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ANALYSIS OF A ONE DIMENSIONAL ENERGY DISSIPATING FREE BOUNDARY MODEL WITH NONLINEAR BOUNDARY CONDITIONS. EXISTENCE OF GLOBAL WEAK SOLUTIONS

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Abstract. This work is part of a general study on the long-term safety of the geological repository of nuclear wastes. A diffusion equation with a moving boundary in one dimension is introduced and studied. The model describes some mechanisms involved in corrosion processes at the surface of carbon steel canisters in contact with a claystone formation. The main objective of the paper is to prove the existence of global weak solutions to the problem. For this, a semi-discrete in time minimizing movements scheme *à la De Giorgi* is introduced. First, the existence of solutions to the scheme is established and then, using *a priori* estimates, it is proved that as the time step goes to zero these solutions converge up to extraction towards a weak solution to the free boundary model.

ec.intro

1. Introduction

This work is motivated by the study of the so-called Diffusion Poisson Coupled Model (DPCM) introduced in [4] by Bataillon *et al.* This system was developed to model the corrosion of a steel plate in contact with a solution. In particular, it is relevant for describing the corrosion of steel canisters containing nuclear wastes (confined in a glass matrix) and stored at a depth of several hundred meters in a claystone layer. Since this storage method is considered by various countries, its reliability requires investigations. In particular, wastes stay radioactive for several hundred of years and it is important to understand the *long term* behaviour of the system. Our main concern is about corrosion and the quantity of hydrogen molecules released during the process which can lead to safety issues. As it is not possible to perform physical experiments at these time scales, the use of reliable models (such as DPCM mentioned above) allowing *in silico* experiments are required. However, to design accurate numerical methods capable of predicting the values of the relevant physical quantities over a long time, it is necessary to understand the mathematical properties of the model.

Let us briefly explain the main features of the DPCM. This is a one dimensional free boundary system. The space is decomposed in three regions: the oxide layer is in contact on one side with the claystone, viewed as a aqueous solution and on the other side with the metal. The DPCM is a system of drift-diffusion equations describing the evolution inside the oxide layer of charge carriers (electrons, Fe^{3+} cations and oxygen vacancies) and coupled with a Poisson equation governing the dynamics of the electrical potential. The positions of the solution/oxide layer and oxide layer/metal interfaces evolve along time according to some given ordinary differential equations. Besides, the

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electrochemical reactions only occur at these interfaces (*i.e.* there is no reaction terms in the drift-diffusion equations in the bulk of the three regions). These reversible electrochemical reactions are modeled by some nonlinear Fourier boundary conditions at the interfaces.

Due to the numerous coupling of the equations of the model and its definition on a moving domain the mathemical study of the DPCM is a challenging task. So far only few results are available in the literature. In [10, 11] the well-posedness of the system has been established for a simplified version of the DPCM where the positions of the interfaces are fixed. A finite-volume scheme approximating the solutions to the DPCM has been proposed in [5]. The numerical experiments with relevant physical data presented in [4, 5] suggest the existence of a global solution to the system. In particular, the existence of traveling wave solutions is established: after a transient time both interfaces move at the same speed, the width of the oxide domain remains constant and the charge carriers and the electrical potential admit a stationary profile. The existence of such traveling wave solutions for a reduced model, where the electroneutrality in the oxide layer is assumed, has been proved in [9]. Thanks to a computer-assisted proof, the existence of traveling wave solutions for the "full" DPCM has been obtained in [6]. Recently, in [8] another simplified model (only two species are considered) is proposed with some changes in the nonlinear boundary conditions that correct a thermodynamical inconsistency of the initial model. This modification makes the mathematical study more tractable in the case of a fixed domain but the well-posedness of the complete free boundary problem is still open.

Up to now, no existence result has been proved for the evolutionary DPCM with free boundaries. One of the main difficulty for establishing the existence of a global solution is to justify that the length of the oxide layer, where the equations of the systems are defined, stays positive along time (as numerically suggested in [4, 5]). The structure of the system and in particular its lack of obvious gradient structure prevents the derivation of classical *a priori* estimates which would lead (even formally) to a positive lower bound for the width of the oxide layer along time.[1] To bypass this difficulty we adapt the approach of Portegies and Peletier in [30] which makes use of tools from optimal transport to study a moving boundary problem.

Indeed, in [30], the authors introduce a one dimensional parabolic free boundary model with two moving interfaces describing the variation of the length of a piece of crystal by dissolution/precipitation. The thermodynamical consistency of the model is deeply connected with its gradient-flow structure with respect to some Wasserstein metric. This structure is very used in [30] for establishing the existence of solutions by using a Jordan, Kinderlehrer and Otto (JKO) minimizing scheme [21]. The relevance of this approach in the context of parabolic equations in a fixed domain is well known, see for instance [1, 21, 23, 28, 29, 31] and the idea to recast some free boundary problems in the Wasserstein gradient-flow setting seems promising.

The main goal of this work is to show that this idea is effective for the DPCM model. For this we consider a free boundary model which compared to the DPCM is very simple in the bulk of the oxide layer but retains all the difficulties related to the nonlinear boundary conditions and to the equation of motion of the oxide-metal interface.

1.1. **Presentation of the reduced model.**

Let us first explain the phenomena that our reduced model has to describe. We only consider the evolution of the concentration of oxygen vacancies, denoted by ρ, inside the oxide layer. We neglect the other charged species as well as the existence

[1] This general idea might possibly work but we have not found the right way to implement it.

of an electrical potential in this domain so that ρ satisfies a heat equation defined on a moving domain. We fix the position of the interface solution/oxide layer at $x = 0$, while the interface oxide layer/metal is moving according to a nonlinear ordinary differential equation. Finally, in order to take into account the chemical reactions at the interfaces we impose boundary conditions which model the exchange of matters at the interfaces of each regions, i.e., at the interfaces solution/oxide layer and oxide layer/metal.

More precisely, we denote by ρ the concentration of oxygen vacancies and by $X(t) > 0$ the position of the moving interface at time t. The respective domains of the solution, of the oxide layer and of the metal at time t are $(-\infty, 0]$, $[0, X(t))$ and $[X(t), +\infty)$. The metal domain is viewed as a constant and homogeneous reserve of oxygen vacancies. This constant represents a maximum for ρ. Assuming after normalization that this constant is 1 we extend ρ by $\rho(x,t) = 1$ for $x > X(t)$. With this convention the density of oxygen is then $1 - \rho$. Denoting by $M(t)$ the opposite of the quantity of oxygen in the oxide domain, we have

$$M(t) = \int_0^{X(t)} (\rho(x,t) - 1)\, dx = \int_{\mathbb{R}_+} (\rho(x,t) - 1)\, dx.$$

P We propose to consider for $T > 0$ the following free boundary model

P.a (1a) $$\partial_t \rho(x,t) - \partial_x^2 \rho(x,t) = 0 \quad \text{for } x \in [0, X(t)],\ t \in [0,T],$$

P.b (1b) $$\rho(x,t) = 1 \quad \text{for } x \geq X(t),\ t \in [0,T],$$

P.c (1c) $$\partial_x \rho(X(t)^-, t) + \dot{X}(t)\, \rho(X(t)^-, t) = \dot{X}(t) \quad \text{for } t \in [0,T],$$

P.d (1d) $$\dot{M}(t) = -\partial_x \rho(0^+, t) \quad \text{for } t \in [0,T],$$

P.e (1e) $$\lambda\, \dot{X}(t) = \alpha - \big(1 - \rho(X(t)^-, t)\big) - \ln \rho(X(t)^-, t) \quad \text{for } t \in [0,T],$$

P.f1 (1f) $$\rho(x,0) = \rho^0(x) \quad \text{for } x \geq 0,$$

P.f2 (1g) $$M(0) = \int_{\mathbb{R}_+} \big(\rho^0(x) - 1\big)\, dx, \qquad X(0) = X^0.$$

where λ, α are positive constants and the given initial data is composed of $X^0 > 0$ and $\rho^0 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho^0(x) = 1$ for $x > X^0$. Let us check that the evolution of M by (1d) is consistent with its first definiton. We compute

$$\begin{aligned} \dot{M}(t) &= \int_0^{X(t)} \partial_t \rho(x,t)\, dx + \dot{X}(t)(\rho(X(t)^-, t) - 1) \\ &\overset{(1a)}{=} -\partial_x \rho(0^+, t) + \Big[\partial_x + \dot{X}(t)\Big] \rho(X(t)^-, t) - \dot{X}(t) \\ &\overset{(1c)}{=} -\partial_x \rho(0^+, t), \end{aligned}$$

which is indeed (1d). We observe that the condition (1c) only expresses the conservation of oxygen vacancies as the position of the oxide/metal interface varies. The rate at which the interface moves is given by the nonlinear equation (1e). This does not correspond to a motion of matter but to a change of state, more precisely, a change in the arrangement of iron atoms at the interface (from metal to oxide if $\dot{X} > 0$) and (1e) is the rate of a chemical reaction.

It remains to impose a boundary condition on ρ at the solution/oxide layer interface at $x = 0$. In order to be consistent with the previous simplifications we should take a non-homogeneous *linear* Fourier condition but we would lose an important difficulty in the problem. According to (1d), $\partial_x \rho(0,t)$ represents the flux of oxygen vacancies at time t from the oxide layer into the solution, this quantity should be a nondecreasing function of $\rho(0,t)$ and more generally, we should have

$$\partial_x \rho(0,t) \in \partial F(\rho(0,t)),$$

where F is some (lower semicontinuous) convex function and ∂F denotes its subderivative. We pick a worst case scenario and choose F as the indicatrix of $[\rho_-, \rho_+]$ for some $0 < \rho_- < \rho_+ < 1$, that is $F(\rho) = 0$ if $\rho_- \leq \rho \leq \rho_+$ and $F(\rho) = +\infty$ in the other cases. This leads to the following nonlinear conditions for $t \in [0,T]$.

P.bord

P.bord1 (2a)
$$\rho_- \leq \rho(0,t) \leq \rho_+,$$

P.bord2 (2b) for $t \in [0,T]$
$$\begin{cases} \partial_x \rho(0,t) & \geq 0 \quad \text{if } \rho(0,t) = \rho_+, \\ \partial_x \rho(0,t) & \leq 0 \quad \text{if } \rho(0,t) = \rho_-, \\ \partial_x \rho(0,t) & = 0 \quad \text{if } \rho_- < \rho(0,t) < \rho_+. \end{cases}$$

The conditons in (2) can be seen as a generalization of a Signorini problem. These problems contain a one sided-constraint on the solution and model usually some irreversible phenomena at the boundary of the domain. They are used for instance in some unilateral contact problems in elasticity [20, 22], in some continuum mechanics models to describe a semipermeable membrane [14] or in chemistry to model electrochemical reacting interface [19]. On the contrary the conditions in (2) allow the exchange of matters, in both "directions", at the interface solution/oxide layer with two distinct thresholds ρ_- and ρ_+.
From a physical point of view, the choice of (2) is disputable as the transport of oxygen vacancies is a reversible phenomenon. However, there is another phenomenon, neglected in the simplified model considered here, which is irreversible, namely the dissolution of the oxide layer. Indeed, the iron in the aqueous solution, rather than possibly reconstituting the oxide layer, will form oxide complexes, less organized and with a porous structure (rust). For this reason, considering nonlinear monotonic and *non-smooth* boundary conditions like (2) anticipates future studies on a full model.

1.2. **Preliminary considerations on the model.** Following [30] we use a JKO minimizing movements scheme to prove the existence of weak solutions to (1)–(2). This approach consists in a semi-discrete in time scheme where at each time step the approximated solutions to (1)–(2) are obtained as minimizers of a functional which writes as the sum of a a squared distance (a Wasserstein energy divided by twice the time step) and an energy functional. Let us first identify a Lyapunov functional associated to the system (1)–(2). We first define $\beta, \theta \in \mathbb{R}$ by

atheta1 (3)
$$\exp(\beta + \theta - 1) = \rho_+ \qquad \text{and} \qquad \exp(\beta - \theta - 1) = \rho_-.$$

The conditions $0 < \rho_- < \rho_+ < 1$ are equivalent to

atheta2 (4)
$$\theta > 0 \qquad \text{and} \qquad \beta + \theta < 1.$$

Now, let us assume that (ρ, X, M) is a regular solution to (1)–(2) on $\mathbb{R}_+ \times [0,T)$. We assume moreover that $\rho \geq \rho_{\min}$ on D_T for some $\rho_{\min} > 0$ where

$$D_T := \{(x,t) : 0 \leq t < T,\ 0 \leq x < X\}.$$

Then we claim that, at least formally, the following functional

$$\mathbf{F}(t) = \int_{\mathbb{R}_+} \Big(\rho(x,t)(\ln \rho(x,t) - \beta) + \beta\Big)\, dx - \alpha X(t),$$

is a Lyapunov functional for the system (1)–(2). Let us check this fact.

In the computation below we use the shorthands ρ_{X^-} for $\rho(X^-(t),t)$ and $\partial_x \rho_{X^-}$ for $\partial_x \rho(X^-(t),t)$ and similarly ρ_{0^+} for $\rho(0^+,t)$ and $\partial_x \rho_{0^+}$ for $\partial_x \rho(0^+,t)$. Differentiating $\mathbf{F}$ at time t we get

$$\begin{aligned}\dot{\mathbf{F}}(t) &= \int_0^{X(t)} \partial_t \rho(x,t) \left[\ln \rho(x,t) + 1 - \beta\right]\, dx + \left[\rho_{X^-}(\ln \rho_{X^-} - \beta) + \beta\right] \dot{X}(t) - \alpha \dot{X}(t) \\ &=: A(t) + B(t) + C(t).\end{aligned}$$

We use (1a) and an integration by parts to rewrite the first term as

$$\begin{aligned}A(t) &= -\int_0^{X(t)} \frac{(\partial_x \rho(x,t))^2}{\rho(x,t)}\, dx - \partial_x \rho_{0^+} \left[\ln \rho_{0^+} + 1 - \beta\right] + \partial_x \rho_{X^-} \left[\ln \rho_{X^-} + 1 - \beta\right] \\ &=: A_0(t) + A_1(t) + A_2(t).\end{aligned}$$

The bulk term $A_0(t)$ rewrites as

$$A_0(t) = -4 \int_0^{X(t)} \left(\partial_x \sqrt{\rho(x,t)}\right)^2 dx \ \leq 0.$$

Next, we regroup the boundary terms at $X(t)$, namely $A_2(t)$, $B(t)$, $C(t)$. We compute

$$\begin{aligned}A_2(t) + B(t) + C(t) &= \partial_x \rho_{X^-} \left[\ln \rho_{X^-} + 1 - \beta\right] + \left[\rho_{X^-}(\ln \rho_{X^-} - \beta) + \beta\right] \dot{X}(t) - \alpha \dot{X}(t) \\ &\overset{(1c)}{=} \dot{X}(t) \left((1 - \rho_{X^-}) \left[\ln \rho_{X^-} + 1 - \beta\right] + \left[\rho_{X^-}(\ln \rho_{X^-} - \beta) + \beta\right] - \alpha\right) \\ &= \dot{X}(t) \left(-\rho_{X^-} + \ln \rho_{X^-} + 1 - \alpha\right) \\ &\overset{(1e)}{=} -\lambda \left(\dot{X}(t)\right)^2 \ \leq 0.\end{aligned}$$

For the remaining term $A_1(t)$ we write

$$A_1(t) = -\partial_x \rho_{0^+} \left[\ln \rho_{0^+} + 1 - \beta\right] + \theta |\dot{M}(t)| - \theta |\dot{M}(t)|.$$

We have to distinguish different cases. Indeed, thanks to (1d) and (2b), if $\dot{M}(t) = 0$ then $\partial_x \rho_{0^+} = 0$ and

$$A_1(t) = 0.$$

If $\dot{M}(t) > 0$ then

$$A_1(t) = -\partial_x \rho_{0^+} \left(\ln \rho_{0^+} + 1 - \beta + \theta\right) - \theta \dot{M}(t) \overset{(3)}{=} -\theta \dot{M}(t) \leq 0.$$

Finally, if $\dot{M}(t) < 0$ then

$$A_1(t) = -\partial_x \rho_{0^+} \left(\ln \rho_{0^+} + 1 - \beta - \theta\right) + \theta \dot{M}(t) \overset{(3)}{=} \theta \dot{M}(t) \leq 0.$$

In summary, we have for $t \in [0, T)$,

dissip

$$\dot{\mathbf{F}}(t) = -4 \int_0^{X(t)} \left(\partial_x \sqrt{\rho(x,t)}\right)^2 dx - \lambda \left(\dot{X}(t)\right)^2 - \theta \left|\dot{M}(t)\right|, \tag{5}$$

and the three terms on the right hand side are nonpositive so they can be interpreted as the contributions of different dissipation phenomena in the bulk of the domain, at the left boundary and at the right boundary respectively.

Let us emphasize that in this work we are using techniques similar to those employed to analyze (Wasserstein) gradient flow systems. However, we cannot claim that (1)–(2) admits a gradient flow structure even in the generalized sense introduced by Mielke in [27]. Indeed, in this theory, one has to specify an energy functional (or driving functional) and a dissipation potential with a quadratic form allowing to write it thanks to a scalar product. In our case, using (5), the energy functional and the dissipation potential are clearly identified. But, due to the linear dissipation term $-\theta|\dot{M}|$ in (5), we cannot recast the dynamics of (1)–(2) in an Hilbertian setting and interpret this system as a generalized gradient flow. Nevertheless the methods developed in [21, 29] can be applied in our case in order to prove the existence of weak solutions to (1)–(2).

weaksol

1.3. **Notion of weak solution and main result.** In this subsection we define a notion of weak solution for system (1)–(2a), ignoring the boundary condition (2b) at $x = 0$. Then following the classical approach to deal with Signorini problem [14, 24] this later is expressed separately in a weak form as a variational inequality. Before this we recall some definitions and notation about the spaces of functions with bounded variations.

Due to the constraint (1b), the function $x \mapsto \rho(x,t)$ may admit a jump at the free interface $x = X(t)$. It is then convenient to work in some BV-space, see [2, 15]. Given an open interval $I \subseteq \mathbb{R}$, we denote $BV(I)$ the space of functions with bounded variations in I, i.e., functions $u \in L^1(I)$ such that the distributional derivative Du is a finite Radon measure on I. For every function $u \in BV(I)$ we have the following unique decomposition

$$Du = u'(x)\,dx + D^j u + D^c u,$$

where $Du = u'(x)\,dx + (Du - u'(x)dx)$ is the Radon–Nicodym decomposition of Du with respect to the Lebesgue measure and $u' \in L^1(I)$ is the corresponding Radon–Nicodym derivative and the remaining term $(Du - u'(x)dx)$ decomposes into an atomic part

$$D^j u := \sum_{x_i \in J_u} \left(u(x_i^+) - u(x_i^-)\right) \delta_{x_i},$$

called the jump part and the so called Cantor part $D^c u$ which concentrates on a Lebesgue null set but has no atomic part. The space of special functions with bounded variations $SBV(I)$ is the subspace of $BV(I)$ formed by the elements u such that $D^c u$ vanishes. For a finite exponent $p > 1$, $SBV^p(I) \subseteq SBV(I)$ is defined as

$$SBV^p(I) := \{u \in SBV(I) \ : \ u' \in L^p(I)\}.$$

Similarly, we will look for $t \mapsto M(t)$ in $BV([0,T])$.

Let us now derive a variational identity satisfied by any (sufficiently smooth) solution (ρ, M, X) to (1)–(2a). Let $\varphi \in C_0^\infty(\mathbb{R}_+ \times [0,T))$. Mutiplying (1a) by φ, integrating over

$$D_T := \{(x,t) : 0 \le t \le T, 0 \le x < X(t)\},$$

integrating by parts the first term with respect to time and the second term with respect to space and using the conservative boundary condition (1c), we get

$$\int_0^T \int_{\mathbb{R}_+} [-\rho(x,t)\partial_t\varphi(x,t) + \partial_x\rho(x,)\partial_x\varphi(x,t)]\, dxdt \\ - \int_0^{X(0)} \rho(x,0)\varphi(0,t)\, dx + \int_0^T \partial_x\rho(0,t)\varphi(0,t)\, dt = 0.$$

Subtituting the initial conditions (1f)(1g) and the boundary condition (1d) we obtain

wf_0 (6) $$\int_0^T \int_{\mathbb{R}_+} [-\rho(x,t)\partial_t\varphi(x,t) + \partial_x\rho(x,)\partial_x\varphi(x,t)]\, dxdt \\ - \int_0^{X^0} \rho^0(x)\varphi(0,t)\, dx - \int_0^T \dot{M}(t)\varphi(0,t)\, dt = 0.$$

As we will consider weak solutions such that M has BV-regularity, we rewrite the last term in the left hand side as

$$\int_0^T \dot{\varphi}(0,t)\, dDM(t).$$

The above weak formulation has to be complemented with the initial condition on M and the law of motion (1e) of $X(t)$. These conditions and the weak formulation (6) are equivalent to (1)–(2a) as soon as X is Lipschitz continuous, M is BV and ρ has regularity $L^2_t H^2_x \cap H^1_t L^2_x$ in the domain D_T.

