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Risk arbitrage and hedging to acceptability under transaction costs

Emmanuel Lépinette · Ilya Molchanov

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Abstract The classical discrete-time model of proportional transaction costs relies on the assumption that a feasible portfolio process has solvent increments at each step. We extend this setting in two directions, allowing convex transaction costs and assuming that increments of the portfolio process belong to the sum of a solvency set and a family of multivariate acceptable positions, e.g. with respect to a dynamic risk measure. We describe the sets of superhedging prices, formulate several no (risk) arbitrage conditions and explore connections between them. In the special case when multivariate positions are converted into a single fixed asset, our framework turns into the no-good-deals setting. However, in general, the possibilities of assessing the risk with respect to any asset or a basket of assets lead to a decrease of superhedging prices and the no-arbitrage conditions become stronger. The mathematical techniques rely on results for unbounded and possibly non-closed random sets in Euclidean space.

Keywords Acceptance set · Risk arbitrage · Risk measure · Superhedging · Good deal · Solvency set · Random set · Transaction costs

Mathematics Subject Classification (2010) 91G20, 60D05, 60G42

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1 Introduction

Transaction costs in financial markets are often described using solvency sets, which consist of all financial positions (in physical quantities) regarded as better than the zero

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position or at least equivalent to it. In dynamic discrete-time setting, the solvency sets form a set-valued random process $(K_t)_{t=0,\dots,T}$ adapted to the underlying filtration $(\mathcal{F}_t)_{t=0,\dots,T}$. The no-arbitrage conditions are usually formulated in terms of selections of these solvency sets, that is, for random vectors that a.s. belong to the solvency sets and so correspond to particular choices of solvent portfolios. In many cases, solvency sets are polyhedral cones, and the corresponding model is known as Kabanov's model with proportional transaction costs; see Kabanov et al. [25], Kabanov and Safarian [26, Sect. 3.1], Schachermayer [34], where the no-arbitrage conditions are thoroughly discussed.

If ξ is a claim that matures at time T , then the set of initial positions suitable as a starting value for a self-financing portfolio process $(V_t)_{t=0,\dots,T}$ paying ξ at maturity forms the family of superhedging prices for ξ . In the multivariate setting, the starting values are vectors which are not necessarily comparable to each other, and so instead of comparing them by a single numerical quantity, it is sensible to look at the whole set of superhedging prices. The self-financing requirement amounts to imposing that the (negative) increment $V_{t-1} - V_t$ of the portfolio process is solvent at all times, that is, it a.s. belongs to K_t for all t (in other words, the increment is a *selection* of K_t).

Instead of requiring the superhedging of the terminal payoff with probability one, it is possible to require that the shortfall of the terminal value of the portfolio, in comparison with the claim, is acceptable with respect to a certain risk measure. This yields to a larger set of all superhedging prices and the minimal cash we need to get a vector-valued superhedging price, i.e., the minimal cost, should be reduced. This approach may provide arbitrage opportunities as *good deals*, i.e., terminal claims attainable from zero capital and such that the risk of the claim is strictly negative. The no-good-deal condition, first introduced in Cochrane and Saá-Requejo [11] and then formalised in Černý and Hodges [3], Cherny [8], requires that this is impossible. Unlike the univariate setting, the existence of a good deal in the multivariate setting does not necessarily mean the existence of a claim whose multivariate risk belongs to $-\mathbb{R}_+^d$. Indeed, a vector-valued financial position may be acceptable if some acceptable components compensate for the non-acceptable ones. This may result in various types of arbitrage opportunities.

Indeed, it is possible to strengthen the no-arbitrage requirement by also considering hedging strategies where the self-financing condition is replaced by the acceptability of all intermediate portfolio changes with respect to a dynamic risk measure; see Cherny [7]. The setting of Cherny [7,9] involves at least two assets exchangeable without transaction costs and pinpoints a particular asset that is used as the cash equivalent. A portfolio is converted to its cash equivalent, with the acceptability condition imposed on the increments of these cash values for consecutive time moments. The idea of converting portfolios to a single numerical quantity with acceptable increments in view of superhedging one-dimensional claims has been further explored by Cheridito et al. [5].

However, if there are several currencies (exchangeable with random frictionless rates or with transaction costs), it may well be the case that the position expressed in one currency is acceptable, while the position in another is not; see Molchanov and Cascos [31, Example 1.1]. This may lead to regulatory arbitrages as studied by Willeson [36]. If the regulator is prepared to apply a relaxed acceptability criterion

for one currency, it would be logical to expect the same policy with respect to another currency or a basket of currencies. We show how to handle this in a way that treats all components of a portfolio in the same manner.

The key idea of this work is to extend the family of self-financing portfolio processes by requiring that $V_{t-1} - V_t$ equals the sum of a selection of K_t (a solvent position) and another random vector that is not necessarily solvent, but is acceptable with respect to a dynamic multivariate risk measure. It is worth mentioning that K_t is only supposed to be convex, contrarily to the classical literature of linear transaction costs. With this *hedging to acceptability* approach, all components of the portfolio are treated in the same way. Then $(V_t)_{t=0,\dots,T}$ is called an *acceptable portfolio process*. For example, the classical superhedging setting arises if the componentwise conditional essential infimum is chosen as the risk measure, so that acceptable random vectors necessarily have all a.s. nonnegative components. The hedging to acceptability substantially increases the choice of possible hedging strategies, but in some cases may lead to arbitrage.

Example 1.1 Let ρ be any coherent risk measure. Consider a one-period zero-interest model with two currencies as assets. Assume that the exchange rate π (so that π units of the second asset buy one unit of the first) at time one is lognormally distributed (in the real world) and the exchanges are free from transaction costs. Then the positions $\gamma' = (-a, \pi a)$ and $\gamma'' = (a, -\pi a)$ for $a > 0$ are reachable from $(0, 0)$ at zero cost. Their risks are $(a, a\rho(\pi))$ and $(-a, a\rho(-\pi))$. In order to secure the capital reserves for γ' , the agent has to reserve a of the first currency and $a\rho(\pi)$ of the second one (note that $\rho(\pi) < 0$). If the exchange rate at time zero is π_0 , the initial cost expressed in the second currency is

$$\pi_0 a + a\rho(\pi) = a(\pi_0 + \rho(\pi)).$$

In order to secure γ'' , the initial cost is $a(-\pi_0 + \rho(-\pi))$. If π_0 does not belong to the interval $[-\rho(\pi), \rho(-\pi)]$, then either $\pi_0 + \rho(\pi) < 0$ or $-\pi_0 + \rho(-\pi) < 0$, and we can let a grow to release infinite capital at time zero. Note that this model does not admit financial arbitrage, since there exists a martingale measure. This example can be easily modified by accounting for proportional transaction costs.

It is recognised by now that risks of multivariate positions involving possible exchanges of assets and transaction costs are described as sets; see Cascos and Molchanov [2], Hamel and Heyde [20]. The multiasset setting naturally makes it possible to offset a risky position using various combinations of assets. In this framework, it is also natural to consider the family of all attainable positions as a set-valued portfolio. Treating both arguments and values of a risk measure as random sets leads to law-invariant risk measures and makes it possible to iterate the construction, which is essential to handle dynamic risk measures.

One of the aims of this paper is to introduce a geometric characterisation of superhedging prices. On the way, we suggest a constructive definition of dynamic risk-measure based on the families of acceptable positions, in line with the existing works by Feinstein and Rudloff [14, 15] on consistency in time for dynamic set-valued risk measures taking integrable random variable arguments, by letting the arguments of

risks and their values be sets of random vectors in \mathbb{R}^d . In many instances, these sets may be interpreted as random (possibly non-closed) sets. The necessary background on random sets is provided in the Appendix. In particular, it is shown that the Minkowski (elementwise) sum of two random closed sets is measurable, no matter if the sum is closed or not. Special attention is devoted to the decomposability and infinite decomposability properties, which are the key concepts suitable to relate families of random vectors to selections of random sets.

We refer to Delbaen [12, Sect. 4] and Föllmer and Schied [18, Chap. 4], among others, for the basics of static risk measures and to Acciaio and Penner [1] for a survey of the dynamic L^∞ -setting, further extended by the module approach worked out by Filipović et al. [16, 17].

Static risk measures are usually defined on $L^p(\mathcal{F}; \mathbb{R})$ with $p \in [1, \infty]$. However, in many cases, they are well defined also on larger sets of random variables. For example, $\rho(\xi) = -\text{ess inf } \xi$ makes sense for all random variables essentially bounded from below by a constant. Similarly, if $\rho(\xi) = -E[\xi]$, then the acceptance set is defined as the family of all ξ such that their positive and negative parts satisfy $E[\xi^+] \geq E[\xi^-] > \infty$. The boundedness of $E[\xi^+]$ is not required.

To account for similar effects in relation to multivariate dynamic risk measures, we put forward acceptance sets in place of risk measures. The *acceptance sets* $\mathcal{C}_{t,s}$ with $t \leq s$ are subsets of the sum of the family of \mathcal{F}_s -measurable random vectors in \mathbb{R}^d that admit generalised conditional p -th moment with respect to \mathcal{F}_t and the family of all \mathcal{F}_s -measurable random vectors in \mathbb{R}_+^d . Section 2 introduces basic conditions on the acceptance sets and several optional ones.

The dynamic *selection risk measure* $R_{t,s}(\Xi)$ for a family $\Xi \subseteq L^0(\mathcal{F}_T; \mathbb{R}^d)$ is introduced as the closure in probability of $(\Xi + \mathcal{C}_{t,s}) \cap L^0(\mathcal{F}_t; \mathbb{R}^d)$. If the family of selections for a random closed set X is $\Xi = L^0(\mathcal{F}_T; X)$, then the set $R_{t,s}(X) = R_{t,s}(\Xi)$ itself is an \mathcal{F}_t -measurable random closed set. In comparison with Feinstein and Rudloff [15], this approach explicitly defines a set-valued dynamic risk measure instead of imposing on it some axiomatic properties. This yields a set-valued risk measure with a set-valued argument that can be naturally iterated in the dynamic framework. The conditional convexity of the acceptance sets yields that

$$R_{t,s}(\lambda X + (1 - \lambda)Y) \supseteq \lambda R_{t,s}(X) + (1 - \lambda)R_{t,s}(Y) \quad \text{a.s.}$$

for any $\lambda \in L^0(\mathcal{F}_t; [0, 1])$ and any random closed sets X and Y , meaning that the risk measure is also conditionally convex. The static case of this construction was considered by Molchanov and Cascos [31], where properties of selection risk measures in the coherent case were obtained, some of them easily extendable for the dynamic convex setting. In comparison to [31], we work with solvency sets instead of portfolios available at price zero and also allow the argument of the risk measure to be a rather general family of random vectors.

The hedging to acceptability relies on a sequence $(K_t)_{t=0, \dots, T}$ of solvency sets and the acceptance sets $\mathcal{C}_{t,s}$ for $0 \leq t \leq s \leq T$. Note that the solvency sets are not assumed to be conical, since non-conical models naturally appear, e.g. in the order book setting; see Çetin et al. [4], Pennanen and Penner [32]. An acceptable portfolio process $(V_t)_{t=0, \dots, T}$ introduced in Sect. 3 satisfies $V_{t-1} - V_t = k_t + \eta_t$ for $k_t \in L^0(\mathcal{F}_t; K_t)$, $\eta_t \in \mathcal{C}_{t-1,t}$, and all t . In other words, the available assets do suffice

to pay for the portfolio at the next step up to an amount acceptable with respect to some risk measure. The strongest acceptability condition assumes that $\mathcal{C}_{t-1,t}$ consists of random vectors with nonnegative components and yields the classical arbitrage theory for markets with transaction costs; see Kabanov and Safarian [26, Sect. 3.2]. The weakest acceptability requirement presumes that all η_t from $\mathcal{C}_{t-1,t}$ have nonnegative \mathcal{F}_{t-1} -conditional expectations.

If ξ is a terminal claim on d assets, then Ξ_t^ξ denotes the set of all initial endowments at time t that ensure the existence of an acceptable portfolio process paying ξ at maturity, that is, $V_T \in \xi + K_T$ a.s. Equivalently, Ξ_t^ξ is the family of \mathcal{F}_t -measurable elements of $(\xi - A_{t,T})$, where $A_{t,T}$ is the set of claims attainable at time T starting from zero investment at time t . The family Ξ_t^ξ may be used to assess the risk associated with ξ at time t .

The no-arbitrage conditions we study are imposed on the set of superhedging prices Ξ_t^0 for the zero claim $\xi = 0$. They may be reformulated as no-arbitrage conditions on the set of attainable claims, which are weaker than the usual ones of the literature. These *no-risk-arbitrage* conditions are introduced and analysed in Sect. 4. In contrast to Cherny [7], we do not rely on the weak compactness of the duals to the acceptance sets and we do not need to pinpoint any reference asset. It should be noted that the risk arbitrage only makes sense in the multiasset setting with some trading opportunities between the assets; if $K_t = \mathbb{R}_+^d$ (which is always the case on the line), then all no-risk-arbitrage conditions automatically hold.

It is shown that in some cases, it is possible to represent the families of capital requirements as a set-valued process, and the no-risk-arbitrage conditions for linear transaction costs can be characterised in terms of weakly consistent price systems, thus providing a variant of the fundamental theorem of asset pricing in our framework; see Theorem 4.10.

A comparison of our approach with the no-good-deals setting is provided in Sect. 5. It is shown that our approach imposes stronger no-arbitrage conditions that are more difficult to check, but which result in lower superhedging prices.

Note that the sets $\mathcal{C}_{t,s}$ of acceptable positions always contain the family $L^0(\mathcal{F}_s; \mathbb{R}_+^d)$ of random vectors with a.s. nonnegative components, and in many cases, $\mathcal{C}_{t,s}$ is a subset of the family of random vectors with nonnegative generalised conditional expectation given \mathcal{F}_t . Thus the no-risk-arbitrage conditions are sandwiched between those for the risk measures based on the conditional essential infimum and on the conditional expectation. The first choice corresponds to the classical financial arbitrage with transaction costs, where our no-risk-arbitrage conditions become the classical no-arbitrage conditions.

Section 6 recovers and extends several results from Kabanov [26, Sect. 3.2]. In this classical setting, our approach yields a new geometric interpretation of the sets of superhedging prices with possibly non-conical solvency sets; it is formulated using the concept of the conditional core of a random set elaborated by Lépinette and Molchanov [28]. The result applies also in some cases when the classical characterisation via consistent price systems fails.

In Sect. 7, we characterise no-arbitrage conditions arising by adopting the generalised conditional expectation as the acceptability criterion. These are the strongest no-arbitrage conditions in our framework; their validity ensures the absence of arbitrage

for all acceptance criteria satisfying a dynamic version of the dilatation monotonicity condition from Cherny and Grigoriev [10].

The results from Sects. 6 and 7 are illustrated in a two-asset example in Sect. 8.

2 Dynamic acceptance sets and selection risk measures

2.1 Definition and main properties

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0, \dots, T}, \mathbb{P})$ be a stochastic basis on a complete probability space such that \mathcal{F}_0 is the trivial σ -algebra and $\mathcal{F}_T = \mathcal{F}$. In the following, we endow random vectors and events with a subscript that indicates the σ -algebras they are measurable with respect to. The subscript is often omitted for \mathcal{F}_T -measurable random vectors.

