).$$

Therefore, if we are able to prove that there exists $v \in L^2(0,T)$ such that (B.1) is satisfied, then we deduce the null-controllability. It is obtained by proving that $\{e^{-\lambda_n s}\}_{n\in\mathbb{Z}^*}$ is a Riesz basis of $L^2(0,T)$ using (A.1). To this end, we introduce Haraux's Ingham inequality,

THEOREM B.1 ([21][Théorème 2)**.** *Let $J \subset \mathbb{R}$ be a bounded interval and $\gamma, \omega > 0$. Assume there exists $\{\mu_n\}_{n\in\mathbb{Z}}$ a real sequence such that,*

1. *$|\mu_{n+1} - \mu_n| \geq \omega, \forall n \in \mathbb{Z}$;*
2. *there exists $N \in \mathbb{N}^*$ such that $|\mu_{n+1} - \mu_n| \geq \gamma$, for all $|n| \geq N$;*
3. *$|J| > 2\pi/\gamma$.*

Then, there exist $c, C > 0$ such that for any sequence $\{a_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z};\mathbb{C})$,

$$c\sum_{n\in\mathbb{Z}} |a_n|^2 \leq \int_J \left|\sum_{n\in\mathbb{Z}} a_n e^{i\mu_n t}\right|^2 dt \leq C\sum_{n\in\mathbb{Z}} |a_n|^2.$$

From the behaviour of the eigenvalues λ_n given by (1.15), we easily see that there exists a uniform lower bound $|\lambda_{n+1} - \lambda_n| \geq \omega > 0$, and since $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = \infty$, we are able to choose $\gamma > 0$ arbitrarily small. Hence, the application of Theorem B.1 implies that $\{e^{-\lambda_n s}\}_{n\in\mathbb{N}^*}$ is a Riesz basis of $H := \overline{\{e^{-\lambda_n s}\}_{n\in\mathbb{N}^*}} \subset L^2(0,T)$. Moreover, there exists a bi-orthogonal sequence

$\{g_m\}_{m\in\mathbb{N}^*}$ being a Riesz basis of H (see, for example, [8, Proposition 19]). Therefore, the control can be chosen in such fashion as

$$v(s) := -\sum_{m\in\mathbb{N}^*} \frac{u_0^m}{b_m} g_m$$

satisfying $\|v\|_{L^2(0,T)} \lesssim \|u_0\|_{L^2}$. Hence the null-controllability of the system (1.6).

□

Appendix C. Proof of Lemmas 4.3–4.5

Proof of Lemma 4.3. It suffices to prove the second inequality as the first one is a direct consequence. Suppose that $n \geq m$, then

$$(n-m)n^{1/2} \geq (n-m)^{3/2}.$$

Inspired by the definition of λ_n we define

$$g(x) := \big((g+x^2)x\tanh(bx)\big)^{\frac{1}{2}}, \forall x \in (0,+\infty),$$

which is strictly increasing and verifies that for any $x \in [1,+\infty)$,

$$\begin{aligned} g'(x) &= \frac{1}{2}\big((g+x^2)x\tanh(bx)\big)^{-\frac{1}{2}}\big((g+3x^2)\tanh(bx) + b(g+x^2)x(1-\tanh^2(bx))\big) \\ &\geq Cx^{-\frac{3}{2}}x^2 = Cx^{\frac{1}{2}}. \end{aligned}$$

We also observe that

$$g^2(2x) \geq 2g^2(x),\ \forall x \in [1,+\infty).$$

Let given $m,n \in \mathbb{N}^*$ verifying $m<n$. If $n \geq 2m$, then

$$|\lambda_n - \lambda_m| = g(n) - g(m) \geq g(n) - g(n/2) \geq \frac{2-\sqrt{2}}{2} g(n) \geq Cn^{\frac{3}{2}} \geq C(n-m)n^{\frac{1}{2}}.$$

If $m<n<2m$, then there exists some $y \in [m,n]$ such that

$$|\lambda_n - \lambda_m| = g(n) - g(m) = (n-m)g'(y) \geq C(n-m)m^{\frac{1}{2}} \geq C(\sqrt{2})^{-1}(n-m)n^{\frac{1}{2}}.$$

□

Proof of Lemma 4.4 and Lemma 4.5. Define $f(x) := x^s$ for $x \in [1,+\infty)$. There exists $c_0, C_0 > 0$ such that

$$c_0 n^s \leq f(x) \leq C_0 n^s,\ \forall n \in \mathbb{N}^*, \forall x \in [n,n+1].$$

Concerning Lemma 4.4, suppose that $s \neq -1$, then

$$\sum_{n=1}^{p} n^s \leq c_0^{-1}\int_1^{p+1} f(x)dx = \frac{c_0^{-1}}{s+1}\big((p+1)^{s+1}-1\big) \leq C(1+p^{1+s}),$$

this ends the proof of Lemma 4.4.

Next we turn to the proof of Lemma 4.5. Let given $s \in \mathbb{R}$ and $\varepsilon_1 > 0$. We are able to select $\varepsilon \in (0,\varepsilon_1)$ such that $s+\varepsilon \neq -1$. Because there exists $C>0$ such that

$$\log(n) \leq Cn^{\varepsilon},\ \forall n \in \mathbb{N}^*,$$

we can use Lemma 4.4 as well as the fact that $s+\varepsilon \neq -1$. This yields

$$\sum_{n=1}^{p} n^s \log(n) \leq \sum_{n=1}^{p} n^{s+\varepsilon} \leq C(1+p^{1+s+\varepsilon}).$$

□

Appendix D. Proof of Property (i) in Lemma 4.11

The property must be proved in both the odd and even cases. As in the article, we focus on the odd case, and the proof can be easily adapted to the even case as the resolvant $\mathcal{A}_\lambda$ defined below is well-defined and invertible both on H_1 and H_2.

