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Generalized Nash Fairness solutions for Bi-Objective Minimization Problems

Minh Hieu Nguyen¹ | Mourad Baiou¹ | Viet Hung Nguyen¹ | Thi Quynh Trang Vo¹

¹INP Clermont Auvergne, Univ Clermont Auvergne, Mines Saint-Etienne, CNRS, UMR 6158 LIMOS, 1 Rue de la Chebarde, Aubiere Cedex, France

Correspondence

Viet Hung Nguyen

Email: viet_hung.nguyen@uca.fr

In this paper, we consider a particular case of Bi-Objective Optimization (BOO), called *Bi-Objective Minimization* (BOM), where the two objective functions to be minimized take only positive values. As well as for BOO, most of the methods proposed in the literature for solving BOM focus on computing the Pareto-optimal solutions representing different trade-offs between two objectives. However, it may be difficult for a central decision-maker to determine the preferred solutions due to the huge number of solutions in the Pareto set. We propose a novel criterion for selecting the preferred Pareto-optimal solutions by introducing the concept of ρ -Nash Fairness (ρ -NF) solutions inspired by the definition of proportional fairness. The ρ -NF solutions are the feasible solutions achieving some proportional Nash equilibrium between the two objectives. The positive parameter ρ is introduced to reflect the relative importance of the first objective to the second one. For this work, we will discuss existential and algorithmic questions about the ρ -NF solutions by first showing their existence for BOM. Furthermore, the ρ -NF solution set can be a strict subset of the Pareto set. As there are possibly many ρ -NF solutions, we focus on extreme ρ -NF solutions achieving the smallest values for one of the objectives. Then, we propose

two Newton-based iterative algorithms for finding extreme ρ - NF solutions. Finally, we present computational results on some instances of the Bi-Objective Travelling Salesman Problem and the Bi-Objective Shortest Path Problem.

KEYWORDS

Bi-Objective Optimization, Bi-Criteria Decision Making, Pareto optimal, Weighted Sum Method, Proportional Fairness, Bi-Objective Travelling Salesman Problem, Bi-Objective Shortest Path Problem

1 | INTRODUCTION

In this paper, we consider a particular case of Bi-Objective Optimization (BOO), called *Bi-Objective Minimization* (BOM), where two objective functions to be minimized take only positive values. Many applications in telecommunications, logistics, economics, etc., can be formulated as BOM. We can list here several examples like the Bi-Objective Spanning Tree Problem [10], which aims at finding a spanning tree minimizing the total cost simultaneously and the diameter of the tree, and the Bi-Objective Shortest Path Problem [33] where each arc is associated with a cost and a travel time and one desire to compute the shortest path minimizing the total cost and the total travel time from a given source to a given destination. In a recent conference paper [24], we considered a bi-objective version of the Travelling Salesman Problem where, in addition to the total cost, the difference between the maximum edge cost and the minimum edge cost in the tour is also to be minimized. This problem is a BOM that combines the objective of the classic TSP and the one of the Balanced TSP introduced in [20]. This paper presents a generalized and extended version of [24] to BOM.

Obviously, popular methods for solving BOO can also be applied to solving BOM. Based on the concept of the Pareto-optimal solutions that are non-dominated with respect to each other, such methods usually construct a representation of the Pareto set that represents different trade-offs between the objectives [22]. They can be mainly divided into two classes, i.e., methods with a posterior articulation of preferences and methods with a priori articulation of preferences. In the former methods, such as the Normal Boundary Intersection (NBI) [6], the Non-dominated Sorting Genetic Algorithm-II (NSGA-II) [7], a central decision-maker (CDM) selects his own preferred solutions from the Pareto set manually. Although these methods can provide all the Pareto-optimal solutions, it may be difficult due to the huge number of solutions in the Pareto set [29].

In practice, methods with a priori articulation of preferences [17], which consider the preferences of the CDM before running the optimization algorithm, and then allow the algorithm to determine the solutions that reflect such preferences, which have been used more extensively due to their computational efficiency. They usually formulate a single-objective optimization problem whose optimal solutions are the Pareto-optimal solutions to the bi-objective optimization problem. For instance, the Weighted Sum Method [5], [36] scalarizes two objectives into a single objective by multiplying each objective with a weight supplied by the CDM. The ϵ -constraints Method [15] keeps only one objective and uses a CDM-specified value as an upper bound for the other objective. By changing this value, we can obtain some different optimal solutions. In the lexicographic method [11], the CDM is asked to establish a pre-defined order amongst the objective functions. Then this method solves a sequence of single-objective optimization problems, where each objective is optimized at a single point of time. In the goal programming [2], [3], the CDM determines an aspiration level of the objective function. All the parameters of objective functions are assigned a goal value to be

accomplished. Then the objective of goal-programming is to minimize the deviations from the set goals. There are also many other methods such as Sen's Multi-Objective Programming [31], Hypervolume/Chebyshev scalarization [13], achievement scalarizing problems of Wierzbicki [34], etc. However, most of the above approaches simply outline the methods and show that they can efficiently provide Pareto-optimal solutions. The questions of determining the selection criteria for the preferred solutions and the significance of parameters used for the preferences remain challenging.

In this paper, we propose a novel selection criterion for BOM which can efficiently guide the Weighted Sum Method to find the preferred Pareto-optimal solutions achieving some proportional Nash equilibrium between the two objectives in the context of fair competition based on proportional fairness [1], [19], [25]. The latter aims to provide a compromise between the utilitarian rule - which emphasizes overall system efficiency, and the egalitarian rule - which emphasizes individual fairness. In the context of BOM, proportional fairness means that the sum of proportional changes in objective values when switching from a preferred solution to any other feasible solution is not negative. For our purpose, we consider a more general version of proportional fairness to take into account the relative importance of one objective to the other according to the point of view of the CDM. More precisely, we introduce the concept of the *generalized Nash Fairness* solution, i.e., ρ -*NF* solution, for BOM where $\rho > 0$ is a factor denoting the relative importance of the first objective comparing to the second one. This allows the CDM to consider that ρ percent change of the first objective is comparably equivalent to a one percent change of the second one. Hence, when switching from a ρ -*NF* solution to any other feasible solution, the sum of the factor ρ of the proportional change in the first objective and the proportional change in the second objective is not negative. In this paper, we show the existence of ρ -*NF* solutions and design efficient algorithms for finding some particular ρ -*NF* solutions.

Note that in two recent conference papers [23] and [24], we introduced the notion of *NF* solution, which is a particular case of ρ -*NF* solution when $\rho = 1$ for the Bi-Objective Assignment Problem (BOAP) and the BOTSP. This paper introduces the ρ -*NF* solution concept and generalizes the theory to BOM.

The paper is organized as follows. In Section 2, we first show that for BOM, ρ -*NF* solutions always exist, and the ρ -*NF* solution set is a subset of the Pareto set. Then, we give an example of BOM for which this inclusion can be strict. In Section 3, we focus on designing algorithms for finding ρ -*NF* solutions. In particular, we are interested in extreme ρ -*NF* solutions, which are ρ -*NF* solutions having the smallest values for one of two objectives. We propose Newton-based iterative algorithms assuring the convergence to extreme ρ -*NF* solutions in a finite number of iterations. Section 4 discusses the effect of ρ on the values of corresponding extreme ρ -*NF* solutions. Finally, in Section 5, we present computational results for the Bi-Objective Travelling Salesman Problem (BOTSP) introduced in [24] and the Bi-Objective Shortest Path Problem (BOSPP) introduced in [33].

2 | GENERALIZED NASH FAIRNESS SOLUTION

2.1 | Characterization of generalized NF solution

BOM can be generally formulated as

$$\min_{x \in \mathcal{X}} (P(x), Q(x)),$$

where P, Q represent two objective functions, \mathcal{X} is the feasible set of decision vectors and $P(x), Q(x) > 0, \forall x \in \mathcal{X}$. Let us assume that \mathcal{X} is finite. This property is verified for many practical examples of BOM, especially for bi-objective combinatorial optimization problems.

As a particular case of BOO, popular methods for solving BOM provide Pareto-optimal solutions representing different trade-offs between two objectives. However, in practice, it may be difficult for the CDM to determine a clear selection criterion for accurate a priori articulation of preferences in a representation of the Pareto set. We propose in this section a novel criterion for selecting the preferred Pareto-optimal solutions. For this purpose, we introduce the concept of *generalized Nash Fairness* solution, i.e., ρ -*NF* solution, inspired from the definition of proportional fairness [1], [19], [25]. The parameter $\rho > 0$ reflects the relative importance of the first objective over the second one.

Let $(P, Q) = (P(x), Q(x))$ denote the objective values corresponding to a decision vector $x \in X$. In the standard definition of proportional fairness, a solution with the greater values of two objectives is preferred, i.e., we have two utility functions to be maximized. However, by contrast, in BOM, we prefer the solution with smaller values of P and Q , i.e., P, Q are two cost functions to be minimized. Furthermore, as $\rho > 0$ is a factor denoting the relative importance of P over Q , ρ percent change of P is now comparably equivalent to a one percent change of Q . Thus, the ρ -*NF* solution should be such that, when compared to any other feasible solutions, the sum of the factor ρ of the proportional change of P and the proportional change of Q is not negative. Notice that it is easy for the CDM to explain the ρ -*NF* solutions using the signification of proportional fairness.

Let S be the set of all the couples (P, Q) such that $P = P(x)$ and $Q = Q(x)$ for $x \in X$. As in this paper, the ρ -*NF* solutions will be characterized only by the couples (P, Q) and not by the decision vectors; we will consider the set S instead of the set X . More precisely, by using the term "feasible solution", we refer to a couple (P, Q) in S . Note that a solution (P, Q) in S may represent many decision vectors x in X . Hence, $|S| \leq |X|$, which implies that S is finite since X is finite.

