



HAL
open science

A Nitsche method for the elastoplastic torsion problem

Franz Chouly, Tom Gustafsson, Patrick Hild

► **To cite this version:**

Franz Chouly, Tom Gustafsson, Patrick Hild. A Nitsche method for the elastoplastic torsion problem. ESAIM: Mathematical Modelling and Numerical Analysis, 2023, 57 (3), pp.1731 - 1746. 10.1051/m2an/2023034 . hal-03891827

HAL Id: hal-03891827

<https://hal.science/hal-03891827v1>

Submitted on 9 Dec 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Nitsche method for the elastoplastic torsion problem

Franz Chouly^{a,b,c,*}, Tom Gustafsson^e, Patrick Hild^d

^a*Université de Bourgogne, Institut de Mathématiques de Bourgogne, 21078 Dijon, France*

^b*Center for Mathematical Modeling and Department of Mathematical Engineering, University of Chile and IRL
2807 – CNRS, Santiago, Chile*

^c*Departamento de Ingeniería Matemática, CI²MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile*

^d*Institut de Mathématiques de Toulouse - UMR CNRS 5219, Université Paul Sabatier, 118 route de Narbonne,
31062 Toulouse Cedex 9, France*

^e*Department of Mathematics and Systems Analysis, Aalto University, Otakaari 1 F, Espoo, Finland*

Abstract

This study is concerned with the elastoplastic torsion problem, in dimension $n \geq 1$, and in a polytopal, convex or not, domain. In the physically relevant case where the source term is a constant, this problem can be reformulated using the distance function to the boundary. We combine the aforementioned reformulation with a Nitsche-type discretization as in [Burman, Erik, et al. Computer Methods in Applied Mechanics and Engineering 313 (2017): 362-374]. This has two advantages: 1) it leads to optimal error bounds in the natural norm, even for nonconvex domains; 2) it is easy to implement within most of finite element libraries. We establish the well-posedness and convergence properties of the method, and illustrate its behavior with numerical experiments.

Keywords: variational inequalities; elastoplastic torsion problem; finite elements; Nitsche; error estimates.

2020 MSC: 65N15, 65N30, 74C05.

1. Introduction

Problems written with weak formulations involving variational inequalities represent various nonlinear phenomena which occur in mechanics and physics [12, 22]. We focus on the elastoplastic torsion problem, as presented in, *e.g.*, [16] (see also [6, 17]). In the aforementioned reference, a direct piecewise affine Lagrange finite element approximation of the variational inequality is also presented, as well as a convergence result (Theorem 3.3), and two error estimates in the H^1 -norm, in dimension one (Theorem 3.4) and in dimension two (Theorem 3.5). The error estimate in one dimension is optimal ($\mathcal{O}(h)$), whereas it remained suboptimal in dimension two, as it is of order $\mathcal{O}(h^{\frac{1}{2} - \frac{1}{p}})$ for a source term in L^p , $p > 2$. Among the first and few existing results are weak and strong convergence results [25], and error estimates of $\mathcal{O}(h)$ for the L^2 -norm of the gradient of the solution and under suitable restrictive assumptions, for mixed finite element approximations, using $\mathbb{P}_1/\mathbb{P}_0$ finite elements [14] or Raviart-Thomas finite elements [7].

In this paper, we focus on the torsion problem with a positive constant source term, corresponding to constant shear modulus and angle of twist. In this case the variational inequality can be reformulated as an “obstacle” problem where the constraint involves the distance to the boundary, so the obstacle is nonsmooth and the usual techniques from the obstacle problem cannot be

*Corresponding author

Email addresses: franz.chouly@u-bourgogne.fr (Franz Chouly), tom.gustafsson@aalto.fi (Tom Gustafsson), patrick.hild@math.univ-toulouse.fr (Patrick Hild)

directly applied: for instance in [11, Theorem 5.1.2], the obstacle is supposed of Sobolev regularity H^2 . In a previous paper [9], a direct finite element approximation of the variational inequality has been proposed, that makes use of piecewise affine, continuous, Lagrange finite elements, and in which the constraint involving the distance function is imposed at each node. When the domain is convex, error estimates have been established in any dimension $n = 1, 2, 3$, with an optimal error bound of $\mathcal{O}(h)$, for a regular enough continuous solution. In the case of a nonconvex domain, an error bound of $\mathcal{O}(h^{\frac{3}{4}})$ has been proven for a solution of Sobolev regularity H^α , $\alpha \geq 7/4$.

In the present paper, we propose a new method that combines both the reformulation with the distance function, as in [9], and a Nitsche term that allows to incorporate weakly the inequality constraint, following [5] and related works on Nitsche's method for variational inequalities, see, *e.g.*, [8] and references therein. For this discretization, we manage to derive optimal error estimates, for linear and quadratic finite elements, and even in the nonconvex situation, which improves the result of [9]. Moreover, this method is easy to implement into modern finite element libraries, and we provide also some numerical experiments, that allow to confirm the expected theoretical convergence rates.

As usual, we denote by $H^s(\cdot)$, $s \in \mathbb{R}$, the Sobolev spaces. The usual norm of $H^s(D)$ is denoted by $\|\cdot\|_{s,D}$, and the corresponding semi-norm is denoted by $|\cdot|_{s,D}$. The space $H_0^1(D)$ is the subspace of functions in $H^1(D)$ with vanishing trace on ∂D . The letter C stands for a generic constant, independent of the mesh size, which value can changes at different occurrences.

2. The elastoplastic torsion problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open bounded polytope, connected and with Lipschitz boundary. We consider the variational inequality which, for $n = 2$, models the torsion of an infinitely long elastoplastic cylinder of cross section Ω and plasticity yield $r > 0$. To simplify we assume that $r = 1$. The problem is to find the stress potential u such that

$$u \in K_1 : \quad a(u, v - u) \geq L(v - u) \quad \forall v \in K_1, \quad (1)$$

where $a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ is the bilinear form given by:

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v, \quad \forall u, v \in H_0^1(\Omega),$$

and

$$L(v) := \int_{\Omega} f v, \quad \forall v \in H_0^1(\Omega),$$

with $f \in L^2(\Omega)$. The notation K_1 represents the nonempty closed convex set of admissible stress potentials:

$$K_1 := \{v \in H_0^1(\Omega) : |\nabla v| \leq 1 \text{ a.e. in } \Omega\},$$

where $|\cdot|$ denotes the euclidian norm in \mathbb{R}^n . From Stampacchia's theorem we deduce that Problem (1) admits a unique solution (see also, *e.g.*, [12, 16, 17, 22]).