Recall that $q_n \in H^{1-\varepsilon}$ for any $\varepsilon > 0$. By defining $r_n = (\lambda_n + \lambda/2)^{-1}$ we obtain, by definition (3.8) of the (q_n),

$$(A - \lambda - \lambda_n)q_n = -\sum_{p\in\mathbb{N}^*} \varphi_p = -\frac{1}{2}\cot\frac{x}{2} =: h \text{ in } H^{-1}, \tag{D.1}$$

which becomes, defining $\mathcal{A}_\lambda := (A - \lambda/2)^{-1}$,

$$\mathcal{A}_\lambda q_n = r_n q_n - r_n \mathcal{A}_\lambda h.$$

Now, suppose that $\{q_n\}_{n\in\mathbb{N}^*}$ is not ω-independent, then there exists a nontrivial sequence $\{c_n\}_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$ such that

$$\sum_{n\in\mathbb{N}^*} c_n q_n = 0 \text{ in } L^2, \tag{D.2}$$

which is well-defined thanks to Remark 4.8.

Next, by applying $\mathcal{A}_\lambda$ to (D.2), we conclude

$$\sum_{n\in\mathbb{N}^*} c_n r_n q_n = \left(\sum_{n\in\mathbb{N}^*} c_n r_n\right) \mathcal{A}_\lambda h \quad \text{ in } L^2,$$

where we have used the fact $r_n \xrightarrow{n\to\infty} 0$.

Applying again $\mathcal{A}_\lambda$ we get

$$\sum_{n\in\mathbb{N}^*} c_n r_n^2 q_n = \left(\sum_{n\in\mathbb{N}^*} c_n r_n^2\right) \mathcal{A}_\lambda h + \left(\sum_{n\in\mathbb{N}^*} c_n r_n\right) \mathcal{A}_\lambda^2 h \quad \text{ in } L^2,$$

By induction we easily derive

$$\sum_{n\in\mathbb{N}^*} c_n r_n^m q_n = \sum_{i=1}^{m} \left(\sum_{n\in\mathbb{N}^*} c_n r_n^{m+1-i}\right) \mathcal{A}_\lambda^i h = \sum_{i=1}^{m} C_{m+1-i} \mathcal{A}_\lambda^i h, \quad m \in \mathbb{N}^*, \tag{D.3}$$

where

$$C_l := \sum_{n\in\mathbb{N}^*} c_n r_n^l < +\infty, \quad l \in \mathbb{N}^*. \tag{D.4}$$

Let us now distinguish two cases:

- First case: the $\{C_m\}$ are not identically zero. We note

$$m_0 = \inf\{n \in \mathbb{N}^*, \quad C_n \neq 0\}.$$

Then, starting with (D.3) with $m = m_0$, we have by induction,

$$\mathcal{A}_\lambda^m h \in \operatorname{span}\{q_n\}_{n\in\mathbb{N}^*}, \quad m \geq 1.$$

Suppose that $\operatorname{span}\{q_n\}_{n\in\mathbb{N}^*}$ is not dense in L^2, then there exists a nonzero $d = \sum_n d_n \varphi_n \in L^2$ such that

$$\langle g, d\rangle_{L^2} = 0, \quad \forall g \in \operatorname{span}\{q_n\}_{n\in\mathbb{N}^*}, \tag{D.5}$$

which in particular yields,

$$\langle \mathcal{A}_\lambda^m h, d\rangle_{L^2} = 0, \quad \forall m \in \mathbb{N}^*.$$

Recalling that $h = -\sum \varphi_n \in H^{-1}$, we get that

$$\sum_n \overline{d}_n r_n^m = 0, \quad \forall m \in \mathbb{N}^*. \tag{D.6}$$

By defining the complex variable function

$$G(z) := \sum_{n\in\mathbb{N}^*} \overline{d}_n r_n e^{r_n z}, \forall z \in \mathbb{C}.$$

By checking that the series expansion of the right-hand side is absolutely convergent, we deduce that this function is holomorphic.

From (D.6) we know that $G^{(m)}(0) = 0, m \in \mathbb{N}$. Thus $G = 0$, and further $d_n = 0$, which leads to a contradiction. Therefore

$$\text{span}\{q_n\}_{n\in\mathbb{N}^*} \text{ is dense in } L^2. \tag{D.7}$$

- Second case: the $\{C_m\}$ are identically zero. Then we define the complex variable function

$$\tilde{G}(z) := \sum_{n\in\mathbb{N}^*} c_n r_n e^{r_n z}.$$

This function is holomorphic. Moreover, as the (C_m) are identically, it satisfies

$$\tilde{G}^{(m)}(0) = 0, \quad \forall m \in \mathbb{N},$$

thus as previously $\tilde{G} = 0$ and therefore

$$c_n = 0, \quad \forall n \in \mathbb{N}^*,$$

which is in contradiction with the definition of the $\{c_n\}_{n\in\mathbb{N}^*}$.

This ends the proof of the property (i) in Lemma 4.11.

Appendix E. Proof that $ker(T^*) = \{0\}$

We proceed as presented in Section 5.4. Let $\rho \in \mathbb{C}$ to be chosen later on and let us look at $A + BK + \lambda Id + \rho Id$.

1) Let us denote $z := \lambda + \rho$, we try to investigate the invertibility of $Id + A^{-1}BK + zA^{-1}$ in the $H^{3/4}$ space. As ρ can be chosen arbitrarily, $z \in \mathbb{C}$ can be as well. Remark here that A is trivially invertible when looking at odd functions. When looking at even functions A the kernel of A is reduced to constant functions. In both case, we can replace A by $\tilde{A} := A + \delta$ with $\delta \neq 0$ sufficiently close to 0 such that $\tilde{A}$ is invertible. We now consider two cases:

- If $K(A^{-1}B) \neq -1$, then we know that the bounded operator $Id + A^{-1}BK$ is invertible. In fact, for any $f \in H^{3/4}$, we can check that

$$\varphi := f - \frac{A^{-1}B(Kf)}{1 + K(A^{-1}B)} \in H^{3/4},$$

is the unique solution to

$$(Id + A^{-1}BK)\varphi = f.$$

Note that A^{-1} is a compact operator in $H^{3/4}$ (since A is a differential operator) thus a continuous operator in $H^{3/4}$ and $Id + A^{-1}BK$ is invertible, thus thanks to the openness of invertible operators, there exists $\varepsilon > 0$ such that for any $|z| < \varepsilon$

$$(Id + A^{-1}BK) + zA^{-1} \tag{E.1}$$

is invertible in $H^{-3/4}$.

