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Abstract

In this paper, we give estimates of the quadratic transportation cost in the conditional central limit theorem for a large class of dependent sequences. Applications to irreducible Markov chains, dynamical systems generated by intermittent maps and τ-mixing sequences are given.

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Running head: Quadratic transportation cost in the conditional CLT

1 Introduction

Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of real-valued random variables (r.v.) with mean zero and finite variance. Set $S_n = X_1 + X_2 + \cdots + X_n$. By $P_{n^{-1/2}S_n}$ we denote the law of $n^{-1/2}S_n$ and by G_{σ^2} the normal distribution $N(0,\sigma^2)$. In this paper, we assume furthermore that the series $\sigma^2 = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0, X_k)$ is convergent (under this assumption $\lim_n n^{-1}\mathrm{Var}S_n = \sigma^2$) and we shall give quantitative estimates of the approximation of $P_{n^{-1/2}S_n}$ by G_{σ^2} in terms of the quadratic cost, which is the square of the $\mathbb{L}^2$-minimal distance. With this aim, we first recall the definition of the $\mathbb{L}^p$-minimal metrics.

*Jérôme Dedecker, Université de Paris, CNRS, MAP5, UMR 8145, 45 rue des Saints-Pères, F-75006 Paris, France.

†Florence Merlevède, LAMA, Univ Gustave Eiffel, Univ Paris Est Créteil, UMR 8050 CNRS, F-77454 Marne-La-Vallée, France.

‡Emmanuel Rio, Université de Versailles, LMV UMR 8100 CNRS, 45 avenue des Etats-Unis, F-78035 Versailles, France.

Let $\mathcal{L}(\mu,\nu)$ be the set of probability laws on $\mathbb{R}^2$ with marginals μ and ν. For $p \geq 1$, let

$$W_p(\mu,\nu) = \inf\Big\{\Big(\int |x-y|^p P(dx,dy)\Big)^{1/p} : P \in \mathcal{L}(\mu,\nu)\Big\}\,.$$

W_p is usually called the $\mathbb{L}^p$-minimal distance, and sometimes the Wasserstein distance of order p. It is well known that for probability laws μ and ν on $\mathbb{R}$ with respective distributions functions (d.f.) F and G,

$$W_p(\mu,\nu) = \Big(\int_0^1 |F^{-1}(u) - G^{-1}(u)|^p du\Big)^{1/p}\,, \tag{1.1}$$

where F^{-1} and G^{-1} denote respectively the generalized inverse functions of F and G. We refer to Chapter 6 in Villani [25] for the properties of this metric.

For $(X_i)_{i\in\mathbb{Z}}$ a sequence of independent and identically distributed (iid) centered real valued random variables in $\mathbb{L}^4$, with variance σ^2, inequality (1.7) in Rio [21] states that there exists a universal constant c such that for any positive integer n

$$nW_2^2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \leq c\,\sigma^{-2}\|X_1\|_4^4\,. \tag{1.2}$$

In addition, it is also shown in the same paper that this upper bound is optimal. More precisely, for any $\kappa \geq 1$, let $\mathcal{M}(4,\kappa)$ be the class of the probability measures μ on the real line such that $\int x d\mu(x) = 0$, $\int x^2 d\mu(x) = 1$ and $\int x^4 d\mu(x) = \kappa$. In case where $(X_i)_{i\in\mathbb{Z}}$ is a sequence of iid random variables with common law μ in $\mathcal{M}(4,\kappa)$, Theorem 5.1 in [21] asserts that

$$\sup_{\mu\in\mathcal{M}(4,\kappa)} \liminf_{n\to\infty} nW_2^2(P_{S_n/\sqrt{n}}, G_1) \geq \kappa/12\,. \tag{1.3}$$

We refer to Bobkov [2] for another proof of (1.2) based on relative entropy and Talagrand's entropy-transport inequality. Actually, the following more general result holds: for any $p \geq 1$, there exists a universal constant c_p such that for any positive integer n,

$$n^{p/2}W_p^p(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \leq c_p\,\sigma^{-p}\|X_1\|_{p+2}^{p+2}\,.$$

(see Rio [21] for $p \in [1,2]$ and Bobkov [3] for $p > 2$). Extensions to random vectors in $\mathbb{R}^d$ are given in Bonis [5]. We also mention the extensions of the upper bound (1.2) to the m-dependent case and to U-statistics obtained by Fang [12].

In this paper, one of our motivations is to relax the independence assumption and to find sufficient conditions in case of dependent sequences ensuring that

$$W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) = O(n^{-1/2})\,. \tag{1.4}$$

In the dependent setting, a well known class is the class of irreducible aperiodic and positively recurrent Markov chains (ξ_n) with an atom denoted by A (see the definition

page 286 in [4]). Let π be the unique invariant distribution of the Markov chain. From now on, (ξ_n) will be the Markov chain starting from π. Let us then consider the strictly stationary sequence (X_k) defined by $X_k = f(\xi_k)$ with f a bounded function such that $\pi(f) = 0$. In view of the regeneration scheme and the upper bound (1.2), one can conjecture that (1.4) holds for $S_n = \sum_{k=1}^n X_k$ provided that $\mathbb{E}_A(\tau_A^4) < \infty$ where τ_A is the first return time in A and $\mathbb{E}_A$ stands for the expectation under $\mathbb{P}_x$ for $x \in A$. Next, from [4, Lemma 3] and [22, page 165], it is known that $\mathbb{E}_A(\tau_A^4) < \infty$ is equivalent to

$$\sum_{n>0} n^2 \alpha_n < \infty \,, \tag{1.5}$$

where $\alpha_n = \frac{1}{4} \sup_{\|f\|_\infty \le 1} \|\mathbb{E}(f(\xi_n)|\xi_0) - \mathbb{E}(f(\xi_n))\|_1$.

In this paper we shall prove that (1.4) holds true for any stationary sequence $(X_k)_{k\in\mathbb{Z}}$ of bounded real-valued random variables satisfying (1.5) for the sequence $(\alpha_n)_{n\ge 0}$ of strong mixing coefficients in the sense of Rosenblatt (see for instance [17, Section 5.1.1.] for a definition of these coefficients in the general case), which includes the case of Markov chains described above. This will be a consequence of a more general result also valid for a class of weakly dependent sequences, which may fail to be strongly mixing. In order to give more precise statements of our results, let us now introduce the dependence coefficients that we will use in this paper.

Definition 1.1. Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of bounded real-valued random variables and $\mathcal{F}_0 = \sigma(X_i, i \le 0)$. Let $\Gamma_{p,q} = \{(a_i)_{1\le i\le p} \in \mathbb{N}^p : a_1 \ge 1 \text{ and } \sum_{i=1}^p a_i \le q\}$, for p and q positive integers. For $k \ge 0$, set

$$\theta_{X,p,q}(k) = \sup_{\substack{k_p > k_{p-1} > \ldots > k_2 > k_1 \ge k \\ (a_1,\ldots,a_p)\in\Gamma_{p,q}}} \Big\| \mathbb{E}\Big(\prod_{i=1}^p X_{k_i}^{a_i} | \mathcal{F}_0 \Big) - \mathbb{E}\Big(\prod_{i=1}^p X_{k_i}^{a_i} \Big) \Big\|_1 .$$

As a consequence of our Theorem 2.1, we will obtain that if

$$\sum_{k\ge 1} k^2 \theta_{X,4,4}(k) < \infty \,, \tag{1.6}$$

then (1.4) holds, which immediately implies that (1.4) holds for additive bounded functionals of a Markov chain satisfying (1.5). In fact we shall give a conditional version of (1.4) and show that when $(X_k)_{k\in\mathbb{Z}}$ is a stationary sequence of centered and bounded real-valued random variables satisfying (1.6) then

$$\mathbb{E}\big(W_2^2(P_{S_n/\sqrt{n}|\mathcal{F}_0}, G_{\sigma^2})\big) = O(n^{-1}) \,. \tag{1.7}$$

Note that in case of bounded functions of a Markov chain $(\xi_k)_k$ satisfying $\mathbb{E}_A(\tau^4) < \infty$, with invariant distribution π, the Schwarz inequality together with (1.7) imply that

$$\mathbb{E}_\mu\big(W_2(P_{S_n/\sqrt{n}|\xi_0}, G_{\sigma^2})\big) = O(n^{-1/2})$$

for any positive measure μ such that $d\mu = f d\pi$ with $\int f^2 d\pi < \infty$. Above $\mathbb{E}_\mu$ stands for the expectation of the chain under the initial law μ.

It is noteworthy to indicate that (1.7) implies (1.4). Indeed the following fact is valid.

Fact 1.1. *Let X and Y be two random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{F}$ be a sub σ-algebra of $\mathcal{A}$. Then $W_2^2(P_X, P_Y) \le \mathbb{E}\big(W_2^2(P_{X|\mathcal{F}}, P_{Y|\mathcal{F}})\big)$.*

To see this, let U be a random variable with uniform distribution over $[0,1]$, independent of $\mathcal{F}$, and let $F_{X|\mathcal{F}}$ and $F_{Y|\mathcal{F}}$ denote respectively the conditional distribution functions of X and Y given $\mathcal{F}$. Set $X^* = F^{-1}_{X|\mathcal{F}}(U)$ and $Y^* = F^{-1}_{Y|\mathcal{F}}(U)$. Then X^* has the law P_X, Y^* has the law P_Y and, by (1.1), $W_2^2(P_{X|\mathcal{F}}, P_{Y|\mathcal{F}}) = \mathbb{E}\big(|X^* - Y^*|^2 | \mathcal{F}\big)$. Taking the expectation, it implies the above fact, since W_2^2 is the minimal quadratic cost.

To prove Theorem 2.1, we shall apply Lindeberg's method, which was used by Billingsley [1] and Ibragimov [13] in the case of martingales with stationary differences to prove the central limit theorem (we also consider this particular case in our Theorem 2.2). Note that this method was adapted to a large class of dependent sequences (non necessarily martingale differences) to evaluate the $\mathbb{L}^1$-minimal distance between $P_{S_n/\sqrt{n}}$ and G_{σ^2}, by Pène [19] in the bounded multidimensional case, and next by Dedecker and Rio [10] in the unbounded case (under conditions involving some coefficients similar to $\theta_{X,4,3}$, or weak mixing coefficients such as those described in Definition 3.1 below). Recently, estimates of the $\mathbb{L}^1$-minimal distance between $P_{S_n/\sqrt{n}}$ and G_{σ^2} when the underlying process is a function of iid random variables are given in Theorem 3.1 in [14]. Their conditions are expressed in terms of some coupling coefficients.

Our paper is organized as follows. Section 2 is devoted to the statements of upper bounds concerning the quadratic transportation cost in the conditional central limit theorem and their applications to pointwise estimates for the distribution function of the normalized sums and its generalized inverse. Applications to α-dependent sequences, τ-mixing sequences and symmetric random walk in the circle are given in Section 3. The proofs are postponed to Section 4. Links between $|F^{-1}_{S_n/\sigma_n}(u) - \Phi^{-1}(u)|$ and $W_2(P_{S_n/\sigma_n}, G_1)$ are given in Section 5, where $\sigma_n = \sqrt{\mathrm{Var} S_n}$, Φ^{-1} is the inverse of the distribution function of the standard normal distribution and $F^{-1}_{S_n/\sigma_n}$ is the generalized inverse of the distribution function of S_n/σ_n.

In the rest of the paper, we shall use the following notation: for two sequences $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ of positive reals, $a_n \ll b_n$ means there exists a positive constant C not depending on n such that $a_n \le C b_n$ for any $n \ge 1$. Moreover, for a real-valued random variable X in $\mathbb{L}^1$, the notation $X^{(0)}$ means $X - \mathbb{E}(X)$.

2 Quadratic cost in the conditional CLT

The main result of this paper is Theorem 2.1 below.

Theorem 2.1. *Assume that* $\|X_0\|_\infty \le M$ *and that* $\sum_{k\ge1}\theta_{X,2,2}(k) < \infty$. *Then* $\sigma^2 = \mathbb{E}(X_0^2) + 2\sum_{k\ge1}\mathbb{E}(X_0X_k)$ *converges and*

$$\mathbb{E}\big(W_2^2(P_{S_n/\sqrt{n}|\mathcal{F}_0}, G_{\sigma^2})\big) \ll n^{-1/2}\Big(1 + \sum_{k\ge1}(k\wedge\sqrt{n})\theta_{X,2,2}(k)\Big)\,. \tag{a}$$

If furthermore $\sum_{k\ge1} k\theta_{X,4,4}(k) < \infty$, *then*

$$\mathbb{E}\big(W_2^2(P_{S_n/\sqrt{n}|\mathcal{F}_0}, G_{\sigma^2})\big) \ll n^{-1}\Big(1 + \sum_{k\ge1}k(k\wedge\sqrt{n})\theta_{X,4,4}(k)\Big)\,. \tag{b}$$

Comment 2.1. Item (a) provides a rate in the CLT for the W_2-metric as soon as $\sum_{k\ge1}\theta_{X,2,2}(k) < \infty$. In addition, if $\sum_{k\ge1} k\theta_{X,2,2}(k) < \infty$, then the rate in the W_2-metric is of order $n^{-1/4}$. Furthermore, by Item (b), if $\sum_{k\ge1} k\theta_{X,4,4}(k) < \infty$, then the rate in the CLT for the W_2-metric is $o(n^{-1/4})$. For example, if $\theta_{X,4,4}(k) = O(k^{-a})$ with $a \in]1,3[$ and $a \ne 2$, Theorem 2.1 implies that $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-(a-1)/4}$. Moreover $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-1/2}$ as soon as $\sum_{k\ge1} k^2\theta_{X,4,4}(k) < \infty$.

Comment 2.2. Assume $\sigma > 0$. Set $\sigma_n = \sqrt{\mathrm{Var}S_n}$. If $\sigma > 0$, then $\sigma_n > 0$ for any positive n. Set $\kappa_2 = \mathbb{E}\big(W_2^2(P_{S_n/\sigma\sqrt{n}|\mathcal{F}_0}, P_{S_n/\sigma_n|\mathcal{F}_0})\big)$:

$$\kappa_2 = \Big(\frac{\sigma_n}{\sigma\sqrt{n}} - 1\Big)^2 \le \Big(\frac{\sigma_n^2}{n\sigma^2} - 1\Big)^2 = (n\sigma^2)^{-1}\Big|\frac{\sigma_n^2}{n\sigma^2} - 1\Big|\,\Big|2\sum_{k\ge1}(k\wedge n)\mathrm{Cov}(X_0, X_k)\Big|.$$

Now, from the definition of the coefficients $\theta_{X,1,1}(k)$,

$$\sum_{k\ge1}(k\wedge n)|\mathrm{Cov}(X_0, X_k)| \le \|X_0\|_\infty \sum_{k\ge1}(k\wedge n)\theta_{X,1,1}(k). \tag{2.1}$$

Therefore, if in addition $\|X_0\|_\infty \le M$, $\kappa_2 \ll n^{-1}M\|_\infty\sum_{k\ge1}(k\wedge n)\theta_{X,1,1}(k)$, which is always of a smaller order than the upper bounds (a) and (b). Hence Theorem 2.1 also holds for $\mathbb{E}\big(W_2^2(P_{S_n/\sigma_n|\mathcal{F}_0}, G_1)\big)$.

We now give applications of Theorem 2.1 to pointwise estimates. We start by Berry-Esseen type estimates. Arguing for instance as in [8, Remark 2.4], Theorem 2.1 together with Comment 2.2 imply the following upper bound.

Corollary 2.1. *Assume that* $\sigma > 0$, $\|X_0\|_\infty \le M$ *and* $\sum_{k\ge1} k^2\theta_{X,4,4}(k) < \infty$. *Then*

$$\Delta_n = \sup_{x\in\mathbb{R}}\big|\mathbb{P}(S_n/\sigma_n \le x) - \Phi(x)\big| \ll n^{-1/3}\,.$$

We now give applications of our main result to estimates of the quantiles and the superquantiles of S_n/σ_n in the nondegenerate case. Define the 1-risk $Q_{1,X}$ of X, as in Pinelis [20], by

$$Q_{1,X}(u) = \frac{1}{u}\int_0^u F_X^{-1}(1-t)dt. \tag{2.2}$$

Then $Q_{1,X}(u)$ is the value of the superquantile of X at point $(1-u)$. The corollary below, which is a consequence of Theorem 2.1 and Proposition 5.1 provides estimates of the accuracy in the central limit theorem for $F^{-1}_{S_n/\sigma_n}$ and $Q_{1,S_n/\sigma_n}$. Its proof is given in Section 5.

