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RATIONAL MAPS WITH INTEGER MULTIPLIERS

XAVIER BUFF, THOMAS GAUTHIER, VALENTIN HUGUIN, AND JASMIN RAISSY

ABSTRACT. Let O_K be the ring of integers of an imaginary quadratic field. Recently, Zhuchao Ji and Junyi Xie proved that rational maps whose multipliers at all periodic points belong to O_K are power maps, Chebyshev maps or Lattès maps. Their proof relies on a non-archimedean result by Benedetto, Ingram, Jones and Levy. In this note, we show that one may avoid using this non-archimedean result by considering a differential equation instead.

INTRODUCTION

Let $d \geq 2$ be an integer. Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map of degree d , where $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. Consider a sequence $(z_n)_{n \geq 0}$ satisfying $z_0 \in \widehat{\mathbb{C}}$ and $z_{n+1} = f(z_n)$ for $n \geq 0$. The point z_0 is a *periodic point of period k* if $z_k = z_0$ for some minimal $k \geq 1$. In that case, the *multiplier of f at z_0* is the eigenvalue of $D_{z_0} f^{\circ k} : T_{z_0} \widehat{\mathbb{C}} \rightarrow T_{z_0} \widehat{\mathbb{C}}$.

The rational map f is

- a *power map* if it is conjugate to $z \mapsto z^{\pm d}$;
- a *Chebyshev map* if it is conjugate to $\pm T_d$ where T_d is the unique polynomial of degree d satisfying $T_d(z + z^{-1}) = z^d + z^{-d}$;
- a *Lattès map* if there exist a torus $\mathbb{T} = \mathbb{C}/\Lambda$, with $\Lambda \subset \mathbb{C}$ a lattice of rank 2, a holomorphic endomorphism $L : \mathbb{T} \rightarrow \mathbb{T}$ and a nonconstant holomorphic map $\Theta : \mathbb{T} \rightarrow \widehat{\mathbb{C}}$ such that the following diagram commutes:

$$\begin{array}{ccc}
 \mathbb{T} & \xrightarrow{L} & \mathbb{T} \\
 \Theta \downarrow & & \downarrow \Theta \\
 \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}}.
 \end{array}$$

Power maps, Chebyshev maps and Lattès maps are called *finite quotients of affine maps* by Milnor [M] and *exceptional maps* by Ji and Xie [JX]. In this note, we will use the second terminology.

As observed by Milnor in [M], if f is exceptional, the multipliers of f at all periodic points are contained in a discrete subring of \mathbb{C} , thus in the ring of integers of some imaginary quadratic field. Milnor conjectured that the converse is true. In [H1] the third author proved the conjecture when $d = 2$ and in [JX] Ji and Xie proved the conjecture for all $d \geq 2$.

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Theorem 1 (Ji-Xie). *Assume that O_K is the ring of integers of some imaginary quadratic field and $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$ whose multipliers all lie in O_K . Then f is a power map, a Chebyshev map or a Lattès map.*

In this note, we present their proof with a minor modification for one of the arguments. More precisely, our main contribution is Proposition 1.

After writing this note, the third author [H2] proved the following stronger result.

Theorem 2 (Huguin). *Assume that K is a number field and $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$ whose multipliers all lie in K . Then f is a power map, a Chebyshev map or a Lattès map.*

1. EXCEPTIONAL MAPS

Ritt [R] gave the following characterization of exceptional maps.

Lemma 1 (Ritt). *Assume that $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$, $\phi : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ is a nonconstant holomorphic map, $\alpha : \mathbb{C} \rightarrow \mathbb{C}$ is an affine map and $\tau : \mathbb{C} \rightarrow \mathbb{C}$ is a nontrivial translation such that*

- $\phi \circ \alpha = f \circ \phi$ and
- $\phi \circ \tau = \phi$.

Then, f is an exceptional map.

The following generalization is essentially due to Ji and Xie (compare with [JX, Lemma 2.9]).