- If $K(A^{-1}B) = -1$, then one can check that 0 is an eigenvalue of $Id + A^{-1}BK$ with multiplicity 1 and the eigenspace is generated by $A^{-1}B$. Indeed, it is clear that $A^{-1}B$ is an eigenvector of $Id + A^{-1}BK$ with eigenvalue 0. On the other hand, suppose that for some $v \in H^{3/4}$ we have $(Id + A^{-1}BK)v = 0$, then we immediately conclude that $v = -(A^{-1}B)(Kv) \in \text{ span}\{A^{-1}B\}$.

  Therefore, there exist small open neighborhoods Ω and $\widetilde{\Omega}$ of 0 in $\mathbb{C}$ satisfying (see for instance [34])

$$(Id + A^{-1}BK + zA^{-1})y(z) = \lambda(z)y(z), \tag{E.2}$$

$$y(z) : z \in \Omega \mapsto y(z) \in H^{3/4} \text{ is holomorphic}, \tag{E.3}$$

$$\lambda(z) : z \in \Omega \mapsto \lambda(z) \in \widetilde{\Omega} \subset \mathbb{C} \text{ is holomorphic}, \tag{E.4}$$

$$\lambda(0) = 0, \quad y_0 := y(0) = A^{-1}B, \tag{E.5}$$

  in such fashion that for any $z \in \Omega$, $\lambda(z)$ is the unique eigenvalue inside $\widetilde{\Omega}$. Recall that $\lambda(0) = 0$, therefore only two cases are possible: either λ is identically 0 in Ω, or there exists a smaller neighborhood ω such that for any z in $\omega \setminus \{0\}$ there is $\lambda(z) \neq 0$. Let us show by contradiction that λ is not identically 0. Assume that it is. We now that there exists a sequence $(y_k)_{k\in\mathbb{N}^*}$ in $H^{3/4}$ such that

$$y(z) = \sum_{k=0}^{+\infty} y_k z^k,$$

  with $y_0 = A^{-1}B$. From (E.2) and the fact that $\lambda(z) = 0$ in Ω,

$$(Id + A^{-1}BK + zA^{-1}) \sum_{k=0}^{+\infty} y_k z^k = 0 \text{ in } H^{3/4},$$

  by unicity of the development in entire series we get

$$(Id + y_0K)y_k + A^{-1}y_{k-1} = 0 \text{ in } H^{3/4}, \ \forall k \in \mathbb{N}^*. \tag{E.6}$$

  Recall that $K(y_0) = -1$, by applying K to the preceding equation we conclude that

$$K(A^{-1}y_{k-1}) = 0, \ \forall k \in \mathbb{N}^* \Longrightarrow K(A^{-1}y_k) = 0, \ \forall k \geq 0.$$

  By applying KA^{-1} to Equation (E.6) we arrive at

$$K(A^{-1}y_k) + K(A^{-1}y_0)(Ky_k) + K(A^{-2}y_{k-1}) = 0, \ \forall k \in \mathbb{N}^*,$$

  thus

$$K(A^{-2}y_{k-1}) = 0, \ \forall k \in \mathbb{N}^* \Longrightarrow K(A^{-2}y_k) = 0, \ \forall k \geq 0.$$

  Then by successively applying $KA^{-(n-1)}$ to the same equation we arrive at

$$K(A^{-n}y_k) = 0, \ \forall k \geq 0, \forall n \geq 1,$$

  which in particular yields

$$K(A^{-n}y_0) = 0, \forall n \geq 1.$$

  The preceding equality implies

$$\sum_{m\in\mathbb{N}^*} \frac{b_m K_m}{\lambda_m^l} = 0, \forall l \geq 2.$$

  Again using the holomorphic function technique in Appendix D, we conclude that $b_m K_m = 0$, which is a contradiction. Therefore, we know that there exists $\varepsilon > 0$ such that for any

$z \in \mathbb{C}$ with $|z| < \varepsilon$, $z \in \Omega$ and $\lambda(z) \neq 0$. Since $\lambda(z)$ is the unique eigenvalue inside $\widetilde{\Omega}$, $Id + A^{-1}BK + zA^{-1}$ is invertible.

In both cases there exists at least a sequence of $\{z_k\}$ converging to 0 such that $Id + A^{-1}BK + \lambda A^{-1} + (z_k - \lambda)A^{-1}$ is invertible in $H^{3/4}$. As the spectrum of $A + \rho Id$ is discrete, we can find some $\rho := z_k - \lambda$, such that both

$$A + \rho Id \quad \text{and} \quad A + BK + \lambda Id + \rho Id = A(Id + A^{-1}BK + \lambda A^{-1} + \rho A^{-1})$$

are invertible operators from $H^{3/4}$ to $H^{-3/4}$. This ends the proof of the first point.

1)* Alternatively, this first step can be proved using the following direct method. We show that there exists some effectively computable real number $\rho_0 > 0$ such that for any $\rho \in [\rho_0, +\infty)$ the operators

$$A + \rho Id \quad \text{and} \quad A + BK + \lambda Id + \rho Id : H^{3/4} \to H^{-3/4}$$

are invertible. Notice that the spectrum of A belongs to $i\mathbb{R}$, it is straightforward that $A + \rho Id$ is invertible. It suffices to show that $A + BK + \lambda Id + \rho Id$ is invertible.
First observe the following Lemma

LEMMA E.1. *There exists some effectively computable $\rho_0 > 0$ such that for any $\rho \in [\rho_0, +\infty)$ there is*

$$c_\rho := \sum_{m \in \mathbb{N}^*} \frac{b_m K_m}{\rho + \lambda + \lambda_m} \quad \textit{satisfying} \quad |c_\rho| \leq \frac{1}{2}.$$

Proof of Lemma E.1. We know from Lemma 5.2 that $|K_m|$ is uniformly bounded, thus there exists some $C_0 > 0$ such that $|b_m K_m| \leq C_0$ for $\forall m \in \mathbb{N}^*$, hence

$$|c_\rho| \leq \sum_{m \in \mathbb{N}^*} \frac{|b_m K_m|}{|\rho + \lambda + \lambda_m|} \leq \sum_{m \in \mathbb{N}^*} \frac{C_0}{|\rho + \lambda + \lambda_m|}.$$