Remark 2.1. *We recall some regularity results for (1): if $\Omega \subset \mathbb{R}^n$ is open, bounded and convex, with Lipschitz boundary, and for $f \in L^p(\Omega)$ with $n < p < +\infty$, then $u \in W^{2,p}(\Omega) \cap \mathcal{C}^{1,\alpha}(\overline{\Omega})$, where $\alpha = 1 - n/p$ [4]. When the domain is nonconvex the $W^{2,p}(\Omega)$ regularity can be obtained but the boundary needs to be more regular ($\mathcal{C}^{1,1}$ more precisely, see [15]) so reentrant corners of polytopes are not allowed. When reentrant corners of polytopes are considered, the loss of $W^{2,p}$ -regularity is only located near these corners [6].*

Next we suppose that $f = \mathcal{C}$ is a constant function. In this case and according to [3] (see also [21]) the problem (1) can be rewritten as follows: find the stress potential u such that

$$u \in \overline{K} : \quad a(u, v - u) \geq \mathcal{C} \int_{\Omega} (v - u) \quad \forall v \in \overline{K}, \quad (2)$$

with

$$\overline{K} := \{v \in H_0^1(\Omega) : |v| \leq d_{\partial\Omega} \text{ a.e. in } \Omega\},$$

and $d_{\partial\Omega}$ denotes the (interior) distance function with respect to the boundary $\partial\Omega$:

$$d_{\partial\Omega}(x) := \inf_{y \in \partial\Omega} |x - y|, \forall x \in \overline{\Omega}.$$

Note that (2) still admits a unique solution from Stampacchia's theorem. To lighten the discussion we can suppose without loss of generality that $\mathcal{C} > 0$ (see [9, Remark 2.2]), so problem (2) can be rewritten as follows: find the stress potential u such that

$$u \in K : \quad a(u, v - u) \geq \mathcal{C} \int_{\Omega} (v - u) \quad \forall v \in K, \quad (3)$$

with

$$K := \{v \in H_0^1(\Omega) : v \leq d_{\partial\Omega} \text{ a.e. in } \Omega\}.$$

Again (3) admits a unique solution from Stampacchia's theorem. Moreover, Problem (2) and Problem (3) are equivalent, when $\mathcal{C} > 0$ [9, Proposition 2.1 and Remark 2.3]. So the torsion problem can be seen as an obstacle problem where the distance function plays the role of the obstacle. Generally speaking, for a polytope, the distance function does not lie in $H^2(\Omega)$. This implies that the classical finite element error analysis for the obstacle problem can not be directly applied.

Problem (3) in strong form, reads: find $u : \Omega \rightarrow \mathbb{R}$ solution to:

$$\begin{cases} -\Delta u \leq \mathcal{C} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \leq d_{\partial\Omega} & \text{in } \Omega, \\ (u - d_{\partial\Omega})(\Delta u + \mathcal{C}) = 0 & \text{in } \Omega. \end{cases} \quad (4)$$

We reformulate (4) using a Lagrange multiplier λ , and get:

$$\begin{cases} -\Delta u + \lambda = \mathcal{C} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \lambda \geq 0 & \text{in } \Omega, \\ u \leq d_{\partial\Omega} & \text{in } \Omega, \\ (u - d_{\partial\Omega})\lambda = 0 & \text{in } \Omega. \end{cases} \quad (5)$$

We introduce the following notation for the positive part: $[a]_+ := \max(0, a)$, for $a \in \mathbb{R}$, and recall the relationship

$$([a]_+ - [b]_+)(a - b) \geq ([a]_+ - [b]_+)^2, \quad (6)$$

for $a, b \in \mathbb{R}$.

Following [5], the Kuhn-Tucker condition (5)₃₋₅ can equivalently be reformulated as

$$\lambda = \gamma [-d_{\partial\Omega} + u + \gamma^{-1}\lambda]_+, \quad (7)$$

with γ an arbitrary positive function on the domain Ω .

3. A Nitsche finite element method

Let V_h^k be a family of Lagrange finite element spaces of degree $k \geq 1$ indexed by h , and coming from a family T_h of simplicial meshes of the domain Ω ($h := \max_{T \in T_h} h_T$ where h_T is the diameter of $T \in T_h$). The family of meshes is assumed regular. More precisely we have:

$$V_h^k = \{v_h \in \mathcal{C}(\bar{\Omega}) \cap H_0^1(\Omega) : v_h|_T \in \mathbb{P}_k(T) \ \forall T \in T_h\}.$$

Each simplex T of the mesh T_h is supposed to be closed, and we denote by $\overset{\circ}{T}$ the interior of T . We define a piecewise polynomial discrete Laplacian as follows, for every v_h in V_h^k , and every simplex $T \in T_h$:

$$(\Delta_h v_h)|_{\overset{\circ}{T}} := \Delta(v_h|_{\overset{\circ}{T}}).$$

The value of $\Delta_h v_h$ on the facets of the mesh is of no importance, and can be set in practice to 0, for instance. We define also:

$$R_h(v_h) := \Delta_h v_h + \mathcal{C}.$$

Remark that, for $k = 1$, $\Delta_h v_h = 0$ and $R_h(v_h) = \mathcal{C}$. The Nitsche-type method proposed for the discretization of the elastoplastic torsion problem (5) reads: find $u_h \in V_h^k$ such that

$$a(u_h, v_h) + (\gamma_h [-d_{\partial\Omega} + u_h + \gamma_h^{-1} R_h(u_h)]_+, v_h) = (\mathcal{C}, v_h) \quad (8)$$

for all $v_h \in V_h^k$. Above the notation (\cdot, \cdot) stands for the $L^2(\Omega)$ -scalar product, and the function γ_h is defined cell-wise as follows:

$$\gamma_h|_{\overset{\circ}{T}} := \frac{\gamma_0}{h_T^2},$$

where $\gamma_0 > 0$ is the Nitsche parameter. Again, the value of γ_h on the facets of the mesh is of no importance, and can be set in practice to 0, for instance.

Remark 3.1. *As in [10, 20], for any parameter $\theta \in \mathbb{R}$, we can write a whole family of methods:*

$$\begin{aligned} & a(u_h, v_h) - \theta(\gamma_h^{-1} \Delta_h u_h, \Delta_h v_h) \\ & + (\gamma_h [-d_{\partial\Omega} + u_h + \gamma_h^{-1} R_h(u_h)]_+, v_h + \theta \gamma_h^{-1} \Delta_h v_h) = (\mathcal{C}, v_h + \theta \gamma_h^{-1} \Delta_h v_h). \end{aligned} \quad (9)$$

Method (8) corresponds to $\theta = 0$ and can be called an incomplete method, using the terminology widespread for discontinuous Galerkin methods. This method involves less terms and is the easiest to extend to more complex problems [24]. A symmetric method is recovered when $\theta = 1$, that corresponds to the Galerkin Least Squares technique of [5] and the Nitsche method of [19]: this symmetric method can be recovered thanks to a minimization argument, and the tangent system has a symmetric Jacobian. Provided that the Nitsche parameter γ_0 is large enough, the analysis below, for $\theta = 0$, can be extended without difficulty to other values of θ .