If $(P^*, Q^*) \in S$ is a ρ -*NF* solution for BOM, we have

$$\rho \frac{P - P^*}{P^*} + \frac{Q - Q^*}{Q^*} \geq 0, \forall (P, Q) \in S, \quad (1)$$

which is equivalent to

$$\rho \frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1, \forall (P, Q) \in S,$$

Hence, the ρ -*NF* solution for BOM can be defined as follows.

Definition 1 [ρ -*NF* solution] For a given value $\rho > 0$, $(P^*, Q^*) \in S$ is a ρ -*NF* solution for BOM provided that

$$\rho \frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1, \forall (P, Q) \in S,$$

The ρ -*NF* solution is a generalized concept of the *NF* solution we introduced in our conference papers [23] and [24]. In the following, we will discuss some existential questions about the ρ -*NF* solution.

2.2 | Existence of ρ -*NF* solution

In case $\rho = 1$, it is known that there always exists a ρ -*NF* solution which can be obtained by minimizing the product of P and Q [23], [24]. In the following, we show the existence of a ρ -*NF* solution for every BOM.

Theorem 1 For all $\rho > 0$, there always exists a ρ -*NF* solution for BOM.

Proof We aim to show that there exists a feasible solution $(P^*, Q^*) \in S$ such that

$$\rho \frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1, \forall (P, Q) \in S,$$

which is equivalent to

$$\frac{\rho}{\rho+1} \times \frac{P}{P^*} + \frac{1}{\rho+1} \times \frac{Q}{Q^*} \geq 1, \forall (P, Q) \in S,$$

Since S is finite, there always exists a feasible solution $(P^*, Q^*) \in S$ such that

$$(P^*, Q^*) = \arg \min_{(P, Q) \in S} P^\rho Q,$$

Let us show that (P^*, Q^*) is a ρ -NF solution. $\forall (P, Q) \in S$, we have $P^\rho Q \geq P^{*\rho} Q^* > 0$. Using the generalization of Young's inequality for products [35] (see Appendix 1), we get

$$\frac{\rho}{\rho+1} \times \frac{P}{P^*} + \frac{1}{\rho+1} \times \frac{Q}{Q^*} \geq \left(\frac{P}{P^*}\right)^{\frac{\rho}{\rho+1}} \left(\frac{Q}{Q^*}\right)^{\frac{1}{\rho+1}} = \left(\frac{P^\rho Q}{P^{*\rho} Q^*}\right)^{\frac{1}{\rho+1}} \geq 1,$$

Hence, (P^*, Q^*) is a ρ -NF solution. □

Theorem 1 proves the existence of one ρ -NF solution for BOM that minimizes $P^\rho Q$, or equivalently minimizes $(\rho \log P + \log Q)$ where \log denotes the natural logarithmic function. However, finding such a solution may be difficult as it requires minimizing a concave function. In the following, we show that every ρ -NF solution can be found by minimizing an appropriate strict convex combination of ρP and Q . More precisely, every ρ -NF solution can be obtained by solving the following optimization problem

$$\mathcal{P}_\rho(\alpha, \beta) = \min_{(P, Q) \in S} \rho \alpha P + \beta Q,$$

where $(\alpha, \beta) \in C := \{(\alpha, \beta) \mid \alpha, \beta > 0 \text{ and } \alpha + \beta = 1\}$ are the coefficients to be determined. Notice that in this paper, we assume the existence of the algorithms for minimizing the convex combinations of P and Q (i.e., including the problem $\mathcal{P}_\rho(\alpha, \beta)$ with $(\alpha, \beta) \in C$, the problem minimizing P and the problem minimizing Q). For example, when P and Q are linear functions, $\rho \alpha P + \beta Q$ is also linear. The BOTSP and the BOSPP that we consider in the "computational results" section belong to this category.

Theorem 2 $(P^*, Q^*) \in S$ is a ρ -NF solution if and only if (P^*, Q^*) is an optimal solution of $\mathcal{P}_\rho(\alpha^*, \beta^*)$ with $\alpha^* = \frac{Q^*}{P^* + Q^*}$ and $\beta^* = \frac{P^*}{P^* + Q^*}$.

Proof \implies Let (P^*, Q^*) be a ρ -NF solution. Let $\alpha^* = \frac{Q^*}{P^* + Q^*}$ and $\beta^* = \frac{P^*}{P^* + Q^*}$. Hence, $(\alpha^*, \beta^*) \in C$. We will show that (P^*, Q^*) is an optimal solution of $\mathcal{P}_\rho(\alpha^*, \beta^*)$.

As (P^*, Q^*) is a ρ -NF solution, we have

$$\rho \frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1, \forall (P, Q) \in S, \tag{2}$$

Multiplying (2) by $\beta^* Q^* > 0$ gives

$$\rho \frac{\beta^* Q^*}{P^*} P + \beta^* Q \geq \rho \beta^* Q^* + \beta^* Q^*, \quad \forall (P, Q) \in S, \quad (3)$$

From the values of α^* and β^* , we have $\alpha^* P^* = \beta^* Q^*$. Thus, we deduce from (3)

$$\rho \alpha^* P + \beta^* Q \geq \rho \alpha^* P^* + \beta^* Q^*, \quad \forall (P, Q) \in S,$$

Hence, (P^*, Q^*) is an optimal solution of $\mathcal{P}_\rho(\alpha^*, \beta^*)$.

\Leftarrow Now suppose that (P^*, Q^*) is an optimal solution of $\mathcal{P}_\rho(\alpha^*, \beta^*)$ with $\alpha^* = \frac{Q^*}{P^* + Q^*}$ and $\beta^* = \frac{P^*}{P^* + Q^*}$, we show that (P^*, Q^*) is a ρ -NF solution.

As (P^*, Q^*) is an optimal solution of $\mathcal{P}_\rho(\alpha^*, \beta^*)$, we have

$$\rho \alpha^* P + \beta^* Q \geq \rho \alpha^* P^* + \beta^* Q^*, \quad \forall (P, Q) \in S,$$

Replacing α^* by $\frac{Q^*}{P^* + Q^*}$ and β^* by $\frac{P^*}{P^* + Q^*}$, we obtain

$$\rho \frac{P}{P^*} + \frac{Q}{Q^*} \geq \rho + 1, \quad \forall (P, Q) \in S,$$

That means (P^*, Q^*) is a ρ -NF solution. \square

Theorem 2 states a necessary and sufficient condition for the ρ -NF solutions. We are interested now in the following question: For a given solution $(P', Q') \in S$, how to assert that (P', Q') is a ρ -NF solution or not?

Proposition 3 Given a feasible solution $(P', Q') \in S$. Let $\alpha' = \frac{Q'}{P' + Q'}$, $\beta' = \frac{P'}{P' + Q'}$ and (P^*, Q^*) be an optimal solution of $\mathcal{P}_\rho(\alpha', \beta')$, then (P', Q') is a ρ -NF solution if and only if $\rho \alpha' P^* + \beta' Q^* - \rho \alpha' P' - \beta' Q' = 0$.

Proof \Rightarrow If $\rho \alpha' P^* + \beta' Q^* - \rho \alpha' P' - \beta' Q' = 0$ then (P', Q') is also an optimal solution of $\mathcal{P}_\rho(\alpha', \beta')$. Since $(\alpha', \beta') \in C$, $\alpha' = \frac{Q'}{P' + Q'}$ and $\beta' = \frac{P'}{P' + Q'}$, (P', Q') is a ρ -NF solution due to Theorem 2.

\Leftarrow If (P', Q') is a ρ -NF solution then (P', Q') is also an optimal solution of $\mathcal{P}_\rho(\alpha', \beta')$ due to Theorem 2. Thus, $\rho \alpha' P^* + \beta' Q^* = \rho \alpha' P' + \beta' Q'$ which leads to $\rho \alpha' P^* + \beta' Q^* - \rho \alpha' P' - \beta' Q' = 0$. \square

Note that BOM is a special case of multi-objective optimization where the concept of the Pareto-optimal (non-dominated, non-inferior) solution is widely applied for describing the set of efficient solutions [22]. For two feasible solutions $(P', Q'), (P'', Q'') \in S$, let us denote $(P', Q') < (P'', Q'')$, i.e., (P'', Q'') is strictly dominated by (P', Q') , if $P' \leq P'', Q' \leq Q''$ and $(P', Q') \neq (P'', Q'')$, i.e., we do not have $P' = P''$ and $Q' = Q''$ simultaneously. We recall the definition of the Pareto-optimal solution for BOM.

Definition 2 [Pareto-optimal solution] $(P', Q') \in S$ is a Pareto-optimal solution for BOM provided that there does not exist $(P'', Q'') \in S$ such that $(P'', Q'') < (P', Q')$. In other words, it is not strictly dominated by other feasible solutions in S .

In the following, we will prove that each ρ -NF solution is necessarily a Pareto-optimal solution.

Proposition 4 Each ρ -NF solution is necessarily a Pareto-optimal solution.

Proof Let $(P^*, Q^*) \in S$ be an optimal solution of $\mathcal{P}_\rho(\alpha^*, \beta^*)$ where $(\alpha^*, \beta^*) \in C$. We first show that (P^*, Q^*) is a Pareto-optimal solution.

Suppose that there exists a feasible solution $(P', Q') \in S$ satisfying $(P', Q') < (P^*, Q^*)$. Since $(\alpha^*, \beta^*) \in C$, $\rho > 0$ and $(P', Q') < (P^*, Q^*)$, we have $\rho\alpha^*P' + \beta^*Q' < \rho\alpha^*P^* + \beta^*Q^*$.

On the other hand, the optimality of (P^*, Q^*) gives

$$\rho\alpha^*P^* + \beta^*Q^* \leq \rho\alpha^*P' + \beta^*Q',$$

which leads to a contradiction. Thus, (P^*, Q^*) is a Pareto-optimal solution.

