

Tensor rectifiable G -flat chains

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Abstract

We prove a rigidity result for k -rectifiable sets Σ in \mathbb{R}^n (that is, up to a \mathcal{H}^k -negligible set, Σ is covered by a countable union of k -manifolds of class C^1). Given some decompositions $k = k_1 + k_2$, $n = n_1 + n_2$, we consider the following properties.

(1) For \mathcal{H}^k -almost every $x \in \Sigma$, the tangent planes to Σ split as $T_x \Sigma = L^1(x) \times L^2(x)$ for some k_1 -plane $L^1(x) \subset \mathbb{R}^{n_1}$ and some k_2 -plane $L^2(x) \subset \mathbb{R}^{n_2}$.

(2) Up to a \mathcal{H}^k -negligible set, $\Sigma \subset \Sigma^1 \times \Sigma^2$ for some sets $\Sigma^1 \subset \mathbb{R}^{n_1}$, $\Sigma^2 \subset \mathbb{R}^{n_2}$ such that Σ^1 is k_1 -rectifiable and Σ^2 is k_2 -rectifiable (we say that Σ is (k_1, k_2) -rectifiable).

We always have (2) \implies (1). We establish a partial converse: if $A = A \llcorner \Sigma$ for some *normal* rectifiable G -flat k -chain A , then (1) \implies (2) in the sense that $A = A \llcorner \Sigma^1 \times \Sigma^2$ with Σ^1, Σ^2 as in (2).

In the proof we introduce the new groups of *tensor flat chains* (or (k_1, k_2) -chains) in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ generalizing Fleming's G -flat chains. The other main tool is White's rectifiable slices theorem.

1 Introduction

Let n, n_1, n_2, k, k_1, k_2 be nonnegative integers such that

$$0 \leq k_1 \leq n_1, \quad 0 \leq k_2 \leq n_2, \quad k_1 + k_2 = k, \quad n_1 + n_2 = n.$$

Let $\Sigma \subset \mathbb{R}^n$ be a k -manifold of class C^1 . If $\Sigma = \Sigma^1 \times \Sigma^2$ for some k_l -manifolds $\Sigma^l \subset \mathbb{R}^{n_l}$, $l = 1, 2$ then

$$T_x \Sigma = T_{x^1} \Sigma^1 \times T_{x^2} \Sigma^2 \quad \text{for every } x \in \Sigma.$$

Here we write $x = (x^1, x^2)$ with $x^l \in \mathbb{R}^{n_l}$ for $l = 1, 2$. A partial converse holds true in the following sense. If $\Sigma \subset \mathbb{R}^n$ is a k -manifold of class C^1 such that

$$T_x \Sigma = L^1(x) \times L^2(x) \quad \text{for every } x \in \Sigma, \tag{1.1}$$

where, for $l = 1, 2$, $L^l(x)$ is a k_l -plane in \mathbb{R}^{n_l} then

$$\Sigma \subset \Sigma^1 \times \Sigma^2 \tag{1.2}$$

where, for $l = 1, 2$, $\Sigma^l \subset \mathbb{R}^{n_l}$ is a countable union of k_l -manifolds of class C^1 . If moreover Σ is a compact manifold then we have the stronger result $\Sigma = \Sigma^1 \times \Sigma^2$ – see Examples 2&3 of Section 2.

The main question we investigate in this paper is whether the analogue of the implication (1.1) \implies (1.2) holds true when Σ is a more general k -rectifiable set. Denoting by \mathcal{H}^k the

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Hausdorff measure of dimension k , we recall that a set $\Sigma \subset \mathbb{R}^n$ is k -rectifiable if, up to a \mathcal{H}^k -negligible set, Σ is covered by a countable family of k -manifolds of class C^1 .

More precisely, we study the analogue of (1.1) \implies (1.2) for a k -rectifiable Radon measure μ . In this context the implication does not hold in general (see Example 5) but we have a positive result if, in addition, μ is the measure associated to an integral k -current, or more generally to a *normal* rectifiable flat k -chain (with coefficients in a complete abelian normed group).

For the reader's convenience, we first state the result for integral k -currents (rather than general k -chains) and we provide the necessary definitions. We only assume that the reader is familiar with the integration of k -forms on a k -manifolds. First, given a positive measure μ and a set S (or a Radon measure and a Borel set S), the restriction of μ to S is the measure defined by $\mu \llcorner S(E) := \mu(S \cap E)$.

A real rectifiable k -current R in \mathbb{R}^n with integer coefficients writes as

$$R = m\xi\mathcal{H}^k \llcorner \Sigma, \quad (1.3)$$

where:

- (a) $\Sigma \subset \mathbb{R}^n$ is a Borel set which is k -rectifiable, that is, up to a \mathcal{H}^k -negligible set, $\Sigma \subset \cup_{j \geq 1} \Gamma_j$ for some sequence Γ_j of k -manifolds of class C^1 ,
- (b) $m : \Sigma \rightarrow \mathbb{Z} \setminus \{0\}$ is a Borel function, called the multiplicity of R , such that the mass of R is finite, that is, $\int_{\Sigma} |m| d\mathcal{H}^k < \infty$,

For \mathcal{H}^k -almost every $x \in \Sigma$, there exists an (approximate) tangent k -plane $T_x\Sigma$ which is just the tangent plane at x of one of the Γ_j 's covering Σ .¹

- (c) $\xi : \Sigma \rightarrow \bigwedge_k \mathbb{R}^n$ is a Borel mapping with values in the space of k -vectors of \mathbb{R}^n such that for \mathcal{H}^k -almost every x , $\xi(x)$ is a unit, simple k -vector spanning $T_x\Sigma$, that is, $\xi(x)$ writes as $f_1(x) \wedge \dots \wedge f_k(x)$ for some orthonormal basis $(f_1(x), \dots, f_k(x))$ of $T_x\Sigma$.
Consequently, $\xi(x)$ gives an orientation to the k -plane $T_x\Sigma$.

According to (1.3), R is a continuous linear form on the space of smooth and compactly supported k -forms ω given by

$$\langle R, \omega \rangle = \int_{\Sigma} m \langle \xi(x), \omega(x) \rangle d\mathcal{H}^k(x).$$

In other words, R is the integration of k -forms over the oriented " k -manifold" (Σ, ξ) with the multiplicity coefficient m . By definition the boundary of R is the continuous linear form on smooth and compactly supported $(k-1)$ -forms ω' given by

$$\langle \partial R, \omega' \rangle = \int_{\Sigma} m \langle \xi(x), d\omega'(x) \rangle d\mathcal{H}^k(x).$$

When Σ is a smooth k -manifold with boundary and $m \equiv 1$, the current ∂R is the integration of $(k-1)$ -forms over the oriented $(k-1)$ -manifold $\text{Bdry } \Sigma$.

We make a last assumption on R (which in particular implies that ∂R has finite mass).

- (d) We assume that ∂R is a $(k-1)$ -rectifiable current.

A current R satisfying assumptions (a) to (d) is called an integral k -current in \mathbb{R}^n , see [Fed69, 4.1.24].

In this context our main result states as follows.

¹Rigorously, using Definition 1.1 with $\mu := |m|\mathcal{H}^k \llcorner \Sigma$, there holds $T_x\Sigma = T_x\mu$ for \mathcal{H}^k -almost every $x \in \Sigma$.

Theorem. *Let R be an integral k -current in \mathbb{R}^n of the form 1.3. The following are equivalent.*

- (i) *for \mathcal{H}^k -almost every $x \in \Sigma$, $T_x \Sigma = L^1(x) \times L^2(x)$ for some k_l -planes $L^l(x) \subset \mathbb{R}^{n_l}$, $l = 1, 2$.*
- (ii) *$\Sigma \subset \Sigma^1 \times \Sigma^2$ for some k_l -rectifiable sets $\Sigma^l \subset \mathbb{R}^{n_l}$, $l = 1, 2$.*

This result is a *first order* rigidity property in the sense that a local information about the tangent planes to Σ yields a global behavior of Σ , compared for instance to Alexandrov's rigidity theorem [Ale62] where an information about *second order* derivatives (the compact manifold has constant mean curvature) yields a global information (it is a sphere).

Let us now state the result in its full generality. We are naturally led to work in the framework of flat chains rather than currents (see Step 2 of the sketch of proof of Theorem 1.3 below). Rectifiable flat chains generalize rectifiable currents in the sense that the coefficients $m(x)$ are now taken in a given complete abelian normed group $(G, +, \|\cdot\|)$. For instance $G = \mathbb{R}$, $G = \mathbb{R}^n$, G is a Banach space or $G = \mathbb{T}^m$, $G = \mathbb{Z}/m\mathbb{Z}$, G is a group of p -adic numbers and so on. The construction of flat chains proposed in [Fle66] is described in Section 3, together with the results used thereafter, but let us now introduce some vocabulary.

Given A , a flat k -chain in \mathbb{R}^n with coefficients in G , we denote its mass by $\mathbb{M}(A)$ and its boundary by ∂A . We write $\mathbb{N}(A) = \mathbb{M}(A) + \mathbb{M}(\partial A)$ and we say that A is *normal* whenever $\mathbb{N}(A) < \infty$. The positive measure associated to A is defined for Borel sets $S \subset \mathbb{R}^n$ by $\mu_A(S) = \mathbb{M}(A \llcorner S)$ where $A \llcorner S$ is the k -chain obtained by restricting A to the set S .

Definition 1.1. A positive Radon measure μ in \mathbb{R}^n is k -rectifiable if $\mu \ll \mathcal{H}^k$ and there exists a k -rectifiable set Σ such that $\mu = \mu \llcorner \Sigma$.

We have the following property [AFP00, §2.9 & §2.11]. If μ is a k -rectifiable positive Radon measure in \mathbb{R}^n then for μ -almost every x there exist a k -plane $T_x \mu \subset \mathbb{R}^n$ and $\rho(x) > 0$ such that

$$\forall \varphi \in C_c(\mathbb{R}^n) \quad \frac{1}{r^k} \int \varphi \left(\frac{1}{r}(y-x) \right) d\mu(y) \xrightarrow{r \uparrow \infty} \rho(x) \int_{T_x \mu} \varphi d\mathcal{H}^k.$$

Besides, we have $\mu = \rho \mathcal{H}^k \llcorner \Sigma$. $T_x \mu$ is called the (approximate) tangent k -plane to μ at x .

We now translate the properties (1)&(2) of the abstract in terms of rectifiable measures.

Definition 1.2. Let μ be a positive k -rectifiable Radon measure in \mathbb{R}^n .

- (1) We say that $T\mu$ is (k_1, k_2) -split if for μ -almost every $x \in \mathbb{R}^n$, $T_x \mu = L^1(x) \times L^2(x)$ for some k_l -planes $L^l(x) \subset \mathbb{R}^{n_l}$, $l = 1, 2$.
- (2) We say that μ is (k_1, k_2) -rectifiable if $\mu = \mu \llcorner \Sigma^1 \times \Sigma^2$ for some k_l -rectifiable sets $\Sigma^l \subset \mathbb{R}^{n_l}$, $l = 1, 2$. Equivalently, $\mu(\mathbb{R}^n \setminus \Sigma^1 \times \Sigma^2) = 0$.

Notice that the two definitions depend on (n_1, n_2) and not only on (k_1, k_2) but to lighten notation we avoid explicit references to n_1, n_2 . We say that a flat chain A is rectifiable if the measure μ_A is rectifiable. The main result of the article is the following.

Theorem 1.3. *Let G be a complete Abelian normed group and let A be a normal and rectifiable k -chain with coefficients in G . The following are equivalent.*

- (i) *$T\mu_A$ is (k_1, k_2) -split.*
- (ii) *μ_A is (k_1, k_2) -rectifiable.*

The latter is equivalent to $A = A \llcorner \Sigma^1 \times \Sigma^2$ for some k_l -rectifiable sets $\Sigma^l \subset \mathbb{R}^{n_l}$, $l = 1, 2$.

We also say that A is (k_1, k_2) -split when $T\mu_A$ is and that A is (k_1, k_2) -rectifiable when μ_A is.

The theorem concerns rectifiable chains and not general rectifiable measures. However, given a k -rectifiable finite positive Borel measure μ on \mathbb{R}^n , we have $\mu_A = \mu$ for any real flat chain $A = \xi\mu$ where $x \mapsto \xi(x)$ is any Borel measurable choice of orientation of the tangent planes $T_x\mu$. The difficulty is that the theorem applies only to chains with finite boundary mass and except in the cases $\mu = 0$ and $k = 0$, there holds $\mathbb{M}(\partial A) = \infty$ for most (and sometimes all) of the choices of ξ . Such situation is illustrated by the counterexample 5 of Section 2.² This example indicates that the hypothesis on ∂A (or at least some weaker condition) is necessary: without further assumption, given a k -rectifiable measure μ , the fact that $T\mu$ is (k_1, k_2) -split does not imply in general that μ is (k_1, k_2) -rectifiable. On the bright side, we can pick A with coefficients in any complete Abelian normed group, which might help for measures with branching structure, see e.g. [MM16, MMT19, BOO18].

This article is motivated by former and current works by the authors. More precisely, in [GM21] we introduce some functional $\mathcal{E}(u)$ that penalizes oblique oscillations of a function $u : (0, 1)^{n_1} \times (0, 1)^{n_2} \rightarrow \mathbb{R}$. The functional vanishes on functions $u(x^1, x^2)$ of the form $u(x) = u_1(x^1)$ or $u(x) = u_2(x^2)$. The sequel [GM23] studies the $\mathbb{R}^{n_1 \times n_2}$ -valued distribution $\mu[u]$ obtained by extracting the block $D_{x^1}D_{x^2}u$ from the Hessian matrix D^2u . In particular $\mu[u]$ vanishes if and only if u decomposes as $u_1(x^1) + u_2(x^2)$. Theorem 1.3 is used in [GM23] to establish that when u is bounded and such that $\mathcal{E}(u) < \infty$ then $\mu[u]$ is a $(n_1 - 1, n_2 - 1)$ -rectifiable measure. Indeed, since $\mu[u]$ is defined by differentiating u , it identifies naturally with a $(n - 2)$ -current. Using White's rectifiability criterion a first time we can prove that this current is actually $(n - 2)$ -rectifiable. By the cross-derivative structure of $\mu[u]$ it follows that $T\mu[u]$ is $(n_1 - 1, n_2 - 1)$ -split so that Theorem 1.3 applies. As pointed out in [GM23], the differential inclusion given by the condition that $D_{x^1}D_{x^2}u$ is a measure fits in the general framework treated in [ARDPHR19, Theorem 1.5]. Using this result would however yield a much weaker result namely that the most singular part of $\mu[u]$ is $(n - 2)$ -rectifiable.

Outline of the proof of Theorem 1.3

First, the direction (ii) \implies (i) is the easy part and in fact holds for general rectifiable measures.

Let us give a quick idea of the proof of (i) \implies (ii).

Step 1. The case $k_1 = 0$ (Proposition 6.2). We prove that if A is a normal rectifiable k -chain in \mathbb{R}^n such that $T\mu_A$ is $(0, k)$ -split then, there exist sequences $y_j^1 \in \mathbb{R}^{n_1}$ and A_j^2 normal and rectifiable k -chains in \mathbb{R}^{n_2} such that

$$A = \sum \llbracket y_j^1 \rrbracket \times A_j^2. \quad (1.4)$$

By [GM22b, Theorem 1.2], given a normal rectifiable k -chain A in \mathbb{R}^n , there exists a partition S_j of \mathbb{R}^n in Borel sets such that setting $A_j := A \llcorner S_j$, the A_j 's are normal rectifiable chains and

$$\mathbb{N}(A) = \sum \mathbb{N}(A_j).$$

Moreover this decomposition is maximal in the sense that for each j and for every Borel set $S \subset \mathbb{R}^n$ there holds

$$\mathbb{N}(A_j) = \mathbb{N}(A_j \llcorner S) + \mathbb{N}(A_j \llcorner (\mathbb{R}^n \setminus S)) \implies A_j \llcorner S = A_j \text{ or } A_j \llcorner S = 0.$$

In this case, we say that A_j is set-indecomposable. Since we argue separately on each A_j , we may assume that $A = A_j$ is set-indecomposable and prove that $A = \llbracket y \rrbracket \times A^2$.

²Indeed, there is no change of orientation of the chain A_1 of Figure 2 such that the new chain has finite boundary mass.

Next, given $i \in \{1, \dots, n_1\}$, we consider the family of half-spaces $H_i(s) = \{x \in \mathbb{R}^n : x_i > s\}$. The fact that $T\mu_A$ is $(0, k)$ -split implies that $T_x\mu_A$ and $\text{Bdry } H_i(s) = \{x : x_i = s\}$ are parallel so that, roughly speaking, the intersection of A and $\{x : x_i = s\}$ is not transverse (A does not cross the hyperplane $\{x : x_i = s\}$). With this idea, we establish rigorously that the slice of A by $\{x : x_i = s\}$ vanishes. Equivalently, the hyperplane $\{x : x_i = s\}$ splits A in $A = A \llcorner H_i(s) + A \llcorner (\mathbb{R}^n \setminus H_i(s))$ and without creating additional boundary, we have:

$$\mathbb{N}(A) = \mathbb{N}(A \llcorner H_i(s)) + \mathbb{N}(A \llcorner (\mathbb{R}^n \setminus H_i(s))).$$

Since A is indecomposable this implies that $A \llcorner H_i(s) = 0$ or $A \llcorner (\mathbb{R}^n \setminus H_i(s)) = 0$. Proceeding by dichotomy we obtain that $A = A \llcorner \{x : x_i = y_i\}$ for some $y_i \in \mathbb{R}$ and using all the directions e_i for $i = 1, \dots, n_1$ we deduce that A is supported in $\{y^1\} \times \mathbb{R}^{n_2}$ for some $y^1 \in \mathbb{R}^{n_1}$. This concludes the proof of (1.4).

Step 2. The general case (Theorem 6.4). If $P \in \mathcal{P}_k^G(\mathbb{R}^n)$ is a polyhedral chain (see (3.1)) and $T\mu_P$ is (k_1, k_2) -split it is not hard to see that it must be of the form

$$P = \sum g_i p_i^1 \times p_i^2,$$

where $g_j \in G$ and for $l = 1, 2$, p_i^l is an oriented polyhedral k_l -cell in \mathbb{R}^{n_l} . In particular P is in an element of the direct group product $\mathcal{P}_{k_1}^G(\mathbb{R}^{n_1}) \otimes \mathcal{P}_{k_2}^G(\mathbb{R}^{n_2})$. Moreover, it can be alternatively interpreted as an element of $\mathcal{F}_{k_1}(\mathbb{R}^{n_1}, \mathcal{F}_{k_2}^G(\mathbb{R}^{n_2}))$ (the group of k_1 -flat chains with values in the group $\mathcal{F}_{k_2}^G(\mathbb{R}^{n_2})$ of k_2 -chains with values in G) through the identification

$$\iota P := \sum [g_i p_i^2] p_i^1.$$

In this notation we consider $g_i p_i^2$ as an element of $\mathcal{F}_{k_2}^G(\mathbb{R}^{n_2})$. One of the main points of the proof of Theorem 6.4 is to show that every normal and rectifiable chain A for which $T\mu_A$ is (k_1, k_2) -split identifies with an element ιA of $\mathcal{F}_{k_1}(\mathbb{R}^{n_1}, \mathcal{F}_{k_2}^G(\mathbb{R}^{n_2}))$. The advantage is that we can now apply the rectifiable slices theorem [Whi99b] of White to ιA (White's result is recalled in Theorem 3.11). Indeed, at least formally, 0-slices of ιA correspond to $(0, k_2)$ -slices of A and the fact that A is (k_1, k_2) -split implies that the 0 slices of ιA are $(0, k_2)$ -split. Using *Step 1* we can write these slices in the form (1.4). In particular this means that the 0-slices of ιA should write as

$$\sum A_j^2 \llcorner [y_j^1]$$

which exactly means that they are 0-rectifiable. By White's rectifiable slices theorem we then obtain that $\iota A = (\iota A) \llcorner \Sigma_1$ for some k_1 -rectifiable set $\Sigma_1 \subset \mathbb{R}^{n_1}$. This yields $A = A \llcorner \Sigma_1 \times \mathbb{R}^{n_2}$. Eventually, exchanging the roles of n_1 and n_2 concludes the proof.

In order to make this sketch of proof rigorous, an essential step is to extend the operator ι . As usual for flat chains, this is done by continuity. Setting for $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{R}^n)$

$$\mathbb{F}^\wedge(P) := \mathbb{F}(\iota P), \tag{1.5}$$

we see that \mathbb{F}^\wedge is a norm on $\mathcal{P}_{k_1, k_2}^G(\mathbb{R}^n)$. The operator ι then naturally extends to the completion of this space for the \mathbb{F}^\wedge norm. It turns out (see Proposition 4.11) that this space is exactly the same as the space obtained as completion by the projective norm (which coincides with \mathbb{F}^\wedge in this case) of the tensor product $\mathcal{F}_{k_1}^{\mathbb{Z}}(\mathbb{R}^{n_1}) \otimes \mathcal{F}_{k_2}^G(\mathbb{R}^{n_2})$. For this reason we call this (new) group, the group of tensor flat chains (or (k_1, k_2) -chains). One of the contributions of the paper is to start developing the theory of these tensor flat chains. The overall theory resembles the one for flat chains. For instance, \mathbb{F}^\wedge has a Whitney type representation (see Proposition 4.8). In order to clarify the analogies with the classical theory of flat chain, rather than (1.5), we actually

take this definition as the starting point of our construction, see (4.5). An important role in the theory is played by the two partial boundary operators ∂_1 and ∂_2 which for a tensor flat chain $A' = A^1 \otimes A^2$ are given by

$$\partial_1 A' := (\partial A^1) \otimes A^2, \quad \partial_2 A' := (-1)^{k_1} A^1 \otimes (\partial A^2).$$

Because of this extra structure, in the core of the paper, we use to the notation $A^1 \wedge A^2$ rather than $A^1 \otimes A^2$.

Let us point out that another source of difficulties comes from the fact that a tensor chain does not always identify with a classical chain. There are two reasons for that. The first one is of *geometrical nature*: if $n_1 = n_2 = 1$ and S is the segment with vertices $(0,0)$ and $(1,1)$, the two 1-currents $T_1 = e_1 \mathcal{H}^1 \llcorner S$ and $T_2 = e_2 \mathcal{H}^1 \llcorner S$ are $(1,0)$ - and $(0,1)$ -chains respectively but they are not flat chains (however, their sum is a flat chain). The other reason is of *Functional Analysis* nature: the norm on the groups of tensor chains is weaker than the classical flat norm and the groups of tensor chains are “larger”. Indeed, the usual flat norm involves only boundary operators of first order (see e.g. Proposition 3.1) and is thus reminiscent of the Sobolev space with negative regularity index $W^{-1,1}(\Omega)$, see [AF03] for the definition of Sobolev spaces. The tensor flat norm \mathbb{F}^\wedge instead, involves not only ∂_1 and ∂_2 but also $\partial_1 \partial_2$, see (4.5), and is thus closer in spirit to $W^{-2,1}(\Omega)$.

Conversely, in general a k -chain cannot be interpreted as a tensor chain. For instance if $n_1 = n_2 = 1$, the “transverse” 1-chain associated with the integration over the segment S defined above is not a tensor chain (but writes as the sum of a $(1,0)$ -chain and a $(0,1)$ -chain).

This example shows that we need to be careful when considering the embeddings of subgroups of k -chains in the group of tensor chains (which is a necessary step in light of the sketch of proof above). We do this through the operator

$$\begin{aligned} j : \mathcal{F}_k^G(\mathbb{R}^n) &\longrightarrow \left(\mathcal{F}_{k'_1, k-k'_1}^G(\mathbb{R}^n) \right)_{0 \leq k'_1 \leq k}, \\ A &\longmapsto (j_{k'_1, k-k'_1} A)_{k'_1}, \end{aligned}$$

which morally speaking decomposes a flat chain into its $(k'_1, k - k'_1)$ -components.

The following properties are essential in the proof of the main results.

- (0) The operators ι and j satisfy the expected commutation properties with slicing and restrictions (see Proposition 4.14 and Proposition 4.16).
- (1) (Proposition 4.9) We have the identities $j_{k'_1, k'_2} \partial A = \partial_1 j_{k'_1+1, k'_2} A + \partial_2 j_{k'_1, k'_2+1} A$.
- (2) (Theorem 5.15) Normal $(0, k_2)$ -chains identify with the subgroup of normal k_2 -chains such that

$$\forall (k'_1, k'_2) \neq (0, k_2) \quad j_{k'_1, k'_2} A = 0.$$

- (3) (Theorem 5.7) j is one-to-one.
- (4) (Corollary 5.12) If A is a normal rectifiable k -chain in \mathbb{R}^n ,

$$T\mu_A \text{ is } (k_1, k_2)\text{-split} \quad \iff \quad \forall (k'_1, k'_2) \neq (k_1, k_2) \quad j_{k'_1, k'_2} A = 0.$$

Further remarks

(a) Building on the existing theory of flat chains, the axiomatic definition of tensor chains and the proof of their basic properties is mostly routine though lengthy. On the contrary, and maybe surprisingly, the isomorphism stated in (2) and the fact (3) that j is one-to-one require more work. In the complementary note [GM22a], we generalize Theorem 5.15 to all dimensions: the group of *normal* (k_1, k_2) -chains identifies with a subgroup of the group of *normal* k -chains. This is a delicate question: the counterpart for *finite mass* chains is not true, see [GM22a].

(b) In the case $k_1 = 0$, as in Step 1, a normal $(0, k)$ -rectifiable chain writes as (see Section 6.1)

$$A = \sum \llbracket y_j^1 \rrbracket \times A_j^2 \quad \text{with} \quad \mathbb{N}(A) = \sum \mathbb{N}(A_j^2),$$

for some sequence $y_j^1 \in \mathbb{R}^{n_1}$ and some sequence A_j^2 of normal k -rectifiable G -flat chains in \mathbb{R}^{n_2} . One may wonder whether such property generalizes in higher dimension. Let us restrict ourselves to cycles for simplicity. The question is: given a (k_1, k_2) -rectifiable cycle A in \mathbb{R}^n , is it possible to find sequences A_j^l for $l = 1, 2$ with A_j^l a k_l -cycle in \mathbb{R}^{n_l} such that

$$A = \sum A_j^1 \times A_j^2, \quad \mathbb{M}(A) = \sum \mathbb{M}(A_j^1) \mathbb{M}(A_j^2). \quad (1.6)$$

As already noticed, the answer is *yes* if $k_1 = 0$ (or $k_2 = 0$). In the other limit case $k_1 = n_1$ (or $k_2 = n_2$) the question is trivial as the only (n_1, k_2) -rectifiable cycle is 0 by the constancy theorem (see [Fle66, Lemma 7.2]) applied to $\iota_{j_{n_1, k_2}} A$. On the contrary, in the intermediate cases $1 \leq k_1 \leq n_1 - 1$, $1 \leq k_2 \leq n_2 - 1$, the answer is *no* in general for any non trivial group. This is established in the final subsection 6.1 by exhibiting a counterexample. We do not know whether the answer is still *no* if we relax the identity constraint on masses and demand only

$$\sum \mathbb{M}(A_j^1) \mathbb{M}(A_j^2) \leq C \mathbb{M}(A),$$

for some constant C possibly depending on the n_l 's and k_l 's.

(c) We point out here a link between tensor chains and the theory of *cartesian currents* developed by Giaquinta, Modica and Souček in [GMS98a, GMS98b]. Let $\Omega \subset \mathbb{R}^{n_1}$ be an open set. A cartesian current T in $\Omega \times \mathbb{R}^{n_2}$ is a particular n_1 -current:

there exists a sequence $u_j : \Omega \rightarrow \mathbb{R}^{n_2}$ such that, denoting G_{u_j} the integration of n_1 -forms over the graph of u_j , the G_{u_j} 's have uniformly bounded masses and weakly converge to T .

The limit current T still have a part which corresponds to the graph of a mapping $\Omega \rightarrow \mathbb{R}^{n_2}$ but it may also contain “vertical parts” supported in $\Gamma \times \mathbb{R}^{n_2}$ where $\Gamma \subset \Omega$ has dimension less than n_1 . Under appropriate assumptions this vertical part is a tensor flat chain.

(α) For instance, taking $n_1 = n_2 = 1$, we can consider the currents G_{u_j} corresponding to the integration over the graph of the function $u_j : x \in \mathbb{R} \mapsto \arctan(jx)$ oriented towards positive x 's. As $j \uparrow \infty$, G_{u_j} converges towards a 1-current T . Denoting $u : x \mapsto \text{sign}(x)$, T is the sum of G_u and of the vertical part $\llbracket \{0\} \times [-1, 1] \rrbracket$ which is the integration of 1-forms in \mathbb{R}^2 over the vertical segment $\{0\} \times [-1, 1]$ oriented upwards (so that $\partial T = 0$). This vertical part is a rectifiable $(0, 1)$ -chain.

(β) In the case $n_1 \geq 2$, $n_2 = 3$, the structure of the limits of currents associated with mappings $u_j : \Omega \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ with uniformly bounded Dirichlet energy is described in [GMS98b, §5.2.3]. Namely, it is assumed that

$$\sup_{j \geq 1} \int_{\Omega} |Du_j|^2 < \infty. \quad (1.7)$$

As above, we consider the n_1 -current G_{u_j} associated with the integration over the graph of u_j . Namely, for any smooth and compactly supported n -form ω in $\Omega \times \mathbb{R}^3$,

$$\langle G_{u_j}, \omega \rangle := \int_{\mathcal{G}_{u_j}} \langle \xi_j(x), \omega(x) \rangle d\mathcal{H}^{n_1}(x),$$

where $\mathcal{G}_{u_j} \subset \Omega \times \mathbb{S}^2$ is the graph of u_j and $\xi_j(x)$ is the unit simple n_1 -vector orienting $T_x \mathcal{G}_{u_j}$ such that $\langle \xi_j(x), dx_1 \wedge \dots \wedge dx_{n_1} \rangle > 0$. Theorem 1 of [GMS98b, §5.2.3] states that if T is the limit of the G_{u_j} 's then

$$T = G_u + L_T \times \llbracket \mathbb{S}^2 \rrbracket$$

where $u : \Omega \rightarrow \mathbb{S}^2$ has finite Dirichlet energy, L_T is a $(n_1 - 2)$ -rectifiable current in \mathbb{R}^{n_1} with integer coefficients and $\llbracket \mathbb{S}^2 \rrbracket$ is the standard integration of 2-forms on \mathbb{S}^2 . Here the vertical part $L_T \times \llbracket \mathbb{S}^2 \rrbracket$ is a rectifiable $(n_1 - 2, 2)$ -chain in $\mathbb{R}^{n_1} \times \mathbb{R}^3$ with integer coefficients.

- (γ) In more general cases, more precisely, when $n_1 \geq 3$, $n_2 \geq 4$, the limits of a sequence of currents G_{u_j} such that $u_j : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{S}^{n_2-1}$ satisfies the uniform bound (1.7) do not have such a nice structure (see the counterexamples and the partial results of [GMS98b, §5.4]).

The result cited in (β) is based on different ideas than the ones of the present paper.³

Organization of the article

In Section 2, we construct a few examples which show both why the question is delicate and to which extent our result is sharp.

In Section 3, we recall the theory of G -flat chains in \mathbb{R}^n as introduced by Fleming and developed by White. We also give a characterization of the (k_1, k_2) -splitting property in terms of slices and state Theorem 3.14 from [GM22b] about the set-decomposition of normal rectifiable chains.

The groups of tensor flat chains are defined in Section 4. We introduce various operations involving these objects (slices, restrictions, the morphisms ι and j) and study their basic properties.

In Section 5, we focus on identifications of chains with tensor chains *via* the operator j . We prove Theorem 5.7, Corollary 5.12 and Theorem 5.15, (the points (2)(3)&(4) in the above sketch).