Let us now derive a weak formulation of the boundary conditions (2). Let us assume formally that (ρ, M, X) is a smooth solution to (1)–(2) with $t \mapsto X(t)$ a nondecreasing function. Let $\chi \in C_0^\infty(\mathbb{R}_+, \mathbb{R}_+)$ such that $\chi \equiv 1$ on $[0, X^0/2)$ and $\mathrm{supp}(\chi) \subseteq [0, 3X^0/4)$ and let us set $u(x,t) := \chi(x)\rho(x,t)$. We have:

eg.var1 (7) $$\partial_t u(x,t) - \partial_x^2 u(x,t) = g(x,t) \quad \text{for } (x,t) \in [0, X^0] \times [0,T],$$

where the source term g is given by

$$g(x,t) := -\chi''(x)\, \rho(x,t) - 2\chi'(x)\, \partial_x\rho(x,t) \quad \text{for } (x,t) \in [0, X^0] \times [0,T].$$

Now let $\eta \in C_0^\infty(\mathbb{R}_+ \times [0,T))$ with $\eta(0,t) \in [\rho_-, \rho_+]$ for all $t \in [0,T]$ and let $\phi \in C_0^\infty([0,T), \mathbb{R}_+)$. Thanks to the boundary conditions (2) we have

P.bordw (8) $$\partial_x\rho(0,t)\, (\rho(0,t) - \eta(0,t)) \ge 0, \quad \forall t \in [0,T].$$

In fact it is easily seen that (2) holds true if and only if (8) holds true for every η such that $\eta(0,t) \in [\rho_-, \rho_+]$.

Next, since $\phi \geq 0$ and $u \equiv \rho$ in the neighborhood of $x = 0$ we have

neg.var2 (9) $$\phi(t)\partial_x u(0,t)\,(u(0,t)-\eta(0,t)) \geq 0, \quad \forall t \in [0,T].$$

Multiplying (7) by $\phi(u-\eta)$, integrating in space and time, integrating by parts and using inequality (9), we obtain

ineg.var (10) $$-\int_0^T \dot\phi \int_0^{X^0} \left(\frac{u^2}{2} - \eta\, u\right) dxdt + \int_0^T \int_0^{X^0} \phi\, u\, \partial_t \eta\, dxdt + \int_0^T \int_0^{X^0} \phi \partial_x u \partial_x (u-\eta)\, dxdt \leq \phi(0) \int_0^{X^0} \left(\frac{u^2}{2} - u\eta\right)(x,0)\, dx + \int_0^T \int_0^{X^0} \phi g (u-\eta)\, dxdt.$$

On the one hand this computation is valid for ρ such that $u = \chi\rho \in H^1(0,T;H^2(\mathbb{R}_+))$ and in this case (10) implies (2). On the other hand (10) has a meaning as soon as $\rho \in L^2(0,T;H^1(0,3X_0/4))$. In this sense, (10) is a weak formulation of the boundary conditions (2).

.faible **Definition 1.1.** *Let $T > 0$ be finite, $X^0 > 0$ and $\rho^0 \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$.*

We say that (ρ, M, X) is a weak solution to (1)–(2a) if the following conditions are satisfied:

(a) X is a nondecreasing function in $H^1(0,T)$.

(b) $M \in BV(0,T)$.

(c) $\rho \in L^2_{\mathrm{loc}}(\mathbb{R}_+ \times (0,T)) \cap L^\infty(\mathbb{R}_+ \times (0,T))$ and $\partial_x \rho \in L^2(D_T)$ with $\rho(0,t) \in [\rho_-, \rho_+]$ for a.e. $t \in (0,T)$ and $\rho(x,t) = 1$ for a.e. $x \geq X(t)$ and $t \in (0,T)$.

(d) For all $\varphi \in C_0^\infty(\mathbb{R}_+ \times [0,T))$

imit_rho (11) $$-\int_0^T \int_{\mathbb{R}_+} \rho(x,t)\, \partial_t \varphi(x,t) dx\, dt - \int_{\mathbb{R}_+} \rho^0(x)\, \varphi(x,0)\, dx - \int_0^T \varphi(0,t)\, dDM(t) + \int_0^T \int_{\mathbb{R}_+} \partial_x \rho(x,t)\, \partial_x \varphi(x,t)\, dx\, dt = 0.$$

(e) For all $\xi \in \mathcal{C}(0,T)$

_limit_X (12) $$\lambda \int_0^T \dot X(t)\, \xi(t)\, dt = \alpha \int_0^T \xi(t)\, dt - \int_0^T \left(1 - \rho(X(t)^-, t)\right) \xi(t)\, dt - \int_0^T \ln \rho(X(t)^-, t)\, \xi(t)\, dt.$$

The triplet (ρ, M, X) is a weak solution to (1)–(2) if these conditions are satisfied as well as the variational inequality (10) for all $\phi \in C_0^\infty([0,T), \mathbb{R}_+)$ and every $\eta \in C_0^\infty(\mathbb{R}_+ \times [0,T))$ such that $\eta(0,t) \in [\rho_-, \rho_+]$ for $t \in [0,T]$. Eventually, we say that (ρ, M, X) is a global in time weak solution to (1)–(2) if the functions ρ, M and X satisfy the above conditions for all $T > 0$.

We are now in position to state the main result of this paper:

th.main **Theorem 1.1.** *Let the following assumptions hold*

(H1) Given data: Let α, λ, β, θ and T some positive constants with $\beta + \theta < 1$.

(H2) Initial data: Let $X^0 > 0$ and $\rho^0 \in L^\infty(\mathbb{R}_+)$ be a positive function with $\rho^0_{|[0,X^0]} \in C^{1,1}([0,X^0])$, $\rho^0(x) = 1$ for every $x > X^0$, $\rho^0(0) \in [\rho_-, \rho_+]$ and

$$0 < \rho_{\min} \le \rho^0(x) \le \rho_{\max} \le 1, \quad \forall x \in (0, X^0],$$

with $\rho_{\min}$ and $\rho_{\max}$ some positive constants.

Then, there exists (at least) one weak solution (ρ, M, X) to the system (1)–(2) in the sense of Definition 1.1.

In order to prove Theorem 1.1, we study a JKO minimizing scheme: the problem is semi-discretized in time and the solution at time-step $k+1$ is defined as a minimizer of some functional depending on the time step and on the solution at step k. This scheme is defined in Section 2 and its properties as welll as the properties of the minimizers are studied in Section 3. Section 4 is concerned with the proof of Theorem 1.1. The proof is based on some uniform (w.r.t. the time step) estimates satisfied by the sequences $(\rho^k)_k$, $(M^k)_k$ and $(X^k)_k$ solving the JKO-like scheme. These estimates provide sufficient compactness properties on $(\rho^k, M^k, X^k)_k$ to pass to the limit (up to extraction) and obtain a triple (ρ, M, X) solution to (1)–(2) in the sense of Definition 1.1.

c.intJKO

2. Introduction of the JKO minimizing scheme

In this section we define the minimizing-movements scheme. We first recall the definition of the Wasserstein metric and define the energy functional, then we introduce the JKO scheme and prove the existence of (at least) one solution to this scheme. In Subsection 2.3, we introduce some notations used in the sequel.

sec.dist

2.1. Wasserstein metric and energy functional.

Let $\mathcal{M}^+(I)$ be the set of positive measures defined on I, a bounded interval of $\mathbb{R}$. For two given measures μ and $\tilde\mu \in \mathcal{M}_+(I)$ with $\mu(I) = \tilde\mu(I) = \mathrm{m}$, for some $\mathrm{m} > 0$, we define the squared Wasserstein distance for the quadratic cost $\mathbf{W}_2$ as

def.wass

$$\mathbf{W}_2^2(\mu, \tilde\mu) := \inf_{\gamma \in \Gamma(\mu,\tilde\mu)} \int_{I\times I} (x-y)^2 \, d\gamma(x,y), \tag{13}$$

where $\Gamma(\mu, \tilde\mu)$ denotes the set of transport plans between μ and $\tilde\mu$ defined as

$$\Gamma(\mu,\tilde\mu) := \Big\{ \gamma \in \mathcal{M}^+(I\times I) \ : \ \gamma(I\times I) = \mathrm{m}, \ \pi_{1\#}\gamma = \mu, \ \pi_{2\#}\gamma = \tilde\mu \Big\},$$

with π_1 and π_2 the projections into the first and second component respectively. For the sake of completeness let us recall a classical result in optimal transport theory:

n.rappel

Theorem 2.1. [31, Theorem 1.17] *Let μ and $\tilde\mu$ be two positive measures on a bounded interval I of $\mathbb{R}$ with $\mu(I) = \tilde\mu(I) = \mathrm{m}$. Then there exists a unique optimal transport plan $\gamma \in \Gamma(\mu,\tilde\mu)$ associated to the minization problem in the definition of (13). Moreover, if μ is atomless then this optimal transport plan γ is induced by a map T such that $\gamma = (\mathrm{id}, T)$. In this case there exists an unique (up to an additive constant) Lipschitz function Ψ, called Kantorovich potential, such that it holds*

$$\Psi'(x) = x - T(x) \quad \text{for a.e. } x \in I.$$

Let us define the set $\mathbb{A}$ given by

$$\mathbb{A} := \Big\{ \rho \in L^1_{\rm loc}(\mathbb{R}_+;\mathbb{R}_+) \,:\, \rho \ln \rho \in L^1_{\rm loc}(\mathbb{R}_+;\mathbb{R}) \text{ and } \exists x>0 \text{ s.t. } \rho \equiv 1 \text{ a.e. on } (x,+\infty) \Big\}.$$

We denote

$$x_\rho := \inf \left\{ x>0 \,:\, \rho \equiv 1 \text{ a.e. on } (x,+\infty) \right\}.$$

Given $\rho, \tilde\rho \in \mathbb{A}$, we define the measure

2.mu
$$\mu(\rho,\tilde\rho) := \rho \mathcal{L} \llcorner \mathbb{R}_+ + (\mathbf{M}(\tilde\rho) - \mathbf{M}(\rho))_+ \delta_0, \tag{14}$$

where $(x)_+ = \max(x,0)$, δ_0 denotes the Dirac measure at point $x=0$ and the functional $\mathbf{M} : \mathbb{A} \to \mathbb{R}$ is defined by

2.mapM
$$\mathbf{M}(\rho) := -\int_{\mathbb{R}_+} (1-\rho) < \infty. \tag{15}$$

In the sequel, for two measures of the type (14), we will write

$$\mathbf{W}_2^2(\rho,\tilde\rho) := \mathbf{W}_2^2\Big(\mu(\rho,\tilde\rho), \mu(\tilde\rho,\rho)\Big).$$

Let us now determine this distance. We first notice that these measures do not enter directly in the framework of Theorem 2.1. Indeed, the measure $\mu(\rho,\tilde\rho)$ admits an "infinite" mass. In order to bypass this difficulty, defining Λ as $\Lambda := \max(x_\rho, x_{\tilde\rho})$ we rewrite these measures as

$$\mu(\rho,\tilde\rho) = \nu(\rho,\tilde\rho) + \mathcal{L} \llcorner (\Lambda,\infty) \quad \text{with} \quad \nu(\rho,\tilde\rho) = \rho \mathcal{L} \llcorner (0,\Lambda) + (\mathbf{M}(\tilde\rho) - \mathbf{M}(\rho))_+ \delta_0.$$

Hence, since $\nu(\rho,\tilde\rho)([0,\Lambda]) = \nu(\tilde\rho,\rho)([0,\Lambda])$ and thanks to Theorem 2.1, we define the unique optimal transport plan

2.OT
$$\gamma = \hat\gamma + (\mathrm{Id},\mathrm{Id})_\# \, \mathcal{L} \llcorner (\Lambda,\infty) \in \Gamma(\mu(\rho,\tilde\rho), \mu(\tilde\rho,\rho)), \tag{16}$$

where $\hat\gamma \in \Gamma(\nu(\rho,\tilde\rho), \nu(\tilde\rho,\rho))$. Thus

$$\mathbf{W}_2^2(\rho,\tilde\rho) = \int_{(0,\Lambda)\times(0,\Lambda)} (x-y)^2 d\hat\gamma(x,y).$$

Let us now introduce the energy functional considered in this paper. First of all, we define the set

$$\mathcal{A} := \{ (X,\rho) \,:\, \rho \in \mathbb{A},\, X \geq x_\rho \}.$$

as well as the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with

2.def.f
$$f(r) = r\left(\ln(r) - \beta\right) + \beta. \tag{17}$$

For a given $\theta > 0$ and a given $(X^0,\rho^0) \in \mathcal{A}$, we define the functional $\mathbf{E}_{(X^0,\rho^0)} : \mathcal{A} \to \mathbb{R}$ by

def.NRJ
$$\mathbf{E}_{(X^0,\rho^0)}(X,\rho) := \int_{\mathbb{R}_+} f(\rho(x))\,dx + \theta \left| \mathbf{M}(\rho) - \mathbf{M}\left(\rho^0\right) \right| - \alpha X. \tag{18}$$

Let us notice that the functional $\mathbf{E}_{(X^0,\rho^0)}$ is well-defined since we can write for any $(X,\rho) \in \mathcal{A}$

$$\mathbf{E}_{(X^0,\rho^0)}(X,\rho) = \int_0^X f(\rho(x))\,dx + \theta \left| \mathbf{M}(\rho) - \mathbf{M}\left(\rho^0\right) \right| - \alpha X.$$

In the following we will use the notation

2.notM
$$M\left(\rho,\rho^0\right) := \mathbf{M}(\rho) - \mathbf{M}\left(\rho^0\right) = \int_{\mathbb{R}_+} \left(\rho-\rho^0\right)\, dx. \tag{19}$$

sec.JKO
2.2. **The JKO minimizing scheme.** Let $\tau>0$ be a time step of $(0,T)$, we define $p_\tau:\mathbb{R}\to\mathbb{R}_+$, the function given by

penalty
$$p_\tau(m) := \frac{K_\tau}{2}\left(-m-m_\tau\right)_+^2, \tag{20}$$

with K_τ and m_τ some positive parameters depending on τ and defined in Section 3.3.

Then, starting from the initial configuration $(X^0,\rho^0)\in\mathcal{A}$ we want to determine the existence of at least one $(X,\rho)\in\mathcal{A}$ such that

2.JKO
$$(X,\rho)\in \operatorname{argmin}_{(Y,\tilde\rho)\in\mathcal{A}}\left\{\frac{1}{2\tau}\mathbf{d}^2\Big((Y,\tilde\rho),(X^0,\rho^0)\Big) + \mathbf{E}_{(X^0,\rho^0)}(Y,\tilde\rho) + p_\tau\left(M\left(\tilde\rho,\rho^0\right)\right)\right\}, \tag{21}$$

where $\mathbf{d}$ denotes the tensorized metric given by

$$\mathbf{d}^2\Big((Y,\tilde\rho),(X^0,\rho^0)\Big) := \mathbf{W}_2^2\left(\tilde\rho,\rho^0\right) + \lambda\,\left(Y-X^0\right)^2.$$

The function p_τ is a technical penalization term which will allow us to derive an upper bound on the derivative of the function ρ solution to (21) (see Proposition 3.5).

Finally, for all $(X^0,\rho^0)\in\mathcal{A}$ we introduce the functional $\mathbf{J}_{(X^0,\rho^0)}:\mathcal{A}\to\mathbb{R}$ defined as

$$\mathbf{J}_{(X^0,\rho^0)}(X,\rho) := \frac{1}{2\tau}\mathbf{d}^2\Big((X,\rho),(X^0,\rho^0)\Big) + \mathbf{E}_{(X^0,\rho^0)}(X,\rho) + p_\tau\left(M(\rho,\rho^0)\right),$$

and we rewrite the minimization problem (21) as follows: starting from $(X^0,\rho^0)\in\mathcal{A}$ find

2.JKO1
$$(X,\rho)\in \operatorname{argmin}_{(Y,\tilde\rho)\in\mathcal{A}}\mathbf{J}_{(X^0,\rho^0)}(Y,\tilde\rho). \tag{22}$$

tenceJKO
Theorem 2.2 (Existence of a minimizer)**.** *Assume that the assumptions (H1)-(H2) hold, then for $0<\tau<1$ the minimizing problem (22) admits at least one solution $(X,\rho)\in\mathcal{A}$ where X satisfies $X\geq X^0$.*

Proof. Bearing in mind definition (17) of f we notice that it holds $f(x)\geq -\exp(\beta-1)+\beta$ for all $x\geq 0$. Then, thanks to the definition of the functional $\mathbf{J}_{(X^0,\rho^0)}$ we have for all $(X,\rho)\in\mathcal{A}$

$$\mathbf{J}_{(X^0,\rho^0)}(X,\rho) \geq \frac{\lambda}{2\tau}\left(X-X^0\right)^2 - \left(\alpha+\exp(\beta-1)-\beta\right)X.$$

Thus, a meticulous but rather straightforward analysis of the function in the right hand side leads to

$$\begin{aligned}\mathbf{J}_{(X^0,\rho^0)}(X,\rho) &\geq -\frac{\tau}{2\lambda}\left(\alpha+\exp(\beta-1)-\beta\right)^2 - X^0\left(\alpha+\exp(\beta-1)-\beta\right)\\ &\geq -\frac{1}{2\lambda}\left(\alpha+\exp(\beta-1)-\beta\right)^2 - X^0\left(\alpha+\exp(\beta-1)-\beta\right).\end{aligned}$$

In particular we deduce the existence of a constant $c\in\mathbb{R}$ such that $\mathbf{J}_{(X^0,\rho^0)}(X,\rho)\geq c$. Now, let $(X_k,\rho_k)_{k\in\mathbb{N}}$ be a minimizing sequence in $\mathcal{A}$ of $\mathbf{J}_{(X^0,\rho^0)}$, i.e.,

$$\mathbf{J}_{(X^0,\rho^0)}(X_k,\rho_k)\to \inf_{(Y,\tilde\rho)\in\mathcal{A}}\mathbf{J}_{(X^0,\rho^0)}(Y,\tilde\rho),\quad \text{as } k\uparrow\infty.$$

Hence we deduce that there exists a constant $C \in \mathbb{R}$ and $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ we have

rnes.exi

$$c \leq \mathbf{J}_{(X^0,\rho^0)}(X_k, \rho_k) \leq C. \tag{23}$$

We conclude from this inequality that the sequence $(X_k)_{k\in\mathbb{N}}$ is bounded and converges, up to a subsequence, towards $X \geq 0$ as $k \uparrow \infty$.

Furthermore, using the definition of the functional $\mathbf{J}_{(X^0,\rho^0)}$ we have

$$\theta|M(\rho_k, \rho^0)| \leq C + \alpha X_k - \int_0^{X_k} f(\rho_k(x))\,dx, \quad \forall k \geq k_0.$$

Using the bound $f(x) \geq -\exp(\beta - 1) + \beta$ for all $x \geq 0$, we obtain

$$\theta|M(\rho_k, \rho^0)| \leq C + X_k\,(\alpha + \exp(\beta - 1) - \beta).$$

Hence, since the sequence $(X_k)_{k\in\mathbb{N}}$ is bounded we conclude that there exists $M \in \mathbb{R}$ such that, up to a subsequence, $(M(\rho_k, \rho^0))_{k\in\mathbb{N}}$ converges towards M as $k \uparrow \infty$.

Let us now prove that the sequence $(\rho_k)_{k\geq 0}$ is weakly compact in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$. First, let $\Lambda \geq 1 + \sup_k X_k$, thanks to (23), we have

$$\int_0^{\Lambda} f(\rho_k(x))\,dx \leq C + \alpha\, X_k, \quad \forall k \geq k_0,$$

Thus, since the sequence $(X_k)_{k\in\mathbb{N}}$ is bounded we apply the Dunford-Pettis theorem and we conclude that there exists a nonnegative function $\rho \in L^1(0, \Lambda)$ such that, up to a subsequence,

$$\rho_k \rightharpoonup \rho \quad \text{weakly in } L^1(0, \Lambda).$$

Setting $\rho \equiv 1$ a.e. on (Λ, ∞), we get the weak convergence (up to a subsequence) in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ of $(\rho_k)_{k\in\mathbb{N}}$. Now we have to prove that $\rho(x) = 1$ for a.e. $x \in (X, \Lambda)$. In this purpose, let $\varphi \in C_0^\infty(0, \Lambda)$, then applying the weak convergence in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ of $(\rho_k)_{k\in\mathbb{N}}$ and the convergence of the sequence $(X_k)_{k\in\mathbb{N}}$ we deduce that, as $k \uparrow \infty$,

$$\int_0^{\Lambda} \varphi(\rho_k - 1)\,dx \to \int_0^{\Lambda} \varphi(\rho - 1)\,dx,$$

and

$$\int_0^{\Lambda} \varphi(\rho_k - 1)\,dx = \int_0^{X_k} \varphi(\rho_k - 1)\,dx \to \int_0^{X} \varphi(\rho - 1)\,dx.$$

Subtracting there holds

$$\int_X^{\Lambda} \varphi(\rho - 1)\,dx = 0, \quad \forall \varphi \in C_0^\infty(0, \Lambda),$$

which implies that $\rho(x) = 1$ for a.e. $x \in (X, \Lambda)$ and we readily deduce that $\int_0^X \rho \ln(\rho)\,dx < \infty$ which implies that $(X, \rho) \in \mathbb{A}$.