Corollary 2.2. *Assume that $\|X_0\|_\infty \le M$, $\sum_{k\ge 1} k^2\theta_{X,4,4}(k) < \infty$ and $\sigma^2 > 0$. Let Y be a standard normal. Then there exists some constant $C > 0$ such that, for any $n \ge 1$ and any u in $(0,1)$,*

$$|F^{-1}_{S_n/\sigma_n}(u) - \Phi^{-1}(u)| \le C\max\big((nu(1-u))^{-1/2}, (nu(1-u))^{-1/3}|\log(u(1-u))|^{-1/6}\big) \quad (a)$$

and

$$|Q_{1,S_n/\sigma_n}(u) - Q_{1,Y}(u)| \le C(nu)^{-1/2}\sqrt{1-u}. \tag{b}$$

Comment 2.3. From Corollary 2.2(a), for any sequence $(\varepsilon_n)_n$ of reals in $(0,1/2)$ such that $\lim_n \varepsilon_n = 0$ and $\lim_n n\varepsilon_n = \infty$,

$$\lim_{n\to\infty} \sup_{u\in[\varepsilon_n, 1-\varepsilon_n]} |F^{-1}_{S_n/\sigma_n}(u) - \Phi^{-1}(u)| = 0,$$

which can not be deduced from a Berry-Esseen type bound with the rate $n^{-1/2}$. Indeed, if Δ_n is defined as in Corollary 2.1, one can only get that

$$|F^{-1}_{S_n/\sigma_n}(u) - \Phi^{-1}(u)| \le \Phi^{-1}(\min(1, u+\Delta_n)) - \Phi^{-1}(u)$$

for $u \ge 1/2$, which is of interest only if $u < 1 - \Delta_n$.

If furthermore the sequence $(X_i)_{i\in\mathbb{Z}}$ is a sequence of martingale differences, then the conditions on the dependence coefficients can be weakened as follows (the proof being less intricate is left to the reader).

Theorem 2.2. *Assume that $(X_i)_{i\in\mathbb{Z}}$ is a sequence of martingale differences such that $\|X_0\|_\infty \le M$ and $\mathbb{E}(X_0^2) = \sigma^2$. Then*

$$\mathbb{E}\big(W_2^2(P_{S_n/\sqrt{n}|\mathcal{F}_0}, G_{\sigma^2})\big) \ll n^{-1/2}\Big(1 + \sum_{k=1}^{[\sqrt{n}]}\theta_{X,1,2}(k)\Big). \tag{a}$$

If furthermore $\sum_{k\ge1}\theta_{X,3,4}(k) < \infty$, then

$$\mathbb{E}\big(W_2^2(P_{S_n/\sqrt{n}|\mathcal{F}_0}, G_{\sigma^2})\big) \ll n^{-1}\Big(1 + \sum_{k\ge1}(k\wedge\sqrt{n})\theta_{X,3,4}(k)\Big). \tag{b}$$

Comment 2.4. Item (a) provides a rate in the CLT as soon as $\theta_{X,1,2}(k) = o(1)$. If $\theta_{X,1,2}(k) = O(k^{-a})$ with a in $(0,1)$, (a) ensures that $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-a/4}$. If $\sum_{k\geq 1} \theta_{X,1,2}(k) < \infty$, then the rate is of order $n^{-1/4}$. Item (b) provides faster rates under the condition $\sum_{k\geq 1} \theta_{X,3,4}(k) < \infty$. Indeed the rate of convergence under this condition is $o(n^{-1/4})$. If $\theta_{X,3,4}(k) = O(k^{-a})$ with a in $(1,2)$, (b) ensures that $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-a/4}$. Moreover $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-1/2}$ as soon as $\sum_{k\geq 1} k\theta_{X,3,4}(k) < \infty$.

3 Examples

3.1 α-mixing sequences

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $\mathcal{U}$ and $\mathcal{V}$ be two σ-algebras of $\mathcal{A}$. The strong mixing coefficient $\alpha(\mathcal{U}, \mathcal{V})$ between these σ-algebras is defined as follows:

$$\alpha(\mathcal{U}, \mathcal{V}) = \sup\{|\ \mathbb{P}(U \cap V) - \mathbb{P}(U)\mathbb{P}(V)| : U \in \mathcal{U}, V \in \mathcal{V}\}\,.$$

Next, for a stationary sequence $(Y_i)_{i\in\mathbb{Z}}$ of random variables with values in a Polish space S, define its strong mixing (or α-mixing) coefficients of order 4 as follows: Let

$$\alpha_{\infty,4}(n) = \sup_{i_4 > i_3 > i_2 > i_1 \geq n} \alpha(\mathcal{F}_0, \sigma(Y_{i_1}, Y_{i_2}, Y_{i_3}, Y_{i_4}))\,.$$

where $\mathcal{F}_0 = \sigma(Y_i, i \leq 0)$. As, page 146 in [17], these coefficients can be rewritten in the following form: Let B_1 be the class of measurable functions from S^4 to $\mathbb{R}$ and bounded by one. Then

$$\alpha_{\infty,4}(n) = \frac{1}{4} \sup_{f\in B_1} \sup_{i_4 > i_3 > i_2 > i_1 \geq n} \left\| \mathbb{E}(f(Y_{i_1}, Y_{i_2}, Y_{i_3}, Y_{i_4})|\mathcal{F}_0) - \mathbb{E}(f(Y_{i_1}, Y_{i_2}, Y_{i_3}, Y_{i_4})) \right\|_1 .$$

Hence, an application of Item (b) of Theorem 2.1 provides the following result.

Corollary 3.1. *Let $(Y_k)_{k\in\mathbb{Z}}$ be a stationary sequence of random variables with values in a Polish space and such that $\sum_{k\geq 1} k^2 \alpha_{\infty,4}(k) < \infty$. Let f be a bounded measurable numerical function and $X_k = f(Y_k) - \mathbb{E}(f(Y_k))$. Set $S_n = \sum_{k=1}^n X_k$. Then $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-1/2}$.*

As mentioned in the introduction, this results applies to the class of irreducible aperiodic and positively recurrent Markov (ξ_n) with an atom denoted by A, under the condition $\mathbb{E}_A(\tau_A^4) < \infty$. Here τ_A is the first return time in A and $\mathbb{E}_A$ stands for the expectation under $\mathbb{P}_x$ for $x \in A$.

3.2 α-dependent sequences and τ-mixing sequences

We start by recalling the definition of the α-dependence coefficients as considered in [6].

Definition 3.1. For any random variable $Y = (Y_1, \cdots, Y_k)$ with values in $\mathbb{R}^k$ and any σ-algebra $\mathcal{F}$, let

$$\alpha(\mathcal{F}, Y) = \sup_{(x_1,\ldots,x_k)\in\mathbb{R}^k} \left\| \mathbb{E}\Big(\prod_{j=1}^{k} (\mathbb{1}_{Y_j \le x_j})^{(0)} \Big| \mathcal{F} \Big)^{(0)} \right\|_1 .$$

For the sequence $\mathbf{Y} = (Y_i)_{i\in\mathbb{Z}}$, let

$$\alpha_{k,\mathbf{Y}}(0) = 1 \text{ and } \alpha_{k,\mathbf{Y}}(n) = \max_{1\le l\le k} \sup_{n\le i_1\le\ldots\le i_l} \alpha(\mathcal{F}_0, (Y_{i_1}, \ldots, Y_{i_l})) \text{ for } n > 0\,, \tag{3.1}$$

where $\mathcal{F}_0 = \sigma(Y_i, i \le 0)$.

Item (b) of Theorem 2.1 together with equality (A.4) in [10] (with $\mathbb{E}(\cdot|\mathcal{F}_0)$ instead of $\mathbb{E}$) provide the following result.

Corollary 3.2. *Let f be a bounded variation (BV) function and $X_k = f(Y_k) - \mathbb{E}(f(Y_k))$ where $(Y_k)_{k\in\mathbb{Z}}$ is a stationary sequence of real-valued random variables. Let $S_n = \sum_{k=1}^n X_k$. If $\sum_{k\ge1} k^2 \alpha_{4,\mathbf{Y}}(k) < \infty$, then $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-1/2}$.*

From this result, we can derive rates in the CLT for the partial sums associated with BV observables of the LSV map. More precisely, for $\gamma \in]0,1[$, let T_γ defined from $[0,1]$ to $[0,1]$ by

$$T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1]\,. \end{cases}$$

This is the so-called LSV [16] map with parameter γ. Recall, that there exists a unique T_γ-invariant measure ν_γ on $[0,1]$, which is absolutely continuous with respect to the Lebesgue measure with positive density denoted by h_γ. From Corollary 3.2 above and [6, Prop. 1.17], we derive that $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-1/2}$ for any $\gamma < 1/4$, where f is a bounded variation function and $S_n = \sum_{k=1}^n (f(T_\gamma^k) - \nu_\gamma(f))$.

We now apply Theorem 2.1 to functions of τ-dependent sequences. Before stating the result, some definitions are needed.

Definition 3.2. Let $\eta \in]0,1]$, ℓ be a positive integer and let $\Lambda_\eta(\mathbb{R}^\ell)$ be the set of functions f from $\mathbb{R}^\ell$ to $\mathbb{R}$ such that for $x = (x_1, \ldots, x_\ell)$ and $y = (y_1, \ldots, y_\ell)$,

$$|f(x) - f(y)| \le \frac{1}{\ell} \sum_{i=1}^{\ell} |x_i - y_i|^\eta\,.$$

Define the dependence coefficients $(\tau_{\eta,\ell,\mathbf{Y}}(k))_{k\ge1}$ of the sequence $(Y_i)_{i\in\mathbb{Z}}$ by

$$\tau_{\eta,\ell,\mathbf{Y}}(k) = \max_{1\le j\le\ell} \sup_{i_j>\ldots>i_1\ge k} \Big\| \sup_{f\in\Lambda_\eta(\mathbb{R}^j)} \Big| \mathbb{E}(f(Y_{i_1}, \ldots, Y_{i_j})|\mathcal{F}_0) - \mathbb{E}(f(Y_{i_1}, \ldots, Y_{i_j})) \Big| \Big\|_1 .$$

Examples of τ_η-dependent sequences are given in [9].

Let $(Y_k)_{k\in\mathbb{Z}}$ be a stationary sequence of real-valued random variables and f be a bounded and η-Hölder function, with $\eta \in]0,1]$. Define $X_k = f(Y_k) - \mathbb{E}(f(Y_k))$. Then, for any positive integers p, q and k, $\theta_{X,p,q}(k) \leq C\tau_{\eta,p,\mathbf{Y}}(k)$ where C is a positive constant depending only on p, q and $\|f\|_\infty$. Hence the following result holds.

Corollary 3.3. *Let f be a bounded and η-Hölder function with $\eta \in]0,1]$ and $X_k = f(Y_k) - \mathbb{E}(f(Y_k))$ where $(Y_k)_{k\in\mathbb{Z}}$ is a stationary sequence of real-valued random variables. Let $S_n = \sum_{k=1}^n X_k$. If $\sum_{k\geq 1} k^2 \tau_{\eta,4,\mathbf{Y}}(k) < \infty$, then $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-1/2}$.*

From this result, we can derive rates in the CLT for the partial sums associated with Hölder functions of the LSV map above. Starting from Corollary 3.3 and taking into account [7, Prop. 5.3 and Inequality (4.2)], we derive that if $\gamma < 1/4$, then $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-1/2}$, where $S_n = \sum_{k=1}^n (f(T_\gamma^k) - \nu_\gamma(f))$ and f is an η-Hölder observable with $\eta \in]0,1]$.

We now define another class of functions which are well adapted to τ-dependence.

Definition 3.3. Let c be any concave function from $\mathbb{R}^+$ to $\mathbb{R}^+$, with $c(0) = 0$. Let $\mathcal{L}_c$ be the set of functions g from $\mathbb{R}$ to $\mathbb{R}$ such that

$$|g(x) - g(y)| \leq Kc(|x-y|), \quad \text{for some positive } K.$$

Let $g \in \mathcal{L}_c$ and $X_k = g(Y_k) - \mathbb{E}(g(Y_k))$ where $(Y_k)_{k\in\mathbb{Z}}$ be a stationary sequence of bounded real-valued random variables. Then, for any positive integers ℓ and k, $\tau_{1,\ell,\mathbf{X}}(k) \leq Kc(\tau_{1,\ell,\mathbf{Y}}(k))$. As a consequence of Corollary 3.3, the following result holds:

Corollary 3.4. *Let $g \in \mathcal{L}_c$ and $X_k = g(Y_k) - \mathbb{E}(g(Y_k))$ where $(Y_k)_{k\in\mathbb{Z}}$ is a stationary sequence of bounded real-valued random variables such that $\tau_{1,4,\mathbf{Y}}(k)) = O(\rho^k)$ for some ρ in $]0,1[$. Let $S_n = \sum_{i=1}^n X_i$. If*

$$\int_0^1 \frac{(\log t)^2}{t} c(t)dt < \infty\,,$$

then $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-1/2}$.

Corollary 3.4 applies in particular to $X_k = g(T^k) - \nu(g)$ where T is a map from $[0,1]$ to $[0,1]$ that can be modelled by a Young tower with exponential tails of the return times and ν is the usual invariant measure (see Section 4 in [7] adapted to the case of exponential tails of the return times).

3.3 Symmetric random walk on the circle

Let K be the Markov kernel defined by $Kf(x) = (f(x+a)+f(x-a))/2$ on the torus $\mathbb{R}/\mathbb{Z}$, with a irrational in $[0,1]$. The Lebesgue-Haar measure m is the unique probability which is invariant by K. Let $(\xi_i)_{i\in\mathbb{Z}}$ be the stationary Markov chain with transition kernel K and invariant distribution m. For $f \in \mathbb{L}^2(m)$, let

$$X_k = f(\xi_k) - m(f)\,. \tag{3.2}$$

This example has been considered by Derriennic and Lin [11] who showed that the central limit theorem holds with the normalization $\sqrt{n}$ as soon as

$$\sum_{k\in\mathbb{Z}^*} \frac{|\hat{f}(k)|^2}{d(ka,\mathbb{Z})^2} < \infty\,, \tag{3.3}$$

where $\hat{f}(k)$ are the Fourier coefficients of f and $d(ka,\mathbb{Z}) = \min_{i\in\mathbb{Z}} |ka-i|$. The aim in this section is to give additional conditions on f and on the properties of the irrational number a ensuring rates of convergence in the CLT. Let us then introduce the following definition: a is said to be *badly approximable in the weak sense* by rationals if for any positive ε,

$$\text{the inequality } d(ka,\mathbb{Z}) < |k|^{-1-\varepsilon} \text{ has only finitely many solutions for } k \in \mathbb{Z}^*. \tag{3.4}$$

From Roth's theorem the algebraic numbers are badly approximable in the weak sense (cf. Schmidt [24]). Note also that the set of badly approximable numbers in $[0,1]$ has Lebesgue measure 1.

An application of Theorem 2.1 together with Lemma 5.2 in [10] and their inequality (5.18) give the following corollary.

Corollary 3.5. *Let X_k be defined by* (3.2)*. Suppose that the irrational number a satisfies* (3.4)*. Assume that for some positive ε,*

$$\sup_{k\neq 0} |k|^{6+\varepsilon} |\hat{f}(k)| < \infty\,.$$

Then $W_2(P_{S_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-1/2}$.

4 Proofs

4.1 Proof of Theorem 2.1

Assume first that $\sigma^2 = 0$. In this case $G_{\sigma^2} = \delta_0$ and

$$\mathbb{E}\big(W_2^2(P_{S_n/\sqrt{n}|\mathcal{F}_0}, \delta_0)\big) = n^{-1}\mathbb{E}(S_n^2) - \sigma^2 = -2n^{-1}\sum_{k\geq 1}(k\wedge n)\mathrm{Cov}(X_0, X_k),$$

which, combined with (2.1), shows that the upper bounds (a) and (b) hold.

We turn now to the case $\sigma^2 > 0$. Let δ be a random variable with uniform distribution over $[0,1]$ independent of $(X_k)_{k\in\mathbb{Z}}$. Define $\mathcal{G}_\ell = \sigma((X_i)_{i\le\ell}, \delta)$ and $\mathcal{G}_\infty = \sigma((X_i)_{i\in\mathbb{Z}}, \delta)$. Define also the conditional expectation operator $\mathbb{E}_0$ by $\mathbb{E}_0(\cdot) = \mathbb{E}(\cdot|\mathcal{G}_0)$.

In what follows $(Y_k)_{k\ge1}$ will be a sequence of iid random variables independent of $\mathcal{G}_\infty$. In case of Item (a), their common law will be the normal law $\mathcal{N}(0,\sigma^2)$ whereas in case of Item (b), we will have to prescribe also their third moment as it is described below.