The main results, namely Proposition 6.2, Theorem 6.3 and Theorem 1.3, are established successively in the last section. In a final subsection, we exhibit some counterexamples for (1.6).

Conventions and notation

In the article $(G, +, |\cdot|_G)$ is a complete Abelian normed group, that is $(G, +)$ is a commutative group, $|\cdot|_G : G \rightarrow \mathbb{R}_+$ satisfies for $g, g' \in G$,

$$|g + g'|_G \leq |g|_G + |g'|_G, \quad |g|_G = 0 \iff g = 0,$$

and denoting $d_G(g, g') = |g' - g|_G$, the metric space (G, d_G) is complete.

The integers $n_1, n_2 \geq 0$ and $n \geq 1$ are fixed and satisfy $n_1 + n_2 = n$. The integers k_1, k_2, k are non negative and we always assume $0 \leq k_1 \leq n_1$, $0 \leq k_2 \leq n_2$ and $k = k_1 + k_2$.

The standard basis of \mathbb{R}^n is (e_1, \dots, e_n) .

³In our opinion, the main tools are: (i) the closure theorem for integral currents [GMS98a, §2.2.4, Theorem 1], (ii) the fact that the bound (1.7) implies that the “less vertical” part of T vanishes [GMS98a, §5.5.2, Proposition 3], (iii) the constancy theorem (for 2-currents over \mathbb{S}^2) [GMS98a, §2.2.3, Theorem 4].

We denote by \mathcal{H}^k the k -dimensional Hausdorff measure in \mathbb{R}^n .

The topological boundary of $S \subset \mathbb{R}^n$ is denoted $\text{Bdry } S := \overline{S} \setminus \text{Int } S$.

Unless otherwise specified, the sequences are indexed on $j \geq 1$. Most of the time we use the index i for finite sequences.

We denote by I_k^n the set of subsets of $\{1, \dots, n\}$ with k elements. We also denote $I^n := \bigcup I_k^n$ the collection of subsets of $\{1, \dots, n\}$. We simply write j for the singleton $\{j\} \in I_1^n$ and $\beta \setminus j$ for $\beta \setminus \{j\}$ and so on.

For $\beta, \gamma \in I^n$,

- (*) $\overline{\beta}$ is the complement of β in $\{1, \dots, n\}$, In particular $\overline{\emptyset} = \{1, \dots, n\}$ and $\overline{j} = \overline{\emptyset} \setminus j$;
- (*) $\mathbb{X}_\beta := \text{span}\{e_j : j \in \beta\}$ and for $x \in \mathbb{R}^n$, $\mathbb{X}_\beta(x)$ is the affine space $x + \mathbb{X}_\beta$. These spaces are called coordinate spaces;
- (*) $|\beta|$ is the number of elements of β . Setting $m := |\beta|$, we denote by β_1, \dots, β_m the elements of β arranged in increasing order and we define the m -vector,

$$e_\beta := e_{\beta_1} \wedge \dots \wedge e_{\beta_m} \in \bigwedge_m \mathbb{X}_\beta. \quad (1.8)$$

Here $\bigwedge_m \mathbb{X}$ denotes the space of m -vectors in \mathbb{X} .

Given $\beta \in I^n$, we denote by

$$\mathcal{P}_r^G(\mathbb{X}_\beta) \subset \mathcal{N}_r^G(\mathbb{X}_\beta) \subset \mathcal{M}_r^G(\mathbb{X}_\beta) \subset \mathcal{F}_r^G(\mathbb{X}_\beta)$$

the groups of polyhedral r -chains, normal r -chains, finite mass r -chains and flat r -chains in \mathbb{X}_β with coefficients in G . See Section 3 for the precise definitions. By convention, for $r < 0$ and $r > |\beta|$, these groups are identified with the trivial group $\{0\}$. Moreover, the boundary operator is denoted ∂ , the flat norm of a chain A is denoted $\mathbb{F}(A)$ and its mass $\mathbb{M}(A)$. We also set $\mathbb{N}(A) := \mathbb{M}(A) + \mathbb{M}(\partial A)$.

We have some specific notation for tensor flat chains. For $0 \leq k \leq n$, we denote

$$D_k := \{(k'_1, k'_2) : 0 \leq k'_1 \leq n_1, 0 \leq k'_2 \leq n_2, k'_1 + k'_2 = k\}.$$

We introduce

$$\alpha := \{1, \dots, n_1\} \in I^n,$$

so that $\overline{\alpha} = \{n_1 + 1, \dots, n\}$.

For $\beta \in I^n$ we denote $\beta^1 := \beta \cap \alpha$ and $\beta^2 := \beta \setminus \alpha = \beta \cap \overline{\alpha}$.

For $x \in \mathbb{R}^n$, we denote x^1 (resp. x^2) the orthogonal projection of x on \mathbb{X}_α (resp. on $\mathbb{X}_{\overline{\alpha}}$), so that x decomposes as $x^1 + x^2$.

The tensor chains with respect to the decomposition $\mathbb{X}_\alpha + \mathbb{X}_{\overline{\alpha}}$ of \mathbb{R}^n are introduced in Section 4. The groups of tensor chains in \mathbb{X}_β are always defined with respect to the decomposition $\mathbb{X}_{\beta^1} + \mathbb{X}_{\beta^2}$. We denote respectively

$$\mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta) \subset \mathcal{N}_{k_1, k_2}^G(\mathbb{X}_\beta) \subset \mathcal{M}_{k_1, k_2}^G(\mathbb{X}_\beta) \subset \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$$

the groups of polyhedral, normal, finite mass and flat (k_1, k_2) -chains in \mathbb{X}_β with G coefficients.

We sometimes deal with functions in $L^1(\Omega, G)$ or $L_{\text{loc}}^1(\Omega, G)$ where Ω is a measure space and $(G, +, |\cdot|_G)$ is a complete Abelian normed group (in fact Ω is always a finite dimensional space of the form $\mathbb{X}_\beta(x)$). We use the integration in the sense of Bochner, that is, $f \in L^1(\Omega, G)$ if there exist a sequence of measurable simple functions $f_j : \Omega \rightarrow G$ and a sequence of integrable functions $g_j : \Omega \rightarrow \mathbb{R}_+$ such that $|f_j - f|_G \leq g_j$ almost everywhere in Ω and $\int g_j \rightarrow 0$. In this case $\int |f|_G := \lim \int |f_j|_G$.

2 Examples and counterexamples

Example 1.

(a) If A is the 1-chain with multiplicity 1 associated with the graph of a Lipschitz continuous function $f : (0, 1) \rightarrow \mathbb{R}$, assuming that $T\mu_A$ is $(1, 0)$ -split in $\mathbb{R} \times \mathbb{R}$ is equivalent to $f' \equiv 0$, hence f is constant. Using Theorem 1.3 alone, we only obtain that f takes at most countably many values but not that it is constant. However, in a second step, we deduce from the intermediate value theorem that f is indeed constant.

(b) Let us build a slightly more complex example, again with $k = n_1 = n_2 = 1$. Let $S_j \subset \mathbb{R}^2$ be a sequence of disjoint oriented segments of length 2^{-j} and let us define A as the sum of the 1-chains associated with the integration along the S_j 's. If we assume that $T\mu_A$ is $(1, 0)$ -split, then for every j the approximate tangent direction to μ_A is horizontal (except at the end points of the S_j 's where it is not defined). We deduce that each S_j is an horizontal segment as above so that, up to a \mathcal{H}^1 -negligible set, $\cup S_j \subset \mathbb{R} \times \Sigma^2$ with $\Sigma^2 \subset \mathbb{R}$ finite or countable. This is exactly the conclusion of the theorem.

Example 2. Let us now discuss the case of smooth chains. Let \mathcal{M} be a k -manifold of class C^1 in \mathbb{R}^n and assume that \mathcal{M} is *connected* and *compact*. We consider the property:

$$\forall x \in \mathcal{M} \exists k'_l(x)\text{-planes } L^l(x) \subset \mathbb{R}^{n_l} \text{ for } l = 1, 2 \text{ such that } T_x\mathcal{M} = L^1(x) \times L^2(x). \quad (2.1)$$

This condition is weaker than the (k_1, k_2) -splitting property in the sense that the values of k_1 and k_2 might depend on x .

Lemma 2.1. *Let \mathcal{M} be a connected k -manifold of class C^1 without boundary and satisfying (2.1). Then \mathcal{M} is of the form $\mathcal{M}^1 \times \mathcal{M}^2$ where for $l = 1, 2$, \mathcal{M}^l is a k_l -manifold in \mathbb{R}^{n_l} for some k_l 's such that $k = k_1 + k_2$.*

Proof.

Step 1. Local decomposition. Assuming without loss of generality that $0 \in \mathcal{M}$ we set $(k_1, k_2) := (k'_1(0), k'_2(0))$. Up to a change of orthonormal frames in \mathbb{X}_α and $\mathbb{X}_{\bar{\alpha}}$ we may assume that

$$V^1 := L^1(0) = \text{span}\{e_1, \dots, e_{k_1}\}, \quad V^2 := L^2(0) = \text{span}\{e_{n_1+1}, \dots, e_{n_1+k_2}\},$$

Let us also denote,

$$W^1 := \text{span}\{e_{k_1+1}, \dots, e_{n_1}\} \quad \text{and} \quad W^2 := \text{span}\{e_{n_1+k_2+1}, \dots, e_n\},$$

so that $V^1 + W^1 = \mathbb{X}_\alpha$ and $V^2 + W^2 = \mathbb{X}_{\bar{\alpha}}$. In some nonempty open ball centered at 0, \mathcal{M} is the graph of a C^1 mapping

$$f : V^1 + V^2 \longrightarrow W^1 + W^2,$$

such that $f(0) = 0$ and $Df(0) = 0$. Writing $x = x^1 + x^2 \in V^1 + V^2$ and

$$f(x) = f^1(x) + f^2(x) \in W^1 + W^2,$$

the linear operator

$$\text{Id} + Df(x) : V^1 + V^2 \rightarrow V^1 + W^1 + V^2 + W^2,$$

writes as the block matrix,

$$M(x) := \text{Id} + Df(x) = \begin{pmatrix} \text{Id}_{k_1} & 0 \\ D_{x^1}f^1(x) & D_{x^2}f^1(x) \\ 0 & \text{Id}_{k_2} \\ D_{x^1}f^2(x) & D_{x^2}f^2(x) \end{pmatrix}.$$

For x in some neighborhood of 0 in \mathcal{M} , the tangent space to \mathcal{M} at point $x + f(x)$ is the space $L(x)$ spanned by the columns of $M(x)$. The condition (2.1) is then equivalent to the following property:

$$\text{For } l = 1, 2 \quad \dim [L(x) \cap (V^l + W^l)] = k'_l(x). \quad (2.2)$$

We first remark that the left-hand side of these identities are upper semi-continuous and integer valued and since $k'_1(x) + k'_2(x) = k$ is constant, we must have $k'_l(x) = k_l$ in some neighborhood of 0.

Next, for $l = 1$, the condition (2.2) and the form (0 Id_{k_2}) of the third block of rows of $M(x)$ enforces that $L(x) \cap (V^1 + W^1)$ is the space spanned by the first k_1 columns of $M(x)$. Furthermore, all these columns belong to $V^1 + W^1$ if and only if $D_{x^1} f^2(x) = 0$. Similarly, for $l = 2$, we obtain $D_{x^2} f^1(x) = 0$. Therefore $f(x) = (f^1(x^1), f^2(x^2))$ in some neighborhood of 0.

We have just established that \mathcal{M} is locally a cartesian product. Namely, for $x = (x^1, x^2) \in \mathcal{M}$ there exists $\varepsilon_x > 0$ such that

$$\mathcal{M} \cap \{y = (y^1, y^2) \in \mathbb{R}^n : |y^1 - x^1|, |y^2 - x^2| < \varepsilon_x\} = \Sigma_x^1 \times \Sigma_x^2, \quad (2.3)$$

where for $l = 1, 2$, $\Sigma_x^l \subset \mathbb{R}^{n_l}$ is a k_l -manifold of class C^1 up to its boundary with $\Sigma^l \subset B_\varepsilon(x^l)$ and $\text{Bdry } \Sigma^l \subset \text{Bdry } B_\varepsilon(x^l)$.

Step 2. From local to global decomposition. Notice that since \mathcal{M} is connected there holds $k_1(x) = k_1$, $k_2(x) = k_2$ for every $x \in \mathcal{M}$. Let us fix $\hat{x} = (\hat{x}^1, \hat{x}^2) \in \mathcal{M}$ and with the notation of (2.3), let us define set

$$\mathcal{M}^2 := \left\{ x^2 \in \mathbb{R}^{n_2} : \overline{\Sigma_{\hat{x}}^1} \times \{x^2\} \subset \mathcal{M} \right\} \subset \mathbb{R}^{n_2}.$$

Since \mathcal{M} has no boundary, \mathcal{M}^2 is closed. Besides, by (2.3), $\overline{\Sigma_{\hat{x}}^2} \subset \mathcal{M}^2$. Let us fix $x^2 \in \mathcal{M}^2$. For every $x^1 \in \overline{\Sigma_{\hat{x}}^1}$, using (2.3) at $x = (x^1, x^2)$ and identifying, we get,

$$\Sigma_{(x^1, x^2)}^1 = \Sigma_{\hat{x}}^1 \text{ in } B_{\hat{\varepsilon}}(\hat{x}^1) \cap B_{\eta(x^1)}(x^1) \quad \text{and} \quad \Sigma_{(x^1, x^2)}^2 = \Sigma_{\hat{x}}^2 \text{ in } B_{\hat{\varepsilon}}(\hat{x}^2) \cap B_{\eta(x^1)}(x^2),$$

where we denote $\hat{\varepsilon} := \varepsilon_{\hat{x}}$ and $\eta(x^1) := \varepsilon_{(x^1, x^2)}$. Using a finite covering $\{B_{\eta(x_i^1)}(x_i^1)\}_i$ of $\overline{\Sigma_{\hat{x}}^1}$ and denoting $\eta := \min_i \eta(x_i^1)$, we see that

$$\mathcal{M} \cap B_{\hat{\varepsilon}}(\hat{x}^1) \times B_{\eta}(x^2) = \Sigma_{\hat{x}}^1 \times \Gamma_{x^2}^2 \quad \text{where} \quad \Gamma_{x^2}^2 := \bigcap_i \Sigma_{x_i^1, x^2}^2.$$

It follows that $\Gamma_{x^2}^2 \subset \mathcal{M}^2$. By construction, $\Gamma_{x^2}^2$ is a k_2 -manifold which contains x^2 and whose topological boundary lies in $\text{Bdry } B(x^2, \eta)$. We deduce that \mathcal{M}^2 is a nonempty k_2 -manifold without boundary. Recalling that \mathcal{M} is compact, we have in fact that \mathcal{M}^2 is compact.

Next, we define

$$\mathcal{M}^1 := \{x^1 \in \mathbb{R}^{n_1} : \{x^1\} \times \mathcal{M}^2 \subset \mathcal{M}\} \subset \mathbb{R}^{n_1}.$$

We have $\overline{\Sigma_{\hat{x}}^1} \subset \mathcal{M}^1$ and we prove similarly that \mathcal{M}^1 is a nonempty compact k_1 -manifold in \mathbb{R}^{n_1} . By construction $\mathcal{M}^1 \times \mathcal{M}^2 \subset \mathcal{M}^2$ and since \mathcal{M} is connected, we have in fact, $\mathcal{M} = \mathcal{M}^1 \times \mathcal{M}^2$. \square

As a conclusion, in the case of a connected C^1 manifold without boundary, the theorem is rather elementary. Moreover:

- (*) The (k_1, k_2) -splitting can be replaced by the weaker $(k'_1(x), k'_2(x))$ -splitting assumption (2.1).
- (*) The conclusion can be strengthened by replacing the inclusion $\mathcal{M} \subset \Sigma^1 \times \Sigma^2$ by an equality.

Example 3. When \mathcal{M} is a manifold with boundary, the local analysis of Step 1 in the proof of the lemma of Example 2 is still valid locally but Step 2 fails. It turns out that for manifolds with boundary the conclusion of the theorem is sharp in the sense that in some cases, we do need a *countable* union of Cartesian products of connected manifolds to cover \mathcal{M} . Let us build a counterexample in \mathbb{R}^3 with $n_1 = 2$ and $n_2 = k_1 = k_2 = 1$.

Let $f \in C^\infty([-1/2, 1], \mathbb{R})$ such that $f \equiv 0$ on $[-1/2, 0]$ and $f > 0$ on $(0, 1]$. For $j \geq 0$ we set $f_j := 2^{-j}f$ and we denote \mathcal{G}_j the graph of f_j . By construction, these graphs are the same in $[-1, 0] \times \mathbb{R}$ but do not meet in $(0, 1] \times \mathbb{R}$. Let us define,

$$\Sigma := \bigcup_{j \geq 0} \mathcal{G}_j \times [2^{-j-1}, 2^{-j}] \subset \mathbb{R}^3.$$

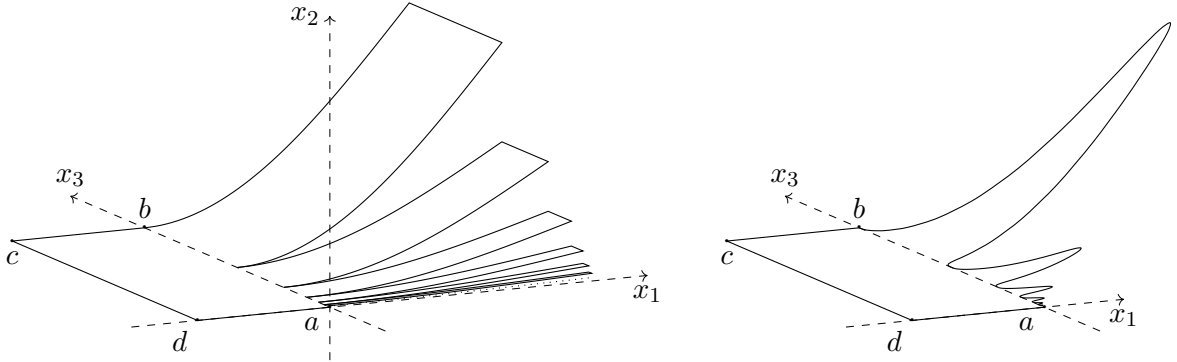


Figure 1: On the left, the set Σ , on the right, the 2-manifold \mathcal{M} of Example 3.

The set Σ is of dimension 2, contains the rectangle $[-1/2, 0] \times \{0\} \times [0, 1]$ and has Lipschitz regularity away from the cusp points $(0, 0, 2^{-j})$ for $j \geq 1$. To obtain a Lipschitz manifold, we cut just below these points in the first coordinate. We set for $j \geq 1$,

$$C_j := \{(x_1, x_2, x_3) : x_2 \in \mathbb{R}, 2^{-j} \leq |x_3| < 2^{1-j}, x_1 < 3^{-j} \sin^2(2^{-j} \pi x_3)\},$$

and then $\mathcal{M} := \Sigma \cap \bigcup_j C_j$, see Figure 1.

We have built a Lipschitz connected 2-manifold with boundary in \mathbb{R}^3 . Moreover, \mathcal{M} is of class C^1 away from the four vertices a, b, c, d of the rectangle $[-1/2, 0] \times \{0\} \times [0, 1]$. By construction $\mathcal{M} \subset \Sigma^1 \times \mathbb{R}$ where Σ^1 is the countable union of the graphs \mathcal{G}_j . This time, we do not have $\mathcal{M} = \mathcal{M}^1 \times \mathcal{M}^2$, neither $\mathcal{M} \subset \mathcal{M}^1 \times \mathcal{M}^2$ with $\mathcal{M}^1 \subset \mathbb{R}^2$, $\mathcal{M}^2 \subset \mathbb{R}$ connected 1-manifolds.

The tangent planes of \mathcal{M} satisfy the $(1, 1)$ -splitting property with $n_1 = 2$ and $n_2 = 1$. Moreover, orienting \mathcal{M} arbitrarily, the 2-chain A with multiplicity 1 associated with the integration on \mathcal{M} has finite boundary mass (the boundary of \mathcal{M} has finite length) so the theorem applies to A .

Example 4. We now turn to a more complex example. Let $f_1, f_2 \in C^\infty(\mathbb{R}, [0, 1])$ with

$$f_1 \equiv 0 \text{ on } \mathbb{R} \setminus (-2, 2), \quad f_1 \equiv 1 \text{ on } [-1, 1], \quad f_2 \equiv 0 \text{ on } \mathbb{R} \setminus (-1, 1), \quad f_2 > 0 \text{ on } (-1, 1).$$

We define for $x \in \mathbb{R}^2$, $f(x_1, x_2) := f_1(x_1)f_2(x_2)$ and we denote $Q := ([-2, -1] \cup [1, 2]) \times [-1, 1]$. This set is the union of two closed rectangles and $\partial_{x_1} f \equiv 0$ on $\mathbb{R}^2 \setminus Q$. Next, let $R \geq 10$ and let $x^{(j)}$ be a dense sequence of points in a ball B of \mathbb{R}^2 with radius R . We set, for $x \in \mathbb{R}^2$,

$$F(x) := \sum 8^{-j} f\left(2^j(x - x^{(j)})\right),$$

so that F is of class C^2 (observe that the second order differential of the terms of the series are uniformly bounded by $2^{-j}\|D^2f\|_\infty$). Denoting

$$E := \bigcup \left[x^{(j)} + 2^{-j}Q \right],$$

there holds, $\mathcal{H}^2(E) \leq 8/3$ and $\partial_{x_1}F = 0$ in $\mathbb{R}^2 \setminus E$. However, remark that for every open set $\omega \subset \mathbb{R}^2$ intersecting $B \setminus E$, there exists a set of real numbers y^2 with positive measure such that F is not constant on $\omega \cap \{x : x^2 = y^2\}$. Let us define \mathcal{G}_F as the graph of $F|_B$ and

$$\Sigma := \{(x, F(x)) : x \in B \setminus E\} = \mathcal{G}_f \cap [(\mathbb{R}^2 \setminus E) \times \mathbb{R}].$$

Orienting arbitrarily \mathcal{G}_f , we define A as the 2-chain with multiplicity 1 corresponding to the integration over Σ . This chain has finite positive mass and finite boundary mass and with $n_1 = 1$ and $n_2 = 2$, $T\mu_A$ is $(1, 1)$ -split. The theorem applies and we get that $\Sigma \subset \mathbb{R} \times \Sigma^2$ where, up to negligible set, $\Sigma^2 \subset \mathbb{R}^2$ is a countable union of C^1 -curves.

The point of this example is that even though, like in Example 3, $A = A \llcorner \mathcal{G}_F$ with \mathcal{G}_F diffeomorphic to the 2-dimensional ball, we cannot cover \mathbb{R}^3 by countably many balls B_j such that $A \llcorner B_j$ is the integration over $(\mathcal{M}_j^1 \times \mathcal{M}_j^2) \cap B_j$ for some C^1 curves $\mathcal{M}_j^1 \subset \mathbb{R}$, $\mathcal{M}_j^2 \subset \mathbb{R}^2$.

Example 5. To stress the necessity of the assumption $\mathbb{M}(\partial A) < \infty$ in Theorem 1.3, we build an example for which the conclusion of the theorem fails. More precisely, A is rectifiable, $T\mu_A$ is (k_1, k_2) -split but A is not (k_1, k_2) -rectifiable. Here, $k = n_1 = n_2 = 1$. We first introduce the following positive atomic measure on \mathbb{R} .

$$\nu := \frac{1}{2} \sum_{j \geq 0} \frac{1}{4^j} \sum_{i=1}^{2^j} \delta_{i/2^j},$$

We have $\nu(\mathbb{R}) = 1$ and $\nu = \nu \llcorner D$ where D is the set of dyadic numbers in $(0, 1]$. More precisely:

$$\nu = \frac{2}{3} \sum_{x \in D} \frac{1}{4^{j(x)}} \delta_x,$$

where for $x \in D$, $j(x) := j$ is the nonnegative integer such that $x = i2^{-j}$ for some odd number $0 < i < 2^j$. Next, for $x_1 \in \mathbb{R}$, we set $f(x_1) := \nu((-\infty, x_1))$. We have that $f' = \nu$ in the sense of distributions, f is left-continuous, increasing on $[0, 1]$ and differentiable on $\mathbb{R} \setminus D$ with $f'(x_1) = 0$ in the classical sense. We denote \mathcal{G}_f the graph of the restriction of f to $(0, 1)$. Remark that the closure of the projection of \mathcal{G}_f on the vertical axis is a classical Cantor set. Since $f' = 0$ almost everywhere, the 1-chain A_1 with support $\overline{\mathcal{G}_f}$, orientation e_1 and multiplicity 1 is rectifiable, see Figure 2.

Now, on the one hand, since $f' \equiv 0$ on $\mathbb{R} \setminus D$, $T\mu_{A_1}$ is $(1, 0)$ -split (at every point $(x_1, f(x_1))$ with $x_1 \in (0, 1) \setminus D$). On the other hand, f being increasing on $(0, 1)$, for any countable set $\Sigma^2 \subset \mathbb{R}$, the intersection $\mathcal{G}_f \cap \mathbb{R} \times \Sigma^2$ is at most countable and therefore $\mathcal{H}^1(\mathcal{G}_f \cap (\mathbb{R} \times \Sigma^2)) = 0$. Since $\mu_{A_1} \ll \mathcal{H}^1 \llcorner \mathcal{G}_f$, we also have $\mu_{A_1}(\mathbb{R} \times \Sigma^2) = 0 \neq 1 = \mu_{A_1}(\mathbb{R}^2)$. This implies that μ_{A_1} is not $(1, 0)$ -rectifiable (the conclusion of the theorem is wrong). Here the theorem does not apply because A has infinite boundary mass. This example shows that some control on ∂A is necessary.

Example 6. Let us rule out a possible generalization of the theorem suggested by the weaker assumption (2.1). We consider the following variant of the splitting condition where the dimensions k_1, k_2 may vary.

$$\text{for } \mu\text{-a.e. } x, T_x\mu = L^1(x) \times L^2(x) \text{ for some } k'_l(x)\text{-planes } L^l(x) \subset \mathbb{R}^{m_l}, l = 1, 2. \quad (2.4)$$

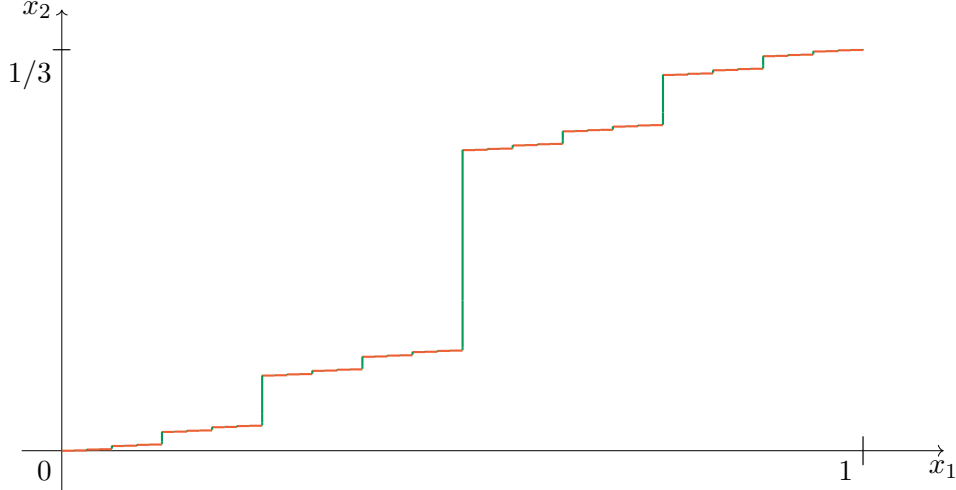


Figure 2: An approximate representation of the “horizontal” chain A_1 (in red) and of the “vertical” chain A_2 (in green) of Examples 5 and 6.

A natural question is whether, when A is a rectifiable normal k -chain such that $\mu = \mu_A$ satisfies (2.4), there holds,

$$\mu_A = \mu_A \llcorner \bigcup_{k'_1} \Sigma_{k'_1}^1 \times \Sigma_{k-k'_1}^2, \quad (2.5)$$

for some k'_l -rectifiable sets $\Sigma_{k'_l}^l \subset \mathbb{R}^{n_l}$ for $0 \leq k'_l \leq k$ and $l = 1, 2$. The answer is *no* and we get a counterexample by building on the construction of Example 5.

Let A_2 be the sum of the 1-chains with multiplicity 1, orientation e_2 and supports $V_{x_1} := \{x_1\} \times [f(x_1), f(x_1^+)]$ for $x_1 \in D \setminus \{1\}$, see again Figure 2. The chain A_2 “fills the holes” of A_1 in the sense that $\partial(A_1 + A_2) = \llbracket(1, 1/3)\rrbracket - \llbracket(0, 0)\rrbracket$. Besides A_2 is $(0, 1)$ -rectifiable so that $T\mu_{A_2}$ is $(0, 1)$ -split. Summing up, the rectifiable chain $A = A_1 + A_2$ has finite boundary mass and μ_A satisfies (2.4). However, for any countable sets $\Sigma^1, \Sigma^2 \subset \mathbb{R}$ we have,

$$\mu_A(\mathbb{R}^2 \setminus (\Sigma^1 \times \mathbb{R} \cup \mathbb{R} \times \Sigma^2)) \geq \mu_{A_1}(\mathbb{R}^2 \setminus (\Sigma^1 \times \mathbb{R} \cup \mathbb{R} \times \Sigma^2)) = \mu_{A_1}(\mathbb{R}^2 \setminus \Sigma^1 \times \mathbb{R}) = 1.$$

Consequently, (2.5) does not hold.

Remark 2.2. The proof of the main result presented in this paper takes a long detour by introducing tensor flat chains. As building a theory with a single application might be seen as overkill we could look for a more direct method. However, mimicking the smooth case arguments of Example 2 in the general case seems hopeless. As illustrated by Examples 3 & 4, the presence of a boundary with a possibly dense support is an obstacle. Even in the very favorable situation where A is a rectifiable cycle with multiplicity 1, the assumptions do not ensure that the restriction of A to a k -surface of class C^1 has a reasonable boundary.

3 G -flat chains

Here we recall the theory of G -flat chains in \mathbb{R}^n as introduced by Fleming in [Fle66]. We also recall the notion of slices of G -flat chains by m -planes and the rectifiable slices theorem of White [Whi99b]. As it is now standard and contrarily to [Fle66], we use the notation \llcorner and not \cap for restrictions, the notation \cap is used for slicings. Two remarks are in order. First, to avoid the introduction of compact sets which are not relevant here, the flat chains we consider are not assumed to be compactly supported. Second, in addition to the classical definition of

the slice $A \cap \mathbb{X}_{\bar{\gamma}}(x)$ which is a flat chain in \mathbb{R}^n supported in $\mathbb{X}_{\bar{\gamma}}(x)$, we introduce in (3.5) the object $\text{Sl}_{\bar{\gamma}}^x A$ which is the flat chain in $\mathbb{X}_{\bar{\gamma}}$ obtained by projecting $A \cap \mathbb{X}_{\bar{\gamma}}(x)$ on $\mathbb{X}_{\bar{\gamma}} = \mathbb{X}_{\bar{\gamma}}(0)$. In anticipation of the generalization of Section 4 we describe the basic theory in detail. The expert reader can skip most of the section, however, besides the unusual slicing operators $\text{Sl}_{\bar{\gamma}}$, let us point out the following less standard or new results:

- (1) Proposition 3.6 gives a formula for the boundary of the restriction of a chain on a half-space.
- (2) Proposition 3.12 expresses the (k_1, k_2) -splitting property (recall Definition 1.2) by the vanishing of some family of slices.
- (3) Theorem 3.14 from [GM22b] asserts that normal rectifiable chains can be decomposed in set-indecomposable subchains.

Let $(G, +, |\cdot|_G)$ be a complete Abelian normed group, let $\beta \in I^n$ and $0 \leq k \leq |\beta|$.