Remark 3.2. *Remark that, for the distance function, there holds $d_{\partial\Omega} \in H^1(\Omega) \cap \mathcal{C}^{0,1}(\bar{\Omega})$, see [9] and references therein. In [5] the assumption made on the obstacle function is stronger and this function is supposed to be $\mathcal{C}^{1,1}(\bar{\Omega})$. In fact, such Nitsche or Galerkin Least Squares formulations do not require so much regularity on the obstacle function.*

The following local inverse inequality will be helpful in the sequel, that holds for an arbitrary $v_h \in V_h^k$ and every $T \in T_h$:

$$\|\nabla v_h\|_{0,T} \leq C_1 h_T^{-1} \|v_h\|_{0,T}, \quad (10)$$

where $C_1 > 0$ is a constant that depends on the shape regularity of the mesh and of the polynomial order k , but not of h and $T \in T_h$. See, e.g., [1, 13] for the proof.

4. Numerical analysis: well-posedness and error estimate

We first state a preliminary consistency result:

$$a(u, v_h) + (\gamma_h [-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda]_+, v_h) = (C, v_h) \quad \forall v_h \in V_h^k. \quad (11)$$

The above result is a direct consequence of (5)–(7) and the inclusion $V_h^k \subset H_0^1(\Omega)$.

4.1. Well-posedness

We make use of the results from Brezis (see, *e.g.*, [2]) for M-type and pseudo-monotone operators in vector spaces. It consists in showing that the operator associated to Problem (8) is one-to-one.

Theorem 4.1. *For $\gamma_0 \geq C_1^2$, Problem (8) admits one unique solution $u_h \in V_h^k$.*

Proof: We introduce the nonlinear operator $B_h : V_h^k \rightarrow V_h^k$ defined as follows:

$$(B_h z_h, v_h)_{1,\Omega} \stackrel{\text{def}}{=} a(z_h, v_h) + (\gamma_h [-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)]_+, v_h) \quad (12)$$

for all $z_h \in V_h^k$ and where $(\cdot, \cdot)_{1,\Omega}$ is the scalar product in $H^1(\Omega)$. Note that the right-hand side of (12) is linear with respect to v_h and, hence, B_h is well defined thanks to Riesz' Theorem. The existence and uniqueness of solution for problem (8) is equivalent to the property of B_h to be one-to-one. To this purpose, according to [2], it suffices to show that by B_h is monotone and hemicontinuous.

For the monotonicity we first note that, owing to (12), we have

$$\begin{aligned} & (B_h z_h - B_h v_h, z_h - v_h)_{1,\Omega} \\ &= \|\nabla(z_h - v_h)\|_{0,\Omega}^2 \\ & \quad + \underbrace{(\gamma_h [-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)]_+ - \gamma_h [-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)]_+, z_h - v_h)}_{T_1} \end{aligned} \quad (13)$$

for all $z_h, v_h \in V_h^k$. On the other hand, by adding and subtracting suitable terms into T_1 , we get

$$\begin{aligned} & T_1 \\ &= (\gamma_h [-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)]_+ - \gamma_h [-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)]_+, \\ & \quad -d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h) - (-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h))) \\ & \quad + (\gamma_h^{\frac{1}{2}} [-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)]_+ - \gamma_h^{\frac{1}{2}} [-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)]_+, \gamma_h^{-\frac{1}{2}} (\Delta_h v_h - \Delta_h z_h)). \end{aligned}$$

Hence, using the inequality (6), Cauchy-Schwarz, Young inequalities and the inverse inequality (10), it follows that

$$\begin{aligned} & T_1 \\ & \geq \left\| \gamma_h^{\frac{1}{2}} \left([-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)]_+ - [-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)]_+ \right) \right\|_{0,\Omega}^2 \\ & \quad - \frac{1}{2} \left\| \gamma_h^{\frac{1}{2}} \left([-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)]_+ - [-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)]_+ \right) \right\|_{0,\Omega}^2 \\ & \quad - \frac{1}{2} \left\| \gamma_h^{-\frac{1}{2}} (\Delta_h z_h - \Delta_h v_h) \right\|_{0,\Omega}^2 \\ & \geq \frac{1}{2} \left\| \gamma_h^{\frac{1}{2}} \left([-d_{\partial\Omega} + z_h + \gamma_h^{-1}R_h(z_h)]_+ - [-d_{\partial\Omega} + v_h + \gamma_h^{-1}R_h(v_h)]_+ \right) \right\|_{0,\Omega}^2 \\ & \quad - \frac{C_1^2}{2\gamma_0} \|\nabla(z_h - v_h)\|_{0,\Omega}^2. \end{aligned}$$

The monotonicity of B_h then follows by inserting this estimate into (13) under the condition $\gamma_0 \geq C_I^2$.

For the hemicontinuity, we must show that the real function $\varphi : [0, 1] \rightarrow \mathbb{R}$ defined as

$$\varphi(t) \stackrel{\text{def}}{=} (B_h(z_h - tv_h), v_h)$$

is continuous for all $z_h, v_h \in V_h^k$.

We bound $\varphi(t) - \varphi(s)$ using the inequality $|[a]_+ - [b]_+| \leq |a - b|$, for all $a, b \in \mathbb{R}$. This gives

$$\begin{aligned} & |\varphi(t) - \varphi(s)| \\ &= |(s-t)a(v_h, v_h) + (\gamma_h[-d_{\partial\Omega} + (z_h - tv_h) + \gamma_h^{-1}R_h(z_h - tv_h)]_+ \\ &\quad - \gamma_h[-d_{\partial\Omega} + (z_h - sv_h) + \gamma_h^{-1}R_h(z_h - sv_h)]_+, v_h)| \\ &\leq |t-s|a(v_h, v_h) + (\gamma_h|(z_h - tv_h) + \gamma_h^{-1}R_h(z_h - tv_h) - (z_h - sv_h) - \gamma_h^{-1}R_h(z_h - sv_h)|, |v_h|) \\ &= |t-s| \left[a(v_h, v_h) + (\gamma_h|v_h + \gamma_h^{-1}\Delta_h v_h|, |v_h|) \right] \end{aligned}$$

which means that φ is Lipschitz and, thus, B_h is hemicontinuous. \square

4.2. A priori error estimate

We provide first an abstract estimate, as in [5].