According to Theorem 2, each ρ -NF solution is necessarily a Pareto-optimal solution. □

Let S_N be the set of Pareto-optimal solutions which will be shortened as the Pareto set.

Remark 1 The Pareto set S_N is finite.

Proof As S is finite and $S_N \subseteq S$, we conclude that S_N is also finite. □

Let S_ρ be the set of ρ -NF solutions. In the following, we will establish the relationship between S_N and S_ρ .

Theorem 5 For all $\rho > 0$, $S_\rho \subseteq S_N$ and this inclusion can be strict.

Proof According to Proposition 4, we have $S_\rho \subseteq S_N$.

To show that the ρ -NF solution set can be a strict subset of the Pareto set, i.e., $S_\rho \subset S_N$, let us consider an example of the Bi-Objective Travelling Salesman Problem (BOTSP) where we look for a tour minimizing simultaneously the total edge cost and the max-min distance which is the difference between the maximum edge cost and the minimum one [24]. This example of BOTSP is represented by an undirected graph with each edge cost illustrated in Figure 1.

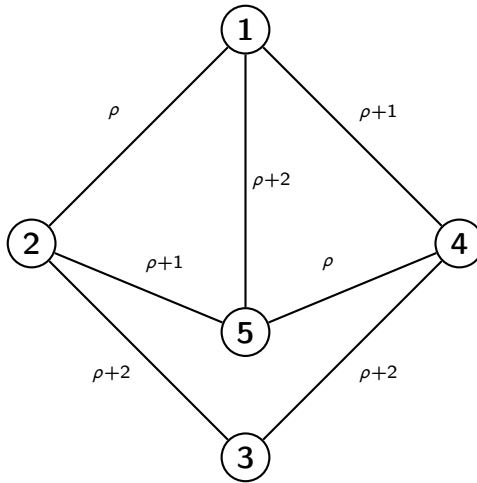


FIGURE 1 An example of BOTSP with $S_\rho \subset S_N$

Let (P, Q) denote the solution for the total cost and the max-min distance corresponding to a solution tour. We see that there are two different Hamiltonian tours $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$ and $1 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 5 \rightarrow 1$ in G_0 corresponding respectively to two feasible solutions $(P', Q') = (5\rho + 6, 2)$ and $(P'', Q'') = (5\rho + 8, 1)$. Since $P' < P''$ and $Q' > Q''$, (P', Q') and (P'', Q'') are non-dominated solutions with respect to each other. Thus, they are two Pareto-optimal solutions. We have

$$\rho \frac{P''}{P'} + \frac{Q''}{Q'} = \rho \frac{5\rho + 8}{5\rho + 6} + \frac{1}{2} = \rho + 1 + \frac{2\rho}{5\rho + 6} - \frac{1}{2} = \rho + 1 - \frac{\rho + 6}{2(5\rho + 6)} < \rho + 1,$$

Hence, (P', Q') is a Pareto-optimal solution but not a ρ -NF solution.

By contrast, (P'', Q'') is a ρ -NF solution because

$$\rho \frac{P'}{P''} + \frac{Q'}{Q''} = \rho \frac{5\rho + 6}{5\rho + 8} + \frac{2}{1} = \rho + 1 + 1 - \frac{2\rho}{5\rho + 8} = \rho + 1 + \frac{3\rho + 8}{5\rho + 8} > \rho + 1,$$

There exist some particular cases where $S_\rho = S_N$. For example, if there is only one Pareto-optimal solution in S_N , it is also a ρ -NF solution. \square

As shown by Theorem 1, one ρ -NF solution for BOM always exists that minimizes $P^\rho Q$. In the following, we assert that there may be more than one ρ -NF solution for BOM.

Remark 2 *There may be more than one ρ -NF solution for BOM.*

Proof To prove this conclusion, we consider an example of the Bi-Objective Shortest Path Problem (BOSPP) introduced in [33]. We first restate the definition of the BOSPP. Let $G = (V, A)$ be a directed graph where V is the set of nodes ($|V| = n$) and A is the set of arcs ($|A| = m$). Let $c_{ij}, t_{ij} > 0$ denote respectively the cost and the time associated with each arc (i, j) . The BOSPP consists in finding a path H from a source node s to a target node t minimizing both criteria $\left\{ \sum_{(i,j) \in H} c_{ij}, \sum_{(i,j) \in H} t_{ij} \right\}$.

Let G_0 be a directed graph with a source node s and a target node t illustrated in Figure 2.

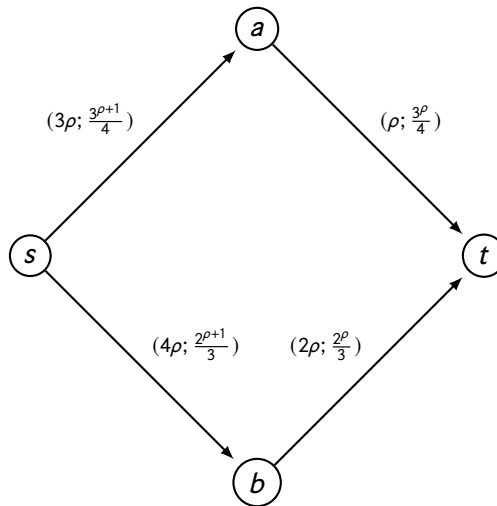


FIGURE 2 An example of BOSPP with two ρ -NF solutions

Note that each arc of G_0 carries two attributes: cost and time values. For example, the cost and time associated with arc (s, a) are respectively 3ρ and $\frac{3^{\rho+1}}{4}$. Let (P, Q) denote the solution corresponding to the total cost and time associated with a path from s to t . We can see that there are two different paths $s \rightarrow a \rightarrow t$ and $s \rightarrow b \rightarrow t$ in G_0 corresponding respectively to two feasible solutions $(P', Q') = (4\rho, 3^\rho)$ and $(P'', Q'') = (6\rho, 2^\rho)$. Using Young's inequality for products (see Appendix 1), we have

$$\rho \frac{P''}{P'} + \frac{Q''}{Q'} = \rho \frac{6\rho}{4\rho} + \frac{2^\rho}{3^\rho} = (\rho+1) \left(\frac{\rho}{\rho+1} \times \frac{3}{2} + \frac{1}{\rho+1} \times \left(\frac{2}{3}\right)^\rho \right) \geq (\rho+1) \left(\frac{3}{2}\right)^{\frac{\rho}{\rho+1}} \left(\frac{2}{3}\right)^{\frac{\rho}{\rho+1}} = \rho+1,$$

Thus, (P', Q') is a ρ -NF solution. Moreover, we also have

$$\rho \frac{P'}{P''} + \frac{Q'}{Q''} = \rho \frac{4\rho}{6\rho} + \frac{3^\rho}{2^\rho} = (\rho+1) \left(\frac{\rho}{\rho+1} \times \frac{2}{3} + \frac{1}{\rho+1} \times \left(\frac{3}{2}\right)^\rho \right) \geq (\rho+1) \left(\frac{2}{3}\right)^{\frac{\rho}{\rho+1}} \left(\frac{3}{2}\right)^{\frac{\rho}{\rho+1}} = \rho+1,$$

Thus, (P'', Q'') is also a ρ -NF solution. Hence, there are two ρ -NF solutions for this example of BOSPP. \square

As shown by Remark 2, there are possibly many ρ -NF solutions for BOM. Among these solutions, the CDM may naturally prefer two solutions: the one with the smallest value of P and the other with the smallest value of Q . Let us refer to these solutions as the *extreme ρ -NF solutions*. In the next section, we present two iterative algorithms for finding extreme ρ -NF solutions in a finite number of iterations.

3 | ALGORITHMS FOR FINDING EXTREME ρ -NF SOLUTIONS

Let P -extreme ρ -NF solution denote the ρ -NF solution with the smallest value of P . Similarly, let Q -extreme ρ -NF solution denote the ρ -NF solution with the smallest value of Q . Notice that P -extreme ρ -NF solution and Q -extreme ρ -NF solution are unique with respect to the values of P and Q since they are both Pareto-optimal solutions.

In the following, we will focus first on the P -extreme ρ -NF solution. We will see that all the subsequent results applied to the P -extreme ρ -NF solution can also be applied to the Q -extreme ρ -NF solution with minor changes.

We propose an iterative algorithm to find the P -extreme ρ -NF solution, inspired by the works of Isbell [18], and Dinkelbach [9] on the application of the Newton-Raphson method for solving linear and nonlinear fractional programming. Starting from $(\alpha_0, \beta_0) = (1, 0)$, we determine the initial solution (P_0, Q_0) which is an optimal solution of $\mathcal{P}_\rho(\alpha_0, \beta_0)$, i.e., the problem minimizing P . Notice that (P_0, Q_0) is not necessarily a Pareto-optimal solution due to $(\alpha_0, \beta_0) = (1, 0) \notin C$. Our algorithm can be stated as follows.

Algorithm 1

Input: An instance of BOM, $\rho > 0$, the initial solution (P_0, Q_0) and a corresponding decision vector X_0 .

Output: A decision vector corresponding to the P -extreme ρ -NF solution.