We say that the pair $p = (q, \xi)$ is a k -cell of \mathbb{X}_{β} if $q \subset \mathbb{X}_{\beta}$ is a closed convex k -polyhedron and $\xi \in \bigwedge_k \mathbb{X}_{\beta}$ is a unit simple k -vector orienting the affine k -plane spanned by q . There are two possible choices of orientation and the k -cell $(q, -\xi)$ is denoted $-p$. The multivector ξ is the orientation of p and is denoted ξ_p . The polyhedron q , denoted $\text{supp } p$, is the support of p . However, for set operations such as unions and intersections, we usually write p for $\text{supp } p$. As usual the 0-cell with support x and orientation 1 is denoted $\llbracket x \rrbracket$ and for $x \neq y \in \mathbb{X}_{\beta}$, the 1-cell supported by $[x, y]$ and with orientation $(y - x)/|y - x|$ is denoted $\llbracket (x, y) \rrbracket$. If q is a $|\beta|$ -polyhedron, we denote $\llbracket \mathbf{1}_q \rrbracket$ the $|\beta|$ -cell supported by q and with orientation $e_{\beta_1} \wedge e_{\beta_2} \wedge \dots \wedge e_{\beta_{|\beta|}}$ (referred to as *positive* orientation).

The group $\mathcal{P}_k^G(\mathbb{X}_{\beta})$ of G -polyhedral k -chains in \mathbb{X}_{β} is the group of formal finite sums,

$$P = \sum g_i p_i, \quad g_i \in G, \quad p_i \text{ } k\text{-cells in } \mathbb{X}_{\beta}, \quad (3.1)$$

quotiented by the following relations : for $g, g' \in G$ and p, p', p'' , k -cells,

- (i) $gp + g(-p) = 0$;
- (ii) $gp + g'p = (g + g')p$;
- (iii) $gp' + gp'' = gp$ if $\xi_p = \xi_{p'} = \xi_{p''}$ and, as sets, $p = p' \cup p''$ with p' and p'' intersecting on a common $(k - 1)$ -face.

The representation (3.1) of $P \in \mathcal{P}_k^G(\mathbb{X}_{\beta})$ is not unique. The mass of P is defined as,

$$\mathbb{M}(P) := \min \left\{ \sum |g_i|_G \mathcal{H}^k(p_i) \right\}, \quad (3.2)$$

where the infimum runs over all the representations of P .⁴

The boundary ∂p of a k -cell p is the sum of its $(k - 1)$ -faces p'_j with orientation ξ'_j such that $\nu_j \wedge \xi'_j = \xi_p$ where ν_j is the exterior normal to p on p'_j . Coming back to (3.1), it is easy to see that the $(k - 1)$ -chain $\sum g_i \partial p_i$ does not depend on the particular representation of P and we can safely define the boundary of P as

$$\partial P := \sum g_i \partial p_i.$$

For instance for $g \in G$ and $x, y \in \mathbb{X}_{\beta}$,

$$\partial(g \llbracket x \rrbracket) = 0 \quad \text{and} \quad \partial(g \llbracket (x, y) \rrbracket) = g \llbracket y \rrbracket - g \llbracket x \rrbracket.$$

⁴This infimum is reached at any representation such that $|g_i|_G |g_j|_G \mathcal{H}^k(p_i \cap p_j) = 0$ whenever $i \neq j$.

We obtain a group morphism $\partial : \mathcal{P}_k^G(\mathbb{X}_\beta) \rightarrow \mathcal{P}_{k-1}^G(\mathbb{X}_\beta)$ satisfying $\partial^2 = 0$. Whitney's flat norm is then given for polyhedral chains by:

$$\mathbb{F}(P) := \inf \{ \mathbb{M}(Q) + \mathbb{M}(R) : Q \in \mathcal{P}_k^G(\mathbb{X}_\beta), R \in \mathcal{P}_{k+1}^G(\mathbb{X}_\beta), P = Q + \partial R \}.$$

It turns out that \mathbb{F} is a norm on $(\mathcal{P}_k(\mathbb{X}_\beta), +)$ ([Fle66, Theorem 2.2]) moreover ∂ is continuous with respect to the \mathbb{F} -norm. Indeed writing $P = Q + \partial R$, we have $\partial P = \partial Q$ and $\mathbb{F}(\partial P) \leq \mathbb{M}(Q)$ which yields $\mathbb{F}(\partial P) \leq \mathbb{F}(P)$ by taking the infimum over the possible representations. The group $\mathcal{F}_k^G(\mathbb{X}_\beta)$ of k -chains in \mathbb{X}_β and with coefficients in G is the completion of $(\mathcal{P}_k^G(\mathbb{X}_\beta), +, \mathbb{F})$. Unless specified otherwise, we always intend convergence of flat chains in \mathbb{F} -norm. The boundary operator extends by continuity as a 1-Lipschitz continuous morphism $\partial : \mathcal{F}_k^G(\mathbb{X}_\beta) \rightarrow \mathcal{F}_{k-1}^G(\mathbb{X}_\beta)$. In summary, the sequence $(\mathcal{F}_k^G(\mathbb{X}_\beta))_{k \in \mathbb{Z}}$ endowed with the norm \mathbb{F} and the operator ∂ forms a chain complex of normed groups:

$$\dots \xrightarrow{\partial} \mathcal{F}_{k+1}^G(\mathbb{X}_\beta) \xrightarrow{\partial} \mathcal{F}_k^G(\mathbb{X}_\beta) \xrightarrow{\partial} \mathcal{F}_{k-1}^G(\mathbb{X}_\beta) \xrightarrow{\partial} \dots$$

The mass of $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$ is then defined as,

$$\mathbb{M}(A) := \inf \left\{ \liminf \mathbb{M}(P_j) : \mathcal{P}_k^G(\mathbb{X}_\beta) \ni P_j \xrightarrow{\mathbb{F}} A \right\}.$$

By [Fle66, Theorem 2.3], the function \mathbb{M} first introduced in (3.2) is lower semi-continuous with respect to \mathbb{F} -convergence in the group of polyhedral chains, so the definition of $\mathbb{M}(P)$ is not ambiguous. We have the following properties.

Proposition 3.1 ([Fle66, Theorem 3.1] for (ii), [Fle66, Theorem 5.6.] for (iii)). *Let $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$.*

(i) $\mathbb{F}(\partial A) \leq \mathbb{F}(A) \leq \mathbb{M}(A)$ with possibly $\mathbb{M}(A) = \infty$.

(ii) $\mathbb{F}(A) = \inf \{ \mathbb{M}(B) + \mathbb{M}(C) : B \in \mathcal{F}_k^G(\mathbb{X}_\beta), C \in \mathcal{F}_{k+1}^G(\mathbb{X}_\beta) \text{ such that } A = B + \partial C \}$.

The limit case $k = |\beta|$

We assume here that $k = |\beta|$. Since $\mathcal{F}_{k+1}^G(\mathbb{X}_\beta) = \{0\}$, there holds $\mathbb{F} = \mathbb{M}$ on $\mathcal{F}_k^G(\mathbb{X}_\beta)$. Moreover, a G -polyhedral k -chain of the form (3.1) identifies with the function $f_P : \mathbb{X}_\beta \rightarrow G$ given by

$$f_P := \sum \langle \xi_{p_i}, e_\beta \rangle g_i \mathbf{1}_{\text{supp } p_i},$$

(here the factors $\langle \xi_{p_i}, e_\beta \rangle = \pm 1$ account for the orientation). There holds $\|f_P\|_{L^1} = \mathbb{M}(P)$ and as the f_P 's form a dense subgroup of $L^1(\mathbb{X}_\beta, G)$, we have in fact the isometry,

$$(\mathcal{F}_{|\beta|}^G(\mathbb{X}_\beta), \mathbb{F}) = (\mathcal{F}_{|\beta|}^G(\mathbb{X}_\beta), \mathbb{M}) \sim (L^1(\mathbb{X}_\beta, G), \|\cdot\|_{L^1}).$$

Subgroups of chains

We denote by $\mathcal{M}_k^G(\mathbb{X}_\beta)$, the subgroup of finite mass k -chains in \mathbb{X}_β with coefficients in G . This group, endowed with \mathbb{M} is complete but the family $(\mathcal{M}_*^G(\mathbb{X}_\beta), \partial)$ does not form a chain complex. Denoting $\mathbb{N}(A) := \mathbb{M}(A) + \mathbb{M}(\partial A)$, we define the group of normal flat k -chains,

$$\mathcal{N}_k^G(\mathbb{X}_\beta) := \{ A \in \mathcal{F}_k^G(\mathbb{X}_\beta) : \mathbb{N}(A) < \infty \}.$$

Again, $(\mathcal{N}_k^G(\mathbb{X}_\beta), \mathbb{N})$ is a complete normed group. Moreover, since $\mathbb{N}(\partial A) = \mathbb{M}(\partial A) \leq \mathbb{N}(A)$, the family $(\mathcal{N}_*^G(\mathbb{X}_\beta), \partial)$ forms a chain complex of normed groups.

A chain such that $\partial A = 0$ is called a cycle (the finite mass k -cycles form a closed subgroup of $\mathcal{N}_k^G(\mathbb{X}_\beta)$ and the sequence of subgroups of finite mass cycles also form a chain complex).

Supports

We say that the flat chain $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$ is supported in the closed set $S \subset \mathbb{X}_\beta$ if for any neighborhood U of S , there exists a sequence $P_j \in \mathcal{P}_k^G(\mathbb{X}_\beta)$ which admits representations $P_j = \sum g_i^j p_i^j$ with $\text{supp } p_i^j \subset U$, such that $P_j \rightarrow A$ in $\mathcal{F}_k^G(\mathbb{X}_\beta)$. If there exists a smallest closed subset S supporting A , we write $\text{supp } A = S$. Finite mass flat chains admit a support and moreover for $A \in \mathcal{M}_k^G(\mathbb{X}_\beta)$ and any neighborhood U of $\text{supp } A$ there exists a sequence $P_j \in \mathcal{P}_k^G(\mathbb{X}_\beta)$ supported in U such that $P_j \rightarrow A$ with $\mathbb{M}(P_j) \rightarrow \mathbb{M}(A)$.

Tensor products of chains (see [Fle66, Section 6])

Let $\beta, \gamma \in I^n$ such that $\beta \cap \gamma = \emptyset$ and assume for simplicity that $\max \beta < \min \gamma$. Given two polyhedral cells $p^1 = (q^1, \xi^1)$ and $p^2 = (q^2, \xi^2)$ respectively in \mathbb{X}_β and \mathbb{X}_γ and $g \in G$, we define the tensor product $p^1 \wedge (gp^2)$ of p^1 and gp^2 as the chain $gp \in \mathcal{P}_*^G(\mathbb{X}_{\beta \cup \gamma})$ where p is the polyhedral chain with support $q^1 + q^2$ and orientation $\xi^1 \wedge \xi^2$. Given $0 \leq k_1, k_2 \leq n$, the definition extends by \mathbb{Z} bilinearity as a bilinear mapping

$$\begin{aligned} \mathcal{P}_{k_1}^{\mathbb{Z}}(\mathbb{X}_\beta) \times \mathcal{P}_{k_2}^G(\mathbb{X}_\beta) &\longrightarrow \mathcal{P}_{k_1+k_2}^G(\mathbb{X}_{\beta \cup \gamma}), \\ (P^1, P^2) &\longmapsto P^1 \times P^2. \end{aligned}$$

There holds,

$$\partial(P^1 \times P^2) = (\partial P^1) \times P^2 + (-1)^{k_1} P^1 \times (\partial P^2),$$

and we have the estimates,

$$\mathbb{M}(P^1 \times P^2) \leq \mathbb{M}(P^1)\mathbb{M}(P^2), \quad \mathbb{F}(P^1 \times P^2) \leq \mathbb{N}(P^1)\mathbb{F}(P^2), \quad \mathbb{F}(P^1 \times P^2) \leq \mathbb{F}(P^1)\mathbb{N}(P^2).$$

Using these estimates, and a density argument, we can define $A^1 \times A^2$ for $A^1 \in \mathcal{F}_{k_1}^{\mathbb{Z}}(\mathbb{X}_\beta)$, $A^2 \in \mathcal{F}_{k_2}^G(\mathbb{X}_\gamma)$ as soon as both A^1 and A^2 have finite mass or at least one of them is normal.

This construction foreshadows the definition of tensor chains in Section 4. It has the disadvantage of not being well defined for all pairs of flat chains A^1, A^2 . The groups of tensor flat chains correct this defect but inevitably contain objects which are not flat chains.

Push-forwards by Lipschitz maps and homotopies (see [Fle66, Sections 5&6])

Let $\gamma \in I^n$. A Lipschitz mapping $f : \mathbb{X}_\beta \rightarrow \mathbb{X}_\gamma$ induces a morphism of chain complexes of normed groups, $f_\# : \mathcal{F}_*^G(\mathbb{X}_\beta) \rightarrow \mathcal{F}_*^G(\mathbb{X}_\gamma)$. In particular, for $A \in \mathcal{F}_*^G(\mathbb{X}_\beta)$, there holds $\partial f_\# A = f_\# \partial A$. Moreover the following estimates hold true.

$$\mathbb{M}(f_\# A) \leq C^k \mathbb{M}(A), \quad \mathbb{F}(f_\# A) \leq \max(C^k, C^{k+1}) \mathbb{F}(A),$$

where C is a Lipschitz constant of f on $\text{supp } A$. This mapping has the following additional properties for $\gamma, \delta \in I^n$ and $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$.

- (1) If $f \in \text{Lip}(\mathbb{X}_\beta, \mathbb{X}_\gamma)$, and A is supported in $\{x \in \mathbb{X}_\beta : f(x) = 0\}$ then $f_\# A = 0$.
- (2) For $f \in \text{Lip}(\mathbb{X}_\beta, \mathbb{X}_\gamma)$ and $g \in \text{Lip}(\mathbb{X}_\gamma, \mathbb{X}_\delta)$ then $(g \circ f)_\# A = g_\# (f_\# A)$.
- (3) $\text{Id}_\# A = A$ and it follows from the previous points that if $g \circ f = \text{Id}$ on some closed set S supporting A then $g_\# (f_\# A) = A$.
- (4) If the image of $f \in \text{Lip}(\mathbb{X}_\beta, \mathbb{X}_\gamma)$ lies in some closed set S then $f_\# A$ is supported in S . If moreover $S = L$ is an affine subspace of \mathbb{X}_γ we have a stronger property: there exists a sequence of G -polyhedral chains P_j all supported in L such that $P_j \rightarrow f_\# A$ so that $f_\# A$ identifies with a chain in L .

Given, $f, g \in \text{Lip}(\mathbb{X}_\beta, \mathbb{X}_\gamma)$ and $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$, we define a linear homotopy from $f\# A$ to $g\# A$ as follows. Let $e_0 := (1, 0_{\mathbb{R}^n}) \in \mathbb{R} \times \mathbb{R}^n$ and consider the mapping:

$$h : te_0 + x \in \mathbb{R}e_0 + \mathbb{X}_\beta \longmapsto tf(x) + (1-t)g(x) \in \mathbb{X}_\gamma,$$

We have the identity

$$f\# A - g\# A = \partial [h\# (\llbracket(0, e_0)\rrbracket \times A)] + h\# (\llbracket(0, e_0)\rrbracket \times \partial A), \quad (3.3)$$

and the estimate

$$\mathbb{M}(h\# (\llbracket(0, e_0)\rrbracket \times A)) \leq \|f - g\|_{L^\infty(\text{supp } A)} C^k \mathbb{M}(A), \quad (3.4)$$

where C is a common Lipschitz constant of f and g on $\text{supp } A$.

Slicing of G -flat chains

In this part, we recall the definition and properties of slices as introduced in [Whi99b, Section 3] with the small addition of the new operators Sl_γ (see below). We only consider slices by affine coordinate planes.

Let $\gamma \subset \beta$ and set $m := |\beta|$, $r := |\gamma|$. Given a G -polyhedral k -chain P with representation (3.1) and $x \in \mathbb{X}_\gamma$, if $r \leq k$, the slice of P with respect to $\mathbb{X}_{\beta \setminus \gamma}(x)$ is the $(k-r)$ -chain defined as,

$$P \cap \mathbb{X}_{\beta \setminus \gamma}(x) := \sum g_i q_i,$$

where $q_i = 0$ if the affine subspace spanned $p_i \cap \mathbb{X}_{\beta \setminus \gamma}(x)$ is not of dimension $k-r$ and in the other cases q_i is the $(k-r)$ -cell of $\mathbb{X}_{\beta \setminus \gamma}(x)$ described by (recall the notation (1.8)):

$$\begin{cases} \text{supp } q_i := \text{supp } p_i \cap \mathbb{X}_{\beta \setminus \gamma}(x) \\ \xi_{q_i} \text{ is the unit simple } (k-r)\text{-vector orienting } q_i \text{ such that } \langle e_\gamma \wedge \xi_{q_i}, \xi_{p_i} \rangle > 0. \end{cases}$$

When $r > k$, we set $P \cap \mathbb{X}_{\beta \setminus \gamma}(x) := 0$. Notice that for $x \in \mathbb{X}_\gamma$, the chain $P \cap \mathbb{X}_{\beta \setminus \gamma}(x)$ is supported in $\mathbb{X}_{\beta \setminus \gamma}(x)$. Identifying each $\mathbb{X}_{\beta \setminus \gamma}(x)$ with $\mathbb{X}_{\beta \setminus \gamma}$, it will be convenient to see all these chains as chains in $\mathbb{X}_{\beta \setminus \gamma}$. For this reason we define,

$$\text{Sl}_\gamma^x P := \pi\# (P \cap \mathbb{X}_{\beta \setminus \gamma}(x)) \in \mathcal{P}_k^G(\mathbb{X}_{\beta \setminus \gamma}), \quad (3.5)$$

where π is the orthogonal projection on $\mathbb{X}_{\beta \setminus \gamma}$. Denoting $\pi_x := x + \pi$ the orthogonal projection on $\mathbb{X}_\gamma(x)$, we see that $\pi_x : \mathbb{X}_{\beta \setminus \gamma} \rightarrow \mathbb{X}_{\beta \setminus \gamma}(x)$ is an affine isometry with inverse π . We deduce a reverse formula for (3.5),

$$P \cap \mathbb{X}_{\beta \setminus \gamma}(x) = \pi_{x\#} \text{Sl}_\gamma^x P.$$

The usual properties of $P \cap \mathbb{X}_\gamma(x)$ transfer to Sl_γ^x . We have in particular,

$$\partial \text{Sl}_\gamma^x P = \text{Sl}_\gamma^x \partial P \quad \text{for a.e. } x \quad \text{and} \quad \int_{\mathbb{X}_\gamma} \mathbb{M}(\text{Sl}_\gamma^x P) d\mathcal{H}^r(x) \leq \mathbb{M}(P),$$

so that Sl_γ extends on $\mathcal{F}_k^G(\mathbb{X}_\beta)$ as a Lipschitz continuous group morphism:

$$\begin{aligned} \text{Sl}_\gamma : \mathcal{F}_k^G(\mathbb{X}_\beta) &\longrightarrow L^1(\mathbb{X}_\gamma, \mathcal{F}_{k-r}^G(\mathbb{X}_{\beta \setminus \gamma})), \\ A &\longmapsto x \mapsto \text{Sl}_\gamma^x A. \end{aligned}$$

For the usual slicing operator, we have:

$$\begin{aligned} \mathcal{F}_k^G(\mathbb{X}_\beta) &\longrightarrow L^1(\mathbb{X}_\gamma, \mathcal{F}_{k-r}^G(\mathbb{X}_\beta)), \\ A &\longmapsto x \mapsto A \cap (x + \mathbb{X}_{\beta \setminus \gamma}). \end{aligned}$$

The two definitions of slicing contain the same information (same mass, same \mathbb{F} -norm, *etc*). The main difference is a translation by x and the fact that $A \cap (x + \mathbb{X}_{\beta \setminus \gamma})$ is a chain in \mathbb{X}_β supported in $x + \mathbb{X}_{\beta \setminus \gamma}$ whereas $\text{Sl}_\gamma^x A$ is a chain in $\mathbb{X}_{\beta \setminus \gamma}$. More precisely,

$$A \cap (x + \mathbb{X}_{\beta \setminus \gamma}) = \tau_{x\#} I\# \text{Sl}_\gamma^x A,$$

where I is the inclusion of $\mathbb{X}_{\beta \setminus \gamma}$ in \mathbb{X}_β and τ_x is the translation $y \in \mathbb{X}_\beta \mapsto x + y \in \mathbb{X}_\beta$. Conversely,

$$\text{Sl}_\gamma^x A = \pi\# \tau_{-x\#} (A \cap (x + \mathbb{X}_{\beta \setminus \gamma})),$$

where π is the projection on $\mathbb{X}_{\beta \setminus \gamma}$. The second definition (3.5) is not standard but useful in the proof of the main result.

Proposition 3.2 ([Whi99b, Proposition 3.1 (1,2)]). *Let $\gamma \subset \beta$. The operator Sl_γ is a group morphism satisfying the following properties for $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$.*

- (i) *There holds $\text{Sl}_\gamma^x \partial A = \partial \text{Sl}_\gamma^x A$ for almost every $x \in \mathbb{X}_\gamma$.*
- (ii) *For $\zeta \subset \gamma$ and almost every $x \in \mathbb{X}_\gamma$, writing $x = x' + \bar{x}$ with $x' \in \mathbb{X}_\zeta$, $\bar{x} \in \mathbb{X}_{\gamma \setminus \zeta}$, we have,*

$$\text{Sl}_\gamma^x A = \text{Sl}_\gamma^{\bar{x}} \text{Sl}_\zeta^{x'} A.$$

With the usual operator, this translates as,

$$A \cap \mathbb{X}_{\beta \setminus \gamma}(x) = [A \cap \mathbb{X}_{\beta \setminus \zeta}(x')] \cap \mathbb{X}_{\beta \setminus \gamma}(x).$$

- (iii) *Moreover, denoting $r := |\gamma|$, we have the estimates:*

$$\int_{\mathbb{X}_\gamma} \mathbb{M}(\text{Sl}_\gamma^x A) d\mathcal{H}^r(x) \leq \mathbb{M}(A), \quad \int_{\mathbb{X}_\gamma} \mathbb{F}(\text{Sl}_\gamma^x A) d\mathcal{H}^r(x) \leq \mathbb{F}(A).$$

In particular, if $A \in \mathcal{M}_k^G(\mathbb{X}_\beta)$ then $\text{Sl}_\gamma^x A \in \mathcal{M}_{k-r}^G(\mathbb{X}_{\beta \setminus \gamma})$ for almost every $x \in \mathbb{X}_\gamma$.

The collection of the slices of k -chain by coordinate planes of codimension k completely characterizes this chain.

Theorem 3.3 ([Whi99b, Theorem 3.2]). *If $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$ is such that $\text{Sl}_\gamma A = 0$ for every $\gamma \subset \beta$ with $|\gamma| = k$, then $A = 0$.*

Remark 3.4 (and vocabulary). For $0 \leq m \leq k$, the slices by planes of codimension $k - m$ are of dimension m (if A is a k -chain in \mathbb{X}_β , for $\gamma \subset \beta$ with $|\gamma| = m$, $\text{Sl}_\gamma^x A$ and $A \cap \mathbb{X}_{\beta \setminus \gamma}(x)$ are m -chains). They are called its m -slices. Theorem 3.3 states that a chain is determined by its 0-slices (and, using iterate slicing, in fact by its m -slices for any $m \in \{0, \dots, k\}$). Since finite mass 0-chains have a nice structure (as we see below they are measures), this result is of particular interest.

Finite mass G -flat chains

Let us now turn our attention to finite mass G -flat chains. To every $A \in \mathcal{M}_k^G(\mathbb{X}_\beta)$ is associated a mapping

$$S \in \{\text{Borel subsets of } \mathbb{X}_\beta\} \longmapsto A \llcorner S \in \mathcal{M}_k^G(\mathbb{X}_\beta),$$

and a finite Borel measure μ_A defined by

$$\mu_A(S) := \mathbb{M}(A \llcorner S).$$

The restriction $A \llcorner S$ corresponds to “the part of A in S ”. For a polyhedral chain P and a n -interval I , $P \llcorner I$ is defined in the natural way. The restriction enjoys the following properties for $A \in \mathcal{M}_k^G(\mathbb{X}_\beta)$ and S Borel subset of \mathbb{X}_β (see⁵ [Fle66, Lemma 4.2, Theorem 4.3])

⁵Beware that in Fleming’s paper the restriction is denoted \cap and not \llcorner .

- (i) If $A_j \in \mathcal{M}_k^G(\mathbb{X}_\beta)$ is such that $A_j \rightarrow A$ and $\mathbb{M}(A_j) \rightarrow \mathbb{M}(A)$ then $A_j \llcorner S \rightarrow A \llcorner S$ whenever $\mu_A(\text{Bdry } S) = 0$.
- (ii) A admits a support and if S is closed, $\text{supp}(A \llcorner S) = S \cap \text{supp } \mu_A$. In particular, $\text{supp } A = \text{supp } \mu_A$. If S is merely a Borel set, we still have $\text{supp}(A \llcorner S) \subset \bar{S} \cap \text{supp } \mu_A$.

We recall that the slicing and restriction operators commute.

Proposition 3.5 ([Whi99b, Proposition 3.1 (3)]). *Let $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$, let $S \subset \mathbb{X}_\beta$ be a Borel set and let $\gamma \subset \beta$. For almost every $x \in \mathbb{X}_\gamma$, there holds*

$$(A \llcorner S) \cap \mathbb{X}_{\beta \setminus \gamma}(x) = (A \cap \mathbb{X}_{\beta \setminus \gamma}(x)) \llcorner S = (A \cap \mathbb{X}_{\beta \setminus \gamma}(x)) \llcorner (S \cap \mathbb{X}_{\beta \setminus \gamma}(x)).$$

Using the Sl_γ operator, this rewrites as,

$$\text{Sl}_\gamma^x(A \llcorner S) = (\text{Sl}_\gamma^x A) \llcorner (S - x) = (\text{Sl}_\gamma^x A) \llcorner [(S - x) \cap \mathbb{X}_{\beta \setminus \gamma}(x)].$$

The next proposition is a key tool in the proof of Proposition 6.2. We recall that $\bar{1} = \{2, \dots, n\}$.

Proposition 3.6. *Let $A \in \mathcal{N}_k^G(\mathbb{R}^n)$. Denoting $H(s) := \{x \in \mathbb{R}^n : x_1 > s\}$ then $A \llcorner H(s) \in \mathcal{N}_k^G(\mathbb{R}^n)$ for almost every $s \in \mathbb{R}$ and the following identity holds true.*

$$\partial(A \llcorner H(s)) = (\partial A) \llcorner H(s) + A \cap \mathbb{X}_{\bar{1}}(se_1).$$

Notice that $\mathbb{X}_{\bar{1}}(se_1) = \text{Bdry } H(s)$.

Proof. Let $A \in \mathcal{N}_k^G(\mathbb{R}^n)$. By [Fle66, Theorem 5.6], there exists $Q_j \in \mathcal{P}_k^G(\mathbb{R}^n)$ such that $Q_j \rightarrow A$, $\mathbb{M}(Q_j) \rightarrow \mathbb{M}(A)$ and $\mathbb{M}(\partial Q_j) \rightarrow \mathbb{M}(\partial A)$. By [Fle66, Lemma 4.2] we have for almost every $s \in \mathbb{R}$ (recall that convergence is intended in \mathbb{F} -norm),

$$Q_j \llcorner H(s) \rightarrow A \llcorner H(s), \quad (\partial Q_j) \llcorner H(s) \rightarrow (\partial A) \llcorner H(s). \quad (3.6)$$

Moreover, the proposition is immediate for polyhedral chains, so,

$$\partial(Q_j \llcorner H(s)) - (\partial Q_j) \llcorner H(s) = Q_j \cap \mathbb{X}_{\bar{1}}(se_1). \quad (3.7)$$

By Proposition 3.2, the mapping $s \mapsto Q_j \cap \mathbb{X}_{\bar{1}}(se_1)$ converges towards $s \mapsto A \cap \mathbb{X}_{\bar{1}}(se_1)$ in $L^1(\mathbb{R}, \mathcal{F}_{k-1}^G(\mathbb{R}^n))$. Consequently, up to extraction, $Q_j \cap \mathbb{X}_{\bar{1}}(se_1) \rightarrow A \cap \mathbb{X}_{\bar{1}}(se_1)$. Passing to the limit in (3.7) and using (3.6) yield the result. \square

Let $\mathcal{M}(\mathbb{X}_\beta, G)$ be the group of G -valued Borel measures in \mathbb{X}_β . The total variation of $\nu \in \mathcal{M}(\mathbb{X}_\beta, G)$ is defined as,

$$|\nu| := \sup \left\{ \sum |\nu(S_j)|_G : S_j \text{ Borel partition of } \mathbb{X}_\beta \right\}.$$

We recall that $(\mathcal{M}(\mathbb{X}_\beta, G), +, |\cdot|)$ is a complete Abelian normed group and that finite mass 0-chains identify with G -valued Borel measures.

Theorem 3.7 ([Whi99b, Theorem 2.1]). *There exists an isometric isomorphism,*

$$\psi : (\mathcal{M}_0^G(\mathbb{X}_\beta), +, \mathbb{M}) \longrightarrow (\mathcal{M}(\mathbb{X}_\beta, G), +, |\cdot|).$$

Remark 3.8. Using this identification, given finite sequences $x_i \in \mathbb{X}_\beta$ and $g_i \in G$, we may write $\sum g_i \delta_{x_i}$ for the polyhedral 0-chain $\sum g_i \llbracket x_i \rrbracket$.

Morphisms of the group of coefficients

Let $(G^a, +, |\cdot|_{G^a})$ and $(G^b, +, |\cdot|_{G^b})$ be two complete Abelian normed groups and $\phi : G^a \rightarrow G^b$ be a Lipschitz continuous group morphism. For $P^a \in \mathcal{P}_k(\mathbb{X}_\beta, G^a)$ with representation,

$$P^a = \sum g_j^a p_j, \quad (3.8)$$

we define

$$\phi_* P^a := \sum \phi(g_j^a) p_j.$$

This mapping extends as a Lipschitz continuous morphism:

$$\phi_* : \mathcal{F}_k(\mathbb{X}_\beta, G^a) \longrightarrow \mathcal{F}_k(\mathbb{X}_\beta, G^b).$$

Moreover, for every $A \in \mathcal{F}_k(\mathbb{X}_\beta, G^a)$, there holds

$$\partial \phi_* A = \phi_* \partial A \quad \text{and} \quad \mathbb{M}(\phi_* A) \leq \text{Lip}(\phi) \mathbb{M}(A),$$

where $\text{Lip}(\phi)$ is the Lipschitz constant of ϕ . More generally, if (Ω, μ) is a measured space and

$$\Phi : G^a \rightarrow L^1(\Omega, G^b),$$

is a Lipschitz continuous group morphism, the mapping defined, for P^a of the form (3.8) and $\omega \in \Omega$, by

$$(\Phi_* P^a)(\omega) := \sum [\Phi(g_j^a)](\omega) p_j,$$

extends as a Lipschitz continuous morphism:

$$\Phi_* : \mathcal{F}_k(\mathbb{X}_\beta, G^a) \longrightarrow L^1\left(\Omega, \mathcal{F}_k(\mathbb{X}_\beta, G^b)\right).$$

Proposition 3.9. *With the above notation, for every $A \in \mathcal{F}_k(\mathbb{X}_\beta, G^a)$, there hold*

$$\phi_* \text{Sl}_\gamma A = \text{Sl}_\gamma \phi_* A, \quad (\Phi_* \text{Sl}_\gamma A)(\omega) = \text{Sl}_\gamma [\Phi_* A(\omega)] \quad \text{for almost every } \omega \in \Omega.$$

Similarly, for every $A \in \mathcal{M}_k(\mathbb{X}_\beta, G^a)$ and every Borel set $S \subset \mathbb{X}_\beta$, there hold

$$\phi_* (A \llcorner S) = (\phi_* A) \llcorner S, \quad [\Phi_* (A \llcorner S)](\omega) = [\Phi_* A(\omega)] \llcorner S \quad \text{for almost every } \omega \in \Omega.$$

Proof. These identities hold true for polyhedral chains and we get the general cases by a continuity/density argument. \square

Rectifiable chains

It follows from the deformation theorem of White (see [Whi99a, Theorem 3.1]) that given a k -chain $A \in \mathcal{M}_k^G(\mathbb{X}_\beta)$ and a Borel set $S \subset \mathbb{X}_\beta$ if $\mathcal{H}^k(S) = 0$ or even if S has vanishing k -integral geometric measure then $A \llcorner S = 0$.⁶ This means that μ_A cannot concentrate more than \mathcal{H}^k . The limit case corresponds to rectifiable chains defined below.