Moreover, the weak convergence in $L^1_{\rm loc}(\mathbb{R}_+)$ of $(\rho_k)_{k\in\mathbb{N}}$ and the convergence (up to a subsequence) of $(X_k)_{k\in\mathbb{N}}$ lead to

$$M(\rho_k,\rho^0)=\int_0^{X_k}\rho_k\,dx-\int_0^{X^0}\rho^0\,dx+(X_0-X_k)\to\int_{\mathbb{R}_+}(\rho-\rho^0)\,dx,\quad\text{as } k\uparrow\infty,$$

and, using the convergence $M(\rho_k,\rho^0)\to M$, we obtain

$$M=\int_{\mathbb{R}_+}(\rho-\rho^0)\,dx=M(\rho,\rho^0).$$

Furthermore, thanks to the lower semicontinuity for the weak convergence in L^1 of the metric $\mathbf{W}_2$ [31, Proposition 7.4] and the functional $\rho\mapsto\int_0^\infty[\rho(\ln(\rho)-\beta)+\beta]\,dx$ [31, Proposition 7.7] and the continuity of the other terms, we conclude that $(X,\rho)\in\mathbb{A}$ is a minimizer of the functional $\mathbf{J}_{(X^0,\rho^0)}$.

It remains to establish that $X\geq X^0$. Assuming by contradiction that $X^0>X$, (X^0,ρ) is an admissible competitor. Since (X,ρ) is a minimizer of $\mathbf{J}_{(X^0,\rho^0)}$ we have

$$\mathbf{J}_{(X^0,\rho^0)}(X,\rho)\leq\mathbf{J}_{(X^0,\rho^0)}(X^0,\rho),$$

which yields

$$\frac{\lambda}{2\tau}(X-X^0)^2+\mathbf{E}_{(X^0,\rho^0)}(X,\rho)\leq\mathbf{E}_{(X^0,\rho^0)}(X^0,\rho).$$

Using $\mathbf{E}_{(X^0,\rho^0)}(X,\rho)=\mathbf{E}_{(X^0,\rho^0)}(X^0,\rho)+\alpha(X^0-X)$, we deduce that

$$\alpha\,(X^0-X)\leq 0.$$

which contradicts the hypothesis $\alpha>0$. This concludes the proof of Theorem 2.2. □

sec.not

2.3. **Notations for the optimal transport plan.** Theorem 2.2 implies the existence of (at least) one solution, denoted $(X,\rho)\in\mathcal{A}$, to the JKO scheme (22). Since $X\geq X^0$, we can specified, in terms of optimal tranport map, the construction of the transport plan γ given by (16) between the measures

$$\mu(\rho,\rho^0)=\rho\,\mathcal{L}\llcorner\mathbb{R}_+ + \big(-M(\rho,\rho^0)\big)_+\,\delta_0,$$

and

$$\mu(\rho^0,\rho)=\rho^0\,\mathcal{L}\llcorner\mathbb{R}_+ + \big(M(\rho,\rho^0)\big)_+\,\delta_0,$$

recalling definition (19) of $M(\rho,\rho^0)$. The construction of this map depends on the sign of the quantity $M(\rho,\rho^0)$. From now on in order to simplify the notation and if no confusion can occur we simply write μ and μ^0 instead of $\mu(\rho,\rho^0)$ and $\mu(\rho^0,\rho)$ respectively and M instead of $M(\rho,\rho^0)$.

Case $M\geq 0$**.** In this case there exists an increasing map $T_+:[0,X]\to[0,X]$ and a constant $\ell_+\geq 0$ such that

$$T_+(x)=0\quad\text{for } 0\leq x\leq\ell_+,\quad T_+(X)=X,\quad T_{+\#}\,\rho\mathcal{L}\llcorner(\ell_+,X)=\rho^0\mathcal{L}\llcorner(0,X).$$

Then, since $M=\int_0^{\ell_+}\rho\,dx$, we define the unique optimal transport plan $\gamma_+\in\Gamma(\mu,\mu^0)$ by

$$\gamma_+:=(\mathrm{Id},T_+)_\#\,\rho\,\mathcal{L}\llcorner(0,X)+(\mathrm{Id},\mathrm{Id})_\#\,\mathcal{L}\llcorner(X,+\infty),$$

which implies

Mass.R.1 (24) $$\mathbf{W}_2^2(\rho,\rho^0)=\int_0^{\ell_+} x^2\,\rho(x)\,dx+\int_{\ell_+}^{X}(x-T_+(x))^2\,\rho(x)\,dx=\int_0^X (x-T_+(x))^2\,\rho(x)\,dx.$$

Case $M<0$. In this case there exists an increasing map T_-, a constant $\ell_->0$ and a constant map $S_-\equiv 0$ on $[0,\ell_-]$ such that $T_-:[0,X]\to[\ell_-,X]$ with

$$T_-(0)=\ell_-,\quad T_-(X)=X,\quad T_{-\#}\,\rho\mathcal{L}\llcorner(0,X)=\rho^0\mathcal{L}\llcorner(\ell_-,X).$$

Then, since $-M=\int_0^{\ell_-}\rho^0\,dx$ we define the unique optimal transport plan $\gamma_-\in\Gamma(\mu,\mu^0)$ by

$$\gamma_-:=(\mathrm{Id},T_-)_{\#}\,\rho\,\mathcal{L}\llcorner(0,X)+(S_-,\mathrm{Id})_{\#}\,\rho^0\,\mathcal{L}\llcorner(0,\ell_-)+(\mathrm{Id},\mathrm{Id})_{\#}\,\mathcal{L}\llcorner(X,+\infty),$$

which yields

Mass.R.2 (25) $$\mathbf{W}_2^2(\rho,\rho^0)=\int_0^X (x-T_-(x))^2\,\rho(x)\,dx+\int_0^{\ell_-} y^2\,\rho^0(y)\,dy.$$

prop_min

3. Study of the minimizers

In this section we investigate the properties satisfied by a minimizer $(X,\rho)\in\mathcal{A}$ of (22) obtained in Theorem 2.2. In particular, we establish the Euler-Lagrange equations fulfilled by ρ and X, the behavior at the fixed interface $x=0$ of ρ and the positivity of ρ. In this purpose, following a classical approach, we will construct some admissible perturbations $(X_\varepsilon,\rho_\varepsilon)\in\mathcal{A}$ of (X,ρ) in order to study the variations of the functional $\mathbf{J}_{(X^0,\rho^0)}$.

c:couche

3.1. The Euler-Lagrange equation in the oxide layer.

Let us now establish the equation satisfied by ρ in the oxide layer.

nt_oxyde **Proposition 3.1.** *Let the assumptions of Theorem 2.2 hold. Then ρ satisfies the following equation*

3.Oxyde (26) $$\int_0^X \xi(x)\,\frac{(x-T(x))}{\tau}\,\rho(x)\,dx-\int_0^X \rho(x)\,\xi'(x)\,dx=0,\quad\forall\xi\in C_0^\infty(0,X),$$

where T denotes the optimal transport map, i.e. either T_+ or T_-, between μ and μ^0 defined in Section 2.3.

Proof. We follow the proof of [21, Theorem 5.1]. Let $\gamma\in\Gamma(\mu,\mu^0)$, $\xi\in C_0^\infty(0,X)$ and $\varepsilon>0$ small enough such that $(\mathrm{Id}+\varepsilon\,\xi)(\mathbb{R}_+)\subseteq\mathbb{R}_+$, we define the following transport plan

$$\gamma_\varepsilon:=((\mathrm{Id}+\varepsilon\,\xi)\circ\pi_1,\pi_2)_{\#}\,\gamma.$$

We denote by μ_ε the measure given by $\mu_\varepsilon:=\pi_{1\#}\,\gamma_\varepsilon=(\mathrm{Id}+\varepsilon\,\xi)_{\#}\,\mu$ and since $\pi_{2\#}\,\gamma_\varepsilon=\mu^0$ we have $\gamma_\varepsilon\in\Gamma(\mu_\varepsilon,\mu^0)$. Using the optimality of (X,ρ) for $\mathbf{J}_{(X^0,\rho^0)}$ we have

Oxyde_1 (27) $$0\le\frac{\mathbf{J}_{(X^0,\rho^0)}(X,\rho_\varepsilon)-\mathbf{J}_{(X^0,\rho^0)}(X,\rho)}{\varepsilon}=\frac{\mathbf{W}_2^2(\rho_\varepsilon,\rho^0)-\mathbf{W}_2^2(\rho,\rho^0)}{2\varepsilon\tau}+\frac{1}{\varepsilon}\int_{\mathbb{R}_+}(f(\rho_\varepsilon)-f(\rho))\,dx,$$

where we recall that f is given by $f(x) = x(\ln(x) - \beta) + \beta$ for all $x \geq 0$. Applying the definition of γ_ε we obtain

$$\begin{aligned}\frac{\mathbf{W}_2^2(\rho_\varepsilon, \rho^0) - \mathbf{W}_2^2(\rho, \rho^0)}{2\varepsilon\tau} &\leq \frac{1}{2\varepsilon\tau}\int_{\mathbb{R}^2} (x + \varepsilon\,\xi(x) - y)^2\, d\gamma(x,y) - \frac{1}{2\varepsilon\tau}\int_{\mathbb{R}^2} (x-y)^2\, d\gamma(x,y)\\ &\leq \frac{1}{\tau}\int_{\mathbb{R}^2} \xi(x)(x-y)\, d\gamma(x,y) + \frac{\varepsilon}{2\tau}\int_{\mathbb{R}^2} \xi(x)^2\, d\gamma(x,y).\end{aligned}$$

Then, using either (24) or (25), depending on the sign of M, we obtain

$$\frac{\mathbf{W}_2^2(\rho_\varepsilon, \rho^0) - \mathbf{W}_2^2(\rho, \rho^0)}{2\varepsilon\tau} \leq \int_0^X \xi(x)\frac{(x - T(x))}{\tau}\rho(x)\, dx + \frac{\varepsilon||\xi^2||_\infty\, ||\rho||_{L^1(0,X)}}{2\tau}.$$

Thus

$$\limsup_{\varepsilon\downarrow 0}\left\{\frac{\mathbf{W}_2^2(\rho_\varepsilon, \rho^0) - \mathbf{W}_2^2(\rho, \rho^0)}{2\varepsilon\tau}\right\} \leq \int_0^X \xi(x)\,\frac{(x - T(x))}{\tau}\,\rho(x)\, dx.$$

Furthermore, following the proof of [21, Theorem 5.1], passing to the limit $\varepsilon \downarrow 0$ in the last term of the right hand side of (27) leads to

$$\lim_{\varepsilon\downarrow 0}\left\{\frac{1}{\varepsilon}\int_{\mathbb{R}_+} (f(\rho_\varepsilon) - f(\rho))\, dx\right\} = -\int_0^X \rho(x)\,\xi'(x)\, dx.$$

Now we pass to the limit $\varepsilon \downarrow 0$ in (27) and thanks to the above inequalities we conclude that

$$\int_0^X \xi(x)\,\frac{(x - T(x))}{\tau}\,\rho(x)\, dx - \int_0^X \rho(x)\,\xi'(x) \geq 0.$$

Finally, replacing ξ by $-\xi$ we deduce (26) which concludes the proof of Proposition 3.1. □

hini_W11 **Corollary 3.1.** *Let the assumptions of Theorem 2.2 hold. Then, the function ρ satisfying (26) belongs to $SBV^2_{\mathrm{loc}}(\mathbb{R}_+)$ and fulfills the following estimates*

grad_L2 (28) $$\int_0^X |\rho'(x)|^2\, dx \leq ||\rho||_{L^\infty(0,X)}\,\frac{\mathbf{W}_2^2(\rho, \rho^0)}{\tau^2},$$

and

grad_L1 (29) $$\int_0^X |\rho'(x)|\, dx \leq X^{1/2}\,||\rho||^{1/2}_{L^\infty(0,X)}\,\frac{\mathbf{W}_2(\rho, \rho^0)}{\tau}.$$

Moreover $\rho_{|(0,X)} \in H^1(0,X)$ which implies that ρ is continuous on $[0,X]$.

Proof. Let us first notice that thanks to the proof of Theorem 2.2 we have $(\rho, X) \in \mathcal{A}$ which implies that ρ is a nonnegative function in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$. Then, we deduce from (26) and the Cauchy-Schwarz inequality the following estimate

$$\left|\int_0^X \rho(x)\,\xi'(x)\, dx\right| \leq \frac{||\xi||_{L^\infty(0,X)}}{\tau}\,||\rho||^{1/2}_{L^1(0,X)}\,\mathbf{W}_2(\rho, \rho^0), \quad \forall \xi \in C_0^\infty(0,X).$$

As a consequence, since $\rho(x) = 1$ for a.e. $x \geq X$, we have $\rho \in BV_{\mathrm{loc}}(\mathbb{R}_+)$. Hence, thanks to the continuity of the embedding $BV_{\mathrm{loc}}(\mathbb{R}_+) \hookrightarrow L^\infty(\mathbb{R}_+)$, we deduce that $\rho \in L^\infty(\mathbb{R}_+)$. Besides, applying (26) and the Cauchy-Schwarz inequality we obtain

$$\left| \int_0^X \rho(x)\, \xi'(x)\, dx \right| \leq \frac{||\xi||_{L^2(0,X)}}{\tau}\, ||\rho||^{1/2}_{L^\infty(0,X)}\, \mathbf{W}_2(\rho, \rho^0), \quad \forall \xi \in C_0^\infty(0, X).$$

We conclude that

$$\int_0^X |\rho'(x)|^2\, dx \leq ||\rho||_{L^\infty(0,X)}\, \frac{\mathbf{W}_2^2(\rho, \rho^0)}{\tau^2}.$$

Moreover, applying again the Cauchy-Schwarz inequality we have

$$\int_0^X |\rho'(x)|\, dx \leq X^{1/2}\, ||\rho||^{1/2}_{L^\infty(0,X)}\, \frac{\mathbf{W}_2(\rho, \rho^0)}{\tau}.$$

This completes the proof of Corollary 3.1. □

1_Euler1 **Proposition 3.2.** *Let the assumptions of Theorem 2.2 hold. Then, for all $\psi \in C_0^\infty(\mathbb{R}_+)$ we have*

3.EL0
$$\int_{\mathbb{R}_+} \frac{\rho(x) - \rho^0(x)}{\tau}\, \psi(x)\, dx - \frac{M}{\tau}\psi(0) + \int_0^X \rho'(x)\, \psi'(x)\, dx = Q_\tau(\psi), \tag{30}$$

where the right hand side is linear in ψ and satisfies

EL_reste
$$|Q_\tau(\psi)| \leq \frac{||\psi''||_{L^\infty(\mathbb{R}_+)}}{\tau}\, \mathbf{W}_2^2(\rho, \rho^0) + \frac{||\psi'||_{L^\infty(\mathbb{R}_+)}}{\tau}\, \left(\int_0^{\ell_-} y\, \rho^0(y)\, dy \right), \tag{31}$$

with convention $\ell_- = 0$ if $M \geq 0$.

Proof. Thanks to Corollary 3.1 we use an integration by parts in (26) and we obtain

$$\int_0^X \xi(x)\, \frac{(x - T(x))}{\tau}\, \rho(x)\, dx + \int_0^X \rho'(x)\, \xi(x)\, dx = 0, \quad \forall \xi \in C_0^\infty(0, X).$$

Then we extend by density this equality to all functions ξ in $C_0^\infty(\mathbb{R}_+)$ and we set $\xi = \psi'$ such that:

$$\int_0^X \psi'(x)\, \frac{(x - T(x))}{\tau}\, \rho(x)\, dx + \int_0^X \rho'(x)\, \psi'(x)\, dx = 0, \quad \forall \psi \in C_0^\infty(\mathbb{R}_+).$$

Now our main objective is to show that

3.EL3
$$\int_0^X \psi'(x)\, \frac{(x - T(x))}{\tau}\, \rho(x)\, dx = \int_{\mathbb{R}_+} \frac{\rho(x) - \rho^0(x)}{\tau} \psi(x)\, dx - \frac{M}{\tau}\psi(0) + Q_\tau(\psi). \tag{32}$$

In this purpose we split the proof in two different cases depending on M.

Case $M \geq 0$**.** First, using the relation

$$\psi'(x)(x - T_+(x)) = \psi(x) - \psi(T_+(x)) + O(\|\psi''\|_{L^\infty(\mathbb{R}_+)}|x - T_+(x)|^2),$$

we obtain

$$\int_0^X \psi'(x)\, \frac{(x - T_+(x))}{\tau}\, \rho(x)\, dx = \frac{1}{\tau}\int_0^X \rho(x)\, \psi(x)\, dx - \frac{1}{\tau}\int_0^X \rho(x)\, \psi(T_+(x))\, dx + Q_{1,\tau}(\psi),$$

where $Q_{1,\tau}$ is a remaining term satisfying

$$|Q_{1,\tau}(\psi)| \leq \frac{||\psi''||_{L^\infty(\mathbb{R}_+)}}{\tau}\mathbf{W}_2^2(\rho,\rho^0).$$

We notice that

$$\begin{aligned}\int_0^X \rho(x)\,\psi(T_+(x))\,dx &= \int_0^{\ell_+} \psi(0)\,\rho(x)\,dx + \int_{\ell_+}^X \psi(T_+(x))\,\rho(x)\,dx\\ &= M\psi(0) + \int_0^X \rho^0(y)\,\psi(y)\,dy.\end{aligned}$$

Hence, we obtain

$$\int_0^X \psi'(x)\,\frac{(x-T_+(x))}{\tau}\,\rho(x)\,dx = \frac{1}{\tau}\int_0^X (\rho(x)-\rho^0(x))\,\psi(x)\,dx - \frac{M}{\tau}\,\psi(0) + Q_{1,\tau}(\psi).$$

Thus, we conclude that (32) holds.

Case $M<0$. Applying one more time the relation

$$\psi'(x)(x-T_-(x)) = \psi(x) - \psi(T_-(x)) + O\left(\|\psi''\|_{L^\infty(\mathbb{R}_+)}|x-T_-(x)|^2\right),$$

we obtain that

$$\int_0^X \psi'(x)\,\frac{(x-T_-(x))}{\tau}\,\rho(x)\,dx = \frac{1}{\tau}\int_0^X \rho(x)\,\psi(x)\,dx - \frac{1}{\tau}\int_0^X \rho(x)\,\psi(T_-(x))\,dx + Q_{2,\tau}(\psi),$$

with

$$|Q_{2,\tau}(\psi)| \leq \frac{||\psi''||_{L^\infty(\mathbb{R}_+)}}{\tau}\mathbf{W}_2^2(\rho,\rho^0).$$

Thus

$$\begin{aligned}\int_0^X \psi'(x)\,\frac{(x-T_-(x))}{\tau}\,\rho(x)\,dx &= \int_{\mathbb{R}_+} \psi(x)\,\frac{\rho(x)-\rho^0(x)}{\tau}\,dx - \frac{M}{\tau}\psi(0)\\ &\quad + \frac{1}{\tau}\int_0^{\ell_-} \rho^0(x)\,(\psi(x)-\psi(0))dx + Q_{2,\tau}(\psi),\end{aligned}$$

and thanks to the regularity of ψ we deduce that (32) holds which completes the proof of Proposition 3.2. □

er_fixe

3.2. **Behavior of the minimizers at the fixed interface.** Thanks to Corollary 3.1 the function ρ admits a trace at the fixed interface $x=0$ and in this section we study its behavior. Let us first recall the definition of ρ_- and ρ_+

$$\rho_- = \exp(\beta-\theta-1) \quad \text{and} \quad \rho_+ = \exp(\beta+\theta-1).$$

prop_0 **Proposition 3.3.** *Let the assumptions of Theorem 2.2 hold. Then, ρ satisfies either*

rop_0_1 (33) $$\rho(0) = \rho_+\exp(-p'_\tau(M)), \quad \text{if } M<0,$$

rop_0_2 (34) $$\rho(0) = \rho_-, \quad \text{if } M>0,$$

or

prop_0_3 (35) $$\rho_- \leq \rho(0) \leq \rho_+, \quad \textit{if } M = 0.$$

Proof. Depending on the sign of M, we construct (X, ρ_ε) some admissible perturbations of (X, μ). Roughly speaking, the first one corresponds to the case where some oxygen vacancies are transfered from the "solution", i.e. from $x = 0$, towards the oxide layer $(0, X)$ (**Case 1**) and the second one corresponds to the transfer of oxygen vacancies from the oxide layer towards the solution (**Case 2**). Then, for each case, we study

$$\liminf_{\varepsilon \downarrow 0} \left\{ \frac{\mathbf{J}_{(X^0,\rho^0)}(X, \rho_\varepsilon) - \mathbf{J}_{(X^0,\rho^0)}(X, \rho)}{\varepsilon} \right\},$$

and we deduce the relations (33)–(35).