Theorem 4.2. *Assume that the solution (u, λ) ($\lambda = \Delta u + \mathcal{C}$) to the elastoplastic torsion problem (5) belongs to $K \times L^2(\Omega)$. For $\gamma_0 > 0$ large enough, the approximation u_h provided by (8) satisfies the following error estimate:*

$$\begin{aligned} & \|u - u_h\|_{1,\Omega} \\ &+ \left\| \gamma_h^{\frac{1}{2}} \left([-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda]_+ - [-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)]_+ \right) \right\|_{0,\Omega} + \left\| \gamma_h^{-\frac{1}{2}} (\Delta u - \Delta_h u_h) \right\|_{0,\Omega} \\ &\leq C \inf_{v_h \in V_h^k} \left\{ \|u - v_h\|_{1,\Omega} + \left\| \gamma_h^{\frac{1}{2}} (u - v_h) \right\|_{0,\Omega} + \left\| \gamma_h^{-\frac{1}{2}} (\Delta u - \Delta_h v_h) \right\|_{0,\Omega} \right\}. \quad (14) \end{aligned}$$

Proof: Let $v_h \in V_h^k$. We first use the V -ellipticity and the continuity of $a(\cdot, \cdot)$, as well as Young's inequality, to obtain:

$$\begin{aligned} \alpha \|u - u_h\|_{1,\Omega}^2 &\leq a(u - u_h, u - u_h) \\ &= a(u - u_h, (u - v_h) + (v_h - u_h)) \\ &\leq C \|u - u_h\|_{1,\Omega} \|u - v_h\|_{1,\Omega} + a(u - u_h, v_h - u_h) \\ &\leq \frac{\alpha}{2} \|u - u_h\|_{1,\Omega}^2 + \frac{C^2}{2\alpha} \|u - v_h\|_{1,\Omega}^2 + a(u - u_h, v_h - u_h), \end{aligned} \quad (15)$$

with $\alpha > 0$ the ellipticity constant. We can transform the last term using the consistency property (11) for u and the finite element formulation (8) for u_h . This yields

$$\begin{aligned} & a(u - u_h, v_h - u_h) \\ &= a(u, v_h - u_h) - a(u_h, v_h - u_h) \\ &= (\gamma_h[-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda]_+ - \gamma_h[-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)]_+, u_h - v_h). \end{aligned} \quad (16)$$

We now transform the expression $u_h - v_h$:

$$\begin{aligned} u_h - v_h &= u_h - v_h + \gamma_h^{-1}R_h(u_h) - \gamma_h^{-1}R_h(u_h) \\ &\quad - (-d_{\partial\Omega} - u - \gamma_h^{-1}\lambda + d_{\partial\Omega} + u + \gamma_h^{-1}\lambda) \\ &= -[(-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda) - (-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h))] - v_h + u - \gamma_h^{-1}(R_h(u_h) - \lambda). \end{aligned}$$

Hence, by inserting this expression into (16), we get

$$\begin{aligned}
& a(u - u_h, v_h - u_h) \\
&= - \underbrace{(\gamma_h [-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda]_+ - \gamma_h [-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)]_+)}_{T_1}, \\
& \quad \underbrace{(-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda) - (-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h))}_{T_1} \\
& \quad + \underbrace{(\gamma_h [-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda]_+ - \gamma_h [-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)]_+, -v_h + u - \gamma_h^{-1}(R_h(u_h) - \lambda))}_{T_2}.
\end{aligned} \tag{17}$$

The first term is estimated by using the inequality (6), which yields

$$T_1 \leq - \left\| \gamma_h^{\frac{1}{2}} \left([-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda]_+ - [-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)]_+ \right) \right\|_{0,\Omega}^2. \tag{18}$$

For the term T_2 we use Cauchy-Schwarz and Young's inequality, to obtain

$$\begin{aligned}
T_2 &\leq \frac{1}{2} \left\| \gamma_h^{\frac{1}{2}} \left([-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda]_+ - [-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)]_+ \right) \right\|_{0,\Omega}^2 \\
&\quad + \left\| \gamma_h^{\frac{1}{2}} (u - v_h) \right\|_{0,\Omega}^2 + \left\| \gamma_h^{-\frac{1}{2}} (\lambda - R_h(u_h)) \right\|_{0,\Omega}^2.
\end{aligned} \tag{19}$$

There remains to bound the last term above. Let $T \in T_h$ be a mesh cell. Using a triangular inequality, we bound first

$$\|R_h(u_h) - \lambda\|_{0,T} = \|\Delta_h u_h - \Delta u\|_{0,T} \leq \|\Delta_h u_h - \Delta_h v_h\|_{0,T} + \|\Delta u - \Delta_h v_h\|_{0,T}.$$

With the inverse inequality (10) and triangular inequality, we bound

$$\|\Delta_h u_h - \Delta_h v_h\|_{0,T} \leq C_I h_T^{-1} \|\nabla u_h - \nabla v_h\|_{0,T} \leq C_I h_T^{-1} (\|\nabla u_h - \nabla u\|_{0,T} + \|\nabla u - \nabla v_h\|_{0,T}).$$

Therefore the last term in (19) can be bounded as

$$\begin{aligned}
\left\| \gamma_h^{-\frac{1}{2}} (R_h(u_h) - \lambda) \right\|_{0,\Omega}^2 &\leq C \gamma_0^{-1} (\|\nabla u_h - \nabla u\|_{0,\Omega}^2 + \|\nabla u - \nabla v_h\|_{0,\Omega}^2) \\
&\quad + C \left\| \gamma_h^{-\frac{1}{2}} (\Delta u - \Delta_h v_h) \right\|_{0,\Omega}^2.
\end{aligned} \tag{20}$$

We combine estimates (15)–(17)–(18)–(19)–(20), which yields

$$\begin{aligned}
& \left(\frac{\alpha}{2} - \frac{C}{\gamma_0} \right) \|u - u_h\|_{1,\Omega}^2 \\
& + \frac{1}{2} \left\| \gamma_h^{\frac{1}{2}} \left([-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda]_+ - [-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)]_+ \right) \right\|_{0,\Omega}^2 \\
& \leq \left(\frac{C^2}{2\alpha} + \frac{C}{\gamma_0} \right) \|u - v_h\|_{1,\Omega}^2 + \left\| \gamma_h^{\frac{1}{2}} (u - v_h) \right\|_{0,\Omega}^2 + C \left\| \gamma_h^{-\frac{1}{2}} (\Delta u - \Delta_h v_h) \right\|_{0,\Omega}^2.
\end{aligned} \tag{21}$$

Choosing γ_0 large enough, we obtain the desired bound on the first two terms in (14). The bound on the error on the multiplier λ comes from combination of (20) and (21). \square

The optimal convergence of the method for \mathbb{P}_1 and \mathbb{P}_2 Lagrange finite elements is stated below, for a Sobolev regularity $\alpha \leq 2$.

Theorem 4.3. *Suppose that $k = 1, 2$, and that the solution u belongs to $H_0^1(\Omega) \cap H^\alpha(\Omega)$, with the Sobolev regularity α that satisfies $\max(1, n/2) < \alpha \leq 2$. Suppose that λ belongs to $L^2(\Omega)$. Suppose finally that the Nitsche parameter γ_0 is large enough. The solution u_h to Problem (8) satisfies the following error estimate:*

$$\begin{aligned} & \|u - u_h\|_{1,\Omega} \\ & + \left\| \gamma_h^{\frac{1}{2}} \left([-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda]_+ - [-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)]_+ \right) \right\|_{0,\Omega} + \left\| \gamma_h^{-\frac{1}{2}} (\Delta u - \Delta_h u_h) \right\|_{0,\Omega} \\ & \leq C(h^{\alpha-1}|u|_{\alpha,\Omega} + h\|\Delta u\|_{0,\Omega}) \end{aligned} \tag{22}$$

with $C > 0$ a constant, independent of h and u , but not of γ_0 .