Let us recall that a Borel set $S \subset \mathbb{X}_\beta$ is (countably) k -rectifiable if $S \subset \Sigma_0 \cup (\cup_{j \geq 1} \Sigma_j)$ where the Σ_j 's are Borel subsets of \mathbb{X}_β such that $\mathcal{H}^k(\Sigma_0) = 0$ and for $j \geq 1$, Σ_j is a C^1 k -manifold.

Definition 3.10.

⁶In [Whi99a, Theorem 3.1] the result is stated under the stronger assumption $\mathcal{H}^k(\text{supp } A) = 0$, but the present statement transpires from some arguments of [Whi99b] and follows rather directly from [Whi99b, Corollary 6.1].

- (1) A finite Borel measure μ on \mathbb{X}_β is k -rectifiable if $\mu = \mu \llcorner S$ for some k -rectifiable set $S \subset \mathbb{X}_\beta$ and $\mu \ll \mathcal{H}^k \llcorner S$. Observe that the set Σ_0 does not play any role here.
- (2) A G -flat chain $A \in \mathcal{M}_k^G(\mathbb{X}_\beta)$ is rectifiable if $A = A \llcorner S$ for some k -rectifiable set $S \subset \mathbb{X}_\beta$. Equivalently μ_A is a k -rectifiable measure.

Remark that if by definition rectifiable chains have finite mass they are not assumed to be compactly supported.

By Theorem 3.7, finite mass 0-chains identify with finite measures with values in G and rectifiable 0-chains in \mathbb{X}_β are the atomic measures (of the form $\sum g_j \llcorner x_j$ with $x_j \in \mathbb{X}_\beta$, $g_j \in G$ and $\sum |g_j|_G < \infty$). Given a rectifiable k -chain A in \mathbb{X}_β and $\gamma \subset \beta$ with $|\gamma| = k$, for almost every $x \in \mathbb{X}_\gamma$, $\text{Sl}_\gamma^x A$ is a rectifiable 0-chain in $\mathbb{X}_{\bar{\gamma}}$. White's rectifiable slices theorem (recalled below) states that the converse is also true. This is a crucial tool in our proof of the main result.

Theorem 3.11 ([Whi99b]). *Let $A \in \mathcal{M}_k^G(\mathbb{X}_\beta)$. The following are equivalent.*

- (i) A is rectifiable.
- (ii) For every $\gamma \subset \beta$ with $|\gamma| = k$ and almost every $x \in \mathbb{X}_\gamma$, $\text{Sl}_\gamma^x A$ is a rectifiable 0-chain.

The (k_1, k_2) -splitting property (recall Definition 1.2) can be characterized in terms of slices.

Proposition 3.12. *Let $A \in \mathcal{M}_k^G(\mathbb{R}^n)$ be rectifiable and let $n_1, n_2, k_1, k_2 \geq 0$ with $n_1 + n_2 = n$ and $k_1 + k_2 = k$. The following statements are equivalent.*

- (i) $T\mu_A$ is (k_1, k_2) -split.
- (ii) $\text{Sl}_\gamma A = 0$ for every $\gamma \in I_k^n$ such that $(|\gamma^1|, |\gamma^2|) \neq (k_1, k_2)$.

Proof. Without loss of generality, we assume that $A \in \mathcal{M}_k^G(\mathbb{R}^n)$ is such that $A = A \llcorner \Sigma$ where Σ is a compact k -manifold of class C^1 (with boundary).

Step 1, (i) \implies (ii). Let $x \in \Sigma \mapsto \xi(x) \in \bigwedge_k(\mathbb{R}^n)$ be a Borel mapping such that for every $x \in \Sigma$, $\xi(x)$ is a unit k -vector orienting $T_x \Sigma$ and let $\gamma \in I_k^n$. Let us define,

$$D := \{x \in \mathbb{X}_\gamma : A \cap \mathbb{X}_{\bar{\gamma}}(x) \neq \emptyset\}.$$

Denoting by π the orthogonal projection on \mathbb{X}_γ we have that, up to a \mathcal{H}^k -negligible set, $D \subset \pi\Sigma$. Indeed, let $x \in \mathbb{X}_\gamma \setminus \pi\Sigma$. Since Σ is compact, there exists $r > 0$ such that denoting $B'_r(x) := \{y \in \mathbb{X}_\gamma : |y| < r\}$ we have $\Sigma \cap B'_r(x) + \mathbb{X}_{\bar{\gamma}} = \emptyset$. It follows that

$$A \llcorner [B'_r(x) + \mathbb{X}_{\bar{\gamma}}] = (A \llcorner \Sigma) \llcorner [B'_r(x) + \mathbb{X}_{\bar{\gamma}}] = A \llcorner (\Sigma \cap [B'_r(x) + \mathbb{X}_{\bar{\gamma}}]) = 0.$$

Hence $A \cap \mathbb{X}_{\bar{\gamma}}(y) = 0$ \mathcal{H}^k -almost everywhere in $B'_r(x)$ and $D \cap B'_r(x)$ is negligible. We conclude that $D \subset \pi\Sigma$ as claimed.

Denoting $S := \Sigma \cap (D + \mathbb{X}_{\bar{\gamma}})$, we have $D = \pi S$ by the previous inclusion and using the co-area formula [AFP00, Theorem 2.93], we compute,

$$\mathcal{H}^k(D) \leq \int_{\mathbb{X}_\gamma} \mathcal{H}^0(\mathbb{X}_{\bar{\gamma}}(y) \cap S) dy = \int_S |\langle e_\gamma, \xi(x) \rangle| d\mathcal{H}^k(x). \quad (3.9)$$

Next, let $S' \subset \mathbb{R}^n$ be a Borel set such that $\mu_A(S') = 0$. We have $A \llcorner S' = 0$ and using Proposition 3.5 we see that for almost every $x \in \mathbb{X}_\gamma$,

$$[A \cap \mathbb{X}_{\bar{\gamma}}(x)] \llcorner S' = [A \llcorner S'] \cap \mathbb{X}_{\bar{\gamma}}(x) = 0.$$

It follows that $\mathcal{H}^k(S \cap S') = 0$. We deduce that,

$$\mathcal{H}^k \llcorner S \ll \mu_A. \quad (3.10)$$

Let us now assume that (i) holds true. For γ such that $(|\gamma^1|, |\gamma^2|) \neq (k_1, k_2)$, we have by assumption $\langle e_\gamma, \xi \rangle = 0$ μ_A -almost everywhere. By (3.10), this holds true $\mathcal{H}^k \llcorner S$ -almost everywhere and the right-hand side of (3.9) vanishes. We conclude that $\text{Sl}_\gamma A = 0$ so (ii) holds true.

Step 2, (ii) \implies (i). Conversely, let us assume that A satisfies (ii) and let us suppose by contradiction that $T\mu_A$ is not (k_1, k_2) -split. In this case there exist a k -manifold Σ of class C^1 , a point x on Σ and $r > 0$ such that $T_x \Sigma$ is well defined and *not* (k_1, k_2) -split and denoting $A' := A \llcorner \Sigma$, there holds,

$$\mu_{A'}(B_s(x)) > 0 \quad \text{for } s \in (0, r). \quad (3.11)$$

Let us assume without loss of generality that $x = 0$. Since $T_0 \Sigma$ is not (k_1, k_2) -split there exists $\gamma \in I_k^n$ with $(|\gamma^1|, |\gamma^2|) \neq (k_1, k_2)$ such that, in some neighborhood of 0, Σ is the graph of a Lipschitz mapping over \mathbb{X}_γ . Denoting $B'_r := B_r(0) \cap \mathbb{X}_\gamma$, we define the cylinder $C_r := B'_r + \mathbb{X}_{\bar{\gamma}}$. Reducing r if necessary, we may assume that the connected component of $\Sigma \cap C_r$ which contains 0 is the graph of a Lipschitz continuous function over B'_r . We denote Σ_* this component and write

$$\Sigma_* = \{y + f(y) : y \in B'_r\}, \quad (3.12)$$

with $f \in \text{Lip}(\mathbb{X}_\gamma, \mathbb{X}_{\bar{\gamma}})$.

Step 2.a. Let $A_* := A \llcorner \Sigma_*$. Notice that this chain also satisfies (3.11). Let π be the orthogonal projection on \mathbb{X}_γ . Let us establish the identity:

$$\text{Sl}_\gamma \pi_\# A' = \pi_\# \text{Sl}_\gamma A' \quad \text{for every } A' \in \mathcal{F}_k^G(\mathbb{R}^n). \quad (3.13)$$

We first prove it for $A' = gp$ with $g \in G$ and $p = (q, \xi)$ a polyhedral k -cell in \mathbb{R}^n . If $\mathcal{H}^k(\pi q) = 0$ then both $\pi_\#(gp)$ and $\text{Sl}_\gamma(gp)$ vanish. In the other case, denoting $q' := \pi q$, q' is a convex k -polyhedron in \mathbb{X}_γ and q is the restriction on q' of the graph of a linear mapping $\ell : \mathbb{X}_\gamma \rightarrow \mathbb{X}_{\bar{\gamma}}$. We observe that:

- (*) $\pi_\# A'$ is the polyhedral cell $\sigma g(q', e_\gamma)$ where $\sigma = \pm 1$ is the sign of $\langle e_\gamma, \xi \rangle$.
- (*) $\text{Sl}_\gamma^y A' = \begin{cases} 0 & \text{for } y \in \mathbb{X}_\gamma \setminus \bar{q}', \\ \sigma g[\ell(y)] & \text{for } y \text{ in the relative interior of } q'. \end{cases}$

It follows that $\text{Sl}_\gamma^y \pi_\# A' = \pi_\# \text{Sl}_\gamma^y A' = \begin{cases} 0 & \text{for } y \in \mathbb{X}_\gamma \setminus \bar{q}', \\ \sigma g[0] & \text{for } y \text{ in the relative interior of } q'. \end{cases}$

The relation (3.13) is then true for $A' = gp$. The case of polyhedral k -chains follows by \mathbb{Z} linearity and the general case by a density/continuity argument.

Next, applying (3.13) with $A' = A_*$, we get by assumption,

$$\text{Sl}_\gamma \pi_\# A_* = \pi_\# \text{Sl}_\gamma A_* = \pi_\# (\text{Sl}_\gamma(A \llcorner S_*)) = \pi_\# ((\text{Sl}_\gamma A) \llcorner S_*) = \pi_\# (0 \llcorner S_*) = 0. \quad (3.14)$$

Since $\pi_\# A_*$ has maximal dimension in \mathbb{X}_γ , it writes as a L^1 function $g_* : \mathbb{X}_\gamma \rightarrow G$. With this notation, $\text{Sl}_\gamma^y \pi_\# A_* = g_*(y)[0]$ for almost every $y \in \mathbb{X}_\gamma$. We deduce from (3.14) that $g_* \equiv 0$ and consequently, by Theorem 3.3

$$\pi_\# A_* = 0. \quad (3.15)$$

Step 2.b. Recalling (3.12), we define $F \in \text{Lip}(C_r, \mathbb{R}^n)$ by,

$$F(x) = y + f(y) \quad \text{with the decomposition } x = y + z, \quad y \in L \cap C_r, \quad z \in \mathbb{X}_{\bar{\gamma}}.$$

Noticing that $F \circ \pi$ is the identity on Σ_* and using $A_* = A_* \perp \Sigma_*$, we have

$$A_* = (F \circ \pi)_\# A_* = F_\# (\pi_\# A_*) \stackrel{(3.15)}{=} F_\# 0 = 0.$$

This contradicts (3.11). \square

To end this section we recall the notion of set-decomposition of normal chains introduced in [GM22b] (see also [Fed69, §4.2.25] for related definitions) and state the existence of maximal set-decompositions for normal *rectifiable* chains. This result and the formula of Proposition 3.6 are the main ingredients of the proof of Proposition 6.2.

Definition 3.13 ([GM22b, Definition 1.1]). Let $A \in \mathcal{N}_k^G(\mathbb{R}^n)$.

(1) A set-decomposition of A is a sequence of normal chains A_j such that there exists a Borel partition S_j of \mathbb{R}^n such that $A_j = A_j \perp S_j$ for every j and $\mathbb{N}(A) = \sum \mathbb{N}(A_j)$.

(2) We say that A is set-indecomposable if the only set-decompositions of A are trivial, that is, for any such decomposition A_j there holds $A_j = A$ for some index j and $A_j = 0$ for the others.

Theorem 3.14 ([GM22b, Theorem 1.2]). *Let $A \in \mathcal{N}_k^G(\mathbb{R}^n)$, if A is rectifiable then it admits a set-decomposition in set-indecomposable components.*

4 Tensor flat chains

Let us assume $n = n_1 + n_2$ with $n_1, n_2 \geq 0$. We present a generalization of the theory of G -flat chains adapted to the decomposition $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and to the partial boundary operators ∂_1, ∂_2 defined below. The case $\{n_1, n_2\} = \{0, n\}$ corresponds to the classical G -flat chains.

We recall some notation: we set $\alpha := \{1, \dots, n_1\}$ and for $\beta \in I^n$, $\beta^1 := \beta \cap \alpha$ and $\beta^2 := \beta \setminus \alpha = \beta \cap \bar{\alpha}$. Finally, for $k \geq 0$,

$$D_k := \{(k'_1, k'_2) : 0 \leq k'_1 \leq n_1, 0 \leq k'_2 \leq n_2, k'_1 + k'_2 = k\}.$$

Throughout the section, $k_1, k_2, k \in \mathbb{Z}$ and $\beta \in I^n$. If not indicated otherwise we assume $k = k_1 + k_2$, $0 \leq k_1 \leq |\beta^1|$ and $0 \leq k_2 \leq |\beta^2|$.

4.1 Tensor polyhedral chains and \mathbb{F}^\times -norm

We define the group $\mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ of polyhedral (k_1, k_2) -chains with coefficients in G as the subgroup of $\mathcal{P}_k^G(\mathbb{X}_\beta)$ formed by elements with representations:

$$P = \sum g_i p_i^1 \times p_i^2, \tag{4.1}$$

where for every i , $g_i \in G$, p_i^1 is a k_1 -cell in \mathbb{X}_{β^1} and p_i^2 is a k_2 -cell in \mathbb{X}_{β^2} .

We denote by $\mathcal{TP}_k^G(\mathbb{X}_\beta)$ the subgroup of $\mathcal{P}_k^G(\mathbb{X}_\beta)$ spanned by the union of the groups $\mathcal{P}_{k'_1, k'_2}^G(\mathbb{X}_\beta)$ for (k'_1, k'_2) ranging in D_k . The decomposition of $P \in \mathcal{TP}_k^G(\mathbb{X}_\beta)$ as $\sum P_{k'_1, k'_2}$ with $P_{k'_1, k'_2} \in \mathcal{P}_{k'_1, k'_2}^G(\mathbb{X}_\beta)$ is unique. Moreover

$$\mathbb{M}(P) = \sum \mathbb{M}(P_{k'_1, k'_2}). \tag{4.2}$$

We denote this decomposition,

$$\begin{aligned} J : \mathcal{TP}_k^G(\mathbb{X}_\beta) &\longrightarrow \left(\mathcal{P}_{k'_1, k'_2}^G(\mathbb{X}_\beta) \right)_{(k'_1, k'_2) \in D_k}, \\ P &\longmapsto J_{k'_1, k'_2} P = P_{k'_1, k'_2}. \end{aligned} \tag{4.3}$$

As a consequence of the deformation theorem [Whi99a, Corollary 1.3], the group $\mathcal{TP}_k^G(\mathbb{X}_\beta)$ is dense in $\mathcal{F}_k^G(\mathbb{X}_\beta)$. For $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$, we define

$$\mathbb{M}^\times(A) := \inf\{\liminf \mathbb{M}(P_j) : P_j \in \mathcal{TP}_k^G(\mathbb{X}_\beta), P_j \rightarrow A\}.$$

There holds,

$$\mathbb{M}(A) \leq \mathbb{M}^\times(A) \leq C(k, n_1, n_2)\mathbb{M}(A), \quad (4.4)$$

for some constant $C(k, n_1, n_2) \geq 1$. The left inequality follows directly from the definitions as $\mathcal{TP}_k^G(\mathbb{X}_\beta) \subset \mathcal{P}_k^G(\mathbb{X}_\beta)$. The right inequality is a consequence of White's deformation theorem [Whi99a]. We define the corresponding flat norm as,

$$\mathbb{F}^\times(A) := \inf\{\mathbb{M}^\times(B) + \mathbb{M}^\times(C) : B \in \mathcal{F}_k^G(\mathbb{X}_\beta), C \in \mathcal{F}_{k+1}^G(\mathbb{X}_\beta), A = B + \partial C\}.$$

With the convention $C(n+1, n_1, n_2) = 1$, we have

$$\mathbb{F}(A) \leq \mathbb{F}^\times(A) \leq \max(C(k, n_1, n_2), C(k+1, n_1, n_2))\mathbb{F}(A) \quad \text{for every } A \in \mathcal{F}_k^G(\mathbb{R}^n),$$

so that \mathbb{F}^\times is a norm on $\mathcal{F}_k^G(\mathbb{X}_\beta)$, equivalent to \mathbb{F} .

Notice that for $k = |\beta|$, $\mathcal{TP}_{|\beta|}^G(\mathbb{X}_\beta)$ is \mathbb{M} -dense in $\mathcal{F}_{|\beta|}^G(\mathbb{X}_\beta)$ and $\mathbb{F}^\times = \mathbb{M}^\times = \mathbb{F} = \mathbb{M}$. For $k = 0$, $\mathcal{TP}_0^G(\mathbb{X}_\beta) = \mathcal{P}_0^G(\mathbb{X}_\beta)$ and $\mathbb{M}^\times = \mathbb{M}$.

4.2 Partial boundary operators and \mathbb{F}^\wedge -norm

Let us now introduce the *partial* boundary operators ∂_1, ∂_2 . For $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ of the form (4.1) we set,

$$\partial_1 P := \sum g_i(\partial p_i^1) \times p_i^2, \quad \partial_2 P := (-1)^{k_1} \sum g_i p_i^1 \times (\partial p_i^2).$$

This defines group morphisms,

$$\partial_1 : \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta) \rightarrow \mathcal{P}_{k_1-1, k_2}^G(\mathbb{X}_\beta), \quad \partial_2 : \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta) \rightarrow \mathcal{P}_{k_1, k_2-1}^G(\mathbb{X}_\beta),$$

which satisfy the relations,

$$\partial_1^2 = 0, \quad \partial_2^2 = 0, \quad \partial_2 \partial_1 = -\partial_1 \partial_2 \quad \text{and} \quad \partial = \partial_1 + \partial_2.$$

With these operators we build a *tensor flat norm* \mathbb{F}^\wedge defined for $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ by

$$\mathbb{F}^\wedge(P) := \inf \{ \mathbb{M}(Q^{0,0}) + \mathbb{M}(Q^{1,0}) + \mathbb{M}(Q^{0,1}) + \mathbb{M}(Q^{1,1}) : \\ Q^{i_1, i_2} \in \mathcal{P}_{k_1+i_1, k_2+i_2}^G(\mathbb{X}_\beta), P = Q^{0,0} + \partial_1 Q^{1,0} + \partial_2 Q^{0,1} + \partial_1 \partial_2 Q^{1,1} \}. \quad (4.5)$$

At this point, we could continue as in [Fle66] and first show that \mathbb{F}^\wedge is a norm on $\mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ before defining the groups of (k_1, k_2) -chains by completion. However, we take a detour by first identifying the elements of $\mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ with flat chains on \mathbb{X}_{β^1} .

We now interpret $\mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ as the tensor product $\mathcal{P}_{k_1}^Z(\mathbb{X}_{\beta^1}) \otimes \mathcal{P}_{k_2}^G(\mathbb{X}_{\beta^2})$ and in fact as⁷ $\mathcal{P}_{k_1}(\mathbb{X}_{\beta^1}, G'_{k_2})$ where here and below $G'_{k_2} := \mathcal{P}_{k_2}^G(\mathbb{X}_{\beta^2})$. Notice that this breaks the symmetry between \mathbb{X}_{β^1} and \mathbb{X}_{β^2} : we consider here chains in \mathbb{X}_{β^1} whereas \mathbb{X}_{β^2} only appears in the definition of the coefficient group G'_{k_2} . For $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ with representation of the form (4.1), we denote ιP the corresponding element of $\mathcal{P}_{k_1}(\mathbb{X}_{\beta^1}, G'_{k_2})$:

$$\iota P := \sum [g_i p_i^2] p_i^1,$$

⁷For a better readability, we sometimes write $\mathcal{P}_{k_1}(\mathbb{X}_\beta, G)$ for $\mathcal{P}_{k_1}^G(\mathbb{X}_\beta)$.

the terms $g_i p_i^2$ being understood as coefficients in G'_{k_2} .

Let us denote $G_{k_2}^* := \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})$ and let us endow the group $(G_{k_2}^*, +)$ with the \mathbb{F} -norm. Since $G'_{k_2} \subset G_{k_2}^*$, we have

$$\iota P \in \mathcal{P}_{k_1}(\mathbb{X}_{\beta^1}, G_{k_2}^*)$$

and now $(G_{k_2}^*, +, \mathbb{F})$ is a complete Abelian normed group. There holds $\mathbb{M}(\iota P) \leq \mathbb{M}(P)$ but equality does not hold in general, except in the limit case $k_2 = |\beta^2|$. We recover isometry when passing to the flat norms.

Proposition 4.1. *Let $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_{\beta})$, there hold,*

$$(i) \quad \mathbb{M}(\iota P) = \inf \mathbb{M}(Q^{0,0}) + \mathbb{M}(Q^{0,1}),$$

where the infimum is over the decompositions $P = Q^{0,0} + \partial_2 Q^{0,1}$ with $Q^{0,0} \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_{\beta})$, $Q^{0,1} \in \mathcal{P}_{k_1, k_2+1}^G(\mathbb{X}_{\beta})$.

$$(ii) \quad \mathbb{F}(\iota P) = \mathbb{F}^\wedge(P).$$

Proof. We first prove (i). By definition,

$$\mathbb{M}(\iota P) = \inf \left\{ \sum \mathbb{M}(p_j^1) \mathbb{F}(P_j^2) \right\},$$

where the infimum runs over all the representations $P = \sum p_j^1 \wedge P_j^2$, with p_j^1 , k_1 -cell in \mathbb{X}_{β^1} and $P_j^2 \in G'_{k_2}$. Then, by definition of \mathbb{F} in $G_{k_2}^*$,

$$\mathbb{M}(\iota P) = \inf \left\{ \sum \mathbb{M}(p_j^1) (\mathbb{M}(Q_j^2) + \mathbb{M}(R_j^2)) \right\},$$

where now the infimum runs over the representations $P = \sum p_j^1 \times (Q_j^2 + \partial R_j^2)$ with p_j^1 , k_1 -cell in \mathbb{X}_{β^1} , $Q_j^2 \in G'_{k_2}$ and $R_j^2 \in G'_{k_2+1}$. Using the identities $\mathbb{M}(p_j^1) \mathbb{M}(Q_j^2) = \mathbb{M}(p_j^1 \times Q_j^2)$, $\mathbb{M}(p_j^1) \mathbb{M}(R_j^2) = \mathbb{M}(p_j^1 \times R_j^2)$, we get,

$$\mathbb{M}(\iota P) = \inf \left\{ \sum \mathbb{M}(S_j) + \mathbb{M}(T_j) \right\},$$

where $S_j \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_{\beta})$, $T_j \in \mathcal{P}_{k_1, k_2+1}^G(\mathbb{X}_{\beta})$ are such that $P = \sum S_j + \partial_2 T_j$. Eventually setting $Q^{0,0} := \sum S_j$ and $Q^{0,1} := \sum T_j$, we have established the identity (i).

Let us prove the second point. By definition,

$$\mathbb{F}(\iota P) = \inf \left\{ \mathbb{M}(P^0) + \mathbb{M}(P^1) : P^{i_1} \in \mathcal{P}_{k_1+i_1}(\mathbb{X}_{\beta^1}, G'_{k_2}) \text{ for } i_1 = 0, 1, \iota P = P^0 + \partial P^1 \right\},$$

and since ι is a bijection and $\iota(\partial_1 P) = \partial(\iota P)$, this rewrites as

$$\mathbb{F}(\iota P) = \inf \left\{ \mathbb{M}(\iota Q^0) + \mathbb{M}(\iota Q^1) : Q^{i_1} \in \mathcal{P}_{k_1+i_1, k_2}^G(\mathbb{X}_{\beta}) \text{ for } i_1 = 0, 1, P = Q^0 + \partial_1 Q^1 \right\}.$$

The result then follows by applying point (i) to Q^0 and Q^1 . □

4.3 Groups of tensor flat chains

As a consequence of Proposition 4.1, \mathbb{F}^\wedge is a norm on $\mathcal{P}_{k_1, k_2}^G(\mathbb{X}_{\beta})$. The group of (k_1, k_2) -chains is defined as the completion of $(\mathcal{P}_{k_1, k_2}^G(\mathbb{X}_{\beta}), \mathbb{F}^\wedge)$ and is denoted $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_{\beta})$. For pairs $(k_1, k_2) \in \mathbb{Z}$ with $\min(k_1, k_2) < 0$ or $\max(k_1 - |\beta^1|, k_2 - |\beta^2|) > 0$ we set by convention $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_{\beta}) = \{0\}$. Let $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_{\beta})$. Given a decomposition,

$$P = Q^{0,0} + \partial_1 Q^{1,0} + \partial_2 Q^{0,1} + \partial_1 \partial_2 Q^{1,1}, \tag{4.6}$$

since

$$\partial_1^2 = 0, \quad \partial_2^2 = 0, \quad \partial_2 \partial_1 = -\partial_1 \partial_2, \quad (4.7)$$

we have $\partial_1 P = \partial_1 Q^{0,0} + \partial_1 \partial_2 Q^{0,1}$ and $\partial_2 P = \partial_2 Q^{0,0} - \partial_1 \partial_2 Q^{1,0}$, so that $\mathbb{F}^\wedge(\partial_1 P) \leq \mathbb{M}(Q^{0,0}) + \mathbb{M}(Q^{0,1})$ and $\mathbb{F}^\wedge(\partial_2 P) \leq \mathbb{M}(Q^{0,0}) + \mathbb{M}(Q^{1,0})$. Optimizing over the decompositions of P , we obtain

$$\mathbb{F}^\wedge(\partial_l P) \leq \mathbb{F}^\wedge(P) \quad \text{for } l = 1, 2.$$

With these inequalities, we can extend the partial differential operators ∂_1, ∂_2 by continuity on $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$. Besides,

$$\partial_1 : \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta) \longrightarrow \mathcal{F}_{k_1-1, k_2}^G(\mathbb{X}_\beta) \quad \text{and} \quad \partial_2 : \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta) \longrightarrow \mathcal{F}_{k_1, k_2-1}^G(\mathbb{X}_\beta)$$

are 1-Lipschitz group morphisms. Notice that (4.7) extends to $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$.

In summary, the family $(\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta))_{k_1, k_2}$ forms a two-dimensional chain complex of normed groups:

$$\begin{array}{ccccccc} & & \partial_2 \downarrow & & \partial_2 \downarrow & & \\ \dots & \xrightarrow{\partial_1} & \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta) & \xrightarrow{\partial_1} & \mathcal{F}_{k_1-1, k_2}^G(\mathbb{X}_\beta) & \xrightarrow{\partial_1} & \dots \\ & & \partial_2 \downarrow & & \partial_2 \downarrow & & \\ \dots & \xrightarrow{\partial_1} & \mathcal{F}_{k_1, k_2-1}^G(\mathbb{X}_\beta) & \xrightarrow{\partial_1} & \mathcal{F}_{k_1-1, k_2-1}^G(\mathbb{X}_\beta) & \xrightarrow{\partial_1} & \dots \\ & & \partial_2 \downarrow & & \partial_2 \downarrow & & \end{array}$$

As for flat chains, we say that $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ is supported in the closed set S if for every neighborhood $S \subset U \subset \mathbb{X}_\beta$, there exists a sequence $P_j \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ with each P_j supported in U such that $P_j \rightarrow A$ (from now on for $A_j, A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$, $A_j \rightarrow A$ means that A_j converges to A in \mathbb{F}^\wedge -norm, that is $\mathbb{F}^\wedge(A_j - A) \rightarrow 0$).

When $\{|\beta^1|, |\beta^2|\} = \{0, |\beta|\}$, $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ and $\mathcal{F}_k^G(\mathbb{X}_\beta)$ are exactly the same groups by construction. In the other cases, any element $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ can be considered either as an element of $\mathcal{F}_k^G(\mathbb{X}_\beta)$ or as an element of $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$. For clarity, in the later case we use from now on the “wedge product” notation,

$$P = \sum g_i p_i^1 \wedge p_i^2,$$

in place of the “cross product” (4.1) even though these expressions refer to the same object.

Remark 4.2.

(a) If $\beta^2 = \emptyset$, $\mathcal{F}_{k, 0}^G(\mathbb{X}_\beta)$ is just $\mathcal{F}_k^G(\mathbb{X}_\beta)$ and on these groups $\mathbb{F}^\wedge = \mathbb{F}$, $\partial_1 = \partial$ and $\partial_2 = 0$. Symmetrically, if $\beta^1 = \emptyset$, $\mathcal{F}_{0, k}^G(\mathbb{X}_\beta) = \mathcal{F}_k^G(\mathbb{X}_\beta)$ and $\partial_2 = \partial$, $\partial_1 = 0$.

(b) In the extreme case $(k_1, k_2) = (|\beta^1|, |\beta^2|)$, there holds $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta) = \mathcal{F}_k^G(\mathbb{X}_\beta) = L^1(\mathbb{X}_\beta, G)$ with $\mathbb{F}^\wedge = \mathbb{F} = \mathbb{M}$.

(c) In the other cases, tensor flat chains are not flat chains in general. For instance let $n_1 = n_2 = 1$ and let us consider the triangle

$$T = \{(x^1, x^2) \in \mathbb{R}^2 : x^1, x^2 \geq 0, x^1 + x^2 \leq 1\}.$$

As a current, $\partial_1 \llbracket T \rrbracket$ is a measure supported on the frontier $\text{Bdry } T$ of T but it is oriented along e_2 which is not tangent to $\text{Bdry } T$ on the segment $\{(t, 1-t) : 0 \leq t \leq 1\}$, so $\partial_1 \llbracket T \rrbracket$ is not a

flat chain. This example is detailed in the setting of general complete Abelian normed groups $G \neq \{0\}$ in the proof of [GM22a, Proposition A.5].

(d) Another phenomenon arises from the second order term $\partial_1 \partial_2 Q^{1,1}$ in the decomposition (4.6). Let us consider again $n_1 = n_2 = 1$ and $G = \mathbb{Z}$. Let $S \subset \mathbb{R}^2$ be a (non degenerate) closed square with sides parallel to the axes and let us set $P := \partial_1 \partial_2 \llbracket S \rrbracket \in \mathcal{F}_{0,0}^{\mathbb{Z}}(\mathbb{R}^2)$. We easily see that $P = -\llbracket a \rrbracket + \llbracket b \rrbracket - \llbracket c \rrbracket + \llbracket d \rrbracket$ where a, b, c, d are the vertices of S labeled counterclockwise with a at the bottom left corner. Denoting ℓ the side-length of S , on the one hand we have of course $\mathbb{F}^\wedge(P) \leq \mathbb{M}(\llbracket S \rrbracket) = \ell^2$ and in fact this inequality is an identity if $\ell \leq 2$. On the other hand, $\mathbb{F}(P) = 2\ell$ if $\ell \leq 2$.