Let β_3 be a fixed real number. Let Z be a r.v. with distribution $\mathcal{N}(0,\sigma^2/2)$. There exists a random variable B independent of Z, taking only 2 values and such that $Y = Z + B$ satisfies

$$\mathbb{E}(Y) = 0\,,\ \mathbb{E}(Y^2) = \sigma^2 \text{ and } \mathbb{E}(Y^3) = \beta_3\,. \tag{4.1}$$

We refer to Lemma 5.1 in [10] for more details. For the proof of Item (b),

$$\beta_3 = \mathbb{E}(X_0^3) + 3\sum_{i\ge1}\{\mathbb{E}(X_0^2X_i) + \mathbb{E}(X_0X_i^2)\} + 6\sum_{u\ge1}\sum_{v\ge u+1}\mathbb{E}(X_0X_uX_v)\,, \tag{4.2}$$

which is the limit of $n^{-1}\mathbb{E}(S_n^3)$, as $n\to\infty$, under the conditions of Item (b) of Theorem 2.1.

Let $(Z_k)_{k\ge1}$ be a sequence of independent r.v.'s distributed as Z and let $(B_k)_{k\ge1}$ be a sequence of independent r.v.'s distributed as B and independent of $(Z_k)_{k\ge1}$. Suppose furthermore that the sequence $(Z_k,B_k)_{k\ge1}$ is independent of $\mathcal{G}_\infty$. For any $k\ge1$, set $Y_k = Z_k + B_k$.

Next, in case of both items, we define $T_n = Y_1 + Y_2 + \cdots + Y_n$. Note that

$$W_2(P_{S_n/\sqrt{n}|\mathcal{F}_0}, G_{\sigma^2}) \le W_2(P_{S_n/\sqrt{n}|\mathcal{F}_0}, P_{T_n/\sqrt{n}}) + W_2(P_{T_n/\sqrt{n}}, G_{\sigma^2})\,.$$

According to Theorem 4.1 in [21], since $Y\in\mathbb{L}^4$, $W_2(P_{T_n/\sqrt{n}}, G_{\sigma^2}) \ll n^{-1/2}$. Since $P_{S_n/\sqrt{n}|\mathcal{F}_0} = P_{S_n/\sqrt{n}|\mathcal{G}_0}$, the theorem will follow if one can prove that the upper bounds (a) and (b) still hold with $P_{S_n/\sqrt{n}|\mathcal{G}_0}$ replacing $P_{S_n/\sqrt{n}|\mathcal{F}_0}$. With this aim, we shall apply Lemma 5.1 in [18]. We start by introducing some notations. Let Λ_2 be the class of real functions f which are continuously differentiable and such that $|f'(x) - f'(y)| \le |x-y|$ for any $(x,y)\in\mathbb{R}\times\mathbb{R}$. Let also $\Lambda_2(E)$ be the set of measurable functions $f:\mathbb{R}\times E\to\mathbb{R}$ wrt the σ-fields $\mathcal{L}(\mathbb{R}\times E)$ and $\mathcal{B}(\mathbb{R})$, such that $f(\cdot,w)\in\Lambda_2$ and $f(0,w) = f'(0,w) = 0$ for any $w\in E$. Next, let $W = ((X_i)_{i\in\mathbb{Z}^-}, \delta)$ and $E = \mathbb{R}^{\mathbb{Z}^-}\times[0,1]$. According to [18, Lemma 5.1], and denoting by N a $\mathcal{N}(0,\sigma^2)$-distributed random variable, independent of all the above sequences (so independent of $(X_k,Y_k)_k$), the upper bound (a) will follow if one can prove that

$$\sup_{f\in\Lambda_2(E)}\mathbb{E}\big(f(S_n+N,W) - f(T_n+N,W)\big) \ll \sqrt{n}\Big(1 + \sum_{k\ge1}(k\wedge\sqrt{n})\theta_{X,2,2}(k)\Big)\,, \tag{4.3}$$

whereas the upper bound (b) will follow if

$$\sup_{f\in\Lambda_2(E)} \mathbb{E}\big(f(S_n+N,W)-f(T_n+N,W)\big) \ll 1+\sum_{k\geq 1} k(k\wedge\sqrt{n})\theta_{X,4,4}(k)\,. \tag{4.4}$$

In what follows to soothe the notations we omit the subscripts for the coefficients $\theta(k)$.

Proof of Item (a). We shall apply the Lindeberg method. Let us first introduce some notations.

Notation 4.1. Set $f_{n-k}(x)=\mathbb{E}_0(f(x+N+T_n-T_k,W))$.

Notice that

$$f_{n-k}(x)=\int_{\mathbb{R}} f(x-t,W)\varphi_{\sigma^2(n-k+1)}(t)dt\,,$$

where φ_{t^2} is the density of a $\mathcal{N}(0,t^2)$. Hence, according to Lemma 6.1 in [8],

$$\|f^{(i)}_{n-k}\|_\infty := b_i \ll (n-k+1)^{(2-i)/2}\,. \tag{4.5}$$

Since the sequence $(N,(Y_i)_{i\geq 1})$ is independent of the sequence $((X_i)_{i\in\mathbb{Z}},W)$,

$$\mathbb{E}(f(S_n+N,W)-f(T_n+N,W))=\sum_{k=1}^{n}\mathbb{E}(f_{n-k}(S_{k-1}+X_k)-f_{n-k}(S_{k-1}+Y_k))\,. \tag{4.6}$$

By the Taylor formula at order 3 and using (4.5), we get

$$\Big|\mathbb{E}\big(f_{n-k}(S_{k-1}+Y_k)-f_{n-k}(S_{k-1})-\frac{\sigma^2}{2}f''_{n-k}(S_{k-1})\big)\Big|\leq C(n-k+1)^{-1/2}\,. \tag{4.7}$$

Similarly

$$\big|\mathbb{E}(f_{n-k}(S_k)-f_{n-k}(S_{k-1})-f'_{n-k}(S_{k-1})X_k-\frac{1}{2}f''_{n-k}(S_{k-1})X_k^2)\big|\leq C(n-k+1)^{-1/2}\,. \tag{4.8}$$

Now we control the second order term. Let

$$\Gamma_k(k,i)=f''_{n-k}(S_{k-i})-f''_{n-k}(S_{k-i-1})\,. \tag{4.9}$$

Clearly

$$f''_{n-k}(S_{k-1})X_k^2=\sum_{i=1}^{[\sqrt{k}]-1}\Gamma_k(k,i)X_k^2+f''_{n-k}(S_{k-[\sqrt{k}]})X_k^2\,.$$

Since $|\Gamma_k(k,i)|\leq b_3|X_{k-i}|$, by stationarity we get that for any $i\leq k-1$,

$$\big|\operatorname{Cov}(\Gamma_k(k,i),X_k^2)\big|\leq b_3\|X_0\big(\mathbb{E}_0(X_i^2)-\mathbb{E}(X_i^2)\big)\|_1\ll (n-k+1)^{-1/2}\theta(i)\,.$$

Since $\|f''_{n-k}\|_\infty\leq b_2$ a.s., we also get by stationarity that

$$\big|\operatorname{Cov}(f''_{n-k}(S_{k-[\sqrt{k}]}),X_k^2)\big|\leq b_2\|\mathbb{E}_0(X^2_{[\sqrt{k}]})-\mathbb{E}(X^2_{[\sqrt{k}]})\|_1\ll\theta([\sqrt{k}])\,.$$

Starting from (4.8), it follows that

$$\big|\mathbb{E}(f_{n-k}(S_k) - f_{n-k}(S_{k-1}) - f'_{n-k}(S_{k-1})X_k) - \frac{1}{2}\mathbb{E}(f''_{n-k}(S_{k-1}))\mathbb{E}(X_k^2)\big| \\ \ll \theta([\sqrt{k}]) + (n-k+1)^{-1/2}\big(1 + \sum_{i=1}^{[\sqrt{k}]}\theta(i)\big)\,. \quad (4.10)$$

Starting from (4.6) and taking into account (4.7) and (4.10) we derive that

$$\big|\mathbb{E}(f(S_n + Y, W) - f(T_n + Y, W))\big| \\ \ll \sqrt{n}\Big(1 + \sum_{i=1}^{[\sqrt{n}]}\theta(i)\Big) + \Big|\sum_{k=1}^{n}\Big\{\mathbb{E}(f'_{n-k}(S_{k-1})X_k) - \mathbb{E}(f''_{n-k}(S_{k-1}))\sum_{j\geq 1}\mathbb{E}(X_0X_j)\Big\}\Big|\,. \quad (4.11)$$

To give now an estimate of $\mathbb{E}(f'_{n-k}(S_{k-1})X_k)$, we write

$$f'_{n-k}(S_{k-1}) = f'_{n-k}(0) + \sum_{i=1}^{k-1}(f'_{n-k}(S_{k-i}) - f'_{n-k}(S_{k-i-1}))\,.$$

Hence

$$\mathbb{E}(f'_{n-k}(S_{k-1})X_k) = \sum_{i=1}^{k-1}\mathrm{Cov}\big(f'_{n-k}(S_{k-i}) - f'_{n-k}(S_{k-i-1}), X_k\big) + \mathbb{E}(f'_{n-k}(0)X_k)\,. \quad (4.12)$$

Now $f'_{n-k}(0)$ is a $\mathcal{G}_0$-measurable random variable. Since $f \in \Lambda_2(E)$ then $f'(0,w) = 0$ and $f'(\cdot, w)$ is 1-Lipschitz. Therefore

$$|f'_{n-k}(0)| \leq \int_{\mathbb{R}}|f'(u,W) - f'(0,W)|\varphi_{\sigma^2(n-k+1)}(-u)du \leq \sigma\sqrt{n-k+1}\ \text{ a.s.}$$

It follows that

$$\sum_{k=1}^{n}\big|\mathbb{E}(f'_{n-k}(0)X_k)\big| \ll \sum_{k=1}^{n}\sqrt{n-k+1}\|\mathbb{E}_0(X_k)\|_1 \ll \sqrt{n}\sum_{k=1}^{n}\theta(k)\,. \quad (4.13)$$

We give now an estimate of $\sum_{i=1}^{k-1}\mathrm{Cov}\big(f'_{n-k}(S_{k-i}) - f'_{n-k}(S_{k-i-1}), X_k\big)$. Using the stationarity and noting that $|f'_{n-k}(S_{k-i}) - f'_{n-k}(S_{k-i-1})| \leq b_2|X_{k-i}|$, we have

$$|\,\mathrm{Cov}(f'_{n-k}(S_{k-i}) - f'_{n-k}(S_{k-i-1}), X_k)| \leq b_2M\|\mathbb{E}_0(X_i)\|_1 \ll \theta(i)\,.$$

Hence

$$\sum_{k=1}^{n}\sum_{i=[\sqrt{k}]}^{k}|\,\mathrm{Cov}(f'_{n-k}(S_{k-i}) - f'_{n-k}(S_{k-i-1}), X_k)| \\ \ll \sum_{i=1}^{n}(i\wedge\sqrt{n})^2\theta(i) \ll \sqrt{n}\sum_{i\geq 1}(i\wedge\sqrt{n})\theta(i)\,. \quad (4.14)$$

From now on, we assume that $i < [\sqrt{k}]$. We first write

$$f'_{n-k}(S_{k-i}) - f'_{n-k}(S_{k-i-1}) = f''_{n-k}(S_{k-i-1})X_{k-i} + R_{k,i}\,,$$

where $R_{k,i}$ is $\mathcal{F}_{k-i}$-measurable and $|R_{k,i}| \le b_3 X_{k-i}^2/2$. Hence, by stationarity,

$$|\operatorname{Cov}(R_{k,i}, X_k)| \le b_3 \|X_0^2 \mathbb{E}_0(X_i)\|_1/2 \ll (n-k+1)^{-1/2}\theta(i)\,.$$

implying that

$$\sum_{k=1}^{n}\sum_{i=1}^{[\sqrt{k}]} |\operatorname{Cov}(R_{k,i}, X_k)| \ll \sqrt{n}\sum_{i=1}^{[\sqrt{n}]}\theta(i)\,. \tag{4.15}$$

In order to estimate the term $\mathbb{E}(f''_{n-k}(S_{k-i-1})X_{k-i}X_k)$, we introduce the decomposition below:

$$f''_{n-k}(S_{k-i-1}) = \sum_{\ell=1}^{i-1}(f''_{n-k}(S_{k-i-\ell}) - f''_{n-k}(S_{k-i-\ell-1})) + f''_{n-k}(S_{k-2i})\,,$$

where by convention we set $S_p = 0$ if $p \le 0$. For any $\ell \in \{1, \cdots, i-1\}$, by using the notation (4.9) and the stationarity, we get that

$$|\operatorname{Cov}(\Gamma_k(k, \ell+i)X_{k-i}, X_k)| \le b_3 \|X_{-\ell}X_0\mathbb{E}_0(X_i)\|_1 \ll (n-k+1)^{-1/2}\theta(i)\,.$$

Hence

$$\sum_{k=1}^{n}\sum_{i=1}^{[\sqrt{k}]}\sum_{\ell=1}^{i-1} \operatorname{Cov}(\Gamma_k(k, \ell+i)X_{k-i}, X_k)| \ll \sqrt{n}\sum_{i=1}^{[\sqrt{n}]} i\theta(i)\,. \tag{4.16}$$

As a second step, we bound up $|\operatorname{Cov}(f''_{n-k}(S_{k-2i}), X_{k-i}X_k)|$. Clearly,

$$f''_{n-k}(S_{k-2i}) = \sum_{\ell=i}^{k-i-1}\Gamma_k(k, \ell+i) + f''_{n-k}(0)\,.$$

Now for any $\ell \in \{i, \cdots, (k-i-1)\}$, by stationarity,

$$|\operatorname{Cov}(\Gamma_k(k, \ell+i), X_{k-i}X_k)| \le b_3 \|X_{-\ell}\big(\mathbb{E}_{-\ell}(X_0X_i) - \mathbb{E}(X_0X_i)\big)\|_1 \ll (n-k+1)^{-1/2}\theta(\ell)\,.$$

Hence

$$\sum_{k=1}^{n}\sum_{i=1}^{[\sqrt{k}]}\sum_{\ell=i}^{k-i-1} |\operatorname{Cov}(\Gamma_k(k, \ell+i), X_{k-i}X_k)| \ll \sqrt{n}\sum_{\ell=1}^{n}(\ell \wedge \sqrt{n})\theta(\ell)\,. \tag{4.17}$$

Next, note that

$$|\operatorname{Cov}(f''_{n-k}(0), X_{k-i}X_k)| \ll b_2 \min(\theta(k-i), \theta(i)) \ll \theta([k/2])\,,$$

implying that

$$\sum_{k=1}^{n}\sum_{i=1}^{[\sqrt{k}]}|\operatorname{Cov}(f''_{n-k}(0),X_{k-i}X_k)|\ll\sum_{k=1}^{n}\sum_{i=1}^{[\sqrt{k}]}\theta([k/2])\ll\sqrt{n}\sum_{k=1}^{n}\theta(k)\,. \tag{4.18}$$

Taking into account the inequalities (4.13)-(4.18), and using that $\sum_{k\geq 1}\theta(k)<\infty$, we get

$$\sum_{k=1}^{n}\Big|\mathbb{E}(f'_{n-k}(S_{k-1})X_k)-\sum_{i=1}^{[\sqrt{k}]}\mathbb{E}(f''_{n-k}(S_{k-2i}))\mathbb{E}(X_{k-i}X_k)\Big|\ll\sqrt{n}\Big(1+\sum_{\ell\geq 1}(\ell\wedge\sqrt{n})\theta(\ell)\Big)\,. \tag{4.19}$$

We handle now the quantity

$$A_k:=\sum_{i=1}^{[\sqrt{k}]}\mathbb{E}(f''_{n-k}(S_{k-2i}))\mathbb{E}(X_{k-i}X_k)-\sum_{i=1}^{\infty}\mathbb{E}(f''_{n-k}(S_{k-1}))\mathbb{E}(X_{k-i}X_k)\,.$$

We first note that by stationarity,

$$\sum_{i\geq[\sqrt{k}]+1}|\mathbb{E}(f''_{n-k}(S_{k-1}))\mathbb{E}(X_{k-i}X_k)|\leq b_2\sum_{i\geq[\sqrt{k}]+1}|\mathbb{E}(X_0\mathbb{E}_0(X_i))|\ll\sum_{i\geq[\sqrt{k}]+1}\theta(i)\,.$$

Hence

$$\sum_{k=1}^{n}\sum_{i\geq[\sqrt{k}]+1}|\mathbb{E}(f''_{n-k}(S_{k-1}))\mathbb{E}(X_{k-i}X_k)|\ll\sum_{i\geq 1}(i\wedge\sqrt{n})^2\theta(i)\ll\sqrt{n}\sum_{i\geq 1}(i\wedge\sqrt{n})\theta(i)\,. \tag{4.20}$$

On another hand, we write

$$\mathbb{E}(f''_{n-k}(S_{k-1})-f''_{n-k}(S_{k-2i}))\mathbb{E}(X_{k-i}X_k)=\sum_{\ell=1}^{2i-1}\mathbb{E}(\Gamma_k(k,\ell)\mathbb{E}(X_0\mathbb{E}_0(X_i))\,.$$

Therefore

$$\sum_{i=1}^{[\sqrt{k}]}|\mathbb{E}(f''_{n-k}(S_{k-1})-f''_{n-k}(S_{k-2i}))\mathbb{E}(X_{k-i}X_k)|\leq(n-k+1)^{-1/2}\sum_{i=1}^{[\sqrt{k}]}i\theta(i)\,,$$

implying that

$$\sum_{k=1}^{n}\sum_{i=1}^{[\sqrt{k}]}|\mathbb{E}(f''_{n-k}(S_{k-1})-f''_{n-k}(S_{k-2i}))\mathbb{E}(X_{k-i}X_k)|\ll\sqrt{n}\sum_{i=1}^{[\sqrt{n}]}i\theta(i)\,. \tag{4.21}$$

Hence (4.20) and (4.21) entail that

$$\sum_{k=1}^{n}|A_k|\ll\sqrt{n}\sum_{i\geq 1}(i\wedge\sqrt{n})\theta(i)\,. \tag{4.22}$$

The estimates (4.19) and (4.22) yield to

$$\sum_{k=1}^{n}\Big|\mathbb{E}(f'_{n-k}(S_{k-1})X_k)-\sum_{i=1}^{\infty}\mathbb{E}(f''_{n-k}(S_{k-1}))\mathbb{E}(X_0X_i)\Big| \ll \sqrt{n}\Big(1+\sum_{\ell\geq 1}(\ell\wedge\sqrt{n})\theta(\ell)\Big)\,. \quad (4.23)$$

Taking into account the estimates (4.11) and (4.23), Item (a) follows.