Let $L^p(\mathcal{F}; \mathbb{R}^d)$ with $p \in [1, \infty]$ be the family of p -integrable random vectors (essentially bounded if $p = \infty$), and let $L^0(\mathcal{F}; \mathbb{R}^d)$ be the family of all random vectors in \mathbb{R}^d . The closure in the strong topology on $L^p(\mathcal{F}; \mathbb{R}^d)$ for $p \in [1, \infty)$ is denoted by cl_p , and cl_0 is the closure in probability in $L^0(\mathcal{F}; \mathbb{R}^d)$. If $p = \infty$, the closure is considered with respect to the a.s. convergence of uniformly bounded sequences.

For a sub- σ -algebra $\mathcal{H} \subseteq \mathcal{F}$, denote by $L^p_{\mathcal{H}}(\mathcal{F}; \mathbb{R}^d)$ the module of \mathcal{F} -measurable random vectors represented as $\gamma\xi$ with $\xi \in L^p(\mathcal{F}; \mathbb{R}^d)$ and $\gamma \in L^0(\mathcal{H}; \mathbb{R})$; see [16, Example 2.5]. In particular, $L^1_{\mathcal{H}}(\mathcal{F}; \mathbb{R}^d)$ is the family of all ξ that admit a generalised conditional expectation $E^g[\xi|\mathcal{H}]$ with respect to \mathcal{H} ; see [28, Lemma B.3]. Following [16, Example 2.5], the module norm is defined by

$$\|\xi\|_{p, \mathcal{H}} = \begin{cases} E[|\xi|^p|\mathcal{H}]^{1/p}, & p \in [1, \infty), \\ \text{ess sup}_{\mathcal{H}}|\xi|, & p = \infty, \end{cases}$$

where $\text{ess sup}_{\mathcal{H}}|\xi|$ is the \mathcal{H} -measurable essential supremum of $|\xi|$; see [18, Appendix A.5]. We endow the space $L^p_{\mathcal{H}}(\mathcal{F}; \mathbb{R}^d)$ with the topology of $L^p_{\mathcal{H}}$ -convergence by saying that (ξ^n) converges to ξ if $\|\xi^n - \xi\|_{p, \mathcal{H}} \rightarrow 0$ in probability if $p \in [1, \infty)$. For $p = \infty$, we use the \mathcal{H} -bounded convergence in probability, meaning that $\sup_n \text{ess sup}_{\mathcal{H}}|\xi^n|$ is finite a.s. and $\|(\xi^n - \xi) \wedge 1\|_{1, \mathcal{H}} \rightarrow 0$ in probability as $n \rightarrow \infty$.

Denote shortly $L^p = L^p(\mathcal{F}; \mathbb{R}^d)$, $L^p_{t,s} = L^p_{\mathcal{F}_t}(\mathcal{F}_s; \mathbb{R}^d)$ for $t \leq s$, and let $\hat{L}^p_{t,s} = \hat{L}^p_{\mathcal{F}_t}(\mathcal{F}_s; \mathbb{R}^d)$ be the family of random vectors ξ_s that can be decomposed as $\xi_s = \xi'_s + \xi''_s$, where $\xi'_s \in L^p_{t,s}$ and $\xi''_s \in L^0(\mathcal{F}_s; \mathbb{R}^d_+)$. Following the classical definition of an acceptance set in the theory of risk measures, we introduce the acceptance set $\mathcal{C}_{t,s}$ for $t \leq s$ as the collection of all \mathcal{F}_s -measurable financial positions regarded as acceptable at time t .

Definition 2.1 Discrete-time L^p -dynamic convex acceptance sets are defined as a family $\{\mathcal{C}_{t,s}, 0 \leq t \leq s \leq T\}$ such that $\mathcal{C}_{t,s} \subseteq \hat{L}^p_{t,s}$ and the following properties hold for all $0 \leq t \leq s \leq T$:

(i) *Normalisation*: we have $\mathcal{C}_{t,t} = L^0(\mathcal{F}_t; \mathbb{R}^d_+)$, $\mathcal{C}_{t,s} \supseteq L^0(\mathcal{F}_s; \mathbb{R}^d_+)$ and $\mathcal{C}_{t,s} \cap L^0(\mathcal{F}_t; \mathbb{R}^d_-) = \{0\}$.

(ii) *Integrability*:

$$\mathcal{C}_{t,s} = (\mathcal{C}_{t,s} \cap L^p_{t,s}) + L^0(\mathcal{F}_s; \mathbb{R}^d_+).$$

(iii) *Closedness*: $\mathcal{C}_{t,s} \cap L^p_{t,T}$ is closed in $L^p_{t,T}$.

(iv) *Conditional convexity*: for all $\alpha_t \in L^0(\mathcal{F}_t; [0, 1])$ and $\eta'_s, \eta''_s \in \mathcal{C}_{t,s}$,

$$\alpha_t \eta'_s + (1 - \alpha_t) \eta''_s \in \mathcal{C}_{t,s}.$$

(v) *Weak time-consistency*: $\mathcal{C}_{t,s} \cap L^0(\mathcal{F}_u; \mathbb{R}^d) = \mathcal{C}_{t,u}$ for all $0 \leq t \leq u \leq s \leq T$.

(vi) *Compensation*: if $\xi_s \in L^p_{t,s}$, then $(\xi_s + \mathcal{C}_{t,s}) \cap L^0(\mathcal{F}_t; \mathbb{R}^d) \neq \emptyset$.

The integrability property implies that $\mathcal{C}_{t,s}$ is an *upper set*, i.e., $\eta_s \in \mathcal{C}_{t,s}$ and $\eta_s \leq \eta'_s$ a.s. (all inequalities are understood coordinatewise) implies that $\eta'_s \in \mathcal{C}_{t,s}$. The compensation property implies that for all $\xi_s \in L^p_{t,s}$, there exists $\gamma_t \in L^0(\mathcal{F}_t; \mathbb{R}^d)$ such that $\gamma_t + \xi_s \in \mathcal{C}_{t,s}$, i.e., it is possible to make the financial position ξ_s acceptable by adding the position γ_t . In the following, we consider a fixed family of such acceptance sets $\mathcal{C}_{t-1,t}$ for $t = 1, \dots, T$.

Example 2.2 The simplest example is given by a static univariate convex risk measure. Consider the one-period setting in one dimension with $t = 0, 1$. If ρ is a convex L^p -risk measure, where $p \in [1, \infty)$, then its acceptance set $\mathcal{C}_{0,1} \cap L^p(\mathcal{F}_1; \mathbb{R})$ is the set of $\eta_1 \in L^p(\mathcal{F}_1; \mathbb{R})$ such that $\rho(\eta_1) \leq 0$. The lower semicontinuity of ρ is equivalent to the closedness of the acceptance set. The conditional convexity property of the acceptance set is equivalent to the convexity property of the risk measure. The compensation property corresponds to the finiteness of ρ .

Definition 2.3 A family $\Xi \subseteq L^0(\mathcal{F}; \mathbb{R}^d)$ is said to be *infinitely \mathcal{H} -decomposable* if $\sum_n \xi_n \mathbf{1}_{A_n} \in \Xi$ for all sequences $(\xi_n)_{n \geq 1}$ from Ξ and all \mathcal{H} -measurable partitions $(A_n)_{n \geq 1}$ of Ω . If this holds for finite partitions, we say that Ξ is *\mathcal{H} -decomposable*.

The following result shows the infinite decomposability property (see Definition 2.3), also known as the countable concatenation property [16] or σ -stability [17]. Decomposability means that the family is stable under partitioning.

Lemma 2.4 *For every $0 \leq t \leq s \leq T$, the family $\mathcal{C}_{t,s}$ is infinitely \mathcal{F}_t -decomposable.*

Proof If $\eta_s^i \in \mathcal{C}_{t,s} \cap L^p_{t,s}$ and $B_t^i \in \mathcal{F}_t$, $i \geq 1$, then

$$\bar{\eta}_s^n = \sum_{i=1}^n \mathbf{1}_{B_t^i} \eta_s^i + \eta_s^1 \mathbf{1}_{\Omega \setminus \cup_{i=1}^n B_t^i} \in \mathcal{C}_{t,s}$$

by the conditional convexity property, so that $\mathcal{C}_{t,s} \cap L^p_{t,s}$ is \mathcal{F}_t -decomposable. Since $\bar{\eta}_s^n \rightarrow \bar{\eta}_s = \sum_{i=1}^{\infty} \mathbf{1}_{B_t^i} \eta_s^i$ in the $\|\cdot\|_{p, \mathcal{F}_t}$ -norm if $p \in [1, \infty)$ and \mathcal{F}_t -boundedly in probability if $p = \infty$, we have $\bar{\eta}_s \in \mathcal{C}_{t,s} \cap L^p_{t,s}$. By the integrability property, $\mathcal{C}_{t,s}$ is also infinitely decomposable. \square

Definition 2.5 A family of dynamic convex acceptance sets is called

(i) *coherent* if $\alpha_t \eta_s \in \mathcal{C}_{t,s}$ for all $t \leq s$, $\alpha_t \in L^0(\mathcal{F}_t; \mathbb{R}_+)$ and $\eta_s \in \mathcal{C}_{t,s}$;

(ii) *continuous from below at zero* if for every $t \leq s$ and any sequence $(\xi_s^n)_{n \in \mathbb{N}}$ in $L^p_{\mathcal{F}_t}(\mathcal{F}_s; \mathbb{R}_+^d)$ with $\|\xi_s^n\|_{p, \mathcal{F}_t} \rightarrow 0$ in probability as $n \rightarrow \infty$, there exist a sequence $(\gamma_t^n)_{n \in \mathbb{N}}$ in $L^0(\mathcal{F}_t; \mathbb{R}_+^d)$ and $k \in \mathbb{R}_+$ such that $\gamma_t^n + \xi_s^n \in \mathcal{C}_{t,s}$ and $|\gamma_t^n| \leq k \|\xi_s^n\|_{p, \mathcal{F}_t}$ a.s. for all n .

If $p = \infty$, then continuity from below at zero always holds and is easily verified by choosing γ_t^n with all components being $\|\xi_s^n\|_{\infty, \mathcal{F}_t}$.

Example 2.6 The acceptance sets can be defined using a univariate convex dynamic L^p -risk measure $(\rho_t)_{t=0, \dots, T}$, so that $\mathcal{C}_{t,s} \cap L_{t,s}^p$ is the d -th Cartesian power of the acceptance set for ρ_t . Equivalently, $\xi_s \in \mathcal{C}_{t,s} \cap L_{t,s}^p$ if and only if all components of ξ_s are acceptable under ρ_t . The continuity from below at zero (with $p \in [1, \infty)$) holds if ρ_t is lower semicontinuous in the $\|\cdot\|_{p, \mathcal{F}_t}$ -norm and continuous from below, which is the case if ρ_t is convex and a.s. finite; see Vogelpoth [35, Theorem 4.1.4].

Example 2.7 For a dual construction of conditional acceptance sets, let $p = \infty$ and consider families $\mathcal{Z}_{t,s} \subseteq L_{\mathcal{F}_t}^1(\mathcal{F}_s; \mathbb{R}_+^d)$ with $0 \leq t \leq s \leq T$ such that $\mathcal{Z}_{t,u} \subseteq \mathcal{Z}_{t,s}$ for all $t \leq u \leq s$ and $\mathbb{E}^g[\zeta_s | \mathcal{F}_t] = (1, \dots, 1)$ for all $\zeta_s \in \mathcal{Z}_{t,s}$. Note that we do not assume that $\mathcal{Z}_{t,s}$ is weakly compact. Define

$$\mathcal{C}_{t,s} = L^0(\mathcal{F}_s; \mathbb{R}_+^d) + \bigcap_{\zeta_s \in \mathcal{Z}_{t,s}} \{\eta_s \in L_{t,s}^\infty : \mathbb{E}^g[\langle \zeta_s, \eta_s \rangle | \mathcal{F}_t] \geq 0\},$$

where $\langle \zeta_s, \eta_s \rangle$ is the scalar product. It is easily seen that conditions (i), (ii), (iv) and (v) of Definition 2.1 hold and these acceptance sets are coherent.

If $\xi_s^n \rightarrow \xi_s$ in probability with $\xi_s^n \in \mathcal{C}_{t,s}$ and all ξ_s^n are bounded in the Euclidean norm by $\gamma_t \in L^0(\mathcal{F}_t; \mathbb{R}_+)$, then $\mathbb{E}^g[\langle \zeta_s, \xi_s \rangle | \mathcal{F}_t] \geq 0$ by the dominated convergence theorem for generalised conditional expectations. Thus condition (iii) also holds.

If $\xi_s \in L_{t,s}^\infty$, the components of ξ_s are bounded in absolute value by $\eta_t \in L^0(\mathcal{F}_t; \mathbb{R}_+^d)$. Then $\eta_t - \xi_s$ is nonnegative and so belongs to $\mathcal{C}_{t,s}$, and $\xi_s + (\eta_t - \xi_s)$ is in $L^0(\mathcal{F}_t; \mathbb{R}^d)$. Thus (vi) also holds.

2.2 Dynamic selection risk measures

Let Ξ_T be an *upper* subset of $L^0(\mathcal{F}_T; \mathbb{R}^d)$, that is, with each $\xi \in \Xi_T$, the family Ξ_T also contains all $\xi' \in L^0(\mathcal{F}_T; \xi + \mathbb{R}_+^d)$. The most important example of such a family is the family of selections $L^0(\mathcal{F}_T; X_T)$ for an \mathcal{F}_T -measurable upper random set X_T in \mathbb{R}^d , that is, $X_T + \mathbb{R}_+^d \subseteq X_T$ a.s. The (graph) measurability of a random set is defined in the Appendix. If X_T is also closed, then its so-called centrally symmetric version $-X_T := x \in L^0 : -x \in X_T$ is a *set-valued portfolio* in the terminology of Molchanov and Cascos [31].

Definition 2.8 Let $\Xi_T \subseteq L^0(\mathcal{F}_T; \mathbb{R}^d)$ be an upper set. For $t \leq s \leq T$,

$$\mathbb{R}_{t,s}^0(\Xi_T) := (\Xi_T + \mathcal{C}_{t,s}) \cap L^0(\mathcal{F}_t; \mathbb{R}^d) \quad (2.1)$$

denotes the family of all $\gamma_t \in L^0(\mathcal{F}_t; \mathbb{R}^d)$ such that $\gamma_t - \xi \in \mathcal{C}_{t,s}$ for some $\xi \in \Xi_T$. Let $\mathbb{R}_{t,s}(\Xi_T)$ denote the closure in probability of $\mathbb{R}_{t,s}^0(\Xi_T)$. If $\Xi_T = L^0(\mathcal{F}_T; X_T)$ is the family of selections of an upper random set X_T , we write $\mathbb{R}_{t,s}^0(X_T)$ and $\mathbb{R}_{t,s}(X_T)$ instead of $\mathbb{R}_{t,s}^0(\Xi_T)$ and $\mathbb{R}_{t,s}(\Xi_T)$. In view of this, $\mathbb{R}_{t,s}(X_T)$ (and also $\mathbb{R}_{t,s}(\Xi_T)$) is called a *dynamic selection risk measure*.