- 1: $i \leftarrow 0$
 - 2: **repeat**
 - 3: $\alpha_{i+1} \leftarrow \frac{Q_i}{P_i + Q_i}, \beta_{i+1} \leftarrow \frac{P_i}{P_i + Q_i}$
 - 4: solve $\mathcal{P}_\rho(\alpha_{i+1}, \beta_{i+1})$ to obtain an optimal solution (P_{i+1}, Q_{i+1}) and a corresponding decision vector X_{i+1}
 - 5: $T_i \leftarrow \rho\alpha_{i+1}P_{i+1} + \beta_{i+1}Q_{i+1} - \rho\alpha_{i+1}P_i - \beta_{i+1}Q_i$
 - 6: $i \leftarrow i + 1$
 - 7: **until** $T_i = 0$
 - 8: **return** $\alpha_i, (P_i, Q_i)$ and X_i .
-

Let $\{\alpha_i\}_{i \geq 0}$ (including $\alpha_0 = 1$) be the sequence of $\{\alpha\}$ constructed by Algorithm 1 and $T_i = \rho\alpha_{i+1}P_{i+1} + \beta_{i+1}Q_{i+1} - \rho\alpha_{i+1}P_i - \beta_{i+1}Q_i$ where $(P_i, Q_i), \forall i \geq 0$ is the optimal solution of $\mathcal{P}_\rho(\alpha_i, \beta_i)$ in the execution of Algorithm 1.

We will prove that Algorithm 1 terminates in a finite number of iterations, and the obtained solution is exactly the P -extreme ρ -NF solution. Our proof will use the following lemmas and theorems.

Lemma 6 Let $\alpha', \alpha'' \in [0, 1], \beta' = 1 - \alpha', \beta'' = 1 - \alpha''$ and $(P', Q'), (P'', Q'')$ be respectively the optimal solutions of $\mathcal{P}_\rho(\alpha', \beta')$ and $\mathcal{P}_\rho(\alpha'', \beta'')$. If $\alpha' > \alpha''$ then $P' \leq P''$ and $Q' \geq Q''$.

Proof Since $\alpha', \alpha'' \in [0, 1]$, we also have $\beta', \beta'' \in [0, 1]$.

The optimality of (P', Q') and (P'', Q'') gives

$$\rho\alpha'P' + \beta'Q' \leq \rho\alpha'P'' + \beta'Q'', \quad (4a)$$

$$\rho\alpha''P'' + \beta''Q'' \leq \rho\alpha''P' + \beta''Q', \quad (4b)$$

Multiplying (4a) and (4b) by respectively $\beta'', \beta' \geq 0$ gives

$$\rho\alpha'\beta''P' + \beta'\beta''Q' \leq \rho\alpha'\beta''P'' + \beta'\beta''Q'', \quad (5a)$$

$$\rho\alpha''\beta'P'' + \beta'\beta''Q'' \leq \rho\alpha''\beta'P' + \beta'\beta''Q', \quad (5b)$$

By adding (5a) and (5b), we obtain $\rho(\alpha'\beta'' - \alpha''\beta')(P' - P'') \leq 0$. Since $\alpha' > \alpha''$, we have $\beta'' > \beta' \geq 0$. Thus, $\alpha'\beta'' > \alpha''\beta'$ and it follows that $P' \leq P''$.

In addition, the inequality (4b) implies $\beta''(Q' - Q'') \geq \rho\alpha''(P'' - P') \geq 0$. Since $\beta'' > 0$, we get $Q' \geq Q''$. \square

Lemma 7 During the execution of Algorithm 1, $(\alpha_{i+1}, \beta_{i+1}) \in C, T_i \leq 0$ and the sequence $\{\alpha_i\}$ is strictly decreasing, $\forall i \geq 0$.

Proof Since $P_i, Q_i > 0, \forall i \geq 0$, we have $\alpha_{i+1}, \beta_{i+1} > 0$ and $\alpha_{i+1} + \beta_{i+1} = 1$. Thus, $(\alpha_{i+1}, \beta_{i+1}) \in C, \forall i \geq 0$. It follows that (P_{i+1}, Q_{i+1}) is a Pareto-optimal solution, $\forall i \geq 0$ due to the proof of Proposition 4.

The optimality of (P_{i+1}, Q_{i+1}) gives

$$\rho\alpha_{i+1}P_{i+1} + \beta_{i+1}Q_{i+1} \leq \rho\alpha_{i+1}P_i + \beta_{i+1}Q_i, \forall i \geq 0,$$

Thus, $T_i = \rho\alpha_{i+1}P_{i+1} + \beta_{i+1}Q_{i+1} - \rho\alpha_{i+1}P_i - \beta_{i+1}Q_i \leq 0, \forall i \geq 0$.

We will prove $\alpha_i - \alpha_{i+1} > 0$ by induction on i . If $i = 0$ then $\alpha_0 - \alpha_1 = 1 - \frac{Q_0}{P_0 + Q_0} = \frac{P_0}{P_0 + Q_0} > 0$.

Suppose our hypothesis is true until $i = k \geq 0$. Since $\alpha_k > \alpha_{k+1}$, we have $P_{k+1} \geq P_k > 0$ and $Q_k \geq Q_{k+1} > 0$ due to Lemma 6. It leads to $Q_k P_{k+1} \geq P_k Q_{k+1}$ and $Q_k P_{k+1} = P_k Q_{k+1} \iff P_k = P_{k+1}$ and $Q_k = Q_{k+1}$, i.e., $(P_k, Q_k) \equiv (P_{k+1}, Q_{k+1})$.

If $T_k = 0$ then Algorithm 1 returns the value α_k . Hence, the sequence $\{\alpha_i\}$ for $i = 0, 1, \dots, k$ is strictly decreasing.

If $T_k < 0$ then we have $(P_k, Q_k) \neq (P_{k+1}, Q_{k+1})$. It implies $Q_k P_{k+1} > P_k Q_{k+1}$. We get

$$\alpha_{k+1} = \frac{Q_k}{P_k + Q_k} > \frac{Q_{k+1}}{P_{k+1} + Q_{k+1}} = \alpha_{k+2}.$$

Thus, in this case, our hypothesis is also true with $i = k + 1$. Consequently, $\{\alpha_i\}$ is strictly decreasing, $\forall i \geq 0$. \square

Lemma 8 *Algorithm 1 terminates in a finite number of iterations. More precisely, it terminates in at most K iterations where $K \in \mathbb{N}_+ \setminus \{0\}$ represents the number of Pareto-optimal solutions.*

Proof Since $\alpha_i > \alpha_{i+1}$, we have $Q_i \geq Q_{i+1}$ and $P_i \leq P_{i+1}, \forall i \geq 0$ due to Lemma 6.

We first prove that if $T_i < 0$ then $Q_i > Q_{i+1}, \forall i \geq 0$.

Suppose that $T_i < 0$ but $Q_i = Q_{i+1}$. The optimality of (P_{i+1}, Q_{i+1}) gives

$$\rho\alpha_{i+1}P_{i+1} + \beta_{i+1}Q_{i+1} \leq \rho\alpha_{i+1}P_i + \beta_{i+1}Q_i, \quad (6)$$

Since $Q_i = Q_{i+1}$ and $\rho\alpha_{i+1} > 0$, (6) yields $P_{i+1} \leq P_i$, which implies that $P_i = P_{i+1}$. Consequently, we have $T_i = 0$, leading to a contradiction.

Hence, if $T_i < 0$ then $Q_i > Q_{i+1}$. It implies that for any $k \geq 0$ such that $T_k < 0$, we have $T_i < 0, \forall 0 \leq i \leq k$. Hence, the sequence of solutions (P_{i+1}, Q_{i+1}) with $0 \leq i \leq k$ obtained by the execution of Algorithm 1 is strictly decreasing in Q . Consequently, (P_{i+1}, Q_{i+1}) are distinct Pareto-optimal solutions, $\forall 0 \leq i \leq k$.

As a result, we obtain $T = 0$ after at most K iterations where $K \in \mathbb{N}_+ \setminus \{0\}$ represents the number of Pareto-optimal solutions. Moreover, according to Remark 1, the Pareto set S_N is finite, which implies that K is finite. Thus, Algorithm 1 terminates in a finite number of iterations. \square

Theorem 9 *The solution given by Algorithm 1 is the P -extreme ρ -NF solution.*

Proof Let (P_n, Q_n) be the solution given by Algorithm 1. Since $T_n = 0$, (P_n, Q_n) is a ρ -NF solution due to Proposition 3.

If $n = 0$, then (P_0, Q_0) is obviously the P -extreme ρ -NF solution as it is an optimal solution of the problem minimizing P . We consider $n \geq 1$ that implies $T_n = 0$ and $T_i < 0, \forall 0 \leq i \leq n - 1$.

We will prove that (P_n, Q_n) is the ρ -NF solution with the smallest value of P . Suppose that it is not true. Thus, there exists a ρ -NF solution (P^*, Q^*) such that $P^* < P_n$. According to Theorem 2, there exists $(\alpha^*, \beta^*) \in C$ such that (P^*, Q^*) is an optimal solution of $\mathcal{P}_\rho(\alpha^*, \beta^*)$ and $\alpha^*P^* = \beta^*Q^*$. Since $P^* < P_n$, we have $\alpha^* \geq \alpha_n$ due to Lemma 4. Since the sequence $\{\alpha\}$ is strictly decreasing and $\alpha^* < \alpha_0 = 1$, there exists $0 \leq i \leq n - 1$ such that $\alpha^* \in [\alpha_{i+1}, \alpha_i]$.

We first show that $\alpha^* \neq \alpha_{i+1}$. Let us suppose that $\alpha^* = \alpha_{i+1}$. We also have $\beta^* = \beta_{i+1}$ that leads to $\alpha_{i+1}P^* = \beta_{i+1}Q^*$. Thus, we get

$$\frac{P^*}{Q^*} = \frac{\beta_{i+1}}{\alpha_{i+1}} = \frac{P_i}{Q_i} \implies \frac{P^*}{P_i} = \frac{Q^*}{Q_i},$$

If $i = 0$ then $\frac{P^*}{Q_0} = \frac{Q^*}{Q_0}$. Since $P^* \geq P_0$, we get $Q^* \geq Q_0$. Thus, $\rho\alpha_1P^* + \beta_1Q^* \geq \rho\alpha_1P_0 + \beta_1Q_0$.