The arguments used to study rigorously this limit are similar when $M < 0$, $M > 0$ or $M = 0$. For this reason in the sequel we only give the full details of our computations when $M < 0$. We refer to [35, Part 3] for the other cases.

Proof of (33). Let us first consider the case when some oxygen vacancies are transfered from the solution towards the oxide layer.

Case 1. Consider $\tilde{\rho} > 0$ and $0 < \varepsilon < 1$ such that $\tilde{\rho} \leq \rho^0(x)$ for $x \in (0, \varepsilon)$ and $\varepsilon < \ell_-$. Then we introduce the transport plan γ_ε as

$$\begin{aligned}\gamma_\varepsilon := (\mathrm{Id}, T_-)_\# \, \rho \, \mathcal{L} \llcorner \mathbb{R}_+ + (\mathrm{Id}, \mathrm{Id})_\# \, \tilde{\rho} \, \mathcal{L} \llcorner (0, \varepsilon) \\ + (S_-, \mathrm{Id})_\# \, (\rho^0 - \tilde{\rho}) \, \mathcal{L} \llcorner (0, \varepsilon) + (S_-, \mathrm{Id})_\# \, \rho^0 \, \mathcal{L} \llcorner (\varepsilon, \ell_-),\end{aligned}$$

where we set $T_- = \mathrm{Id}$ on $(X, +\infty)$. We define the measure $\mu_\varepsilon := \pi_{1\#} \, \gamma_\varepsilon$ and by construction of γ_ε we notice that $\pi_{2\#} \, \gamma_\varepsilon = \mu^0$ such that $\gamma_\varepsilon \in \Gamma(\mu_\varepsilon, \mu^0)$. We rewrite μ_ε as

$$\mu_\varepsilon = \rho_\varepsilon \, \mathcal{L} \llcorner \mathbb{R}_+ + \left(\int_0^{\ell_-} \rho^0 \, dx - \tilde{\rho} \varepsilon \right) \delta_0,$$

where

$$\rho_\varepsilon(x) := \begin{cases} \rho(x) + \tilde{\rho} & \text{for } x \in [0, \varepsilon), \\ \rho(x) & \text{for } x \in [\varepsilon, \infty). \end{cases}$$

Now we consider

$$\begin{aligned}\frac{\mathbf{J}_{(X^0,\rho^0)}(X, \rho_\varepsilon) - \mathbf{J}_{(X^0,\rho^0)}(X, \rho)}{\varepsilon} = \frac{\mathbf{W}_2^2(\rho_\varepsilon, \rho^0) - \mathbf{W}_2^2(\rho, \rho^0)}{2 \, \varepsilon \, \tau} + \frac{1}{\varepsilon} \int_{\mathbb{R}_+} \left(f(\rho_\varepsilon) - f(\rho) \right) dx \\ + \frac{\theta}{\varepsilon} \left(|M(\rho_\varepsilon, \rho^0)| - |M(\rho, \rho^0)| \right) + \frac{1}{\varepsilon} \left(p_\tau(M(\rho_\varepsilon, \rho^0)) - p_\tau(M(\rho, \rho^0)) \right),\end{aligned}$$

where f is defined by (17). Using the definition of $\gamma_\varepsilon \in \Gamma(\mu_\varepsilon, \mu^0)$ we obtain

0.cas1.1 (36) $$\frac{\mathbf{W}_2^2(\rho_\varepsilon, \rho^0) - \mathbf{W}_2^2(\rho, \rho^0)}{2 \, \varepsilon \, \tau} \leq -\frac{1}{2\varepsilon\tau} \int_0^\varepsilon x^2 \, \tilde{\rho} \, dx \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Now, using the definition of ρ_ε we notice that the following relations hold

0.cas1.2 (37) $$\frac{1}{\varepsilon} \int_{\mathbb{R}_+} \Big(f(\rho_\varepsilon) - f(\rho) \Big) \, dx = \frac{1}{\varepsilon} \int_0^\varepsilon \Big(f(\rho + \tilde{\rho}) - f(\rho) \Big) \, dx \to f(\rho(0) + \tilde{\rho}) - f(\rho(0)) \quad \text{as } \varepsilon \downarrow 0,$$

and

0.cas1.3 (38) $$\frac{\theta}{\varepsilon}\Big(|M(\rho_\varepsilon,\rho^0)|-|M(\rho,\rho^0)|\Big)=-\theta\tilde\rho.$$

Finally, we have

0.cas1.4 (39) $$\frac{1}{\varepsilon}\Big(p_\tau\left(M(\rho_\varepsilon,\rho^0)\right)-p_\tau\left(M(\rho,\rho^0)\right)\Big)\to\tilde\rho p_\tau'\left(M(\rho,\rho^0)\right)\quad\text{as }\varepsilon\downarrow 0.$$

Thus, gathering (36)–(39), we conclude that

$$0\le\liminf_{\varepsilon\downarrow0}\left\{\frac{\mathbf{J}_{(X^0,\rho^0)}(X,\rho_\varepsilon)-\mathbf{J}_{(X^0,\rho^0)}(X,\rho)}{\varepsilon}\right\}\le f(\rho(0)+\tilde\rho)-f(\rho(0))-\theta\tilde\rho+\tilde\rho p_\tau'(M(\rho,\rho^0)),$$

which implies

$$f'(\rho(0))-\theta+p_\tau'(M(\rho,\rho^0))\ge 0.$$

In other words we obtain

clusion1 (40) $$\ln(\rho(0))\ge\ln(\rho_+)-p_\tau'(M(\rho,\rho^0)).$$

Case 2. For $\tilde\rho>0$ we consider $0<\varepsilon<1$ with $\tilde\rho\le\rho(x)$ for $x\in[0,\varepsilon]$ and $\varepsilon<\ell_-$. Let us notice that such $\tilde\rho$ and ε exist since thanks to the previous case we know that $\rho(0)\ge\rho_+\exp(-p_\tau'(M(\rho,\rho^0)))>0$ and according to Corollary 3.1 ρ is continuous near $x=0$. We introduce the transport plan γ_ε as

$$\begin{aligned}\gamma_\varepsilon:=&\,(\mathrm{Id},T_-)_\#\left(\rho-\tilde\rho\right)\mathcal{L}\llcorner(0,\varepsilon)+(S_-,T_-)_\#\,\tilde\rho\,\mathcal{L}\llcorner(0,\varepsilon)\\&+(\mathrm{Id},T_-)_\#\,\rho\,\mathcal{L}\llcorner(\varepsilon,\infty)+(S_-,\mathrm{Id})_\#\,\rho^0\,\mathcal{L}\llcorner(0,\ell_-),\end{aligned}$$

with convention $T_-=\mathrm{Id}$ on $(X,+\infty)$. We set $\mu_\varepsilon:=\pi_{1\#}\,\gamma_\varepsilon$ and we notice that $\pi_{2\#}\,\gamma_\varepsilon=\mu^0$ such that $\gamma_\varepsilon\in\Gamma(\mu_\varepsilon,\mu^0)$. We rewrite μ_ε as

$$\mu_\varepsilon=\rho_\varepsilon\,\mathcal{L}\llcorner\mathbb{R}_+\,+\left(\int_0^{\ell_-}\rho^0\,dx+\varepsilon\tilde\rho\right)\delta_0,$$

where

$$\rho_\varepsilon(x):=\begin{cases}\rho(x)-\tilde\rho & \text{for } x\in[0,\varepsilon),\\ \rho(x) & \text{for } x\in(\varepsilon,\infty).\end{cases}$$

Similarly to the previous case we obtain

0.cas2.1 (41) $$\frac{\mathbf{W}_2^2(\rho_\varepsilon,\rho^0)-\mathbf{W}_2^2(\rho,\rho^0)}{2\,\varepsilon\,\tau}\le-\frac{1}{2\,\varepsilon\,\tau}\int_0^\varepsilon(x-T_-(x))^2\,\tilde\rho\,dx+\frac{1}{2\,\varepsilon\,\tau}\int_0^\varepsilon T_-(x)^2\,\tilde\rho\,dx\to0\quad\text{as }\varepsilon\downarrow0.$$

Applying the definition of ρ_ε we have

0.cas2.2 (42) $$\frac{1}{\varepsilon}\left(\int_{\mathbb{R}_+}(f(\rho_\varepsilon)-f(\rho))\,dx\right)=\frac{1}{\varepsilon}\int_0^\varepsilon(f(\rho-\tilde\rho)-f(\rho))\,dx\to f(\rho(0)-\tilde\rho)-f(\rho(0))\quad\text{as }\varepsilon\downarrow0,$$

and

0.cas2.3 (43) $$\frac{\theta}{\varepsilon}\Big(|M(\rho_\varepsilon,\rho^0)|-|M(\rho,\rho^0)|\Big)=\theta\tilde\rho,$$

and eventually

0.cas2.4 (44) $$\frac{1}{\varepsilon}\left(p_\tau(M(\rho_\varepsilon,\rho^0))-p_\tau(M(\rho,\rho^0))\right)\to-\tilde\rho\,p'_\tau(M(\rho,\rho^0))\quad\text{as }\varepsilon\downarrow 0.$$

Now, gathering (41)–(44) yields

$$0\le\liminf_{\varepsilon\downarrow0}\left\{\frac{\mathbf{J}_{(X^0,\rho^0)}(X,\rho_\varepsilon)-\mathbf{J}_{(X^0,\rho^0)}(X,\rho)}{\varepsilon}\right\}\le f(\rho(0)-\tilde\rho)-f(\rho(0))+\theta\tilde\rho-\tilde\rho p'_\tau(M(\rho,\rho^0)),$$

such that $-f'(\rho(0))+\theta-p'_\tau(M(\rho,\rho^0))\ge 0$ and then

$$\ln(\rho(0))\le\ln(\rho_+)-p'_\tau(M(\rho,\rho^0)).$$

Hence, bearing in mind (40), we conclude that (33) holds. □

unif.Lip 3.3. **Positivity and Lipschitz estimates.** In (H2) we assume that $\rho_{\min}\le\rho^0(x)\le 1$ for $x\in(0,X^0)$. In this section we prove that these bounds are preserved for the solution $(X,\rho)\in\mathcal{A}$ to the JKO scheme (22) and we deduce that ρ is a Lipschitz continuous function. We first need to establish the following result:

sitivity **Proposition 3.4.** *Let the assumptions of Theorem 2.2 hold. Then, the density $\rho\in SBV^2_{\rm loc}(\mathbb{R}_+)$ satisfies $\rho>0$ for all $x\in\mathbb{R}_+$ and $\ln\rho\in L^1(\mathbb{R}_+)$.*

Proof. In this purpose we follow the proof of Lemma 8.6 in [31]. The key argument of this proof is to build an admissible perturbation ρ_ε of ρ in order to deduce the existence of a constant C such that

perturb (45) $$\int_0^X\Big(f(\rho)-f(\rho_\varepsilon)\Big)\,dx\le C\varepsilon.$$

In our case we will make explicit the construction of these admissible perturbations such that (45) holds (if $M\ge0$ or $M<0$). Then, when (45) will be established, we will refer to the remaining of the proof of [31, Lemma 8.6] to conclude that $\rho>0$ for all $x\in\mathbb{R}_+$ and $\ln\rho\in L^1(\mathbb{R}_+)$.
case $M\ge0$**.** In this case we introduce the following piecewise constant function

$$\tilde\rho(x):=\begin{cases}\frac{1}{\ell_+}\int_0^{\ell_+}\rho(x)\,dx & \text{if } 0\le x\le\ell_+,\\ \frac{1}{X-\ell_+}\int_{\ell_+}^X\rho(x)\,dx & \text{if } \ell_+<x\le X,\\ 1 & \text{if } X<x.\end{cases}$$

We also introduce $\rho_\varepsilon(x)=(1-\varepsilon)\rho(x)+\varepsilon\tilde\rho(x)$ for all $x\in\mathbb{R}_+$ and we notice that $(X,\rho_\varepsilon)\in\mathcal{A}$. Then, denoting by $\tilde\mu:=\tilde\rho\,\mathcal{L}\llcorner\mathbb{R}_+$, we consider the measure

$$\mu_\varepsilon:=\rho_\varepsilon\,\mathcal{L}\llcorner\mathbb{R}_+=(1-\varepsilon)\mu+\varepsilon\tilde\mu.$$

Let us now notice, by construction of $\tilde\rho$, that it holds $M(\tilde\rho,\rho^0)=M(\rho,\rho^0)$. Besides, by linearity of the integral it also holds $M(\rho_\varepsilon,\rho^0)=M(\rho,\rho^0)$. In particular, we have

$$\mu^0=\rho^0\,\mathcal{L}\llcorner\mathbb{R}_+ + M(\rho,\rho^0)\,\delta_0=\rho^0\,\mathcal{L}\llcorner\mathbb{R}_+ + M(\tilde\rho,\rho^0)\,\delta_0.$$

Thus, following Section 2.3, there exists an optimal transport map $\tilde T_+$ such that

$$\mathbf{W}_2^2(\tilde\rho,\rho^0)=\int_0^X(x-\tilde T_+(x))^2\,\tilde\rho(x)\,dx.$$

We now define the transport plan γ_ε by

$$\gamma_\varepsilon := (\mathrm{Id}, T_+)_\# \, (1-\varepsilon)\rho\mathcal{L}\llcorner(0,X) + (\mathrm{Id}, \tilde{T}_+)_\# \, \varepsilon\tilde{\rho}\mathcal{L}\llcorner(0,X) + (\mathrm{Id},\mathrm{Id})_\# \, \mathcal{L}\llcorner(X,+\infty),$$

and we have $\gamma_\varepsilon \in \Gamma(\mu_\varepsilon, \mu^0)$. Therefore, we get

3.pos1 (46) $$\mathbf{W}_2^2(\rho_\varepsilon, \rho^0) \leq (1-\varepsilon)\, \mathbf{W}_2^2(\rho, \rho^0) + \varepsilon \mathbf{W}_2^2(\tilde{\rho}, \rho^0).$$

It remains to notice that

3.pos2 (47) $$0 \geq \mathbf{J}_{(X^0,\rho^0)}(X,\rho) - \mathbf{J}_{(X^0,\rho^0)}(X,\rho_\varepsilon) = \frac{1}{2\tau}\mathbf{W}_2^2(\rho,\rho^0) - \frac{1}{2\tau}\mathbf{W}_2^2(\rho_\varepsilon,\rho^0) + \int_0^X \Big(f(\rho) - f(\rho_\varepsilon)\Big)\, dx.$$

Hence, combining (46) and (47), we conclude that there exists a constant C such that (45) holds.
Case $M < 0$**.** Similarly to the previous case we introduce the following piecewise constant function

$$\tilde{\rho}(x) := \begin{cases} \frac{1}{X}\int_0^X \rho(x)\, dx & \text{if } 0 \leq x \leq X, \\ 1 & \text{if } X < x. \end{cases}$$

We also introduce $\rho_\varepsilon(x) = (1-\varepsilon)\rho(x) + \varepsilon\tilde{\rho}(x)$ for all $x \in \mathbb{R}_+$ and we have $(X, \rho_\varepsilon) \in \mathcal{A}$. Then, we consider the measure

$$\mu_\varepsilon := \rho_\varepsilon \, \mathcal{L}\llcorner\mathbb{R}_+ - M(\rho_\varepsilon, \rho^0)\, \delta_0 = (1-\varepsilon)\mu + \varepsilon\tilde{\mu},$$

with

$$\tilde{\mu} = \tilde{\rho}\, \mathcal{L}\llcorner\mathbb{R}_+ - M(\tilde{\rho}, \rho^0).$$

Since $M(\tilde{\rho}, \rho^0) = M(\rho, \rho^0)$, there exists an optimal transport map $\tilde{T}_-$ such that

$$\mathbf{W}_2^2(\tilde{\rho}, \rho^0) = \int_0^X (x - \tilde{T}_-(x))^2\, \tilde{\rho}(x)\, dx + \int_0^{\ell_-} y^2\, \rho^0(y)\, dy,$$

and a transport plan γ_ε defined by

$$\begin{aligned}\gamma_\varepsilon :&= (\mathrm{Id}, T_-)_\# \, (1-\varepsilon)\rho\mathcal{L}\llcorner(0,X) + (\mathrm{Id}, \tilde{T}_-)_\# \, \varepsilon\tilde{\rho}\mathcal{L}\llcorner(0,X) \\ &\quad + (S_-, \mathrm{Id})_\# \, \rho^0 \, \mathcal{L}\llcorner(0,\ell_-) + (\mathrm{Id},\mathrm{Id})_\# \, \mathcal{L}\llcorner(X,+\infty).\end{aligned}$$

We notice that γ_ε satisfies $\gamma_\varepsilon \in \Gamma(\mu_\varepsilon, \mu^0)$ and we also get in this case

$$\begin{aligned}\mathbf{W}_2^2(\rho_\varepsilon, \rho^0) &\leq (1-\varepsilon) \int_0^X (x - T_-(x))^2\, \rho(x)\, dx + \varepsilon \int_0^X (x - \tilde{T}_-(x))^2\, \tilde{\rho}(x)\, dx + \int_0^{\ell_-} y^2\, \rho^0(y)\, dy \\ &\leq (1-\varepsilon)\, \mathbf{W}_2^2(\rho, \rho^0) + \varepsilon\, \mathbf{W}_2^2(\tilde{\rho}, \rho^0).\end{aligned}$$

Thus, we deduce the existence of a constant C such that (45) holds. This ends the proof of Proposition 3.4. □

Thanks to Corollary 3.1 and classical Sobolev embedding the function ρ restricted to $(0, X)$ is continuous. Moreover, Corollary 3.1 and Proposition 3.4 imply that the function $\ln(\rho)$ is in $SBV^2_{\mathrm{loc}}(\mathbb{R}_+)$. Then, we deduce from Proposition 3.1 that it holds

$$\frac{\Psi'(x)}{\tau} + (\ln\rho)'(x) = 0, \quad \text{for a.e. } x \in (0,X),$$

where we recall that Ψ denotes the unique Kantorovich potential associated to the optimal transport map T (with T either given by T_+ or T_- depending on the sign of M). In particular, we conclude that there exists a constant C such that ρ satisfies the optimality condition:

$$\frac{\Psi(x)}{\tau} + \ln \rho(x) = C, \quad \text{for every } x \in [0, X].$$

Hence, ρ is a Lipschitz continuous function since

$$\rho(x) = \exp\left(C - \frac{\Psi(x)}{\tau}\right), \quad \text{for every } x \in [0, X].$$

We gather these remarks in the following result.

coro.lip **Corollary 3.2.** *Let the assumptions of Theorem 2.2 hold. Then, there exists a constant C such that ρ satisfies*

opt.cond (48) $$\frac{\Psi(x)}{\tau} + \ln \rho(x) = C, \quad \textit{for every } x \in [0, X],$$

where Ψ denotes the (unique) Kantorovich potential associated to the optimal transport map T constructed in Section 2.3. Moreover, ρ is Lipschitz continuous and fulfills

$$\rho(x) = \exp\left(C - \frac{\Psi(x)}{\tau}\right), \quad \textit{for every } x \in [0, X].$$

Corollary 3.2 implies that $(\ln \rho)'(x) \lesssim \tau^{-1}$ for all $x \in [0, X]$. However, for later use, see the proof of Lemma 4.2 and Proposition 4.8, this upper bound is too coarse and we need to derive a finer upper bound of the type $\tau^{-\vartheta}$ for some $\vartheta \in (0, 1)$. In this purpose, the penalty function p_τ will play a crucial role. In particular, by defining properly the parameters m_τ and K_τ introduced in the definition of p_τ we prove the desired estimate. But first, let us introduce the following notations

$$a := \min_{x \in [0, X^0]} \ln \rho^0(x), \quad b := \max_{x \in [0, X^0]} \ln \rho^0(x),$$

and

$$A := -\min\left(0, \inf_{x \in [0, X^0]} \left(\ln \rho^0\right)'(x)\right), \quad B_0 := \sup_{x \in [0, X^0]} \left(\ln \rho^0\right)'(x).$$

Then, for $\vartheta \in (0, 1)$ and $\delta_0 > 0$, we define $B_\tau := \max(B_0, \delta_0)\, \tau^{-\vartheta}$ such that $B_\tau \geq \max(B_0, \delta_0)$ and

$$a \leq \ln \rho^0(x) \leq b, \quad -A \leq \left(\ln \rho^0\right)'(x) \leq B_\tau \quad \text{for } x \in [0, X^0].$$

In the following statement we claim that these bounds are preserved for $\ln \rho$.

unif.Lip **Proposition 3.5.** *Let the assumptions of Theorem 2.2 hold. For $\delta_0 > 0$ and $\vartheta \in (0, 1)$ fixed, we set $B_0' = \max(B_0, \delta_0)$, $B_\tau = B_0'\, \tau^{-\vartheta}$ and*

onstants (49) $$m_\tau := \frac{B_0' \exp(a)}{2}\, \tau^{1-\vartheta}, \quad K_\tau := \frac{2(b - \ln \rho_+)}{B_0' \exp(a)}\, \tau^{\vartheta - 1},$$

where m_τ and K_τ denote the parameters involved in the definition (20) of the penalty function p_τ. Then, the Lipschitz continuous function ρ satisfies

s.logrho
$$a \le \ln \rho(x) \le b \quad \text{for } x \in [0, X], \tag{50}$$

r.logrho
$$-A \le (\ln \rho)'(x) \le B_\tau \quad \text{for } x \in [0, X]. \tag{51}$$

Proof. We split the proof in two cases, the first one corresponds to $M \ge 0$ and the second one to $M < 0$.