Proof of Item (b). Recall that in this case the iid random variables $(Y_k)_{k\geq 1}$ have their first three moments defined by (4.1) and (4.2)

Notation 4.2. For any integer $k \geq 0$, let $\tilde{X}_k = X_k - \mathbb{E}_0(X_k)$ and $\tilde{S}_k = S_k - \mathbb{E}_0(S_k)$, with the convention $S_0 = 0$.

Note that, since we assume that $\sum_{j\geq 1} j\theta(j) < \infty$,

$$\|\mathbb{E}_0(S_n)\|_2^2 \leq 2\sum_{i=1}^{n}\sum_{j=i}^{n}\big|\mathbb{E}(\mathbb{E}_0(X_i)\mathbb{E}_0(X_j))\big| \leq 2M\sum_{j=1}^{n} j\theta(j) \ll 1\,.$$

Therefore, using that $f'(0,W) = 0$ and that $|f'(x,W) - f'(y,W)| \leq |x-y|$, we infer that to prove (4.4), it is enough to show that for any $f \in \Lambda_2(E)$ and any positive n,

$$\sup_{f\in\Lambda_2(E)}\mathbb{E}\big(f(\tilde{S}_n+N,W) - f(T_n+NN,W)\big) \ll 1+\sum_{k\geq 1}k(k\wedge\sqrt{n})\theta(k)\,. \quad (4.24)$$

This will be done by using again the Lindeberg method. Let us introduce some additional notations.

Notation 4.3. For any positive integer k, let $\Delta_{n,k} = f_{n-k}(\tilde{S}_{k-1}+\tilde{X}_k) - f_{n-k}(\tilde{S}_{k-1}+Y_k)$ where f_{n-k} is defined in Notation 4.1.

All along the proof, the following lemma will be used (the proof is postponed to the Appendix and is based on the fact that the common distribution of the random variables $(Y_k)_{k\geq 1}$ is smooth).

Lemma 4.1. *Let $f \in \Lambda_2(E)$.*

1. *For any $i \geq 2$, there exists a positive constant κ_1 depending on σ^2 and i and such that $\|f^{(i)}_{n-k}\|_\infty \leq \kappa_1(n-k+1)^{(2-i)/2}$.*
2. *Assume that $\sum_{k\geq 1} k\theta_{X,3,4}(k) < \infty$. Then, for any $i \geq 2$, there exists a constant $\kappa_2 > 0$ depending on σ^2 and i such that, for any integer $\ell > 0$,*

$$\big|\mathbb{E}(f^{(i)}_{n-k}(\tilde{S}_{\ell-1}))\big| \leq \kappa_2(n-k+1)^{(1-i)/2} + \kappa_2(n-k+\ell)^{(2-i)/2}\,.$$

Remark 4.1. If $(X_k)_{k\in\mathbb{Z}}$ is a stationary sequence of martingale differences, Item 2. is valid under the condition $\sum_{k\geq 1}\theta_{X,2,3}(k) < \infty$.

Since the sequence $(N,(Y_i)_{i\geq 1})$ is independent of $((X_i)_{i\in\mathbb{N}},W)$,

$$\mathbb{E}(f(\tilde{S}_n+N,W)-f(T_n+N,W))=\sum_{k=1}^{n}\mathbb{E}(\Delta_{n,k})\,. \tag{4.25}$$

Next the functions f_{n-k} are C^∞. Consequently, from the Taylor integral formula at order 5,

$$\Delta_{n,k}=\sum_{j=1}^{4}\frac{1}{j!}\,f_{n-k}^{(j)}(\tilde{S}_{k-1})(\tilde{X}_k^j-Y_k^j)+R_{n,k}\,, \tag{4.26}$$

with

$$R_{n,k}=\frac{1}{24}\tilde{X}_k^5\int_0^1(1-s)^4f_{n-k}^{(5)}(\tilde{S}_{k-1}+s\tilde{X}_k)ds-\frac{1}{24}Y_k^5\int_0^1(1-s)^4f_{n-k}^{(5)}(\tilde{S}_{k-1}+sY_k)ds\,.$$

Taking into account the fact that $\|X_k\|_\infty\leq M$ and Item 1 of Lemma 4.1, we derive that

$$\|R_{n,k}\|_1\ll(M^5+\mathbb{E}(|Y_1|^5))\|f_{n-k}^{(5)}\|_\infty\ll(n-k+1)^{-3/2}\,.$$

Therefore,

$$\sum_{k\in[1,n]}\|R_{n,k}\|_1\ll 1\,. \tag{4.27}$$

Let $\beta_4=\sigma^2=\mathbb{E}(Y_k^2)$ and $\beta_4=\mathbb{E}(Y_k^4)$. Since the sequence $(Y_i)_{i\geq 1}$ is independent of the sequence $(X_i)_{i\geq 1}$,

$$\begin{aligned}\mathbb{E}(\Delta_{n,k}-R_{n,k})&=\mathbb{E}\Big(f_{n-k}'(\tilde{S}_{k-1})\tilde{X}_k+\sum_{\ell=2}^{4}\frac{1}{\ell!}f_{n-k}^{(\ell)}(\tilde{S}_{k-1})(\tilde{X}_k^\ell-\beta_\ell)\Big)\\&=\mathbb{E}\Big(f_{n-k}'(\tilde{S}_{k-1})\tilde{X}_k+\sum_{\ell=2}^{4}\frac{1}{\ell!}f_{n-k}^{(\ell)}(\tilde{S}_{k-1})(X_k^\ell-\beta_\ell)\Big)+\tilde{B}_{n,k}\\&:=\mathbb{E}\Big(\Delta_{n,k}^{(1)}+\frac{1}{2}\Delta_{n,k}^{(2)}+\frac{1}{6}\Delta_{n,k}^{(3)}+\frac{1}{24}\Delta_{n,k}^{(4)}\Big)+\tilde{B}_{n,k}\,.\end{aligned} \tag{4.28}$$

Using Item 1 of Lemma 4.1, we first notice that

$$\sum_{k\in[1,n]}|\tilde{B}_{n,k}|\ll\sum_{k\in[1,n]}\|\mathbb{E}_0(X_k)\|_1\ll 1. \tag{4.29}$$

Next we develop the first four terms in the right-hand side of the decomposition (4.28) with the help of the Lindeberg method. From now on, to soothe the notation, we shall omit most of the time the index n in all the $\Delta_{n,k}^{(i)}$ and the related quantities, and then rather write $\Delta_k^{(i)}$. Let us start with the term $\Delta_k^{(4)}$. Using Item 2 of Lemma 4.1, note first that

$$\sum_{k=1}^{n}\big|\mathbb{E}(f_{n-k}^{(4)}(\tilde{S}_{k-1}))(\mathbb{E}(X_k^4)+\beta_4)\big|\ll\sum_{k=1}^{n}\Big(\frac{1}{(n-k+1)^{3/2}}+\frac{1}{n}\Big)\ll 1\,. \tag{4.30}$$

Next, we write

$$f^{(4)}_{n-k}(\tilde{S}_{k-1})(X_k^4 - \mathbb{E}(X_k^4)) = f^{(4)}_{n-k}(0)(X_k^4 - \mathbb{E}(X_k^4)) + \sum_{i=1}^{k-1} \big(f^{(4)}_{n-k}(\tilde{S}_{k-i}) - f^{(4)}_{n-k}(\tilde{S}_{k-i-1})\big)(X_k^4 - \mathbb{E}(X_k^4))\,.$$

By Item 1 of Lemma 4.1 we get

$$\sum_{k=1}^{n} \big|\mathrm{Cov}(f^{(4)}_{n-k}(0), X_k^4)\big| \ll \sum_{k=1}^{n} (n-k+1)^{-1}\theta(k) \ll \sum_{k=1}^{n} \theta(k)\,, \tag{4.31}$$

and

$$\sum_{k=1}^{n}\sum_{i=1}^{k-1} \big|\mathrm{Cov}\big(f^{(4)}_{n-k}(\tilde{S}_{k-i}) - f^{(4)}_{n-k}(\tilde{S}_{k-i-1}), X_k^4\big)\big| \ll M^5 \sum_{k=1}^{n} (n-k+1)^{-3/2} \sum_{i=1}^{k} \theta(i) \ll \sum_{i=1}^{n} \theta(i)\,. \tag{4.32}$$

Taking into account (4.30), (4.31), (4.32) and the fact that $\sum_{k\geq 1}\theta(k) < \infty$, it follows that

$$\sum_{k=1}^{n} \big|\mathbb{E}(\Delta^{(4)}_{n,k})\big| \ll 1\,. \tag{4.33}$$

Now, concerning the first term in the right-hand side of (4.28), letting $\ell_k = [k/2]$, we write

$$\begin{aligned}\mathbb{E}(\Delta^{(1)}_k) &= \mathbb{E}(f'_{n-k}(\tilde{S}_{k-\ell_k-1})\tilde{X}_k) + \sum_{i=1}^{\ell_k} \mathbb{E}\big(\{f'_{n-k}(\tilde{S}_{k-i}) - f'_{n-k}(\tilde{S}_{k-i-1})\}\tilde{X}_k\big) \\ &= \mathbb{E}(\tilde{\Delta}^{(1)}_{k,2}) + \frac{1}{2}\mathbb{E}(\tilde{\Delta}^{(1)}_{k,3}) + \frac{1}{6}\mathbb{E}(\tilde{\Delta}^{(1)}_{k,4}) + \tilde{B}^{(1)}_{n,k}\,,\end{aligned} \tag{4.34}$$

where, for $j = 2, 3, 4$,

$$\tilde{\Delta}^{(1)}_{k,j} = \sum_{i=1}^{\ell_k} f^{(j)}_{n-k}(\tilde{S}_{k-i-1})\tilde{X}_{k-i}^{j-1}\tilde{X}_k\,,$$

and

$$\tilde{B}^{(1)}_{n,k} = \mathbb{E}(f'_{n-k}(S_{k-\ell_k-1})\tilde{X}_k) + \frac{1}{6}\sum_{i=1}^{\ell_k}\int_0^1 (1-s)^3 \mathbb{E}\big(f^{(5)}_{n-k}(\tilde{S}_{k-i-1} + s\tilde{X}_{k-i})\tilde{X}_{k-i}^4\tilde{X}_k\big)ds.$$

We start by noticing that, by Item 1 of Lemma 4.1, for any $m \geq 2$ and any s in $[0,1]$,

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{i=1}^{\ell_k} \|f^{(m)}_{n-k}(\tilde{S}_{k-i-1} + s\tilde{X}_{k-i})(\tilde{X}_{k-i}^{m-1}\tilde{X}_k - X_{k-i}^{m-1}X_k)\|_1 \\ &\ll M^{m-1}\sum_{k=1}^{n}(n-k+1)^{(2-m)/2}\sum_{i=1}^{\ell_k}\big(\|\mathbb{E}_0(X_k)\|_1 + \|\mathbb{E}_0(X_{k-i})\|_1\big) \ll \sum_{k\geq 1} k\theta(k)\,.\end{aligned} \tag{4.35}$$

On another hand, since $f'_{n-k}(0)$ is $\mathcal{F}_0$-measurable, $\mathbb{E}(f'_{n-k}(0)\tilde{X}_k) = 0$. Therefore

$$|\mathbb{E}(f'_{n-k}(\tilde{S}_{k-\ell_k-1})\tilde{X}_k)| = |\mathbb{E}\big(\{f'_{n-k}(\tilde{S}_{k-\ell_k-1}) - f'_{n-k}(0)\}\tilde{X}_k\big)| \\ \leq \int_0^1 |\mathbb{E}\big(f''_{n-k}(t\tilde{S}_{k-\ell_k-1})\tilde{S}_{k-\ell_k-1}\tilde{X}_k\big)|dt \leq 4M\|f^{(2)}_{n-k}\|_\infty(k-\ell_k)\theta(\ell_k)\,.$$

Since $\|f^{(2)}_{n-k}\|_\infty \ll 1$ and $\sum_{k\geq 1} k\theta(k) < \infty$,

$$\sum_{k\in[1,n]} |\mathbb{E}(f'_{n-k}(\tilde{S}_{k-\ell_k-1})\tilde{X}_k)| \ll 1\,. \tag{4.36}$$

Next, Item 1 of Lemma 4.1 implies that

$$|\mathbb{E}\big(f^{(5)}_{n-k}(\tilde{S}_{k-i-1} + s\tilde{X}_{k-i})X^4_{k-i}X_k\big)| \leq M^4\|f^{(5)}_{n-k}\|_\infty\theta(i) \ll (n-k+1)^{-3/2}\theta(i)\,.$$

Hence

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k} |\mathbb{E}\big(f^{(5)}_{n-k}(S_{k-i-1} + sX_{k-i})X^4_{k-i}X_k\big)| \ll \sum_{i=1}^{n}\theta(i) \ll 1\,. \tag{4.37}$$

The upper bounds (4.35), (4.36) and (4.37) imply that

$$\sum_{k\in[1,n]} |\tilde{B}^{(1)}_{n,k}| \ll 1\,. \tag{4.38}$$

Next, taking into account Item 2 of Lemma 4.1 and the fact that $|\mathbb{E}(X^3_{k-i}X_k)| \leq M^3\theta(i)$, we derive that

$$|\mathbb{E}\{f^{(4)}_{n-k}(\tilde{S}_{k-i-1})\}\mathbb{E}(X^3_{k-i}X_k)| \ll ((n-k+1)^{-3/2} + (n-i)^{-1})\theta(i)\,.$$

Therefore

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k} |\mathbb{E}\{f^{(4)}_{n-k}(\tilde{S}_{k-i-1})\}\mathbb{E}(X^3_{k-i}X_k)| \ll \sum_{i\geq 1}\theta(i) \ll 1\,. \tag{4.39}$$

So, overall, starting from (4.34) and taking into account (4.35), (4.38) and (4.39) we get

$$\mathbb{E}(\Delta^{(1)}_k) = \mathbb{E}(\Delta^{(1)}_{k,2}) + \frac{1}{2}\mathbb{E}(\Delta^{(1)}_{k,3}) + \frac{1}{6}\mathbb{E}(\Delta^{(1)}_{k,4}) + A^{(1)}_{k,2} + \frac{1}{2}A^{(1)}_{k,3} + B^{(1)}_{n,k}\,, \tag{4.40}$$

where $B^{(1)}_{n,k}$ is such that

$$\sum_{k\in[1,n]} |B^{(1)}_{n,k}| \ll 1\,, \tag{4.41}$$

and the following notations have been used: for $j = 2, 3, 4$,

$$\Delta^{(1)}_{k,j} = \sum_{i=1}^{\ell_k}\{f^{(j)}_{n-k}(\tilde{S}_{k-i-1})\}(X^{j-1}_{k-i}X_k)^{(0)}, A^{(1)}_{k,j} = \sum_{i=1}^{\ell_k}\mathbb{E}\{f^{(j)}_{n-k}(\tilde{S}_{k-i-1})\}\mathbb{E}(X^{j-1}_{k-i}X_k). \tag{4.42}$$

Introduce now the following additional notations.