Lemma 2. *Assume that $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$, $\phi : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ is a nonconstant holomorphic map and $\alpha_1 : \mathbb{C} \rightarrow \mathbb{C}$ and $\alpha_2 : \mathbb{C} \rightarrow \mathbb{C}$ are affine maps such that*

- α_1 and α_2 do not commute and
- $\phi \circ \alpha_1 = f \circ \phi = \phi \circ \alpha_2$.

Then, f is an exceptional rational map.

Proof. First, note that the affine map

$$\tau := \alpha_1 \alpha_2^{-1} \alpha_1 \alpha_2 \alpha_1^{-2} : \mathbb{C} \rightarrow \mathbb{C}$$

is a nontrivial translation. Indeed, the differentials of α_1 and α_2 are linear maps, thus commute. Therefore, we have $D\tau = \text{id} : \mathbb{C} \rightarrow \mathbb{C}$. In addition, $\tau \neq \text{id}$ since otherwise

$$\begin{aligned} \alpha_1 \alpha_2^{-1} \alpha_1 \alpha_2 \alpha_1^{-2} = \text{id} &\implies \alpha_1 \alpha_2^{-1} \alpha_1 \alpha_2 = \alpha_1^2 \\ &\implies \alpha_2^{-1} \alpha_1 \alpha_2 = \alpha_1 \\ &\implies \alpha_1 \alpha_2 = \alpha_2 \alpha_1, \end{aligned}$$

contradicting the fact that α_1 and α_2 do not commute.

Second, observe that

$$\phi \circ \alpha_1 \alpha_2 = f \circ \phi \circ \alpha_2 = f^{\circ 2} \circ \phi = f \circ \phi \circ \alpha_1 = \phi \circ \alpha_1^2,$$

so that

$$\phi \circ \tau = \phi \circ \alpha_1 \alpha_2^{-1} \alpha_1 \alpha_2 \alpha_1^{-2} = \phi \circ \alpha_2 \alpha_2^{-1} \alpha_1 \alpha_2 \alpha_1^{-2} = \phi \circ \alpha_1^2 \alpha_1^{-2} = \phi.$$

The result then follows from Lemma 1. □

2. ESCAPING QUADRATIC-LIKE MAPS

An *escaping quadratic-like map* is a covering map $g : U \rightarrow V$ of degree 2 between open subsets of \mathbb{C} , with V simply connected, and U compactly contained in V . If $g : U \rightarrow V$ is such a map, then U has two connected components U_1 and U_2 , each of which is simply connected.

Lemma 3. *If $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$, there exist an integer $n \geq 1$ and open sets $U \Subset V \subset \mathbb{C}$, such that the restriction $f^{\circ n} : U \rightarrow V$ is an escaping quadratic-like map.*

We say that such a restriction of $f^{\circ n}$ is an *escaping quadratic-like map associated to f* .

Proof. Let $z_1 \in \mathbb{C}$ be a repelling periodic point for f of period $r \geq 1$ which is not contained in the forward orbit of a critical point of f . Let $V_1 \subset \mathbb{C}$ be a simply connected neighborhood of z_1 such that the inverse branch h_1 of $f^{\circ r}$ fixing z_1 is defined on V_1 with $h_1(V_1) \Subset V_1$. Since z_1 is in the Julia set of f and since the iterated preimages of z_1 are dense in the Julia set of f which contains no isolated point, there exist $z_2 \in V_1 \setminus \{z_1\}$ and an integer $s \geq 1$ such that $f^{\circ s}(z_2) = z_1$. Let $V \subset V_1$ be a simply connected neighborhood of z_1 such that

- $h_1(V) \Subset V$,
- the inverse branch h_2 of $f^{\circ s}$ sending z_1 to z_2 is defined on V and
- $h_2(V) \Subset V_1 \setminus \{z_1\}$.