Also notice the existence of $N_0 \in \mathbb{N}^*$ such that

$$\sum_{m > N_0} \frac{C_0}{|\lambda_m|} \leq \frac{1}{4}.$$

By choosing $\rho_0 > 0$ large enough there is

$$\sum_{m \in \mathbb{N}^*} \frac{|b_m K_m|}{|\rho + \lambda + \lambda_m|} \leq \sum_{m \leq N_0} \frac{C_0}{|\rho_0 + \lambda|} + \sum_{m > N_0} \frac{C_0}{|\lambda_m|} \leq \frac{1}{2}.$$

□

Now we come back and prove that for any $\rho \in [\rho_0, +\infty)$ the operator $A + BK + \lambda Id + \rho Id$ is invertible. It suffices to show that for any $g \in H^{-3/4}$ there exists a unique $f \in H^{3/4}$ such that

$$(A + BK + \lambda Id + \rho Id) f = g, \tag{E.7}$$

$$\|f\|_{H^{3/4}} \lesssim \|g\|_{H^{-3/4}}. \tag{E.8}$$

Let us denote by

$$g := \sum_{n \in \mathbb{N}^*} g_n n^{\frac{3}{4}} \sin nx, \quad f := \sum_{n \in \mathbb{N}^*} f_n n^{-\frac{3}{4}} \sin nx.$$

By comparing the coefficients in both side of the equation (E.7) we get

$$f_n n^{-\frac{3}{4}} (\lambda_n + \lambda + \rho) + b_n K(f) = n^{\frac{3}{4}} g_n,$$

hence $(f_n)_n$ is implicitly solved by

$$f_n = \frac{n^{\frac{3}{4}} g_n - b_n K(f)}{n^{-\frac{3}{4}}(\lambda_n + \lambda + \rho)}.$$

The preceding implicit formula of f yields

$$K(f) = \sum_{m\in\mathbb{N}^*} K_m f_m m^{-\frac{3}{4}} = \sum_{m\in\mathbb{N}^*} \frac{K_m g_m m^{\frac{3}{4}}}{\lambda_m + \lambda + \rho} - K(f)\left(\sum_{m\in\mathbb{N}^*} \frac{b_m K_m}{\lambda_m + \lambda + \rho}\right)$$

Thanks to Lemma E.1, the coefficients $(f_n)_n$ are uniquely determined by

$$f_n = \frac{n^{\frac{3}{2}} g_n}{\lambda_n + \lambda + \rho} - \frac{n^{\frac{3}{4}} b_n}{(1+c_\rho)(\lambda_n + \lambda + \rho)}\left(\sum_{m\in\mathbb{N}^*} \frac{K_m g_m m^{\frac{3}{4}}}{\lambda_m + \lambda + \rho}\right). \tag{E.9}$$

Since

$$\left|\sum_{m\in\mathbb{N}^*} \frac{K_m g_m m^{\frac{3}{4}}}{\lambda_m + \lambda + \rho}\right| \lesssim \sum_{m\in\mathbb{N}^*} \frac{m^{\frac{3}{4}}|g_m|}{|\lambda_m|} \lesssim \|(g_m)_m\|_{l^2},$$

we have

$$\|(f_n)_n\|_{l^2} \lesssim \|(g_n)_n\|_{l^2} + \|(g_m)_m\|_{l^2}\left\|\left(\frac{n^{\frac{3}{4}} b_n}{(1+c_\rho)(\lambda_n + \lambda + \rho)}\right)_n\right\|_{l^2} \lesssim \|(g_n)_n\|_{l^2}.$$

This finishes the proof of the inequality (E.8).

2) Let us assume that $\ker T^* \neq \{0\}$ and let ρ be defined as in 1). We are going to show that there exists an eigenvector of $A + \rho Id$ in $ker(T^*)$ and deduce that there exists $n \in \mathbb{N}^*$ such that $n^{3/4}\varphi_n \in ker(T^*)$. We know from the operator equality (5.13) that

$$T(A + BK + \lambda Id + \rho Id) = AT + \rho T$$

holds when the operators are seen as acting on $H^{3/4}$ to $H^{-3/4}$. Thus from the invertibility of the two operators (from point 1)),

$$(A + \rho Id)^{-1} T = T(A + BK + \lambda Id + \rho Id)^{-1}, \tag{E.10}$$

where the operator are seen as acting on $H^{-3/4}$ to $H^{3/4}$. Since $ker(T^*) \neq \{0\}$ we can take $h \neq 0$ such that $h \in \ker T^*$ and $h \in H^{-3/4}$. We deduce from (E.10) that for any $\varphi \in H^{-3/4}$,

$$\begin{aligned} 0 &= \langle (A + \rho Id)^{-1} T\varphi - T(A + BK + \lambda Id + \rho Id)^{-1}\varphi, h\rangle_{H^{-3/4}}, \\ &= \langle \varphi, T^*(A^* + \overline{\rho} Id)^{-1} h\rangle_{H^{-3/4}} - \langle (A + BK + \lambda Id + \rho Id)^{-1}\varphi, T^* h\rangle_{H^{-3/4}}, \\ &= \langle \varphi, T^*(A^* + \overline{\rho} Id)^{-1} h\rangle_{H^{-3/4}}, \end{aligned}$$

where A^* is the adjoint of A.