Now, let S_j be a sequence of such squares with side lengths $\ell_j = 1/j$ for $j \geq 1$ and such that $d(S_i, S_j) \geq 2\ell_i$ for $1 \leq i < j$. Let us denote $P_j := \partial_1 \partial_2 \llbracket S_j \rrbracket$. Since $\mathbb{F}^\wedge(P_j) = 1/j^2$, the sequence $A_j := \sum_{i \leq j} P_i$ converges in \mathbb{F}^\wedge -norm towards some $(0, 0)$ -chain A_∞ which identifies with a 0-current. On the contrary, for $j \geq 1$ and $m \geq 2j$,

$$\mathbb{F}(A_m - A_j) = 2 \sum_{r=j+1}^m \frac{1}{r} \geq 2 \sum_{r=j+1}^{2j} \frac{1}{r} \sim 2 \ln 2 \quad \text{as } j \uparrow \infty.$$

Hence A_j does not admit any Cauchy subsequence in $(\mathcal{F}_0^{\mathbb{Z}}(\mathbb{R}^2), \mathbb{F})$. Moreover, as a current, A_∞ is not a flat chain.⁸

(e) However, as mentioned in the introduction, we establish a positive result, namely: the groups of normal tensor chains (defined further on) identify with subgroups of normal chains, see [GM22a, Theorem A.3].

(f) The basic theory of tensor chains is similar to its counterpart for flat chains, however some results do not generalize. For instance, if A is a flat chain, we can write $A = B + \partial C$ for some *finite mass* flat chains B, C and we have the direct implication

$$\mathbb{M}(A) < \infty \implies C \text{ is normal.} \quad (4.8)$$

In the case of a tensor chain A' , we can still write (see Proposition 4.8 below) $A' = B'_{0,0} + \partial_1 B'_{1,0} + \partial_2 B'_{0,1} + \partial_1 \partial_2 B'_{1,1}$ for some finite mass tensor chains B'_{i_1, i_2} but for the corresponding \mathbb{M}^\wedge -mass (introduced in Subsection 4.5 below) the additional information $\mathbb{M}^\wedge(A') < \infty$ does not imply that $\partial_1 B'_{1,0}$, $\partial_2 B'_{0,1}$ or $\partial_1 \partial_2 B'_{1,1}$ have finite \mathbb{M}^\wedge -mass.

This seems innocent but (4.8) is used to establish that the spaces of flat chains $\mathcal{F}_k^{\mathbb{R}}(\mathbb{R}^n)$, as defined above with $G = \mathbb{R}$, are the same as the spaces of real valued flat chains defined as subspaces of currents in [FF60, Fed69] (namely the closure of the space of polyhedral currents with respect to the dual of the norm defined by

$$\|\omega\|_{\mathbb{F}} := \max(\|\omega\|_{\infty}, \|d\omega\|_{\infty})$$

for ω smooth and compactly supported k -form on \mathbb{R}^n).⁹ Here, it is not clear whether the spaces of tensor chains identify with the closures of the spaces of tensor polyhedral currents with respect to the dual of the norm defined (with the natural definition of the partial exterior derivatives d_1, d_2) by $\|\omega\|_{\mathbb{F}^\wedge} := \max(\|\omega\|_{\infty}, \|d_1 \omega\|_{\infty}, \|d_2 \omega\|_{\infty}, \|d_1 d_2 \omega\|_{\infty})$.

⁸Indeed, if A_∞ were a flat chain we would have $\sum_{j \geq 1} |\varphi(a_j) - \varphi(b_j) + \varphi(c_j) - \varphi(d_j)| < \infty$ for every compactly supported Lipschitz function φ , where (a_j, b_j, c_j, d_j) are the vertices of S_j listed counterclockwise. Setting $\varphi(x) := \sum_{j \geq 1} (\ell_j - |x - a_j|)_+$ we obtain the contradiction $\sum_{j \geq 1} 1/j < \infty$.

⁹See [Fed69, §2.1.12]. Only a partial result is established there but the proof can be easily completed.

4.4 Orthogonal projection of tensor chains

We only consider orthogonal projections of tensor chains on affine subspaces $L \subset \mathbb{X}_\beta$ which write as $L = L^1 + L^2$ with L^l affine m_l -plane of \mathbb{X}^{β^l} for $l = 1, 2$. Denoting π the orthogonal projection on such L and π^l the orthogonal projection on L^l for $l = 1, 2$, we have for $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ with representation (4.1),

$$\pi_{\#} P = \sum g_i(\pi^1_{\#} p_i^1) \wedge (\pi^2_{\#} p_i^2).$$

Consequently,

$$\partial_1 \pi_{\#} P = \sum g_i(\partial \pi^1_{\#} p_i^1) \wedge (\pi^2_{\#} p_i^2) = \sum g_i(\pi^1_{\#} \partial p_i^1) \wedge (\pi^2_{\#} p_i^2) = \pi_{\#} \partial_1 P.$$

Similarly, $\partial_2 \pi_{\#} P = \pi_{\#} \partial_2 P$. It follows that for a representation of P of the form (4.6),

$$\pi_{\#} P = \pi_{\#} Q^{0,0} + \partial_1 \pi_{\#} Q^{1,0} + \partial_2 \pi_{\#} Q^{0,1} + \partial_1 \partial_2 \pi_{\#} Q^{1,1}.$$

Observe that for a tensor polyhedral chain Q there holds $\mathbb{M}(\pi_{\#} Q) \leq \mathbb{M}(Q)$. Using this with $Q = Q^{i_1, i_2}$ for $i_1, i_2 \in \{0, 1\}$ and optimizing over the decompositions of P , we get $\mathbb{F}^\wedge(\pi_{\#} P) \leq \mathbb{F}^\wedge(P)$. We can then extend the operator $\pi_{\#}$ by continuity.

Proposition 4.3. *With the above notation the orthogonal projection on $L = L^1 + L^2$ with L^l affine m_l -plane of \mathbb{X}^{β^l} for $l = 1, 2$ extends as a continuous group morphism $\pi_{\#} : \mathcal{F}_{k_1, k_2}^G(\mathbb{X}^\beta) \rightarrow \mathcal{F}_{k_1, k_2}^G(L)$. Moreover, for $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}^\beta)$, there hold*

$$\partial_l \pi_{\#} A = \pi_{\#} \partial_l A \quad \text{for } l = 1, 2 \quad \text{and} \quad \mathbb{F}^\wedge(\pi_{\#} A) \leq \mathbb{F}^\wedge(A).$$

4.5 Mass of tensor chains

We are now almost ready to define the mass of a tensor chain, see (4.14) after the proof of the next lemma. Later on we consider restrictions to Borel sets of tensor chains of finite mass. This analysis is almost identical to the one of [Fle66] but we reproduce it in details because on the one hand the necessary adaptations might not be completely obvious and on the other hand the estimate (4.9) of the next lemma is used in [GM22a]. As in the case of classical chains, we start with restriction on intervals.

Definition 4.4. We call intervals of \mathbb{X}_β the sets of the form $I_1 \times I_2 \times \dots \times I_n$ where for $j \in \beta$, $I_j \subset \mathbb{R}$ is any interval and for $j \in \bar{\beta}$, $I_j = \{0\}$.

Lemma 4.5 (Counterpart of [Fle66, Lemma 2.1 & Theorem 2.3]).

(i) *Let $P_j \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ such that $\sum \mathbb{F}^\wedge(P_j) < \infty$, then for every interval $I \subset \mathbb{X}_\beta$, there holds*

$$\sum \mathbb{F}^\wedge(P_j \llcorner (x + I)) < \infty \quad \text{for almost every } x \in \mathbb{X}_\beta.$$

The intervals $x + I$ for which $\sum \mathbb{F}^\wedge(P_j \llcorner (x + I)) < \infty$ are called non-exceptional with respect to the sequence P_j .

More precisely, for any interval I of \mathbb{X}_β and any Cartesian product $\omega = \omega_1 \times \omega_2 \times \dots \times \omega_n \subset \mathbb{X}^\beta$ such that for $i \in \beta$, $\omega_i \subset \mathbb{R}$ is measurable with finite length (and $\omega_i = \{0\}$ for $i \in \bar{\beta}$), there holds

$$\int_\omega \sum \mathbb{F}^\wedge(P_j \llcorner (x + I)) d\mathcal{H}^{|\beta|}(x) \leq c_\omega \sum \mathbb{F}^\wedge(P_j) < \infty, \quad (4.9)$$

where $c_\omega \geq 0$ depends on ω .

(ii) *The mapping $P \mapsto \mathbb{M}(P)$ is lower semi-continuous in $(\mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta), \mathbb{F}^\wedge)$.*

Proof. Without loss of generality, we assume that $\beta = \{1, \dots, n\}$. We proceed exactly as in [Fle66].

Proof of (i). We start with I being a half-space. Lets $i \in \{1, \dots, n\}$. For $s \in \mathbb{R}$ we denote,

$$H_i(s) := \{x \in \mathbb{R}^n : x_1 < s\}.$$

Let $P_j \in \mathcal{P}_{k_1, k_2}^G(\mathbb{R}^n)$ such that $\sum \mathbb{F}^\wedge(P_j) < \infty$. By definition of \mathbb{F}^\wedge , for every $j \geq 1$, we can write $P_j = Q_j^{0,0} + \partial_1 Q_j^{1,0} + \partial_2 Q_j^{0,1} + \partial_1 \partial_2 Q_j^{1,1}$ with $Q_j^{i_1, i_2} \in \mathcal{P}_{k_1+i_1, k_2+i_2}^G(\mathbb{R}^n)$ such that $\sum_{i_1, i_2} \mathbb{M}(Q_j^{i_1, i_2}) \leq 2\mathbb{F}^\wedge(P_j)$. Consequently,

$$\sum_j \sum_{i_1, i_2} \mathbb{M}(Q_j^{i_1, i_2}) \leq 2 \sum_j \mathbb{F}^\wedge(P_j).$$

Next, there holds for almost every $s \in \mathbb{R}$ and $j \geq 1$,

$$P_j \lrcorner H_i(s) = Q_j^{0,0} \lrcorner H_i(s) + \partial_1(Q_j^{1,0} \lrcorner H_i(s)) + \partial_2(Q_j^{0,1} \lrcorner H_i(s)) + \partial_1 \partial_2(Q_j^{1,1} \lrcorner H_i(s)) + R_j(s).$$

where the remaining term decomposes as $R_j(s) := S_j^{1,0}(s) + S_j^{0,1}(s) + S_j^{1,1}(s)$ with,

$$\begin{aligned} S_j^{1,0}(s) &= (\partial_1 Q_j^{1,0}) \lrcorner H_i(s) - \partial_1(Q_j^{1,0} \lrcorner H_i(s)), \\ S_j^{0,1}(s) &= (\partial_2 Q_j^{0,1}) \lrcorner H_i(s) - \partial_2(Q_j^{0,1} \lrcorner H_i(s)), \\ S_j^{1,1}(s) &= (\partial_1 \partial_2 Q_j^{1,1}) \lrcorner H_i(s) - \partial_1 \partial_2(Q_j^{1,1} \lrcorner H_i(s)). \end{aligned}$$

We have,

$$\sum \mathbb{F}^\wedge(P_j \lrcorner H_i(s) - R_j(s)) \leq \sum_{i_1, i_2 \in \{0,1\}} \sum_j \mathbb{M}(Q_j^{i_1, i_2} \lrcorner H_i(s)).$$

Since $\mathbb{M}(Q_j^{i_1, i_2} \lrcorner H_i(s)) \leq \mathbb{M}(Q_j^{i_1, i_2})$ for every $j \geq 1$ and $i_1, i_2 \in \{0, 1\}$, we deduce,

$$\sum \mathbb{F}^\wedge(P_j \lrcorner H_i(s) - R_j(s)) \leq 2 \sum_j \mathbb{F}^\wedge(P_j) < \infty. \quad (4.10)$$

Let us treat the remaining terms $R_j(s)$. We assume that $i \in \alpha$ (the other case is similar). In this case, ∂_2 commutes with the restriction on $H_i(s)$, hence $S_j^{0,1}(s) = 0$ and $S_j^{1,1}(s)$ rewrites as $S_j^{1,1}(s) = \partial_2 T_j(s)$, with

$$T_j(s) = -(\partial_1 Q_j^{1,1}) \lrcorner H_i(s) + \partial_1(Q_j^{1,1} \lrcorner H_i(s)).$$

By Proposition 3.6, we have

$$S_j^{1,0}(s) = \sigma Q_j^{1,0} \cap \mathbb{X}_{\bar{i}}(se_i), \quad T_j(s) = \sigma' Q_j^{1,1} \cap \mathbb{X}_{\bar{i}}(se_i),$$

where the factors $\sigma, \sigma' = \pm 1$ depend on the orientation conventions. From Proposition 3.2, we have the estimates,

$$\int_{\mathbb{R}} \mathbb{M}(S_j^{1,0}(s)) ds \leq \mathbb{M}(Q_j^{1,0}), \quad \int_{\mathbb{R}} \mathbb{M}(T_j(s)) ds \leq \mathbb{M}(Q_j^{1,1}).$$

Writing $R_j(s) = S_j^{1,0}(s) + \partial_2 T_j(s)$. We deduce that

$$\int_{\mathbb{R}} \sum \mathbb{F}^\wedge(R_j(s)) ds \leq 2 \sum_j \mathbb{F}^\wedge(P_j) < \infty. \quad (4.11)$$

Combining (4.10)&(4.11) we get that for every measurable set $\omega_1 \subset \mathbb{R}$ with finite length ℓ_1 , there holds

$$\int_{\omega_1} \sum \mathbb{F}^\wedge (P_j \llcorner H_i(s)) ds \leq 2(1 + \ell_1) \sum_j \mathbb{F}^\wedge (P_j) < \infty. \quad (4.12)$$

The first point of the lemma in the case of an open half-space follows from (4.12) (notice that $x + H_i(0) = H_i(x_i)$). From $P \llcorner (\mathbb{R}^n \setminus H_i(s)) = P - P \llcorner H_i(s)$, we see that the result also holds for the closed half-space $\mathbb{R}^n \setminus H_i(0)$. By symmetry the result holds for any closed or open half-space with boundary normal to some e_i . For a general interval I , writing I as the intersection of at most $2n$ coordinate half-spaces and applying the result recursively, we obtain the estimate (4.9). This proves the first point.

Proof of (ii). Let $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ and let $P_j \rightarrow P$ in \mathbb{F}^\wedge -norm such that the sequence $\mathbb{M}(P_j)$ has a finite limit. Up to extraction, we assume moreover $\sum \mathbb{F}^\wedge (P_i - P) < \infty$.

First we consider an interval I of \mathbb{X}_β which is non exceptional with respect to the sequence $P_j - P$ and is such that $P^0 := P \llcorner I$ is supported in some affine $(k_1 + k_2)$ -plane $L \subset \mathbb{X}_\beta$ of the form $L = L^1 + L^2$ (with L^l affine k_l -plane of \mathbb{X}^{β^l} for $l = 1, 2$). Let us set $P_j^0 := \pi_\# (P_j \llcorner I)$ where π is the orthogonal projection on L . The chain $P_j^0 - P^0$ being of maximal dimension in L , there holds

$$\mathbb{M}(P_j^0 - P^0) = \mathbb{F}^\wedge (P_j^0 - P^0) \leq \mathbb{F}^\wedge ((P_j - P) \llcorner I).$$

Since I is non exceptional, we have $\sum \mathbb{F}^\wedge ((P_j - P) \llcorner I) < \infty$ and the right-hand side tends to 0. We conclude that,

$$\mathbb{M}(P^0) = \lim \mathbb{M}(P_j^0) \leq \liminf \mathbb{M}(P_j \llcorner I). \quad (4.13)$$

Eventually, given $\varepsilon > 0$, there exists a finite set I^1, \dots, I^m of disjoint intervals of \mathbb{X}_β of the form above such that $\mathbb{M}(P \llcorner (\mathbb{X}_\beta \setminus \cup I^r)) < \varepsilon$. The result then follows from (4.13) applied to the I^r 's and the fact that if P and P' have disjoint supports then $\mathbb{M}(P + P') = \mathbb{M}(P) + \mathbb{M}(P')$. \square

We define the mass of $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ in the same way as the mass of classical chains. Namely,

$$\mathbb{M}^\wedge(A) := \inf \left\{ \liminf \mathbb{M}(P_j) : P_j \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta), P_j \rightarrow A \text{ in } \mathbb{F}^\wedge\text{-norm} \right\}. \quad (4.14)$$

Thanks to Lemma 4.5(ii), we have $\mathbb{M}^\wedge(P) = \mathbb{M}(P)$ for $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ and for such chains we use the two notations interchangeably.

The elements of $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ with finite mass form a subgroup denoted $\mathcal{M}_{k_1, k_2}^G(\mathbb{X}_\beta)$ and

$$(\mathcal{M}_{k_1, k_2}^G(\mathbb{X}_\beta), +, \mathbb{M}^\wedge) \text{ is a complete normed group.}$$

As in the case of flat chains, if $|\beta| \geq 1$ these groups do not form a chain complex

Remark 4.6. In the extreme case $(k_1, k_2) = (|\beta^1|, |\beta^2|)$, we have $\mathbb{M}^\wedge = \mathbb{F}^\wedge$ on $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ and $\mathcal{M}_{k_1, k_2}^G(\mathbb{X}_\beta) = \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$. Recalling Remark 4.2(b) we have in fact $\mathcal{M}_{k_1, k_2}^G(\mathbb{X}_\beta) = \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta) = \mathcal{M}_k^G(\mathbb{X}_\beta) = \mathcal{F}_k^G(\mathbb{X}_\beta) = L^1(\mathbb{X}_\beta, G)$ with $\mathbb{M}^\wedge = \mathbb{F}^\wedge = \mathbb{M} = \mathbb{F}$.

Before continuing let us come back to the push-forwards by orthogonal projections and complete Proposition 4.3.

Proposition 4.7. *Given a projection π on $L = L^1 + L^2 \subset \mathbb{X}_\beta$ as in Proposition 4.3, there holds $\mathbb{M}^\wedge(\pi_\# A) \leq \mathbb{M}^\wedge(A)$ for every $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$.*

Proof. The result holds true for $A \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$. The general case follows from the continuity of $\pi_\#$ in $(\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta), \mathbb{F}^\wedge)$ and the definition of \mathbb{M}^\wedge . \square

Now that we have defined the mass, we can express $\mathbb{F}^\wedge(A)$ with a formula similar to (4.5).

Proposition 4.8 (Counterpart of [Fle66, Theorem 3.1]). *Let $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$, there holds*

$$\mathbb{F}^\wedge(A) = \inf \sum_{i_1, i_2 \in \{0, 1\}} \mathbb{M}^\wedge(B^{i_1, i_2}), \quad (4.15)$$

where the infimum runs over the decompositions

$$A = B^{0,0} + \partial_1 B^{1,0} + \partial_2 B^{0,1} + \partial_1 \partial_2 B^{1,1}, \quad (4.16)$$

with $B^{i_1, i_2} \in \mathcal{F}_{k_1+i_1, k_2+i_2}^G(\mathbb{X}_\beta)$ for $i_1, i_2 \in \{0, 1\}$.

Proof. Let $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ and let us denote by $\tilde{\mathbb{F}}(A)$ the right-hand side of (4.15).

Let us consider a decomposition of A of the form (4.16). For $i_1, i_2 \in \{0, 1\}$, there exist sequences of tensor polyhedral chains $Q_j^{i_1, i_2}$ with $Q_j^{i_1, i_2} \rightarrow B^{i_1, i_2}$ in \mathbb{F}^\wedge -norm and $\mathbb{M}^\wedge(Q_j^{i_1, i_2}) \rightarrow \mathbb{M}^\wedge(B^{i_1, i_2})$. The sequence

$$P_j := Q_j^{0,0} + \partial_1 Q_j^{1,0} + \partial_2 Q_j^{0,1} + \partial_1 \partial_2 Q_j^{1,1},$$

converges to A in \mathbb{F}^\wedge -norm. Hence

$$\mathbb{F}^\wedge(A) \leq \liminf_j \mathbb{F}^\wedge(P_j) \leq \lim_j \sum_{i_1, i_2 \in \{0, 1\}} \mathbb{M}^\wedge(Q_j^{i_1, i_2}) = \sum_{i_1, i_2 \in \{0, 1\}} \mathbb{M}^\wedge(B^{i_1, i_2}).$$

We deduce that $\mathbb{F}^\wedge(A) \leq \tilde{\mathbb{F}}(A)$.

Let us prove the converse inequality. Let $P_j \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ converging rapidly to A (that is $\sum \mathbb{F}^\wedge(P_j - A) < \infty$) and let $\varepsilon_j > 0$ converging to 0. For $j \geq 1$, there exist decompositions

$$\begin{aligned} P_j &= Q_j^{0,0} + \partial_1 Q_j^{1,0} + \partial_2 Q_j^{0,1} + \partial_1 \partial_2 Q_j^{1,1}, \\ P_{j+1} - P_j &= R_j^{0,0} + \partial_1 R_j^{1,0} + \partial_2 R_j^{0,1} + \partial_1 \partial_2 R_j^{1,1}, \end{aligned}$$

with

$$\sum_{i_1, i_2} \mathbb{M}^\wedge(Q_j^{i_1, i_2}) < \mathbb{F}^\wedge(P_j) + \varepsilon_j \quad \text{for } j \geq 1, \quad (4.17)$$

$$\sum_j \mathbb{M}^\wedge(R_j^{i_1, i_2}) < \infty \quad \text{for } i_1, i_2 \in \{0, 1\}. \quad (4.18)$$

For $i_1, i_2 \in \{0, 1\}$, we set $B_j^{i_1, i_2} := Q_j^{i_1, i_2} + \sum_{i \geq j} R_i^{i_1, i_2}$. There holds for every $j \geq 1$,

$$A = B_j^{0,0} + \partial_1 B_j^{1,0} + \partial_2 B_j^{0,1} + \partial_1 \partial_2 B_j^{1,1}, \quad (4.19)$$

so that we obtain,

$$\tilde{\mathbb{F}}(A) \stackrel{(4.19)}{\leq} \sum_{i_1, i_2} \mathbb{M}^\wedge(B_j^{i_1, i_2}) \leq \sum_{i_1, i_2} \mathbb{M}^\wedge(Q_j^{i_1, i_2}) + \sum_{i \geq j} \left(\sum_{i_1, i_2} \mathbb{M}^\wedge(R_i^{i_1, i_2}) \right) \stackrel{(4.17) \& (4.18)}{\leq} \mathbb{F}^\wedge(P_j) + \varepsilon'_j,$$

with $\varepsilon'_j \rightarrow 0$. Passing to the limit, we get $\tilde{\mathbb{F}}(A) \leq \mathbb{F}^\wedge(A)$. \square

4.6 The operators ι and j

We introduce here operations that involve both tensor and classical chains.

Let us first extend the morphism ι . Recall that $\iota : \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta) \rightarrow \mathcal{P}_{k_1}(\mathbb{X}_{\beta^1}, \mathcal{P}_{k_2}^G(\mathbb{X}_{\beta^2}))$ is a group isomorphism which is moreover an isometry by the identity of Proposition 4.1(ii). By density of $\mathcal{P}_{k_2}^G(\mathbb{X}_{\beta^2})$ in $\mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})$ and a diagonal extraction argument we deduce that ι extends as the following group isomorphism which is again an isometry.

$$\iota : (\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta), +, \mathbb{F}^\wedge) \longrightarrow (\mathcal{F}_{k_1}(\mathbb{X}_{\beta^1}, \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})), +, \mathbb{F}).$$

Besides, by continuity, we still have the identity $\iota\partial_1 A = \partial_1 \iota A$ for $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$. Moreover

$$\mathbb{M}(\iota A) \leq \mathbb{M}^\wedge(A). \quad (4.20)$$

We now come back to the operator j of (4.3). Let $P \in \mathcal{TP}_k^G(\mathbb{X}_\beta)$ and let us consider a decomposition $P = Q + \partial R$ with $Q \in \mathcal{TP}_k^G(\mathbb{X}_\beta)$, $R \in \mathcal{TP}_{k+1}^G(\mathbb{X}_\beta)$. By identification, we have for $(k'_1, k'_2) \in D_k$,

$$P_{k'_1, k'_2} = j_{k'_1, k'_2} P = j_{k'_1, k'_2} Q + j_{k'_1, k'_2}(\partial R) = Q_{k'_1, k'_2} + \partial_1 R_{k'_1+1, k'_2} + \partial_2 R_{k'_1, k'_2+1}.$$

We deduce $\mathbb{F}^\wedge(P_{k'_1, k'_2}) \leq \mathbb{M}(Q_{k'_1, k'_2}) + \mathbb{M}(R_{k'_1+1, k'_2}) + \mathbb{M}(R_{k'_1, k'_2+1})$ (remark that $Q_{k'_1, k'_2}$, $R_{k'_1+1, k'_2}$ and $R_{k'_1, k'_2+1}$ are tensor polyhedral chains). Summing over $(k'_1, k'_2) \in D_k$ and optimizing, with respect to the decompositions of P , we get

$$\sum_{(k'_1, k'_2) \in D_k} \mathbb{F}^\wedge(P_{k'_1, k'_2}) \leq 2\mathbb{F}^\times(P).$$

This inequality and the density of $\mathcal{TP}_k^G(\mathbb{X}_\beta)$ in $\mathcal{F}_k^G(\mathbb{X}_\beta)$ allows to extend j on the latter. We obtain a continuous group morphism

$$\begin{aligned} j : \mathcal{F}_k^G(\mathbb{X}_\beta) &\longrightarrow \left(\mathcal{F}_{k'_1, k'_2}^G(\mathbb{X}_\beta) \right)_{(k'_1, k'_2) \in D_k}, \\ A &\longmapsto j_{k'_1, k'_2} A = A_{k'_1, k'_2}. \end{aligned}$$

Moreover, for $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$, we have the estimate

$$\sum_{(k'_1, k'_2) \in D_k} \mathbb{F}^\wedge(j_{k'_1, k'_2} A) \leq 2\mathbb{F}^\times(A). \quad (4.21)$$

In particular given $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$ a sequence of polyhedral chains $P_j \in \mathcal{P}_k^G(\mathbb{X}_\beta)$ such that $P_j \rightarrow A$, we have $j_{k'_1, k'_2} P_j \rightarrow j_{k'_1, k'_2} A$ for every $(k'_1, k'_2) \in D_k$. We deduce from (4.2), (4.4) and the lower semicontinuity of \mathbb{M}^\wedge in $(\mathcal{F}_{k'_1, k'_2}^G(\mathbb{X}_\beta), \mathbb{F}^\wedge)$ that,

$$\sum_{(k'_1, k'_2) \in D_k} \mathbb{M}^\wedge(j_{k'_1, k'_2} A) \leq \mathbb{M}^\times(A) \leq C(k, n_1, n_2) \mathbb{M}^\wedge(A). \quad (4.22)$$

For later use, we state the following relations.

Proposition 4.9. *For every $A \in \mathcal{F}_k^G(\mathbb{R}^n)$ and every $(k'_1, k'_2) \in D_{k-1}$,*

$$j_{k'_1, k'_2}(\partial A) = \partial_1 j_{k'_1+1, k'_2} A + \partial_2 j_{k'_1, k'_2+1} A.$$

Proof. The identity holds true by identification for polyhedral (k_1, k_2) -chains. Recalling that by the deformation theorem of White [Whi99a], $\mathcal{P}_{k_1, k_2}^G(\mathbb{R}^n)$ is dense in $\mathcal{F}_k^G(\mathbb{R}^n)$, the general case follows by continuity of j and of the (partial) boundary operators. \square

4.7 Tensor products of chains as tensor chains

Before introducing the restrictions and slices of tensor chains, we pause to discuss the structure of groups of tensor chains as (completion of) tensor products.

We have seen in Section 3 that given $P_1 \in \mathcal{P}_{k_1}^{\mathbb{Z}}(\mathbb{X}_{\beta^1})$ and $P_2 \in \mathcal{P}_{k_1}^G(\mathbb{X}_{\beta^2})$, we can form a chain $P := P^1 \times P^2 \in \mathcal{P}_k^G(\mathbb{X}_{\beta})$. More precisely, P is of the form (4.1) and we have $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_{\beta}) \subset \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_{\beta})$. Recall that when we consider P as an element of $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_{\beta})$ we write $P = P^1 \wedge P^2$. We have

$$\iota P = [P^2]P^1 \in \mathcal{P}_{k_1}(\mathbb{X}_{\beta^1}, \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})),$$

where here P^2 is a coefficient. It follows that

$$\mathbb{F}^{\wedge}(P^1 \wedge P^2) = \mathbb{F}(\iota P) \leq \mathbb{F}(P^1)\mathbb{F}(P^2).$$

We also have

$$\mathbb{M}^{\wedge}(P^1 \wedge P^2) \leq \mathbb{M}(P^1)\mathbb{M}(P^2).$$

In general, these inequalities are not identities. Indeed, if $G = \mathbb{Z}/2\mathbb{Z} = \{0, \bar{1}\}$ and p^1, p^2 are non zero polyhedral cells in \mathbb{X}_{β^1} and \mathbb{X}_{β^2} respectively, taking $P^1 = 2p^1$ and $P^2 = \bar{1}p^2$, the chains P^1 and P^2 do not vanish but

$$P^1 \wedge P^2 = (2p^1) \wedge (\bar{1}p^2) = p^1 \wedge (2 \cdot \bar{1}p^2) = p^1 \wedge (0p^2) = 0. \quad (4.23)$$

Anyway, we can extend the tensor product by density as a continuous \mathbb{Z} -bilinear mapping,

$$\begin{aligned} \mathcal{F}_{k_1}^{\mathbb{Z}}(\mathbb{X}_{\beta^1}) \times \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2}) &\longrightarrow \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_{\beta}), \\ (A^1, A^2) &\longmapsto A^1 \wedge A^2. \end{aligned} \quad (4.24)$$

Moreover, there hold

$$\partial_1(A^1 \wedge A^2) = (\partial A^1) \wedge A^2, \quad \partial_2(A^1 \wedge A^2) = (-1)^{k_1} A^1 \wedge (\partial A^2),$$

and

$$\mathbb{M}^{\wedge}(A^1 \wedge A^2) \leq \mathbb{M}(A^1)\mathbb{M}(A^2), \quad \mathbb{F}^{\wedge}(A^1 \wedge A^2) \leq \mathbb{F}(A^1)\mathbb{F}(A^2). \quad (4.25)$$

Taking finite sums,

$$\sum A_i^1 \wedge A_i^2 \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_{\beta}),$$

we obtain the tensor product of the groups $\mathcal{F}_{k_1}^{\mathbb{Z}}(\mathbb{X}_{\beta^1})$ and $\mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})$ that we denote $\mathcal{F}_{k_1}^{\mathbb{Z}}(\mathbb{X}_{\beta^1}) \otimes \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})$ and (4.24) extends as

$$\begin{aligned} \phi : \mathcal{F}_{k_1}^{\mathbb{Z}}(\mathbb{X}_{\beta^1}) \otimes \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2}) &\longrightarrow \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_{\beta}), \\ \sum A_i^1 \otimes A_i^2 &\longmapsto \sum A_i^1 \wedge A_i^2. \end{aligned} \quad (4.26)$$

By convention we use the index i for finite sums and the index j for (at most) countable sums. We observe that by density of tensor polyhedral chains, the image of ϕ is dense in $(\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_{\beta}), \mathbb{F}^{\wedge})$. Let us first state more precisely this density result and give another expression for $\mathbb{F}^{\wedge}(A)$. Apart from (4.27) the rest of this subsection is not used in the proof of the main results and can be safely skipped.