Let $m_1 \geq 1$ be sufficiently large so that $W_2 := h_1^{\circ m_1} \circ h_2(V) \Subset V$. Let $m_2 \geq 1$ be sufficiently large so that $W_1 := h_1^{\circ m_2}(V) \Subset V \setminus W_2$. Set

$$k_1 := h_1^{\circ m_2} : V \rightarrow W_1 \quad \text{and} \quad k_2 := h_1^{\circ m_1} \circ h_2 : V \rightarrow W_2.$$

Note that k_1 is an inverse branch of $f^{\circ n_1}$ with $n_1 := m_2 r$ and k_2 is an inverse branch of $f^{\circ n_2}$ with $n_2 := m_1 r + s$. Set

$$n := n_1 n_2 = m_2 r (m_1 r + s) \quad \text{and} \quad U := k_1^{\circ n_2}(V) \cup k_2^{\circ n_1}(V).$$

Then, the restriction of $f^{\circ n}$ from U to V is an escaping quadratic-like map associated to f . \square

3. AFFINE ESCAPING QUADRATIC-LIKE MAPS

An escaping quadratic-like map $g : U \rightarrow V$ is *affine* if the restriction of g to each connected component of U coincides with the restriction of an affine map.

In addition, two escaping quadratic-like maps $g_1 : U_1 \rightarrow V_1$ and $g_2 : U_2 \rightarrow V_2$ are *conjugate* if there exists a holomorphic isomorphism $\phi : V_2 \rightarrow V_1$ such that the relation $\phi \circ g_2 = g_1 \circ \phi$ holds on U_2 .

Lemma 4. *Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. If an escaping quadratic-like map associated to f is conjugate to an affine escaping quadratic-like map, then f is an exceptional map.*

Proof. Assume that $f^{\circ n} : U \rightarrow V$ is an escaping quadratic-like map associated to f , that $g : U' \rightarrow V'$ is an affine escaping quadratic-like map and that $\phi : V' \rightarrow V$ conjugates $g : U' \rightarrow V'$ to $f^{\circ n} : U \rightarrow V$, i.e.,

$$\phi \circ g = f^{\circ n} \circ \phi.$$

By assumption, U' has two connected components U'_1 and U'_2 and the restrictions of g to U'_1 and U'_2 coincide with the restrictions of affine maps $\alpha_1 : \mathbb{C} \rightarrow \mathbb{C}$ and $\alpha_2 : \mathbb{C} \rightarrow \mathbb{C}$ to U'_1 and U'_2 . The relations $\phi \circ \alpha_1 = f^{\circ n} \circ \phi$ and $\phi \circ \alpha_2 = f^{\circ n} \circ \phi$ hold on U' . Since the affine maps $\alpha_1 : \mathbb{C} \rightarrow \mathbb{C}$ and $\alpha_2 : \mathbb{C} \rightarrow \mathbb{C}$ are repelling, we may use any of those two relations to extend $\phi : V' \rightarrow V$ to a global meromorphic map $\hat{\phi} : \mathbb{C} \rightarrow \hat{\mathbb{C}}$. We then have

$$\hat{\phi} \circ \alpha_1 = f^{\circ n} \circ \hat{\phi} = \hat{\phi} \circ \alpha_2 \quad \text{on } \mathbb{C}.$$

The affine maps α_1 and α_2 have distinct fixed points, respectively in U'_1 and U'_2 . Thus, they do not commute. It follows from Lemma 2 that $f^{\circ n}$ is an exceptional map, and so, f is an exceptional map (a rational map is exceptional if and only if its iterates are exceptional). \square

4. THE PROOF OF THEOREM 1

It follows from Lemma 3 and Lemma 4 that Theorem 1 is a consequence of the following result.