The above implies that $T^*(A^* + \overline{\rho} Id)^{-1} h = 0$ in $H^{-3/4}$, thus $(A^* + \overline{\rho} Id)^{-1} h \in \ker T^*$. Namely, we have deduced that

$$(A^* + \overline{\rho} Id)^{-1} : \ker T^* \to \ker T^*.$$

Because $\ker T^*$ is of finite dimension (recall that T is Fredholm, hence T^* is) and not reduced to $\{0\}$ there exists an eigenfunction $h \in ker(T^*)$ of $(A^* + \overline{\rho} Id)^{-1}$, associated to an eigenvalue $\mu \neq 0$ (since the operator $(A^* + \overline{\rho} Id)^{-1}$ is invertible). Thus

$$(A^* + \overline{\rho} Id)^{-1} h = \mu h, \tag{E.11}$$

which in particular implies that $h \in H^{3/4}$. We immediately deduce that h is an eigenfunction of A^* in $H^{-3/4}$. Moreover, we know from (E.11) that

$$A^* h = \frac{1-\overline{\rho}\mu}{\mu} h.$$

Now we would like to conclude that there exists $n \in \mathbb{N}^*$ and $C \neq 0$ such that $h = Cn^{3/4}\varphi_n$. Note that since $(n^{3/4}\varphi_n)_{n\in\mathbb{N}^*}$ is an orthonormal basis of eigenvectors of A, it is also a basis of eigenvector of A^* (associated to eigenvalues $(\overline{\lambda_n})_{n\in\mathbb{N}^*}$). To obtain such a conclusion, we notice that the eigenspaces of A^* (in $H^{-3/4}$) have dimension 1, in particular the dimension of the eigenspace associated to $(1-\overline{\rho}\mu)/\mu$ is one, and therefore there exist some $n \in \mathbb{N}^*$ and $C \neq 0$ such that $h = Cn^{3/4}\varphi_n$.

3) From point 2), we know that if $ker(T^*) \neq \{0\}$ there exists $n \in \mathbb{N}^*$ such that $n^{3/4}\varphi_n \in ker(T^*)$, thus

$$\langle T\varphi, n^{3/4}\varphi_n\rangle_{H^{-3/4}} = 0, \ \forall \varphi \in H^{-3/4}.$$

We know that this is impossible: as $TB = B$ holds in $H^{-3/4}$, we can take $\varphi := B$ to achieve

$$0 = \langle TB, n^{3/4}\varphi_n\rangle_{H^{-3/4}} = \langle B, n^{3/4}\varphi_n\rangle_{H^{-3/4}} = \frac{b_n}{n^{3/4}},$$

which is in contradiction with the fact that $|b_n|$ is uniformly bounded by below. Hence $ker(T^*) = \{0\}$.

Appendix F. Adapting the proof of Theorem 2.2 to Theorem 2.4

In this Appendix we briefly comment on the proof of Theorem 2.4 concerning Fredholm-type backstepping of the general skew-adjoint operators, namely the Fourier multiplier $A := i\,h(|D_x|)$ (one can simply regard it as $i|D_x|^\alpha$ with $\alpha > 1$ even if we deal with the general case for completeness).

Observe that

$$A\sin(nx) = \lambda_n \sin(nx), A\cos(nx) = \lambda_n \cos(nx),$$

in the following we shall only work with odd functions to simplify the notation,

$$(\varphi_n, \lambda_n) := (\sin(nx), -ih(n)).$$

Follow the strategy given in Section 3. The operator equality leads to $TB = B$ condition as well as the formal expression

$$T\varphi_n = \sum_{p\in\mathbb{N}^*} \langle (T\varphi_n), \varphi_p\rangle \varphi_p = -K_n \sum_{p\in\mathbb{N}^*} \frac{b_p\varphi_p}{\lambda_n - \lambda_p + \lambda}. \tag{F.1}$$

This further leads to study

- The families
$$q_n := \sum_{p\in\mathbb{N}^*} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda}.$$
- The operators
$$S : n^{-r}\varphi_n \mapsto n^{-r} q_n \ \text{ and } \ \tau : \varphi_n \mapsto b_n\varphi_n.$$
from H^r to H^r for any $r \in \mathbb{R}$, as well as
$$\tau_K : n^{-r}\tau q_n \to (-K_n) n^{-r}\tau q_n,$$
for any $r \in (1/2 - \alpha, \alpha - 1/2)$.

Thus

$$T = \tau_K \circ \tau \circ S,$$

and we shall show that the operators τ_K, τ and S are isomorphisms from H^r to itself for any $r \in (1/2-\alpha, \alpha-1/2)$, which indeed is equivalent to show that

- for any $r \in (1/2-\alpha, \alpha-1/2)$, the sequence $(n^{-r}q_n)_n$ is a Riesz basis of H^r;
- the unique candidate $(K_n)_n$ verifying $TB = B$ condition satisfies $c \leq |K_n| \leq C$ for $\forall n \in \mathbb{N}^*$.

As proposed in Section 3 for any $r \in (1/2-\alpha, \alpha-1/2)$ we follow a 8-step strategy to prove the preceding properties: the first one is based on the compactness/duality method (see *Steps (1)-(3)* and Section 4), while the rest of the steps are devoted to defining $(K_n)_n$ (see Section 5). Finally, combining this choice of $(K_n)_n$, and the corresponding T, with the Riesz basis properties as well as the operator equality, leads to well-posedness of the closed-loop system with the desired decay property.

The compactness/duality method for Riesz basis properties.

By decomposing $S = \mathrm{Id}/\lambda + S_c$ we are able to prove that S is a Fredholm operator (of index 0) from H^r to itself, using the following key estimates

LEMMA F.1. *For any $s < \alpha - 1$ we have*

$$\sum_{n\in\mathbb{N}^*\setminus\{p\}} \frac{n^s}{|\lambda_n - \lambda_p|} \lesssim p^{1-\alpha+s}\log(p) + p^{-\alpha}, \ \forall p \in \mathbb{N}^*. \tag{F.2}$$

LEMMA F.2. *For any $r \in (1/2-\alpha, \alpha-1/2)$, there exists $\varepsilon = \varepsilon(r) > 0$ such that the operator S_c defined by*

$$S_c : \sum_{n\in\mathbb{N}^*} a_n n^{-r}\varphi_n \mapsto \sum_{n\in\mathbb{N}^*} a_n n^{-r}\left(\sum_{p\in\mathbb{N}^*\setminus\{n\}} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda}\right)$$

is continuous from H^r to $H^{r+\varepsilon}$. In particular this operator is compact from H^r in itself.

These Lemmas are proved exactly as in the case $\alpha = 3/2$ described in Section 4.