Proposition 4.10. *Let $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_{\beta})$, there exists (infinite) sequences $A_j^1 \in \mathcal{F}_{k_1}^{\mathbb{Z}}(\mathbb{X}_{\beta^1})$, $A_j^2 \in \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})$ such that,*

$$A = \sum A_j^1 \wedge A_j^2, \quad (4.27)$$

with a normal convergence of the series in \mathbb{F}^{\wedge} norm. Moreover,

$$\mathbb{F}^{\wedge}(A) = \inf \sum \mathbb{F}(A_j^1)\mathbb{F}(A_j^2), \quad (4.28)$$

where the infimum runs over the decompositions (4.27).

Proof. The proof is reminiscent of the proof of Proposition 4.8, we use similar notation. Let $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$. We denote by $\widetilde{\mathbb{F}}(A)$ the right-hand side of (4.28). For the moment we do not know whether A admits a decomposition of the form (4.27), if not we set $\widetilde{\mathbb{F}}(A) := +\infty$.

Let us consider a decomposition of A of the form (4.27). By the triangle inequality and (4.25), we have

$$\mathbb{F}^\wedge(A) \leq \sum \mathbb{F}^\wedge(A_j^1 \wedge A_j^2) \leq \sum \mathbb{F}(A_j^1) \mathbb{F}(A_j^2),$$

and taking the infimum with respect to all the decompositions, we obtain,

$$\mathbb{F}^\wedge(A) \leq \widetilde{\mathbb{F}}(A). \quad (4.29)$$

Let us establish the converse inequality. Let $P_j \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ converging rapidly to A and let $\varepsilon_j > 0$ converging to 0. From the proof of Proposition 4.8, there exist sequences of polyhedral tensor chains $Q_j^{i_1, i_2}, R_j^{i_1, i_2}$ for $i_1, i_2 \in \{0, 1\}$ such that,

$$\begin{aligned} P_j &= Q_j^{0,0} + \partial_1 Q_j^{1,0} + \partial_2 Q_j^{0,1} + \partial_1 \partial_2 Q_j^{1,1}, \\ P_{j+1} - P_j &= R_j^{0,0} + \partial_1 R_j^{1,0} + \partial_2 R_j^{0,1} + \partial_1 \partial_2 R_j^{1,1}, \end{aligned}$$

with

$$\begin{aligned} \sum_{i_1, i_2} \mathbb{M}(Q_j^{i_1, i_2}) &\leq \mathbb{F}^\wedge(P_j) + \varepsilon_j \quad \text{for } j \geq 1, \\ \sum_j \mathbb{M}(R_j^{i_1, i_2}) &< \infty \quad \text{for } i_1, i_2 \in \{0, 1\}. \end{aligned}$$

By construction, each tensor chain $Q = Q_j^{0,0}$ or $Q = R_j^{0,0}$ writes as a finite sum $\sum_i p_i^1 \wedge (g_i p_i^2)$ with $g_i \in G$ and p_i^ℓ oriented k_ℓ -cell of \mathbb{X}_{β^ℓ} such that

$$\mathbb{M}(Q) = \sum_i \mathbb{M}(p_i^1) \mathbb{M}(g_i p_i^2). \quad (4.30)$$

Denoting, $A_j := Q_j^{0,0} + \sum_{r \geq j} R_r^{0,0}$, we see that

$$A_j = \sum_s A_{j,s}^1 \wedge A_{j,s}^2,$$

for some sequence of polyhedral chains $A_{j,s}^1 \in \mathcal{P}_{k_1}^Z(\mathbb{X}_{\beta^1})$, $A_{j,s}^2 \in \mathcal{P}_{k_2}^G(\mathbb{X}_{\beta^2})$, with

$$\sum_s \mathbb{F}(A_{j,s}^1) \mathbb{F}(A_{j,s}^2) \leq \sum_s \mathbb{M}(A_{j,s}^1) \mathbb{M}(A_{j,s}^2) \leq \mathbb{M}(Q_j^{0,0}) + \sum_{r \geq j} \mathbb{M}(R_r^{0,0}). \quad (4.31)$$

Next, each chain $Q = Q_j^{1,0}$ or $Q = R_j^{1,0}$ also writes as a finite sum $\sum_i p_i^1 \wedge (g_i p_i^2)$ with $g_i \in G$ and p_i^ℓ oriented $(k_\ell + 1)$ -cell in \mathbb{X}_{β^ℓ} such that (4.30) holds. Writing

$$\partial_1 Q = \sum_i (\partial p_i^1) \wedge (g_i p_i^2),$$

and defining $B_j := \partial_1 Q_j^{1,0} + \sum_{r \geq j} \partial_1 R_r^{1,0}$, we see that B_j writes as

$$B_j = \sum_s (\partial B_{j,s}^1) \wedge B_{j,s}^2,$$

for some sequence of polyhedral chains $B_{j,s}^1 \in \mathcal{P}_{k_1+1}^{\mathbb{Z}}(\mathbb{X}_{\beta^1})$, $B_{j,s}^2 \in \mathcal{P}_{k_2}^G(\mathbb{X}_{\beta^2})$ and we have the estimate,

$$\sum_s \mathbb{F}(\partial B_{j,s}^1) \mathbb{F}(B_{j,s}^2) \leq \sum_s \mathbb{M}(B_{j,s}^1) \mathbb{M}(B_{j,s}^2) \leq \mathbb{M}(Q_j^{1,0}) + \sum_{r \geq j} \mathbb{M}(R_r^{1,0}). \quad (4.32)$$

Similarly, we define $C_j := \partial_2 Q_j^{0,1} + \sum_{r \geq j} \partial_2 R_r^{0,1}$ and $D_j := \partial_1 \partial_2 Q_j^{1,1} + \sum_{r \geq j} \partial_1 \partial_2 R_r^{1,1}$ which write as

$$C_j = \sum_s C_{j,s}^1 \wedge (\partial C_{j,s}^2), \quad D_j = \sum_s (\partial D_{j,s}^1) \wedge (\partial D_{j,s}^2),$$

with the estimates,

$$\sum_s \mathbb{F}(C_{j,s}^1) \mathbb{F}(\partial C_{j,s}^2) \leq \mathbb{M}(Q_j^{0,1}) + \sum_{r \geq j} \mathbb{M}(R_r^{0,1}), \quad (4.33)$$

$$\sum_s \mathbb{F}(\partial D_{j,s}^1) \mathbb{F}(\partial D_{j,s}^2) \leq \mathbb{M}(Q_j^{1,1}) + \sum_{r \geq j} \mathbb{M}(R_r^{1,1}). \quad (4.34)$$

By construction, there holds, for every $j \geq 1$,

$$\begin{aligned} A &= A_j + B_j + C_j + D_j \\ &= \sum_s A_{j,s}^1 \wedge A_{j,s}^2 + \sum_s (\partial B_{j,s}^1) \wedge B_{j,s}^2 + (-1)^{k_1} \sum_s C_{j,s}^1 \wedge (\partial C_{j,s}^2) + (-1)^{k_1} \sum_s (\partial D_{j,s}^1) \wedge (\partial D_{j,s}^2), \end{aligned}$$

which is a series of the form (4.27). Collecting (4.31)–(4.34), we obtain for $j \geq 1$,

$$\tilde{\mathbb{F}}(A) \leq \sum_{i_1, i_2 \in \{0,1\}} \left(\mathbb{M}(Q_j^{i_1, i_2}) + \sum_{r \geq j} \mathbb{M}(R_r^{i_1, i_2}) \right) \leq \mathbb{F}^\wedge(A) + \varepsilon'_j,$$

with $\varepsilon'_j \rightarrow 0$ as $j \uparrow \infty$. We conclude that $\tilde{\mathbb{F}}(A) \leq \mathbb{F}^\wedge(A)$ and with (4.29) we get (4.28). \square

In the context of Banach spaces, given two (say real) Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, the natural norms on their algebraic tensor product $X \otimes Y$ are the so called cross norms which satisfy $\|x \otimes y\| = \|x\|_X \|y\|_Y$ for every $x \in X$, $y \in Y$. These norms have been classified by Grothendieck in [Gro53] (see [Pis12, Section 3] or [Rya02] for an account in english). Among these norms the projective norm is defined for

$$z = \sum_i x_i \otimes y_i \in X \otimes Y, \quad (4.35)$$

by

$$\pi(z) := \inf \sum_i \|x_i\|_X \|y_i\|_Y,$$

where we take the infimum over all representations of z as finite sums of the form (4.35). With this definition, π is a seminorm on $X \otimes Y$ and it turns out that it is also a norm. Moreover, by the triangle inequality, we have $\|\cdot\| \leq \pi$ for any cross norm $\|\cdot\|$ so the projective norm is the largest natural norm and the space $X \widehat{\otimes}_\pi Y$, obtained by taking the completion of $(X \otimes Y, \pi)$ is the smallest natural completion of $X \otimes Y$.

Let us extend these notions to the tensor product $\mathcal{G} := \mathcal{G}_1 \otimes \mathcal{G}_2$ of the two complete normed Abelian groups

$$\mathcal{G}_1 := (\mathcal{F}_{k_1}^{\mathbb{Z}}(\mathbb{X}_{\beta^1}), +, \mathbb{F}), \quad \mathcal{G}_2 := (\mathcal{F}_{k_1}^G(\mathbb{X}_{\beta^2}), +, \mathbb{F}).$$

More precisely, \mathcal{G} is the group of formal finite sums $\sum (A_i^1, A_i^2)$ of pairs $(A_i^1, A_i^2) \in \mathcal{G}_1 \times \mathcal{G}_2$ quotiented by the relations

$$(A_a^1 + A_b^1, A^2) = (A_a^1, A^2) + (A_b^1, A^2), \quad (A^1, A_a^2 + A_b^2) = (A^1, A_a^2) + (A^1, A_b^2),$$

for every $A^l, A_a^l, A_b^l \in \mathcal{G}_l, l \in \{1, 2\}$. The coset of $\sum(A_i^1, A_i^2)$ is denoted $\sum A_i^1 \otimes A_i^2$. Next, we say that a seminorm $\|\cdot\|$ on \mathcal{G} is a cross seminorm if

$$\|A^1 \otimes A^2\| \leq \mathbb{F}(A^1)\mathbb{F}(A^2) \quad \text{for every } A^1 \in \mathcal{G}_1, A^2 \in \mathcal{G}_2. \quad (4.36)$$

By the second estimate of (4.25) the mapping defined by $\|\sum A_i^1 \otimes A_i^2\|_{\mathbb{F}^\wedge} := \mathbb{F}^\wedge(\sum A_i^1 \wedge A_i^2)$ defines a cross seminorm on \mathcal{G} (notice that in view of (4.23) it would be too optimistic to ask for equality in (4.36)).

Next, we define the projective seminorm of $A \in \mathcal{G}$ as

$$\pi(A) := \inf \sum_i \mathbb{F}(A_i^1)\mathbb{F}(A_i^2),$$

where the infimum runs over the representation of A as *finite* sums $\sum_i A_i^1 \otimes A_i^2$. Again, by the triangle inequality, we have $\|\cdot\| \leq \pi$ for any cross seminorm $\|\cdot\|$ on \mathcal{G} .

By Proposition 4.10 the morphism of (4.26) satisfies

$$\mathbb{F}^\wedge(\phi(A)) \leq \pi(A) \quad \text{for every } A \in \mathcal{G}.$$

In particular $\phi \equiv 0$ on the subgroup

$$\mathcal{G}_0 := \{A \in \mathcal{G} : \pi(A) = 0\}.$$

Denoting $\mathcal{H} := \mathcal{G}/\mathcal{G}_0$ and setting $\pi(A + \mathcal{G}_0) := \pi(A)$ for $A \in \mathcal{G}$, we have that $(\mathcal{H}, +, \pi)$ is an Abelian normed group and taking the quotient in (4.26), we obtain a 1-Lipschitz group morphism

$$\begin{aligned} \tilde{\phi} : (\mathcal{H}, +, \pi) &\longrightarrow (\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta), +, \mathbb{F}^\wedge) \\ A + \mathcal{G}_0 &\longmapsto \phi(A). \end{aligned}$$

Eventually, denoting $\mathcal{G}_1 \widehat{\otimes}_\pi \mathcal{G}_2$ the completion of $(\mathcal{G}/\mathcal{G}_0, \pi)$ and extending $\tilde{\phi}$ by continuity we obtain a 1-Lipschitz group morphism

$$\widehat{\phi} : (\mathcal{G}_1 \widehat{\otimes}_\pi \mathcal{G}_2, +, \pi) \longrightarrow (\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta), +, \mathbb{F}^\wedge).$$

In the introduction, we claimed that $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ could be viewed as the complement of $\mathcal{G}_1 \otimes \mathcal{G}_2$ with respect to some appropriate norm. This is justified by the following result up to the operation of taking the quotient of $\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2$ with respect to \mathcal{G}_0 . We do not know whether in general $\mathcal{G}_0 = \{0\}$ or equivalently whether π is a norm on \mathcal{G} .

Proposition 4.11. *With the above notations, the mapping $\widehat{\phi}$ is an isometric group isomorphism from $(\mathcal{G}_1 \widehat{\otimes}_\pi \mathcal{G}_2, +, \pi)$ onto $(\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta), +, \mathbb{F}^\wedge)$.*

Remark 4.12. With this proposition, we see that the norm we choosed on $\mathcal{F}_{k_1}^Z(\mathbb{X}_{\beta^1}) \otimes \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})$ is the largest among “natural” norms and that our definition of $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ corresponds to the smallest complete normed group that extends the definition of the product $(A^1, A^2) \mapsto A^1 \times A^2$ to any pair of chains $(A^1, A^2) \in \mathcal{F}_{k_1}^Z(\mathbb{X}_{\beta^1}) \times \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})$ as a Lipschitz continuous group morphism.

Proof of Proposition 4.11. Since the image of ϕ is dense in $(\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta), \mathbb{F}^\wedge)$ it is enough to establish that $\mathbb{F}^\wedge(A) = \mathbb{F}^\wedge(\pi(A))$ for every $\widehat{A} \in \mathcal{G}_1 \widehat{\otimes}_\pi \mathcal{G}_2$. Equivalently, in view of Proposition 4.10 it is enough to establish that for every $\widehat{A} \in \mathcal{G}_1 \widehat{\otimes}_\pi \mathcal{G}_2$ there holds

$$\pi(\widehat{A}) = \inf \left\{ \sum_{j \geq 1} \mathbb{F}(A_j^1)\mathbb{F}(A_j^2) : \widehat{A} = \sum_{j \geq 1} [A_j^1 \wedge A_j^2 + \mathcal{G}_0] \right\}, \quad (4.37)$$

where the convergence of the series is in π norm. In fact this identity is the analogue in the context of tensor products of Abelian normed groups of [Rya02, Proposition 2.8] which deals with tensor products of Banach spaces.

Let us prove (4.37). Let $\widehat{A} \in \mathcal{G}_1 \widehat{\otimes}_\pi \mathcal{G}_2$. We denote by $\widetilde{\pi}$ the right-hand side of (4.37). First given a decomposition $\widehat{A} = \sum_{j \geq 1} [A_j^1 \wedge A_j^2 + \mathcal{G}_0]$ with $\sum \mathbb{F}(A_j^1) \mathbb{F}(A_j^2) < \infty$ we introduce the partial sums

$$\widehat{A}_j := \sum_{i=1}^j [A_i^1 \wedge A_i^2 + \mathcal{G}_0].$$

Since $\pi(A_j^1 \wedge A_j^2) \leq \mathbb{F}(A_j^1) \mathbb{F}(A_j^2)$, \widehat{A}_j is a Cauchy sequence in $(\mathcal{G}_1 \widehat{\otimes}_\pi \mathcal{G}_2, \pi)$. Its limit is \widehat{A} and we have $\pi(\widehat{A}) \leq \sum \mathbb{F}(A_j^1) \mathbb{F}(A_j^2)$. Taking the infimum with respect to the decompositions, we get

$$\pi(\widehat{A}) \leq \widetilde{\pi}(\widehat{A}). \quad (4.38)$$

Let us establish the converse inequality. Let $\widehat{A}_j = A_j + \mathcal{G}_0 \in \mathcal{G}/\mathcal{G}_0$ with $\pi(\widehat{A}_j - \widehat{A}) \rightarrow 0$. Let $\varepsilon > 0$, up to extraction we may assume that $\pi(\widehat{A}_1 - \widehat{A}) < \varepsilon/3$ and denoting $B_j = A_{j+1} - A_j$,

$$\sum \pi(B_j) < \varepsilon/3.$$

By definition of π , there exist finite sequences $A_r^l \in \mathcal{G}_l$ and (for $j \geq 1$) $B_{j,s}^l \in \mathcal{G}_l$ such that

$$A_1 = \sum_r A_r^1 \otimes A_r^2, \quad B_j = \sum_s B_{j,s}^1 \otimes B_{j,s}^2$$

and

$$\sum_r \mathbb{F}(A_r^1) \mathbb{F}(A_r^2) + \sum_j \sum_s \mathbb{F}(B_{j,s}^1) \mathbb{F}(B_{j,s}^2) < \pi(A_1) + \sum \pi(B_j) + \varepsilon/3 < \pi(\widehat{A}) + \varepsilon. \quad (4.39)$$

Writing

$$\widehat{A} = \sum_r [A_r^1 \otimes A_r^2 + \mathcal{G}_0] + \sum_j \sum_s [B_{j,s}^1 \otimes B_{j,s}^2 + \mathcal{G}_0] =: \sum_t [C_t^1 \otimes C_t^2 + \mathcal{G}_0],$$

we deduce

$$\widetilde{\pi}(\widehat{A}) \leq \sum_t \mathbb{F}^\wedge(C_t^1) \mathbb{F}^\wedge(C_t^2) \stackrel{(4.39)}{<} \pi(\widehat{A}) + \varepsilon.$$

Since ε is arbitrary we get $\widetilde{\pi}(\widehat{A}) \leq \pi(\widehat{A})$ and together with (4.38), this establishes (4.37) and the proposition. \square

Let us consider eventually the particular case when $G = V$ is a Banach space. In this case we have

$$\mathcal{F}_{k_1}^{\mathbb{Z}}(\mathbb{X}_{\beta^1}) \otimes \mathcal{F}_{k_2}^V(\mathbb{X}_{\beta^2}) = \mathcal{F}_{k_1}^{\mathbb{R}}(\mathbb{X}_{\beta^1}) \otimes \mathcal{F}_{k_2}^V(\mathbb{X}_{\beta^2}),$$

that is: the ‘‘group tensor product’’ is the same as the ‘‘vector tensor product’’. As a direct consequence of Proposition 4.11 and [Rya02, Proposition 2.1] we have the following result.

Proposition 4.13.

\mathbb{F}^\wedge is the projective norm on the tensor product of Banach spaces $\mathcal{F}_{k_1}^{\mathbb{R}}(\mathbb{X}_{\beta^1}) \otimes \mathcal{F}_{k_2}^V(\mathbb{X}_{\beta^2})$.

In particular, when G is a Banach space, the second inequality of (4.25) is an identity. We also have in this case $\mathcal{G}_0 = \{0\}$.

4.8 Slicing of tensor chains

The slicing operators Sl_γ have been introduced in Section 3. Let us check that these operators have a good behavior on tensor polyhedral chains. Let $\gamma \subset \beta$ and let $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$ with representation (4.1). From the definition of Sl_γ we have that for almost every $x = x^1 + x^2 \in \mathbb{X}_\gamma$ with $x^l \in \mathbb{X}_{\gamma^l}$,

$$\text{Sl}_\gamma^x P = \sum g_i(\text{Sl}_{\gamma^1}^{x^1} p_i^1) \wedge (\text{Sl}_{\gamma^2}^{x^2} p_i^2).$$

We deduce from this formula that

$$\partial_1 \text{Sl}_\gamma^x P = \text{Sl}_\gamma^x \partial_1 P, \quad \partial_2 \text{Sl}_\gamma^x P = (-1)^{|\gamma^1|} \text{Sl}_\gamma^x \partial_2 P \quad \text{for almost every } x \in \mathbb{X}_\gamma.$$

Together with the bound

$$\int_{\mathbb{X}_\gamma} \mathbb{M}^\wedge(\text{Sl}_\gamma^x P) dx \leq \mathbb{M}^\wedge(P),$$

this leads to

$$\int_{\mathbb{X}_\gamma} \mathbb{F}^\wedge(\text{Sl}_\gamma^x P) dx \leq \mathbb{F}^\wedge(P).$$

Moreover, denoting $r_1 = |\gamma^1|$, $r_2 = |\gamma^2|$, there holds for $(k'_1, k'_2) \in D_k$ (recall the definition (4.3) of j), and $P \in \mathcal{P}_k^G(\mathbb{X}_\beta)$,

$$j_{k'_1 - r_1, k'_2 - r_2} \text{Sl}_\gamma^x P = \text{Sl}_\gamma^x j_{k'_1, k'_2} P.$$

Notice that if $P \in \mathcal{P}_{k_1, k_2}^G(\mathbb{X}_\beta)$, then $j_{k'_1, k'_2} P = 0$ if $(k'_1, k'_2) \neq (k_1, k_2)$. If $\gamma \subset \beta^1$, we have the identity

$$\text{Sl}_\gamma^x \iota P = \iota \text{Sl}_\gamma^x P.$$

Eventually, by construction $\text{Sl}_\gamma^x P$ is a tensor chain in $\mathbb{X}_{\beta \setminus \gamma}$.

We deduce the following result by continuity and density arguments.

Proposition 4.14. *Let $\gamma \subset \beta$, the slicing operators Sl_γ extend as continuous group morphisms,*

$$\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta) \longrightarrow L^1(\mathbb{X}_\gamma, \mathcal{F}_{k_1 - |\gamma^1|, k_2 - |\gamma^2|}^G(\mathbb{X}_{\beta \setminus \gamma})),$$

with the following properties, for $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ and almost every $x \in \mathbb{X}_\gamma$.

(i) $\partial_1 \text{Sl}_\gamma^x A = \text{Sl}_\gamma^x \partial_1 A$ and $\partial_2 \text{Sl}_\gamma^x A = (-1)^{|\gamma^1|} \text{Sl}_\gamma^x \partial_2 A$.

(ii) We have the estimates

$$\int_{\mathbb{X}_\gamma} \mathbb{M}^\wedge(\text{Sl}_\gamma^x A) dx \leq \mathbb{M}^\wedge(A) \quad \text{and} \quad \int_{\mathbb{X}_\gamma} \mathbb{F}^\wedge(\text{Sl}_\gamma^x A) dx \leq \mathbb{F}^\wedge(A).$$

(iii) For $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$ and $(k'_1, k'_2) \in D_k$, there holds $j_{k'_1 - |\gamma^1|, k'_2 - |\gamma^2|} \text{Sl}_\gamma^x A = \text{Sl}_\gamma^x j_{k'_1, k'_2} A$.

(iv) If $\gamma \subset \beta^1$, there holds $\text{Sl}_\gamma^x \iota A = \iota \text{Sl}_\gamma^x A$.

Let us establish the counterpart of Theorem 3.3 for tensor chains, namely that if the $(0, 0)$ -slices of a tensor chain vanish then it also vanishes.

Proposition 4.15. *If $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ is such that $\text{Sl}_\gamma A = 0$ for every $\gamma \subset \beta$ with $(|\gamma^1|, |\gamma^2|) = (k_1, k_2)$, then $A = 0$.*

Proof. Let $A \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ and let us assume that $\text{Sl}_\gamma A = 0$ for every $\gamma \subset \beta$ such that $(|\gamma^1|, |\gamma^2|) = (k_1, k_2)$. We fix $\gamma^2 \in I_{k_2}^n$ with $\gamma^2 \subset \beta^2$ and denote $A_{\gamma^2} := \iota \text{Sl}_{\gamma^2} A$. Recalling Proposition 4.14(iv) (commutativity of ι with slicing) and Proposition 3.2(ii) (iterate slicing), we compute, for every $\gamma^1 \in I_{k_1}^n$ with $\gamma^1 \subset \beta^1$,

$$\text{Sl}_{\gamma^1} A_{\gamma^2} = \text{Sl}_{\gamma^1} \iota \text{Sl}_{\gamma^2} A = \iota \text{Sl}_{\gamma^1} \text{Sl}_{\gamma^2} A = \iota \text{Sl}_\gamma A = 0.$$

Since this holds true for every $\gamma^1 \in I_{k_1}^n$, we deduce from Theorem 3.3 that $A_{\gamma^2} = 0$. By injectivity of ι (since it is an isometry) we get

$$\text{Sl}_{\gamma^2} A = 0 \quad \text{for every } \gamma^2 \in I_{k_2}^n \text{ such that } \gamma^2 \subset \beta^2. \quad (4.40)$$

Exchanging the roles of β^1 and β^2 and denoting $\bar{\iota} : \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta) \rightarrow \mathcal{F}_{k_2}(\mathbb{X}_{\beta^2}, \mathcal{F}_{k_1}^G(\mathbb{X}_{\beta^1}))$ the corresponding operator, we compute for $\gamma^2 \in I_{k_2}^n$ with $\gamma^2 \subset \beta^2$,

$$\text{Sl}_{\gamma^2} \bar{\iota} A = \bar{\iota} \text{Sl}_{\gamma^2} A \stackrel{(4.40)}{=} 0.$$

Again, Theorem 3.3 leads to $\bar{\iota} A = 0$, that is $A = 0$. \square

4.9 Restriction of finite mass tensor chains

To define the restrictions of finite mass tensor chains on Borel subsets, we can proceed exactly as in [Fle66] using Lemma 4.5(i) in place of [Fle66, Lemma 2.1] and the \mathbb{F}^\wedge -convergence in place of the \mathbb{F} -convergence. Given $A \in \mathcal{M}_{k_1, k_2}^G(\mathbb{X}_\beta)$, we obtain a finite positive Borel measure μ_A and a $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ -valued measure $A \llcorner \cdot$ satisfying the relation $\mathbb{M}^\wedge(A \llcorner S) = \mu_A(S)$ for S Borel subset of \mathbb{X}_β . Besides $A \llcorner S$ is supported in \bar{S} and for a sequence $A_j \in \mathcal{M}_{k_1, k_2}^G(\mathbb{X}_\beta)$ with $\sum \mathbb{M}^\wedge(A_j) < \infty$, there holds

$$\left(\sum A_j \right) \llcorner S = \sum (A_j \llcorner S).$$

Eventually, given a sequence $A_j \rightarrow A$ with $\mathbb{M}^\wedge(A_j) \rightarrow \mathbb{M}^\wedge(A)$ we have $A_j \llcorner S \rightarrow A \llcorner S$ for any Borel subset such that $\mu_A(\text{Bdry } S) = 0$.

With the aim of establishing the natural identities involving the restrictions of A , ιA and jA , we recall the main lines of the construction of μ_A and $A \llcorner \cdot$.

Let $A \in \mathcal{M}_{k_1, k_2}^G(\mathbb{X}_\beta)$. We first define the measure μ_A . Let P_j converging to A rapidly and such that $\mathbb{M}^\wedge(P_j) \rightarrow \mathbb{M}^\wedge(A)$. The sequence μ_{P_j} is bounded, hence there exists a finite positive Borel measure μ_A such that up to extraction $\mu_{P_j} \xrightarrow{*} \mu_A$.

Let I be an interval of \mathbb{X}_β . By Lemma 4.5(i), for almost every $x \in \mathbb{X}_\beta$, the series $\sum \mathbb{F}^\wedge([P_{j+1} - P_j] \llcorner (x + I))$ is converging. We define for such x ,

$$A \llcorner (x + I) := P_1 \llcorner (x + I) + \sum (P_{j+1} - P_j) \llcorner (x + I).$$

The construction then proceeds as follows. We first show a counterpart of [Fle66, Lemma 4.1], that is, for almost every $x \in \mathbb{X}_\beta$, there hold

$$\mathbb{M}^\wedge(A \llcorner (x + I)) = \mu_A(x + I) \quad \text{and} \quad \mathbb{M}^\wedge(A - A \llcorner (x + I)) = \mu_A(\mathbb{X}_\beta \setminus (x + I)).$$

This allows us to define $A \llcorner J$ for J finite union of disjoint non-exceptional intervals (with respect to the sequence P_j). This restriction is an additive set functions and we still have $\mathbb{M}^\wedge(A \llcorner J) = \mu_A(J)$. Since the sets of this form span the Borel algebra, we can extend $A \llcorner \cdot$ as a Borel measure with values in $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$. Notice that arguing as in [Fle66], both μ_A and $A \llcorner \cdot$ do not depend on the choice of sequence P_j .

In the proof of Proposition 4.16 below we use the following facts:

- (1) The set function $A \llcorner \cdot$ is determined by its values on any collection of sets $x + I$ where I ranges over the intervals of \mathbb{X}_β and x ranges over $\mathbb{X}_\beta \setminus E_I$ for some negligible set $E_I \subset \mathbb{X}_\beta$.
- (2) If $I \subset \mathbb{X}_\beta$ is an interval and $Q_j \rightarrow A$ rapidly, then $Q_j \llcorner (x + I) \rightarrow A \llcorner (x + I)$ for almost every $x \in \mathbb{X}_\beta$.

Proposition 4.16. *Let $A' \in \mathcal{M}_{k_1, k_2}^G(\mathbb{X}_\beta)$, $A \in \mathcal{M}_k^G(\mathbb{X}_\beta)$ and let $S, \tilde{S} \subset \mathbb{X}_\beta$ be Borel sets. We have the following properties.*

- (i) For $\gamma \subset \beta$ and almost every $x \in \mathbb{X}_\gamma$, $[A' \cap \mathbb{X}_{\beta \setminus \gamma}(x)] \llcorner S = [A' \llcorner S] \cap \mathbb{X}_{\beta \setminus \gamma}(x)$.
With the Sl_γ operator, this translates into $\text{Sl}_\gamma^x(A' \llcorner S) = (\text{Sl}_\gamma^x A') \llcorner ([S - x] \cap \mathbb{X}_{\beta \setminus \gamma})$.
- (ii) $(J_{k'_1, k'_2} A) \llcorner S = J_{k'_1, k'_2}(A \llcorner S)$ for any $(k'_1, k'_2) \in D_k$.
- (iii) If $S \subset \mathbb{X}_{\beta^1}$ then $(\iota A') \llcorner S = \iota(A' \llcorner [S + \mathbb{X}_{\beta^2}])$.
- (iv) $(A' \llcorner S) \llcorner \tilde{S} = A' \llcorner (S \cap \tilde{S})$.

Proof. These identities hold true when A' is a tensor polyhedral chain (resp. A is a polyhedral chain) and when S and \tilde{S} are intervals of \mathbb{X}_β or of \mathbb{X}_{β^1} for point (iii). Taking a sequence of tensor polyhedral chains P'_j converging rapidly to A' (resp. a sequence of polyhedral chains P_j converging rapidly to A for (ii)) and using the continuity properties of the operators Sl_γ , J and ι , the result follows from the two facts stated above. \square

5 Identification of normal tensor chains with normal chains

In this section, we first identify finite mass $(0, 0)$ -chains with finite mass 0 -chains and deduce that J is one-to-one. Next, we introduce the subgroups of k -chains $\mathcal{N}_{k, (k_1, k_2)}(\mathbb{R}^n)$ formed by the elements of $\mathcal{N}_k^G(\mathbb{R}^n)$ which are in the kernel of $J_{k'_1, k'_2}$ for every $(k'_1, k'_2) \in D_k \setminus \{(k_1, k_2)\}$. For our purpose, the interest of these groups lies in the fact that a normal *rectifiable* k -chain belongs to $\mathcal{N}_{k, (k_1, k_2)}^G(\mathbb{R}^n)$ if and only if it is (k_1, k_2) -split (see Corollary 5.12). We also introduce the subgroup of normal (k_1, k_2) -chains $\mathcal{N}_{k_1, k_2}^G(\mathbb{R}^n)$ and show that J_{k_1, k_2} maps (injectively) $\mathcal{N}_{k, (k_1, k_2)}^G(\mathbb{R}^n)$ into $\mathcal{N}_{k_1, k_2}^G(\mathbb{R}^n)$. We also prove that for $k_1 = 0$ this mapping is onto (Theorem 5.15).¹⁰ These results allow us to identify normal rectifiable k -chains which are (k_1, k_2) -split with some normal rectifiable (k_1, k_2) -chains and deduce the main result, Theorem 1.3 (reformulated as Theorem 6.4) from its counterpart for normal rectifiable (k_1, k_2) -chains (Theorem 6.3).