Notation 4.4. Let $\gamma_i = \mathbb{E}(X_0 X_i)$ and $\gamma_i^{(2)} = \mathbb{E}(X_0^2 X_i)$. Define $\beta_{2,\ell_k} = 2\sum_{i=1}^{\ell_k} \gamma_i$, $\beta_2^{(\ell_k)} = 2\sum_{i\geq \ell_k+1} \gamma_i$ and $\beta_{3,1,\ell_k} = 3\sum_{i=1}^{\ell_k} \gamma_i^{(2)}$.

Next note that, since $\mathbb{E}(X_{k-i}X_k) = \gamma_i$,

$$\frac{1}{2}\mathbb{E}\{f''_{n-k}(\tilde{S}_{k-1})\}\beta_{2,\ell_k} - A^{(1)}_{k,2} = \sum_{i=1}^{\ell_k}\gamma_i \sum_{j=1}^{i}\mathbb{E}\{f''_{n-k}(\tilde{S}_{k-j}) - f''_{n-k}(\tilde{S}_{k-j-1})\} =$$

$$\sum_{i=1}^{\ell_k}\gamma_i \sum_{j=1}^{i}\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-j-1})\tilde{X}_{k-j}\big\} + \sum_{i=1}^{\ell_k}\frac{\gamma_i}{2}\sum_{j=1}^{i}\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-j-1})\tilde{X}^2_{k-j}\big\} + r^{(1)}_{n,k,2}\,, \quad (4.43)$$

where

$$r^{(1)}_{n,k,2} := \frac{1}{2}\int_0^1 (1-t)^2 \sum_{i=1}^{\ell_k}\gamma_i \sum_{j=1}^{i}\mathbb{E}\Big\{f^{(5)}_{n-k}(S_{k-j-1} + t\tilde{X}_{k-j})\tilde{X}^3_{k-j}\Big\}dt\,.$$

By Item 1 of Lemma 4.1, it follows that

$$|r^{(1)}_{n,k,2}| \ll M^4(n-k+1)^{-3/2}\sum_{i=1}^{\ell_k} i\theta(i)\,.$$

Since $\sum_{i\geq 1} i\theta(i) < \infty$, this implies that

$$\sum_{k\in[1,n]} |r^{(1)}_{n,k,2}| \ll 1\,. \quad (4.44)$$

Next, taking into account Item 1 of Lemma 4.1, we get

$$\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\Big|\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-j-1})\mathbb{E}_0(X_{k-j})\big\}\gamma_i\Big| \ll (n-k+1)^{-1/2}\theta(k-j)\theta(i)\,.$$

Hence, since $\ell_k = [k/2]$ and $\sum_{i\geq 1} i\theta(i) < \infty$,

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\Big|\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-j-1})\mathbb{E}_0(X_{k-j})\big\}\gamma_i\Big| \ll \sum_{k=1}^{n}\theta([k/2])\sum_{i=1}^{\ell_k} i\theta(i) \ll 1\,. \quad (4.45)$$

With similar arguments, we have

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\Big|\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-j-1})(\tilde{X}^2_{k-j} - X^2_{k-j})\big\}\gamma_i\Big| \ll 1\,. \quad (4.46)$$

In addition, by taking into account Item 2 of Lemma 4.1, we get

$$\Big|\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-j-1})\mathbb{E}(X^2_{k-j})\big\}\gamma_i\Big| \ll M^3((n-k+1)^{-3/2} + (n-j)^{-1})\theta(i)\,.$$

Hence,

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\Big|\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-j-1})\mathbb{E}(X^2_{k-j})\big\}\gamma_i\Big| \\ \ll \sum_{k=1}^{n}((n-k+1)^{-3/2}+(n-\ell_k)^{-1})\sum_{i=1}^{\ell_k} i\ \theta(i) \ll \sum_{i=1}^{n} i\theta(i) \ll 1\,. \quad (4.47)$$

So overall, starting from (4.43) and taking into account the upper bounds (4.44)-(4.47), we derive that

$$\frac{1}{2}\mathbb{E}\{f''_{n-k}(\tilde{S}_{k-1})\}\beta_{2,\ell_k} - A^{(1)}_{k,2} = \sum_{i=1}^{\ell_k}\gamma_i\sum_{j=1}^{i}\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-j-1})X_{k-j}\big\} \\ + \sum_{i=1}^{\ell_k}\frac{\gamma_i}{2}\sum_{j=1}^{i}\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-j-1})(X^2_{k-j})^{(0)}\big\} + R^{(1)}_{n,k,2}\,, \quad (4.48)$$

where $R^{(1)}_{n,k,2}$ is such that

$$\sum_{k\in[1,n]}|R^{(1)}_{n,k,2}| \ll 1\,. \quad (4.49)$$

Now, let $r^{(1)}_{n,k,3} = \frac{1}{3}\mathbb{E}\{f^{(3)}_{n-k}(\tilde{S}_{k-1})\}\beta_{3,1,\ell_k} - A^{(1)}_{k,3}$. Then, recalling the notation $\gamma^{(2)}_i = \mathbb{E}(X^2_0X_i)$,

$$r^{(1)}_{n,k,3} = \sum_{i=1}^{\ell_k}\gamma^{(2)}_i\mathbb{E}\{f^{(3)}_{n-k}(\tilde{S}_{k-1})\} - A^{(1)}_{k,3} = \sum_{i=1}^{\ell_k}\gamma^{(2)}_i\sum_{j=1}^{i}\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-j}) - f^{(3)}_{n-k}(\tilde{S}_{k-j-1})\big\} \\ = \sum_{i=1}^{\ell_k}\gamma^{(2)}_i\sum_{j=1}^{i}\Big(\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-j-1})\tilde{X}_{k-j}\big\} + \int_0^1(1-t)\mathbb{E}\big\{f^{(5)}_{n-k}(\tilde{S}_{k-j-1}+t\tilde{X}_{k-j})\tilde{X}^2_{k-j}\big\}dt\Big) \\ := r^{(1)}_{n,k,3}(1) + r^{(1)}_{n,k,3}(2)\,. \quad (4.50)$$

Taking into account Item 1 of Lemma 4.1 and the fact that $|\gamma^{(2)}_i| \le M^2\theta(i)$ and $\|\tilde{X}_{k-j}\|_\infty \le 2M$, it follows that

$$|r^{(1)}_{n,k,3}(2)| \ll \sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\frac{\theta(i)}{(n-k+1)^{3/2}} \ll \frac{\sum_{i=1}^{\ell_k} i\ \theta(i)}{(n-k+1)^{3/2}}\,.$$

Therefore, since $\sum_{i\ge1} i\theta(i) < \infty$,

$$\sum_{k\in[1,n]}|r^{(1)}_{n,k,3}(2)| \ll 1\,. \quad (4.51)$$

On another hand, by Item 1 of Lemma 4.1,

$$\big|\gamma^{(2)}_i\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-j-1})\mathbb{E}_0(X_{k-j})\big\}\big| \ll (n-k+1)^{-1}\theta(k-j)\theta(i)\,.$$

Hence, since $\sum_{i\geq 1} i\theta(i) < \infty$,

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\big|\gamma_i^{(2)}\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-j-1})\mathbb{E}_0(X_{k-j})\big\}\big| \ll \sum_{k=1}^{n}\theta([k/2])\sum_{i=1}^{n} i\theta(i) \ll 1\,. \tag{4.52}$$

Starting from (4.26) and taking into account (4.27), (4.28), (4.29), (4.33), (4.40), (4.41), (4.48)-(4.52) and the fact that $\beta_2 = \sigma^2 = \mathbb{E}(X_0^2) + \beta_{2,\ell_k} + \beta_2^{(\ell_k)}$, we get

$$\begin{aligned}
\mathbb{E}(\Delta_{n,k}) = {} & \mathbb{E}(\Delta_{k,2}^{(1)}) + \frac{1}{2}\mathbb{E}\big(f_{n-k}''(\tilde{S}_{k-1})(X_k^2)^{(0)}\big) - \frac{1}{2}\mathbb{E}\big(f_{n-k}''(\tilde{S}_{k-1})\big)\beta_2^{(\ell_k)} + \frac{1}{2}\mathbb{E}(\Delta_{k,3}^{(1)}) \\
& - \sum_{i=1}^{\ell_k}\gamma_i\sum_{j=1}^{i}\mathbb{E}\Big\{f_{n-k}^{(3)}(\tilde{S}_{k-j-1})X_{k-j}\Big\} - \sum_{i=1}^{\ell_k}\frac{\gamma_i}{2}\sum_{j=1}^{i}\mathbb{E}\Big\{f_{n-k}^{(4)}(\tilde{S}_{k-j-1})(X_{k-j}^2)^{(0)}\Big\} \\
& - \sum_{i=1}^{\ell_k}\frac{\gamma_i^{(2)}}{2}\sum_{j=1}^{i}\Big(\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-j-1})X_{k-j}\big\} + \frac{1}{6}\mathbb{E}\big(f_{n-k}^{(3)}(\tilde{S}_{k-1})(X_k^3 - (\beta_3 - \beta_{3,1,\ell_k}))\big)\Big) \\
& + \frac{1}{6}\mathbb{E}(\Delta_{k,4}^{(1)}) + \Gamma_{n,k}^{(1)}\,,
\end{aligned} \tag{4.53}$$

where $\Gamma_{n,k}^{(1)}$ satisfies

$$\sum_{k\in[1,n]}|\Gamma_{n,k}^{(1)}| \ll 1\,. \tag{4.54}$$

Note first that

$$\sum_{k=1}^{n}|\mathbb{E}\big(f_{n-k}''(\tilde{S}_{k-1})\big)\beta_2^{(\ell_k)}| \ll \sum_{k=1}^{n}|\beta_2^{(\ell_k)}| \ll \sum_{k=1}^{n}\sum_{i\geq\ell_k+1}\theta(i) \ll \sum_{i\geq 1} i\theta(i) \ll 1\,. \tag{4.55}$$

To handle the first two terms in the right hand side of (4.53), define

$$m_k = [\sqrt{n-k}],\ m_{k,i} = \min(m_k, k-i-1) \text{ and } D_{k,i,2}^{(1)} = \mathbb{E}\big\{f_{n-k}''(\tilde{S}_{k-i-1})(X_{k-i}X_k)^{(0)}\big\}. \tag{4.56}$$

Then, for any integer i in $[0,\ell_k]$, with the convention that $\tilde{S}_u = 0$ for any $u \leq 0$, we write

$$D_{k,i,2}^{(1)} = \mathbb{E}\Big\{\Big(f_{n-k}''(\tilde{S}_{k-i-m_{k,i}-1}) + \sum_{j=i+1}^{i+m_{k,i}}(f_{n-k}''(\tilde{S}_{k-j}) - f_{n-k}''(\tilde{S}_{k-j-1})\Big)(X_{k-i}X_k)^{(0)}\Big\}.$$

Let then, for $\ell = 3,4,5$ and t in $[0,1]$,

$$\tilde{\Delta}_{k,i,2}^{(1,\ell)}(t) = \sum_{j=i+1}^{i+m_{k,i}} f_{n-k}^{(\ell)}(\tilde{S}_{k-j-1} + t\tilde{X}_{k-j})\tilde{X}_{k-j}^{\ell-2}(X_{k-i}X_k)^{(0)}. \tag{4.57}$$

By the Taylor integral formula,

$$\begin{aligned}
D_{k,i,2}^{(1)} = {} & \mathbb{E}\Big\{f_{n-k}''(\tilde{S}_{k-i-m_{k,i}-1})(X_{k-i}X_k)^{(0)} + \tilde{\Delta}_{k,i,2}^{(1,3)}(0) + \frac{1}{2}\tilde{\Delta}_{k,i,2}^{(1,4)}(0)\Big\} \\
& + \frac{1}{2}\int_0^1(1-t)^2\mathbb{E}\big\{\tilde{\Delta}_{k,i,2}^{(1,5)}(t)\big\}dt.
\end{aligned} \tag{4.58}$$

But, since $\|f''_{n-k}\|_\infty \ll 1$,

$$\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\big|\mathbb{E}\big\{f''_{n-k}(\tilde{S}_{k-i-m_{k,i}-1})(X_{k-i}X_k)^{(0)}\big\}\big| \ll \sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\big(\theta(m_k)+\theta(k-i)\big)\wedge\theta(i)$$
$$\ll \sum_{k=1}^{n}\Big(m_k\theta(m_k)+\sum_{i=m_k}^{\ell_k}\theta(i)+k\theta([k/2])\Big) \ll 1+\sum_{k=1}^{[\sqrt{n}]}k^2\theta(k)+n\sum_{k\geq[\sqrt{n}]}\theta(k)\,.$$

Hence

$$\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\big|\mathbb{E}\big\{f''_{n-k}(\tilde{S}_{k-i-m_{k,i}-1})(X_{k-i}X_k)^{(0)}\big\}\big| \ll 1+\sum_{k\geq1}k(k\wedge\sqrt{n})\theta(k)\,. \tag{4.59}$$

On another hand, by using Item 1 of Lemma 4.1,

$$\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\big|\mathbb{E}\big\{\tilde{\Delta}^{(1,5)}_{k,i,2}(t)\big\}\big| \ll \sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\frac{\theta(j-i)\wedge\theta(i)}{(n-k+1)^{3/2}} \ll \sum_{j=1}^{n}j\theta([j/2]) \ll 1\,. \tag{4.60}$$

For $\ell=3,4$, set

$$\Delta^{(1,\ell)}_{k,i,2} := \sum_{j=i+1}^{i+m_{k,i}} f^{(\ell)}_{n-k}(\tilde{S}_{k-j-1})X^{\ell-2}_{k-j}(X_{k-i}X_k)^{(0)}. \tag{4.61}$$

Applying Item 1 of Lemma 4.1 and using that $m_{k,i}\leq\sqrt{n-k+1}$, we get

$$\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\big|\mathbb{E}\big\{\tilde{\Delta}^{(1,3)}_{k,i,2}(0)-\Delta^{(1,3)}_{k,i,2}\big\}\big| \leq \sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\|\mathbb{E}_0(X_{k-j})f^{(3)}_{n-k}(\tilde{S}_{k-j-1})(X_{k-i}X_k)^{(0)}\|_1$$
$$\ll \sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\frac{\theta(k-j)\wedge\theta(j-i)\wedge\theta(i)}{(n-k+1)^{1/2}} \ll \sum_{k=1}^{n}k\theta([k/3]) \ll 1\,. \tag{4.62}$$

Similarly, since $\|f^{(4)}_{n-k}\|_\infty \ll (n-k+1)^{-1}$, we derive

$$\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\big|\mathbb{E}\big\{\tilde{\Delta}^{(1,4)}_{k,i,2}(0)-\Delta^{(1,4)}_{k,i,2}\big\}\big| \ll 1\,. \tag{4.63}$$

Starting from (4.53) and taking into account (4.54), (4.55), (4.58), (4.59), (4.60), (4.62) and (4.63) , we then derive that

$$\begin{aligned}\mathbb{E}(\Delta_{n,k}) &= \frac{1}{2}\sum_{i=0}^{\ell_k}(1+\mathbf{1}_{\{i\neq0\}})\mathbb{E}\Big\{\Delta^{(1,3)}_{k,i,2}+\frac{1}{2}\Delta^{(1,4)}_{k,i,2}\Big\}+\frac{1}{2}\mathbb{E}(\Delta^{(1)}_{k,3})\\
&\quad-\sum_{i=1}^{\ell_k}\gamma_i\sum_{j=1}^{i}\mathbb{E}\Big\{f^{(3)}_{n-k}(\tilde{S}_{k-j-1})X_{k-j}\Big\}-\sum_{i=1}^{\ell_k}\frac{\gamma_i}{2}\sum_{j=1}^{i}\mathbb{E}\Big\{f^{(4)}_{n-k}(\tilde{S}_{k-j-1})(X^2_{k-j})^{(0)}\Big\}\\
&\quad-\sum_{i=1}^{\ell_k}\frac{\gamma^{(2)}_i}{2}\sum_{j=1}^{i}\Big(\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-j-1})X_{k-j}\big\}+\frac{1}{6}\mathbb{E}\big(f^{(3)}_{n-k}(\tilde{S}_{k-1})(X^3_k-(\beta_3-\beta_{3,1,\ell_k}))\big)\\
&\quad+\frac{1}{6}\mathbb{E}(\Delta^{(1)}_{k,4})+\Gamma^{(2)}_{n,k}\,,\end{aligned} \tag{4.64}$$

where $\Gamma^{(2)}_{n,k}$ satisfies $\sum_{k=1}^n |\Gamma^{(2)}_{n,k}| \ll 1 + \sum_{k\geq 1} k(k\wedge\sqrt{n})\theta(k)$. Introduce now the following notations.