Proof: We consider first the case $k = 1$. We start from the estimate (14). We take $v_h = \mathcal{I}_h^1 u$, the Lagrange interpolant of u onto V_h^1 . So, first, from standard interpolation estimates, we obtain

$$\|u - \mathcal{I}_h^1 u\|_{1,\Omega} \leq Ch^{\alpha-1}|u|_{\alpha,\Omega},$$

and

$$\left\| \gamma_h^{\frac{1}{2}} (u - \mathcal{I}_h^1 u) \right\|_{0,\Omega} \leq Ch^{\alpha-1}|u|_{\alpha,\Omega}.$$

And then, for $k = 1$, we can simply proceed as follows:

$$\left\| \gamma_h^{-\frac{1}{2}} (\Delta u - \Delta_h(\mathcal{I}_h^1 u)) \right\|_{0,\Omega} \leq \gamma_0^{-\frac{1}{2}} h \|\Delta u - \underbrace{\Delta_h(\mathcal{I}_h^1 u)}_{=0}\|_{0,\Omega} = \gamma_0^{-\frac{1}{2}} h \|\Delta u\|_{0,\Omega}.$$

For $k = 2$, we proceed exactly as above. We start from the estimate (14). Since $V_h^1 \subset V_h^2$, we take $v_h = \mathcal{I}_h^1 u \in V_h^1 \subset V_h^2$, the Lagrange interpolant of u onto V_h^1 , and get the same estimates as above for the three terms. This ends the proof. \square

The next statement is for \mathbb{P}_2 Lagrange finite elements, where a better convergence rate than $\mathcal{O}(h)$ can be expected if the solution u is regular enough.

Theorem 4.4. *Suppose that $k = 2$, and that the solution u belongs to $H_0^1(\Omega) \cap H^\alpha(\Omega)$, with the Sobolev regularity α that satisfies $\max(2, n/2) < \alpha \leq 3$. Suppose that the Nitsche parameter γ_0 is large enough. The solution u_h to Problem (8) satisfies the following error estimate:*

$$\begin{aligned} & \|u - u_h\|_{1,\Omega} \\ & + \left\| \gamma_h^{\frac{1}{2}} \left([-d_{\partial\Omega} + u + \gamma_h^{-1}\lambda]_+ - [-d_{\partial\Omega} + u_h + \gamma_h^{-1}R_h(u_h)]_+ \right) \right\|_{0,\Omega} + \left\| \gamma_h^{-\frac{1}{2}} (\Delta u - \Delta_h u_h) \right\|_{0,\Omega} \\ & \leq Ch^{\alpha-1}|u|_{\alpha,\Omega}, \end{aligned} \tag{23}$$

with $C > 0$ a constant, independent of h and u , but not of γ_0 .

Proof: We proceed as previously, in Theorem 4.3, but with the following modifications. We start from the estimate (14). We take $v_h = \mathcal{I}_h^2 u$, the Lagrange interpolant of u onto V_h^2 . Still from standard interpolation estimates, we obtain

$$\|u - \mathcal{I}_h^2 u\|_{1,\Omega} \leq Ch^{\alpha-1}|u|_{\alpha,\Omega},$$

and

$$\left\| \gamma_h^{\frac{1}{2}} (u - \mathcal{I}_h^2 u) \right\|_{0,\Omega} \leq Ch^{\alpha-1}|u|_{\alpha,\Omega}.$$

For the last term, we need the following local error bound, for each $T \in T_h$:

$$\left\| \gamma_h^{-\frac{1}{2}} (\Delta u - \Delta_h(\mathcal{I}_h^2 u)) \right\|_{0,T} = \gamma_0^{-\frac{1}{2}} h_T \|\Delta u - \Delta(\mathcal{I}_h^2 u|_T)\|_{0,T} \leq C \gamma_0^{-\frac{1}{2}} h_T \|u - \mathcal{I}_h^2 u|_T\|_{2,T}$$

which is possible since $\alpha > 2$. Then we use standard (local) interpolation error estimates to get:

$$\left\| \gamma_h^{-\frac{1}{2}} (\Delta u - \Delta_h(\mathcal{I}_h^2 u)) \right\|_{0,T} \leq C \gamma_0^{-\frac{1}{2}} h_T h_T^{\alpha-2} |u|_{\alpha,T} \leq C h_T^{\alpha-1} |u|_{\alpha,T}.$$

By summation on the simplices T of the mesh T_h , we get finally

$$\left\| \gamma_h^{-\frac{1}{2}} (\Delta u - \Delta_h(\mathcal{I}_h^2 u)) \right\|_{0,\Omega} \leq C h^{\alpha-1} |u|_{\alpha,\Omega}.$$

This ends the proof. □

5. Numerical experiments

The following results are computed with the help of scikit-fem [18] for finite element assembly and autograd [23] for automatic differentiation. We consider two experiments introduced in [17] with convex and nonconvex domains. The parameters are chosen as $\mathcal{C} = \gamma_0 = 10$. First, a convergence study is performed using linear elements and the mesh sequences are depicted in Figure 1. Error in the stress potential u is computed against a reference solution which is obtained using quadratic finite elements and a sufficiently refined mesh. The resulting discrete solutions u_h and the magnitudes of the gradient $|\nabla u_h|$ are depicted in Figures 2 and 4, and the error in Figures 3 and 5 for the convex and nonconvex examples, respectively. The convergence rate of the H^1 error is $\mathcal{O}(h)$ which is also an expected consequence of Theorem 4.3. Moreover, the convergence rate of the L^2 -error is $\mathcal{O}(h^2)$, a rate we are not able to prove. It is well-known that L^2 -error estimates are difficult to prove for problems modelled by variational inequalities.

We continue by solving the same experiments using quadratic elements. For this purpose, a reference solution is computed using cubic elements and a suitably refined mesh. The resulting discrete solutions are given in Figure 6 with the convergence rates in Figure 7. This time we observe that the H^1 error is approximately $\mathcal{O}(h^{1.5})$. This is in agreement with Theorem 4.4. However this is lower than the rate $\mathcal{O}(h^2)$ for quadratic elements and a smooth solution. The reduction in the convergence rate is explained by the Sobolev regularity of the exact solution as $u \in H^2(\Omega)$ while $u \notin H^3(\Omega)$. In one dimension, it is easy to check that the second derivative is expected to have a step discontinuity at the free boundary between elastic and plastic zones which suggests $u \in H^s(\Omega)$, $s < 5/2$ (see also Remark 2.1). This also means that adaptive techniques could be used to improve the convergence rate with respect to the number of degrees-of-freedom.