If $i \geq 1$ then $P^* = P_i, Q^* = Q_i$ since they are both Pareto-optimal solutions and $\frac{P^*}{P_i} = \frac{Q^*}{Q_i}$. Thus, $\rho\alpha_{i+1}P^* + \beta_{i+1}Q^* = \rho\alpha_{i+1}P_i + \beta_{i+1}Q_i$.

Consequently, we always have $\rho\alpha_{i+1}P^* + \beta_{i+1}Q^* \geq \rho\alpha_{i+1}P_i + \beta_{i+1}Q_i$. Moreover, $\rho\alpha_{i+1}P_{i+1} + \beta_{i+1}Q_{i+1} = \rho\alpha_{i+1}P^* + \beta_{i+1}Q^*$ because both (P_{i+1}, Q_{i+1}) and (P^*, Q^*) are the optimal solutions of $\mathcal{P}_\rho(\alpha_{i+1}, \beta_{i+1})$. Thus, $T_i = \rho\alpha_{i+1}P_{i+1} + \beta_{i+1}Q_{i+1} - \rho\alpha_{i+1}P_i - \beta_{i+1}Q_i \geq 0$ which leads to a contradiction due to $T_i < 0, \forall 0 \leq i \leq n - 1$. Hence, we have $\alpha^* \neq \alpha_{i+1}$.

It follows that $\alpha^* \in (\alpha_{i+1}, \alpha_i)$. Since $\alpha^* < \alpha_i$, we have $P^* \geq P_i$ and $Q^* \leq Q_i$ due to Lemma 6. Thus, we get

$$\alpha^* = \frac{Q^*}{P^* + Q^*} \leq \frac{Q_i}{P_i + Q_i} = \alpha_{i+1},$$

which leads to a contradiction due to the fact that $\alpha^* > \alpha_{i+1}$. Hence, (P_n, Q_n) is a ρ -NF solution having the smallest value of P , i.e., the P -extreme ρ -NF solution. \square

For finding the Q -extreme ρ -NF solution, we use a similar algorithm to Algorithm 1. However, we start from $(\alpha_0, \beta_0) = (0, 1)$ instead of $(1, 0)$ and (P_0, Q_0) as an optimal solution of $\mathcal{P}_\rho(\alpha_0, \beta_0)$, i.e., the problem minimizing Q . It can be stated as follows.

Algorithm 2

Input: An instance of BOM, $\rho > 0$, the initial solution (P_0, Q_0) and a corresponding decision vector X_0 .

Output: A decision vector corresponding to the Q -extreme ρ -NF solution.

```

1:  $i \leftarrow 0$ 
2: repeat
3:    $\alpha_{i+1} \leftarrow \frac{Q_i}{P_i+Q_i}, \beta_{i+1} \leftarrow \frac{P_i}{P_i+Q_i}$ 
4:   solve  $\mathcal{P}_\rho(\alpha_{i+1}, \beta_{i+1})$  to obtain an optimal solution  $(P_{i+1}, Q_{i+1})$  and a corresponding decision vector  $X_{i+1}$ 
5:    $T_i \leftarrow \rho\alpha_{i+1}P_{i+1} + \beta_{i+1}Q_{i+1} - \rho\alpha_{i+1}P_i - \beta_{i+1}Q_i$ 
6:    $i \leftarrow i + 1$ 
7: until  $T_i = 0$ 
8: return  $\alpha_i, (P_i, Q_i)$  and  $X_i$ .
```

We also state some lemmas and theorems to prove that we obtain the Q -extreme ρ -NF solution by using Algorithm 2.

Lemma 10 *During the execution of Algorithm 2, $(\alpha_{i+1}, \beta_{i+1}) \in C$, $T_i \leq 0$ and the sequence $\{\alpha_i\}$ is strictly increasing, $\forall i \geq 0$.*

Lemma 11 *Algorithm 2 terminates in a finite number of iterations. More precisely, it terminates in at most K iterations where $K \in \mathbb{N}_+ \setminus \{0\}$ represents the number of Pareto-optimal solutions.*

Theorem 12 *The solution given by Algorithm 2 is the Q -extreme ρ -NF solution.*

4 | EFFECT OF ρ ON THE VALUES OF EXTREME ρ -NF SOLUTIONS

In this section, we show the effect of ρ on the values of the corresponding P -extreme ρ -NF solution and Q -extreme ρ -NF solution. When ρ increases, the CDM prefers a solution with a smaller value of P , and the value of P in the P -extreme ρ -NF solution becomes closer to the value of P in the optimal solution of the problem minimizing the single objective P . Furthermore, when ρ is large enough, they are identical. Conversely, when ρ decreases, a solution with a smaller value of Q is preferred, and the value of Q in the Q -extreme ρ -NF solution becomes closer to the value of Q in the optimal solution of the problem minimizing the single objective Q . When ρ is small enough, they are also identical. Hence, this helps the CDM choose an appropriate parameter ρ corresponding to his own preferred solutions based on the values of P and Q . To prove this conclusion, we state the following lemma and theorems.

Lemma 13 *Given $\rho', \rho'' > 0$. Let $\alpha', \alpha'' \in [0, 1]$, $\beta' = 1 - \alpha', \beta'' = 1 - \alpha''$ and $(P', Q'), (P'', Q'')$ be respectively the optimal solutions of $\mathcal{P}_{\rho'}(\alpha', \beta')$ and $\mathcal{P}_{\rho''}(\alpha'', \beta'')$. If $\rho' \alpha' \beta'' > \rho'' \alpha'' \beta'$ then $P' \leq P''$ and $Q' \geq Q''$.*

Proof In fact, this Lemma is a generalized version of Lemma 6 where we consider two different values of ρ . Using the same procedures as Lemma 6, we obtain $(\rho' \alpha' \beta'' - \rho'' \alpha'' \beta')(P' - P'') \leq 0$. Since $\rho' \alpha' \beta'' > \rho'' \alpha'' \beta'$, it follows that $P' \leq P''$.

In addition, we also have $\beta''(Q' - Q'') \geq \rho'' \alpha''(P'' - P') \geq 0$. Since $\rho' \alpha' \beta'' > \rho'' \alpha'' \beta' \geq 0$, we get $\beta'' > 0$. Hence, $Q' \geq Q''$. \square

Theorem 14 Given $\rho' > \rho'' > 0$. Let $(P', Q'), (P'', Q'') \in S$ be respectively the P -extreme ρ' -NF solution and the P -extreme ρ'' -NF solution. Then $P' \leq P''$.

Proof Suppose that Algorithm 1 returns (P', Q') and (P'', Q'') in respectively n' and n'' iterations where $n', n'' \in \mathbb{N}_+ \setminus \{0\}$. Denote $\{\alpha'_i\}_{0 \leq i \leq n'}$ and $\{\alpha''_i\}_{0 \leq i \leq n''}$ as two sequences constructed by Algorithm 1. Without loss of generality, we assume that $n' \geq n''$. We first extend the sequence $\{\alpha''\}$ to have the same number of elements as $\{\alpha'\}$. In case $n'' < n'$, we add the element $\alpha''_{n''+1} = \frac{Q''}{P''+Q''}$ to $\{\alpha''\}$ and repeat $\alpha''_{n''+1}$ such that both the sequence $\{\alpha''\}$ and the sequence $\{\alpha'\}$ have $n' + 1$ elements. We also repeat the optimal solution (P'', Q'') corresponding to each α''_i where $i \geq n'' + 1$. Let (P'_i, Q'_i) and (P''_i, Q''_i) be respectively the optimal solutions of $\mathcal{P}_{\rho'}(\alpha'_i, \beta'_i)$ and $\mathcal{P}_{\rho''}(\alpha''_i, \beta''_i)$, $\forall 0 \leq i \leq n' + 1$ as constructed above. Then, $(P'_{n'}, Q'_{n'})$ and $(P''_{n'}, Q''_{n'})$ are respectively the P -extreme ρ' -NF solution and the P -extreme ρ'' -NF solution. (In fact, we have $P'_{n'} = P', Q'_{n'} = Q'$, $\forall i \geq n'' + 1$ and $P''_{n'} = P', Q''_{n'} = Q'$)

As both (α'_0, β'_0) and (α''_0, β''_0) are the initial coefficients for using Algorithm 1, we have $\alpha'_0 = \alpha''_0 = 1$ and $\beta'_0 = \beta''_0 = 0$. Note that starting from an arbitrary optimal solution of the problem minimizing P , Algorithm 1 always returns the P -extreme ρ -NF solution. Thus, we can assume that $P'_0 = P''_0$ and $Q'_0 = Q''_0$. We prove $\rho' \alpha'_i \beta'_i > \rho'' \alpha''_i \beta''_i$, $\forall 1 \leq i \leq N$ by induction on i . For $i = 1$, since $P'_0 = P''_0$ and $Q'_0 = Q''_0$, we have $\alpha'_1 = \alpha''_1 > 0$ and $\beta'_1 = \beta''_1 > 0$. Hence, $\rho' \alpha'_1 \beta'_1 > \rho'' \alpha''_1 \beta''_1$. Suppose that our hypothesis is true until $i = k \geq 1$. We will show that it is also true with $i = k + 1$.

The inductive hypothesis gives $\rho' \alpha'_k \beta'_k > \rho'' \alpha''_k \beta''_k$ that implies $P'_k \leq P''_k$ and $Q'_k \geq Q''_k$ due to Lemma 13. Thus, we get

$$\alpha'_{k+1} = \frac{Q'_k}{P'_k + Q'_k} \geq \frac{Q''_k}{P''_k + Q''_k} = \alpha''_{k+1}, \quad (7)$$

Since $\alpha'_{k+1} \geq \alpha''_{k+1} > 0$, we obtain $0 < \beta'_{k+1} \leq \beta''_{k+1}$. Consequently, $\rho' \alpha'_{k+1} \beta'_{k+1} > \rho'' \alpha''_{k+1} \beta''_{k+1}$ and our hypothesis is true with $i = k + 1$.