By the Fredholm alternative for Fredholm operators of index 0, the Riesz basis property of $\{n^{-r}q_n\}$is equivalent to the ω-independence. We adapt the 3-step approach proposed in Section 4.3–4.4 to show that $(n^{-r}q_n)_n$ is a Riesz basis of H^r:

- If $r = 0$, then we prove Lemma 4.11 which further leads to the required property.
- If $r \in (0, \alpha-1/2)$, then the ω-independence of $(n^{-r}q_n)_n$ in H^r can be deduced from the ω-independence of $(q_n)_n$ in L^2.
- If $r \in (1/2-\alpha, 0)$, then by duality the ω-independence of $(n^{-r}q_n)_n$ in H^r is equivalent to the dense property of $(n^{-r}q_n)_n$ in H^{-r}.

This finishes the first part of the proof for Riesz basis properties.

On the construction of $(K_n)_n$ for $TB = B$.

We follow Steps (4)-(8) proposed in Section 3 (see Section 5 for more details). Indeed, Steps (5)-(8) are quite similar to the case that $\alpha = 3/2$, we mainly comment on Step (4). It suffices to deal with the cases that $\alpha \in (1, 3/2)$.

Step (4): Since B belongs to $H^{-\alpha/2}$ and $(n^{\alpha/2}q_n)_n$ is a Riesz basis of $H^{-\alpha/2}$, we are able to solve $TB = B$ in $H^{-\alpha/2}$ space. Indeed, by further separating $-b_nK_n$ into a singular part λ and a regular part k_n, as proposed in Lemma 5.2, we are able prove that

$$(k_n n^{-\varepsilon})_n \in l^2 \text{ for any } \varepsilon > s_0 := 3/2 - \alpha. \tag{F.3}$$

Notice that in the case that $\alpha = 3/2$ the value of ε can be chosen sufficiently close to 0, which then leads to the required estimates together with the expression (5.11). However, in the case where α is close to 1, this expression does not lead to an estimate on $(k_n n^{\varepsilon})_n \in l^{\infty}$ with some ε positive. In order to obtain our result, we derive by induction an asymptotic analysis of K_n at a higher order which depends on α. From now on, in order to simplify the notations, we will assume that $b_n = 1$.

Denote $e_n^0 = \lambda$ and $k_n^0 = k_n$. Thus $-K_n = e_n^0 + k_n^0$, and the expression (5.11) becomes

$$k_n = \lambda \sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{K_m}{\lambda_m - \lambda_n + \lambda} = -\lambda \sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{e_m^0 + k_m^0}{\lambda_m - \lambda_n + \lambda}. \tag{F.4}$$

We shall decompose $k_n = k_n^0 = e_n^1 + k_n^1$ as follows

$$e_n^1 = -\lambda \sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{e_m^0}{\lambda_m - \lambda_n + \lambda},\ k_n^1 = -\lambda \sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{k_m^0}{\lambda_m - \lambda_n + \lambda}. \tag{F.5}$$

It is easy to observe that

$$|e_n^1| \lesssim n^{1-\alpha}\log n \lesssim 1.$$

We further get estimates on $(n^{-\varepsilon}k_n^1)_n$ for $\varepsilon > s_1 = (1-\alpha) + s_0$,

$$\begin{aligned}
\|(n^{-\varepsilon}k_n^1)_n\|_{l^2}^2 &\lesssim \sum_{n\in\mathbb{N}^*} n^{-2\varepsilon}\left(\sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{k_m^0}{\lambda_m - \lambda_n + \lambda}\right)^2 \\
&\lesssim \sum_{n\in\mathbb{N}^*} n^{-2\varepsilon}\left(\sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{|k_m^0|^2}{|\lambda_m - \lambda_n|}\right)\left(\sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{1}{|\lambda_m - \lambda_n|}\right) \\
&\lesssim \sum_{n\in\mathbb{N}^*} n^{-2\varepsilon+1-\alpha}\log n\left(\sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{|k_m^0|^2}{|\lambda_m - \lambda_n|}\right) \\
&\lesssim \sum_{m\in\mathbb{N}^*} |k_m^0|^2 \sum_{n\in\mathbb{N}^*\setminus\{m\}} n^{-2\varepsilon+1-\alpha}\log n\left(\frac{1}{|\lambda_m - \lambda_n|}\right) \\
&\lesssim \sum_{m\in\mathbb{N}^*} |k_m^0|^2 m^{2-2\alpha-2\varepsilon+\delta} \\
&\lesssim \|(n^{-s_0-\delta}k_n^0)_n\|_{l^2}^2 < +\infty.
\end{aligned}$$

If $s_1 = 5/2 - 2\alpha < 0$, namely $\alpha > 5/4$, then we can conclude that $(k_n^1)_n$ belongs to l^2. Therefore, $(k_n)_n$ is uniformly bounded, which further yields that $(K_n)_n$ is uniformly bounded, thus we obtain the desired $(n^{\varepsilon}k_n)_n \in l^{\infty}$ for $\varepsilon \in [0, \alpha - 1)$ using the expression (F.4). Otherwise, we have that $s_1 \geq 0$ and need to continue the iteration procedure to further decompose k_n^1 as $e_n^2 + k_n^2$:

$$e_n^2 = -\lambda \sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{e_m^1}{\lambda_m - \lambda_n + \lambda},\ k_n^2 = -\lambda \sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{k_m^1}{\lambda_m - \lambda_n + \lambda}. \tag{F.6}$$

Again it is straightforward to observe that

$$|e_n^2| \lesssim n^{1-\alpha}\log n \lesssim 1.$$

We further get estimates on $(n^{-\varepsilon}k_n^2)_n$ for $\varepsilon > s_2 = (1-\alpha) + s_1$,

$$\begin{aligned}
\|(n^{-\varepsilon}k_n^2)_n\|_{l^2}^2 &\lesssim \sum_{n\in\mathbb{N}^*} n^{-2\varepsilon}\left(\sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{k_m^1}{\lambda_m-\lambda_n+\lambda}\right)^2 \\
&\lesssim \sum_{n\in\mathbb{N}^*} n^{-2\varepsilon}\left(\sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{|k_m^1|^2}{|\lambda_m-\lambda_n|}\right)\left(\sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{1}{|\lambda_m-\lambda_n|}\right) \\
&\lesssim \sum_{m\in\mathbb{N}^*} |k_m^1|^2 \sum_{n\in\mathbb{N}^*\setminus\{m\}} n^{-2\varepsilon+1-\alpha}\log n\left(\frac{1}{|\lambda_m-\lambda_n|}\right) \\
&\lesssim \sum_{m\in\mathbb{N}^*} |k_m^1|^2 m^{2-2\alpha-2\varepsilon+\delta} \lesssim \|(n^{-s_1-\delta}k_n^1)_n\|_{l^2}^2 < +\infty.
\end{aligned}$$

If $s_2 = 7/2 - 3\alpha < 0$, namely $\alpha > 7/6$, then we can conclude that k_n is bounded thus get the desired estimates. Otherwise, $s_2 \geq 0$ and we continue the iteration to decompose k_n^2 as $e_n^3 + k_n^3$.