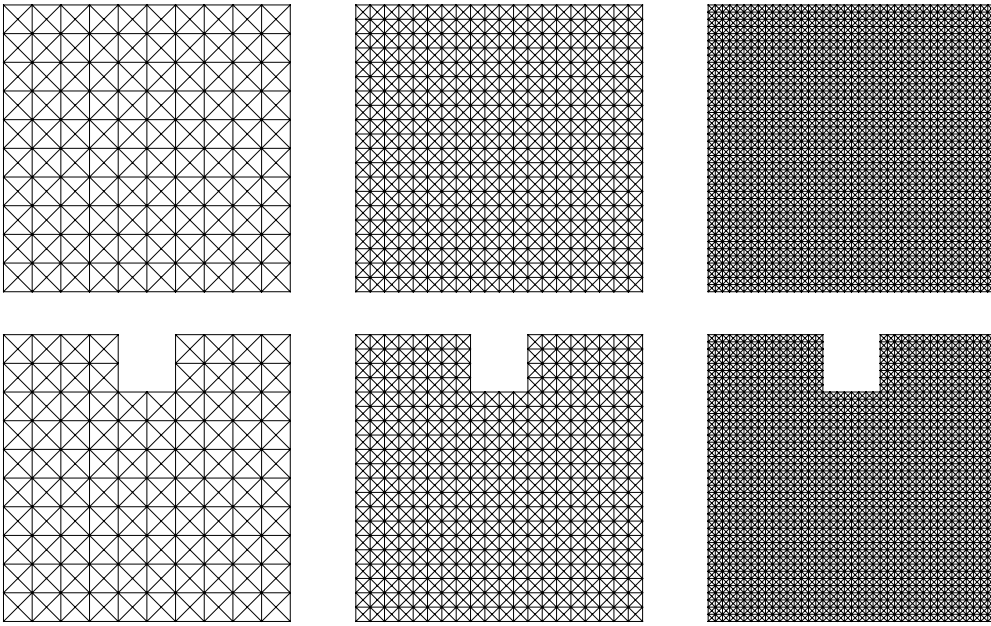


Figure 1: The first three meshes for the convex (top) and nonconvex (bottom) examples from the uniform mesh sequences. The side length of the larger square is 1 while the smaller square in the nonconvex example has a side length of 0.2.

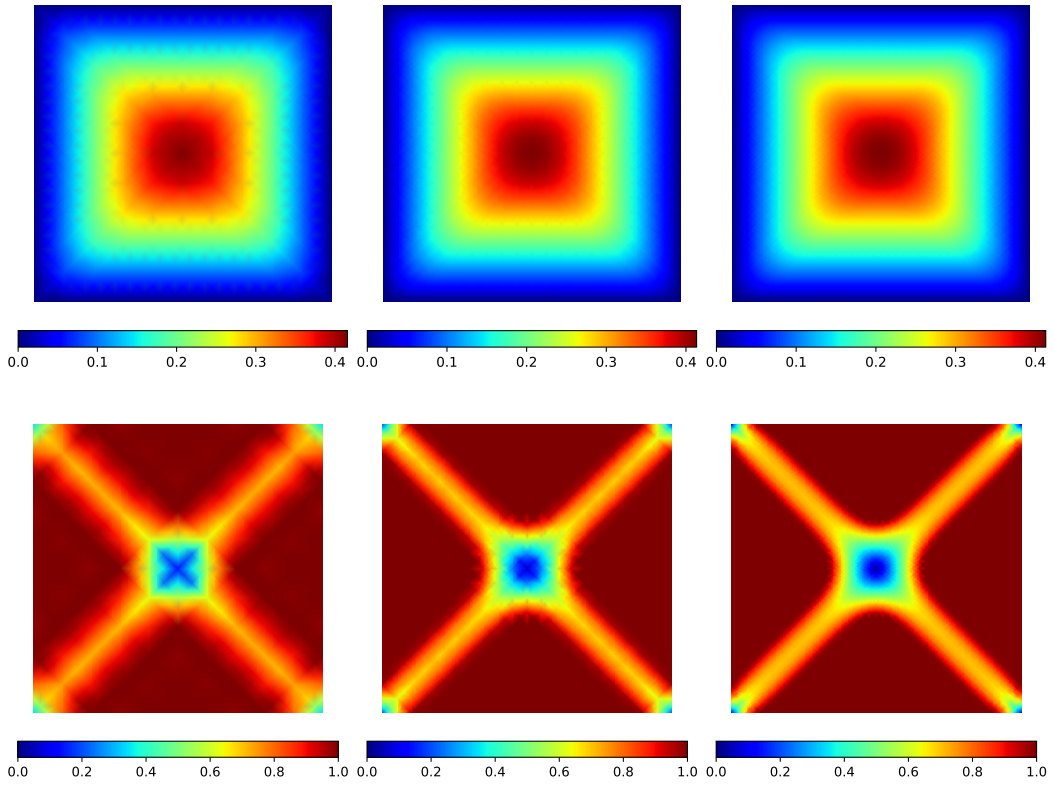


Figure 2: The first three discrete solutions u_h (top) and gradient magnitudes $|\nabla u_h|$ (bottom) for the convex example.

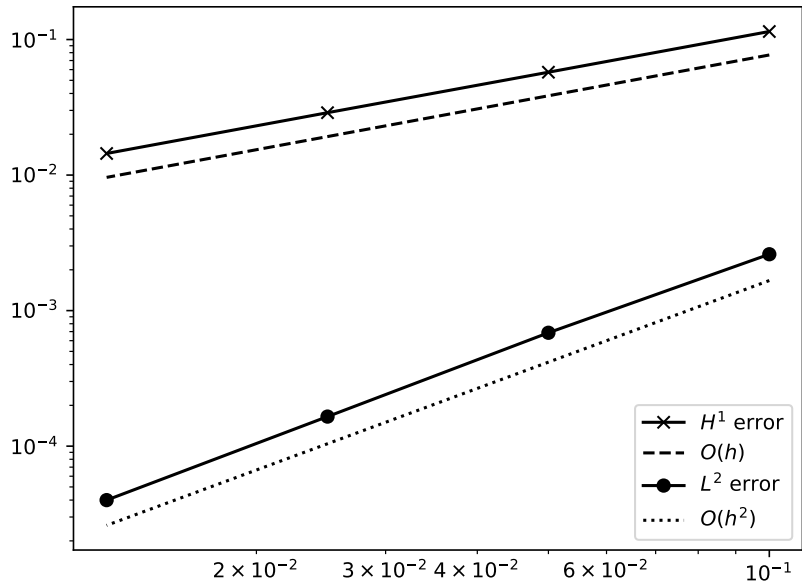


Figure 3: The error between u_h and a reference solution as a function of the mesh parameter h for the convex example. The reference solution is obtained using a quadratic finite element method on a suitably refined mesh.