Case $M \ge 0$, bounds (50)–(51). In this case, thanks to Theorem 2.1 and Corollary 3.2, we notice that the following relations hold

3.Lip1
$$T_+(x) = x - \Psi'_+(x) \quad \text{for } x \in [0, X], \tag{52}$$

3.Lip2
$$(\ln \rho)'(x) = -\frac{\Psi'_+(x)}{\tau} \quad \text{for } x \in [0, X], \tag{53}$$

3.Lip3
$$T'_+(x) = \frac{\rho(x)}{\rho^0(T_+(x))} \quad \text{for } x \in (\ell_+, X], \tag{54}$$

where Ψ_+ denotes the (unique) Kantorovich potential associated to the optimal transport map T_+. Besides, the regularity theory for the solutions to the Monge-Ampère equation done by Caffarelli, see [7, 12, 16, 17], implies that the function Ψ_+ is at least $C^{3,\beta}$ on $(\ell_+, X]$ for some $\beta \in (0, 1)$. Moreover, since $T_+(x) = 0$ for all $x \in [0, \ell_+]$ then using (52) we deduce that Ψ_+ is regular on $[0, \ell_+]$.

Proof of (50). Subcase $M > 0$. Thanks to Proposition 3.3 and (52)–(53) we first notice that $\ln \rho(0) = \ln \rho_-$ and $(\ln \rho)'(x) = -x/\tau \le 0$ for all $x \in [0, \ell_+]$. These relations imply that

$$\ln \rho(x) \le b \quad \forall x \in [0, \ell_+],$$

and, since $\ln \rho$ is strictly decreasing and C^1 on $[0, \ell_+]$, the minimum value of $\ln \rho$ is reached on $(\ell_+, X]$ and its maximum value is less than b or is reached on $(\ell_+, X]$.

Let $x^* \in (\ell_+, X]$ be a point where $\ln \rho$ achieves either its minimum or maximum value on $(\ell_+, X]$. We notice that $x^* = T_+(x^*)$. Indeed, if $x^* < X$ (the case $x^* = X$ being clear) we have with (53) $\Psi'_+(x^*) = 0$ and then $T_+(x^*) = x^*$ is a consequence of (52). Now, using (54) we obtain

$$\frac{\rho(x^*)}{\rho^0(x^*)} = 1 - \Psi''_+(x^*).$$

Applying (53) either $\Psi''_+(x^*) \le 0$ if x^* is a minimum point of $\ln \rho$ or $\Psi''_+(x^*) \ge 0$ if x^* is a maximum point of $\ln \rho$.

If x^* is a *minimum* point of $\ln \rho$ then $x^* < X^0$. Indeed let us assume that $x^* \ge X^0$ thus $\rho(x) \ge \rho(x^*) \ge \rho^0(x^*) = 1$ for all $x \in [x^*, X]$. However, as x^* and X are fixed-points of T_+, we have

$$\int_{x^*}^{X} \rho(x)\, dx = \int_{x^*}^{X} \rho^0(x)\, dx = X - x^*.$$

Thus by conservation of mass and since $\rho(x) \ge 1$ on $[x^*, X]$ we conclude that the set $\{x \in [x^*, X] : \rho(x) > 1\}$ is negligible. Therefore, for every $x \in [x^*, X]$ it holds $\rho(x) = 1$ and a direct computation leads to

$$\mathbf{J}_{(X^0,\rho^0)}(x^*, \rho) < \mathbf{J}_{(X^0,\rho^0)}(X, \rho),$$

which contradicts the optimality of (X,ρ). Thus $x^* < X^0$ and consequently for every $x \in (\ell_+, X)$ we get $\rho(x) \geq \rho(x^*) \geq \rho^0(x^*) \geq e^a$.

If x^* is a *maximum* point of $\ln(\rho)$, then $x^* < X^0$. Indeed let us assume by contradiction that $x^* \geq X^0$ then for every $x \in (x^*, X)$, $\rho(x) \leq \rho(x^*) \leq \rho^0(x^*) = 1$. Arguing as previously we show that $\rho(x) = 1$ for all $x \in [x^*, X]$. However, this fact contradicts the optimality of (X,ρ). Thus we conclude that $x^* < X^0$, and for every $x \in (\ell_+, X)$ we deduce that $\rho(x) \leq \rho(x^*) \leq \rho^0(x^*) \leq e^b$.

Subcase $M = 0$. Here $\ell_+ = 0$ then $\rho_- \leq \rho(0) \leq \rho_+$ (see Proposition 3.3) and $(\ln \rho)'(0) = 0$. Thus arguing as before we show that $\ln \rho$ satisfies the bounds (50).

Proof of (51). Subcase $M > 0$. Since $(\ln \rho)'(x) = -x/\tau$ for all $x \in [0, \ell_+]$ we have

$$-\frac{\ell_+}{\tau} \leq (\ln \rho)'(x) \leq 0 \leq B_\tau \quad \forall x \in [0, \ell_+].$$

Moreover, as $(\ln \rho)'$ is strictly decreasing and C^1 on $[0, \ell_+]$, then the function $(\ln \rho)'$ achieves its minimum value at a point in $(\ell_+, X]$ and its maximum value is either less than zero or reached at a point in $(\ell_+, X]$.

If the function $(\ln \rho)'$ achieves its *minimum or maximum* value at $x^* \in (\ell_+, X]$ then $x^* < X$. Indeed, if $x^* = X$ we have, using (52)–(53), $(\ln \rho)'(X) = -\Psi'_+(X)/\tau = 0$ and (51) still holds true. Thus, if $x^* < X$, we have thanks to (53), $\Psi''_+(x^*) = 0$ such that $T'_+(x^*) = 1$. Besides, equality (53) also implies that $\Psi'''_+(x^*) \leq 0$ if x^* is a minimum point of $(\ln \rho)'$ or $\Psi'''_+(x^*) \geq 0$ if x^* is a maximum point of the function $(\ln \rho)'$. Moreover (52) yields

$$(\ln T'_+)'(x^*) = \frac{T''_+(x^*)}{T'_+(x^*)} = -\Psi'''_+(x^*) \quad \forall x \in (\ell_+, X].$$

Therefore we deduce from (54) that

$$(\ln \rho)'(x^*) = (\ln T'_+)'(x^*) + (\ln \rho^0)'(T_+(x^*)) = -\Psi'''_+(x^*) + (\ln \rho^0)'(T_+(x^*)) \quad \forall x \in (\ell_+, X].$$

Hence, we conclude that either

$$(\ln \rho)'(x^*) \geq (\ln \rho^0)'(T_+(x^*)) \geq -A \quad \text{if } x^* \text{ is a minimum point,}$$

or

$$(\ln \rho)'(x^*) \leq (\ln \rho^0)'(T_+(x^*)) \leq B_\tau \quad \text{if } x^* \text{ is a maximum point.}$$

Subcase $M = 0$. Eventually, if $\ell_+ = 0$, then

$$(\ln \rho)'(0) = -\frac{\Psi'_+(0)}{\tau} = 0.$$

Hence, in any case the relation (51) is fulfilled.

Case $M<0$, bounds (50)–(51). In this case the following relations hold

3.Lip4 (55) $$T_-(x) = x - \Psi'_-(x) \quad \text{for } x\in[0,X],$$

3.Lip5 (56) $$(\ln\rho)'(x) = -\frac{\Psi'_-(x)}{\tau} \quad \text{for } x\in[0,X],$$

3.Lip6 (57) $$T'_-(x) = \frac{\rho(x)}{\rho^0(T_-(x))} \quad \text{for } x\in[0,X],$$

where Ψ_- denotes the Kantorovich potential associated to the optimal map T_-. Furthermore, similarly to the previous case, the function Ψ_- is at least $C^{3,\beta}$, for some $\beta\in(0,1)$, on $[0,X]$.

Arguing as before one can prove that

$$a \le \ln\rho(x) \le b \quad \text{for } x\in(0,X],$$

and

$$-A \le (\ln\rho)'(x) \le B_\tau \quad \text{for } x\in(0,X].$$

It remains to study what happens at $x=0$. In this purpose, using (55) and (56) we notice that

$$(\ln\rho)'(0) = -\frac{\Psi'_-(0)}{\tau} = \frac{T_-(0)}{\tau} = \frac{\ell_-}{\tau} > 0,$$

such that the maximum of $\ln\rho$ is reached on $(0,X]$ and then $\ln\rho(x)\le b$ for all $x\in[0,X]$. Besides $(\ln\rho)'(0)$ is positive and we conclude that $(\ln\rho)'(x)\ge -A$ for all $x\in[0,X]$.

Now thanks to Proposition 3.3 and the assumption $\rho_+\ge\rho^0(0)$ we have $\ln\rho(0)\ge\ln\rho_+\ge a$ which implies that $\ln\rho(x)\ge a$ for every $x\in[0,X]$. Besides, as $\ln\rho(0)=\ln\rho_+ - p'_\tau(M)$ (see (33)), we have $-p'_\tau(M)\le b-\ln\rho_+$ which yields

$$-M \le m_\tau + \frac{b-\ln\rho_+}{K_\tau}.$$

Moreover, since

$$\ell_- = \int_0^{\ell_-}\frac{\rho^0}{\rho^0}\,dx \le \exp(-a)\int_0^{\ell_-}\rho^0\,dx = -M\exp(-a),$$

we conclude, applying the definition (49) of the parameters m_τ and K_τ, that it holds

ellmoins (58) $$(\ln\rho)'(0) = \frac{\ell_-}{\tau} \le \frac{m_\tau\exp(-a)}{\tau} + \frac{b-\ln\rho_+}{\tau\,K_\tau}\exp(-a) = B'_0\tau^{-\vartheta} = B_\tau.$$

In particular $(\ln\rho)'(x)\le B_\tau$ for every $x\in[0,X]$ which concludes the proof of Proposition 3.5. □

vo_inter

3.4. **The Euler-Lagrange equation satisfies by the free interface.** In this section, our main objective is to determine the Euler-Lagrange equation satisfies by X the free interface.

nterface **Proposition 3.6.** *Let the assumptions of Theorem 2.2 hold. Then X satisfies the following equation*

DD_inter (59) $$\lambda\frac{X-X^0}{\tau} = \alpha - (1-\rho(X^-)) - \ln(\rho(X^-)).$$

It is worth mentioning that the right hand side of (59) represents the variation of the "Boltzmann" energy to pass from (X^0, ρ^0) to (X, ρ) and the left hand side the variation in term of the metric $\mathbf{d}^2$ or more precisely in term of the squared euclidean distance between X^0 and X. In particular, $\mathbf{W}_2^2$ does not play any role in (59). Formally, the idea is that the optimal transport plan $\gamma \in \Gamma(\mu, \mu^0)$ only acts on the mass ρ and ρ^0 but not on X or X^0.

Proof. We consider two perturbations of X, the first one corresponds to $X_\varepsilon = X - \varepsilon$ and the second one to $X_\varepsilon = X + \varepsilon$ for $\varepsilon > 0$.
Case $X_\varepsilon = X - \varepsilon$. In this case, in order to construct an admissible perturbation $(X_\varepsilon, \mu_\varepsilon) \in \mathcal{A}$ of (X, μ), we have to ensure that the density ρ_ε of μ_ε is equal to 1 in (X_ε, X). In this aim, for $0 < \varepsilon < 1$ such that $\rho(x) - m(\varepsilon)\sqrt{\varepsilon} \geq 0$ for $x \in [X_\varepsilon - \sqrt{\varepsilon}, X_\varepsilon]$, where

$$m(\varepsilon) := 1 - \fint_{X_\varepsilon}^{X} \rho(x)\, dx,$$

we consider the map $D_1 : (X_\varepsilon - \sqrt{\varepsilon}, X_\varepsilon) \to (X_\varepsilon, X)$ given by $D_1(x) := X_\varepsilon + \sqrt{\varepsilon}(x - X_\varepsilon + \sqrt{\varepsilon})$. This map is defined in such way that

3.X1 (60) $$\mathbf{W}_2^2(m(\varepsilon)\sqrt{\varepsilon}\,\mathcal{L} \llcorner (X_\varepsilon - \sqrt{\varepsilon}, X_\varepsilon), m(\varepsilon)\,\mathcal{L} \llcorner (X_\varepsilon, X)) \leq \int_{X_\varepsilon - \sqrt{\varepsilon}}^{X_\varepsilon} (x - D_1(x))^2 m(\varepsilon)\sqrt{\varepsilon} dx = o(\varepsilon).$$

Then, we consider the transport plan γ_ε defined by

$$\begin{aligned}\gamma_\varepsilon :=\ & \gamma \llcorner \mathbb{R}^2 \setminus (X_\varepsilon - \sqrt{\varepsilon}, X_\varepsilon) \times (T(X_\varepsilon - \sqrt{\varepsilon}), T(X_\varepsilon)) \\ & + (\mathrm{Id}, T)_\# \left(\rho - m(\varepsilon)\sqrt{\varepsilon}\right) \mathcal{L} \llcorner (X_\varepsilon - \sqrt{\varepsilon}, X_\varepsilon) \\ & + (D_1, T)_\# \, m(\varepsilon)\sqrt{\varepsilon}\, \mathcal{L} \llcorner (X_\varepsilon - \sqrt{\varepsilon}, X_\varepsilon),\end{aligned}$$

where T is the optimal transport map introduced in Section 2.3 (we implicitly assume in the case $M \geq 0$ that ε is small enough in order to have $X_\varepsilon - \sqrt{\varepsilon} > \ell_+$). Finally, we set $\mu_\varepsilon := \pi_{1\#}\, \gamma_\varepsilon$ and since $\pi_{2\#}\, \gamma_\varepsilon = \mu^0$ we have $\gamma_\varepsilon \in \Gamma(\mu_\varepsilon, \mu^0)$. Now, let us rewrite μ_ε as

$$\mu_\varepsilon = \rho_\varepsilon\, \mathcal{L} \llcorner \mathbb{R}_+ + (-M)_+\, \delta_0,$$

with

$$\rho_\varepsilon(x) := \begin{cases} \rho(x) & \text{for } x \in \mathbb{R}_+ \setminus [X_\varepsilon - \sqrt{\varepsilon}, X), \\ \rho(x) - m(\varepsilon)\sqrt{\varepsilon} & \text{for } x \in [X_\varepsilon - \sqrt{\varepsilon}, X_\varepsilon], \\ 1 & \text{for } x \in (X_\varepsilon, X). \end{cases}$$

Using the definition of γ_ε yields

$$\begin{aligned}\frac{\mathbf{W}_2^2(\rho_\varepsilon, \rho^0) - \mathbf{W}_2^2(\rho, \rho^0)}{2\,\varepsilon\,\tau} &\leq -\frac{m(\varepsilon)}{2\tau} \fint_{X_\varepsilon - \sqrt{\varepsilon}}^{X_\varepsilon} (x - T(x))^2\, dx + \frac{m(\varepsilon)}{2\tau} \fint_{X_\varepsilon - \sqrt{\varepsilon}}^{X_\varepsilon} (D_1(x) - T(x))^2\, dx \\ &\leq \frac{m(\varepsilon)}{2\tau} \fint_{X_\varepsilon - \sqrt{\varepsilon}}^{X_\varepsilon} (D_1(x) - x)^2\, dx - \frac{m(\varepsilon)}{\tau} \fint_{X_\varepsilon - \sqrt{\varepsilon}}^{X_\varepsilon} (D_1(x) - x)(x - T(x))\, dx.\end{aligned}$$

The two terms in the right hand side converge to zero as $\varepsilon \downarrow 0$ thanks to (60). Thus

$$\limsup_{\varepsilon\downarrow 0}\left\{\frac{\mathbf{W}_2^2(\rho_\varepsilon,\rho^0)-\mathbf{W}_2^2(\rho,\rho^0)}{2\,\varepsilon\,\tau}\right\}\le 0.$$

Now, we notice that

$$\lim_{\varepsilon\downarrow 0}\left\{\frac{1}{\varepsilon}\int_{\mathbb{R}_+}(f(\rho_\varepsilon)-f(\rho))\,dx\right\}=\lim_{\varepsilon\downarrow 0}\left\{\frac{1}{\varepsilon}\int_{X_\varepsilon-\sqrt{\varepsilon}}^{X_\varepsilon}\big(f(\rho-m(\varepsilon)\sqrt{\varepsilon})-f(\rho)\big)\,dx-\fint_{X_\varepsilon}^{X}f(\rho)\,dx\right\},$$

where we recall that $f(x)=x(\ln(x)-\beta)+\beta$. Applying the definition of $m(\varepsilon)$ we get

$$\lim_{\varepsilon\downarrow 0}\left\{\frac{1}{\varepsilon}\int_{\mathbb{R}_+}(f(\rho_\varepsilon)-f(\rho))\,dx\right\}=-(1-\rho(X^-))\,f'(\rho(X^-))-f(\rho(X^-)),$$

and we conclude that

erface_3 (61)

$$0\le\liminf_{\varepsilon\downarrow 0}\left\{\frac{\mathbf{J}_{(X^0,\rho^0)}(X_\varepsilon,\rho_\varepsilon)-\mathbf{J}_{(X^0,\rho^0)}(X,\rho)}{\varepsilon}\right\}\le-\lambda\frac{X-X^0}{\tau}+\alpha-(1-\rho(X^-))-\ln\rho(X^-).$$

Case $X_\varepsilon=X+\varepsilon$. Here to build a perturbation, the idea is to pick up a small amount of mass $\tilde\rho>0$ in $(X,X+\varepsilon)$ with $\varepsilon>0$ and to transfer this mass towards $(X-\sqrt{\varepsilon},X)$ in order to mimic the "collapse" of the mass when the carbon steel canister is consumed by the oxide layer. More precisely, for $0<\varepsilon<1$ and $0\le\tilde\rho\le 1$, we consider the map $D_2:(X-\sqrt{\varepsilon},X)\to(X,X_\varepsilon)$ given by $D_2(x):=X+\sqrt{\varepsilon}(x-X+\sqrt{\varepsilon})$ and we define the transport plan γ_ε as

$$\gamma_\varepsilon:=\gamma\llcorner\mathbb{R}^2\setminus(X,X_\varepsilon)^2+(\mathrm{Id},D_2)_\#\,\sqrt{\varepsilon}\tilde\rho\,\mathcal{L}\llcorner(X-\sqrt{\varepsilon},X)+(\mathrm{Id},\mathrm{Id})_\#\,(1-\tilde\rho)\mathcal{L}\llcorner(X,X_\varepsilon).$$

Then, we set $\mu_\varepsilon:=\pi_{1\#}\,\gamma_\varepsilon$ and we notice that $\pi_{2\#}\,\gamma_\varepsilon=\mu^0$ such that $\gamma_\varepsilon\in\Gamma(\mu_\varepsilon,\mu^0)$. Let us rewrite μ_ε as

$$\mu_\varepsilon=\rho_\varepsilon\,\mathcal{L}\llcorner\mathbb{R}_+ +(-M)_+\,\delta_0,$$

with

$$\rho_\varepsilon(x):=\begin{cases}\rho(x) & \text{for } x\in\mathbb{R}_+\setminus[X-\sqrt{\varepsilon},X_\varepsilon],\\ \rho(x)+\sqrt{\varepsilon}\,\tilde\rho & \text{for } x\in[X-\sqrt{\varepsilon},X),\\ 1-\tilde\rho & \text{for } x\in(X,X_\varepsilon].\end{cases}$$

Thanks to the definition of γ_ε we have

$$\frac{\mathbf{W}_2^2(\rho_\varepsilon,\rho^0)-\mathbf{W}_2^2(\rho,\rho^0)}{2\,\varepsilon\,\tau}\le\frac{1}{2\varepsilon\tau}\int_{X-\sqrt{\varepsilon}}^{X}(x-D_2(x))^2\,\sqrt{\varepsilon}\tilde\rho\,dx=\frac{\tilde\rho}{2\tau}\fint_{X-\sqrt{\varepsilon}}^{X}(x-D_2(x))^2\,dx.$$

Since $(x-D_2(x))^2\le\varepsilon$ for $x\in(X-\sqrt{\varepsilon},X)$ we conclude that

$$\limsup_{\varepsilon\downarrow 0}\left\{\frac{\mathbf{W}_2^2(\rho_\varepsilon,\rho^0)-\mathbf{W}_2^2(\rho,\rho^0)}{2\,\varepsilon\,\tau}\right\}\le 0.$$

Arguing as in the previous case we obtain

$$\lim_{\varepsilon\downarrow 0}\left\{\frac{1}{\varepsilon}\int_{\mathbb{R}_+}(f(\rho_\varepsilon)-f(\rho))\,dx\right\}=\tilde\rho\,f'(\rho(X^-))+f(1-\tilde\rho).$$

Hence

$$0 \leq \liminf_{\varepsilon \downarrow 0} \left\{ \frac{\mathbf{J}_{(X^0,\rho^0)}(X_\varepsilon, \rho_\varepsilon) - \mathbf{J}_{(X^0,\rho^0)}(X,\rho)}{\varepsilon} \right\} \leq \lambda \frac{X - X^0}{\tau} - \alpha + \tilde{\rho} f'(\rho(X^-)) + f(1-\tilde{\rho})$$

Basic computations show that the minimum of the above inequality is reached for $\tilde{\rho} = 1 - \rho(X^-)$ and we obtain

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$$\lambda \frac{X - X^0}{\tau} - \alpha + (1 - \rho(X^-)) + \ln(\rho(X^-)) \geq 0. \tag{62}$$

Finally, collecting (61) and (62) yields (59), which finishes the proof of Proposition 3.6. □

Sec.Th2

4. Existence of weak solutions

In order to prove Theorem 1.1, we first recall the definition of the JKO-iterated scheme (21). Let $\tau > 0$ be a time step of $(0,T)$. Starting from the initial data $(X^0, \rho^0) \in \mathcal{A}$, satisfying assumption (H2), find for all $0 \leq n \leq N_T - 1$ (with N_T an integer such that $N_T \tau = T$) a solution $(X^{n+1}, \rho^{n+1}) \in \mathbb{A}$ to the following minimization problem

$$(X^{n+1}, \rho^{n+1}) \in \operatorname{argmin}_{(Y,\rho)\in\mathcal{A}} \left\{ \frac{1}{2\tau} \mathbf{d}^2((Y,\rho),(X^n,\rho^n)) + \mathbf{E}_{(X^n,\rho^n)}(Y,\rho) + p_\tau(M(\rho,\rho^n)) \right\},$$

where the metric $\mathbf{d}$ and the functional $\mathbf{E}$ are defined in Section 2.3. The existence of a solution $(X^n,\rho^n)_{1\leq n\leq N_T} \in \mathcal{A}$ to this minimization problem is now a consequence of the recursive use of Theorem 2.2.

apriori

4.1. Uniform estimates.

We first quantify the movement of the free interface.

estim_X

Lemma 4.1. *Let the assumptions of Theorem 2.2 hold. Then it holds*

$$X^{N_T} \leq X^0 + \frac{T}{\lambda}(\alpha - a),$$

where we recall that $a = \min_{x\in[0,X^0]} \ln \rho^0(x) \leq 0$.