From now on we use capital letters for chains A, B, C, Q, \dots and capital letters with a prime for tensor chains: A', B', C', Q', \dots . We also use the notation A'' for the chain $\iota A'$.

5.1 Finite mass $(0, 0)$ -chains as measures and injectivity of J

Recall that by Theorem 3.7, there exists an isometric isomorphism $\psi : \mathcal{M}_0^G(\mathbb{X}_\beta) \rightarrow \mathcal{M}(\mathbb{X}_\beta, G)$. We generalize the construction of [Whi99b, Section 2] to obtain a similar result for $(0, 0)$ -chains in Lemma 5.5. For this we need to introduce further material.

As a preliminary we recall the following from [Whi99b].

Theorem 5.1 ([Whi99b, Theorem 2.1]). *There exists a group morphism $\chi : \mathcal{F}_0^G(\mathbb{X}_\beta) \rightarrow G$ such that,*

- (i) $\chi(\sum g_i \llbracket x_i \rrbracket) = \sum g_i$, for any finite sequences $g_i \in G$ and $x_i \in \mathbb{X}_\beta$.
- (ii) $|\chi(A)|_G \leq \mathbb{F}(A)$ for every $A \in \mathcal{F}_0^G(\mathbb{X}_\beta)$.

¹⁰In [GM22a] we establish that in fact $J_{k_1, k_2} : \mathcal{N}_{k, (k_1, k_2)}^G(\mathbb{R}^n) \rightarrow \mathcal{N}_{k_1, k_2}^G(\mathbb{R}^n)$ is onto for every $(k_1, k_2) \in D_k$.

The operator χ is defined on polyhedral 0-chains by the relation (i) and then extended by continuity thanks to (ii). Notice also that given a polyhedral 1-chain B , there holds $\chi(\partial B) = 0$. We need a complement of this result which is not stated in [Whi99b] but which follows from the construction there. Let $(G^a, +, |\cdot|_{G^a})$ and $(G^b, +, |\cdot|_{G^b})$ be two complete Abelian normed groups and let $\phi : G^a \rightarrow G^b$ be a Lipschitz continuous group morphism and $\Phi : G^a \rightarrow L^1(\Omega, G^b)$ as in Proposition 3.9.

Proposition 5.2. *With the notation of Proposition 3.9, we have for every $A \in \mathcal{F}_0(\mathbb{X}_\beta, G^a)$,*

$$\chi(\phi_* A) = \phi(\chi(A)), \quad \chi[(\Phi_* A)(\omega)] = (\Phi[\chi(A)])(\omega) \quad \text{for almost every } \omega \in \Omega.$$

Proof. These identities hold true for polyhedral 0-chains and the general case follows by density of these latter in $\mathcal{F}_0(\mathbb{X}_\beta, G^a)$ and Lipschitz continuity of ϕ , Φ and of the operator χ (point (ii) of Theorem 5.1). \square

We introduce the corresponding operator on $\mathcal{F}_{0,0}^G(\mathbb{X}_\beta)$.

Definition 5.3. We set for $A' \in \mathcal{F}_{0,0}^G(\mathbb{X}_\beta)$,

$$\chi^\wedge(A') := \chi(\chi(\iota A')).$$

Beware that on the right-hand side the notation χ refers both to the morphism $\mathcal{F}_0(\mathbb{X}_{\beta^2}, G) \rightarrow G$ and to the morphism $\mathcal{F}_0(\mathbb{X}_{\beta^1}, \mathcal{F}_0(\mathbb{X}_{\beta^2}, G)) \rightarrow \mathcal{F}_0(\mathbb{X}_{\beta^2}, G)$.

Proposition 5.4. *The group morphism $\chi^\wedge : \mathcal{F}_{0,0}^G(\mathbb{X}_\beta) \rightarrow G$ satisfies the following properties.*

(i) *For every $A' \in \mathcal{F}_{0,0}^G(\mathbb{X}_\beta)$, there holds $|\chi^\wedge(A')|_G \leq \mathbb{F}^\wedge(A') \leq \mathbb{M}^\wedge(A')$.*

(ii) *For every $A \in \mathcal{F}_0^G(\mathbb{X}_\beta)$, there holds $\chi(A) = \chi^\wedge(j_{0,0}A)$.*

Proof. Let $A' \in \mathcal{F}_{0,0}^G(\mathbb{X}_\beta)$. Using point (ii) of Theorem 5.1 twice, we have

$$|\chi^\wedge(A')|_G = \left| \chi \left[\underbrace{\chi(\iota A')}_{\in \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})} \right] \right|_G \leq |\chi(\iota A')|_{\mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})} \leq \mathbb{F}(\iota A') = \mathbb{F}^\wedge(A') \leq \mathbb{M}^\wedge(A'),$$

where we used the notation $|B|_{\mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})} := \mathbb{F}(B)$ for $B \in \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})$. This proves (i). Next, (ii) holds true for 0-polyhedral chains and the general case follows by density and continuity. \square

We can now generalize Theorem 3.7 to tensor chains, with the extra relation (5.1) below.

Lemma 5.5. *There is an isometric isomorphism,*

$$\psi^\wedge : (\mathcal{M}_{0,0}^G(\mathbb{X}_\beta), +, \mathbb{M}^\wedge) \longrightarrow (\mathcal{M}(\mathbb{X}_\beta, G), +, |\cdot|).$$

Moreover, ψ^\wedge is related to the mapping ψ of Theorem 3.7 by

$$\psi = \psi^\wedge \circ j_{0,0} \quad \text{on } \mathcal{M}_0^G(\mathbb{X}_\beta). \quad (5.1)$$

Proof. Let $A' \in \mathcal{M}_{0,0}^G(\mathbb{X}_\beta)$. For every Borel subset $S \subset \mathbb{X}_\beta$, we define,

$$\psi_{A'}^\wedge(S) := \chi^\wedge(A' \llcorner S).$$

Using Proposition 5.4(i) and arguing as in the proof of [Whi99b, Theorem 2.2], we get that $\psi_{A'}^\wedge$ is a finite Borel measure in \mathbb{X}_β with values in G . Moreover,

$$|\psi_{A'}^\wedge| \leq \mathbb{M}^\wedge(A'). \quad (5.2)$$

Let us establish (5.1). Let $A \in \mathcal{M}_0^G(\mathbb{X}_\beta)$. By Proposition 5.4(ii) we have for any Borel set $S \subset \mathbb{X}_\beta$,

$$\psi_{j_{0,0}A}^\wedge(S) = \chi^\wedge((j_{0,0}A) \perp S) = \chi(A \perp S).$$

Now, by definition (see [Whi99b, Theorem 2.2]), $\chi(A \perp S) = \psi_A(S)$. Hence (5.1) holds true.

It remains to show that $A' \mapsto \psi_{A'}^\wedge$ is onto and that

$$\mathbb{M}^\wedge(A') \leq |\psi_{A'}^\wedge|. \quad (5.3)$$

Let $\nu \in \mathcal{M}(\mathbb{X}_\beta, G)$. Since ψ is an isometric isomorphism, there exists $A \in \mathcal{M}_0^G(\mathbb{X}_\beta)$ such that $\psi_A = \nu$ and $\mathbb{M}(A) = |\nu|$. Setting $A' := j_{0,0}A \in \mathcal{M}_{0,0}^G(\mathbb{X}_\beta)$, we have $\psi_{A'}^\wedge = \nu$ by (5.1). This proves that ψ^\wedge is onto. Eventually, there holds

$$\mathbb{M}^\wedge(A') = \mathbb{M}(j_{0,0}A) \leq \mathbb{M}(A) = |\nu|,$$

and we get (5.3). With (5.2), we conclude that ψ^\wedge is an isometry from $(\mathcal{M}_{0,0}^G(\mathbb{X}_\beta), \mathbb{M}^\wedge)$ onto $(\mathcal{M}(\mathbb{X}_\beta, G), |\cdot|)$. \square

The previous lemma allows to identify isomorphically $\mathcal{M}_{0,0}^G(\mathbb{X}_\beta)$ with $\mathcal{M}_0^G(\mathbb{X}_\beta)$ so that, essentially, $\mathcal{M}_{0,0}^G(\mathbb{R}^n)$ does not depend on n_1, n_2 (but as shown by the example of Remark 4.2(d), $\mathcal{F}_{0,0}^G(\mathbb{X}_\beta)$ does!). Identity (5.1) also provides a natural isometric isomorphism.

Theorem 5.6.

(i) *The mapping $j_{0,0} : (\mathcal{M}_0^G(\mathbb{X}_\beta), +, \mathbb{M}) \longrightarrow (\mathcal{M}_{0,0}^G(\mathbb{X}_\beta), +, \mathbb{M}^\wedge)$ is an isometric isomorphism. In the sequel, its inverse is denoted Λ_0 .*

(ii) *Besides, $\mathbb{M}^\wedge(j_{0,0}A) = \mathbb{M}(A)$ for every $A \in \mathcal{F}_0^G(\mathbb{X}_\beta)$. In particular, $j_{0,0}A = 0 \implies A = 0$.*

Proof. From the previous lemma and Theorem 3.7, we have the following commutative diagram of isomorphisms,

$$\begin{array}{ccc} \mathcal{M}_0^G(\mathbb{X}_\beta) & \xrightarrow{j_{0,0}} & \mathcal{M}_{0,0}^G(\mathbb{X}_\beta) \\ & \searrow \psi & \swarrow \psi^\wedge \\ & \mathcal{M}(\mathbb{X}_\beta, G) & \end{array}$$

Moreover ψ and ψ^\wedge are isometries and we conclude that,

$$j_{0,0} : (\mathcal{M}_0^G(\mathbb{X}_\beta), +, \mathbb{M}) \longrightarrow (\mathcal{M}_{0,0}^G(\mathbb{X}_\beta), +, \mathbb{M}^\wedge),$$

is an isometric isomorphism, which is (i). As a consequence, the identity of the second point holds true as soon as $\mathbb{M}(A)$ is finite but we still have to check that $\mathbb{M}^\wedge(j_{0,0}A) = \infty \iff \mathbb{M}(A) = \infty$. This follows again from the diagram above. Indeed we have for $A \in \mathcal{F}_0^G(\mathbb{X}_\beta)$,

$$\mathbb{M}(A) < \infty \iff A \in \text{Im } \psi^{-1} \iff j_{0,0}A \in \text{Im } \psi^{\wedge^{-1}} \iff \mathbb{M}^\wedge(j_{0,0}A) < \infty.$$

This proves the theorem. \square

Using slicing, we deduce that j is one-to-one.

Theorem 5.7. *The group morphism $j : \mathcal{F}_k^G(\mathbb{X}_\beta) \rightarrow (\mathcal{F}_{k'_1, k'_2}^G(\mathbb{X}_\beta))_{(k'_1, k'_2) \in D_k}$ is one-to-one.*

Proof. Let $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$ such that $jA = 0$. By Proposition 4.14(iii), for every $\gamma \subset \beta$ with $|\gamma| = k$, there holds

$$j_{0,0} \text{Sl}_\gamma A = \text{Sl}_\gamma j_{|\gamma^1|, |\gamma^2|} A = \text{Sl}_\gamma 0 = 0.$$

It then follows from Theorem 5.6(ii) that $\text{Sl}_\gamma A = 0$ for every $\gamma \subset \beta$ with $|\gamma| = k$ and by Theorem 3.3 we get $A = 0$. \square

With the same line of arguments and using Proposition 4.15 instead of Theorem 3.3, we deduce the following statement.

Proposition 5.8. *Let $A \in \mathcal{F}_k^G(\mathbb{X}_\beta)$. There holds for every $(k'_1, k'_2) \in D_k$,*

$$j_{k'_1, k'_2} A = 0 \iff \text{Sl}_\gamma A = 0 \text{ for every } \gamma \in I_k^n \text{ such that } (|\gamma^1|, |\gamma^2|) = (k'_1, k'_2).$$

5.2 Normal and rectifiable tensor chains

Let us introduce some subgroups of k -chains which identify through j_{k_1, k_2} with (k_1, k_2) -chains.

Definition 5.9. We set

$$\mathcal{F}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta) := \left\{ A \in \mathcal{F}_k^G(\mathbb{X}_\beta) : j_{k'_1, k'_2} A = 0 \text{ for every } (k'_1, k'_2) \in D_k \setminus \{(k_1, k_2)\} \right\}.$$

Then we define,

$$\mathcal{M}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta) := \mathcal{F}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta) \cap \mathcal{M}_k^G(\mathbb{X}_\beta), \quad \mathcal{N}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta) := \mathcal{F}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta) \cap \mathcal{N}_k^G(\mathbb{X}_\beta).$$

Remark 5.10. Let $A^1 \in \mathcal{F}_{k_1}^{\mathbb{Z}}(\mathbb{X}_{\beta^1})$ and $A^2 \in \mathcal{F}_{k_2}^G(\mathbb{X}_{\beta^2})$. We have seen in Section 3 that when A^1 and A^2 have finite mass or at least one of them is normal, we can define the Cartesian product $A := A^1 \times A^2$ in $\mathcal{F}_k^G(\mathbb{X}_\beta)$. Considering first polyhedral chains, it is easy to see that (recall (4.24))

$$j_{k_1, k_2} A = A^1 \wedge A^2 =: A' \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta),$$

and

$$j_{k'_1, k'_2} A = 0 \quad \text{for } (k'_1, k'_2) \neq (k_1, k_2).$$

It follows that A lies in $\mathcal{F}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta)$ and since j is one-to-one we can identify A with A' . More generally, considering a sequence of pairs (A_j^1, A_j^2) of the above form, if the series $\sum A_j^1 \times A_j^2$ converges in $\mathcal{F}_k^G(\mathbb{X}_\beta)$, we can identify $\sum A_j^1 \times A_j^2 \in \mathcal{F}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta)$ with $\sum A_j^1 \wedge A_j^2 \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$.

Remark 5.11. By Proposition 5.8, we have the alternative definition:

$$\mathcal{F}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta) = \left\{ A \in \mathcal{F}_k^G(\mathbb{X}_\beta) : \text{Sl}_\gamma A = 0 \text{ for every } \gamma \in I_k^n \text{ such that } (|\gamma^1|, |\gamma^2|) \neq (k_1, k_2) \right\},$$

and similarly for $\mathcal{M}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta)$ and $\mathcal{N}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta)$.

As a consequence of this remark, we observe that Proposition 3.12 rephrases as follows.

Corollary 5.12. *The following statements are equivalent for every $A \in \mathcal{M}_k^G(\mathbb{X}_\beta)$ rectifiable.*

- (i) $T\mu_A$ is (k_1, k_2) -split.
- (ii) $\text{Sl}_\gamma A = 0$ for every $\gamma \in I_k^n$ with $(|\gamma^1|, |\gamma^2|) \neq (k_1, k_2)$.
- (iii) $A \in \mathcal{M}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta)$.

Let us now introduce the notion of normal tensor chains.

Definition 5.13. Let $A' \in \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$, we say that A' is normal if

$$\mathbb{N}^\wedge(A') := \mathbb{M}^\wedge(A') + \mathbb{M}^\wedge(\partial_1 A') + \mathbb{M}^\wedge(\partial_2 A') < \infty.$$

We denote by $\mathcal{N}_{k_1, k_2}^G(\mathbb{X}_\beta)$ the group of normal (k_1, k_2) -chains in \mathbb{X}_β .

The normed groups $(\mathcal{N}_{k_1, k_2}^G(\mathbb{X}_\beta), +, \mathbb{N})$ are complete but do not form a two-dimensional chain complex¹¹ (unless $|\beta^1|, |\beta^2| \leq 1$).

Notice also that J_{k_1, k_2} maps $\mathcal{F}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta)$ into $\mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ and that (recalling (4.21)&(4.22)) we have for $A \in \mathcal{F}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta)$,

$$\mathbb{F}^\wedge(J_{k_1, k_2} A) \leq 2\mathbb{F}^\times(A), \quad \mathbb{M}^\wedge(J_{k_1, k_2} A) \leq \mathbb{M}^\times(A).$$

Besides, by Proposition 4.9, for $(k'_1, k'_2) \in D_k$ (and still $A \in \mathcal{F}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta)$), we have the formula,

$$J_{k'_1, k'_2} \partial A = \begin{cases} \partial_1 J_{k_1, k_2} A & \text{if } (k'_1, k'_2) = (k_1 - 1, k_2), \\ \partial_2 J_{k_1, k_2} A & \text{if } (k'_1, k'_2) = (k_1, k_2 - 1), \\ 0 & \text{in the other cases.} \end{cases}$$

We deduce that $\mathbb{N}^\wedge(J_{k_1, k_2} A) \leq \mathbb{N}^\times(A) = \mathbb{M}^\times(A) + \mathbb{M}^\times(\partial A)$. Let us summarize the above statements.

Proposition 5.14. *We have the continuous embeddings:*

$$\begin{aligned} J_{k_1, k_2} &: \mathcal{F}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta) \hookrightarrow \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta), \\ J_{k_1, k_2} &: \mathcal{M}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta) \hookrightarrow \mathcal{M}_{k_1, k_2}^G(\mathbb{X}_\beta), \\ J_{k_1, k_2} &: \mathcal{N}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta) \hookrightarrow \mathcal{N}_{k_1, k_2}^G(\mathbb{X}_\beta). \end{aligned}$$

A natural question is whether some of these mappings are onto. In a complementary paper, we establish that J_{k_1, k_2} defines an isometric isomorphisms from $\mathcal{N}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta)$ onto $\mathcal{N}_{k_1, k_2}^G(\mathbb{X}_\beta)$, (see [GM22a, Theorems A.1]) where the isometry holds with respect to a \mathbb{N} -norm built on the so called *coordinate slicing mass* rather than on the mass itself. We also show there, that except in some limit cases, $J_{k_1, k_2} : \mathcal{M}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta) \rightarrow \mathcal{M}_{k_1, k_2}^G(\mathbb{X}_\beta)$ and $J_{k_1, k_2} : \mathcal{F}_{k, (k_1, k_2)}^G(\mathbb{X}_\beta) \rightarrow \mathcal{F}_{k_1, k_2}^G(\mathbb{X}_\beta)$ are not onto. Here, we only establish the positive result about normal chains in the case $k_1 = 0$ and without the isometry property.

Theorem 5.15. *The mapping $J_{0, k}$ defines a group isomorphism from $\mathcal{N}_{k, (0, k)}^G(\mathbb{X}_\beta)$ onto $\mathcal{N}_{0, k}^G(\mathbb{X}_\beta)$. Its inverse is denoted Λ .*

Proof. Without loss of generality, we assume that $\beta = \{1, \dots, n\}$ so that $\beta^1 = \alpha$ and $\beta^2 = \bar{\alpha}$. In the sequel, we denote $\mathcal{N} := \mathcal{N}_{k, (0, k)}^G(\mathbb{R}^n)$ and $\mathcal{N}' := \mathcal{N}_{0, k}^G(\mathbb{R}^n)$.

The proof is split into two steps. In the first step we define a group morphism $\Lambda : \mathcal{N}' \rightarrow \mathcal{N}$ and show that $J_{0, k} \circ \Lambda = \text{Id}_{|\mathcal{N}'}$. In a second step, we establish that $\Lambda \circ J_{0, k} = \text{Id}_{|\mathcal{N}}$.

Step 1. Definition of Λ and proof of $J_{0, k} \circ \Lambda = \text{Id}$ on \mathcal{N}' .

We start with the following observation. If $A = \sum A_j \llbracket x_j \rrbracket \in \mathcal{P}_0(\mathbb{X}_\alpha, \mathcal{F}_k^G(\mathbb{X}_{\bar{\alpha}}))$ for some sequence $A_j \in \mathcal{F}_k^G(\mathbb{X}_{\bar{\alpha}})$ such that $\sum \mathbb{M}(A_j) < \infty$ and some sequence of pairwise distinct points $x_j \in \mathbb{X}_\alpha$, we have $\chi(A) = \sum_j A_j$ and thus $\mathbb{M}(\chi(A)) \leq \sum_j \mathbb{M}(A_j)$, that is,

$$\mathbb{M}(\chi(A)) \leq \mathbb{M}(A). \tag{5.4}$$

¹¹To get a true analogue of the complex $\mathcal{N}_*^G(\mathbb{X}_\beta)$ we should consider instead $\widetilde{\mathbb{N}}^\wedge(A') := \sum_{0 \leq i_1, i_2 \leq 1} \mathbb{M}^\wedge(\partial_1^{i_1} \partial_2^{i_2} A)$.

By density and continuity of $\chi(A)$, (5.4) extends to all $A \in \mathcal{M}_0(\mathbb{X}_\alpha, \mathcal{F}_k^G(\mathbb{X}_{\bar{\alpha}}))$.

Next, for $j \geq 0$ and $m \in \mathbb{Z}^{n_1}$, we set $y_m^j := 2^{-j}m$ and $Q_m^j := y_m^j + [0, 2^{-j})^{n_1}$. For $j \geq 0$ and $A' \in \mathcal{F}_{0,k}^G(\mathbb{R}^n)$ such that $\iota A' \in \mathcal{M}_0(\mathbb{X}_\alpha, \mathcal{F}_k^G(\mathbb{X}_{\bar{\alpha}}))$ we define

$$\Lambda_j A' := \sum_m \llbracket y_m^j \rrbracket \times \chi([\iota A'] \llcorner Q_m^j).$$

Let $A' \in \mathcal{N}'$ and let us denote $A_j := \Lambda_j A'$ and $A'_j := \jmath_{0,k} A_j$. By construction,

$$\begin{aligned} A_j &= \sum_m \llbracket y_m^j \rrbracket \times \chi([\iota A'] \llcorner Q_m^j) \in \mathcal{F}_k^G(\mathbb{R}^n), \\ A'_j &= \sum_m \llbracket y_m^j \rrbracket \wedge \chi([\iota A'] \llcorner Q_m^j) \in \mathcal{F}_{0,k}^G(\mathbb{R}^n). \end{aligned} \quad (5.5)$$

In the sequel, we show that A_j converges in $\mathcal{F}_k^G(\mathbb{R}^n)$ to some chain $A \in \mathcal{N}$ and we define $\Lambda A' := A$. We then show that A'_j converges to A' in \mathbb{F}^\wedge -norm and finally get $A' = \jmath_{0,k} \Lambda A'$ by continuity of $\jmath_{0,k}$.

Let us fix $j \geq i \geq 1$. Let $m \in \mathbb{Z}^{n_1}$ and denote by \mathcal{Q} the set formed by the $2^{n_1(j-i)}$ cubes Q_r^j contained in Q_m^i . For $Q = Q_r^j \in \mathcal{Q}$, denoting $p_Q := \llbracket (y_m^i, y_r^j) \rrbracket \in \mathcal{P}_1^{\mathbb{Z}}(\mathbb{X}_\alpha)$, we set,

$$P_Q := p_Q \wedge \chi([\iota A'] \llcorner Q) \in \mathcal{P}_1(\mathbb{X}_\alpha, \mathcal{F}_k^G(\mathbb{X}_{\bar{\alpha}})).$$

Using $\partial = \partial_1 + \partial_2$, we have $(\llbracket y_r^j \rrbracket - \llbracket y_m^i \rrbracket) \wedge \chi([\iota A'] \llcorner Q) = \partial P_Q + p_Q \wedge \partial \chi([\iota A'] \llcorner Q)$. Summing on \mathcal{Q} , we get

$$(A_j - A_i) \llcorner [Q_m^i + \mathbb{X}_{\bar{\alpha}}] = \partial \sum_{Q \in \mathcal{Q}} P_Q + \sum_{Q \in \mathcal{Q}} p_Q \wedge \partial \chi([\iota A'] \llcorner Q).$$

This yields,

$$\mathbb{F}((A_j - A_i) \llcorner [Q_m^i + \mathbb{X}_{\bar{\alpha}}]) \leq \sum_{Q \in \mathcal{Q}} \mathbb{M}(P_Q) + \mathbb{M}(p_Q \wedge \partial \chi([\iota A'] \llcorner Q)). \quad (5.6)$$

On the one hand, for every $Q \in \mathcal{Q}$, there holds, $\mathbb{M}(p_Q) \leq \sqrt{n_1} 2^{-i}$ and we deduce,

$$\sum_{Q \in \mathcal{Q}} \mathbb{M}(P_Q) \leq \sqrt{n_1} 2^{-i} \sum_{Q \in \mathcal{Q}} \mathbb{M}(\chi([\iota A'] \llcorner Q)) \stackrel{(5.4)}{\leq} \sqrt{n_1} 2^{-i} \mathbb{M}([\iota A'] \llcorner Q_m^i). \quad (5.7)$$

On the other hand, considering the Lipschitz continuous morphism

$$\bar{\partial} := \partial : \mathcal{F}_k^G(\mathbb{X}_{\bar{\alpha}}) \rightarrow \mathcal{F}_{k-1}^G(\mathbb{X}_{\bar{\alpha}}),$$

we have with the notation of Proposition 3.9,

$$\bar{\partial}_* (\iota A') = \iota(\partial_2 A'),$$

and by Proposition 5.2,

$$\partial(\chi([\iota A'] \llcorner Q)) = \chi(\bar{\partial}_*([\iota A'] \llcorner Q)) = \chi([\iota \partial_2 A'] \llcorner Q). \quad (5.8)$$

We deduce similarly to (5.7),

$$\sum_{Q \in \mathcal{Q}} \mathbb{M}^\wedge(p_Q \wedge \partial \chi([\iota A'] \llcorner Q)) \leq \sqrt{n_1} 2^{-i} \mathbb{M}^\wedge([\iota \partial_2 A'] \llcorner Q_m^i). \quad (5.9)$$

Plugging (5.7)&(5.9) in (5.6), we obtain

$$\mathbb{F}((A_j - A_i) \llcorner [Q_m^i + \mathbb{X}_{\bar{\alpha}}]) \leq \sqrt{n_1} 2^{-i} (\mathbb{M}([\iota A'] \llcorner Q_m^i) + \mathbb{M}(\iota \partial_2 A' \llcorner Q_m^i)).$$

Then, summing over m , we get

$$\mathbb{F}(A_j - A_i) \leq \sqrt{n_1} 2^{-i} \mathbb{N}^\wedge(A'). \quad (5.10)$$

Notice that we cannot substitute the \mathbb{M}^\wedge for \mathbb{N}^\wedge on the right-hand side. Indeed, the corresponding result does not hold for finite mass chains see [GM22a, Proposition A.5]. From (5.10) we conclude that A_j is a Cauchy sequence in $\mathcal{F}_k^G(\mathbb{R}^n)$ and $A_j \rightarrow A$. Moreover, since

$$\mathbb{M}(A_j) \leq \sum_m \mathbb{M}(\chi([\iota A'] \llcorner Q_m^j)) \leq \mathbb{M}(\chi[\iota A']) \leq \mathbb{M}(\iota A'),$$

we have

$$\mathbb{M}(A) \leq \liminf \mathbb{M}(A_j) \leq \mathbb{M}(\iota A') < \infty.$$

Similarly, since

$$\partial A_j = \sum_m \llbracket y_m^j \rrbracket \times \partial \chi([\iota A'] \llcorner Q_m^j) \stackrel{(5.8)}{=} \sum_m \llbracket y_m^j \rrbracket \times \chi([\iota \partial_2 A'] \llcorner Q_m^j)$$

we have

$$\mathbb{M}(\partial A_j) \leq \mathbb{M}(\iota \partial_2 A')$$

and thus

$$\mathbb{M}(\partial A) \leq \liminf \mathbb{M}(\partial A_j) \leq \mathbb{M}(\iota \partial_2 A') < \infty.$$

Besides, every A_j belongs to \mathcal{N} so that by continuity of ι we also have $A \in \mathcal{N}$. Defining $\Lambda A' := A$, the mapping Λ is a group morphism and we have just established that it maps \mathcal{N}' into \mathcal{N} .

Let us now show that $A'_j \rightarrow A'$. Let us fix $j \geq 0$, we have,

$$\mathbb{F}^\wedge(A'_j - A') = \mathbb{F}(\iota(A'_j - A')) \leq \sum_m \mathbb{F}((\iota(A'_j - A')) \llcorner Q_m^j). \quad (5.11)$$

For a fixed $m \in \mathbb{Z}^{n_1}$ we denote $B'' := [\iota A'] \llcorner Q_m^j$. We then have from (5.5),

$$\iota(A'_j - A') \llcorner Q_m^j = \chi(B'') \llbracket y_m^j \rrbracket - B''.$$

Using the homotopy formula (3.3) in \mathbb{X}_α with $f(x) = x$ and $g(x) = y_m^j$ and taking into account that $\partial B'' = 0$ (because B'' is a 0-chain), we get with the notation there,

$$\partial[h_\# (\llbracket (0, e_0) \rrbracket \times B'')] = B'' - g_\# B''. \quad (5.12)$$

Remark that for any polyhedral 0-chain C in \mathbb{X}_α , we have $g_\# C + \chi(C) \llbracket y_m^j \rrbracket$ and this relation extends by continuity to any 0-chain. Hence (5.12) rewrites as

$$\partial[h_\# (\llbracket (0, e_0) \rrbracket \times B'')] = B'' - B'' - \chi(B'') \llbracket y_m^j \rrbracket.$$

We deduce:

$$\begin{aligned} \mathbb{F}(\iota(A'_j - A') \llcorner Q_m^j) &= \mathbb{F}(B'' - \chi(B'') \llbracket y_m^j \rrbracket) \leq \mathbb{M}(h_\# (\llbracket (0, e_0) \rrbracket \times B'')) \\ &\stackrel{(3.4)}{\leq} C \|f - g\|_{L^\infty(\text{supp } B'')} \mathbb{M}(B'') \\ &\leq C \sqrt{n_1} 2^{-j} \mathbb{M}(B'') = C \sqrt{n_1} 2^{-j} \mathbb{M}([\iota A'] \llcorner Q_m^j). \end{aligned}$$

Summing over m and using (5.11), we obtain

$$\mathbb{F}^\wedge(A'_j - A') \leq C\sqrt{n_1}2^{-j}\mathbb{M}(\iota A') \stackrel{(4.20)}{\leq} C\sqrt{n_1}2^{-j}\mathbb{M}^\wedge(A').$$

We conclude that $A'_j \rightarrow A'$ in \mathbb{F}^\wedge -norm.

Eventually, by continuity of $j_{0,k}$, there holds

$$j_{0,k}\Lambda A' = j_{0,k}A = j_{0,k}\lim A_j = \lim j_{0,k}A_j = \lim A'_j = A'.$$

We have established that $j_{0,k} \circ \Lambda = \text{Id}$ on \mathcal{N}' .

Step 2, $\Lambda \circ j_{0,k} = \text{Id}$ on \mathcal{N} . Let us fix $\gamma \in I_k^n$ with $\gamma^1 = \emptyset$. We first show the following identity for $A' \in \mathcal{N}'$,

$$\text{Sl}_\gamma A' = j_{0,0} \text{Sl}_\gamma \Lambda A'. \quad (5.13)$$

Returning to the definition of Λ in the previous step and using the notation there we have $A'_j = j_{0,k}A_j$ so that by (iii) of Proposition 4.14,

$$\text{Sl}_\gamma A'_j = j_{0,0} \text{Sl}_\gamma A_j.$$

By the continuity of Sl_γ and $j_{0,0}$ we can pass to the limit (recall that $A'_j \rightarrow A'$ and $A_j = \Lambda_j A' \rightarrow \Lambda A'$ by the previous step) and obtain (5.13).