Hence, $\rho' \alpha'_{n'} \beta'_{n'} > \rho'' \alpha''_{n'} \beta''_{n'}$ and we obtain $P'_{n'} \leq P''_{n'}$ due to Lemma 13. That concludes the proof. \square

The result for the Q -extreme ρ -NF solution can be proved similarly. We state the following theorem for that.

Theorem 15 Given $\rho' > \rho'' > 0$. Let $(P', Q'), (P'', Q'') \in S$ be respectively the Q -extreme ρ' -NF solution and the Q -extreme ρ'' -NF solution. Then $Q'' \leq Q'$.

As a consequence of Theorem 14, when ρ increases, the value of P in P -extreme ρ -NF solution decreases, and then the value of Q in P -extreme ρ -NF solution increases due to its Pareto efficiency. Furthermore, according to Theorem 15, when ρ increases, the value of Q in Q -extreme ρ -NF solution also increases, and then the value of P in Q -extreme ρ -NF solution decreases. Consequently, when ρ increases, the P -extreme ρ -NF solution is closer to the Q -extreme ρ -NF solution with respect to the values of P and Q . Notice that if the P -extreme ρ -NF solution and the Q -extreme ρ -NF solution are identical, we obtain a ρ -NF solution having the smallest value of P and the smallest value of Q simultaneously. Thus, it is necessarily a unique ρ -NF solution. Consequently, the extreme ρ -NF solutions will coincide with the unique ρ -NF solution when ρ is large enough. Similarly, we also have this conclusion when ρ is small enough.

5 | COMPUTATIONAL RESULTS

5.1 | Bi-Objective Travelling Salesman Problem (BOTSP)

5.1.1 | Definition and modeling

Let us first restate the BOTSP introduced in [24]. The BOTSP is a variant of the TSP where we find a tour simultaneously minimizing the total cost and the max-min distance. The latter is the difference between the maximum edge cost and the minimum one in the tour.

Note that the problem minimizing only the max-min distance, called the Balanced TSP (BTSP), has been introduced by Larusic and Punnen [20] for finding Hamiltonian tours in several cases where the equitable distribution of edges is important, for example, the nozzle guide vane assembly problem [26], and the cyclic workforce scheduling problem [37].

Given an undirected graph $G = (V, E)$ where $|V| = n$, $|E| = m$, $c_e \in \mathbb{R}_+ \setminus \{0\}$ is a cost associated with every edge $e \in E$ and $\Pi(G) \neq \emptyset$ denote the set of all Hamiltonian cycles in G . The BOTSP can be formally formulated as

$$\min_{H \in \Pi(G)} P = \sum_{e \in H} c_e \quad (8a)$$

$$\min_{H \in \Pi(G)} Q = \max_{e \in H} c_e - \min_{e' \in H} c_{e'} \quad (8b)$$

For solving $\mathcal{P}_\rho(\alpha, \beta)$, we present a MIP formulation as follows:

$$\min \rho\alpha P + \beta Q \quad (9a)$$

$$\text{s.t.} \quad \sum_{e \in \delta(v)} x_e = 2 \quad \forall v \in V \quad (9b)$$

$$\sum_{e \in \delta(V')} x_e \geq 2 \quad \forall \emptyset \neq V' \subseteq V \quad (9c)$$

$$P = \sum_{e \in E} c_e x_e \quad (9d)$$

$$Q \geq u - l \quad (9e)$$

$$u \geq c_e x_e \quad \forall e \in E \quad (9f)$$

$$l \leq c_e x_e + (1 - x_e)M \quad \forall e \in E \quad (9g)$$

$$x_e \in \{0, 1\} \quad \forall e \in E \quad (9h)$$

where x_e is the binary variable representing the occurrence of e in the solution tour, $\delta(v)$ is the incident edges set of vertex v , $\delta(V')$ is the set of edges that have exactly one endpoint in V' , M is the largest edge cost (i.e., $M = \max_{e \in E} c_e$). Constraints (9b) are the degree constraints assuring that exactly two edges are incident to every vertex. Constraints (9c) are the subtour elimination constraints. These constraints represent the classical Held-Karp (linear programming) relaxation for TSP. Together with Constraints (9h), they assure that the solution is a tour.

To calculate the max-min distance Q , we determine the maximum and the minimum edge costs u and l in the solution tour. Constraints (9f) allow bounding u from below by the largest weight edge in the solution tour. Constraints (9g) allow bounding l from above by the minimum edge cost in the solution tour. Indeed, in the case where $x_e = 1$, Constraints (9g) guarantee that l is smaller than all the edge costs in the tour. Otherwise, when x_e equals 0, the largest

edge cost M assures the validity of Constraints (9g). As $\rho\alpha P + \beta Q$ is minimized, u and l will take the maximum and minimum edge costs values.

5.1.2 | Computational results on the instances of BOTSP

This section presents computational results on some instances of the above BOTSP. Let us denote NFTSP as the problem of finding extreme ρ -NF solutions for the BOTSP. We conduct several experiments aiming at solving the NFTSP with Algorithms 1 and 2 on rather small size instances from the TSPLIB [30]. We also solve the TSP and the BTSP in the same instances. Notice that they are the problems of minimizing the single objectives P and Q , which are convex combinations of two objectives. Then, the solutions for the three problems will be compared and commented on.

For solving three problems, the TSP, the BTSP, and the NFTSP, we design a simple branch-and-cut algorithm devoted to minimizing a linear objective function over the MIP formulation. Note that Constraints (9f) and (9g) are excluded for the TSP. We use CPLEX 12.10 to implement our branch-and-cut algorithm. Constraints (9c) are set as lazy cuts, which are generated only when being violated by some integer solution. For the BTSP and the NFTSP, we also have some specific branching rules for variable l inspired from some threshold algorithms [20], [21]. For the NFTSP, this branch-and-cut algorithm is used in each iteration of Algorithms 1 and 2 to solve the subproblem $\mathcal{P}_\rho(\alpha, \beta)$. All the experiments are conducted on a PC Intel Core i5-9500 3.00GHz with 6 cores and 6 threads.

TABLE 1 Optimal solutions for TSP and BTSP

Instance	TSP			BTSP		
	P	Q	Time	P	Q	Time
burma14	3323	472	0.06	4986	134	0.03
ulysses16	6859	1452	0.14	13988	868	0.05
gr17	2085	311	0.04	4310	119	0.07
gr21	2707	328	0.01	8645	115	0.10
ulysses22	7013	1490	10.17	19001	868	0.09
gr24	1272	83	0.01	3926	33	0.18
fri26	937	118	0.02	2460	21	0.24
bays29	2020	140	0.04	6810	38	0.62
bayg29	1610	86	0.03	4210	29	0.59

TABLE 2 P -extreme ρ -NF solutions for BOTSP

Instance	$\rho = 1$				$\rho = \log_2 n$				$\rho = 1/\log_2 n$			
	P	Q	Time	Iters	P	Q	Time	Iters	P	Q	Time	Iters
burma14	4986	134	0.17	4	3558	294	0.11	3	4986	134	0.36	3
ulysses16	7047	1399	0.80	3	6859	1452	0.37	2	13670	868	0.72	3
gr17	2227	234	1.43	3	2090	262	0.20	3	4029	119	1.01	4
gr21	2989	278	0.63	3	2709	326	0.05	3	5945	120	0.76	3
ulysses22	7070	1471	67.31	3	7013	1490	26.79	2	18613	868	3.64	4
gr24	1282	81	0.73	3	1272	83	0.10	2	3847	33	11.85	4
fri26	980	82	1.29	3	953	91	0.26	3	2447	21	14.74	4
bays29	3449	59	50.52	5	2020	140	0.24	2	5384	40	75.78	4
bayg29	1817	63	5.40	3	1610	86	0.26	2	4210	29	37.26	4

TABLE 3 Q -extreme ρ -NF solutions for BOTSP

Instance	$\rho = 1$				$\rho = \log_2 n$				$\rho = 1/\log_2 n$			
	P	Q	Time	Iters	P	Q	Time	Iters	P	Q	Time	Iters
burma14	4986	134	0.06	2	4901	142	0.19	3	4986	134	0.33	2
ulysses16	13670	868	0.95	2	6859	1452	0.83	4	13670	868	0.06	2
gr17	3346	139	0.30	3	2090	262	0.61	4	4029	119	0.73	3
gr21	5945	120	0.46	3	2709	326	1.20	4	5945	120	0.42	3
ulysses22	7070	1471	74.37	4	7013	1490	51.37	4	18613	868	0.10	3
gr24	3847	33	2.42	2	1272	83	0.44	4	3847	33	0.04	2
fri26	2447	21	0.25	2	953	91	8.37	5	2447	21	0.10	3
bays29	4558	44	10.05	3	2093	116	29.92	5	6714	38	0.93	3
bay29	3246	35	20.84	4	1610	86	3.20	5	4210	29	0.36	2

TABLE 4 Aggregate proportional changes when switching from the P -extreme ρ -NF solutions (resp. Q -extreme ρ -NF solutions) to the solutions for the TSP (resp. the BTSP)

Instance	P -extreme ρ -NF solutions vs solutions of the TSP			Q -extreme ρ -NF solutions vs solutions of the BTSP		
	$\rho = 1$	$\rho = \log_2 n$	$\rho = 1/\log_2 n$	$\rho = 1$	$\rho = \log_2 n$	$\rho = 1/\log_2 n$
burma14	218.8%	35.4%	243.5%	0%	0.9%	0%
ulysses16	1.1%	0%	54.8%	2.3%	375.5%	0.6%
gr17	26.5%	17.7%	149.5%	14.4%	379.6%	1.7%
gr21	8.5%	0.3%	160.9%	41.2%	897.7%	6.9%
ulysses22	0.4%	0%	57.7%	127.7%	720.5%	0.4%
gr24	1.6%	0%	136.9%	2.1%	896.4%	0.4%
fri26	39.5%	21.7%	448.7%	0.5%	666.3%	0.1%
bays29	95.8%	0%	237.1%	35.7%	1027.6%	0.3%
bay29	25.1%	0%	183.8%	12.5%	718.2%	0%

We assume that the CDM considers three different values of ρ , respectively 1 , $\log_2 n$ and $1/\log_2 n$ where n is the number of nodes in graph G . Notice that the selection for the values of ρ depends on the CDM's own strategy, and it does not affect the Pareto set. In case $\rho = 1$, the two objectives P and Q are equivalently important. In case $\rho = \log_2 n$ (resp. $\rho = 1/\log_2 n$), the CDM gives priority to the value of P (resp. the value of Q) by the relative importance $\log_2 n$. We show the optimal solutions for the TSP and the BTSP in Table 1. Table 2 and Table 3 show respectively the P -extreme ρ -NF solutions and the Q -extreme ρ -NF solutions for NFTSP in several instances from the TSPLIB with a range of nodes from 14 to 29. We provide the number of iterations for the NFTSP in the columns "Iters."