Proof. This result follows directly from Proposition 3.5 and Proposition 3.6. □

Let us now establish some estimates that are uniform with respect to τ .

ont_metr

Proposition 4.1. *Let the assumptions of Theorem 2.2 hold. Then there exists a constant $C > 0$ depending only on X^0, ρ^0, α, β and T such that*

e.a.p

$$\sum_{n=0}^{N_T-1} \frac{\mathbf{d}^2((X^{n+1},\rho^{n+1}),(X^n,\rho^n))}{2\tau} + \theta \sum_{n=0}^{N_T-1} |M(\rho^{n+1},\rho^n)| + \sum_{n=0}^{N_T-1} p_\tau(M(\rho^{n+1},\rho^n)) \leq C. \tag{63}$$

Proof. We use $(X^n,\rho^n) \in \mathbb{A}$ as an admissible competitor in the functional $\mathbf{J}_{(X^n,\rho^n)}$ and we obtain for $0 \leq n \leq N_T - 1$

$$\frac{\mathbf{d}^2((X^{n+1},\rho^{n+1}),(X^n,\rho^n))}{2\tau} + \theta|M(\rho^{n+1},\rho^n)| + p_\tau(M(\rho^{n+1},\rho^n)) \leq \int_0^{X^n} f(\rho^n)\,dx - \int_0^{X^{n+1}} f(\rho^{n+1})\,dx + \alpha(X^{n+1} - X^n).$$

We sum this inequality over n and we get

$$\sum_{n=0}^{N_T-1}\left(\frac{\mathbf{d}^2((X^{n+1},\rho^{n+1}),(X^n,\rho^n))}{2\tau}+\theta|M(\rho^{n+1},\rho^n)|+p_\tau(M(\rho^{n+1},\rho^n))\right)$$
$$\leq\int_0^{X^0} f(\rho^0)\,dx+\left(\alpha-\inf_{x\in[0,1]}f(x)\right)X^{N_T}.$$

It remains to notice that $f(x)\geq\beta-\exp(\beta-1)$, for $x\geq 0$, and to apply Lemma 4.1 in order to deduce the existence of a constant $C>0$ independent of τ such that (63) holds. This finishes the proof of Proposition 4.1. □

Now for $0\leq n\leq N_T-1$, we define the functions ρ^τ, X^τ, $\tilde{X}^\tau$ and M^τ as follows:

$$\rho^\tau(t)=\rho^{n+1},\quad X^\tau(t)=X^{n+1},\quad \text{for } t\in(n\tau,(n+1)\tau],$$

with $\rho^\tau(0)=\rho^0$, $X^\tau(0)=X^0$ and

$$\tilde{X}^\tau(t)=\frac{t-n\tau}{\tau}X^{n+1}+\frac{(n+1)\tau-t}{\tau}X^n,\quad \text{for } t\in(n\tau,(n+1)\tau],$$
$$M^\tau(t)=\frac{t-n\tau}{\tau}\mathbf{M}(\rho^{n+1})+\frac{(n+1)\tau-t}{\tau}\mathbf{M}(\rho^n),\quad \text{for } t\in(n\tau,(n+1)\tau],$$

where we recall definition (15) of $\mathbf{M}$ and with $\tilde{X}^\tau(0)=X^0$ and $M^\tau(0)=\int_0^{X^0}(\rho^0-1)\,dx$. Finally, we introduce the shift operator σ_τ given by

$$\sigma_\tau\rho^\tau(x,t)=\rho^\tau(x,t+\tau)\quad \text{a.e. } (x,t)\in\mathbb{R}_+\times(0,T-\tau).$$

In the following statement our main objective is to establish some uniform (w.r.t. τ) estimates satisfied by the sequences $(\tilde{X}^\tau)_{\tau>0}$, $(M^\tau)_{\tau>0}$ and $(\rho^\tau)_{\tau>0}$.

apriori **Proposition 4.2.** *Let the assumptions of Theorem 1.1 hold. Then, the sequences $(\tilde{X}^\tau)_{\tau>0}$ and $(M^\tau)_{\tau>0}$ are uniformly bounded in $H^1(0,T)$ and $W^{1,1}(0,T)$ respectively. Moreover, the sequence $(\rho^\tau)_{\tau>0}$ is uniformly bounded in $L^2_{\rm loc}(\mathbb{R}_+\times(0,T))\cap L^\infty(\mathbb{R}_+\times(0,T))$ and there exists a constant $C>0$ depending only on X^0, ρ^0, α, β, a, b, θ, λ and T such that the following estimates hold*

rho_H1 (64) $$\int_0^T||\partial_x\rho^\tau(t)||^2_{L^2(\mathbb{R}_+)}\,dt\leq C,$$

and

ranslate (65) $$\int_\tau^T||\rho^\tau(t)-\sigma_{-\tau}\rho^\tau(t)||^2_{H^*}\,dt\leq C\tau,$$

where H^* denotes the dual space of $H^1(\mathbb{R}_+)$.

Proof. Let $\tau > 0$ be fixed. Thanks to Lemma 4.1 and Proposition 4.1 we notice that $\tilde{X}^\tau$ is uniformly bounded in $H^1(0,T)$. Besides, for the function M^τ we have

$$\begin{aligned}\int_0^T |M^\tau(t)|\,dt &= \sum_{n=0}^{N_T-1}\int_{n\tau}^{(n+1)\tau} |M^\tau(t)|\,dt \\ &= \sum_{n=0}^{N_T-1}\int_{n\tau}^{(n+1)\tau} \frac{1}{\tau}\left|(t-n\tau)\left(\mathbf{M}(\rho^{n+1})-\mathbf{M}(\rho^n)\right)+\tau\mathbf{M}(\rho^n)\right| dt \\ &\leq \sum_{n=0}^{N_T-1}\int_{n\tau}^{(n+1)\tau} \frac{1}{\tau}\left|(t-n\tau)\,M\left(\rho^{n+1},\rho^n\right)\right| dt + \sum_{n=0}^{N_T-1}\tau\left|\mathbf{M}(\rho^n)\right|.\end{aligned}$$

Since $|\mathbf{M}(\rho^n)| \leq X^n(\exp(b)+1) \leq X^{N_T}(\exp(b)+1)$ for all $n = 0,\dots,N_T-1$, we deduce from Lemma 4.1

$$\begin{aligned}\int_0^T |M^\tau(t)|\,dt &\leq \sum_{n=0}^{N_T-1}\tau\left|M(\rho^{n+1},\rho^n)\right| + \frac{1}{\lambda}(X^0\lambda + T(\alpha-a))(\exp(b)+1)T \\ &\leq T\sum_{n=0}^{N_T-1}\left|M(\rho^{n+1},\rho^n)\right| + \frac{1}{\lambda}(X^0\lambda + T(\alpha-a))(\exp(b)+1)T\end{aligned}$$

It remains to apply Proposition 4.1 to obtain the existence of a constant $C > 0$ such that

$$\int_0^T |M^\tau(t)|\,dt \leq C,$$

which implies $M^\tau \in L^1(0,T)$. Moreover, we have thanks to (63),

$$\int_0^T \left|\dot{M}^\tau(t)\right| dt = \sum_{n=0}^{N_T-1}\int_{n\tau}^{(n+1)\tau}\frac{|M(\rho^{n+1},\rho^n)|}{\tau}\,dt = \sum_{n=0}^{N_T-1}|M(\rho^{n+1},\rho^n)| \leq C.$$

Therefore, we deduce that $(M^\tau)_{\tau>0}$ is uniformly bounded in $W^{1,1}(0,T)$.

Now, applying Proposition 3.5, the function ρ^τ is uniformly bounded in $L^\infty(\mathbb{R}_+\times(0,T))$ and then bounded in $L^2_{\text{loc}}(\mathbb{R}_+\times(0,T))$. Let us show that $\partial_x\rho^\tau$ is uniformly bounded in $L^2(\mathbb{R}_+\times(0,T))$. In this purpose, applying recursively Corollary 3.1, we deduce that

$$\int_0^T ||\partial_x\rho^\tau(t)||^2_{L^2(\mathbb{R}_+)}\,dt \leq \sum_{n=0}^{N_T-1}\frac{\mathbf{W}_2^2(\rho^{n+1},\rho^n)}{\tau},$$

which implies that the function $\partial_x\rho^\tau$ is uniformly bounded in $L^2(\mathbb{R}_+\times(0,T))$ thanks to (63).

Now we establish (65). Let $0 \leq n \leq N_T-1$ be fixed, $\phi \in H^1(\mathbb{R}_+)$ with $||\phi||_{H^1(\mathbb{R}_+)} \leq 1$. We consider the quantity

$$I_n := \int_{\mathbb{R}_+}(\rho^{n+1}(x)-\rho^n(x))\,\phi(x)\,dx.$$

In the sequel we will assume that $M(\rho^{n+1},\rho^n)\geq 0$ (the case $M(\rho^{n+1},\rho^n)<0$ being similar), then we have

$$I_n=\int_0^{\ell_+^{n+1}}\rho^{n+1}(x)\phi(x)\,dx+\int_{\ell_+^{n+1}}^{+\infty}(\phi(x)-\phi(T_+^{n+1}(x)))\rho^{n+1}(x)\,dx,$$

where the distance ℓ_+^{n+1} and the optimal transport map T_+^{n+1} are defined as in Section 2.3. For the second term of the right hand side we apply (50), the bound $||\rho^{n+1}||_{L^\infty(\mathbb{R}_+)}\leq 1$ and the Cauchy-Schwarz inequality and we obtain

$$\begin{aligned}
\int_{\ell_+^{n+1}}^{+\infty}(\phi(x)-\phi(T_+^{n+1}(x)))\rho^{n+1}(x)\,dx &\leq \left|\int_{\ell_+^{n+1}}^{X^{n+1}}\rho^{n+1}(x)\int_{T_+^{n+1}(x)}^{x}\phi'(s)\,ds\,dx\right|\\
&\leq \left|\int_0^{X^{n+1}}\int_s^{\left(T_+^{n+1}\right)^{-1}(s)}\rho^{n+1}(x)\phi'(s)\,dx\,ds\right|\\
&\leq \exp(b)\int_0^{X^{n+1}}\left|\left(s-\left(T_+^{n+1}\right)^{-1}(s)\right)\phi'(s)\right|\,ds\\
&\leq \frac{\exp(b)}{\exp(a)}\int_0^{X^{n+1}}\rho^n(s)\left|\left(s-\left(T_+^{n+1}\right)^{-1}(s)\right)\phi'(s)\right|\,ds\\
&\leq \frac{\exp(3b/2)\,||\phi'||_{L^2(\mathbb{R}_+)}}{\exp(a)}\,\mathbf{W}_2(\rho^{n+1},\rho^n)
\end{aligned}$$

Hence we get

$$I_n\leq ||\phi||_{L^\infty(\mathbb{R}_+)}\left(\int_0^{\ell_+^{n+1}}\rho^{n+1}(x)\,dx\right)+\frac{\exp(3b/2)\,||\phi'||_{L^2(\mathbb{R}_+)}}{\exp(a)}\,\mathbf{W}_2(\rho^{n+1},\rho^n),$$

and the Sobolev embedding $H^1(\mathbb{R}_+)\hookrightarrow L^\infty(\mathbb{R}_+)$ implies the existence of a constant C_S such that

$$I_n\leq C_S\left|M(\rho^{n+1},\rho^n)\right|+\frac{\exp(3b/2)}{\exp(a)}\,\mathbf{W}_2(\rho^{n+1},\rho^n).$$

Therefore we have

$$||\rho^{n+1}-\rho^n||_{H^*}^2\leq 2\,C_S^2\,|M(\rho^{n+1},\rho^n)|^2+2\frac{\exp(3b)}{\exp(2a)}\,\mathbf{W}_2^2(\rho^{n+1},\rho^n),\quad\forall 0\leq n\leq N_T-1,$$

and since

$$|M(\rho^{n+1},\rho^n)|=\left|\int_0^{\ell_+^{n+1}}\rho^{n+1}(x)\,dx\right|\leq\exp(b)\,X^{N_T},\quad\forall 0\leq n\leq N_T-1,$$

we conclude that it holds

$$||\rho^{n+1}-\rho^n||_{H^*}^2\leq 2\,\exp(b)\,C_S^2\,X^{N_T}|M(\rho^{n+1},\rho^n)|+2\frac{\exp(3b)}{\exp(2a)}\,\mathbf{W}_2^2(\rho^{n+1},\rho^n),\quad\forall 0\leq n\leq N_T-1.$$

Hence, using (63), we end up with

$$\int_\tau^T ||\rho^\tau(t) - \sigma_{-\tau}\rho^\tau(t)||_{H^*}^2 \leq 2\left(\frac{C_S^2\,\exp(b)\,X^{N_T}}{\theta}\,\tau + 2\frac{\exp(3b)}{\exp(2a)}\,\tau^2\right)C.$$

Finally, as $\tau < T$ and thanks to Lemma 4.1, we deduce the existence of a constant, still denoted C, such that (65) holds. This concludes the proof of Proposition 4.2. □

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4.2. **Compactness properties.** In this section we establish the existence of some functions $X \in H^1(0,T)$, $M \in BV(0,T)$ and $\rho \in L^2_{\rm loc}(\mathbb{R}_+ \times (0,T)) \cap L^\infty(\mathbb{R}_+ \times (0,T)) \cap H^1(0,T;H^*)$ with $\partial_x\rho \in L^2(\mathbb{R}_+ \times (0,T))$ obtained as limits, when $\tau \downarrow 0$, of the sequences $(\tilde{X}^\tau)_{\tau>0}$, $(M^\tau)_{\tau>0}$ and $(\rho^\tau)_{\tau>0}$.

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Proposition 4.3. *Let the assumptions of Theorem 2.2 hold. Then, there exists $X \in H^1(0,T)$ such that, up to a subsequence,*

$$\tilde{X}^\tau \to X \quad \textit{strongly in } L^2(0,T), \quad \textit{as } \tau \downarrow 0,$$
$$\dot{\tilde{X}}^\tau \rightharpoonup \dot{X} \quad \textit{weakly in } L^2(0,T), \quad \textit{as } \tau \downarrow 0.$$

It also exists $M \in BV(0,T)$ such that, up to a subsequence,

$$M^\tau \to M \quad \textit{strongly in } L^1(0,T), \quad \textit{as } \tau \downarrow 0,$$
$$\dot{M}^\tau \rightharpoonup DM \quad \textit{weakly in } \mathcal{M}(0,T), \quad \textit{as } \tau \downarrow 0.$$

Moreover, there exists $\rho \in L^2_{\rm loc}(\mathbb{R}_+ \times (0,T)) \cap L^\infty(\mathbb{R}_+ \times (0,T)) \cap H^1(0,T;H^*)$ with $\partial_x\rho \in L^2(\mathbb{R}_+ \times (0,T))$ where $\rho(x,t) = 1$ for a.e. $(x,t) \in (X(t),+\infty) \times (0,T)$ such that, up to a subsequence, as $\tau \downarrow 0$

$$\rho^\tau \to \rho \quad \textit{strongly in } L^p(0,T;L^q_{\rm loc}(\mathbb{R}_+)), \quad \forall 1 \leq p,q < \infty,$$
$$\partial_x\rho^\tau \rightharpoonup \partial_x\rho \quad \textit{weakly in } L^2(\mathbb{R}_+ \times (0,T)),$$
$$\tau^{-1}(\rho^\tau - \sigma_{-\tau}\rho^\tau) \rightharpoonup \partial_t\rho \quad \textit{weakly in } L^2(0,T;H^*).$$

Proof. All the convergence properties stated below occur up to the extraction of a subsequence when $\tau \downarrow 0$. The existence of $X \in H^1(0,T)$ such that the following convergences

$$\tilde{X}^\tau \to X \quad \text{strongly in } L^2(0,T),$$
$$\dot{\tilde{X}}^\tau \rightharpoonup \dot{X} \quad \text{weakly in } L^2(0,T),$$

hold are direct consequences of Proposition 4.2. Moreover, applying again Proposition 4.2 we know that the sequence $(M^\tau)_{\tau>0}$ is uniformly bounded in $W^{1,1}(0,T)$. Then using the compactness criterion [15, Theorem 5.5] for BV functions, we conclude that there exists $M \in BV(0,T)$ such that

$$M^\tau \to M \quad \text{strongly in } L^1(0,T).$$

Furthermore, since (M^τ) is uniformly bounded in $W^{1,1}(0,T) \subseteq BV(0,T)$ we deduce that there exists $P \in \mathcal{M}(0,T)$, such that

$$\dot{M}^\tau \rightharpoonup P \quad \text{weakly in } \mathcal{M}(0,T),$$

and, in the sense of distribution, it holds $P = DM$.

Now, thanks to Proposition 4.2 we apply the compactness argument obtained in [13, Theorem 1] and we deduce the existence of $\rho \in L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}_+))$ such that

forte_Lp
$$\rho^\tau \to \rho \quad \text{strongly in } L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}_+)). \tag{66}$$

Hence, the L^∞ estimates obtained in Proposition 3.5 yields the strong convergence in every $L^p(0,T;L^q_{\mathrm{loc}}(\mathbb{R}_+))$ for all $1 \leq p,q < \infty$.