Notation 4.5. Let $\beta_{3,2,m_k} = 3\sum_{i=1}^{m_{k,0}} \mathbb{E}(X_0X_i^2)$, $\beta^*_{3,\ell_k,m_k} = 6\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}} \mathbb{E}(X_0X_{j-i}X_j)$. Next, let $\tilde{\beta}_3^{(\ell_k,m_k)} = \beta_3 - \beta_{3,1,\ell_k} - \big\{\mathbb{E}(X_0^3) + \beta_{3,2,m_k} + \beta^*_{3,\ell_k,m_k}\big\}$, where we recall that m_k and $m_{k,i}$ have been defined in (4.56).

Since

$$\begin{aligned}\tilde{\beta}_3^{(\ell_k,m_k)} = \;& 3\sum_{i>\ell_k}\mathbb{E}(X_0^2X_i) + 3\sum_{i>m_{k,0}}\mathbb{E}(X_0X_i^2)\\ &+ 6\sum_{i=1}^{\ell_k}\sum_{j>m_{k,i}}\mathbb{E}(X_0X_jX_{j+i}) + 6\sum_{i>\ell_k}\sum_{j\geq 1}\mathbb{E}(X_0X_jX_{j+i}),\end{aligned}$$

by Item 1 of Lemma 4.1,

$$\begin{aligned}\sum_{k=1}^n \|f^{(3)}_{n-k}\|_\infty\big|\tilde{\beta}_3^{(\ell_k,m_k)}\big| \ll\;& \sum_{k=1}^n (n-k+1)^{-1/2}\sum_{i\geq \ell_k\wedge m_k}\theta(i)\\ &+\sum_{k=1}^n (n-k+1)^{-1/2}\Big(\sum_{i=1}^{\ell_k}\sum_{j\geq m_{k,i}+1}\theta(j)\wedge\theta(i) + \sum_{i\geq\ell_k}\sum_{j\geq 1}\theta(j)\wedge\theta(i)\Big).\end{aligned}$$

By simple algebra, and since $\sum_{i\geq 1} i\theta(i) < \infty$, we then derive that

$$\sum_{k=1}^n \|f^{(3)}_{n-k}\|_\infty|\tilde{\beta}_3^{(\ell_k,m_k)}| \ll 1 + \sum_{i\geq 1} i(i\wedge\sqrt{n})\theta(i)\,. \tag{4.65}$$

Next we shall first center the random variables $X_{k-j}(X_{k-i}X_k)^{(0)}$ appearing in the quantity $\Delta^{(1,3)}_{k,i,2}$. Using that $\mathbb{E}\big\{X_{k-j}(X_{k-i}X_k)^{(0)}\big\} = \mathbb{E}\big\{X_{k-j}X_{k-i}X_k\big\}$, an application of Item 2 of Lemma 4.1 gives

$$\begin{aligned}J_{1,k} :=\;& \Big|\frac{1}{6}\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-1})\big\}\beta^*_{3,\ell_k,m_k} - \sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-j-1})\big\}\mathbb{E}\big\{X_{k-j}(X_{k-i}X_k)^{(0)}\big\}\Big|\\ \ll\;& \sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\Big|\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-1}) - f^{(3)}_{n-k}(\tilde{S}_{k-j-1})\big\}\Big|(\theta(j-i)\wedge\theta(i))\,.\end{aligned} \tag{4.66}$$

Let us handle the quantity $\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-1}) - f^{(3)}_{n-k}(\tilde{S}_{k-j-1})\big\}$. By Taylor integral formula,

$$\begin{aligned}\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-1}) - f^{(3)}_{n-k}(\tilde{S}_{k-j-1})\big\} =\;& \sum_{\ell=1}^{j}\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-\ell-1})\tilde{X}_{k-\ell}\big\}\\ &+\int_0^1(1-t)\sum_{\ell=1}^{j}\mathbb{E}\big\{f^{(5)}_{n-k}(\tilde{S}_{k-\ell-1}+t\tilde{X}_{k-\ell})\tilde{X}^2_{k-\ell}\big\}dt\,.\end{aligned}$$

By using Item 1 of Lemma 4.1 and noticing that $\theta(j-i)\wedge\theta(i)\le\theta([j/2])$, we get

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{\ell=1}^{j}\big|\mathbb{E}\big\{f_{n-k}^{(5)}(\tilde S_{k-\ell-1}+t\tilde X_{k-\ell})\tilde X_{k-\ell}^2\big\}(\theta(j-i)\wedge\theta(i))\big|\\
&\ll\sum_{k=1}^{n}(n-k+1)^{-3/2}\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}j\theta([j/2])\\
&\ll\sum_{k=1}^{n}(n-k+1)^{-3/2}\Big\{\sum_{j=1}^{2[\sqrt{n}]}j^2\theta(j)+m_k\sum_{j\ge[\sqrt{n}]}j\theta(j)\Big\}\ll\sum_{i\ge1}i(i\wedge\sqrt{n})\theta(i)\,. \qquad (4.67)
\end{aligned}$$

Next, by Item 1 of Lemma 4.1 again,

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{\ell=1}^{j}\big|\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde S_{k-\ell-1})\mathbb{E}_0(X_{k-\ell})\big\}(\theta(j-i)\wedge\theta(i))\big|\\
&\qquad\ll\sum_{k=1}^{n}(n-k+1)^{-1}\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{\ell=1}^{j}\theta(k-\ell)\theta([j/2])\\
&\qquad\ll\sum_{k=1}^{[n/2]}(n-k+1)^{-1}\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}j\theta([j/2])\\
&\qquad\quad+\sum_{k=[n/2]+1}^{n}(n-k+1)^{-1}\sum_{i=1}^{[k/2]}\sum_{j=i+1}^{i+m_k}\sum_{\ell=1}^{[k/2]+m_k}\theta(k-\ell)\theta([j/2])\,. \qquad (4.68)
\end{aligned}$$

With the computations as given in (4.67) and the fact that

$$\begin{aligned}
&\sum_{k=[n/2]+1}^{n}(n-k+1)^{-1}\sum_{i=1}^{[k/2]}\sum_{j=i+1}^{i+m_k}\sum_{\ell=1}^{[k/2]+m_k}\theta(k-\ell)\theta([j/2])\\
&\qquad\ll\sum_{k=[n/2]+1}^{n}(n-k+1)^{-1}k\theta([k/4])\sum_{i=1}^{[k/2]}\sum_{j=i+1}^{i+m_k}\theta([j/2])\ll\sum_{k\ge1}k\theta(k)\sum_{i\ge1}\theta(i)\,, \qquad (4.69)
\end{aligned}$$

we derive, overall, that

$$J_{1,k}\le\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{\ell=1}^{j}\big|\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde S_{k-\ell-1})X_{k-\ell}\big\}\big|\theta([j/2])+\Gamma_{n,k}^{(3)}\,, \qquad (4.70)$$

where $\Gamma_{n,k}^{(3)}$ satisfies $1+\sum_{k=1}^{n}|\Gamma_{n,k}^{(3)}|\ll\sum_{i\ge1}i(i\wedge\sqrt{n})\theta(i)$. Next, for $m_{k,\ell}$ defined in (4.56), write

$$\begin{aligned}
&\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde S_{k-\ell-1})X_{k-\ell}\big\}\\
&\qquad=\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde S_{k-\ell-m_{k,\ell}-1})X_{k-\ell}\big\}+\sum_{u=\ell+1}^{\ell+m_{k,\ell}}\mathbb{E}\big\{(f_{n-k}^{(4)}(\tilde S_{k-u})-f_{n-k}^{(4)}(\tilde S_{k-u-1})X_{k-\ell}\big\}\,,
\end{aligned}$$

implying, by using Item 1 of Lemma 4.1, that

$$\begin{aligned}&\big|\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-\ell-1})X_{k-\ell}\big\}\big|\\&\qquad\ll (n-k+1)^{-1}(\theta(m_k)+\theta(k-\ell))+(n-k+1)^{-3/2}\sum_{u=\ell+1}^{\ell+m_{k,\ell}}\theta(u-\ell)\,.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{\ell=1}^{j}\big|\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-\ell-1})X_{k-\ell}\big\}\big|\theta([j/2])\\&\qquad\ll\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}(n-k+1)^{-1}\theta([j/2])\Big\{j\theta(m_k)+\sum_{\ell=1}^{j}\theta(k-\ell)\Big\}\\&\qquad\qquad+\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}(n-k+1)^{-3/2}j\theta([j/2])\sum_{u=1}^{\sqrt{n}}\theta(u)\,.\end{aligned}$$

With the computations as given in (4.67)-(4.69) together with the fact that

$$\sum_{k=1}^{n}(n-k+1)^{-1}m_k\theta(m_k)=\sum_{k=1}^{n}k^{-1/2}\theta([\sqrt{k}])\ll\sum_{k=1}^{[\sqrt{n}]}\theta(k)\,,$$

it follows that

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{\ell=1}^{j}\big|\mathbb{E}\big\{f^{(4)}_{n-k}(\tilde{S}_{k-\ell-1})X_{k-\ell}\big\}\big|\theta([j/2])\ll 1+\sum_{i\geq 1}i(i\wedge\sqrt{n})\theta(i)\,.\tag{4.71}$$

Therefore (4.70) together with (4.71) imply

$$\begin{aligned}&\sum_{k=1}^{n}\Big|\frac{1}{6}\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-1})\big\}\beta^{*}_{3,\ell_k,m_k}-\sum_{i=1}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-j-1})\big\}\mathbb{E}\big\{X_{k-j}(X_{k-i}X_k)^{(0)}\big\}\Big|\\&\qquad\ll 1+\sum_{i\geq 1}i(i\wedge\sqrt{n})\theta(i)\,.\end{aligned}\tag{4.72}$$

With similar arguments, we infer that

$$\begin{aligned}&\sum_{k=1}^{n}\Big|\frac{1}{3}\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-1})\big\}\beta_{3,2,m_k}-\sum_{j=1}^{m_{k,0}}\mathbb{E}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-j-1})\big\}\mathbb{E}\big\{X_{k-j}(X_k^2)^{(0)}\big\}\Big|\\&\qquad\ll 1+\sum_{i\geq 1}i(i\wedge\sqrt{n})\theta(i)\,.\end{aligned}\tag{4.73}$$

Now, for any integer $i\in[0,n]$, let

$$\Delta^{(1,3,0)}_{k,i,2}:=\sum_{j=i+1}^{i+m_{k,i}}\big\{f^{(3)}_{n-k}(\tilde{S}_{k-j-1})(X_{k-j}(X_{k-i}X_k)^{(0)})^{(0)}\big\}\,.$$

Starting from (4.64) and taking into account (4.65), (4.72) and (4.73), we then obtain

$$\begin{aligned}\mathbb{E}(\Delta_{n,k}) &= \frac{1}{2}\sum_{i=0}^{\ell_k}(1+\mathbf{1}_{\{i\neq 0\}})\mathbb{E}\big\{\Delta_{k,i,2}^{(1,3,0)}\big\} + \frac{1}{2}\mathbb{E}(\Delta_{k,3}^{(1)}) + \frac{1}{6}\mathbb{E}\big(f_{n-k}^{(3)}(\tilde{S}_{k-1})(X_k^3 - \mathbb{E}(X_0^3))\big)\\ &- \sum_{i=1}^{\ell_k}\gamma_i\sum_{j=1}^{i}\mathbb{E}\big\{f_{n-k}^{(3)}(\tilde{S}_{k-j-1})X_{k-j}\big\} - \sum_{i=1}^{\ell_k}\frac{\gamma_i}{2}\sum_{j=1}^{i}\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-j-1})(X_{k-j}^2)^{(0)}\big\}\\ &- \sum_{i=1}^{\ell_k}\frac{\gamma_i^{(2)}}{2}\sum_{j=1}^{i}\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-j-1})X_{k-j}\big\} + \frac{1}{4}\sum_{i=0}^{\ell_k}(1+\mathbf{1}_{\{i\neq 0\}})\mathbb{E}\big\{\Delta_{k,i,2}^{(1,4)}\big\} + \frac{1}{6}\mathbb{E}(\Delta_{k,4}^{(1)}) + \Gamma_{n,k}^{(4)}\,,\end{aligned} \tag{4.74}$$

where $\Gamma_{n,k}^{(4)}$ satisfies $\sum_{k=1}^{n}|\Gamma_{n,k}^{(3)}| \ll 1 + \sum_{i\geq 1} i(i\wedge\sqrt{n})\theta(i)$.

In what follows we continue the estimation of each term in the right-hand side of (4.74) and show that the sum over k from 1 to n of their absolute values is bounded by a constant times $\{1+\sum_{i\geq 1} i(i\wedge\sqrt{n})\theta(i)\}$. Let us start by dealing with the quantities $\Delta_{k,i,2}^{(1,3,0)}$. With this aim, note first that for $m_{k,j}$ defined in (4.56),

$$\begin{aligned}\big|\mathbb{E}\big\{f_{n-k}^{(3)}(\tilde{S}_{k-j-m_{k,j}-1})(X_{k-j}(X_{k-i}X_k)^{(0)})^{(0)}\big\}\big| &\ll \|f_{n-k}^{(3)}\|_\infty\big(\theta(m_{k,j})\wedge\theta(j-i)\wedge\theta(i)\big)\\ &\ll \|f_{n-k}^{(3)}\|_\infty\big(\theta(m_k)\wedge\theta(j-i)\wedge\theta(i)+\theta(k-j)\wedge\theta(j-i)\wedge\theta(i)\big)\,.\end{aligned}$$

Hence, by Item 1 of Lemma 4.1 and the fact that $m_k \leq \sqrt{n-k+1}$,

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\big|\mathbb{E}\big\{f_{n-k}^{(3)}(\tilde{S}_{k-j-m_{k,j}-1})(X_{k-j}(X_{k-i}X_k)^{(0)})^{(0)}\big\}\big|\\ &\qquad\ll \sum_{k=1}^{n}\frac{m_k}{\sqrt{n-k+1}}\Big(m_k\theta(m_k)+\sum_{i\geq m_k}\theta(i)+\ell_k\theta([k/3]\Big)\\ &\qquad\ll \sum_{k=1}^{n}\sqrt{k}\theta([\sqrt{k}]) + \sum_{k=1}^{n}\sum_{i\geq[\sqrt{k}]}\theta(i) + \sum_{k=1}^{n}k\theta(k) \ll 1+\sum_{i\geq 1}i(i\wedge\sqrt{n})\theta(i)\,.\end{aligned} \tag{4.75}$$

On another hand, by the Taylor integral formula,

$$\begin{aligned}&\mathbb{E}\big\{\big(f_{n-k}^{(3)}(\tilde{S}_{k-j-1}) - f_{n-k}^{(3)}(\tilde{S}_{k-j-m_{k,j}-1})\big)(X_{k-j}(X_{k-i}X_k)^{(0)})^{(0)}\big\}\\ &\quad= \sum_{u=1}^{m_{k,j}}\mathbb{E}\big\{\big(f_{n-k}^{(3)}(\tilde{S}_{k-j-u}) - f_{n-k}^{(3)}(\tilde{S}_{k-j-u-1})\big)(X_{k-j}(X_{k-i}X_k)^{(0)})^{(0)}\big\}\\ &\quad= \sum_{u=1}^{m_{k,j}}\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-j-u-1})\tilde{X}_{k-j-u}(X_{k-j}(X_{k-i}X_k)^{(0)})^{(0)}\big\}\\ &\qquad+\sum_{u=1}^{m_{k,j}}\int_0^1(1-t)\mathbb{E}\big\{f_{n-k}^{(5)}(\tilde{S}_{k-j-u-1}+t\tilde{X}_{k-j-u})\tilde{X}_{k-j-u}^2(X_{k-j}(X_{k-i}X_k)^{(0)})^{(0)}\big\}dt\,.\end{aligned}$$

According to Item 1 of Lemma 4.1,

$$\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{u=1}^{m_{k,j}}|\mathbb{E}\big\{f_{n-k}^{(5)}(\tilde{S}_{k-j-u-1}+t\tilde{X}_{k-j-u})\tilde{X}_{k-j-u}^2(X_{k-j}(X_{k-i}X_k)^{(0)})^{(0)}\big\}|$$
$$\ll\sum_{k=1}^{n}\|f_{n-k}^{(5)}\|_\infty\sum_{i=0}^{\ell_k}\sum_{j=1}^{m_k}\sum_{u=1}^{m_k}\big(\theta(u)\wedge\theta(j)\wedge\theta(i)\big)$$
$$\ll\sum_{k=1}^{n}(n-k+1)^{-3/2}\Big\{\sum_{u=1}^{[\sqrt{n}]}u^2\theta(u)+m_k^2\sum_{i\ge m_k+1}\theta(i)\Big\}$$
$$\ll\sum_{u=1}^{[\sqrt{n}]}u^2\theta(u)+\sum_{i\ge1}i\theta(i)\ll1+\sum_{i\ge1}i(i\wedge\sqrt{n})\theta(i)\,. \tag{4.76}$$