Note that we have $R_{T,T}^0(\Xi_T) = \Xi_T$, $R_{t,s}^0(\Xi_T) = R_{t,s}^0(\Xi_T \cap L^0(\mathcal{F}_s; \mathbb{R}^d))$ and $R_{t,u}^0(\Xi_T) \subseteq R_{t,s}^0(\Xi_T)$ for all $0 \leq t \leq u \leq s \leq T$. If only portfolios from a random set M_t are allowed for compensation at time t , i.e. to be added to a financial position for it to become acceptable as it is the case in Feinstein and Rudloff [15], it is easy to modify (2.1) by intersecting $(\Xi_T + \mathcal{C}_{t,s})$ with $L^0(\mathcal{F}_t; M_t)$.

First, for multi-varied financial positions, there may be several positions that can be added to any position X so that it becomes acceptable. This is why the values of a risk-measure consists of families of random variables (and not a singleton $\{\rho(X)\}$, mainly in the real case where $\rho(X)$ is actually a minimal element, while the family is $[\rho(X), \infty)$). Moreover, a risk-measure is said dynamic because its values are updated from time to time through the available information on the market in the same way than the conditional expectation allows to update the average of all possible values of a random variable accordingly to new information given by the information.

Recall that values of a dynamic risk-measure are families of random variables that are updated from time to time through the available information on the market described by the filtration. A dynamic selection risk measure of the family Ξ_T is a financial position that allows to reach a position of Ξ_T up to an acceptable set. The empty selection risk measure corresponds to completely unacceptable positions. The compensation property of acceptance sets guarantees that $R_{t,s}^0(\Xi_T)$ is not empty if $\Xi_T \cap L_{t,s}^p \neq \emptyset$. The family Ξ_T is said to be *acceptable* for the time horizon s if $0 \in R_{t,s}^0(\Xi_T)$; equivalently, $-\Xi_T$ is the family of all $-x_T$ where $x_T \in \Xi_T$ contains an element from $\mathcal{C}_{t,s}$. A dynamic selection risk measure is conditionally convex, that is,

$$R_{t,s}^0(\alpha_t \Xi_T' + (1 - \alpha_t) \Xi_T'') \supseteq \alpha_t R_{t,s}^0(\Xi_T') + (1 - \alpha_t) R_{t,s}^0(\Xi_T'')$$

for all $\alpha_t \in L^0(\mathcal{F}_t; [0, 1])$, and the same holds for the closures. The next result follows from Lemma 2.4.

Lemma 2.9 *If Ξ_T is infinitely \mathcal{F}_t -decomposable, then $R_{t,s}^0(\Xi_T)$ and $R_{t,s}(\Xi_T)$ are also infinitely \mathcal{F}_t -decomposable for each t and s such that $s \leq t$.*

Lemma 2.10 *Let X_T be an \mathcal{F}_T -measurable random upper closed set for each fixed t and s such that $t \leq s$.*

(i) $R_{t,s}(X_T)$ coincides with the family of measurable selections of an \mathcal{F}_t -measurable random upper set in \mathbb{R}^d , also denoted by $R_{t,s}(X_T)$.

(ii) If X_T is a.s. convex, then $R_{t,s}^0(X_T)$ is a.s. convex. If X_T is a cone and the acceptance sets are coherent, then $R_{t,s}^0(X_T)$ is a cone.

Proof (i) By Lemma 2.9, $R_{t,s}(X_T)$ is an \mathcal{F}_t -decomposable family, and so Theorem A.2 applies.

(ii) If $\gamma_t^1, \gamma_t^2 \in R_{t,s}^0(X_T)$, then $\gamma_t^i - \xi_s^i \in \mathcal{C}_{t,s}$, $i = 1, 2$, for $\xi_s^1, \xi_s^2 \in L^0(\mathcal{F}_s; X_T)$. For any $t \in (0, 1)$, $t\gamma_t^1 + (1-t)\gamma_t^2 - \xi_s \in \mathcal{C}_{t,s}$ by the conditional convexity property with $\xi_s = t\xi_s^1 + (1-t)\xi_s^2 \in L^0(\mathcal{F}_s; X_T)$. The cone property is trivial. \square

Example 2.11

If $X_s = \xi_s + \mathbb{R}_+^d$ for some $\xi_s \in L_{t,s}^p$ and if $\mathcal{C}_{t,s} \cap L_{t,s}^p = \{X \in L_{t,s}^p : r_t(X) \leq 0\}$, then for a vector-valued dynamic risk measure r_t on $L_{t,s}^p$, we have

$$\mathbb{R}_{t,s}^0(X_s) = \mathbb{R}_{t,s}(X_s) = r_t(-\xi_s) + \mathbb{R}_+^d,$$

see Vogelpoth [35, Chap. 5].

3 Hedging to acceptability

3.1 Acceptable portfolio process

Let $(K_t)_{t=0,\dots,T}$ be a sequence of random closed convex sets such that for all t , we have $K_t \cap \mathbb{R}_-^d = \{0\}$, K_t is an upper set and K_t is \mathcal{F}_t -measurable. The set K_t is understood as the family of all solvent positions at time t expressed in physical units and is called a *solvency set*; see Kabanov and Safarian [26, [Sect. 3.1.1]]. If the solvency sets are cones, this model is well studied and called Kabanov's model; it describes the market subject to proportional transaction costs, see [26, Sect. 3.1] and Schachermayer [34]. If the solvency sets are cones and the acceptance sets are coherent, we talk about the *coherent conical* setting.

Let K_t^0 be the largest \mathcal{F}_t -measurable linear subspace contained in K_t , that is,

$$K_t^0 = \bigcap_{c \neq 0} cK_t = \bigcap_{c \in \mathbb{Q} \setminus \{0\}} cK_t,$$

which is also a random closed set. The solvency sets are called *proper* if $K_t^0 = \{0\}$ and *strictly proper* if $\tilde{K}_t := K_t \cap (-K_t) = \{0\}$ for all $t = 0, \dots, T$. If K_t is a cone, then $\tilde{K}_t = K_t^0$, while in general $K_t^0 \subseteq \tilde{K}_t$. Since \tilde{K}_t is convex and origin symmetric, K_t is proper if and only if \tilde{K}_t is bounded.

Definition 3.1 A sequence $(V_t)_{t=0,\dots,T}$ in $L^0(\mathcal{F}_t; \mathbb{R}^d)$ is an *acceptable* portfolio process if

$$V_{t-1} - V_t \in L^0(\mathcal{F}_t; K_t) + \mathcal{C}_{t-1,t}, \quad t = 1, \dots, T. \quad (3.1)$$

By the definition of a selection risk measure, (3.1) is equivalent to

$$V_{t-1} \in \mathbb{R}_{t-1,t}^0(V_t + K_t), \quad t = 1, \dots, T.$$

Thus by paying transaction costs, it is possible to transform $V_{t-1} - V_t$ into an acceptable position for the horizon t . Equivalently, V_{t-1} suffices to purchase $V_t + k_t + \eta_t$ for some $k_t \in K_t$ and $\eta_t \in \mathcal{C}_{t-1,t}$. An *initial endowment* at time t is any $V_{t-} \in L^0(\mathcal{F}_t; V_t + K_t)$, so that it is possible to convert V_{t-} immediately into V_t paying the transaction costs. (In geometrical models like Kabanov's one it is standard to suppose, e.g. at time $t = 0$ for simplicity, that we may immediately rebalance the initial endowment through the initial cone K_t , paying transaction costs.)

3.2 Attainable positions and superhedging

The family of *attainable positions* at time $s > t$ is the family of random vectors that may be obtained as V_s for acceptable portfolio processes starting from zero investment at time t . By (3.1), the family of attainable positions is given by

$$A_{t,s} = \sum_{u=t}^s L^0(\mathcal{F}_u; -K_u) - \sum_{u=t}^{s-1} C_{u,u+1}.$$

Let $\xi \in L^0(\mathcal{F}_T; \mathbb{R}^d)$ be a *terminal claim* (or payoff). Hedging to acceptability aims to come up with an acceptable portfolio process $(V_t)_{t=0,\dots,T}$ that guarantees paying ξ in the sense that the terminal wealth V_T belongs to $\Xi_T^\xi = L^0(\mathcal{F}_T; X_T^\xi)$, the family of selections of the random closed set $X_T^\xi = \xi + K_T$. Define recursively

$$\Xi_t^\xi = L^0(\mathcal{F}_t; K_t) + R_{t,t+1}^0(\Xi_{t+1}^\xi), \quad t = T-1, \dots, 0. \quad (3.2)$$

The family Ξ_t^ξ consists of the time- t superhedging prices for ξ and the corresponding mapping $\xi \mapsto \Xi_t^\xi$ defines a dynamic convex risk measure with values being subsets of $L^0(\mathcal{F}_t; \mathbb{R}^d)$. If $\xi = \xi' - \xi''$ for $\xi' \in L^0(\mathcal{F}_T; \mathbb{R}^d)$ and $\xi'' \in L^0(\mathcal{F}_T; \mathbb{R}_+^d)$, the compensation property of acceptance sets ensures that $\Xi_t^\xi \neq \emptyset$ for all t .

In order to handle the asymptotic version of the risk measure, let $\hat{\Xi}_T^\xi := \Xi_T^\xi$ and further

$$\hat{\Xi}_t^\xi := L^0(\mathcal{F}_t; K_t) + R_{t,t+1}(\Xi_{t+1}^\xi), \quad t = T-1, \dots, 0. \quad (3.3)$$

Note that $\Xi_t^\xi \subseteq \hat{\Xi}_t^\xi \subseteq \text{cl}_0(\Xi_t^\xi)$, whence $\text{cl}_0(\hat{\Xi}_t^\xi) = \text{cl}_0(\Xi_t^\xi)$ for all t . The families Ξ_t^0 and $\hat{\Xi}_T^0$ arise by letting $\xi = 0$ a.s.

Lemma 3.2 (i) *The families $R_{t,s}^0(\Xi_s^\xi)$, $R_{t,s}(\Xi_s^\xi)$ and $\hat{\Xi}_t^\xi$ are convex and infinitely \mathcal{F}_t -decomposable for all $0 \leq t \leq s \leq T$.*

(ii) *For each $t \leq T$, there exists a (possibly non-closed) random set X_t^ξ such that $\hat{\Xi}_t^\xi = L^0(\mathcal{F}_t; X_t^\xi)$.*

(iii) *For any $t \leq T$, the set of all endowments V_{t-} at time t allowing to start an acceptable portfolio process $(V_s)_{t \leq s \leq T}$ such that $V_T \in L^0(\mathcal{F}_T; \xi + K_T)$ a.s. coincides with Ξ_t^ξ , and*

$$\Xi_t^\xi = (-A_{t,T} + \xi) \cap L^0(\mathcal{F}_t; \mathbb{R}^d). \quad (3.4)$$

(iv) *If K_T is a cone, then $\Xi_t^\xi \subseteq \Xi_t^0$ for any $\xi \in L^0(\mathcal{F}_T; K_T)$.*

Proof (i) follows from Lemma 2.9.

(ii) The existence of X_t^ξ is trivial for $t = T$. Suppose that it holds at time t . The result for $t-1$ follows from the induction assumption and (3.3) by Lemma A.3.

(iii) follows from the fact that $(\gamma_T + A_{t,T}) \cap (\xi + K_T) \neq \emptyset$ if and only if $\gamma_T \in (-A_{t,T} + \xi)$.

(iv) follows from (iii), since $\xi + K_T \subseteq K_T$ a.s. \square

Example 3.3 If $K_t = \mathbb{R}_+^d$ a.s. for all t (which is always the case if $d = 1$), then an acceptable portfolio process satisfies $V_{t-1} - V_t \in \mathcal{C}_{t-1,t}$ for all $t = 1, \dots, T$. Then

$$\Xi_t^\xi = \left(\xi + \sum_{s=t}^{T-1} \mathcal{C}_{s,s+1} \right) \cap L^0(\mathcal{F}_t; \mathbb{R}^d).$$

Since $\mathcal{C}_{t-1,t} \cap L^0(\mathcal{F}_{t-1}; \mathbb{R}^d) = L^0(\mathcal{F}_{t-1}; \mathbb{R}_+^d)$ for all $t \geq 1$ by weak time-consistency and normalisation, induction yields that $\Xi_t^0 = L^0(\mathcal{F}_t; \mathbb{R}_+^d)$. If ξ does not a.s. vanish, the set $\mathbb{R}_{t,s}^0(\Xi_t^\xi)$ becomes non-trivial. Its static variant is called a *regulator risk measure* in Hamel et al. [21]; it only takes into account the acceptability requirement and disregards any trading opportunities between the components. In the terminology of [21], $\mathbb{R}_{0,1}(\xi + K_1)$ (in the static setting with a conical K_1) is called the *market extension* of the regulator risk measure.

4 Risk arbitrage

Recall that Ξ_t^0 is the set of time- t superhedging prices for the zero claim. By (3.4),

$$\Xi_t^0 = (-A_{t,T}) \cap L^0(\mathcal{F}_t; \mathbb{R}^d). \quad (4.1)$$

For multivariate financial models, e.g. models with proportional transaction costs, several no-arbitrage conditions have been considered. In Kabanov's model, there is the NA condition, its robust version NA^r , but also the NA2 condition derived using an alternative approach; see the paper by Rásonyi [33] and [24]. All these conditions are formulated in terms of the set $A_{t,T}$ of all terminal claims attainable from zero initial endowment. Here, we consider weaker no-arbitrage conditions imposed on the superhedging prices for the zero claim.

Definition 4.1 The multiperiod model satisfies

- (SNR) (*strict no-risk-arbitrage*) if we have $\hat{\Xi}_t^0 \cap L^0(\mathcal{F}_t; -K_t) \subseteq L^0(\mathcal{F}_t; K_t^0)$ for all $t = 0, \dots, T$;
- (NRA) (*no-risk-arbitrage*) if $\Xi_t^0 \cap L^0(\mathcal{F}_t; \mathbb{R}_-^d) = \{0\}$ for all $t = 0, \dots, T$;
- (NARA) (*no asymptotic risk arbitrage*) if $(\text{cl}_0 \Xi_t^0) \cap L^0(\mathcal{F}_t; \mathbb{R}_-^d) = \{0\}$ for all $t = 0, \dots, T$;
- (NRA2) (*no-risk-arbitrage opportunity of the second kind*) if for $t = 0, \dots, T$ and $\eta_t \in L^0(\mathcal{F}_t; \mathbb{R}^d)$ such that $(\eta_t + A_{t,T}) \cap L^0(\mathcal{F}_T; K_T) \neq \emptyset$, we have $\eta_t \in L^0(\mathcal{F}_t; K_t) + \mathcal{C}_{t-1,t}$;
- (SNRA) (*strong no-risk-arbitrage*) if $\sum_{t=0}^T (k_t + \eta_t) = 0$ for $k_t \in L^0(\mathcal{F}_t; K_t)$ and $\eta_t \in \mathcal{C}_{t-1,t}$ for all t implies that $k_t \in L^0(\mathcal{F}_t; K_t^0)$ and $\eta_t = 0$ a.s. for all t .

Let us comment on the (SNR) condition. If $p_t \in \hat{\Xi}_t^0 \cap L^0(\mathcal{F}_t; -K_t)$, then starting from zero endowment at time t expressed as $0 = p_t - p_t$, we obtain the zero claim at time T from p_t and have an immediate possible profit at time t since the liquidation value of $-p_t \in K_t$ is nonnegative. A similar interpretation applies for the (NRA) condition and its asymptotic version (NARA). The (NRA2) condition may be compared

to the (NA2) condition by Rásonyi [33], while (SNRA) is a version of [26, Sect. 3.2.2, Condition (iii)]. Note that the (NRA) condition reads $A_{t,T} \cap L^0(\mathcal{F}_t; \mathbb{R}_+^d) = \{0\}$, while the usual (NA) condition $A_{t,T} \cap L^0(\mathcal{F}_T; \mathbb{R}_+^d) = \{0\}$ is stronger; see [26, Sect. 3.2.1].