Let now $A \in \mathcal{N}$. Using again (iii) of Proposition 4.14 and applying (5.13) with $A' = j_{0,k}A$, we compute

$$j_{0,0} \text{Sl}_\gamma A = \text{Sl}_\gamma j_{0,k}A = j_{0,0} \text{Sl}_\gamma \Lambda j_{0,k}A.$$

Since $j_{0,0}$ is one-to-one by Theorem 5.6, we deduce that $\text{Sl}_\gamma A = \text{Sl}_\gamma \Lambda j_{0,k}A$. Now if $|\gamma| = k$ but $\gamma^1 \neq \emptyset$ then $\text{Sl}_\gamma A = 0$ since by assumption $A \in \mathcal{N} = \mathcal{N}_{k,(0,k)}^G(\mathbb{R}^n)$. Similarly, since $\Lambda j_{0,k}A \in \mathcal{N}$, we also have $\text{Sl}_\gamma \Lambda j_{0,k}A = 0$ in this case. Hence, $\Lambda j_{0,k}A$ and A have the same 0-slices and it follows from Theorem 3.3 that these chains are equal. We conclude that $\Lambda \circ j_{0,k} = \text{Id}$ on \mathcal{N} and with Step 1, Λ is an isomorphism from \mathcal{N}' onto \mathcal{N} with inverse $j_{0,k}$. \square

6 Proof of the main results and a counterexample for norm preserving decompositions

We obtain Theorem 1.3 as a corollary of the analogous result for rectifiable tensor chains (Theorem 6.3 below). Let us first define this notion.

Definition 6.1. Let $A' \in \mathcal{M}_{k_1, k_2}^G(\mathbb{R}^n)$, we say that:

- (1) A' is rectifiable if $A' = A' \llcorner \Sigma$ for some k -rectifiable set $\Sigma \subset \mathbb{R}^n$.
- (2) A' is (k_1, k_2) -rectifiable if $A' = A' \llcorner (\Sigma^1 \times \Sigma^2)$ for some k_l -rectifiable subsets $\Sigma^l \subset \mathbb{R}^{m_l}$, $l = 1, 2$.

We first consider the case $k_1 = 0$.

Proposition 6.2. *Let $A' \in \mathcal{N}_{0,k}^G(\mathbb{R}^n)$ be a rectifiable tensor chain, then there exists $S \subset \mathbb{X}_\alpha$, at most countable, such that $\iota A' = (\iota A') \llcorner S$.*

More precisely, there exist sequences $y_j^1 \in \mathbb{X}_\alpha$ and $A_j^2 \in \mathcal{N}_k^G(\mathbb{X}_{\bar{\alpha}})$ with $\sum \mathbb{N}(A_j^2) < \infty$ such that (recall (4.24))

$$A' = \sum \llbracket y_j^1 \rrbracket \wedge A_j^2. \quad (6.1)$$

Proof. Let $A' \in \mathcal{N}_{0,k}^G(\mathbb{R}^n)$ be rectifiable and let $\Sigma \subset \mathbb{R}^n$ be a rectifiable set such that $A' = A' \llcorner \Sigma$. Using the isomorphism Λ of Theorem 5.15, we set $A := \Lambda A' \in \mathcal{N}_{k,(0,k)}^G(\mathbb{R}^n)$. Since the inverse of Λ (i.e. the restriction of $\mathcal{J}_{0,k}$ to $\mathcal{N}_{k,(0,k)}^G(\mathbb{R}^n)$) commutes with restrictions (see (ii) of Proposition 4.16), we also have $A \llcorner \Sigma = \Lambda(A' \llcorner \Sigma)$ and A is rectifiable. Besides, by Corollary 5.12,

$$\text{Sl}_\gamma A = 0 \quad \text{for } \gamma \in I_k^n \text{ with } (|\gamma^1|, |\gamma^2|) \neq (0, k). \quad (6.2)$$

Step 1. Let $i \in \alpha$ and let $\gamma \in I_k^n$ such that $i \in \gamma$. By Proposition 3.2, for almost every $s \in \mathbb{R}$, and almost every $x \in \mathbb{X}_\gamma$ with $x_i = s$, there holds

$$A \cap \mathbb{X}_{\bar{\gamma}}(x) = (A \cap \mathbb{X}_{\bar{\gamma}}(se_i)) \cap \mathbb{X}_{\bar{\gamma}}(x).$$

Since $i \in \alpha \cap \gamma$, we have $\gamma^1 \neq \emptyset$ and we deduce from (6.2) that $A \cap \mathbb{X}_{\bar{\gamma}}(x) = 0$ for almost every $x \in \mathbb{X}_\gamma$. Hence, for almost every $s \in \mathbb{R}$ and for $i \in \gamma \in I_k^n$, there holds,

$$(A \cap \mathbb{X}_{\bar{\gamma}}(se_i)) \cap \mathbb{X}_{\bar{\gamma}}(x) = 0.$$

In other words, almost all the coordinate 0-slices of $A \cap \mathbb{X}_{\bar{\gamma}}(se_i)$ vanish. We deduce from Theorem 3.3 that $A \cap \mathbb{X}_{\bar{\gamma}}(se_i) = 0$ for almost every $s \in \mathbb{R}$.

Step 2. Next, by Theorem 3.14, there exists a sequence of set-indecomposable (rectifiable and normal) chains A_j such that $A = \sum A_j$ and such that for each j , $A_j = A \llcorner S_j$ for some Borel set $S_j \subset \mathbb{R}^n$.

Let us fix $j \geq 1$. By Proposition 3.5 (commutation of slicing and restriction), A_j also satisfies (6.2). As a consequence, the analysis of Step 1 applies to A_j and

$$A_j \cap \mathbb{X}_{\bar{\gamma}}(se_i) = 0 \quad \text{for } i \in \alpha \text{ and almost every } s \in \mathbb{R}. \quad (6.3)$$

Let us apply Proposition 3.6 to A_j . For $s \in \mathbb{R}$, we denote $H(s) := \{x \in \mathbb{R}^n : x_1 > s\}$, then for almost every $s \in \mathbb{R}$, $A_j \llcorner H(s) \in \mathcal{N}_k^G(\mathbb{R}^n)$ and

$$\partial(A_j \llcorner H(s)) = (\partial A_j) \llcorner H(s) + A_j \cap \mathbb{X}_{\bar{1}}(se_1) \stackrel{(6.3)}{=} (\partial A_j) \llcorner H(s).$$

Denoting $A_j(s) := A_j \llcorner H(s)$, the chain $A_j - A_j(s) = A_j \llcorner (\mathbb{R}^n \setminus H(s))$ satisfies

$$\partial(A_j \llcorner \mathbb{R}^n \setminus H(s)) = \partial A_j - \partial(A_j(s)) = (\partial A_j) \llcorner (\mathbb{R}^n \setminus H(s)).$$

It follows that

$$\begin{aligned} \mathbb{M}(\partial A_j(s)) + \mathbb{M}(\partial[A_j - A_j(s)]) &= \mathbb{M}((\partial A_j) \llcorner H(s)) + \mathbb{M}((\partial A_j) \llcorner [\mathbb{R}^n \setminus H(s)]) \\ &= \mu_{\partial A_j}(H(s)) + \mu_{\partial A_j}(\mathbb{R}^n \setminus H(s)) \\ &= \mu_{\partial A_j}(\mathbb{R}^n) = \mathbb{M}(\partial A_j). \end{aligned}$$

We also have $\mathbb{M}(A_j(s)) + \mathbb{M}(A_j - A_j(s)) = \mathbb{M}(A_j)$ and therefore

$$(A_j(s), A_j - A_j(s), 0, 0, \dots)$$

is a set-decomposition of A_j . But by construction, A_j is set-indecomposable and we conclude that for almost every $s \in \mathbb{R}$, either $A_j = A_j(s) = A_j \llcorner H(s)$ or $A_j = A_j \llcorner [\mathbb{R}^n \setminus H(s)]$. Proceeding by dichotomy, we obtain that there exists some $s_j \in \mathbb{R}$ such that

$$A_j = A_j \llcorner \{x \in \mathbb{R}^n : x_1 = s_j\}.$$

We can perform the same reasoning with the direction e_1 replaced by e_i for $i \in \alpha$. Taking the intersections, we conclude that there exists $y_j^1 \in \mathbb{X}_\alpha$ such that

$$A_j = A_j \perp \{x \in \mathbb{R}^n : x^1 = y_j^1\}.$$

Eventually, summing over j , we deduce $A = A \perp (S + \mathbb{X}_\alpha)$ with $S = \{y_j^1 : j \geq 1\}$. Now, applying the operators $\mathcal{J}_{0,k}$ and ι , we get $\iota A' = (\iota A') \perp S$. This proves the first part of the proposition.

Step 3. Let us establish (6.1). For $j \geq 1$ and with the notation of *Step 2*, we denote $A'_j := \mathcal{J}_{0,k} A_j$ and set $A_j^2 := \chi(\iota A'_j)$ so that $A'_j = \llbracket y_j^1 \rrbracket \wedge A_j^2$. Indeed, this formula holds true for A'_j polyhedral $(0, k)$ -chain supported on the affine subspace $y_j^1 + \mathbb{X}_{\bar{\alpha}}$ and we deduce the general case by density using the estimate of Theorem 5.1(ii). Summing over j we obtain (6.1) since $A' = \mathcal{J}_{0,k} A = \sum \mathcal{J}_{0,k} A_j$. Moreover, since A_j is normal and rectifiable, A'_j is also normal and rectifiable. From the fact that $A'_j = \llbracket y_j^1 \rrbracket \wedge A_j^2$ we conclude that each A_j^2 is normal and rectifiable. Moreover $\mathbb{N}(A_j^2) = \mathbb{N}(A'_j)$ so that

$$\sum \mathbb{N}(A_j^2) = \sum \mathbb{N}^\wedge(A'_j) \leq \sum \mathbb{N}^\times(A_j) \stackrel{(4.4)}{\leq} C \sum \mathbb{N}(A_j) = CN(A) < \infty.$$

□

The counterpart of Theorem 1.3 for tensor chains states as follows.

Theorem 6.3. *Let $A' \in \mathcal{N}_{k_1, k_2}^G(\mathbb{R}^n)$, The following are equivalent.*

- (i) A' is rectifiable.
- (ii) A' is (k_1, k_2) -rectifiable.

Proof. Since we clearly have (ii) \Rightarrow (i), we only need to prove (i) \Rightarrow (ii). Let $A' \in \mathcal{N}_{k_1, k_2}^G(\mathbb{R}^n)$ be rectifiable and let us set $A'' := \iota A' \in \mathcal{F}_{k_1}(\mathbb{X}_\alpha, \mathcal{F}_{k_2}^G(\mathbb{X}_{\bar{\alpha}}))$. We use the rectifiable slices theorem, Theorem 3.11 to show that A'' is rectifiable. Let $\gamma \subset \alpha$ with $|\gamma| = k_1$, we have, by Proposition 4.14(iv), for a.e. $x \in \mathbb{X}_\gamma$,

$$\text{Sl}_\gamma^x A'' = \text{Sl}_\gamma^x \iota A' = \iota \text{Sl}_\gamma^x A'.$$

Since A' is normal and rectifiable, for almost every $x \in \mathbb{X}_\gamma$, $\text{Sl}_\gamma^x A' \in \mathcal{N}_{0, k_2}^G(\mathbb{X}_{\bar{\gamma}})$ is normal and rectifiable. By Proposition 6.2 there exists a countable set $S_x \subset \mathbb{X}_{\alpha \setminus \gamma}$ such that $\text{Sl}_\gamma^x A'' = (\text{Sl}_\gamma^x A'') \perp S_x$, this means exactly that $\text{Sl}_\gamma^x A''$ is 0-rectifiable for every $\gamma \subset \alpha$ with $|\gamma| = k_1$ and almost every $x \in \mathbb{X}_\gamma$. We conclude from Theorem 3.11 that A'' is rectifiable. Consequently, there exists a k_1 -rectifiable set $\Sigma^1 \subset \mathbb{R}^{n_1}$ such that,

$$A'' = A'' \perp \Sigma^1.$$

By Proposition 4.16(iii) and injectivity of ι this is equivalent to

$$A' = A' \perp \Sigma^1 \times \mathbb{R}^{n_2}.$$

Eventually exchanging the roles of α and $\bar{\alpha}$, there exists a k_2 -rectifiable set $\Sigma^2 \subset \mathbb{R}^{n_2}$ such that $A' = A' \perp \mathbb{R}^{n_1} \times \Sigma^2$. By successive restrictions (Proposition 4.16(iv)), we have,

$$\begin{aligned} A' &= A' \perp \Sigma^1 \times \mathbb{R}^{n_2} = (A' \perp \Sigma^1 \times \mathbb{R}^{n_2}) \perp \mathbb{R}^{n_1} \times \Sigma^2 \\ &= A' \perp (\Sigma^1 \times \mathbb{R}^{n_2} \cap \mathbb{R}^{n_1} \times \Sigma^2) = A' \perp \Sigma^1 \times \Sigma^2. \end{aligned}$$

This proves that A' is (k_1, k_2) -rectifiable. □

We can now prove the main result. Taking into account the characterization of the splitting property of Proposition 3.12, it restates as follows.

Theorem 6.4 (Reformulation of Theorem 1.3). *Let $k_1, k_2 \geq 0$ and $k := k_1 + k_2$. The following statements are equivalent for any rectifiable flat chain $A \in \mathcal{N}_k^G(\mathbb{R}^n)$.*

- (i) *The measure $T\mu_A$ is (k_1, k_2) -split.*
- (ii) *$A \in \mathcal{N}_{k, (k_1, k_2)}^G(\mathbb{R}^n)$, i.e. $J_{k'_1, k'_2} A = 0$ for $(k'_1, k'_2) \in D_k \setminus \{(k_1, k_2)\}$.*
- (iii) *A is (k_1, k_2) -rectifiable.*

Proof. We have already established that (i) \iff (ii) in Corollary 5.12. This equivalence is true even if A is not a normal chain. The implication (iii) \implies (i) is immediate and again, we do not need A to be normal.

To complete the proof of the theorem, we have to show that (ii) \implies (iii).

Let $A \in \mathcal{N}_{k, (k_1, k_2)}^G(\mathbb{R}^n)$ rectifiable and let $\Sigma \subset \mathbb{R}^n$ be k -rectifiable and such that $A \llcorner \Sigma = A$. We define $A' = J_{k_1, k_2} A$. By Proposition 4.16(iii), $A' \llcorner \Sigma = A'$ and by Proposition 4.9 and our assumption (ii), $\partial_1 A' = J_{k_1-1, k_2} \partial A$, $\partial_2 A' = J_{k_1, k_2-1} \partial A$ so that A' is a normal rectifiable (k_1, k_2) -chain. By Theorem 6.3 there exist k_l -rectifiable subsets $\Sigma^l \subset \mathbb{R}^{n_l}$, $l = 1, 2$ such that $A' \llcorner (\Sigma^1 \times \Sigma^2) = A'$, that is

$$J_{k_1, k_2} A = (J_{k_1, k_2} A) \llcorner (\Sigma^1 \times \Sigma^2) = J_{k_1, k_2} [A \llcorner (\Sigma^1 \times \Sigma^2)].$$

Eventually, by assumption, $J_{k'_1, k'_2} A = 0$ for $(k'_1, k'_2) \neq (k_1, k_2)$ and we deduce that $J A = J(A \llcorner (\Sigma^1 \times \Sigma^2))$. Since by Theorem 5.7, J is one-to-one we conclude that $A = A \llcorner (\Sigma^1 \times \Sigma^2)$ so that A is (k_1, k_2) -rectifiable. \square

6.1 Decomposition of (k_1, k_2) -rectifiable chains in tensor products. A counterexample

In this last subsection we consider the group $G = \mathbb{Z}$ endowed with the usual norm. Starting from a $(0, k)$ -rectifiable chain $A \in \mathcal{N}_k^{\mathbb{Z}}(\mathbb{R}^n)$ and arguing as in the proof of Proposition 6.2 (using again that by Theorem 5.15, $J_{0, k}$ is an isomorphism and recalling Remark 5.10) we see that there exist sequences $y_j^1 \in \mathbb{X}_\alpha$ and $A_j^2 \in \mathcal{N}_k^{\mathbb{Z}}(\mathbb{X}_{\bar{\alpha}})$ such that

$$A = \sum \llbracket y_j^1 \rrbracket \times A_j^2 \in \mathcal{F}_k^{\mathbb{Z}}(\mathbb{R}^n) \quad \text{and} \quad \mathbb{N}(A) = \sum \mathbb{N}(A_j^2). \quad (6.4)$$

In the general case, one may ask whether any (k_1, k_2) -rectifiable chain $A \in \mathcal{N}_k^{\mathbb{Z}}(\mathbb{R}^n)$ decomposes as $A = \sum A_j^1 \times A_j^2$ where the A_j^l 's are normal rectifiable k_l -chains in \mathbb{X}_α and $\mathbb{X}_{\bar{\alpha}}$ respectively such that,

$$\mathbb{N}(A) = \sum \mathbb{N}(A_j^1 \times A_j^2) = \sum \mathbb{M}(A_j^1) \mathbb{M}(A_j^2) + \sum \mathbb{M}(\partial A_j^1) \mathbb{M}(A_j^2) + \sum \mathbb{M}(A_j^1) \mathbb{M}(\partial A_j^2). \quad (6.5)$$

Remark that we do not ask for the decomposition to be a set-decomposition. Remark also that for $A^l \in \mathcal{M}_{k_l}^{\mathbb{Z}}(\mathbb{R}^{n_l})$, the identity $\mathbb{M}(A^1 \times A^2) = \mathbb{M}(A^1) \times \mathbb{M}(A^2)$ (which is used in (6.5)) can be readily obtained by considering A^l as integral currents and using the definition of the mass through duality.

Let us assume for simplicity that A is a cycle. In this case (6.5) implies that A_j^1 is a cycle whenever $A_j^2 \neq 0$ and symmetrically A_j^2 is a cycle when $A_j^1 \neq 0$. We may thus assume that the A_j^l 's are cycles so that the identity rewrites as

$$\mathbb{M}(A) = \sum \mathbb{M}(A_j^1) \mathbb{M}(A_j^2). \quad (6.6)$$

The following result shows that in general, one cannot expect the existence of such decompositions.

Proposition 6.5 (Counterexample). *Let $n_1 = n_2 = 2$. For every $\varepsilon > 0$, there exists a $(1, 1)$ -rectifiable cycle $A \in \mathcal{M}_2^{\mathbb{Z}}(\mathbb{R}^4)$ with $\mathbb{M}(A) > 0$ such that for every decomposition $A = \sum A_j^1 \times A_j^2$ with $A_j^1 \in \mathcal{M}_2^{\mathbb{Z}}(\mathbb{X}_\alpha)$ and $A_j^2 \in \mathcal{M}_2^{\mathbb{Z}}(\mathbb{X}_{\bar{\alpha}})$ cycles, there holds*

$$\sum \mathbb{M}(A_j^1) \mathbb{M}(A_j^2) \geq \left(\frac{4}{3} - \varepsilon \right) \mathbb{M}(A).$$

Proof. By assumption, $n_1 = n_2 = 2$ so that $\dim \mathbb{X}_\alpha = \dim \mathbb{X}_{\bar{\alpha}} = 2$. Let us denote $S = 0_{\mathbb{X}_\alpha}$ and $N = e_1$ and let $\ell > 1$. We consider three polyhedral 1-chains in \mathbb{X}_α denoted C_i ($i = 0, 1, 2$). We assume that the C_i 's have multiplicity 1 and are supported by three broken lines, each of which of length ℓ and with end points S and N . We assume that the supports of the C_i 's only intersect at S and N and we orient the edges so that $\partial C_i = \llbracket N \rrbracket - \llbracket S \rrbracket$. Eventually, let $L : \mathbb{X}_\alpha \rightarrow \mathbb{X}_{\bar{\alpha}}$ be the linear isometry defined by $L(e_j) = e_{j+2}$ for $j = 1, 2$. We set $C'_i := L_{\#} C_i$ for $i = 0, 1, 2$. We define:

$$A := C_0 \times (C'_0 - C'_1) + C_1 \times (C'_1 - C'_2) + C_2 \times (C'_2 - C'_0) \in \mathcal{F}_{1,1}^{\mathbb{Z}}(\mathbb{R}^4).$$

This definition has some symmetry. Indeed, A rewrites as

$$A = (C_2 - C_1) \times C'_2 + (C_0 - C_2) \times C'_0 + (C_1 - C_0) \times C'_1.$$

Notice that A is a $(1, 1)$ -rectifiable chain with integer coefficients in \mathbb{R}^4 and $\mathbb{M}(A) = 6\ell^2$. We compute, using the first formula for ∂_1 and the second formula for ∂_2 ,

$$\begin{aligned} \partial A &= (\llbracket N \rrbracket - \llbracket S \rrbracket) \times (C'_0 - C'_1 + C'_1 - C'_2 + C'_2 - C'_0) \\ &\quad - (C_2 - C_1 + C_0 - C_2 + C_1 - C_0) \times (\llbracket L(N) \rrbracket - \llbracket L(S) \rrbracket) = 0. \end{aligned}$$

In conclusion, A is a $(1, 1)$ -rectifiable and normal cycle.

Now, let us consider two sequences of cycles $A_j^1 \in \mathcal{M}_2^{\mathbb{Z}}(\mathbb{X}_\alpha)$, $A_j^2 \in \mathcal{M}_2^{\mathbb{Z}}(\mathbb{X}_{\bar{\alpha}})$ such that $A = \sum A_j^1 \times A_j^2$ and the mass identity (6.6) holds. Let $\gamma = (1, 3)$, we have almost every $x = se_1 + te_3 \in \mathbb{X}_\gamma$,

$$\text{Sl}_\gamma^x A = \sum \text{Sl}_1^{se_1}(A_j^1) \times \text{Sl}_3^{te_3}(A_j^2) \quad (6.7)$$

In order to obtain a lower bound on $\sum_j \mathbb{M}(A_j^1) \mathbb{M}(A_j^2)$ we estimate the masses of the slices $\text{Sl}_1^{se_1}(A_j^1)$, $\text{Sl}_3^{te_3}(A_j^2)$ from below and use the following bounds from Proposition 4.14(ii).

$$\mathbb{M}(A_j^1) \geq \int_0^1 \mathbb{M}(\text{Sl}_1^{se_1}(A_j^1)) ds, \quad \mathbb{M}(A_j^2) \geq \int_0^1 \mathbb{M}(\text{Sl}_3^{te_3}(A_j^2)) dt \quad \text{for } j \geq 1. \quad (6.8)$$

Let us fix $0 < s, t < 1$ and let us denote $se_2 + c_i e_2 := C_i \cap (se_1 + \mathbb{R}e_2)$, $te_3 + c'_i e_4 := C'_i \cap (te_2 + \mathbb{R}e_4)$. We have for every $s, t \in (0, 1)$,

$$\text{Sl}_\gamma^x A = \llbracket c_0 e_2 \rrbracket \times (\llbracket c'_0 e_4 \rrbracket - \llbracket c'_1 e_4 \rrbracket) + \llbracket c_1 e_2 \rrbracket \times (\llbracket c'_1 e_4 \rrbracket - \llbracket c'_2 e_4 \rrbracket) + \llbracket c_2 e_2 \rrbracket \times (\llbracket c'_2 e_4 \rrbracket - \llbracket c'_0 e_4 \rrbracket). \quad (6.9)$$

For $j \geq 1$ and almost every $0 < s < 1$, $\text{Sl}_1^{se_1}(A_j^1)$ is a sum of 0-cells $\sum q_{j,a} \llbracket a e_2 \rrbracket$ with $a \in \mathbb{R}$, $q_{j,a} \in \mathbb{Z}$. We single out the points $a = c_0, c_1, c_2$ and write

$$A_s^1 := \text{Sl}_1^{se_1}(A_j^1) = m_{j,0} \llbracket c_0 e_2 \rrbracket + m_{j,1} \llbracket c_1 e_2 \rrbracket + m_{j,2} \llbracket c_2 e_2 \rrbracket + \sum_{a \notin \{c_0, c_1, c_2\}} q_{j,a} \llbracket a e_2 \rrbracket.$$

Since A_j^1 is a cycle, there holds for $j \geq 1$,

$$m_{j,0} + m_{j,1} + m_{j,2} + \sum_{a \notin \{c_0, c_1, c_2\}} q_{j,a} = 0. \quad (6.10)$$

Similarly, for almost every $0 < t < 1$ we write

$$A_t^2 := \text{Sl}_3^{te_3}(A_j^2) = m'_{j,0} \llbracket c'_0 e_4 \rrbracket + m'_{j,1} \llbracket c'_1 e_4 \rrbracket + m'_{j,2} \llbracket c'_2 e_4 \rrbracket + \sum_{a' \notin \{c'_0, c'_1, c'_2\}} q'_{j,a'} \llbracket a' e_4 \rrbracket,$$

and there holds for $j \geq 1$,

$$m'_{j,0} + m'_{j,1} + m'_{j,2} + \sum_{a' \notin \{c'_0, c'_1, c'_2\}} q_{j,a'} = 0. \quad (6.11)$$

By identification, the identity (6.7) writes as

$$\begin{aligned} \sum_j m_{j,0} m'_{j,0} &= \sum_j m_{j,1} m'_{j,1} = \sum_j m_{j,2} m'_{j,2} = 1, \\ \sum_j m_{j,0} m'_{j,1} &= \sum_j m_{j,1} m'_{j,2} = \sum_j m_{j,2} m'_{j,0} = -1, \end{aligned} \quad (6.12)$$

and

$$\begin{aligned} \sum_j m_{j,i} q'_{j,a'} &= \sum_j q_{j,a} m'_{j,i} = \sum_j q_{j,a} q'_{j,a'} = 0 \\ &\text{for } i = 0, 1, 2, \text{ for } a \in \mathbb{R} \setminus \{c_0, c_1, c_2\} \text{ and for } a' \in \mathbb{R} \setminus \{c'_0, c'_1, c'_2\}. \end{aligned} \quad (6.13)$$

Next, we define $\tilde{m}_{j,0} := m_{j,0} + \sum_{a \notin \{c_0, c_1, c_2\}} q_{j,a}$ and set

$$\tilde{A}_j^1(s) := \tilde{m}_{j,0} \llbracket c_0 e_2 \rrbracket + m_{j,1} \llbracket c_1 e_2 \rrbracket + m_{j,2} \llbracket c_2 e_2 \rrbracket.$$

Similarly, setting $\tilde{m}'_{j,0} := m'_{j,0} + \sum_{a' \notin \{c'_0, c'_1, c'_2\}} q'_{j,a'}$,

$$\tilde{A}_j^2(t) := \tilde{m}'_{j,0} \llbracket c'_0 e_4 \rrbracket + m'_{j,1} \llbracket c'_1 e_4 \rrbracket + m'_{j,2} \llbracket c'_2 e_4 \rrbracket.$$

We have by triangular inequality,

$$\mathbb{M}(\tilde{A}_j^1(s)) \leq \mathbb{M}(\text{Sl}_1^{se_1}), \quad \mathbb{M}(\tilde{A}_j^2(t)) \leq \mathbb{M}(\text{Sl}_1^{te_3}) \quad \text{for almost every } s, t \in (0, 1) \quad (6.14)$$

and taking into account (6.10)–(6.13) there hold

$$\tilde{m}_{j,0} + m_{j,1} + m_{j,2} = 0, \quad \tilde{m}'_{j,0} + m'_{j,1} + m'_{j,2} = 0, \quad (6.15)$$

and

$$\sum \tilde{A}_j^1(s) \times \tilde{A}_j^2(t) = \text{Sl}_\gamma^x A. \quad (6.16)$$

We see from (6.15) that $\mathbb{M}(\tilde{A}_j^1(s)) = |\tilde{m}_{j,0}| + |m_{j,1}| + |m_{j,2}|$ and $\mathbb{M}(\tilde{A}_j^2(t)) = |\tilde{m}'_{j,0}| + |m'_{j,1}| + |m'_{j,2}|$ are even integers, hence

$$\sum_j \mathbb{M}(\tilde{A}_j^1(s)) \mathbb{M}(\tilde{A}_j^2(t)) \in \{0, 4, 8, 12, \dots\}. \quad (6.17)$$

But from (6.16), (6.9) and the triangle inequality we also have

$$6 = \mathbb{M}(\text{Sl}_\gamma^x A) \leq \sum_j \mathbb{M}(\tilde{A}_j^1(s) \times \tilde{A}_j^2(t)) \leq \sum_j \mathbb{M}(\tilde{A}_j^1(s)) \mathbb{M}(\tilde{A}_j^2(t)).$$

Together with (6.17) we get for almost every $s, t \in (0, 1)$,

$$8 \leq \sum_j \mathbb{M}(\tilde{A}_j^1(s)) \mathbb{M}(\tilde{A}_j^2(t)).$$

Eventually, integrating on $\{(s, t) : 0 < s, t < 1\}$ and using (6.8) and (6.14) we conclude that

$$\sum \mathbb{M}(A_j^1) \mathbb{M}(A_j^2) \geq 8.$$

Since $\mathbb{M}(A) = 6\ell^2$, given $\varepsilon > 0$, we can choose $\ell > 1$ such that $\sum \mathbb{M}(A_j^1) \mathbb{M}(A_j^2) \geq 8 \geq 8\ell^2 - 6\varepsilon = (4/3 - \varepsilon)\mathbb{M}(A)$. This proves the proposition. \square

Remark 6.6. (1) If $k_1 = 1$, we have seen that decompositions satisfying (6.4) do exist. In the other extreme case $k_1 = n_1$, the only (n_1, k_2) -rectifiable cycle is 0. In the other cases $1 \leq k_1 \leq n_1 - 1$, $1 \leq k_2 \leq n_2 - 1$, we obtain counterexamples as above by substituting for the C_i 's three multiplicity 1 smooth k_1 -chains close to a unit k_1 -disk whose boundary is a unit $(k_1 - 1)$ -sphere and whose supports only meet on their common frontier. We substitute similarly for the C_i' 's three smooth k_2 -chains with the corresponding properties.

(2) One may wonder whether there exists a universal constant $\kappa \geq 0$ possibly depending on k_1, k_2, n_1, n_2 such that for every (k_1, k_2) -rectifiable cycle $A \in \mathcal{M}_k^{\mathbb{Z}}(\mathbb{R}^n)$,

$$\inf \left\{ \sum \mathbb{M}(A_j^1) \mathbb{M}(A_j^2) : A_j^l \text{ rectifiable } l\text{-cycles s.t. } A = \sum A_j^1 \times A_j^2 \right\} \leq \kappa \mathbb{M}(A). \quad (6.18)$$

From the example considered above we see that for $1 \leq k_1 \leq n_1 - 1$, $1 \leq k_2 \leq n_2 - 1$, we have $\kappa \geq 4/3$. This lower bound is not optimal. Indeed, modifying the construction of the proof of the proposition by considering $N > 3$ non-intersecting 1-chains C_i instead of three and setting $C_i' := L_{\#} C_i$ for $i = 0, 1, \dots, N - 1$ (where $L : \mathbb{X}_{\alpha} \rightarrow \mathbb{X}_{\bar{\alpha}}$ is still the linear isometry defined by $L(e_j) = e_{j+2}$ for $j = 1, 2$) and then

$$A := \sum_{i=0}^{N-1} C_i \times \left(C_i' - C_{(i+1) \pmod{N}}' \right),$$

we obtain a $(1, 1)$ -rectifiable cycle such that $\mathbb{M}(A) = 2N\ell^2$. Reasoning as in the proof of the proposition, we get that the left hand side of (6.18) is bounded from below by $4(N - 1)$ (details are left to the reader). This leads to $\kappa \geq 2$. However, we do not believe that (6.18) holds true in general for $\kappa = 2$ and we do not even know whether there exist $\kappa < \infty$ such that (6.18) holds true. The question remains open.

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