Next, let $Z_{k,j,u,i}:=\tilde{X}_{k-u}(X_{k-j}(X_{k-i}X_k)^{(0)})^{(0)}$ and $Z_{k,j,u,i}^{(0)}:=Z_{k,j,u,i}-\mathbb{E}(Z_{k,j,u,i})$. Since

$$|\mathbb{E}(Z_{k,j,u,i})|\ll\big(\theta(u-j)+\theta(k-j)\big)\wedge\theta(j-i)\wedge\theta(i)\,,$$

by Item 2 of Lemma 4.1,

$$\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{u=j+1}^{j+m_{k,j}}\big|\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-u-1})\big\}\mathbb{E}(Z_{k,j,u,i})\big|$$
$$\ll\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{u=j+1}^{j+m_{k,j}}\frac{\big(\big(\theta(u-j)+\theta(k-j)\big)\wedge\theta(j-i)\wedge\theta(i)\big)}{(n-u)\wedge(n-k+1)^{3/2}}$$
$$\ll\sum_{k=1}^{n}\big((n-k+1)^{-3/2}+n^{-1}\big)\Big(m_k^2\sum_{i=m_k}^{\ell_k}\theta(i)+\sum_{u=1}^{m_k}u^2\theta(u)+m_k^2k\theta([k/3]\Big)$$
$$\ll1+\sum_{u\ge1}u(u\wedge\sqrt{n})\theta(u)\,. \tag{4.77}$$

On another hand, for $m_{k,u}$ defined in (4.56),

$$\big|\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-u-m_{k,u}-1})Z_{k,j,u,i}^{(0)}\big\}\big|\ll\|f_{n-k}^{(4)}\|_\infty\Big\{\big(\theta(m_k)\wedge\theta(u-j)\wedge\theta(j-i)\wedge\theta(i)\big)$$
$$+\big(\theta(k-u)\wedge\theta(u-j)\wedge\theta(j-i)\wedge\theta(i)\big)\Big\}\,.$$

Hence, using Item 1 of Lemma 4.1 and the fact that $m_k^2\le n-k+1$,

$$\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{u=j+1}^{j+m_{k,j}}\big|\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-u-m_{k,u}-1})Z_{k,j,u,i}^{(0)}\big\}\big|$$
$$\ll\sum_{k=1}^{n}\frac{m_k^2}{n-k+1}\Big(m_k\theta(m_k)+\sum_{i\ge m_k}\theta(i)+k\theta([k/4])\Big)$$
$$\ll\sum_{k=1}^{n}\Big(\sqrt{k}\theta([\sqrt{k}])+\sum_{i\ge[\sqrt{k}]}\theta(i)+k\theta(k)\Big)\ll1+\sum_{u\ge1}u(u\wedge\sqrt{n})\theta(u)\,. \tag{4.78}$$

Next

$$\begin{aligned}\mathbb{E}\big\{\big(f^{(4)}_{n-k}(\tilde S_{k-u-1}) - f^{(4)}_{n-k}(\tilde S_{k-u-m_{k,u}-1})\big)Z^{(0)}_{k,j,u,i}\big\}&\\ = \sum_{v=u+1}^{u+m_{k,u}} \mathbb{E}\big\{\big(f^{(4)}_{n-k}(\tilde S_{k-v}) - f^{(4)}_{n-k}(\tilde S_{k-v-1})\big)Z^{(0)}_{k,j,u,i}\big\}&\\ = \sum_{v=u+1}^{u+m_{k,u}} \int_0^1 \mathbb{E}\big\{\big(f^{(5)}_{n-k}(\tilde S_{k-v-1} + t\tilde X_{k-v})\tilde X_{k-v}Z^{(0)}_{k,j,u,i}\big\}dt\,.&\end{aligned}$$

Therefore, by Item 1 of Lemma 4.1,

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{u=j+1}^{j+m_{k,j}} \big|\mathbb{E}\big\{\big(f^{(4)}_{n-k}(\tilde S_{k-u-1}) - f^{(4)}_{n-k}(\tilde S_{k-u-m_{k,u}-1})\big)Z^{(0)}_{k,j,u,i}\big\}\big|\\ &\ll \sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\sum_{u=j+1}^{j+m_{k,j}}\sum_{v=u+1}^{u+m_{k,u}} \|f^{(5)}_{n-k}\|_\infty\big(\theta(v-u)\wedge\theta(u-j)\wedge\theta(j-i)\wedge\theta(i)\big)\\ &\ll \sum_{k=1}^{n}\frac{1}{(n-k+1)^{3/2}}\sum_{\ell=1}^{m_k}\ell^3\theta(\ell) + \sum_{k=1}^{n}\frac{m_k^3}{(n-k+1)^{3/2}}\sum_{i\ge m_k+1}\theta(i)\\ &\ll \sum_{k=1}^{n}\frac{1}{k^{3/2}}\sum_{\ell=1}^{[\sqrt{k}]}\ell^3\theta(\ell) + \sum_{k=1}^{n}\sum_{i\ge[\sqrt{k}]}\theta(i) \ll \sum_{u\ge1}u(u\wedge\sqrt{n})\theta(u)\,.\end{aligned} \tag{4.79}$$

Taking into account (4.75), (4.76), (4.77), (4.78) and (4.79), it follows that

$$\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\big|\mathbb{E}(\Delta^{(1,3,0)}_{k,i,2})\big| \ll 1 + \sum_{u\ge1}u(u\wedge\sqrt{n})\theta(u)\,. \tag{4.80}$$

With similar (but even simpler) arguments, we infer that the sum over k from 1 to n of the second and third terms in the right-hand side of (4.74) are also bounded by a constant times $\{1+\sum_{u\ge1}u(u\wedge\sqrt{n})\theta(u)\}$. More precisely,

$$\sum_{k=1}^{n}\Big\{\big|\mathbb{E}(\Delta^{(1)}_{n,k,3})\big| + \big|\mathbb{E}\big(f^{(3)}_{n-k}(\tilde S_{k-1})(X_k^3 - \mathbb{E}(X_0^3))\big)\big|\Big\} \ll 1 + \sum_{u\ge1}u(u\wedge\sqrt{n})\theta(u)\,. \tag{4.81}$$

We deal now with the fourth term of the right hand side of (4.74). With this aim, recalling the definition (4.56) of $m_{k,j}$, note that

$$\big|\mathbb{E}\Big\{f^{(3)}_{n-k}(\tilde S_{k-j-m_{k,j}-1})X_{k-j}\Big\}\big| \ll \|f^{(3)}_{n-k}\|_\infty\theta(m_{k,j}) \ll \|f^{(3)}_{n-k}\|_\infty\big(\theta(m_k)+\theta(k-j)\big)\,.$$

Hence, by Item 1 of Lemma 4.1 and recalling the notation $\gamma_i = \mathbb{E}(X_0 X_i)$, we get

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\big|\gamma_i \mathbb{E}\big\{f_{n-k}^{(3)}(\tilde{S}_{k-j-m_{k,j}-1})X_{k-j}\big\}\big|$$
$$\ll \sum_{k=1}^{n}(n-k+1)^{-1/2}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\theta(i)(\theta(m_k)+\theta(k-j))$$
$$\ll \sum_{k=1}^{n}k^{-1/2}\theta([\sqrt{k}])\sum_{i\geq 1}i\theta(i) + \sum_{k=1}^{n}\theta([k/2])\sum_{i\geq 1}i\theta(i) \ll 1\,. \tag{4.82}$$

Next, by the Taylor integral formula,

$$\mathbb{E}\big\{\big(f_{n-k}^{(3)}(\tilde{S}_{k-j-1})-f_{n-k}^{(3)}(\tilde{S}_{k-j-m_{k,j}-1})\big)X_{k-j}\big\} = \sum_{u=j+1}^{j+m_{k,j}}\mathbb{E}\big\{\big(f_{n-k}^{(3)}(\tilde{S}_{k-u})-f_{n-k}^{(3)}(\tilde{S}_{k-u-1})\big)X_{k-j}\big\}$$
$$= \sum_{u=j+1}^{j+m_{k,j}}\Big(\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-u-1})\tilde{X}_{k-u}X_{k-j}\big\}+\int_0^1(1-t)\mathbb{E}\big\{f_{n-k}^{(5)}(\tilde{S}_{k-u-1}+t\tilde{X}_{k-u})\tilde{X}_{k-u}^2X_{k-j}\big\}dt\Big)\,.$$

But, by Item 1 of Lemma 4.1,

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\sum_{u=j+1}^{j+m_{k,i}}\big|\gamma_i\mathbb{E}\big\{f_{n-k}^{(5)}(\tilde{S}_{k-u-1}+t\tilde{X}_{k-u})\tilde{X}_{k-u}^2X_{k-j}\big\}\big|$$
$$\ll \sum_{k=1}^{n}(n-k+1)^{-3/2}\sum_{u\geq 1}\theta(u)\sum_{i\geq 1}i\theta(i)\ll 1\,. \tag{4.83}$$

On another hand, by Item 1 of Lemma 4.1 again,

$$\big|\gamma_i\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-u-1})(\tilde{X}_{k-u}-X_{k-u})X_{k-j}\big\}\big| = \big|\gamma_i\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-u-1})\mathbb{E}_0(X_{k-u})X_{k-j}\big\}\big|$$
$$\ll (n-k+1)^{-1}(\theta(k-u)\wedge\theta(u-j))\theta(i) \ll (n-k+1)^{-1}\theta([(k-j)/2])\theta(i)\,.$$

Hence,

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\sum_{u=j+1}^{j+m_{k,j}}\big|\gamma_i\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-u-1})(\tilde{X}_{k-u}-X_{k-u})X_{k-j}\big\}\big|$$
$$\ll \sum_{k=1}^{n}\theta([k/4])\sum_{i=1}^{n}i\theta(i)\ll 1\,. \tag{4.84}$$

Moreover, by Item 2 of Lemma 4.1,

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\sum_{u=j+1}^{j+m_{k,i}}\big|\gamma_i\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-u-1})\big\}\mathbb{E}(X_{k-u}X_{k-j})\big|$$
$$\ll \sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\sum_{u=j+1}^{j+m_{k,i}}\frac{\theta(u-j)\theta(i)}{(n-u)\wedge(n-k+1)^{3/2}} \ll \sum_{u\geq 1}\theta([u/2])\sum_{i\geq 1}i\theta(i)<\infty\,. \tag{4.85}$$

Hence, taking into account (4.82), (4.83), (4.84) and (4.85), we derive that

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\big|\gamma_i\mathbb{E}\big\{f_{n-k}^{(3)}(\tilde{S}_{k-j-1})X_{k-j}\big\}\big| \\ \ll 1+\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\sum_{u=j+1}^{j+m_{k,j}}\big|\gamma_i\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-u-1})(X_{k-u}X_{k-j})^{(0)}\big\}\big|\,. \quad (4.86)$$

Next, recalling the definition (4.56) of $m_{k,u}$, by Item 1 of Lemma 4.1, note that

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\sum_{u=j+1}^{j+m_{k,j}}\big|\gamma_i\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-u-m_{k,u}-1})(X_{k-u}X_{k-j})^{(0)}\big\}\big| \\ &\quad\ll\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\sum_{u=j+1}^{j+m_{k,j}}\|f_{n-k}^{(4)}\|_\infty\big\{\theta(m_{k,u})\wedge\theta(u-j)\big\}\theta(i) \\ &\quad\ll\sum_{k=1}^{n}\frac{m_k\theta(m_k)}{n-k+1}\sum_{i=1}^{n}i\theta(i)+\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\sum_{u=j+1}^{k}\big\{\theta(k-u)\wedge\theta(u-j)\big\}\theta(i) \\ &\quad\ll\Big(\sum_{i=1}^{n}i\theta(i)\Big)^2\ll 1\,. \end{aligned} \quad (4.87)$$

On another hand,

$$\begin{aligned}&\big|\mathbb{E}\big\{\big(f_{n-k}^{(4)}(\tilde{S}_{k-u-1})-f_{n-k}^{(4)}(\tilde{S}_{k-u-m_{k,u}-1})\big)(X_{k-u}X_{k-j})^{(0)}\big\}\big| \\ &\qquad\le\sum_{v=u+1}^{u+m_{k,u}}\big|\mathbb{E}\big\{\big(f_{n-k}^{(4)}(\tilde{S}_{k-v})-f_{n-k}^{(4)}(\tilde{S}_{k-v-1})\big)(X_{k-u}X_{k-j})^{(0)}\big\}\big| \\ &\qquad\le\sum_{v=u+1}^{u+m_{k,u}}\int_0^1\big|\mathbb{E}\big\{f_{n-k}^{(5)}(\tilde{S}_{k-v-1}+tX_{k-v})\tilde{X}_{k-v}(X_{k-u}X_{k-j})^{(0)}\big\}\big|dt\,.\end{aligned}$$

Hence,

$$\begin{aligned}&\big|\gamma_i\mathbb{E}\big\{\big(f_{n-k}^{(4)}(\tilde{S}_{k-u-1})-f_{n-k}^{(4)}(\tilde{S}_{k-u-m_{k,u}-1})\big)(X_{k-u}X_{k-j})^{(0)}\big\}\big| \\ &\qquad\ll\|f_{n-k}^{(5)}\|_\infty\sum_{v=u+1}^{u+m_k}\big(\theta(v-u)\wedge\theta(u-j)\big)\theta(i)\,.\end{aligned} \quad (4.88)$$

Taking into account (4.87) and (4.88) together with Item 1 of Lemma 4.1, it follows that

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\sum_{u=j+1}^{j+m_{k,j}}\big|\gamma_i\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-u-1})(X_{k-u}X_{k-j})^{(0)}\big\}\big| \\ &\qquad\ll 1+\sum_{k=1}^{n}(n-k+1)^{-3/2}\sum_{v=1}^{m_k}v\theta(v)\sum_{i=1}^{n}i\theta(i)\ll 1\,.\end{aligned} \quad (4.89)$$

Starting from (4.86) and taking into account the upper bound (4.89), we get that the sum over k from 1 to n of the fourth term in the right-hand side of (4.74) is uniformly bounded as a function of n. More precisely,

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\left|\gamma_i\mathbb{E}\big\{f_{n-k}^{(3)}(\tilde{S}_{k-j-1})X_{k-j}\big\}\right| \ll 1\,. \tag{4.90}$$

Similar computations (even simpler since we deal with the fourth derivative rather than the third one) give the following upper bound concerning the quantities involved in the fifth and sixth terms in the right-hand side of (4.74):

$$\sum_{k=1}^{n}\sum_{i=1}^{\ell_k}\sum_{j=1}^{i}\Big(\left|\gamma_i\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-j-1})(X_{k-j}^2)^{(0)}\big\}\right| + \left|\gamma_i^{(2)}\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-j-1})X_{k-j}\big\}\right|\Big) \ll 1\,. \tag{4.91}$$

We deal now with the last terms in the decomposition (4.74) and show that

$$\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\left|\mathbb{E}(\Delta_{k,i,2}^{(1,4)})\right| \ll 1 \text{ and } \sum_{k=1}^{n}\left|\mathbb{E}(\Delta_{k,4}^{(1)})\right| \ll 1\,, \tag{4.92}$$

where we recall that $\Delta_{k,i,2}^{(1,4)}$ and $\Delta_{k,4}^{(1)}$ have been respectively defined in (4.61) and (4.42). With this aim, note first that, by Item 2 of Lemma 4.1,

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\left|\mathbb{E}\big\{f_{n-k}^{(4)}(\tilde{S}_{k-j-1})\mathbb{E}\big(X_{k-j}^2(X_{k-i}X_k)^{(0)}\big)\big\}\right| \\
&\quad\ll \sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\big((n-k+1)^{-3/2}+(n-j)^{-1}\big)\big(\theta(j-i)\wedge\theta(i)\ \big) \\
&\quad\ll \sum_{k=1}^{n}\big((n-k+1)^{-3/2}+n^{-1}\big)\Big(m_k\sum_{i\geq m_k}\theta(i)+\sum_{u=1}^{m_k}u\theta(u)\Big) \ll \sum_{u\geq 1}u\theta(u)\,.
\end{aligned} \tag{4.93}$$

Next, let $W_{k,i,j} = \big(X_{k-j}^2(X_{k-i}X_k)^{(0)}\big)^{(0)}$. We start by noticing that