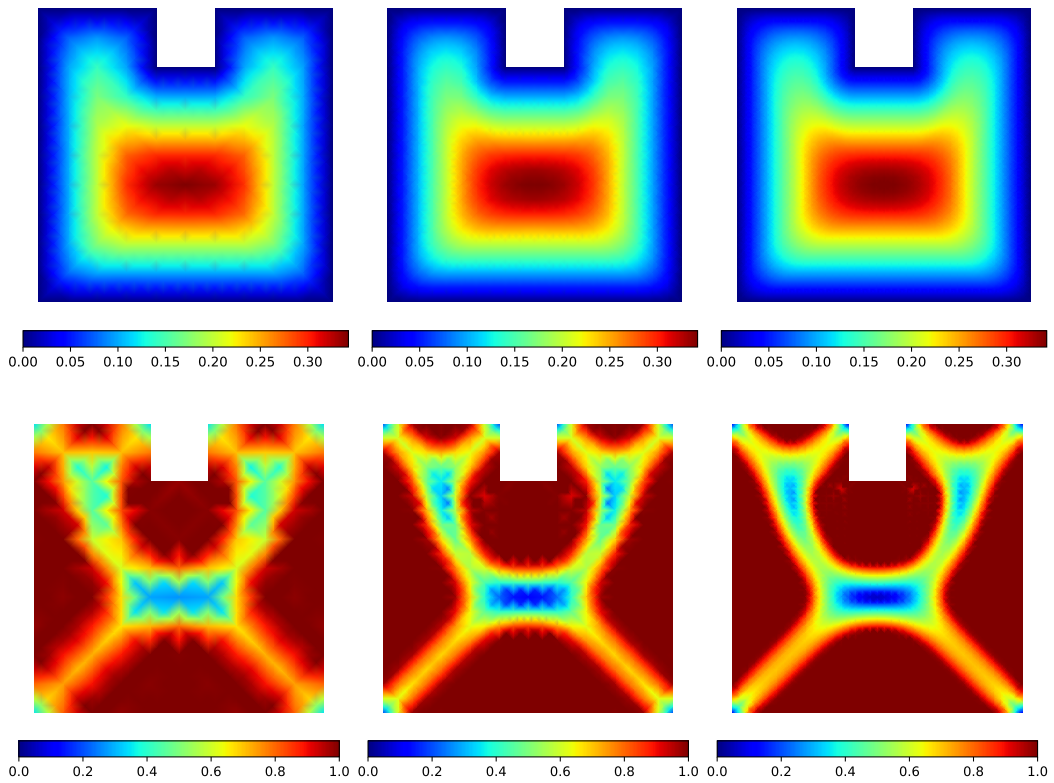


Figure 4: The first three discrete solutions u_h (top) and gradient magnitudes $|\nabla u_h|$ (bottom) for the nonconvex example.

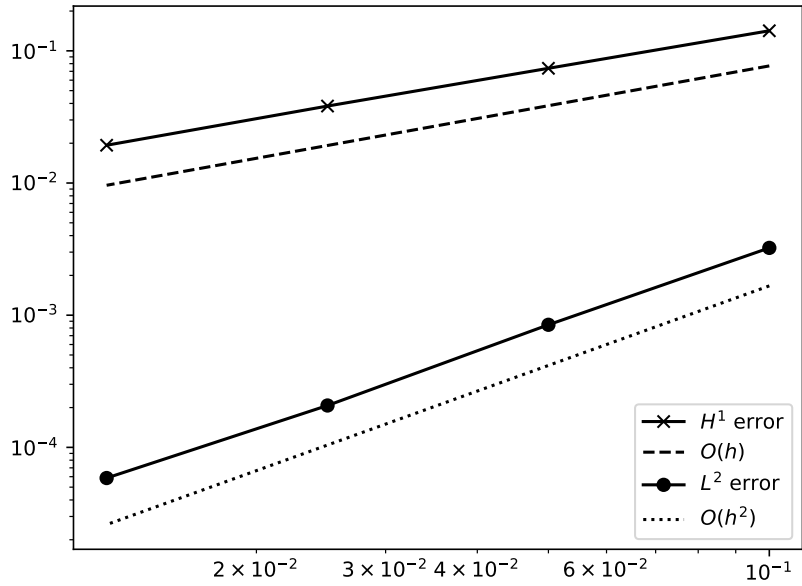


Figure 5: The error between u_h and a reference solution as a function of the mesh parameter h for the nonconvex example. The reference solution is obtained using a quadratic finite element method on a suitably refined mesh.

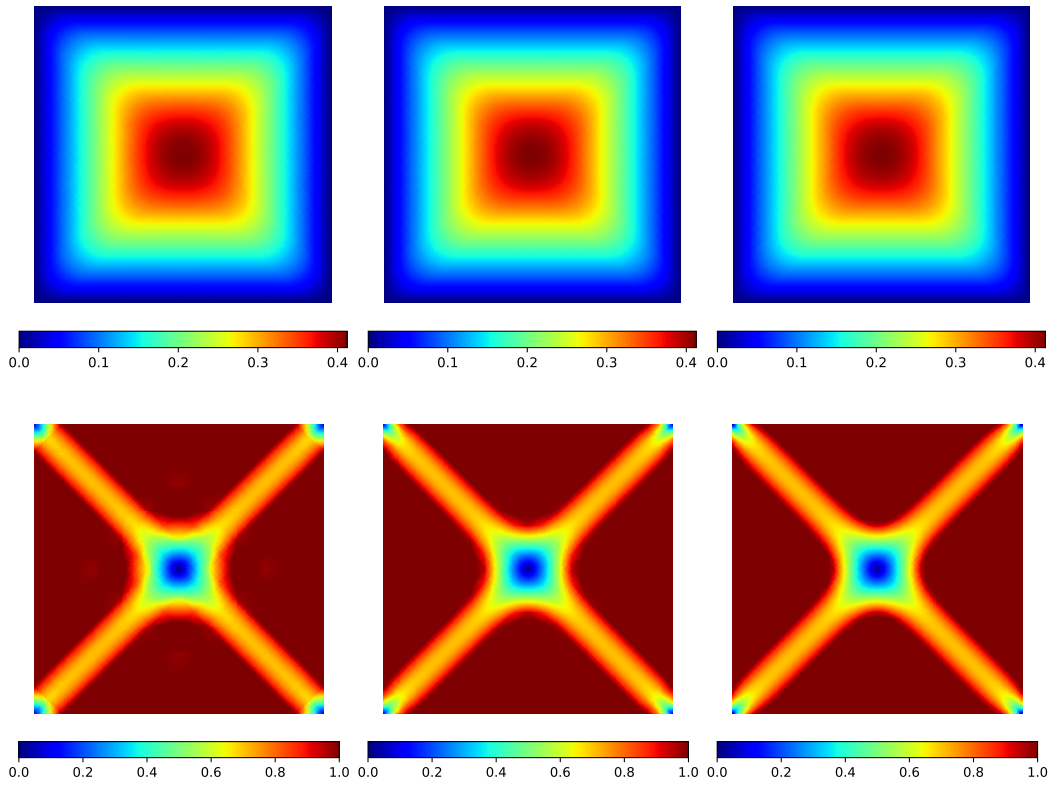


Figure 6: The first three discrete solutions u_h (top) and gradient magnitudes $|\nabla u_h|$ (bottom) for the convex example using quadratic elements.