Proposition 1. *Let O_K be the ring of integers of some quadratic imaginary field. If g is an escaping quadratic-like map whose multipliers at all periodic points belong to O_K , then g is conjugate to an affine escaping quadratic-like map.*

The proof will occupy the rest of the note. From now on, we assume that $g : U \rightarrow V$ is an escaping quadratic-like map whose multipliers at all periodic points belong to O_K . Let U_1 and U_2 be the two connected components of U . Set:

$$g_1 := g|_{U_1}, \quad g_2 := g|_{U_2}, \quad h_1 := g_1^{-1} : V \rightarrow U_1 \quad \text{and} \quad h_2 := g_2^{-1} : V \rightarrow U_2.$$

Let $p_1 \in U_1$ be the unique (repelling) fixed point of $g_1 : U_1 \rightarrow V$ and let λ_1 be its multiplier. Similarly, let $p_2 \in U_2$ be the unique (repelling) fixed point of $g_2 : U_2 \rightarrow V$ and let λ_2 be its multiplier.

The sequence of univalent maps $\psi_n : V \rightarrow \mathbb{C}$ defined by

$$\psi_n(z) := \frac{h_1^{\circ n}(z) - p_1}{h_1^{\circ n}(p_2) - p_1}$$

converges to a univalent map $\psi : V \rightarrow \mathbb{C}$ such that $\psi(p_1) = 0$, $\psi(p_2) = 1$ and $\psi \circ g_1 = \lambda_1 \times \psi$. Replacing g by $\psi \circ g \circ \psi^{-1}$ if necessary, we may therefore assume that

$$p_1 = 0, \quad p_2 = 1 \quad \text{and} \quad g_1(z) = \lambda_1 z.$$

We need to prove that g_2 is an affine map.

4.1. A special sequence of periodic points. In their proof, Ji and Xie consider a particular sequence of periodic points of g . This sequence may be defined as follows. For $n \geq 0$, let z_n be the unique fixed point of the map $h_2 \circ h_1^{\circ n} : V \rightarrow U_2$. Then, z_n is a periodic point of g of period $n + 1$. In particular the multiplier ρ_n of g at z_n belongs to O_K . Note that as $n \rightarrow +\infty$, we have that

$$z_n \rightarrow \alpha := h_2(0)$$

so that

$$z_n = h_2 \left(\frac{z_n}{\lambda_1^n} \right) = \alpha + \frac{\beta}{\lambda_1^n} + o \left(\frac{1}{\lambda_1^n} \right) \quad \text{with} \quad \beta := \alpha h_2'(0).$$

Then,

$$\rho_n = \lambda_1^n g_2'(z_n) = \lambda_1^n a + b + o(1) \quad \text{with} \quad a := g_2'(\alpha) \quad \text{and} \quad b := \beta g_2''(\alpha) = \alpha \frac{g_2''(\alpha)}{g_2'(\alpha)}.$$

Lemma 5. *We have that $\rho_n = a\lambda_1^n + b$ for n large enough.*

Proof. Write $\rho_n = a\lambda_1^n + b + \varepsilon_n$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$. We have that

$$\underbrace{\lambda_1 \rho_n - \rho_{n+1}}_{\in O_K} = (\lambda_1 - 1)b + \underbrace{\lambda_1 \varepsilon_n - \varepsilon_{n+1}}_{\xrightarrow[n \rightarrow +\infty]{} 0}.$$

Thus, $(\lambda_1 - 1)b$ belongs to the closure of O_K , i.e., to O_K .

Since O_K is discrete, we have that $\lambda_1 \rho_n - \rho_{n+1} = (\lambda_1 - 1)b$ for n large enough, i.e., $\varepsilon_{n+1} = \lambda_1 \varepsilon_n$. Since $|\lambda_1| > 1$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, we have that $\varepsilon_n = 0$ for n large enough. \square

4.2. A differential equation.

Lemma 6. *The holomorphic map $g_2 : U_2 \rightarrow V$ satisfies the differential equation*

$$(E) \quad \forall z \in U_2, \quad g_2'(z) = a + b \frac{g_2(z)}{z}.$$

Proof. Note that for n large enough, we have that

$$\lambda_1^n g_2(z_n) = z_n \quad \text{and} \quad g_2'(z_n) = \frac{\rho_n}{\lambda_1^n} = a + \frac{b}{\lambda_1^n} = a + b \frac{g_2(z_n)}{z_n}.$$

Since the sequence $(z_n)_{n \geq 1}$ accumulates at $h_2(0) \in U_2$, the function g_2 satisfies the differential equation (E). \square

Remark. Equation (E) is linear and so, may be easily solved. However, we shall not use the explicit form of the solutions.