We use the induction procedure as follows, suppose that for some $I \in \mathbb{N}^*$ we have the following (recall that $s_0 = 3/2 - \alpha$, $e_n^0 = \lambda$ and $k_n^0 = k_n$):

$$\begin{aligned}
&s_i := s_0 + (1-\alpha)i \geq 0, \quad \forall i \in \{0,1,...,I\}, \\
&-K_n = \sum_{j=0}^{i} e_n^j + k_n^i, \quad \forall i \in \{0,1,...,I\}, \\
&e_n^{i+1} := -\lambda \sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{e_m^i}{\lambda_m-\lambda_n+\lambda}, \quad \forall i \in \{0,1,...,I-1\}, \\
&k_n^{i+1} := -\lambda \sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{k_m^i}{\lambda_m-\lambda_n+\lambda}, \quad \forall i \in \{0,1,...,I-1\},
\end{aligned}$$

satisfying,

$$\begin{aligned}
&|e_n^i| \lesssim n^{1-\alpha}\log n \lesssim 1, \quad \forall i \in \{0,1,...,I\}, \\
&(n^{-\varepsilon}k_n^i)_n \in l^2, \quad \forall \varepsilon > s_i, \forall i \in \{0,1,...,I\}.
\end{aligned}$$

Then we shall define $s_{I+1} := s_I + (1-\alpha)$ and decompose k_n^I as $e_n^{I+1} + k_n^{I+1}$:

$$e_n^{I+1} = -\lambda \sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{e_m^I}{\lambda_m-\lambda_n+\lambda}, \; k_n^{I+1} = -\lambda \sum_{m\in\mathbb{N}^*\setminus\{n\}} \frac{k_m^I}{\lambda_m-\lambda_n+\lambda}, \tag{F.7}$$

which yields

$$|e_n^{I+1}| \lesssim n^{1-\alpha}\log n \lesssim 1,$$

and the estimates on $(n^{-\varepsilon}k_n^{I+1})_n$ for $\varepsilon > s_{I+1} = (1-\alpha) + s_I$,

$$\|(n^{-\varepsilon}k_n^{I+1})_n\|_{l^2}^2 \lesssim \|(n^{-s_I-\delta}k_n^I)_n\|_{l^2}^2 < +\infty.$$

If $s_{I+1} < 0$ then we finish the iteration and conclude the proof. Otherwise we continue the iteration until some $M \in \mathbb{N}^*$ such that $s_M < 0$, which is always possible as $\alpha > 1$.

- Step (5): thanks to Step (4) we know that $(K_n)_n$ is uniformly bounded, thus T is a bounded operator from H^r to itself for any $r \in (1/2-\alpha, \alpha-1/2)$. The same argument as in Section 5.2 yields the operator equality.

- Step (6) can be adapted as follows: we obtain for $f \in H^{-\alpha/2}$

$$Tf - f = \lambda \left(\sum_{n\in\mathbb{N}^*} \frac{f_n}{b_n} n^{\alpha/2} \sum_{p\in\mathbb{N}^*\setminus\{n\}} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} \right) + k(\tau^{-1} f),$$

  where k is still the operator defined in (5.3) which is compact in H^r for any $r \in (1/2 - \alpha, \alpha - 1/2)$.
- Step (7): the 3-step procedure provided in Section 5.4 and Appendix E also adapts here to prove that T is an isomorphism from $H^{-\alpha/2}$.
- Step (8) is exactly the same as in Section 5 with $H^{-\alpha/2}$ instead of $H^{-3/4}$. Indeed, thanks to the preceding step we know that $b_n K_n \neq 0$, which to be combined with the decomposition $-b_n K_n = \lambda + k_n$, yields the fact that $|b_n K_n|$ is uniformly bounded from below.

On the well-posedness of the closed-loop system.

Similar to Section 6 the well-posedness of the closed-loop system

$$\partial_t u = i\, h(|D_x|)u + BK(u) = (A + BK)u$$

in H^r-space is obtained from the semigroup theory.

- If $r \in (1/2 - \alpha, -1/2)$, then the definition domain of $A + BK$ is considered as regular $D_r(A + BK) = H^{r+\alpha}$ and $(A + BK)$ generates a C^0-semigroup in H^r.
- If $r \in [-1/2, \alpha - 1/2)$, then the definition domain of $A + BK$ is considered as singular $D_r(A + BK) = T^{-1}(H^{r+\alpha})$ and $(A + BK)$ also generates a C^0-semigroup in H^r.

The rest is identical to the case $\alpha = 3/2$.

References

[1] T. Alazard. Stabilization of the water-wave equations with surface tension. *Ann. PDE*, 3(2):41, 2017. Id/No 17.

[2] T. Alazard. Stabilization of gravity water waves. *J. Math. Pures Appl. (9)*, 114:51–84, 2018.

[3] T. Alazard, P. Baldi, and D. Han-Kwan. Control of water waves. *J. Eur. Math. Soc. (JEMS)*, 20(3):657–745, 2018.

[4] T. Alazard, N. Burq, and C. Zuily. On the water-wave equations with surface tension. *Duke Math. J.*, 158(3):413–499, 2011.

[5] A. Balogh and M. Krstić. Infinite dimensional backstepping-style feedback transformations for a heat equation with an arbitrary level of instability. *Eur. J. Control*, 8(3):165–175, 2002.