Example 1.1 shows that it may be possible to release infinite capital from a zero position without compromising the acceptability criterion; in particular, it violates the (SNR) condition. By Lemma 3.2, (SNR) can be written as $X_t^0 \cap (-K_t) \subseteq K_t^0$ a.s., and (NARA) as $(\text{cl}X_t^0) \cap \mathbb{R}_-^d = \{0\}$ a.s. It is obvious that (NARA) is stronger than (NRA). By (4.1), the (NRA) condition is equivalent to $A_{t,T} \cap L^0(\mathcal{F}_t; \mathbb{R}_+^d) = \{0\}$ for all t . If $K_t = \mathbb{R}_+^d$ a.s. for all t ; then $\Xi_t^0 = L^0(\mathcal{F}_t; \mathbb{R}_+^d)$ and all no-arbitrage conditions are satisfied, see Example 3.3.

Lemma 4.2 (SNR) implies that

$$R_{t,t+1}(\Xi_{t+1}^0) \cap L^0(\mathcal{F}_t; -K_t) \subseteq L^0(\mathcal{F}_t; K_t^0), \quad t = 0, \dots, T-1.$$

The converse implication holds if the solvency sets are strictly proper.

Proof Denote $M = R_{t,t+1}(\Xi_{t+1}^0)$, $A = L^0(\mathcal{F}_t; K_t)$ and $B = L^0(\mathcal{F}_t; K_t^0)$. It is easily seen that $M \cap (-A) \subseteq B$ if $(M + A) \cap (-A) \subseteq B$ and only if in case $A \cap (-A) = \{0\}$. For the converse implication, if $x \in (M + A) \cap (-A)$, then we have $x = m + a_1 = -a_2$, with $m \in M$ and $a_1, a_2 \in A$. Therefore, $m/2 \in M \cap (-A) \subseteq B$. Then $x/2 \in A \cap (-A)$, so that $x \in B = A \cap (-A) = \{0\}$ if K_t is strictly proper. \square

Lemma 4.3 Assume that the acceptance sets are strictly proper, that is, $\mathcal{C}_{t,s} \cap (-\mathcal{C}_{t,s})$ consists of all random vectors that equal 0 almost surely.

(i) If $K_t^0 = \tilde{K}_t$ for all t , then (SNRA) implies

$$A_{0,t} \cap (L^0(\mathcal{F}_t; K_t) + \mathcal{C}_{t-1,t}) \subseteq L^0(\mathcal{F}_t; K_t^0), \quad t = 0, \dots, T, \quad (4.2)$$

$$A_{t,T} \cap (L^0(\mathcal{F}_t; K_t) + \mathcal{C}_{t-1,t}) \subseteq L^0(\mathcal{F}_t; K_t^0), \quad t = 0, \dots, T. \quad (4.3)$$

(ii) If the solvency sets are strictly proper and

$$L^0(\mathcal{F}_t; -K_t) \cap \mathcal{C}_{t-1,t} = \{0\}, \quad t = 0, \dots, T. \quad (4.4)$$

Then each of the conditions (4.2), (4.3) implies (SNRA).

Proof (i) Motivated by [26, Lemma 3.2.7], assume that

$$-k_0 - \dots - k_t - \eta_0 - \dots - \eta_t = g_t + \zeta_t \in A_{0,t} \cap (L^0(\mathcal{F}_t; K_t) + \mathcal{C}_{t-1,t}),$$

where $k_s \in L^0(\mathcal{F}_s; K_s)$ and $\eta_s \in \mathcal{C}_{s-1,s}$ for $s = 0, \dots, t$, $g_t \in L^0(\mathcal{F}_t; K_t)$ and $\zeta_t \in \mathcal{C}_{t-1,t}$. Since $(\eta_t + \zeta_t)/2 \in \mathcal{C}_{t-1,t}$ by convexity and

$$-k_0/2 - \dots - k_{t-1}/2 - (k_t + g_t)/2 - \eta_0/2 - \dots - (\eta_t + \zeta_t)/2 = 0,$$

we deduce that $(k_t + g_t)/2 \in K_t^0$ and $(\eta_t + \zeta_t)/2 = 0$ by (SNRA). The strict properness of the acceptance sets yields that $\eta_t = \zeta_t = 0$. At last, we observe that $\frac{1}{2}g_t \in -\frac{1}{2}k_t + \frac{1}{2}K_t^0 \subseteq -K_t$ so that $g_t \in K_t^0$, i.e., (4.2) holds.

Property (4.3) is similarly derived from (SNRA).

(ii) In order to show that (4.2) implies (SNRA), proceed by induction as in [26, Lemma 3.2.13]. Let $-k_0 - \dots - k_T - \eta_0 - \dots - \eta_T = 0$. Then

$$k_T + \eta_T = \sum_{s=0}^{T-1} (-k_s - \eta_s) \in A_{0,T-1} \subseteq A_{0,T}.$$

By (4.2), $k_T + \eta_T \in L^0(\mathcal{F}_T; K_T^0)$. Since $k_T + \eta_T$ is \mathcal{F}_{T-1} -measurable and the solvency sets are strictly proper, $k_T + \eta_T \in L^0(\mathcal{F}_{T-1}; K_{T-1}^0)$. Therefore, $k_T + \eta_T$ can be merged with k_{T-1} , and then the induction proceeds with $T - 1$ instead of T .

To show that (4.3) implies (SNRA), proceed by induction starting from time zero. Since

$$k_0 + \eta_0 = \sum_{s=1}^T (-k_s - \eta_s) \in A_{1,T} \subseteq A_{0,T},$$

(4.3) yields that $k_0 + \eta_0 = 0$, and (4.4) implies $k_0 = \eta_0 = 0$. \square

Condition (4.4) can be viewed as a consistency between acceptance sets and solvency sets, namely that $-K_t$ does not contain any acceptable non-trivial selection.

The first part of the following result shows that (NRA) is similar to the weak no-arbitrage property NA^w of Kabanov's model; see [26, Sect. 3.2.1]. Denote by $\text{int}A$ the interior and by ∂A the boundary of $A \subseteq \mathbb{R}^d$.

Proposition 4.4 *Suppose that $\mathbb{R}_+^d \setminus \{0\} \subseteq \text{int}K_t$ a.s. for all t . Then (NRA) is equivalent to each of the following two conditions:*

- (i) $\mathbb{R}_{t,t+1}^0(\Xi_{t+1}^0) \cap L^0(\mathcal{F}_t; -K_t) \subseteq L^0(\mathcal{F}_t; -\partial K_t)$ for all t .
- (ii) $\hat{\Xi}_t^0 \cap L^0(\mathcal{F}_t; \mathbb{R}_-^d) = \{0\}$ for all t .

Proof (i) Consider $x_t = \gamma_t + k_t$ for $\gamma_t \in M = \mathbb{R}_{t,t+1}^0(\Xi_{t+1}^0)$ and $k_t \in L^0(\mathcal{F}_t; K_t)$. Assume that $x_t \in L^0(\mathcal{F}_t; \mathbb{R}_-^d) \setminus \{0\}$. Hence $\gamma_t/2 = x_t/2 - k_t/2 \in L^0(\mathcal{F}_t; -K_t)$ and $\gamma_t/2 \in -\text{int}K_t$ on $\{x_t \neq 0\}$ since $\text{int}K_t$ contains $\mathbb{R}_+^d \setminus \{0\}$. This contradicts the assumption.

Consider any $x_t \in M \cap L^0(\mathcal{F}_t; -K_t)$ such that $x_t = -k_t$ for some $k_t \in L^0(\mathcal{F}_t; K_t)$ with $\mathbb{P}[k_t \in \text{int}K_t] > 0$. By a measurable selection argument, there exists $\gamma_t \in L^0(\mathcal{F}_t; \mathbb{R}_-^d) \setminus \{0\}$ such that $k_t + \gamma_t \in L^0(\mathcal{F}_t; K_t)$. Thus,

$$x_t + k_t + \gamma_t = \gamma_t \in (M + L^0(\mathcal{F}_t; K_t)) \cap L^0(\mathcal{F}_t; \mathbb{R}_-^d),$$

contradicting (NRA).

(ii) It suffices to show that (NRA) implies (ii). Assume that $k_t \in L^0(\mathcal{F}_t; K_t)$ and $\gamma_t \in \mathbb{R}_{t,t+1}^0(\Xi_{t+1}^0)$ are such that $k_t + \gamma_t \in \mathbb{R}_-^d$ a.s. and $k_t + \gamma_t \neq 0$ with positive probability. Since $k_t/2 + \mathbb{R}_+^d \subseteq (\{k_t/2\} \cup \text{int}K_t)$ a.s., with positive probability, the set $(\text{int}K_t + \mathbb{R}_{t,t+1}^0(\Xi_{t+1}^0))$ has a non-trivial intersection with \mathbb{R}_-^d . Applying [28, Prop. 2.10] with $X = \text{int}K_t$ and $\Xi = \mathbb{R}_{t,t+1}^0(\Xi_{t+1}^0)$, the set $(\text{int}K_t + \mathbb{R}_{t,t+1}^0(\Xi_{t+1}^0))$ has a non-trivial intersection with \mathbb{R}_-^d with positive probability, which contradicts (NRA). \square

Theorem 4.5 *If the solvency sets are proper, then (SNR) implies (NARA) plus the closedness of $\hat{\Xi}_t^0$ in probability for all $t = 0, \dots, T$.*

Proof Denote $M = R_{t,t+1}(\Xi_{t+1}^0)$. Recall that we have $\text{cl}_0 \hat{\Xi}_t^0 = \text{cl}_0 \Xi_t^0$. Assume that $k_t^n + \gamma_t^n \rightarrow \zeta_t \in L^0(\mathcal{F}_t; \mathbb{R}_-^d)$ a.s. for $k_t^n \in L^0(\mathcal{F}_t; K_t)$ and $\gamma_t^n \in M$ such that $k_t^n + \gamma_t^n \in \hat{\Xi}_t^0$ for $n \geq 1$. Since M is L^0 -closed and convex, we may assume thanks to [26, Lemma 2.1.2] that $k_t^n \rightarrow k_t \in L^0(\mathcal{F}_t; K_t)$ on the set $A = \{\liminf_{n \rightarrow \infty} |k_t^n| < \infty\}$. Hence $\gamma_t^n \rightarrow \gamma_t \in M$ so that

$$\gamma_t = \zeta_t - k_t \in M \cap L^0(\mathcal{F}_t; -K_t) \subseteq L^0(\mathcal{F}_t; K_t^0).$$

Thus $\gamma_t \in K_t^0$ and $\zeta_t/2 = \gamma_t/2 + k_t/2 \in K_t$. Hence $\zeta_t/2 \in \mathbb{R}_-^d \cap K_t = \{0\}$ and $\zeta_t = 0$ on A .

If $P[\Omega \setminus A] > 0$, assume that $k_t^n = \gamma_t^n = \zeta_t = 0$ on A by \mathcal{F}_t -decomposability, and use a normalisation procedure, i.e., divide $k_t^n, \gamma_t^n, \zeta_t$ by $(1 + |k_t^n|)$. Arguing as previously, we obtain $k_t \in L^0(\mathcal{F}_t; K_t)$ such that $|k_t| = 1$ on $\Omega \setminus A$. Since $0 \in M$, we have $\gamma_t \in M$ by conditional convexity. Moreover, $k_t + \gamma_t = 0$ since $\zeta_t/(1 + |k_t^n|) \rightarrow 0$. Hence, $\gamma_t \neq 0$ belongs to $M \cap L^0(\mathcal{F}_t; -K_t) = \{0\}$, which is a contradiction in view of Lemma 4.2. This argument also yields the closedness of $\hat{\Xi}_t^0 = L^0(\mathcal{F}_t; K_t) + M$. \square

Let $\bar{A}_{t,s}^p$ denote the closure of $A_{t,s}^p = A_{t,s} \cap L^p(\mathcal{F}_T; \mathbb{R}^d)$ with respect to the module norm.

The following theorem states that the (NARA) and (SNR) conditions are weak no-arbitrage conditions of the no-free-lunch type. Recall that the usual NFL condition is $\text{cl}_{\infty,1}(A_{t,T}) \cap L^\infty(\mathcal{F}_T; \mathbb{R}_+^d) = \{0\}$, where $\text{cl}_{\infty,1}$ designates the weak closure in $L^\infty(\mathcal{F}_T; \mathbb{R}_+^d)$ with respect to the (dual) space $L^1(\mathcal{F}_T; \mathbb{R}_+^d)$, see [13, Sect. 5.2].

Theorem 4.6 *Assume that the acceptance sets are continuous from below at zero and $p \in [1, \infty]$.*

(i) (NARA) is equivalent to

$$\bar{A}_{t,T}^p \cap L^0(\mathcal{F}_t; \mathbb{R}_+^d) = \{0\}, \quad t = 0, \dots, T-1. \quad (4.5)$$

(ii) If the solvency sets are proper, then (SNR) is equivalent to

$$\bar{A}_{t,T}^p \cap L^0(\mathcal{F}_t; K_t) = \{0\}, \quad t = 0, \dots, T-1. \quad (4.6)$$

Moreover, properties (4.5) and (4.6) are equivalent to the same ones with $p = 1$ and also to those obtained by taking the closure of $A_{t,T} \cap L^p(\mathcal{F}_T; \mathbb{R}^d)$ with respect to the norm on $L^p(\mathcal{F}_T; \mathbb{R}^d)$.

Proof (i) Assume that (4.5) holds for the closure with respect to the module norm on $\bar{L}_{t,T}^p$. Then (4.5) also holds if the closure is taken with respect to the norm on $L^p(\mathcal{F}_T; \mathbb{R}^d)$. Therefore, given (4.1), it suffices to show that (NARA) follows from

$$\text{cl}_p(\Xi_t^0 \cap L^p(\mathcal{F}_t; \mathbb{R}^d)) \cap L^0(\mathcal{F}_t; \mathbb{R}_-^d) = \{0\}, \quad t = 0, \dots, T-1. \quad (4.7)$$

Assume (4.7) and consider $x_t \in (\text{cl}_0 \Xi_t^0) \cap L^0(\mathcal{F}_t; \mathbb{R}_-^d)$. Then $x_t^n \rightarrow x_t$ a.s. for some $(x_t^n)_{n \in \mathbb{N}}$ in Ξ_t^0 . Hence

$$x_t^n \mathbf{1}_{\{|x_t^n| \leq m+1\}} \mathbf{1}_{\{|x_t| \leq m\}} \rightarrow x_t \mathbf{1}_{\{|x_t| \leq m\}} \quad \text{a.s. as } n \rightarrow \infty$$

for all $m \geq 1$, where $x_t^n \mathbf{1}_{\{|x_t^n| \leq m+1\}} \mathbf{1}_{\{|x_t| \leq m\}} \in \Xi_t^0$ by decomposability and since $0 \in \Xi_t^0$. Dominated convergence therefore yields that $x_t \mathbf{1}_{\{|x_t| \leq m\}}$ belongs to $\text{cl}_p(\Xi_t^0 \cap L^p(\mathcal{F}_t; \mathbb{R}^d)) \cap L^0(\mathcal{F}_t; \mathbb{R}^d) = \{0\}$, where the closure may be taken with respect to the module norm. Letting $m \rightarrow \infty$ yields $x_t = 0$, i.e., (NARA) holds.