Lemma 7. *We have that*

$$\frac{g_2'(\alpha)}{\lambda_2} = 1 + \frac{\nu}{1 - \lambda_2} \quad \text{with} \quad \alpha := h_2(0) \quad \text{and} \quad \nu := p_2 \frac{g_2''(p_2)}{g_2'(p_2)}.$$

Proof. Evaluating Equation (E) at $z = p_2 = g_2(p_2)$, we obtain

$$\lambda_2 = g_2'(p_2) = a + b.$$

In addition, differentiating Equation (E), we obtain

$$\forall z \in U_2, \quad g_2''(z) = b \left(\frac{g_2'(z)}{z} - \frac{g_2(z)}{z^2} \right).$$

Since $\lambda_2 = g_2'(p_2)$ and $a = g_2'(\alpha)$, we have that

$$\nu = p_2 \frac{g_2''(p_2)}{g_2'(p_2)} = (\lambda_2 - g_2'(\alpha)) \left(1 - \frac{1}{\lambda_2} \right).$$

This last equality may be rewritten in the required form. \square

4.3. Conclusion. For each integer $k \geq 1$, consider the escaping quadratic-like map $g^k : U_1 \cup h_2^{\circ k}(V) \rightarrow V$ defined by

$$g^k(z) = \begin{cases} \lambda_1 z & \text{if } z \in U_1 \\ g_2^{\circ k}(z) & \text{if } z \in h_2^{\circ k}(V). \end{cases}$$

Then, all the periodic points of g^k are periodic points of g and their multipliers still belong to O_K . In addition, g^k fixes p_2 with multiplier λ_2^k .

According to Lemma 7, we have that

$$\frac{(g_2^{\circ k})'(\alpha_k)}{\lambda_2^k} = 1 + \frac{\nu_k}{1 - \lambda_2^k} \quad \text{with} \quad \alpha_k := h_2^{\circ k}(0) \quad \text{and} \quad \nu_k := p_2 \frac{(g_2^{\circ k})''(p_2)}{(g_2^{\circ k})'(p_2)}.$$

A rather elementary computation (in fact, it is the composition rule for nonlinearities) yields

$$\nu_k = \nu_1 + \lambda_2 \nu_1 + \lambda_2^2 \nu_1 + \cdots + \lambda_2^{k-1} \nu_1 = \frac{1 - \lambda_2^k}{1 - \lambda_2} \nu_1.$$

In particular,

$$\frac{\nu_k}{1 - \lambda_2^k} = \frac{\nu_1}{1 - \lambda_2}$$

does not depend on $k \geq 1$. As a consequence, for all $k \geq 1$,

$$\frac{g_2'(\alpha_{k+1})}{\lambda_2} \cdot \frac{(g_2^{\circ k})'(\alpha_k)}{\lambda_2^k} = \frac{(g_2^{\circ(k+1)})'(\alpha_{k+1})}{\lambda_2^{k+1}} = \frac{(g_2^{\circ k})'(\alpha_k)}{\lambda_2^k}.$$

Thus,

$$\forall k \geq 1, \quad g_2'(\alpha_{k+1}) = \lambda_2.$$

The sequence $(\alpha_k)_{k \geq 2}$ accumulates at $p_2 \in U_2$. So, $g_2'(z) = \lambda_2$ for all $z \in U_2$ and g_2 is an affine map as required. This completes the proof of Proposition 1.

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