[6] V. Barbu. *Controllability and stabilization of parabolic equations*, volume 90 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2018. Subseries in Control.

[7] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30(5):1024–1065, 1992.

[8] K. Beauchard and C. Laurent. Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control. *J. Math. Pures Appl. (9)*, 94(5):520–554, 2010.

[9] T. Breiten, K. Kunisch, and S. Rodrigues. Feedback stabilization to nonstationary solutions of a class of reaction diffusion equations of FitzHugh-Nagumo type. *SIAM J. Control Optim.*, 55(4):2684–2713, 2017.

[10] O. Christensen et al. *An introduction to frames and Riesz bases*, volume 7. Springer, 2003.

[11] J.-M. Coron. Stabilization of control systems and nonlinearities. In *Proceedings of the 8th International Congress on Industrial and Applied Mathematics*, pages 17–40. Higher Ed. Press, Beijing, 2015.

[12] J.-M. Coron, L. Gagnon, and M. Morancey. Rapid stabilization of a linearized bilinear 1-D Schrödinger equation. *J. Math. Pures Appl. (9)*, 115:24–73, 2018.

[13] J.-M. Coron, A. Hayat, S. Xiang, and C. Zhang. Stabilization of the linearized water tank system. Preprint, March 2021.

[14] J.-M. Coron, L. Hu, and G. Olive. Stabilization and controllability of first-order integro-differential hyperbolic equations. *J. Funct. Anal.*, 271(12):3554–3587, 2016.

[15] J.-M. Coron, L. Hu, G. Olive, and P. Shang. Boundary stabilization in finite time of one-dimensional linear hyperbolic balance laws with coefficients depending on time and space. *J. Differential Equations*, 271:1109–1170, 2021.
[16] J.-M. Coron and Q. Lü. Local rapid stabilization for a Korteweg-de Vries equation with a Neumann boundary control on the right. *J. Math. Pures Appl. (9)*, 102(6):1080–1120, 2014.
[17] J.-M. Coron and Q. Lü. Fredholm transform and local rapid stabilization for a Kuramoto–Sivashinsky equation. *J. Differential Equations*, 259(8):3683–3729, 2015.
[18] J.-M. Coron and E. Trélat. Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM J. Control Optim.*, 43(2):549–569, 2004.
[19] L. Gagnon, A. Hayat, S. Xiang, and C. Zhang. Fredholm transformation on laplacian and rapid stabilization for the heat equation. *preprint arXiv:2110.04028*, 2021.
[20] L. Gagnon, P. Lissy, and S. Marx. Fredholm transformation for the rapid stabilization of a degenerate parabolic equation. Preprint, arXiv:2010.05476, 2020.
[21] A. Haraux. Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire. (Lacunary series and semi-internal control of vibrations of a rectangular plate). *J. Math. Pures Appl. (9)*, 68(4):457–465, 1989.
[22] A. Hayat. PI controllers for the general Saint-Venant equations. 2021. Preprint, arXiv:2108.02703.
[23] Amaury Hayat and Peipei Shang. A quadratic Lyapunov function for Saint-Venant equations with arbitrary friction and space-varying slope. *Automatica J. IFAC*, 100:52–60, 2019.
[24] Amaury Hayat and Peipei Shang. Exponential stability of density-velocity systems with boundary conditions and source term for the H^2 norm. *J. Math. Pures Appl. (9)*, 153:187–212, 2021.
[25] V. Komornik. *Exact controllability and stabilization*. RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994. The multiplier method.
[26] J. Krieger and S. Xiang. Boundary stabilization of focusing nlkg near unstable equilibria: radial case. *Preprint, arXiv: 2004.07616*, 2020.
[27] M. Krstić and A. Smyshlyaev. *Boundary control of PDEs*, volume 16 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. A course on backstepping designs.
[28] D. Lannes. Well-posedness of the water-waves equations. *J. Am. Math. Soc.*, 18(3):605–654, 2005.
[29] D. Lannes. *The water waves problem. Mathematical analysis and asymptotics*, volume 188. Providence, RI: American Mathematical Society (AMS), 2013.
[30] I. Lasiecka and R. Triggiani. *Control theory for partial differential equations: continuous and approximation theories. II*, volume 75 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2000. Abstract hyperbolic-like systems over a finite time horizon.
[31] Russell M. Reid. Control time for gravity-capillary waves on water. *SIAM J. Control Optim.*, 33(5):1577–1586, 1995.
[32] P. Su. Asymptotic behaviour of a linearized water waves system in a rectangle. *preprint arXiv:2104.00286*, 2021.
[33] P. Su, M. Tucsnak, and G. Weiss. Stabilizability properties of a linearized water waves system. *Systems Control Lett.*, 139:104672, 10, 2020.
[34] B. Sz.-Nagy. Perturbations des transformations linéaires fermées. *Acta Sci. Math. (Szeged)*, 14:125–137, 1951.
[35] E. Trélat, G. Wang, and Y. Xu. Characterization by observability inequalities of controllability and stabilization properties. *Pure Appl. Anal.*, 2(1):93–122, 2020.
[36] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2009.
[37] J. M. Urquiza. Rapid exponential feedback stabilization with unbounded control operators. *SIAM J. Control Optim.*, 43(6):2233–2244 (electronic), 2005.
[38] S. Xiang. Quantitative rapid and finite time stabilization of the heat equation. *Preprint, arXiv:2010.04696*, 2020.
[39] S. Xiang. Small-time local stabilization of the two dimensional incompressible Navier-Stokes equations. *Preprint, arXiv:2010.13696*, 2020.
[40] V. E. Zakharov. *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, volume 9. 1968.
[41] C. Zhang. Finite-time internal stabilization of a linear 1-D transport equation. *Systems Control Lett.*, 133:104529, 8, 2019.
[42] C. Zhang. Internal rapid stabilization of a 1-d linear transport equation with a scalar feedback. *Mathematical Control & Related Fields*, 12(1):169–200, 2022.

[43] H. Zhu. Control of three dimensional water waves. *Arch. Ration. Mech. Anal.*, 236(2):893–966, 2020.