Let us now prove that $\rho(x,t) = 1$ for a.e. $(x,t) \in (X(t),+\infty)\times(0,T)$. In this purpose, for $t \in (0,T)$ fixed, we notice that the convergence result (66) implies the convergence almost everywhere of ρ^τ towards ρ and we obtain that $\rho(x,t) = 1$ for a.e. $x > \max(X(t),\tilde{X}^\tau(t))$. Besides, using the strong convergence of $(\tilde{X}^\tau)_\tau$ in $L^2(0,T)$ towards X and the embedding $H^1(0,T) \hookrightarrow C([0,T])$ we conclude that for every $t \in (0,T)$ and up to a subsequence $(X^\tau(t))_{\tau>0}$ converges towards $X(t)$. This allow us to obtain the equality $\rho(x,t) = 1$ for a.e. $x \in (X(t),+\infty)$.
We deduce from estimate (64) that $\partial_x\rho^\tau$ is uniformly bounded in $L^2(\mathbb{R}_+\times(0,T))$. Thus, after identification in the sense of distribution, we obtain that

$$\partial_x\rho^\tau \rightharpoonup \partial_x\rho \quad \text{weakly in } L^2(\mathbb{R}_+\times(0,T)).$$

Moreover, since $\tau^{-1}(\rho^\tau - \sigma_{-\tau}\rho^\tau)$ is uniformly bounded in $L^2(0,T;H^*)$ we also deduce that

$$\tau^{-1}(\rho^\tau - \sigma_{-\tau}\rho^\tau) \rightharpoonup \partial_t\rho \quad \text{weakly in } L^2(0,T;H^*),$$

holds. This concludes the proof of Proposition 4.3. □

In the following two statements we establish some results concerning the convergence of the traces of the sequence $(\rho^\tau)_{\tau>0}$ and the limit function ρ obtained in Proposition 4.3.

o_traces
Proposition 4.4. *Let the assumptions of Theorem 2.2 hold. Then, the limit functions $X \in H^1(0,T)$ and $\rho \in L^2_{\mathrm{loc}}(\mathbb{R}_+\times(0,T)) \cap L^\infty(\mathbb{R}_+\times(0,T)) \cap H^1(0,T;H^*)$ with $\partial_x\rho \in L^2(\mathbb{R}_+\times(0,T))$ obtained in Proposition 4.3 satisfy for all $1 \leq p,q < \infty$*

ce_inter
$$\left(\int_0^T \left|\rho^\tau(X^\tau(t)^-,t) - \rho(X(t)^-,t)\right|^q dt\right)^{1/p} \to 0, \quad \text{as } \tau \downarrow 0, \tag{67}$$

ace_zero
$$\left(\int_0^T |\rho^\tau(0,t) - \rho(0,t)|^q\, dt\right)^{1/p} \to 0, \quad \text{as } \tau \downarrow 0. \tag{68}$$

Proof. Bearing in mind the L^∞ estimates established in Proposition 3.5 it is sufficient to prove (67) and (68) in the case $p = q = 1$. Moreover, since the proofs of (67) and (68) are similar we only establish the convergence result (67).

In this purpose we define for $t \in [0,T]$ and $s \in \mathbb{R}_+$ the function $\check{X}^\tau$ by

$$\check{X}^\tau(t,s) = \min(X^\tau(t), X(t)) - s.$$

Then, for $\varepsilon > 0$ we consider the following splitting

$$\int_0^T \left|\rho^\tau(X^\tau(t)^-,t) - \rho(X(t)^-,t)\right| dt \leq Q_3(\varepsilon) + Q_4(\varepsilon) + Q_5(\varepsilon),$$

where

$$Q_3(\varepsilon) := \int_0^T \fint_0^\varepsilon \left|\rho^\tau(\check{X}^\tau(t,s),t) - \rho(\check{X}^\tau(t,s),t)\right| \, ds\, dt,$$
$$Q_4(\varepsilon) := \int_0^T \fint_0^\varepsilon \left|\rho(\check{X}^\tau(t,s),t) - \rho(X(t)^-,t)\right| \, ds\, dt,$$
$$Q_5(\varepsilon) := \int_0^T \fint_0^\varepsilon \left|\rho^\tau(X^\tau(t)^-,t) - \rho^\tau(\check{X}^\tau(t,s),t)\right| \, ds\, dt.$$

For $Q_3(\varepsilon)$, we use the definition of the function $\check{X}^\tau$, the Cauchy-Schwarz inequality and the fact that $\rho^\tau(x,t) = \rho(x,t) = 1$ for a.e. $(x,t) \in (\max(X^{N_T}, X(T)), +\infty) \times (0,T)$ to deduce that

$$Q_3(\varepsilon) \leq \frac{1}{\sqrt{\varepsilon}} \int_0^T ||\rho^\tau(t) - \rho(t)||_{L^2(\mathbb{R}_+)} \, dt,$$

which implies

E_eps (69) $$Q_3(\varepsilon) \leq \frac{\sqrt{T}}{\sqrt{\varepsilon}} \, ||\rho^\tau - \rho||_{L^2(\mathbb{R}_+\times(0,T))}.$$

For $Q_4(\varepsilon)$ thanks to the regularity of $\rho \in L^2(0,T;H^1(D_T))$ where

$$D_T = \{(x,t) \, : \, 0 \leq t \leq T, \, 0 \leq x \leq X(t)\},$$

and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} Q_4(\varepsilon) &\leq \int_0^T \fint_0^\varepsilon \int_{\check{X}^\tau(t,s)}^{X(t)} |\partial_x \rho(y,t)| \, dy \, ds \, dt \\ &\leq \int_0^T \fint_0^\varepsilon |\check{X}^\tau(t,s) - X(t)|^{1/2} \, ||\partial_x \rho(t)||_{L^2(\mathbb{R}_+)} \, ds \, dt \\ &\leq \int_0^T ||\partial_x \rho(t)||_{L^2(\mathbb{R}_+)} \left(\fint_0^\varepsilon |X^\tau(t) - X(t) - s|^{1/2} \, ds \right) dt. \end{aligned}$$

Applying again the Cauchy-Schwarz inequality we deduce that

$$\begin{aligned} Q_4(\varepsilon) &\leq \frac{1}{\sqrt{\varepsilon}} \int_0^T ||\partial_x \rho(t)||_{L^2(\mathbb{R}_+)} \left(\int_0^\varepsilon |X^\tau(t) - X(t) - s| \, ds \right)^{1/2} dt \\ &\leq \frac{1}{\sqrt{\varepsilon}} \int_0^T ||\partial_x \rho(t)||_{L^2(\mathbb{R}_+)} \left(||X^\tau - X||_{L^1(0,T)} \, \varepsilon + \frac{\varepsilon^2}{2} \right)^{1/2} dt. \end{aligned}$$

Hence

F_eps (70) $$Q_4(\varepsilon) \leq \sqrt{T} \, ||\partial_x \rho||_{L^2(\mathbb{R}_+\times(0,T))} \left(||X^\tau - X||_{L^1(0,T)} + \frac{\varepsilon}{2} \right)^{1/2}.$$

We obtain a similar estimate for the term $Q_5(\varepsilon)$. Now, let $\varepsilon := ||\rho^\tau - \rho||^{2/3}_{L^2(\mathbb{R}_+\times(0,T))}$, then we conclude from (69) and (70) that

$$\int_0^T \left|\rho^\tau(X^\tau(t)^-,t) - \rho(X(t)^-,t)\right| \, dt \to 0, \quad \text{as } \tau \downarrow 0,$$

which concludes the proof of Proposition 4.4. □

As a direct consequence of Proposition 4.4, Proposition 3.5 and the dominated convergence theorem we deduce the following result:

race_log **Corollary 4.1.** *Let the assumptions of Theorem 2.2 hold. Then, the limit functions obtained in Proposition 4.3 satisfy for all* $1 \leq p, q < \infty$,

$$\left(\int_0^T \left| \ln \rho^\tau(X^\tau(t)^-, t) - \ln \rho(X(t)^-, t) \right|^q \, dt \right)^{1/p} \to 0, \quad \text{as } \tau \downarrow 0.$$

It remains to show that the trace at $x = 0$ of the limit function $\rho \in L^2(0,T; H^1(D_T))$ satisfies the inequality $\rho_- \leq \rho(0,t) \leq \rho_+$ for a.e. $t \in (0,T)$.

op_0_rho **Proposition 4.5.** *Let the assumptions of Theorem 2.2 hold. Then the limit function* ρ *satisfies*

esttrace (71) $$\rho_- \leq \rho(0,t) \leq \rho_+ \quad \text{a.e. } t \in (0,T).$$

Proof. First let us notice that the lower bound of (71) holds thanks of Proposition 3.3 and Proposition 4.4. Now our main objective is to prove that

$$\int_0^T (\ln \rho^\tau(0,t) - \ln \rho_+)_+^2 \, dt \to 0, \quad \text{as } \tau \downarrow 0.$$

In this purpose, bearing in mind the result established in Proposition 3.3, we have

$$\left(\ln \rho^{n+1}(0) - \ln \rho_+\right)_+^2 = \left(p'_\tau(M(\rho^{n+1}, \rho^n))\right)^2 = 2\, K_\tau\, p_\tau(M(\rho^{n+1}, \rho^n)).$$

Thus, thanks to (63), we deduce that

$$\int_0^T (\ln \rho^\tau(0,t) - \ln \rho_+)_+^2 \, dt = 2\,\tau\, K_\tau \sum_{n=0}^{N_T-1} p_\tau(M(\rho^{n+1}, \rho^n)) \leq 2\,\tau\, K_\tau\, C.$$

Finally, using the definition (49) of K_τ yields

$$\int_0^T (\ln \rho^\tau(0,t) - \ln \rho_+)_+^2 \, dt \leq \frac{4C(b - \ln \rho_+)}{B'_0 \exp(a)}\, \tau^\vartheta \to 0, \quad \text{as } \tau \downarrow 0.$$

This finishes the proof of Proposition 4.5. □

4.3. **Existence proof of weak solutions for the system (1)–(2).** In this section we prove Theorem 1.1. In this purpose we show that the limit functions ρ, M and X obtained in Proposition 4.3 are weak solutions to (1) in the sense of Definition 1.1. Then, we establish that these functions satisfy the variational inequality (10).

wealsol

4.3.1. *Obtention of* (11) *and* (12). Let us first prove the following statement:

inega_EL **Proposition 4.6.** *Let the assumptions of Theorem 1.1 hold and assume that the parameter ϑ appearing in the definition (49) of m_τ and K_τ satisfies $0 < \vartheta < 1/2$. Then, for all $\varphi \in C_0^\infty(\mathbb{R}_+ \times [0,T))$ the following inequality holds*

ega_EL1 (72)
$$\left| -\int_0^{T-\tau}\int_{\mathbb{R}_+} \rho^\tau(x,t)\,\frac{\sigma_\tau\varphi(x,t)-\varphi(x,t)}{\tau}dx\,dt - \frac{1}{\tau}\int_0^\tau\int_{\mathbb{R}_+}\rho^0(x)\,\varphi(x,t)\,dx\,dt \right.$$
$$+\frac{1}{\tau}\int_{T-\tau}^T\int_{\mathbb{R}_+}\rho^\tau(x,t)\,\varphi(x,t)\,dx\,dt - \int_0^T \dot{M}^\tau(t)\,\varphi(0,t)\,dt$$
$$\left. +\int_0^T\int_{\mathbb{R}_+}\partial_x\rho^\tau(x,t)\,\partial_x\varphi(x,t)\,dx\,dt\right| \le \sum_{n=0}^{N_T-1}\tau\,|Q_\tau^{n+1}(\varphi)|,$$

with

_reste2 (73)
$$\sum_{n=0}^{N_T-1}\tau\,|Q_\tau^{n+1}(\varphi)| \le C\,\left(||\partial_x^2\varphi||_{L^\infty(\mathbb{R}_+\times(0,T))} + ||\partial_x\varphi||_{L^\infty(\mathbb{R}_+\times(0,T))}\right)\,\tau^{1-2\vartheta},$$

where $C > 0$ is a constant independent of τ. Eventually for all $\xi \in \mathcal{C}(0,T)$ we have

00_inter (74)
$$\lambda\int_0^T \dot{\tilde{X}}^\tau(t)\,\xi(t)\,dt = \alpha\int_0^T\xi(t)\,dt - \int_0^T\left(1-\rho^\tau(X^\tau(t)^-,t)\right)\,\xi(t)\,dt$$
$$-\int_0^T \ln\rho^\tau(X^\tau(t)^-,t)\,\xi(t)\,dt.$$

Proof. It is sufficient to establish (73). Indeed, we notice that (72) is a direct consequence of (30) and a rearrangement of the discrete time derivative terms, while (74) is a consequence of (59). Then, in order to establish (73) we notice thanks to Proposition 3.2 that for all $\varphi \in C_0^\infty(\mathbb{R}_+\times(0,T))$ we can write

$$\sum_{n=0}^{N_T-1}\tau\,|Q_\tau^{n+1}(\varphi)| \le 2\,\tau\,||\partial_x^2\varphi||_{L^\infty(\mathbb{R}_+\times(0,T))}\sum_{n=0}^{N_T-1}\frac{\mathbf{d}^2\left((X^{n+1},\rho^{n+1}),(X^n,\rho^n)\right)}{2\tau}$$
$$+||\partial_x\varphi||_{L^\infty(\mathbb{R}_+\times(0,T))}\sum_{n=0}^{N_T-1}\int_0^{\ell_-^{n+1}} y\,\rho^n(y)\,dy,$$

where ℓ_-^{n+1} is defined as in Section 2.3. In particular, if $M(\rho^{n+1},\rho^n) < 0$, then applying (58) we have

$$\ell_-^{n+1} \le B_0'\,\tau^{1-\vartheta},$$

and if $M(\rho^{n+1},\rho^n)\geq 0$ we set $\ell_-^{n+1}=0$. Therefore, we obtain

$$\sum_{n=0}^{N_T-1}\tau\,|Q_\tau^{n+1}(\varphi)|\leq 2\,\tau\,||\partial_x^2\varphi||_{L^\infty(\mathbb{R}_+\times(0,T))}\sum_{n=0}^{N_T-1}\frac{\mathbf{d}^2\left((X^{n+1},\rho^{n+1}),(X^n,\rho^n)\right)}{2\tau}$$
$$+\frac{(B_0')^2\,\exp(b)\,T}{2}||\partial_x\varphi||_{L^\infty(\mathbb{R}_+\times(0,T))}\,\tau^{1-2\vartheta}.$$

Hence, thanks to Proposition 4.1, we conclude that there exists a constant C independent of τ such that

$$\sum_{n=0}^{N_T-1}\tau\,|Q_\tau^{n+1}(\varphi)|\leq C\,\left(||\partial_x^2\varphi||_{L^\infty(\mathbb{R}_+\times(0,T))}+||\partial_x\varphi||_{L^\infty(\mathbb{R}_+\times(0,T))}\right)\,\tau^{1-2\vartheta}.$$

This finishes the proof of Proposition 4.6. □

Thus passing to the limit $\tau\downarrow 0$ in (72) and (74) yields the existence of a weak solution to (1) in the sense of Definition 1.1.

4.3.2. *Obtention of the variational inequality* (10). In order to prove that the weak solution (ρ,M,X) to (1) satifies the variational inequality (10), the main idea is to prove a semi-discrete (in time) counterpart of (10) (see (82) below). More precisely, our objective is to made rigorous the computations done in Section 1.3. In particular, defining $u=\chi\rho$ with $\chi\in C_0^\infty(\mathbb{R}_+)$ a nonnegative function with $\chi(x)=1$ for all $x\in[0,X^0/2)$ and $\mathrm{supp}(\chi)\subseteq[0,3X^0/4]$, we have to be able to define properly the inequality

$$-\dot{M}(t)\,(u(0,t)-\eta(0,t))\geq 0,\quad \text{for a.e. } t\in[0,T],$$

where $\eta\in C_0^\infty(\mathbb{R}_+\times[0,T))$ with $\eta(0,t)\in[\rho_-,\rho_+]$ for every $t\in(0,T)$. However, we only know that $M\in BV(0,T)$. Then, to bypass this regularity issue we establish a semi-discrete variational inequality achieved by the minimizers of our JKO-iterated scheme (21) and then to pass to the limit $\tau\downarrow 0$ in this inequality.

In this purpose, we set

$$u^n(x):=\chi(x)\,\rho^n(x),\quad \forall x\in\mathbb{R}_+,\quad 0\leq n\leq N_T.$$

Besides, as in Section 4.1, we consider the following piecewise in time function

$$u^\tau(t)=u^{n+1},\quad \text{for } t\in(n\tau,(n+1)\tau],\quad \forall 0\leq n\leq N_T-1,$$

with $u^\tau(0)=u^0=\chi\,\rho^0$. Let us now prove the equation satisfies in the weak sense by the sequence $(u^n)_{0\leq n\leq N_T}$:

inegvard **Proposition 4.7.** *Let the assumptions of Theorem 1.1 hold. Then, for all $\varphi\in C_0^\infty([0,X^0])$ and every $0\leq n\leq N_T-1$, it holds*

inegvard (75) $$\int_0^{X^0}\frac{u^{n+1}(x)-u^n(x)}{\tau}\,\varphi(x)\,dx-\frac{M(\rho^{n+1},\rho^n)}{\tau}\varphi(0)$$
$$+\int_0^{X^0}\left(u^{n+1}\right)'(x)\,\varphi'(x)\,dx=\int_0^{X^0}g^{n+1}(x)\,\varphi(x)\,dx+Q_\tau^{n+1}(\chi,\varphi),$$

with

$$g^{n+1}(x) = -2\left(\rho^{n+1}\right)'(x)\,\chi'(x) - \rho^{n+1}(x)\,\chi''(x), \quad \forall 0 \leq n \leq N_T - 1,$$

and where the remaining term Q_τ^{n+1} satisfies

negvard

$$|Q_\tau^{n+1}(\chi,\varphi)| \leq \frac{\left|\left|(\chi\,\varphi)''\right|\right|_{L^\infty([0,X^0])}}{\tau}\,\mathbf{d}^2\left((X^{n+1},\rho^{n+1}),(X^n,\rho^n)\right) + \frac{\left|\left|(\chi\,\varphi)'\right|\right|_{L^\infty([0,X^0])}}{\tau}\left(\int_0^{\ell_-^{n+1}} y\,\rho^n(y)\,dy\right). \tag{76}$$

Proof. For a given function $\varphi \in C_0^\infty(\mathbb{R}_+)$ we consider as a test function $\psi = \chi\varphi$ in (30). Then, as $\chi(0) = 1$ and by definition of the sequence $(u^n)_{0\leq n\leq N_T}$, we obtain

$$\int_0^{X^0} \frac{u^{n+1}(x) - u^n(x)}{\tau}\,\varphi(x)\,dx - \frac{M(\rho^{n+1},\rho^n)}{\tau}\varphi(0) + \int_0^{X^0}\left(\rho^{n+1}\right)'(x)\;(\chi\varphi)'(x)\,dx = Q_\tau^{n+1}(\chi,\varphi),$$

where the bound (76) on Q_τ^{n+1} is directly deduce from (31). Therefore, since

$$\int_0^{X^0}\left(\rho^{n+1}\right)'(x)\,\chi(x)\,\varphi'(x)\,dx = \int_0^{X^0}\left(u^{n+1}\right)'(x)\,\varphi'(x)\,dx - \int_0^{X^0}\rho^{n+1}(x)\,\chi'(x)\,\varphi'(x)\,dx,$$

we notice that it holds

$$\int_0^{X^0}\left(\rho^{n+1}\right)'(x)\;(\chi\varphi)'(x)\,dx = \int_0^{X^0}\left(\rho^{n+1}\right)'(x)\,\chi'(x)\,\varphi(x)\,dx + \int_0^{X^0}\left(u^{n+1}\right)'(x)\,\varphi'(x)\,dx - \int_0^{X^0}\rho^{n+1}(x)\,\chi'(x)\,\varphi'(x)\,dx.$$

Now, applying an integration by parts on the last term of the right hand side yields

$$\int_0^{X^0}\left(\rho^{n+1}\right)'(x)\;(\chi\varphi)'(x)\,dx = \int_0^{X^0}\left(u^{n+1}\right)'(x)\,\varphi'(x)\,dx - \int_0^{X^0} g^{n+1}(x)\varphi(x)\,dx.$$

This completes the proof of Proposition 4.7. □

Now we intend to use, roughly speaking, $(u^{n+1} - \eta)$ for some regular function η with $\eta(0) \in [\rho_-,\rho_+]$, as a test function in (75). If we do this, thanks to Proposition 3.3, we notice that it holds

$$-M(\rho^{n+1},\rho^n)\left(u^{n+1}(0) - \eta(0)\right) \geq 0, \quad \forall 0 \leq n \leq N_T - 1,$$

such that, at least formally, we deduce from (75) the following inequality

$$\int_0^{X^0}\frac{u^{n+1}(x) - u^n(x)}{\tau}\,(u^{n+1} - \eta)(x)\,dx + \int_0^{X^0}\left(u^{n+1}\right)'(x)\;\left(u^{n+1} - \eta\right)'(x)\,dx \leq \int_0^{X^0} g^{n+1}(x)\;\left(u^{n+1} - \eta\right)(x)\,dx + Q_\tau^{n+1}(\chi, u^{n+1} - \eta).$$

This inequality is closed to the semi-discrete variational inequality that we are looking for. But, we notice that in the right hand side, and more precisely in the term Q_τ^{n+1}, we have to be able to define the second derivative in space of u^{n+1}. However, we only know that u^{n+1} belongs to

$H^1(\mathbb{R}_+)$. In order to make rigorous our approach we need to regularize in space the function u^τ thanks to some mollifiers. In this purpose, for $\vartheta' \in (0,1)$ (to be defined later), we set $\delta := \tau^{\vartheta'}$ and we first extend the function u^τ on $\mathbb{R} \times (0,T)$ by

$$\widetilde{u}^\tau(x,t) = \begin{cases} 0 & \text{if } x \geq X^0, \\ u^\tau(x,t) & \text{if } x \in [0, X^0], \\ 2\,u^\tau(0,t) - u^\tau(-x,t) & \text{if } x \in [-\delta, 0], \\ 2\,u^\tau(0,t) - u^\tau(\delta,t) & \text{if } x \leq -\delta. \end{cases}$$

Let ξ be an even nonnegative function in $C_0^\infty(-1/2, 1/2)$ with $\int \xi\, dx = 1$ and $\zeta := \xi * \xi$. Then, ζ is an even nonnegative function in the space $C_0^\infty(-1,1)$ with $\int \zeta\, dx = 1$. We denote by ζ_δ the mollifier function defined as $\zeta_\delta(y) := \zeta(y/\delta)/\delta$ for all $y \in \mathbb{R}$ and

$$u_\delta^\tau(x,t) := \left(\widetilde{u}^\tau * \zeta_\delta\right)(x,t) = \int_{\mathbb{R}} \widetilde{u}^\tau(x-y,t)\, \zeta_\delta(y)\, dy.$$

Finally, for latter use, we also introduce the function

$$\check{u}_\delta^\tau(x,t) := \left(\widetilde{u}^\tau * \xi_\delta\right)(x,t) = \int_{\mathbb{R}} \widetilde{u}^\tau(x-y,t)\, \xi_\delta(y)\, dy,$$

with $\xi_\delta(y) = \xi(y/\delta)/\delta$ for all $y \in \mathbb{R}$. Let us establish in the following result some usefull properties achieved by the function $\widetilde{u}^\tau$ and u_δ^τ.