We can see by the values of P and Q in these tables that the P -extreme ρ -NF and Q -extreme ρ -NF solutions for the BOTSP normally strike a better trade-off between two objectives: the total cost and the max-min distance comparing with those in the optimal solutions for the TSP and the BTSP. For example, when $\rho = 1$, the P -extreme ρ -NF solutions (resp. the Q -extreme ρ -NF solutions) offer a better alternative than the optimal solutions of the TSP (resp. the BTSP) with a significant drop on the values of Q (resp. P) and a slight growth on the values of P (resp. Q). More precisely, Table 4 shows the aggregate proportional changes (in percentage) when switching from the P -extreme ρ -NF solutions (resp. Q -extreme ρ -NF solutions) to the solutions of the classical single objective TSP (resp. the BTSP). Notice that these proportional changes are always not negative, as (1) should be respected. Furthermore, the values much larger than 0 means that the corresponding P -extreme ρ -NF solutions (resp. Q -extreme ρ -NF solutions) are much preferable to the solutions of the TSP (resp. the BTSP) since they have then a better trade-off between P and Q in terms of proportional change. We see that when $\rho = 1/\log_2 n$, the P -extreme ρ -NF solutions are much preferable,

and when $\rho = \log_2 n$, the Q -extreme ρ -NF solutions are much preferable.

Table 2 and Table 3 also indicate that Algorithms 1 and 2 seem to converge quickly after only a maximum number of 5 iterations. In general, they terminate in a polynomial number of iterations. To prove this, we state the following lemma. Note that the result for Algorithm 2 can be proved similarly.

Lemma 16 *For the BOTSP, Algorithm 1 returns the P -extreme ρ -NF solution in a polynomial number of iterations.*

Proof We first show that the number of Pareto-optimal solutions for the BOTSP is at most $C_{C_n^2}^2 = \frac{1}{8}n(n^2 - 1)(n + 2)$ where n is the number of nodes. Let c_i^{max} and c_i^{min} be the maximum and the minimum edge cost in the tour corresponding to the Pareto-optimal solution (P_i, Q_i) then $Q_i = c_i^{max} - c_i^{min}$. For two distinct Pareto-optimal solutions $(P_i, Q_i), (P_j, Q_j)$, we have $Q_i \neq Q_j$ which is equivalent to $c_i^{max} - c_i^{min} \neq c_j^{max} - c_j^{min}$. Thus, $(c_i^{max}, c_i^{min}) \neq (c_j^{max}, c_j^{min})$. Consequently, each Pareto-optimal solution obtained by an iteration of Algorithm 1 has distinct edges corresponding to the maximum and minimum edge costs.

Note that for the graph G with n nodes, we have at most C_n^2 edges, and then the distinct pairs of edges are at most $C_{C_n^2}^2$. Hence, the number of Pareto-optimal solutions for the BOTSP is at most $C_{C_n^2}^2 = \frac{1}{8}n(n^2 - 1)(n + 2)$, which implies that Algorithm 1 terminates in a polynomial number of iterations due to the proof of Lemma 8. \square

When $\rho = \log_2 n$, the values of P in the obtained P -extreme ρ -NF are closer to those in the optimal solutions for the TSP compared to $\rho = 1$ and $\rho = 1/\log_2$. Moreover, we need fewer (resp. more) iterations for finding the P -extreme ρ -NF solutions when ρ becomes larger (resp. smaller). Similarly, when $\rho = 1/\log_2 n$, the values of Q in the obtained Q -extreme ρ -NF are closer to those in the optimal solutions for the BTSP compared to $\rho = \log_2 n$ and $\rho = 1$. We also need fewer (resp. more) iterations for finding the Q -extreme ρ -NF solutions when ρ becomes smaller (resp. larger).

In addition, when $\rho = \log_2 n$ and $\rho = 1/\log_2 n$, the P -extreme ρ -NF solutions and the Q -extreme ρ -NF solutions are identical in most of instances. Thus, we have a unique ρ -NF solution for such instances. Generally, as shown in Section 4, the extreme ρ -NF solutions will coincide with the unique ρ -NF solution when ρ is large enough or small enough. In some instances, the CPU time for solving the NFTSP is more than the CPU time spent for solving the TSP and the BTSP. A deeper analysis of the iterations of Algorithms 1 and 2 tells us that the smaller the value of α , the more time is needed for solving $\mathcal{P}_\rho(\alpha, \beta)$.

5.2 | Bi-Objective Shortest Path Problem (BOSPP)

5.2.1 | Definition and modeling

First, we restate the BOSPP that we utilized in Section 2.2. The BOSPP is a variant of the shortest path problem cited as the most common problem in combinatorial optimization [4] due to its numerous applications as the minimum cost path problem, the quickest path problem, the most reliable path problem, and various routing problems. As one of the most straightforward problems in multicriteria linear integer programming, the BOSPP refers to finding the shortest path between two nodes satisfying two objectives, such as "cheap and quickest", "cheap and more reliable", "cheap and energy efficient," and so on. More precisely, we describe the BOSPP as follows.

Let $G = (V, A)$ be a directed graph where V is the set of nodes ($|V| = n$) and A is the set of arcs ($|A| = m$). Let $c_{ij}, t_{ij} > 0$ denote respectively the cost and the time associated with each arc (i, j) . The BOSPP consists in finding a path H from a source node s to a target node t minimizing both criteria $\{ \sum_{(i,j) \in H} c_{ij}, \sum_{(i,j) \in H} t_{ij} \}$. Notice that although there exist efficient exact algorithms for solving the single-objective shortest path problem, the multi-objective case

(including the bi-objective case) is known to be NP-complete by transformation from a 0-1 knapsack problem [12], [16].

Let P and Q be the objective functions for the total cost and time corresponding to the s - t path. As shown in Algorithm 1, for finding the extreme ρ -NF solutions for the BOSPP, we need to solve $\mathcal{P}_\rho(\alpha, \beta)$ as a sequence of convex combinations of ρP and Q , i.e., including $(\alpha, \beta) = (1, 0), (0, 1)$. For each value of (α, β) , we have $\rho\alpha P + \beta Q = \sum_{(i,j) \in A} (\rho\alpha c_{ij} + \beta t_{ij}) x_{ij}$. Thus, we construct a graph G' with the same sets of nodes and arcs of G , i.e., $G' = (V, A)$, and $\rho\alpha c_{ij} + \beta t_{ij}$ is a unique value of cost associated with each arc (i, j) of G' . Hence, solving $\mathcal{P}_\rho(\alpha, \beta)$ is now equivalent to solving the single-objective shortest path problem in G' .

5.2.2 | Computational results on the instances of BOSPP

Let us denote NFSPP as the problem of finding extreme ρ -NF solutions for the BOSPP. We investigate the performance of the presented algorithms for the NFSPP on random NETMAKER networks [32]. NETMAKER networks generator was first proposed in [32] to test a label-correcting algorithm for the BOSPP. It was also used in [27], [28] to evaluate the performance of algorithms for the BOSPP. To generate a NETMAKER network, a random Hamiltonian cycle is first constructed to ensure that the network is strongly connected. Let $\{1, \dots, n\}$ be the network node set. For each node i , a random number of i 's outgoing arcs, which belongs to the interval $\{a_{\min}, \dots, a_{\max}\}$, is uniformly generated. Moreover, node i is only allowed to reach nodes j with $j \in \{i - \lfloor \frac{I_{node}}{2} \rfloor, i + \lfloor \frac{I_{node}}{2} \rfloor\}$ where I_{node} is the node interval. The cost and time associated with an arc are chosen randomly in two intervals $\{1, \dots, 33\}$ and $\{67, \dots, 100\}$ such that two values belong to different intervals. Table 5 describes the generated NETMAKER networks. Notice that for all the instances, the source node is 1, and the target node is n . To solve the single-objective shortest path problem, we use Dijkstra's algorithm [8] implemented in Networkx package version 2.5.1 [14]. All the experiments are conducted on a PC Intel Core i5-9500 3.00GHz with 6 cores and 6 threads.