Assume (NARA). Consider a sequence $(V_{t,T}^n)_{n \in \mathbb{N}}$ from $A_{t,T}^p$ which converges in $L_{t,T}^p$ to $z_t^+ \in L^p(\mathcal{F}_t; \mathbb{R}_+^d)$. Then $\tilde{V}_{t,T}^n := V_{t,T}^n \wedge z_t^+ \rightarrow z_t^+$ in $L_{u,T}^p$, where the minimum is taken coordinatewise and u is any time between t and T , and $\tilde{V}_{t,T}^n \in A_{t,T}$ so that we may assume without loss of generality that $V_{t,T}^n \leq z_t^+$. Passing to subsequences, assume that $V_{t,T}^n \rightarrow z_t^+$ in $L_{u,T}^p$ and almost surely for each given $u \geq t$.

Define $\xi_T^n = V_{t,T}^n - z_t^+ \leq 0$. Then $\|\xi_T^n\|_{p, \mathcal{F}_{T-1}} \rightarrow 0$ in probability. By the continuity from below at zero, there exists a sequence $(\gamma_{T-1}^n)_{n \in \mathbb{N}}$ in $L^0(\mathcal{F}_{T-1}; \mathbb{R}_+^d)$ such that

$$\eta_T^n = \xi_T^n + \gamma_{T-1}^n \in \mathcal{C}_{T-1, T}$$

and $0 \leq \gamma_{T-1}^n \leq x_T \|\xi_T^n\|_{p, \mathcal{F}_{T-1}}$ for all n and some $x_T \in \mathbb{R}_+^d$. Hence $\gamma_{T-1}^n \rightarrow 0$ in $L_{T-2, T}^p$ if $T-2 \geq t$. Since $-\gamma_{T-1}^n \rightarrow 0$ in $L_{T-2, T}^p$, the continuity from below at zero yields the existence of a sequence $(\gamma_{T-2}^n)_{n \in \mathbb{N}}$ in $L^0(\mathcal{F}_{T-2}; \mathbb{R}_+^d)$ such that

$$\eta_{T-1}^n = -\gamma_{T-1}^n + \gamma_{T-2}^n \in \mathcal{C}_{T-2, T-1}$$

for all n and for some constant $x_{T-1} \in \mathbb{R}_+^d$, we have

$$0 \leq \gamma_{T-2}^n \leq x_{T-1} \|\gamma_{T-1}^n\|_{p, \mathcal{F}_{T-1}} \leq x_T x_{T-1} \|\xi_T^n\|_{p, \mathcal{F}_{T-2}},$$

so that $\gamma_{T-2}^n \rightarrow 0$ in $L_{T-3, T}^p$ if $T-3 \geq t$. Iterate the construction to find $\gamma_{T-3}^n, \dots, \gamma_t^n$ such that $\gamma_t^n \rightarrow 0$ a.s. Then $\eta_{u+1}^n = -\gamma_{u+1}^n + \gamma_u^n \in \mathcal{C}_{u, u+1}$ if $t \leq u \leq T-2$. Hence

$$\xi_t^n + \gamma_t^n = \sum_{u=t}^{T-1} \eta_{u+1}^n \in \mathcal{C}_{t, t+1} + \dots + \mathcal{C}_{T-1, T}.$$

By convexity,

$$\frac{1}{2}(-z_t^+ + \gamma_t^n) = -\frac{1}{2}V_{t,T}^n + \frac{1}{2}(\xi_t^n + \gamma_t^n) \in \Xi_t^0.$$

Letting $n \rightarrow \infty$ yields that $-\frac{1}{2}z_t^+ \in (\text{cl}_0 \Xi_t^0) \cap L^0(\mathcal{F}_t; -\mathbb{R}_+^d)$, so that $z_t^+ = 0$ by (NARA). Thus (4.5) holds with respect to the conditional norm and also with respect to the L^p -norm.

(ii) Recall that $\hat{\Xi}_t^0 = \text{cl}_0(\hat{\Xi}_t^0) = \text{cl}_0(\Xi_t^0)$ by Theorem 4.5. Following the arguments from (i), we obtain that (SNR) is equivalent to

$$\text{cl}_p(\Xi_t^0 \cap L^p(\mathcal{F}_t; \mathbb{R}^d)) \cap L^p(\mathcal{F}_t; -K_t) = \{0\}, \quad t = 0, \dots, T-1. \quad (4.8)$$

In view of (4.1), $\text{cl}_p(\Xi_t^0 \cap L^p) \subseteq -\bar{A}_{t,T}^p$. Therefore, (4.6) implies (4.8) and (SNR) holds.

Now assume (SNR). Consider a sequence $(V_{t,T}^n)_{n \in \mathbb{N}}$ from $A_{t,T}^p$ which converges in L^p to $k_t \in L^0(\mathcal{F}_t; K_t)$. Then follow the proof of (i) with k_t instead of z_t^+ .

Consider a sequence $(V_{t,T}^n)_{n \in \mathbb{N}}$ from $A_{t,T}^1$ which tends to $k_t \in L^0(\mathcal{F}_t; K_t)$ in L^1 . We may assume that $(V_{t,T}^n)_{n \in \mathbb{N}}$ converges a.s. Then for every $M > 0$, the sequence $(V_{t,T}^n \mathbf{1}_{\{|k_t| \leq M\}})_{n \in \mathbb{N}}$ is in $A_{t,T}^1$ and converges to $k_t \mathbf{1}_{\{|k_t| \leq M\}} \in L^p(\mathcal{F}_t; K_t)$ in L^1 so that we may assume without loss of generality that $|k_t|$ is bounded by M . Passing to a subsequence, we may assume that $\mathbb{E}[|V_{t,T}^n - k_t| | \mathcal{F}_t] \rightarrow 0$ a.s. Thus $V_{t,T}^n \mathbf{1}_{\{\mathbb{E}[|V_{t,T}^n| | \mathcal{F}_t] \leq M+1\}} \rightarrow k_t$ almost surely and in $L_{t,T}^1$. So (4.6) holds with $p = 1$. \square

Now consider more general claims ξ . Recall that if K_T is a cone and $\xi \in K_T$ a.s., then $\Xi_t^\xi \subseteq \Xi_t^0$ for all t .

Theorem 4.7 *If the solvency sets are proper and the acceptance sets are continuous from below at zero, then (SNR) yields that $\hat{\Xi}_t^\xi$ is closed in probability for all t and any $\xi \in L^p(\mathcal{F}_T; \mathbb{R}^d)$, so that $\hat{\Xi}_t^\xi = L^0(\mathcal{F}_t; X_t^\xi)$ for a random closed set X_t^ξ , $t = 0, \dots, T$.*

Proof Assume that $k_t^n + \gamma_t^n \rightarrow \zeta_t \in L^0(\mathcal{F}_t; \mathbb{R}^d)$ a.s. for $k_t^n \in L^0(\mathcal{F}_t; K_t)$ and $\gamma_t^n \in M = \mathbb{R}_{t,t+1}(\Xi_{t+1}^\xi)$ such that $k_t^n + \gamma_t^n \in \hat{\Xi}_t^0$. Since M is L^0 -closed and convex, we may assume by [26, Lemma 2.1.2] that $k_t^n \rightarrow k_t \in L^0(\mathcal{F}_t; K_t)$ on the set $A := \{\liminf_{n \rightarrow \infty} |k_t^n| < \infty\}$. Hence $\gamma_t^n \rightarrow \gamma_t \in M$, so that $\zeta_t = k_t + \gamma_t \in \hat{\Xi}_t^\xi$.

If $\mathbb{P}[\Omega \setminus A] > 0$, assume that $k_t^n = \gamma_t^n = \zeta_t = 0$ on A by \mathcal{F}_t -decomposability, and use a normalisation procedure, i.e., obtain \tilde{k}_t^n and $\tilde{\gamma}_t^n$ by scaling k_t^n and γ_t^n with $c_t^n = (1 + |k_t^n|)^{-1}$. We may assume that $|\tilde{\gamma}_t^n| \leq 2$ since $c_t^n \zeta_t \rightarrow 0$, so that $\tilde{k}_t^n + \tilde{\gamma}_t^n \rightarrow 0$. Arguing as previously, $\tilde{k}_t^n \rightarrow \tilde{k}_t \in L^0(\mathcal{F}_t; K_t)$ in L^p , and $|\tilde{k}_t| = 1$ on $\Omega \setminus A$. Therefore $\tilde{\gamma}_t^n \rightarrow \tilde{\gamma}_t = -\tilde{k}_t$ in L^p , so that $\tilde{k}_t + \tilde{\gamma}_t = 0$. Notice that

$$M = \text{cl}_0((-\mathbf{A}_{t+1,T} + \xi) \cap L^0(\mathcal{F}_t; \mathbb{R}^d)).$$

By convexity,

$$\tilde{\gamma}_t^n \in \text{cl}_0(-\mathbf{A}_{t+1,T} + c_t^n \xi) \cap L^0(\mathcal{F}_t; \mathbb{R}^d).$$

Since $|\tilde{\gamma}_t^n| \leq 2$, assume without loss of generality that $\tilde{\gamma}_t^n \in \text{cl}_p(-\mathbf{A}_{t+1,T} + c_t^n \xi)$. Indeed, it suffices to approximate $\tilde{\gamma}_t^n$ by $(\tilde{\gamma}_t^{mn})_{m \in \mathbb{N}}$ in $(-\mathbf{A}_{t+1,T} + c_t^n \xi)$ and multiply the latter by $\mathbf{1}_{\{|\tilde{\gamma}_t^{mn}| \leq 3\}}$. Letting $n \rightarrow \infty$, (4.6) yields that

$$-\tilde{\gamma}_t \in \overline{\mathbf{A}}_{t,T}^p \cap L^0(\mathcal{F}_t; K_t) = \{0\}.$$

Thus $\tilde{\gamma}_t = 0$, so that $\mathbb{P}[\Omega \setminus A] = 0$ and the conclusion follows. \square

Lemma 4.8 (NRA2) *is equivalent to*

$$\Xi_t^\xi \subseteq L^0(\mathcal{F}_t; K_t) + \mathcal{C}_{t-1,t}, \quad t = 0, \dots, T, \quad (4.9)$$

for any $\xi \in L^0(\mathcal{F}_T; K_T)$. If (4.4) holds, then (NRA2) implies (NARA).

Proof Note that $\eta_t + \mathbf{A}_{t,T}$ intersects $L^0(\mathcal{F}_T; K_T)$ if and only if

$$\eta_t \in (-\mathbf{A}_{t,T} + L^0(\mathcal{F}_T; K_T)) \cap L^0(\mathcal{F}_t; \mathbb{R}^d),$$

or equivalently if $\eta_t \in \Xi_t^\xi$ for some $\xi = k_T \in L^0(\mathcal{F}_T; K_T)$. \square

Denote by $K_t^* = \{x : \langle x, u \rangle \geq 0, u \in K_t\}$ the positive dual set to K_t and assume that $K_t^* \setminus \{0\}$ is a subset of the interior of \mathbb{R}_+^d for all t .

Definition 4.9 An adapted process $Z = (Z_s)_{s=t, \dots, T}$, $t \leq T$, is a q -integrable t -weakly consistent price system if it is a \mathbb{Q} -martingale for $\mathbb{Q} \approx \mathbb{P}$ such that Z_s is a q -integrable (under \mathbb{Q}) \mathcal{F}_s -measurable selection of K_s^* for every $s \geq t$ and $Z_t \neq 0$ a.s. We denote by $\mathcal{M}_{t,T}^{q,w}(\mathbb{Q})$ the set of all q -integrable t -weakly consistent price systems under \mathbb{Q} , where $q \in [1, \infty]$.

The following result characterises the prices under the conditions (SNR) and (NARA) conditions and so may be viewed as a fundamental theorem of asset pricing in our framework.

Theorem 4.10 Assume the coherent conical setting and that the solvency sets are continuous from below at zero. Let q be the conjugate of the number p from the definition of the acceptance sets.

(i) (NARA) is equivalent to the existence for each t of $Z \in \mathcal{M}_{t,T}^{q,w}(\mathbb{P})$ such that

$$\mathbb{E}[\langle Z_u, \eta_u \rangle] \geq 0 \quad \text{for all } \eta_u \in \mathcal{C}_{u-1,u}, u = t+1, \dots, T. \quad (4.10)$$

(ii) If $\text{int}K_t^* \neq \emptyset$ a.s. for all t , then (SNR) is equivalent to the existence for each t of $Z \in \mathcal{M}_{t,T}^{q,w}(\mathbb{P})$ such that (4.10) holds and $Z_t \in L^0(\mathcal{F}_t; \text{int}K_t^*)$.

Proof (i) Under (NARA), the existence of $Z \in \mathcal{M}_{t,T}^{q,w}(\mathbb{P})$ such that $\mathbb{E}[\langle Z_T, \eta \rangle] \geq 0$ for every $\eta \in \mathcal{C}_{t,t+1} + \dots + \mathcal{C}_{T-1,T}$ is a direct consequence of the Hahn–Banach separation theorem and Theorem 4.6, (i), since we may take $p = 1$. We then deduce (4.10) as $-\mathcal{C}_{u-1,u} \subseteq \mathcal{A}_{t,T}$ for $u \geq t+1$.

To prove the converse implication, assume the existence of $Z \in \mathcal{M}_{t,T}^{q,w}(\mathbb{P})$ and consider $x_T \in \mathcal{A}_{t,T}^p$. Then

$$x_T = -k_t - (k_{t+1} + \eta_{t+1}) - \dots - (k_T + \eta_T),$$

where $\eta_s \in \mathcal{C}_{s-1,s}$ and $k_s \in L^0(\mathcal{F}_s; K_s)$ for $s \geq t$. Since $\eta_s = \eta'_s + \eta''_s$ with $\eta'_s \in \mathcal{C}_{s-1,s} \cap L_{s-1,s}^p$ and $\eta''_s \in L^0(\mathcal{F}_s; \mathbb{R}_+^d)$, we may merge η''_s and k_s and suppose without loss of generality that $\eta_s = \eta'_s$.