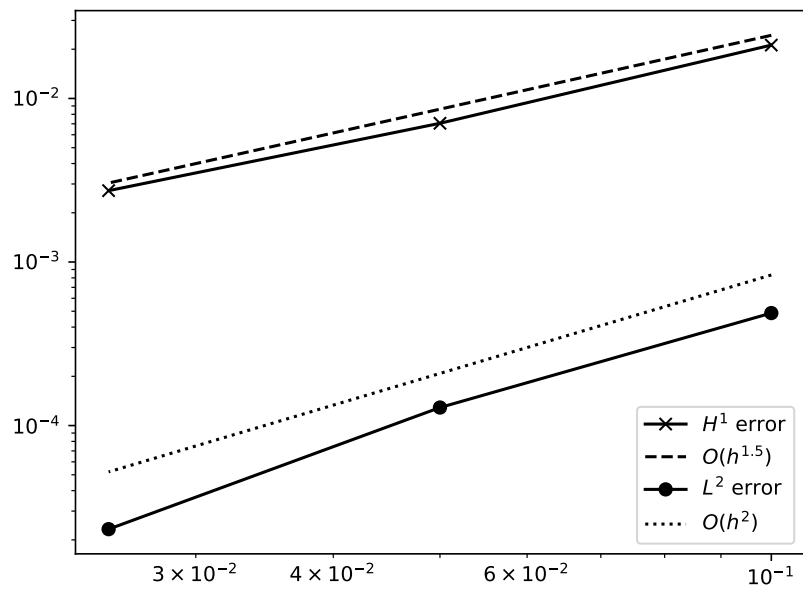


Figure 7: The error between u_h and a reference solution as a function of the mesh parameter h for the convex example using quadratic elements. The reference solution is obtained using a cubic finite element method on a suitably refined mesh.

Acknowledgements

FC’s work is partially supported by the I-Site BFC project NAANoD and the EIPHI Graduate School (contract ANR-17-EURE-0002). FC is grateful of the Center for Mathematical Modeling grant FB20005. The work of TG was supported by the Academy of Finland (decision nr. 338341).

References

- [1] S.C. Brenner, L.R. Scott, The mathematical theory of finite element methods, volume 15 of *Texts in Applied Mathematics*, Springer, New York, third edition, 2008.
- [2] H. Brezis, Équations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier (Grenoble)* 18 (1968) 115–175.
- [3] H. Brezis, M. Sibony, Équivalence de deux inéquations variationnelles et applications, *Arch. Rational Mech. Anal.* 41 (1971) 254–265.
- [4] H. Brezis, G. Stampacchia, Sur la régularité de la solution d’inéquations elliptiques, *Bulletin de la Société Mathématique de France* 96 (1968) 153–180.
- [5] E. Burman, P. Hansbo, M.G. Larson, R. Stenberg, Galerkin least squares finite element method for the obstacle problem, *Comput. Meth. Appl. Mech. Engrg.* 313 (2017) 362–374.
- [6] L.A. Caffarelli, A. Friedman, The free boundary for elastic-plastic torsion problems, *Trans. Amer. Math. Soc.* 252 (1979) 65–97.
- [7] A. Bermúdez de Castro López, A mixed method for the elastoplastic torsion problem, *IMA J. Numer. Anal.* 2 (1982) 325–334.
- [8] F. Chouly, M. Fabre, P. Hild, R. Mlika, J. Pousin, Y. Renard, An overview of recent results on Nitsche’s method for contact problems, in: *Geometrically unfitted finite element methods and applications*, volume 121 of *Lect. Notes Comput. Sci. Eng.*, Springer, Cham, 2017, pp. 93–141.
- [9] F. Chouly, P. Hild, On a finite element approximation for the elastoplastic torsion problem, *Appl. Math. Lett.* 132 (2022) Paper No. 108191, 6.
- [10] F. Chouly, P. Hild, Y. Renard, Symmetric and non-symmetric variants of Nitsche’s method for contact problems in elasticity: theory and numerical experiments, *Math. Comp.* 84 (2015) 1089–1112.
- [11] P.G. Ciarlet, The finite element method for elliptic problems, volume 40 of *Classics in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.
- [12] G. Duvaut, J.L. Lions, Inequalities in mechanics and physics, volume 219 of *Grundlehren der Mathematischen Wissenschaften*, Springer-Verlag, Berlin-New York, 1976.
- [13] A. Ern, J.L. Guermond, Finite elements I—Approximation and interpolation, volume 72 of *Texts in Applied Mathematics*, Springer, Cham, [2021] ©2021.
- [14] R.S. Falk, B. Mercier, Error estimates for elasto-plastic problems, *RAIRO Anal. Numér.* 11 (1977) 135–144, 219.
- [15] C. Gerhardt, Regularity of solutions of nonlinear variational inequalities with a gradient bound as constraint, *Arch. Rational Mech. Anal.* 58 (1975) 309–315.
- [16] R. Glowinski, Lectures on numerical methods for nonlinear variational problems, volume 65 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*, Tata Institute of Fundamental Research, Bombay; Springer-Verlag, Berlin-New York, 1980.
- [17] R. Glowinski, J.L. Lions, R. Trémolieres, Numerical analysis of variational inequalities, volume 8 of *Studies in Mathematics and its Applications*, North-Holland Publishing Co., Amsterdam-New York, 1981.
- [18] T. Gustafsson, G.D. McBain, scikit-fem: A Python package for finite element assembly, *Journal of Open Source Software* 5 (2020) 2369.
- [19] T. Gustafsson, R. Stenberg, J. Videman, Mixed and stabilized finite element methods for the obstacle problem, *SIAM Journal on Numerical Analysis* 55 (2017) 2718–2744.
- [20] Q. Hu, F. Chouly, P. Hu, G. Cheng, S.P.A. Bordas, Skew-symmetric Nitsche’s formulation in isogeometric analysis: Dirichlet and symmetry conditions, patch coupling and frictionless contact, *Comput. Methods Appl. Mech. Engrg.* 341 (2018) 188–220.
- [21] G. Idone, A. Maugeri, C. Vitanza, Variational inequalities and the elastic-plastic torsion problem, *J. Optim. Theory Appl.* 117 (2003) 489–501.
- [22] D. Kinderlehrer, G. Stampacchia, An introduction to variational inequalities and their applications, volume 88 of *Pure and Applied Mathematics*, Academic Press, Inc., New York-London, 1980.
- [23] D. Maclaurin, D. Duvenaud, R.P. Adams, Autograd: Effortless Gradients in Numpy, in: *ICML 2015 AutoML workshop*, volume 238.
- [24] R. Mlika, Y. Renard, F. Chouly, An unbiased Nitsche’s formulation of large deformation frictional contact and self-contact, *Comput. Methods Appl. Mech. Engrg.* 325 (2017) 265–288.
- [25] K. Mouallif, Approximation du problème de la torsion élasto-plastique d’une barre cylindrique par régularisation et discrétisation d’un problème inf-sup sur $H_0^1(\Omega) \times L_+^\infty(\Omega)$, *Travaux Sém. Anal. Convexe* 12 (1982) exp. no. 1, 24.