TABLE 5 Random NETMAKER networks

Name	Nodes	I_{node}	Outgoing arcs		Arcs
			a_{\min}	a_{\max}	
N1.1	1000	20	5	15	8319
N1.2	1000	20	1	20	8290
N1.3	1000	50	5	15	9465
N5.1	5000	20	5	15	41438
N5.2	5000	20	1	20	41787
N10.1	10000	20	5	15	82785
N10.2	10000	20	1	20	83222
N20.1	20000	20	5	15	165702
N20.2	20000	20	1	20	166086

We assume that the CDM considers three different values of ρ , respectively 1, $1/2$, and 2. The optimal solutions for the problem minimizing P and the problem minimizing Q are shown in Table 6. Table 7 and Table 8 present respectively the P -extreme ρ -NF solutions and the Q -extreme ρ -NF solutions. We provide the number of iterations for solving the NFSPP in the columns "Iters."

Similar to the above results for the BOTSP, when $\rho = 1$, the P -extreme ρ -NF solutions (resp. the Q -extreme ρ -NF solutions) offer a better alternative than the optimal solutions of the problem minimizing P (resp. the problem minimizing Q) with a significant drop on the values of Q (resp. P) and a slight growth on the values of P (resp. Q). Table

TABLE 6 Optimal solutions of the problems minimizing P and Q

Instance	Min P			Min Q		
	P	Q	Time	P	Q	Time
N1.1	1521	14482	0.03	13815	1541	0.03
N1.2	1562	14443	0.03	14057	1516	0.03
N1.3	489	5539	0.03	6067	531	0.03
N5.1	7906	71535	0.17	70702	7902	0.18
N5.2	8385	70901	0.17	72431	7899	0.18
N10.1	16158	145440	0.44	139676	15728	0.42
N10.2	15904	139863	0.43	141761	16330	0.44
N20.1	31482	286335	1.56	287844	31645	1.66
N20.2	32559	285324	1.56	285408	33389	1.72

TABLE 7 P -extreme ρ -NF solutions for BOSPP

Instance	$\rho = 1$				$\rho = 1/2$				$\rho = 2$			
	P	Q	Time	iter	P	Q	Time	Iters	P	Q	Time	Iters
N1.1	1629	12339	0.12	4	13089	1564	0.40	13	1556	13097	0.10	3
N1.2	1652	12199	0.12	4	2190	10069	0.17	6	1594	12952	0.12	4
N1.3	496	5166	0.11	3	650	4335	0.17	5	496	5166	0.10	3
N5.1	8655	59520	1.10	6	63466	8083	2.27	13	8141	64727	0.90	5
N5.2	9383	58079	1.06	6	65438	8049	1.92	11	8519	65782	0.71	4
N10.1	17625	117018	2.08	5	126963	16072	5.29	13	16681	126573	2.99	7
N10.2	17127	118037	1.66	4	127784	16777	5.55	14	16237	127635	2.13	5
N20.1	34321	234978	7.15	5	253918	32603	17.62	13	32413	253953	7.68	5
N20.2	35186	239017	9.17	6	255794	34265	18.20	13	33307	258202	6.68	4

9 also shows the aggregate proportional changes (in percentage) when switching from the P -extreme ρ -NF solutions (resp. Q -extreme ρ -NF solutions) to the solutions minimizing P (resp. the solutions minimizing Q). When $\rho = 1/2$, the P -extreme ρ -NF solutions are much preferable to the solution minimizing P , and when $\rho = 2$, the Q -extreme ρ -NF solutions are much preferable to the solutions minimizing Q .

Generally, the number of iterations when using Algorithms 1 and 2 for the BOSPP is more than for the BOTSP. It may be challenging to determine a good upper bound on the number of iterations for Algorithms 1 and 2 since the values of P and Q in the BOSPP are independent (unlike the BOTSP). However, the quick CPU time spent, even for some instances with 20000 nodes, shows our algorithms' efficiency combined with Dijkstra's algorithm for solving the NFSP.

When $\rho = 2$, the values of P in the obtained P -extreme ρ -NF are closer to those in the optimal solutions for the problem minimizing P comparing to $\rho = 1$ and $\rho = 1/2$. In most cases, we need fewer (resp. more) iterations for finding the P -extreme ρ -NF solutions when ρ becomes larger (resp. smaller). Similarly, when $\rho = 1/2$, the values of Q in the obtained Q -extreme ρ -NF are closer to those in the optimal solutions for the problem minimizing Q compared to $\rho = 2$ and $\rho = 1$. We also need fewer (resp. more) iterations for finding the Q -extreme ρ -NF solutions when ρ becomes smaller (resp. larger).

When $\rho = 2$ and $\rho = 1/2$, the P -extreme ρ -NF solutions and the Q -extreme ρ -NF solutions are identical in most of instances. Thus, there is a unique ρ -NF solution for such instances.

While it is generally difficult for the CDM to justify the choice of a particular Pareto optimal solution in the Pareto set, the ρ -NF solutions (in particular, the extreme ρ -NF solutions) offer a justification based on the concept of

TABLE 8 Q -extreme ρ - NF solutions for BOSPP

Instance	$\rho = 1$				$\rho = 1/2$				$\rho = 2$			
	P	Q	Time	Iters	P	Q	Time	Iters	P	Q	Time	Iters
N1.1	11778	1688	0.12	4	13089	1564	0.09	3	1556	13097	0.45	15
N1.2	12506	1609	0.13	4	13438	1525	0.10	3	9786	2366	0.21	7
N1.3	4987	565	0.10	3	5093	558	0.10	3	4713	598	0.10	3
N5.1	59944	8407	0.71	4	63466	8083	0.90	5	8151	64568	2.59	15
N5.2	59608	8661	1.05	6	65438	8049	0.71	4	8519	65782	2.25	13
N10.1	118773	16855	1.99	5	126963	16072	1.67	4	16681	126573	5.13	13
N10.2	118563	17777	2.48	6	127784	16777	2.07	5	16237	127635	5.07	13
N20.1	235196	34504	7.20	5	253918	32603	7.41	5	32413	253953	17.71	13
N20.2	235068	36483	13.48	9	255794	34265	7.67	5	33313	258109	16.86	12

TABLE 9 Aggregate proportional changes when switching from the P -extreme ρ - NF solutions (resp. Q -extreme ρ - NF solutions) to the solutions minimizing P (resp. the solutions minimizing Q)

Instance	P -extreme ρ - NF solutions vs solutions minimizing P			Q -extreme ρ - NF solutions vs solutions minimizing Q		
	$\rho = 1$	$\rho = 1/2$	$\rho = 2$	$\rho = 1$	$\rho = 1/2$	$\rho = 2$
burma14	10.7%	781.7%	6.1%	8.6%	1.3%	1487.4%
ulysses16	12.9%	29.1%	7.5%	6.6%	1.7%	51.3%
gr17	5.8%	15.3%	4.4%	15.6%	4.7%	46.2%
gr21	11.5%	741.2%	4.7%	11.9%	3.4%	1447.1%
ulysses22	11.4%	737.2%	4.6%	12.7%	3.5%	1412.4%
gr24	15.9%	761.3%	8.6%	10.9%	2.8%	1387.1%
fri26	11.3%	689.8%	5.4%	11.4%	2.8%	1458.9%
bays29	13.5%	734.4%	7%	14.1%	3.7%	1488.5%
bayg29	11.9%	689.1%	6%	12.9%	3.2%	1426.4%

proportional fairness. One interesting property of proportional fairness is that it can be used as a standard comparison for two solutions, even when the two objectives have different orders of magnitude. On the computational side, the CDM can use the algorithms proposed in our paper to obtain the extreme NF solutions. Moreover, the CDM can prioritize the value of P (resp. Q) over the one of Q (resp. P) by calibrating a value of ρ to achieve the ρ - NF solutions reflecting this priority.

Notice that the concept of ρ - NF solution can be generalized for multi-objective optimization problems with more than two objectives. In this setting, the ρ - NF solution should be such that, when compared to any other feasible solutions, the aggregate proportional change of the objectives is not negative. The challenge is to find an equivalent concept of extreme ρ - NF solutions that can be efficiently computed, as shown in our paper for the case of bi-objective minimization problems. This will be left for future work.

6 | CONCLUSION

In this paper, we have utilized proportional Nash equilibrium in the context of *Bi-Objective Minimization* (BOM) to propose a novel criterion for selecting the preferred solutions in the Pareto set. For this purpose, we have generalized the concept of *Nash Fairness* (NF) solution introduced in our conference papers, called ρ - NF solution, by adding a parameter $\rho > 0$, which reflects the relative importance of one objective to the other one. We have first proven the existence of ρ - NF solutions for BOM. Significantly, the ρ - NF solution set can be a strict subset of the Pareto set.

Second, we have designed two algorithms to find extreme ρ - NF solutions for BOM. Based on the Weighted Sum Method, these algorithms only require minimizing a sequence of linear combinations of the objectives. In addition, computational results conducted on some instances of the BOTSP and the BOSPP have shown the efficiency of our algorithms as they seem to converge quickly, and the obtained extreme ρ - NF solutions have a better trade-off between the two objectives.

Future work should establish an effective upper bound on the number of iterations in our algorithms and show that the ρ - NF solution set is generally much smaller than the Pareto set. Additionally, we can generalize the ρ - NF solution concept for multi-objective optimization, where the number of objectives is not limited to two, and the objectives can be either maximized or minimized.

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APPENDIX

| Appendix 1: Young's inequality for products

We restate the proof of Young's inequality for products as follows.

Theorem 15 Suppose $a, b > 0, \alpha, \beta \geq 0$ and $\alpha + \beta = 1$ then $\alpha a + \beta b \geq a^\alpha b^\beta$.

Proof Since the natural logarithmic function is concave, using Jensen's inequality, we have

$$\log(\alpha a + \beta b) \geq \alpha \log(a) + \beta \log(b) = \log(a^\alpha b^\beta),$$

Thus, $\alpha a + \beta b \geq a^\alpha b^\beta$. Furthermore, the equality holds if and only if $a = b$ or $(\alpha, \beta) \in \{(0, 1), (1, 0)\}$. □