Using backward induction on $t \leq T$, we now show that $\mathbb{E}[\langle Z_T, x_T \rangle] \leq 0$. If $x_T = -k_{T-1} - k_T - \eta_T$, this is trivial. Since $\eta_{t+1}, k_t \in L_{t,t+1}^p$, there exists a partition $(B_t^i)_{i \geq 1}$ from \mathcal{F}_t such that $\eta_{t+1} \mathbf{1}_{B_t^i}, k_t \mathbf{1}_{B_t^i} \in L^p(\mathcal{F}_t; \mathbb{R}^d)$ for all $i \geq 1$. Then

$$x_{t+1}^i = (-k_{t+1} - \dots - k_T - \eta_{t+2} - \dots - \eta_T) \mathbf{1}_{B_t^i} \in \mathcal{A}_{t+1,T}^p, \quad i \geq 1.$$

Moreover,

$$\begin{aligned} \mathbb{E}[\langle Z_T, x_T \rangle] &= \sum_{i=1}^{\infty} \mathbb{E}[\langle Z_T, x_{t+1}^i \rangle] + \sum_{i=1}^{\infty} \mathbb{E}[\langle Z_T, -k_t \mathbf{1}_{B_t^i} \rangle] + \mathbb{E}[\langle Z_t, -\eta_{t+1} \rangle \mathbf{1}_{B_t^i}] \\ &= \sum_{i=1}^{\infty} \mathbb{E}[\langle Z_T, x_{t+1}^i \rangle] + \sum_{i=1}^{\infty} \mathbb{E}[\langle Z_t, -k_t \mathbf{1}_{B_t^i} \rangle] + \mathbb{E}[\langle Z_{t+1}, -\eta_{t+1} \mathbf{1}_{B_t^i} \rangle] \\ &\leq \sum_{i=1}^{\infty} \mathbb{E}[\langle Z_T, x_{t+1}^i \rangle]. \end{aligned}$$

The induction hypothesis yields $E[\langle Z_T, x_{t+1}^i \rangle] \leq 0$, hence $E[\langle Z_T, x_T \rangle] \leq 0$. Therefore $E[\langle Z_T, x_T \rangle] \leq 0$ for all $x_T \in \overline{A}_{t,T}^p$. In particular, if we consider $x_T = x_t \in L^0(\mathcal{F}_t; \mathbb{R}_+^d)$, then $E[\langle Z_t, x_t \rangle] \leq 0$ and finally $E[\langle Z_t, x_t \rangle] = 0$. Since $Z_t \in \text{int}\mathbb{R}_+^d$, we have $x_T = 0$, i.e., (NARA) holds by Theorem 4.6 (i).

(ii) Replicate the proof of (i) using the Hahn–Banach theorem and following the arguments of [27, Theorem 4.1] in order to construct $Z \in \mathcal{M}_{t,T}^{q,w}(\mathbb{P})$ such that $Z_t \in L^0(\mathcal{F}_t; \text{int}K_t^*)$. \square

Remark 4.11 Condition (4.10) can be equivalently written in terms of the conditional expectation as $E[\langle Z_u, \eta_u \rangle | \mathcal{F}_{u-1}] \geq 0$ a.s., which also corresponds to the duality pairing in modules; see Filipović et al. [16]. Suppose that (4.10) holds. Then $E[\langle Z_u, \eta_u \mathbf{1}_{A_{u-1}} \rangle] \geq 0$ for all $A_{u-1} \in \mathcal{F}_{u-1}$. Therefore, $E[\langle Z_u, \eta_u \rangle | \mathcal{F}_{u-1}] \geq 0$. The opposite implication is obvious. In other words, (4.10) means that Z_u belongs to the positive dual of $\mathcal{C}_{u-1,u}$.

If the acceptance sets with $p = \infty$ are generated by convex families $\mathcal{Z}_{t,s}$ (see Example 2.7), then (4.10) means that Z_t belongs to the closure of $\mathcal{Z}_{t,s}$ with respect to \mathcal{F}_t -bounded convergence in probability. If the K_t are all half-spaces (in the frictionless setting), then (NARA) is equivalent to the existence of a martingale $Z_u = \phi_u S_u$, where S_u is the price vector, such that (4.10) holds.

5 Good-deal hedging

Assume that $\mathcal{C}_{t,s}$ consists of random vectors $(\eta_s, 0, \dots, 0) \in L^p(\mathcal{F}_s; \mathbb{R}^d)$ with all components vanishing except the first one and such that $\rho_{t,s}(\eta_s) \leq 0$ for a univariate dynamic risk measure $\rho_{t,s}$. This corresponds to the case when the acceptability at each step is assessed by calculating the risk of a portfolio expressed in the units of the first asset, most usually cash. An arbitrage opportunity in this setting is called a *good deal*; see Cherny [7].

For simplicity, consider a one-period setting with zero interest rate and two assets exchangeable without transaction costs so that without loss of generality, the first asset is assumed to be cash and the second is a risky asset priced at S_t for $t = 0, 1$. Let ξ be the cash value of a terminal claim. If x_0 is the initial endowment (in cash), then the terminal position of a portfolio is

$$V_1 = (x_0, 0) + (-k_0 S_0, k_0) + (-k_1 S_1, k_1) - (\eta', \eta''),$$

where η' and η'' are acceptable with respect to some static convex risk measure ρ , meaning that $\rho(\eta') \leq 0$ and $\rho(\eta'') \leq 0$. Given the choice of the acceptance set $\mathcal{C}_{0,1}$, we have $\eta'' = 0$ so that V_1 suffices to pay the claim if

$$x_0 + k_0(S_1 - S_0) - \eta' \geq \xi.$$

The smallest value of x_0 which ensures that there exists some acceptable position $\eta' = -\xi + x_0 + k_0(S_1 - S_0)$, satisfying the above inequality equals the infimum of $\rho(k_0(S_1 - S_0) - \xi)$ over all deterministic $k_0 \in \mathbb{R}$. For instance, the zero claim $\xi = 0$ can be hedged with a negative initial capital if $\rho(S_1 - S_0) < 0$ or $\rho(S_0 - S_1) < 0$,

and this means the existence of a good-deal arbitrage. If this is the case and the risk measure is coherent, then also

$$\rho(k_0(S_1 - S_0) - \xi) \leq \rho(k_0(S_1 - S_0)) + \rho(-\xi) < 0 \quad (5.1)$$

for sufficiently large $k_0 > 0$ if $\rho(S_1 - S_0) < 0$ (or negative k_0 if $\rho(S_0 - S_1) < 0$), meaning that any claim with finite $\rho(-\xi)$ can be also hedged with a negative initial investment. In other words, the no-good-deal (NGD) arbitrage condition becomes

$$\rho(S_1/S_0) \geq -1 \quad \text{and} \quad \rho(-S_1/S_0) \geq 1.$$

Our setting is more general than the good-deal hedging since it allows more general acceptance sets and eliminates the prescribed choice of a single asset in order to assess the acceptability. As a result, the no-arbitrage conditions become stronger and the infimum cost of the superhedging price declines. To illustrate this, consider the above two-asset one-period setting with the acceptance set $\mathcal{C}_{0,1}$ that consists of all (η', η'') such that $\rho(\eta') \leq 0$ and $\rho(\eta'') \leq 0$. By allowing a non-trivial η'' , it is possible to decrease the price of a terminal cash claim ξ . For this, note that ξ can be paid if

$$\begin{aligned} x_0 - k_0 S_0 - k_1 S_1 - \eta' &\geq \xi \\ k_0 + k_1 - \eta'' &\geq 0 \end{aligned}$$

for some deterministic k_0 , \mathcal{F}_1 -measurable k_1 and acceptable η', η'' . This increases the hedging possibilities and so leads to a decrease of the superhedging price costs, but also creates extra arbitrage opportunities. In particular, considering $(\eta', \eta'') \in \mathcal{C}_{0,1}$ with $\eta' = 0$, arbitrage becomes possible if $\rho((k_0(S_1 - S_0) + x_0)/S_1) \leq 0$ for some $x_0 < 0$ and $K_0 \in \mathbb{R}$. By letting x_0 increase to zero, we see that the necessary no-arbitrage condition in addition to (5.1) yields that

$$\rho(S_0/S_1) \geq -1 \quad \text{and} \quad \rho(-S_0/S_1) \geq 1.$$

This corresponds to the fact that with two assets, a position expressed in one of them may be not acceptable, while it may be acceptable expressed in the other one. The necessary and sufficient no-arbitrage condition is stronger and should also include all possible combinations of the two assets.

Assume that $\xi = (S_1 - K)^+$ for some $K > 0$ and that the support of S_1 is the whole half-line $(0, \infty)$. If $\rho(X) = \text{ess sup}_{\mathcal{F}_0}(-X)$, i.e., when the acceptable positions are nonnegative random variables, then the minimal price

$$x_0 = \inf_{k_0 \in \mathbb{R}} \rho(k_0(S_1 - S_0) - \xi)$$

equals S_0 . If ρ is non-trivial, we have $x_0 \leq S_0 + \rho(S_1 - \xi)$. Observe that we have $S_1 - \xi = S_1 \wedge K$ so that $x_0 \leq S_0 + \rho(S_1) \wedge K$ and finally $x_0 \leq S_0 - \rho(S_1)$, where $\rho(S_1) < 0$ given that $S_1 > 0$ and ρ is non-trivial. This simple example illustrates the decrease of the superhedging price in presence of a non-trivial risk measure.

Example 5.1 Assume that the risk measure ρ is the negative essential infimum, that is, consider the setting of conditional cores from Sect. 6. Then NGD arbitrage is not possible if $\text{ess inf}_{\mathcal{F}_0} S_1 \leq S_0 \leq \text{ess sup}_{\mathcal{F}_0} S_1$. With this choice of risk measure, the condition NGD coincides with the (SNR) condition; see Theorem 8.1.

6 Conditional core as risk measure

Assume that $p = \infty$ and $\mathcal{C}_{t,s} = L^0(\mathcal{F}_s; \mathbb{R}_+^d)$ for all $0 \leq t \leq s \leq T$, so that $\mathbb{R}_{t,s}^0(\Xi) = \Xi \cap L^0(\mathcal{F}_t; \mathbb{R}^d)$ for any upper set $\Xi \subseteq L^0(\mathcal{F}_s; \mathbb{R}^d)$. If X is an upper random closed set, then

$$\mathbb{R}_{t,s}^0(X) = \mathbb{R}_{t,s}(X) = \mathbf{m}(X|\mathcal{F}_t),$$

where the latter notation designates the largest \mathcal{F}_t -measurable random closed subset of X , called the *conditional core* of X ; see [28, Definition 4.1]. An acceptable portfolio process is characterised by $V_{t-1} - V_t \in K_t$ a.s. for $t = 1, \dots, T$. Then $\mathbb{A}_{t,s}$ becomes the sum of $L^0(\mathcal{F}_u; -K_u)$ for $u = t, \dots, s$, exactly like in the classical theory of markets with transaction costs [26, Sect. 3.2.2]. For a claim ξ , the set Ξ_t^ξ defined in Sect. 3 becomes the set of superhedging prices that was used in Löhne and Rudloff [29] to define a risk measure $\xi \mapsto \Xi_t^\xi$.

The classical no-arbitrage condition (NA^s) (no strict arbitrage opportunity at any time, see [26, Sect. 3.1.4]) then becomes (4.2); (SNA) (strong no arbitrage, see Condition (iii) in [26, Sect. 3.2.2]) then becomes (4.2); the strong no-arbitrage NA^r equivalent to Condition (iii) in [26, Sect. 3.2.2] coincides with (SNRA).

Theorem 6.1 *Suppose that the solvency sets $(K_t)_{t=0,\dots,T}$ are strictly proper. Then (SNR), (NA^s) and (SNA) are all equivalent and are also equivalent to each of the following conditions:*

- (i) $\bar{\mathbb{A}}_{t,T}^p \cap L^0(\mathcal{F}_t; K_t) = \{0\}$ for all $t \leq T - 1$.
- (ii) $\mathbb{A}_{t,T}$ is closed in L^0 and $\mathbb{A}_{t,T} \cap L^0(\mathcal{F}_t; K_t) = \{0\}$ for all $t \leq T - 1$.

Proof (SNR) is equivalent to (i) by Theorem 4.6 (ii). The equivalence of (NA^s) and (SNA) follows from Lemma 4.3 given that (4.4) trivially holds.

The implication (i) \Rightarrow (ii) is simple to show by induction. First, $\mathbb{A}_{T,T}$ is closed. Assume that $-k_t^n - \dots - k_T^n \rightarrow \xi$ a.s. for $k_u^n \in L^0(\mathcal{F}_u; K_u)$, $u \geq t$. On the set $\{\liminf_{n \rightarrow \infty} |k_t^n| = \infty\}$, we use a normalisation procedure to arrive at a contradiction with (i). Otherwise, suppose that $-k_t^n \rightarrow -k_t \in -K_t$, so that we may use the induction hypothesis to conclude.

In order to derive the closedness of $\mathbb{A}_{t,r}$ under (SNA), it suffices to follow the proof of [26, Lemma 3.2.8]. Indeed, since K_t^0 is a linear space, the recession cone

$$K_t^\infty = \bigcap_{\alpha > 0} \alpha K_t = \{x \in \mathbb{R}^d : K_t + \alpha x \subseteq K_t, \forall \alpha > 0\}$$

satisfies $K_t^0 \subseteq K_t^\infty$; see Pennanen and Penner [32]. Therefore $k_t + \alpha x \in K_t$ for all $k_t \in K_t$, $x \in K_t^0$ and $\alpha \in \mathbb{R}$. Furthermore, (SNA) trivially implies that $\mathbb{A}_{t,T} \cap L^0(\mathcal{F}_t; K_t) = \{0\}$ for all t .

In order to show that (ii) implies (NA^s), assume

$$-k_0 - \dots - k_t = \tilde{k}_t \in \mathbb{A}_{0,t} \cap L^0(\mathcal{F}_t; K_t).$$

Then $k_0 \in \mathbb{A}_{0,T} \cap L^0(\mathcal{F}_0; K_0)$, i.e., $k_0 = 0$ by (ii). Similarly, $k_1 = \dots = k_{t-1} = 0$, so that $-k_t = \tilde{k}_t = 0$ since K_t is strictly proper. Thus (NA^s) holds.

At last, (NA^s) yields (SNA) so that $\mathbb{A}_{t,T}$ is closed in L^0 . Finally, (SNA) yields (4.3) and so $\mathbb{A}_{t,T} \cap L^0(\mathcal{F}_t; K_t) = \{0\}$, that is, (i) holds. \square

(NA2) (no-arbitrage opportunity of the second kind) from Rásonyi [33] and Kabanov and Safarian [26, Sect. 3.2.6] has the same formulation as (NRA2).

Lemma 6.2 *Assume that the solvency sets are cones. Then:*

- (i) (NA2) is equivalent to $\Xi_t^0 = L^0(\mathcal{F}_t; K_t)$ for all $t \leq T$.
- (ii) (NA2) is equivalent to

$$\mathbf{m}(K_t | \mathcal{F}_{t-1}) \subseteq K_{t-1}, \quad t = 1, \dots, T. \quad (6.1)$$

- (iii) *If the solvency sets are strictly proper, then (NA2) implies (SNR).*

Proof (i) By Lemma 4.8 and Lemma 3.2 (iv), $\Xi_t^0 = L^0(\mathcal{F}_t; K_t)$ yields (4.9) and so implies (NA2). In the other direction, we have $\Xi_t^0 \subseteq L^0(\mathcal{F}_t; K_t) + \mathcal{C}_{t-1,t}$ by (NA2), while (3.2) and the choice of $\mathcal{C}_{t-1,t}$ yield $\Xi_t^0 \supseteq L^0(\mathcal{F}_t; K_t)$.

- (ii) If $\Xi_t^0 \subseteq L^0(\mathcal{F}_t; K_t)$, then

$$\Xi_{t+1}^0 \cap L^0(\mathcal{F}_t; \mathbb{R}^d) \subseteq L^0(\mathcal{F}_t; K_t)$$

for all t by (3.2). Since $L^0(\mathcal{F}_{t+1}; K_{t+1}) \subseteq \Xi_{t+1}^0$, we obtain (6.1).