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\left|\mathbb{E}\big\{\big(f_{n-k}^{(4)}(\tilde{S}_{k-j-m_{k,j}-1})W_{k,i,j}\big\}\right| \\
&\qquad\qquad\ll \sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}}\|f_{n-k}^{(4)}\|_\infty\big(\theta(m_{k,j})\wedge\theta(j-i)\wedge\theta(i)\ \big)\,.
\end{aligned}$$

But, $\theta(m_{k,j}) = \theta(m_k) \vee \theta(k-j)$. Hence, using Item 1 of Lemma 4.1,

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}} \big|\mathbb{E}\big\{\big(f_{n-k}^{(4)}(\tilde{S}_{k-j-m_{k,j}-1})W_{k,i,j}\big\}\big| \\
&\ll \sum_{k=1}^{n}(n-k+1)^{-1}\Big(m_k^2\theta(m_k) + m_k\sum_{i\geq m_k}\theta(i) + m_k\, k\theta([k/3])\Big) \\
&\ll \sum_{k=1}^{n}\theta(m_k) + \sum_{k=1}^{n}(n-k+1)^{-1/2}\sum_{i\geq m_k}\theta(i) + \sum_{k=1}^{n}k\theta(k) \ll \sum_{u\geq 1}u\theta(u)\,. \qquad (4.94)
\end{aligned}$$

Next, by Item 1 of Lemma 4.1, we derive

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_{k,i}} \big|\mathbb{E}\big\{\big(f_{n-k}^{(4)}(\tilde{S}_{k-j-1}) - f_{n-k}^{(4)}(\tilde{S}_{k-j-m_{k,j}-1})\big)W_{k,i,j}\big\}\big| \\
&\quad\leq \sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_k}\sum_{u=j+1}^{j+m_{k,j}}\int_0^1 \big|\mathbb{E}\big\{\big(f_{n-k}^{(5)}(\tilde{S}_{k-u-1}+t\tilde{X}_{k-u})\big)\tilde{X}_{k-u}W_{k,i,j}\big\}\big|dt \\
&\quad\ll \sum_{k=1}^{n}\sum_{i=0}^{\ell_k}\sum_{j=i+1}^{i+m_k}\sum_{u=j+1}^{j+m_{k,j}}\|f_{n-k}^{(5)}\|_\infty\big(\theta(u-j)\wedge\theta(j-i)\wedge\theta(i)\ \big) \\
&\quad\ll \sum_{k=1}^{n}(n-k+1)^{-3/2}\Big(\sum_{u=1}^{m_k}u^2\theta(u) + m_k^2\sum_{i\geq m_k}\theta(i)\Big) \ll \sum_{u\geq 1}u\theta(u)\,. \qquad (4.95)
\end{aligned}$$

Putting together (4.93), (4.94) and (4.95), the first part of (4.92) follows. Similar (but simpler) arguments lead to the second part of (4.92). Finally, starting from (4.74) and taking into account the upper bounds (4.80), (4.81), (4.90), (4.91) and (4.92), it follows that $\sum_{k=1}^{n}\big|\mathbb{E}(\Delta_{n,k})\big| \ll 1 + \sum_{k\geq 1}k(k\wedge\sqrt{n})\theta(k)$, which combined with (4.25) implies (4.24) and then proves Item (b) of the theorem.

4.2 Proof of Lemma 4.1

Item 1 comes from the smoothing lemma 6.1 in [8]. To prove Item 2, we write

$$\big|\mathbb{E}(f_{n-k}^{(i)}(\tilde{S}_{\ell-1})) - \mathbb{E}(f_{n-k}^{(i)}(S_{\ell-1}))\big| \leq \|f_{n-k}^{(i+1)}\|_\infty\|\mathbb{E}_0(S_{\ell-1})\|_1\,.$$

Hence, since $\|\mathbb{E}_0(S_{\ell-1})\|_1 \leq \sum_{k=1}^{\ell}\theta_{X,1,1} \ll 1$, using Item 1, we derive that for any positive integer ℓ,

$$\big|\mathbb{E}(f_{n-k}^{(i)}(\tilde{S}_{\ell-1})) - \mathbb{E}(f_{n-k}^{(i)}(S_{\ell-1}))\big| \ll (n-k+1)^{-(i-1)/2}\,.$$

Next, let $(G_i)_{i\geq 1}$ be a sequence of iid centered Gaussian random variables with variance σ^2 and independent of $(X_i, B_i, Z_i)_{i\geq 1}$ (recall that the random variables (B_i) and (Z_i) have been defined at the beginning of Section 4.1). Let $N_k = \sum_{i=1}^{k}G_i$. Write that

$$\mathbb{E}(f_{n-k}^{(i)}(S_{\ell-1})) = \mathbb{E}(f_{n-k}^{(i)}(S_{\ell-1})) - \mathbb{E}(f_{n-k}^{(i)}(N_{\ell-1})) + \mathbb{E}(f_{n-k}^{(i)}(N_{\ell-1}))\,.$$

Next, let $t_k = \sigma\sqrt{(n-k)/2+1}$ and let $\varphi_{t_k^2}$ be the density of the law $\mathcal{N}(0,t_k^2)$. Denote also by $H_{k,n} = \sum_{i=k+1}^{n} B_i$ and note that, by definition, $H_{k,n}$ is independent of $S_{\ell-1}$ and of $N_{\ell-1}$. Note that

$$\mathbb{E}(f_{n-k}^{(i)}(N_{\ell-1})) = \mathbb{E}(f * \varphi_{t_k^2}^{(i)}(N_{\ell-1}+H_{k,n})) = \mathbb{E}(f * \varphi_{t_k^2+\sigma^2(\ell-1)}^{(i)}(H_{k,n}))\,.$$

Using Item 1, it follows that

$$|\mathbb{E}(f_{n-k}^{(i)}(N_{\ell-1}))| \ll (n-k+\ell)^{-(i-2)/2}\,.$$

On another hand

$$\begin{aligned}\mathbb{E}(f_{n-k}^{(i)}(S_{\ell-1})) - \mathbb{E}(f_{n-k}^{(i)}(N_{\ell-1})) &= \mathbb{E}(f * \varphi_{t_k^2}^{(i)}(S_{\ell-1}+H_{k,n})) - \mathbb{E}(f * \varphi_{t_k^2}^{(i)}(N_{\ell-1}+H_{k,n}))\\ &= \int_{\mathbb{R}} \mathbb{E}\big\{f'(S_{\ell-1}+H_{k,n}-u) - f'(N_{\ell-1}+H_{k,n}-u)\big\}\varphi_{t_k^2}^{(i-1)}(u)du\,.\end{aligned}$$

Since $f \in \Lambda_2(E)$, $g := f'$ is in $\Lambda_1(E)$ meaning that $g : \mathbb{R}\times E \to \mathbb{R}$ is measurable wrt the σ-fields $\mathcal{L}(\mathbb{R}\times E)$ and $\mathcal{B}(\mathbb{R})$, $g(\cdot,w)$ is 1-Lipschitz and $g(0,w)=0$ for any $w\in E$. Therefore, since it is assumed that $\sum_{k\geq 1} k\theta_{X,3,4}(k) < \infty$, one can use Item a) of Theorem 3.1 in [10] (see also Theorem 1.1 in [19]) which entails that

$$\sup_{v\in\mathbb{R}} \big|\mathbb{E}(f'(S_{\ell-1}+v)) - \mathbb{E}(f'(N_{\ell-1}+v))\big| \ll 1\,.$$

Note that Item a) of Theorem 3.1 in [10] is stated for g a Lipschitz function but following its proof one can show that it holds also if g belongs to $\Lambda_1(E)$. On another hand, $\varphi_{t_k^2}^{(i-1)}(u) = t_k^{-i}\varphi_1^{(i-1)}(u/t_k)$. Therefore

$$\big|\mathbb{E}(f_{n-k}^{(i)}(S_{\ell-1})) - \mathbb{E}(f_{n-k}^{(i)}(N_{\ell-1}))\big| \ll t_k^{1-i}\|\varphi_1^{(i-1)}\|_1 \ll t_k^{1-i}\,.$$

Putting together all the above upper bounds gives Item 2 of Lemma 4.1.

5 Annex: convergence of quantiles in the CLT

In this section, we give an inequality involving the difference between the quantile of a normalized random variable and the quantile of a standard normal, and the Wasserstein distance of order p between the corresponding laws. The main result of this section is Proposition 5.1 below which is a key result to prove Corollary 2.2.

Proposition 5.1. *Let Z be a centered real-valued random variable satisfying $\mathbb{E}(Z^2) \leq 2$. Let F_Z denote the distribution function of Z and Φ denote the distribution function of a standard normal Y. For any $p \geq 1$, let*

$$K_p = \int_0^1 \big|F_Z^{-1}(t) - \Phi^{-1}(t)\big|^p dt.$$

Then, for any u in $(0, 1/2]$,

$$\left|F_Z^{-1}(1-u) - \Phi^{-1}(1-u)\right| \leq \max\Big(\Big(\frac{(p+1)eK_p}{uQ_{1,Y}(u)}\Big)^{1/(p+1)}, \Big(\frac{(p+1)eK_p}{u}\Big)^{1/p}\Big).$$

Proof. Throughout the proof, $H_Y = 1 - \Phi$ and Q_Y is the inverse function of H_Y. With these notations,

$$Q_{1,Y}(u) = u^{-1}\int_0^u Q_Y(t)dt = u^{-1}\mathbb{E}\big(Y\mathbb{I}_{Y\geq Q_Y(u)}\big) = \frac{\exp(-Q_Y^2(u)/2)}{\sqrt{2\pi}\,u}\,. \tag{5.1}$$

We also set $H_Z = 1 - F_Z$ and we denote by Q_Z the generalized inverse function of H_Z. From (5.1), Proposition 5.1 is equivalent to

$$|Q_Z(u) - Q_Y(u)| \leq \max\Big(\big((p+1)e\sqrt{2\pi}\,e^{Q_Y^2(u)/2}K_p\big)^{1/(p+1)}, \big((p+1)eK_p/u\big)^{1/p}\Big) \tag{5.2}$$

for $u \leq 1/2$. We start by proving (5.2) in the case $Q_Z(u) > Q_Y(u)$.

Proof of (5.2) in the case $Q_Z(u) > Q_Y(u)$. Let $\delta = Q_Z(u) - Q_Y(u)$ and let η be the unique real in $(0, u)$ such that $Q_Y(u-\eta) = Q_Y(u) + \delta = Q_Z(u)$. From the convexity of Q_Y on $(0, 1/2]$,

$$Q_Y(u - t\eta) \leq Q_Y(u) + t\delta \quad \text{for any } t \in [0,1]. \tag{5.3}$$

Moreover $Q_Z(u - t\eta) \geq Q_Z(u) \geq Q_Y(u) + \delta$ for t in $[0, 1]$, whence, using the change of variables $s = u - t\eta$,

$$K_p \geq \int_{u-\eta}^u |Q_Z(s) - Q_Y(s)|^p ds \geq \eta\int_0^1 (\delta - \delta t)^p dt = \eta\delta^p/(p+1). \tag{5.4}$$

In view of the above inequality, we have to bound η from below. In order to get a lower bound on η, we will bound up $-Q_Y'$. From the definition of Q_Y,

$$-Q_Y'(s) = -1/H_Y'(Q_Y(s)) = \sqrt{2\pi}\exp(Q_Y^2(s)/2) \leq \sqrt{2\pi}\exp((Q_Y(u)+\delta)^2/2)$$

for any s in $[u-\eta, u]$,

We now separate two cases. If $\delta \leq \sqrt{2 + Q_Y^2(u)} - Q_Y(u)$,

$$-Q_Y'(s) \leq \sqrt{2\pi}\exp((Q_Y(u)+\delta)^2/2) \leq \sqrt{2\pi}\exp(1 + Q_Y^2(u)/2)$$

for any s in $[u-\eta, u]$. Then

$$Q_Y(u-\eta) - Q_Y(u) \leq \eta e\sqrt{2\pi}\exp(Q_Y^2(u)/2). \tag{5.5}$$

In that case, putting the above lower bound on η in (5.4), we obtain that

$$\delta^{p+1} \leq (p+1)e\sqrt{2\pi}\,e^{Q_Y^2(u)/2}K_p. \tag{5.6}$$

If $\delta > \sqrt{2+Q_Y^2(u)} - Q_Y(u)$, let $\delta_0 = \sqrt{2+Q_Y^2(u)} - Q_Y(u)$ and let η_0 be the real in $(0,u)$ such that $Q_Y(u-\eta_0) = Q_Y(u) + \delta_0$. Then $\eta \geq \eta_0$ and (δ_0, η_0) still satisfies (5.5), from which

$$\eta \geq \eta_0 \ \geq (e\sqrt{2\pi})^{-1}\Big(\sqrt{2+Q_Y^2(u)} - Q_Y(u)\Big)\exp(-Q_Y^2(u)/2). \tag{5.7}$$

Putting this lower bound in (5.4), we obtain that

$$\delta^p \leq (p+1)e\,(K_p/u)\frac{\sqrt{2\pi}\exp(Q_Y^2(u)/2)u}{\sqrt{2+Q_Y^2(u)} - Q_Y(u)}. \tag{5.8}$$

Now, setting $u = H_Y(x)$,

$$\sup_{u\in(0,1/2]} \frac{\sqrt{2\pi}\exp(Q_Y^2(u)/2)u}{\sqrt{2+Q_Y^2(u)} - Q_Y(u)} = \sup_{x\geq 0} \frac{\sqrt{2\pi}\exp(x^2/2)H_Y(x)}{\sqrt{2+x^2} - x} \leq 1$$

by an inequality on the Mills ratio of Komatu [15]. The two above inequalities imply that

$$\delta^p \leq (p+1)e\,(K_p/u), \tag{5.9}$$

if $\delta \geq \sqrt{2+Q_Y^2(u)} - Q_Y(u)$. Combining (5.6) and (5.9), we get (5.2) in the case $Q_Z(u) > Q_Y(u)$. It remains to prove (5.2) in the case $Q_Z(u) < Q_Y(u)$.

Proof of (5.2) *in the case* $Q_Z(u) < Q_Y(u)$. Let then $\delta = Q_Y(u) - Q_Z(u)$. From the assumptions $\mathbb{E}(Z) = 0$, $\mathbb{E}(Z^2) \leq 2$ and the Tchebichef-Cantelli inequality, for any $x \leq 0$, $H_Z(x) \geq x^2/(2+x^2)$. This which implies that

$$Q_Z(u) \geq -\sqrt{2u/(1-u)} \ \text{ for any } u \in (0,1). \tag{5.10}$$

In particular, for $u \leq 1/2$, $Q_Z(u) \geq -\sqrt{2} \geq -\sqrt{2+Q_Y^2(u)}$. Let then β be the positive real such that $Q_Y(u+\beta) = Q_Z(u)$. From (5.10), $-Q_Y'(s) \leq \sqrt{2\pi}\exp(1+Q_Y^2(u)/2)$ for any s in $[u, u+\beta]$. It follows that

$$Q_Y(u+s) \geq Q_Y(u) - s\sqrt{2\pi}\exp(1+Q_Y^2(u)/2)$$

for any s in $[0,\beta]$. For $s = \beta$, the above inequality yields

$$\beta \geq (e\sqrt{2\pi})^{-1}\exp(-Q_Y^2(u)/2)\delta := \eta.$$

With the above definition of η, for any t in $[0,1]$,

$$Q_Y(u+t\eta) \geq Q_Y(u) - t\delta \geq Q_Y(u) - \delta \geq Q_Z(u+t\eta).$$

Hence

$$Q_Y(u+t\eta) - Q_Z(u+t\eta) \geq (1-t)\delta$$

for any t in $[0,1]$. It follows that

$$K_p \geq \eta \int_0^1 |Q_Y(u+t\eta) - Q_Z(u+t\eta)|^p dt \geq \eta \int_0^1 (1-t)^p \delta^p dt = \frac{\eta \delta^p}{p+1}.$$

The above inequality together with the definition of η then imply (5.6), which completes the proof of (5.2).

Proof of Corollary 2.2. Recall that from Item (b) of Theorem 2.1 (see also Comment 2.2), under the assumptions of Corollary 2.2, $W_2(P_{S_n/\sigma_n}, G_1) = O(n^{-1/2})$. Hence, Item (a) comes from an application of Proposition 5.1 by taking into account the fact that, if Y is a standard normal r.v., there exists a positive constant η such that

$$\inf_{u \in (0,1/2]} \frac{Q_{1,Y}(u)}{\sqrt{\ln(1/u)}} \geq \eta\,.$$

Indeed, $Q_{1,Y}(u) \sim_{u \to 0} \sqrt{2\ln(1/u)}$ and $Q_{1,Y}(1/2) > 0$.

Item (b) follows again from Item (b) of Theorem 2.1 together with Inequality (2.7) in [23].

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