If (6.1) holds, then

$$\Xi_{T-1}^0 = L^0(\mathcal{F}_{T-1}; K_{T-1}) + \mathbf{m}(K_T | \mathcal{F}_{T-1}) \subseteq L^0(\mathcal{F}_{T-1}; K_{T-1}).$$

Assume that $\Xi_s^0 \subseteq L^0(\mathcal{F}_s; K_s)$ for $s = t+1, \dots, T$. Then

$$\begin{aligned} \Xi_t^0 &= L^0(\mathcal{F}_t; K_t) + (\Xi_{t+1}^0 \cap L^0(\mathcal{F}_t; \mathbb{R}^d)) \\ &\subseteq L^0(\mathcal{F}_t; K_t) + \mathbf{m}(K_{t+1} | \mathcal{F}_t) \subseteq L^0(\mathcal{F}_t; K_t). \end{aligned}$$

The proof is finished by induction.

(iii) Since Ξ_t^0 is closed in probability under (NA2) by (i), we deduce that $\Xi_t^0 = \hat{\Xi}_t^0 = L^0(\mathcal{F}_t; K_t)$. Therefore, $\hat{\Xi}_t^0 \cap L^0(\mathcal{F}_t; -K_t) = \{0\}$, if the solvency sets are strictly proper. We deduce that (SNR) holds. \square

For conical solvency sets satisfying $\mathbf{m}(K_t^0 | \mathcal{F}_{t-1}) \subseteq K_{t-1}^0$, $t \leq T$, and in particular for strictly proper ones, (NA^s) is equivalent to the existence of a Q-martingale evolving in the relative interiors of $(K_t^*)_{t=0, \dots, T}$ for a probability measure Q equivalent to P; see [26, Theorem 3.2.2]. Such a martingale is called a *strictly consistent price system*. If $\text{int} K_t^* \neq \emptyset$ for all t , this result follows from Theorem 4.10 (ii).

Note that $\Xi_T^\xi = L^0(\mathcal{F}_T; X_T^\xi)$ for $X_T = \xi + K_T$, and we also have that $\Xi_{T-1}^\xi = L^0(\mathcal{F}_{T-1}; X_{T-1}^\xi)$ is the family of measurable selections for some (possibly non-closed) random set $X_{T-1}^\xi = K_{T-1} + \mathbf{m}(X_T^\xi | \mathcal{F}_{T-1})$. One needs additional assumptions of no-arbitrage type in order to extend this interpretation for Ξ_t^ξ with $t \leq T-2$. Precisely the sum above should be closed, so that $\mathbf{m}(X_t^\xi | \mathcal{F}_{t-1})$ exists for $t \leq T-1$, which makes it possible to apply Lemma A.3.

Theorem 6.3 *Assume that the solvency sets are strictly proper and (NA^s) (equivalently, (SNR) or (SNA)) holds. Then $\Xi_t^\xi = L^0(\mathcal{F}_t; X_t^\xi)$, where X_t^ξ is an \mathcal{F}_t -measurable random closed convex set, $t = 0, \dots, T$, such that $X_T^\xi = \xi + K_T$ and*

$$X_t^\xi = K_t + \mathbf{m}(X_{t+1}^\xi | \mathcal{F}_t), \quad t = T-1, \dots, 0.$$

Proof It suffices to confirm the statement for $t = T-1$ and then use induction. Indeed, by Theorem 6.1, (NA^s) is equivalent to (SNR) so that Theorem 4.7 applies. Since Ξ_t^ξ is \mathcal{F}_t -decomposable, Theorem A.2 yields the existence of an \mathcal{F}_{T-1} -measurable closed set X_{T-1}^ξ such that $\Xi_{T-1}^\xi = L^0(\mathcal{F}_{T-1}; X_{T-1}^\xi)$. Since X_T^ξ is closed,

$$\begin{aligned}\Xi_{T-1}^\xi &= L^0(\mathcal{F}_{T-1}; K_{T-1}) + L^0(\mathcal{F}_{T-1}; \mathbf{m}(X_T^\xi | \mathcal{F}_{T-1})), \\ &= L^0(\mathcal{F}_{T-1}; K_{T-1} + \mathbf{m}(X_T^\xi | \mathcal{F}_{T-1})) \\ &= L^0(\mathcal{F}_{T-1}; X_{T-1}^\xi),\end{aligned}$$

where X_{T-1}^ξ is a random set by Lemma A.3. \square

Proposition 6.4 *Suppose that the solvency sets are strictly proper. Then (NA^s) holds if and only if $\Xi_t^0 = L^0(\mathcal{F}_t; X_t^0)$ for random closed sets $(X_t^0)_{t=0, \dots, T}$ such that $X_t^0 \cap (-K_t) = \{0\}$ a.s. for all t . In the conical case, the latter condition is equivalent to $\text{int}(X_t^0)^* \neq \emptyset$ for all t , and under (NA^s) ,*

$$A_{t,T} = \sum_{s=t}^T L^0(\mathcal{F}_s; -X_s^0), \quad 0 \leq t \leq T, \quad (6.2)$$

where X_t^0 is a strictly proper random closed convex cone for all t .

Proof Assume (NA^s) ; hence Theorem 6.3 applies. Let $-g_t \in L^0(\mathcal{F}_t; X_t^0 \cap (-K_t))$. Then we have $g_t \in K_t$ a.s., and there exist $k_u \in L^0(\mathcal{F}_u; K_u)$ for $u = t, \dots, T$ and $\tilde{g}_T \in L^0(\mathcal{F}_T; K_T)$ such that $-g_t - k_t - k_{t+1} - \dots - k_T = \tilde{g}_T$. Since (SNA) holds, $g_t + k_t = 0$ and $g_t = 0$. The converse implication is trivial. In the conical case, since $K_t + K_t = K_t$ for all $t \leq T$, (6.2) follows from the inclusions

$$K_t \subseteq X_t^0 \subseteq K_t + \dots + K_T, \quad t \leq T.$$

Note that $X_T^0 = K_T$ is strictly proper by assumption. Since

$$X_{t-1}^0 = K_{t-1} + \mathbf{m}(X_t^0 | \mathcal{F}_{t-1}) \subseteq K_{t-1} + X_t^0,$$

induction yields $X_t^0 \subseteq K_t + \dots + K_T$. Since (SNA) holds under (NA^s) , X_t^0 is strictly proper for all t .

By [26, Lemma 5.1.2], Condition (NA^s) holds if and only if Ξ_t^0 is closed and $\Xi_t^0 = L^0(\mathcal{F}_t; X_t^0)$ with

$$\text{int}K_t^* \cap \text{int}(X_t^0)^* = \text{int}K_t^* \cap \text{int} \mathbf{m}(X_{t+1}^0 | \mathcal{F}_t)^* \neq \emptyset, \quad t \leq T.$$

Finally, observe that

$$\text{int}K_t^* \cap \text{int} \mathbf{m}(X_{t+1}^0 | \mathcal{F}_t)^* = \text{int}K_t^* \cap \mathbf{m}(X_{t+1}^0 | \mathcal{F}_t)^* = \text{int}(X_t^0)^*.$$

\square

Equation (6.2) means that in the superhedging problem, we may replace the solvency sets K_t with X_t^0 . The solvency sets $(X_t^0)_{t=0,\dots,T}$ satisfy the condition (NA2) by Lemma 6.2, which is generally required to obtain a dual characterisation of the superhedging prices; see Condition **B** (equivalent to (NA2)) in [26, Sect. 3.6.3]. Therefore, (NA^s) suffices for [26, Theorem 3.6.3] to hold, provided that we consider the consistent price systems associated to $(X_t^0)_{t=0,\dots,T}$.

Now consider the conditions (NRA) and (NARA) for the chosen acceptance sets. Assume that the solvency sets are conical and satisfy $K_t^* \setminus \{0\} \subseteq \text{int}\mathbb{R}_+^d$. Since $\hat{\Xi}_t^0 = \text{cl}_0(\hat{\Xi}_t^0) = \text{cl}_0(\Xi_t^0) = \Xi_t^0$, (NRA) and (NARA) are equivalent by Proposition 4.4. Denote by $A_{t,T}^p(\mathbb{Q})$ and $\bar{A}_{t,T}^p(\mathbb{Q})$ for $p \in [1, \infty]$ the variants of $A_{t,T}^p$ and $\bar{A}_{t,T}^p$ when the reference probability measure is \mathbb{Q} .

Proposition 6.5 *The following statements are equivalent:*

- (i) $\bar{A}_{t,T}^p(\mathbb{Q}) \cap L^0(\mathcal{F}_t; \mathbb{R}_+^d) = \{0\}$ for all $t \leq T-1$, $p \in [1, \infty)$ and $\mathbb{Q} \approx \mathbb{P}$.
- (ii) $\mathcal{M}_{t,T}^{q,w}(\mathbb{Q}) \neq \emptyset$ for all $t \leq T-1$, $p \in [1, \infty)$ and $\mathbb{Q} \approx \mathbb{P}$.
- (iii) (NARA).
- (iv) $\mathcal{M}_{t,T}^{\infty,w}(\mathbb{P}) \neq \emptyset$ for all $t \leq T-1$.
- (v) $\mathcal{M}_{t,T}^{1,w}(\mathbb{P}) \neq \emptyset$ for all $t \leq T-1$.

Proof By Theorem 4.10, (v) and (iii) are equivalent. We deduce the equivalence of (ii) and (iv) by following the proof of [26, Lemma 3.2.4], which makes it possible to construct a weakly consistent price system (see Definition 4.9) from any consistent price system in L^1 . In particular, (v) implies (iv) and clearly (iv) implies (v). Then (iii) implies (ii), i.e., (NARA) holds for \mathbb{Q} in place of \mathbb{P} . By Theorem 4.6, (i) holds. Finally, (i) implies (NARA) by Theorem 4.6. \square

7 Arbitrage with acceptable expectations

Assume that $p = 1$ and let $\mathcal{C}_{t,s} \cap L_{t,s}^1$ be the set of all $\eta_s \in L_{t,s}^1$ such that $E^g[\eta_s | \mathcal{F}_t]$ has all nonnegative components. In other words, the acceptable positions are those having nonnegative generalised conditional expectation. This is the weakest possible acceptability criterion, which is always the case (in the static setting) if the acceptance sets are dilatation monotonic.

The generalised conditional expectation is well defined for each $\gamma_s \in \hat{L}_{t,s}^1$ by letting $E^g[\gamma_s | \mathcal{F}_t] = E^g[\gamma'_s | \mathcal{F}_t] + E[\gamma''_s | \mathcal{F}_t]$, where $\gamma''_s \in L^0(\mathcal{F}_s; \mathbb{R}_+^d)$ may have an infinite expectation. If X_T is an \mathcal{F}_T -measurable random upper convex set that admits at least one selection from $L_{t,s}^1$, then let

$$R_{t,s}^0(X_T) = \{E^g[\gamma_s | \mathcal{F}_t] : \gamma_s \in L_{\mathcal{F}_t}^1(\mathcal{F}_s; X_T)\},$$

and $R_{t,s}(X_T) = E^g[X_T | \mathcal{F}_t]$ is the generalised conditional expectation of the random closed set X_T ; see [28, Definition 6.3].

Let $\xi \in L^1(\mathcal{F}_T; \mathbb{R}^d)$. By Lemma 3.2, $\hat{\Xi}_{T-1}^\xi$ is the family of selections of the (possibly non-closed) random set

$$X_{T-1}^\xi = K_{T-1} + E[\xi | \mathcal{F}_{T-1}] + E[K_T | \mathcal{F}_{T-1}],$$

which is \mathcal{F}_{T-1} -measurable by Lemma A.3. Note that all solvency sets are integrable and so their generalised conditional expectation coincides with the usual one. Therefore,

$$\begin{aligned} R_{T-2,T-1}(\hat{\Xi}_{T-1}^\xi) &= \mathbb{E}[X_{T-1}^\xi | \mathcal{F}_{T-2}] \\ &= \mathbb{E}[\xi | \mathcal{F}_{T-2}] + \mathbb{E}[K_{T-1} + \mathbb{E}[K_t | \mathcal{F}_{T-1}] | \mathcal{F}_{T-2}], \\ &= \mathbb{E}[\xi | \mathcal{F}_{T-2}] + \mathbb{E}[K_T + K_{T-1} | \mathcal{F}_{T-2}]. \end{aligned}$$

Since $k_T \in L_{\mathcal{F}_{T-1}}^1(\mathcal{F}_T; K_T)$ and

$$\begin{aligned} R_{T-2,T-1}^0(\Xi_{T-1}^\xi) \\ = \{ \mathbb{E}^g[k_{T-1} + \mathbb{E}^g[\xi + k_T | \mathcal{F}_{T-1}] | \mathcal{F}_{T-2}] : k_{T-1} \in L^0(\mathcal{F}_{T-1}; K_{T-1}) \}, \end{aligned}$$

we deduce that $R_{T-2,T-1}(\hat{\Xi}_{T-1}^\xi) \subseteq R_{T-2,T-1}(\Xi_{T-1}^\xi)$. Since Ξ_{T-1}^ξ is a subset of $\hat{\Xi}_{T-1}^\xi$, we have

$$R_{T-2,T-1}(\hat{\Xi}_{T-1}^\xi) = R_{T-2,T-1}(\Xi_{T-1}^\xi).$$

Therefore,

$$X_{T-2}^\xi = K_{T-2} + \mathbb{E}[\xi | \mathcal{F}_{T-2}] + \mathbb{E}[K_T + K_{T-1} | \mathcal{F}_{T-2}],$$

Continuing recursively, we obtain $\hat{\Xi}_t^\xi = L^0(\mathcal{F}_t; X_t^\xi)$ with a not necessarily closed \mathcal{F}_t -measurable random set

$$X_t^\xi = K_t + \mathbb{E}[\xi | \mathcal{F}_t] + \mathbb{E}[K_T + K_{T-1} + \dots + K_{t+1} | \mathcal{F}_t].$$

Notice that $X_t^\xi = X_t^0 + \mathbb{E}[\xi | \mathcal{F}_t]$, i.e., X_t^0 determines all superhedging prices. Reformulating the requirements from Definition 4.1, we arrive at the following result.

Proposition 7.1 *For the risk arbitrage conditions formulated for the conditional expectation as the risk measure, the following hold:*

(i) *If the solvency sets are strictly proper, (SNR) is equivalent to*

$$\mathbb{E}[K_{t+1} + \dots + K_T | \mathcal{F}_t] \cap (-K_t) = \{0\} \quad a.s., t = 0, \dots, T-1.$$

(ii) *(NARA) is equivalent to*

$$(K_t + \mathbb{E}[K_{t+1} + \dots + K_T | \mathcal{F}_t]) \cap \mathbb{R}_-^d = \{0\} \quad a.s., t = 0, \dots, T-1.$$

Note that statement (ii) above follows from (i). Theorem 4.10 yields the following result.

Proposition 7.2 *Assume that the solvency sets $(K_t)_{t=0,\dots,T}$ are cones. Then (NARA) (resp. (SNR)) is equivalent to the existence of a deterministic point $z \neq 0$ that belongs to all K_t^* (resp. $\text{int}K_t^*$), $t = 0, \dots, T$.*

Proof Any acceptable position from $\mathcal{C}_{u-1,u}$ is of the form

$$\eta_u = (\gamma_u - \mathbf{E}^g[\gamma_u|\mathcal{F}_{u-1}]) + \mathbf{E}^g[\gamma_u|\mathcal{F}_{u-1}] + \zeta_u^+$$

with $\mathbf{E}^g[\gamma_u|\mathcal{F}_{u-1}] \in \mathbb{R}_+^d$ and $\zeta_u^+ \in L^0(\mathcal{F}_u; \mathbb{R}_+^d)$. Thus

$$\begin{aligned} \mathbf{E}^g[\langle Z_u, \eta_u \rangle | \mathcal{F}_{u-1}] &\geq \mathbf{E}^g[\langle Z_u, \gamma_u - \mathbf{E}^g[\gamma_u|\mathcal{F}_{u-1}] \rangle | \mathcal{F}_{u-1}] \\ &= \mathbf{E}^g[\langle Z_u, \gamma_u \rangle | \mathcal{F}_{u-1}] - \langle Z_{u-1}, \mathbf{E}^g[\gamma_u|\mathcal{F}_{u-1}] \rangle. \end{aligned}$$

Hence $\mathbf{E}^g[\langle Z_u, \eta_u \rangle | \mathcal{F}_{u-1}] \geq 0$ if there exists $Z \in \mathcal{M}_{t,T}^{\infty,w}(\mathbb{P})$ such that

$$\mathbf{E}^g[\langle Z_u, \gamma_u \rangle | \mathcal{F}_{u-1}] = \langle Z_{u-1}, \mathbf{E}^g[\gamma_u|\mathcal{F}_{u-1}] \rangle \quad \text{a.s.}$$

for all $\gamma_u \in L_{\mathcal{F}_{u-1}}^1(\mathcal{F}_u; \mathbb{R}^d)$. The equality follows from (4.10) by taking unconditional expectations (restricting to a partition if necessary) and applying the same reasoning with $-\gamma_u$. Given that Z_u is essentially bounded, one can let γ_u be equal to one of the components of Z_u multiplied by the corresponding basis vector. Thus the square of every component of $(Z_u)_{u=0,\dots,T}$ is a martingale, whence Z_u must be deterministic z for all u .

The converse implication follows from Theorem 4.10 (i) applied to

$$\eta_u = \gamma_u - \mathbf{E}^g[\gamma_u|\mathcal{F}_{u-1}] \in \mathcal{C}_{u-1,u}.$$

The proof for (SNR) follows from the same argument and Theorem 4.10 (ii). \square

Remark 7.3 It is possible to derive the result of Proposition 7.2 from Proposition 7.1 by using the fact that the expectation of the cone K_t is the whole space unless K_t^* contains a deterministic point z distinct from the origin, and then $\mathbf{E}[K_t]$ is a subset of the half-space with outer normal $-z$. In order that X_t^0 does not intersect \mathbb{R}_-^d , all cones K_t should have non-trivial expectation and the sum of these expectations must be non-trivial. This amounts to the existence of a deterministic point $z \neq 0$ that belongs to $K_0^* \cap \dots \cap K_T^*$.

8 Application to a two-dimensional model

Consider a financial market model composed of two assets. The first one has constant value 1 and the second is a risky asset modelled by a bid-ask spread $Y_t = [S_t^b, S_t^a]$ such that $0 < S_t^b \leq S_t^a$ a.s. for all $t \leq T$. This is Kabanov's model with the conical solvency set $K_t = C(Y_t)$, where $C([s', s''])$ is the positive dual to the smallest cone in \mathbb{R}^2 containing the set $\{1\} \times [s', s'']$.

Consider the acceptance sets from Sect. 6 so that the conditional core is the risk measure. Then X_{T-1}^0 is the sum of K_{T-1} and $\mathbf{m}(X_T^0, \mathcal{F}_{T-1}) = C(\mathbf{M}(Y_T|\mathcal{F}_{T-1}))$, where $\mathbf{M}(Y_T|\mathcal{F}_{T-1})$ is the conditional convex hull of Y_T , that is, the smallest \mathcal{F}_{T-1} -measurable random closed convex set that contains Y_{T-1} ; see [28, Definition 5.1].

Since X_{T-1}^0 is a random closed set, iterating this argument yields that $X_t^0 = C(\tilde{Y}_t)$ for $t = 0, \dots, T$, where $\tilde{Y}_T = Y_T$ and

$$\tilde{Y}_t = \mathbf{M}(\tilde{Y}_{t+1}|\mathcal{F}_t) \cap Y_t, \quad t = T-1, \dots, 0.$$

Note that we do not make any no-arbitrage assumption to obtain X_t^0 . Observe that $\tilde{Y}_t = [\tilde{S}_t^b, \tilde{S}_t^a]$, where $\tilde{S}_T^a = S_T^a$, $\tilde{S}_T^b = S_T^b$ and

$$\tilde{S}_t^a = S_t^a \wedge \text{ess sup}_{\mathcal{F}_t} \tilde{S}_{t+1}^a, \quad \tilde{S}_t^b = S_t^b \vee \text{ess inf}_{\mathcal{F}_t} \tilde{S}_{t+1}^b$$

for $t = T - 1, \dots, 1$. Since $0 < \tilde{S}_t^b \leq \tilde{S}_t^a$ a.s. for all t , (NRA) always holds. By Definition 4.1 and Lemma 4.8, we easily deduce the following result.

Theorem 8.1 (i) (SNR) holds if and only if $S_t^b \leq \text{ess sup}_{\mathcal{F}_t} \tilde{S}_{t+1}^a$ and $S_t^a \geq \text{ess inf}_{\mathcal{F}_t} \tilde{S}_{t+1}^b$ a.s. with strict inequality P -a.s. when we have the strict inequality $S_t^b < S_t^a$ a.s. for all $t \leq T - 1$.
(ii) (NA2) holds if and only if $\text{ess sup}_{\mathcal{F}_t} S_{t+1}^a \geq S_t^a$ and $S_t^b \geq \text{ess inf}_{\mathcal{F}_t} S_{t+1}^b$ a.s. for all $t \leq T - 1$.

Remark 8.2 (NA^s) is equivalent to (SNR) in the proper case, but also to the existence of a strictly consistent price system; see [26, Theorem 3.2.2]. In the two-asset case, the Grigoriev theorem, see [26, Theorem 3.2.15] and Grigoriev [19], asserts that (NA^s) is equivalent to the existence of a (possibly non-strict) consistent price system, i.e., the existence of a martingale (Z_t) with respect to a probability measure Q equivalent to P such that $S_t^b \leq Z_t \leq S_t^a$ for all t . By Theorem 8.1, the existence of a consistent price system, or equivalently (NA^s), implies (SNR). Indeed, $\text{ess sup}_{\mathcal{F}_t} \tilde{S}_{t+1}^a \geq E_Q[Z_{t+1} | \mathcal{F}_t] \geq S_t^b$ and similarly we have $\text{ess inf}_{\mathcal{F}_t} \tilde{S}_{t+1}^b \leq E_Q[Z_{t+1} | \mathcal{F}_t] \leq S_t^a$, the inequalities being strict when $S_t^b < S_t^a$ a.s.

Corollary 8.3 If there exist probability measures Q^a, Q^b which are equivalent to P such that S^a is a Q^a -submartingale and S^b is a Q^b -supermartingale, then (NA2) holds.

The condition in the following corollary means that the two quantities $\delta_t^b = S_t^b / S_{t-1}^b$ and $\delta_t^a = S_t^a / S_{t-1}^a$ admit conditional full supports on \mathbb{R}_+ for all $t \leq T$.

Corollary 8.4 If $P[\delta_t^b \leq c | \mathcal{F}_{t-1}] P[\delta_t^a \geq c | \mathcal{F}_{t-1}] > 0$ a.s. for all $t = 1, \dots, T$ and all $c > 0$, then (NA2) holds.

Proof Let $\gamma = \text{ess sup}_{\mathcal{F}_{t-1}} S_t^a$. Then $\gamma \mathbf{1}_{\{\delta_t^a \geq c\}} \geq S_t^a \mathbf{1}_{\{\delta_t^a \geq c\}} \geq c S_{t-1}^a \mathbf{1}_{\{\delta_t^a \geq c\}}$. Taking conditional expectations yields that

$$\gamma P[\delta_t^a \geq c | \mathcal{F}_{t-1}] \geq c S_{t-1}^a P[\delta_t^a \geq c | \mathcal{F}_{t-1}].$$

Then $\gamma \geq c S_{t-1}^a$, and letting $c \rightarrow \infty$ yields that $\gamma = +\infty$ a.s. Similarly, we obtain $\text{ess inf}_{\mathcal{F}_{t-1}} S_t^b = 0$ a.s., and so Theorem 8.1 (ii) applies. \square

Assume now that the acceptability criterion is based on the generalised conditional expectation as in Sect. 7. By Proposition 7.2, (NARA) holds if and only if there is a deterministic z that belongs to all Y_t , $t = 0, \dots, T$, and (SNR) additionally requires that this point belongs to the interiors of the Y_t .

Example 8.5 Consider a two-asset setting where it is allowed to perform transactions up to one cash unit amount. This is a simple *limit order book* setting with only one break point. Then K_t is the sum $[0, \alpha_t] + [0, \beta_t] + \mathbb{R}_+^2$, where $\alpha_t = (1, -S_t^a)$ and $\beta_t = (-1, S_t^b)$. By Proposition 7.1, (NARA) (with acceptability based on conditional expectation) holds if and only if

$$\mathbb{E} \left[\sum_{s=t+1}^T ([0, \alpha_s] + [0, \beta_s]) \middle| \mathcal{F}_t \right] \cap \mathbb{R}_-^d = \{0\}$$

for all $t = 0, \dots, T-1$. The sum of the segments $[0, \alpha_s]$ and $[0, \beta_s]$ is a random convex compact set called a *zonotope*. The setting can be easily extended to the case of limit order books with several break points.

A Random sets and their selections

Let \mathbb{R}^d be the Euclidean space with norm $|\cdot|$ and the Borel σ -algebra $\mathcal{B}(\mathbb{R}^d)$. The closure of a set $A \subseteq \mathbb{R}^d$ is denoted by $\text{cl}A$. A set-valued function $\omega \mapsto X(\omega)$ from a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the family of all subsets of \mathbb{R}^d is called *\mathcal{F} -measurable* (or *graph-measurable*) if its graph

$$\text{Gr}X = \{(\omega, x) \in \Omega \times \mathbb{R}^d : x \in X(\omega)\} \subseteq \Omega \times \mathbb{R}^d$$

belongs to the product σ -algebra $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$. In this case, X is said to be a *random set*. In the same way, the \mathcal{H} -measurability of X with respect to a sub- σ -algebra \mathcal{H} of \mathcal{F} is defined. The random set X is said to be *closed* (*convex*, *open*) if $X(\omega)$ is a closed (convex, open) set for almost all ω .

Definition A.1 An \mathcal{F} -measurable random element ξ in \mathbb{R}^d is said to be an \mathcal{F} -measurable *selection* (selection in short) of X if $\xi(\omega) \in X(\omega)$ for almost all $\omega \in \Omega$. We denote by $L^0(\mathcal{F}; X)$ the family of all \mathcal{F} -measurable selections of X , and $L^p(\mathcal{F}; X)$ is the family of p -integrable ones.

It is known that an a.s. non-empty random set has at least one selection; see Hess [22, Theorem 4.4]. Let \mathcal{H} be a sub- σ -algebra of \mathcal{F} and recall Definition 2.3.

The decomposable subsets of $L^0(\mathcal{F}; \mathbb{R}^d)$ are called *stable* and infinitely decomposable ones are called *σ -stable* in Cheridito et al. [6]. The following result for $\mathcal{H} = \mathcal{F}$ is well known in case $p = 1$ (see Hiai and Umegaki [23] where the decomposability concept was first introduced); see also Molchanov [30, Theorem 2.1.10] for $\mathcal{H} = \mathcal{F}$ and Kabanov and Safarian [26, Prop. 5.4.3] for $p = 0$.

Theorem A.2 (see [28, Theorem 2.4] and [30, Theorem 2.1.10]) *Let Ξ be a non-empty subset of $L^p(\mathcal{F}; \mathbb{R}^d)$ for $p = 0$ or $p \in [1, \infty]$. Then*

$$\Xi \cap L^p(\mathcal{H}; \mathbb{R}^d) = L^p(\mathcal{H}; X)$$

for an \mathcal{H} -measurable random closed set X if and only if Ξ is \mathcal{H} -decomposable and closed.

For $A_1, A_2 \subseteq \mathbb{R}^d$, define their elementwise (Minkowski) sum as

$$A_1 + A_2 = \{x_1 + x_2 : x_1 \in A_1, x_2 \in A_2\}.$$

The same definition applies for the sum of subsets of $L^0(\mathcal{F}; \mathbb{R}^d)$. The set of pairwise differences of points from A_1 and A_2 is obtained as $A_1 + (-A_2)$, or shortly $A_1 - A_2$, where $-A_2 = \{-x : x \in A_2\}$ is the centrally symmetric variant of A_2 . For the sum $A + \{x\}$ of a set and a singleton we write shortly $A + x$. Note that the sum of two closed sets is not necessarily closed unless at least one of the closed summands is compact. The following result differs from [30, Theorem 1.3.25] in considering the possibly non-closed sum of two random closed sets.

Lemma A.3 *Let X and Y be two random sets. Then*

$$L^0(\mathcal{F}; X) + L^0(\mathcal{F}; Y) = L^0(\mathcal{F}; X + Y).$$

If both X and Y are random closed sets, then $X + Y$ is measurable.

Proof It is trivial that $L^0(\mathcal{F}; X) + L^0(\mathcal{F}; Y) \subseteq L^0(\mathcal{F}; X + Y)$. To prove the converse inclusion, consider $\xi \in L^0(\mathcal{F}; X + Y)$. Since X and Y are \mathcal{F} -measurable, the measurable selection theorem [26, Theorem 5.4.1] yields that there exist \mathcal{F} -measurable selections $\xi' \in L^0(\mathcal{F}; X)$ and $\xi'' \in L^0(\mathcal{F}; Y)$ such that $\xi = \xi' + \xi''$.

Assume that X and Y are closed. Consider their Castaing representations (see [30, Definition 1.3.6]) as $X(\omega) = \text{cl}\{\xi'_i(\omega), i \geq 1\}$ and $Y(\omega) = \text{cl}\{\xi''_i(\omega), i \geq 1\}$. The measurability of $X + Y$ follows from the representation

$$\text{Gr}(X + Y) = \bigcup_{k \geq 1} \bigcap_{m \geq 1} \bigcup_{i, j \geq 1} \left\{ (\omega, x) : |x - \xi'_i(\omega) - \xi''_j(\omega)| \leq \frac{1}{m}, |\xi'_i(\omega)| \leq k \right\}.$$

Indeed, if $(\omega, x) \in \text{Gr}(X_1 + X_2)$, then $x = a + b$ for $a \in X_1(\omega)$ and $b \in X_2(\omega)$. Let $k \geq 1$ be such that $|a| + 1 \leq k$. Since $a \in X_1$, there exists a subsequence $(\xi'_{n_l})_{l \geq 1}$ such that $\xi'_{n_l}(\omega) \rightarrow a$. We may assume without loss of generality that $|\xi'_{n_l}(\omega)| \leq k$. Similarly, $\xi''_{n_l}(\omega) \rightarrow b$. Therefore $|x - \xi'_{n_l}(\omega) - \xi''_{n_l}(\omega)| \leq \frac{1}{m}$ if $m > 0$, and $|\xi'_{n_l}(\omega)| \leq k$ for some i, j . \square

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