_inegvar **Lemma 4.2.** *Let the assumptions of Theorem 1.1 hold and, without loss of generality, assume that $0 < \tau < 1$ is small enough such that $\delta < X^0/2$. Then, the functions $\widetilde{u}^\tau$ and u_δ^τ satisfy the properties:*

(i) *There exists constants C_1, only depending on ρ^0, and C_2, only depending on χ, such that*

nd1.ext (77) $$||\widetilde{u}^\tau||_{L^\infty(\mathbb{R}\times(0,T))} \leq 2\,\exp(b) - \exp(a),$$

nd2.ext (78) $$||\partial_x \widetilde{u}^\tau||_{L^\infty(\mathbb{R}\times(0,T))} \leq C_1\, \tau^{-\vartheta},$$

nd3.ext (79) $$||\partial_x \widetilde{u}^\tau||_{L^2(\mathbb{R}\times(0,T))} \leq C_2\, ||\rho^\tau||_{L^2(0,T;H^1(0,X^0))}.$$

(ii) *Moreover, the function u_δ^τ (as well as $\check{u}_\delta^\tau$) satisfies the estimates (77)–(79), the equality $u_\delta^\tau(0,t) = u^\tau(0,t)$ for a.e. $t \in (0,T)$ and there exists a constant $C_3 > 0$, only depending on ρ^0, X^0, T and ζ, such that*

.bound1 (80) $$\int_0^T \int_0^{X^0} \frac{(u_\delta^\tau - u^\tau)^2}{\tau}\, dxdt \leq C_3\, \tau^{2\vartheta'-2\vartheta-1},$$

and a constant $C_4 > 0$, only depending on ρ^0 and ζ, such that

.bound2 (81) $$\left|\left|\partial_x^2 u_\delta^\tau\right|\right|_{L^\infty(\mathbb{R}\times(0,T))} \leq C_4\, \tau^{-\vartheta-\vartheta'},$$

where we recall that the parameter $\vartheta \in (0, 1/2)$ appears in the definition (49) of m_τ and K_τ.

Proof. For the point (i), the estimates (77) and (78) are direct consequences of the definition of the function $\widetilde{u}^\tau$ and Proposition 3.5. For the estimate (79), we notice that

$$\begin{aligned}\left|\left|\partial_x\widetilde{u}^\tau\right|\right|^2_{L^2(\mathbb{R}\times(0,T))} &= \int_0^T\int_{-\delta}^0 \left|\partial_x u^\tau(-x,t)\right|^2\,dxdt + \int_0^T\int_0^{X^0}\left|\partial_x u^\tau(x,t)\right|^2\,dxdt\\ &= \int_0^T\int_{-\delta}^0 \left|\partial_x \rho^\tau(-x,t)\right|^2\,dxdt + \int_0^T\int_0^{X^0}\left|\partial_x \rho^\tau(x,t)\,\chi(x)+\rho^\tau(x,t)\,\chi'(x)\right|^2\,dxdt.\end{aligned}$$

Now, we directly obtain

$$\left|\left|\partial_x\widetilde{u}^\tau\right|\right|^2_{L^2(\mathbb{R}\times(0,T))} \leq \left(3+2\left|\left|\chi'\right|\right|^2_{L^\infty(0,X^0)}\right)\,\left|\left|\rho^\tau\right|\right|^2_{L^2(0,T;H^1(0,X^0))},$$

and applying Proposition 4.2 yields the existence of the constant C_2 such that (79) holds.
Now for the point (ii), using similar arguments and standard properties achieved by the convolution product we deduce that the function u^τ_δ satisfies the estimates (77)–(79). Furthermore, since ζ is an even function, we directly deduce the equality $u^\tau_\delta(0,t)=u^\tau(0,t)$ for a.e. $t\in(0,T)$. In order to establish estimate (80), we first notice that

$$\begin{aligned}\int_0^T\int_0^{X^0}\frac{(u^\tau_\delta-u^\tau)^2}{\tau} &= \sum_{n=0}^{N_T-1}\int_0^{X^0}\left(u^{n+1}_\delta-u^{n+1}\right)^2(x)\,dx\\ &= \sum_{n=0}^{N_T-1}\int_0^{X^0}\left[\int_{-\delta}^{\delta}\left(\widetilde{u}^{n+1}(x-y)-u^{n+1}(x)\right)\,\zeta_\delta(y)\,dy\right]^2 dx\\ &= \sum_{n=0}^{N_T-1}\int_0^{X^0}\left[\int_{-\delta}^{\delta}\left(\widetilde{u}^{n+1}(x-y)-\widetilde{u}^{n+1}(x)\right)\,\zeta_\delta(y)\,dy\right]^2 dx.\end{aligned}$$

Then, thanks to the Cauchy-Schwarz inequality and the regularity of the function ζ, we get

$$\begin{aligned}\int_0^T\int_0^{X^0}\frac{(u^\tau_\delta-u^\tau)^2}{\tau} &\leq \sum_{n=0}^{N_T-1}\left(\int_0^{X^0}\int_{-\delta}^{\delta}\left|\widetilde{u}^{n+1}(x-y)-\widetilde{u}^{n+1}(x)\right|^2\,dy\,dx\right)\left(\int_{-\delta}^{\delta}\left|\zeta_\delta(y)\right|^2\,dy\right)\\ &\leq \frac{\left|\left|\zeta\right|\right|^2_{L^\infty(-1,1)}}{\delta}\sum_{n=0}^{N_T-1}\int_0^{X^0}\int_{-\delta}^{\delta}\left|\widetilde{u}^{n+1}(x-y)-\widetilde{u}^{n+1}(x)\right|^2\,dy\,dx.\end{aligned}$$

Then, since the function u^τ and its extension $\widetilde{u}^\tau$ are Lipchitz continuous we have

$$\begin{aligned}\int_0^T\int_0^{X^0}\frac{(u^\tau_\delta-u^\tau)^2}{\tau} &\leq \frac{\left|\left|\zeta\right|\right|^2_{L^\infty(-1,1)}}{\delta}\sum_{n=0}^{N_T-1}\left|\left|\partial_x\widetilde{u}^\tau\right|\right|^2_{L^\infty(\mathbb{R}\times(0,T))}\int_0^{X^0}\int_{-\delta}^{\delta}y^2\,dy\,dx\\ &\leq \frac{2\,X^0\left|\left|\zeta\right|\right|^2_{L^\infty(-1,1)}}{3}\sum_{n=0}^{N_T-1}\left|\left|\partial_x\widetilde{u}^\tau\right|\right|^2_{L^\infty(\mathbb{R}\times(0,T))}\delta^2.\end{aligned}$$

Hence, applying (78), we deduce that

$$\int_0^T\int_0^{X^0}\frac{(u^\tau_\delta-u^\tau)^2}{\tau} \leq \frac{2\,C_1\,X^0\,T\left|\left|\zeta\right|\right|^2_{L^\infty(-1,1)}}{3}\,\tau^{2\vartheta'-2\vartheta-1},$$

and we conclude the existence of the constant C_3 such that (80) holds. Finally, the estimate (81) is consequence of (78) and the fact that $\int \zeta_\delta'(y)\,dy$ is bounded by C/δ for some constant C independent of δ which yields the existence of a constant C_4, depending only on ρ^0 and ζ, such that

$$\left|\left|\partial_x^2 u_\delta^\tau\right|\right|_{L^\infty(\mathbb{R}\times(0,T))} \leq C_4 \frac{\tau^{-\vartheta}}{\delta} = C_4\,\tau^{-\vartheta-\vartheta'}.$$

This completes the proof of Lemma 4.2. □

Let us now establish the semi-discrete variational inequality. In this purpose, we consider a nonnegative function $\phi \in C_0^\infty([0,T))$ and $\eta \in C_0^\infty(\mathbb{R}_+ \times [0,T))$ with $\eta(0,t) \in [\rho_-,\rho_+]$ for all $t \in [0,T)$. Then, taking $u_\delta^\tau - \eta$ as a test function in (75), multiplying this equation by ϕ and integrating in time we obtain

$$\begin{aligned}\int_\tau^T \int_0^{X^0} \phi(t) \frac{u^\tau(x,t) - \sigma_{-\tau} u^\tau(x,t)}{\tau}\ (u_\delta^\tau - \eta)\,(x,t)\,dxdt - \int_0^T \phi(t)\,\dot{M}^\tau(t)\ (u_\delta^\tau - \eta)\,(0,t)\,dt \\ + \int_0^T \int_0^{X^0} \phi(t)\partial_x u^\tau(x,t)\,\partial_x\,(u_\delta^\tau - \eta)\,(x,t)\,dxdt = \int_0^T \int_0^{X^0} \phi(t)\,g^\tau(x,t)\ (u_\delta^\tau - \eta)\,(x,t)\,dxdt \\ + \sum_{n=0}^{N_T-1} \int_{n\tau}^{(n+1)\tau} \phi(t)\left(Q_\tau^{n+1}(\chi, u_\delta^{n+1}) - Q_\tau^{n+1}(\chi,\eta(t))\right)\,dt,\end{aligned}$$

where we have used the linearity of the remaining term Q_τ^{n+1} (recall Proposition 3.2). Bearing in mind point (ii) of Lemma 4.2 we have $u_\delta^\tau(0,t) = u^\tau(0,t) = \rho^\tau(0,t)$ for a.e. $t \in [0,T]$. Hence, thanks to Proposition 3.3, we notice that

$$-\int_0^T \phi(t)\,\dot{M}^\tau(t)\ (u_\delta^\tau - \eta)\,(0,t)\,dt = -\sum_{n=0}^{N_T-1} \int_{n\tau}^{(n+1)\tau} \phi(t)\,\frac{M\left(\rho^{n+1},\rho^n\right)}{\tau}\ \left(\rho^{n+1}(0) - \eta(0,t)\right)\,dt \geq 0.$$

Therefore, it holds the following semi-discrete variational inequality:

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$$\begin{aligned}(82)\quad \int_\tau^T \int_0^{X^0} \phi(t)\,\frac{u^\tau(x,t) - \sigma_{-\tau} u^\tau(x,t)}{\tau}\ (u_\delta^\tau - \eta)\,(x,t)\,dxdt \\ + \int_0^T \int_0^{X^0} \phi(t)\,\partial_x u^\tau(x,t)\,\partial_x\,(u_\delta^\tau - \eta)\,(x,t)\,dxdt \leq \int_0^T \int_0^{X^0} \phi(t)\,g^\tau(x,t)\ (u_\delta^\tau - \eta)\,(x,t)\,dxdt \\ + \sum_{n=0}^{N_T-1} \int_{n\tau}^{(n+1)\tau} \phi(t)\ \left(Q_\tau^{n+1}(\chi, u_\delta^{n+1}) - Q_\tau^{n+1}(\chi,\eta(t))\right)\,dt.\end{aligned}$$

We rewrite this inequality as

$$A_1^\tau - A_2^\tau + A_3^\tau - A_4^\tau \leq A_5^\tau + A_6^\tau - A_7^\tau,$$

with

$$A_1^\tau = \int_\tau^T \int_0^{X^0} \phi(t)\, \frac{u^\tau(x,t) - \sigma_{-\tau} u^\tau(x,t)}{\tau}\, u_\delta^\tau(x,t)\, dxdt,$$
$$A_2^\tau = \int_\tau^T \int_0^{X^0} \phi(t)\, \frac{u^\tau(x,t) - \sigma_{-\tau} u^\tau(x,t)}{\tau}\, \eta(x,t)\, dxdt,$$
$$A_3^\tau = \int_0^T \int_0^{X^0} \phi(t)\, \partial_x u^\tau(x,t)\, \partial_x u_\delta^\tau(x,t)\, dxdt,$$
$$A_4^\tau = \int_0^T \int_0^{X^0} \phi(t)\, \partial_x u^\tau(x,t)\, \partial_x \eta(x,t)\, dxdt,$$
$$A_5^\tau = \int_0^T \int_0^{X^0} \phi(t)\, g^\tau(x,t)\, \left(u_\delta^\tau - \eta\right)(x,t)\, dxdt,$$
$$A_6^\tau = \sum_{n=0}^{N_T-1} \int_{n\tau}^{(n+1)\tau} \phi(t)\, Q_\tau^{n+1}(\chi, u_\delta^{n+1})\, dt,$$
$$A_7^\tau = \sum_{n=0}^{N_T-1} \int_{n\tau}^{(n+1)\tau} \phi(t)\, Q_\tau^{n+1}(\chi, \eta(t))\, dt.$$

In order to establish the variational inequality (10) it remains to pass to the limit $\tau \downarrow 0$ in the above terms. This is the main objective of the next result.

inegvar **Proposition 4.8.** *Let the assumptions of Theorem 1.1 hold and assume that $0 < \tau < 1$ is small enough such that $\delta < X^0/2$. Moreover, we impose the following conditions on ϑ and ϑ'*

vartheta (83) $$3\vartheta + \vartheta' < 1, \quad \vartheta' - \vartheta > 1/2, \quad \vartheta' > 2\vartheta, \quad \vartheta < 1/2.$$

Then, for any nonnegative function $\phi \in C_0^\infty([0,T))$ and $\eta \in C_0^\infty(\mathbb{R}_+ \times [0,T))$ with $\eta(0,t) \in [\rho_-, \rho_+]$ for all $t \in [0,T)$, it holds

lim.A1 (84) $$\liminf_{\tau\downarrow 0} A_1^\tau \geq -\int_0^T \dot\phi \int_0^{X^0} \frac{u^2}{2}\, dxdt - \phi(0) \int_0^{X^0} \frac{u^2}{2}(x,0)\, dt,$$

lim.A2 (85) $$\lim_{\tau\downarrow 0} A_2^\tau = -\int_0^T \dot\phi \int_0^{X^0} u\,\eta\, dxdt - \int_0^T \int_0^{X^0} \phi\, u\, \partial_t \eta\, dxdt - \phi(0) \int_0^{X^0} u(x,0)\, \eta(x,0)\, dx,$$

lim.A3 (86) $$\liminf_{\tau\downarrow 0} A_3^\tau \geq \int_0^T \int_0^{X^0} \phi\, \left(\partial_x u\right)^2\, dxdt,$$

lim.A4 (87) $$\lim_{\tau\downarrow 0} A_4^\tau = \int_0^T \int_0^{X^0} \phi\, \partial_x u\, \partial_x \eta\, dxdt,$$

lim.A5 (88) $$\lim_{\tau\downarrow 0} A_5^\tau = \int_0^T \int_0^{X^0} \phi\, g\, (u - \eta)\, dxdt,$$

and

lim.A6A7 (89) $$\lim_{\tau\downarrow 0} A_6^\tau = \lim_{\tau\downarrow 0} A_7^\tau = 0.$$

Proof. Let us first notice that $\vartheta = 1/15$ and $\vartheta' = 3/4$ satisfy the conditions (83). Let us also notice that the limits (85), (87), (88) and $\lim_{\tau\downarrow 0} A_7^\tau = 0$ are directly deduced thanks to Proposition 4.3 and the techniques used in Section 4.3.1. Now, we rewrite the term A_1^τ as

$$A_1^\tau = \int_\tau^T \int_0^{X^0} \phi\, \frac{u^\tau - \sigma_{-\tau} u^\tau}{\tau} u^\tau \, dxdt + \int_\tau^T \int_0^{X^0} \phi\, \frac{u^\tau - \sigma_{-\tau} u^\tau}{\tau} \left(u_\delta^\tau - u^\tau\right) dxdt = A_{11}^\tau + A_{12}^\tau.$$

Then, we split the term A_{11}^τ as

$$A_{11}^\tau = \frac{1}{2} \int_\tau^T \int_0^{X^0} \phi\, \frac{(u^\tau)^2 - (\sigma_{-\tau} u^\tau)^2}{\tau} \, dxdt + \frac{1}{2} \int_\tau^T \int_0^{X^0} \phi\, \frac{(u^\tau - \sigma_{-\tau} u^\tau)^2}{\tau} \, dxdt = A_{111}^\tau + A_{112}^\tau.$$

Rearranging the terms in A_{111}^τ we obtain

$$A_{111}^\tau = -\frac{1}{2} \int_0^{T-\tau} \int_0^{X^0} \frac{\sigma_\tau \phi - \phi}{\tau} \, (u^\tau)^2 \, dxdt - \frac{1}{2\tau} \int_0^\tau \int_0^{X^0} \phi \, (u^\tau)^2 \, dxdt + \frac{1}{2\tau} \int_{T-\tau}^T \int_0^{X^0} \phi \, (u^\tau)^2 \, dxdt.$$

Thus, as $\phi \in C_0^\infty([0,T))$ and since, as $\tau \downarrow 0$, we have (up to a subsequence) $u^\tau \to u := \chi\, \rho$ strongly in $L^2(\mathbb{R}_+ \times (0,T))$, we deduce that

lim.A111 (90) $$\lim_{\tau\downarrow 0} A_{111}^\tau = -\int_0^T \dot\phi \int_0^{X^0} \frac{u^2}{2} \, dxdt - \phi(0) \int_0^{X^0} \frac{u^2}{2}(x,0) \, dt.$$

Now, as ϕ is a nonnegative function and thanks to (80), we have

A12.A112 (91) $$-A_{12} - A_{112} \le \frac{1}{2} \int_0^T \int_0^{X^0} \phi\, \frac{(u^\tau - u_\delta^\tau)^2}{\tau} \, dxdt \le \frac{C_3\, ||\phi||_{L^\infty([0,T))}}{2}\, \tau^{2\vartheta' - 2\vartheta - 1} \to 0, \quad \text{as } \tau \downarrow 0,$$

where we have used the conditions (83) which imply that $\vartheta' - \vartheta > 1/2$. Therefore, the limits (90)–(91) yield (84). For A_3^τ we write

$$A_3^\tau = \int_0^T \int_0^{X^0} \phi \, \left(\partial_x u_\delta^\tau\right)^2 \, dxdt + \int_0^T \int_0^{X^0} \phi\, \partial_x u_\delta^\tau \left(\partial_x u^\tau - \partial_x u_\delta^\tau\right) \, dxdt = A_{31}^\tau + A_{32}^\tau.$$

We first notice, thanks to Lemma 4.2, that $\partial_x u_\delta^\tau \rightharpoonup \partial_x u = \partial_x(\chi\rho)$ weakly in $L^2(\mathbb{R}\times(0,T))$ (where ρ is the function obtained in Proposition 4.3 which we extend continuously outside of $[0, X^0]$). Then, using the lower semicontinuity of the functional $v \in L^2((0,X^0)\times(0,T)) \mapsto \int_0^T \int_0^{X^0} v^2(x,t)\phi(t)\, dxdt$ for the weak topology in L^2 we have

lim.A31 (92) $$\liminf_{\tau\downarrow 0} A_{31}^\tau \ge \int_0^T \int_0^{X^0} \phi \, (\partial_x u)^2 \, dxdt.$$

Then, we also rewrite the term A_{32}^τ as

$$A_{32}^\tau = \int_0^T \int_{\mathbb{R}} \phi\, \partial_x u_\delta^\tau \, \left(\partial_x \widetilde{u}^\tau - \partial_x u_\delta^\tau\right) \, dxdt - \int_0^T \int_{\mathbb{R}\setminus[0,X^0]} \phi\, \partial_x u_\delta^\tau \, \left(\partial_x \widetilde{u}^\tau - \partial_x u_\delta^\tau\right) \, dxdt = A_{321}^\tau + A_{322}^\tau.$$

For A_{321}^τ we will use the following equality

$$\int_{\mathbb{R}} h(x) \, (w * \zeta_\delta)(x) \, dx = \int_{\mathbb{R}} (h * \xi_\delta)(x) \, (w * \xi_\delta)(x) \, dx, \quad \forall h, w \in L^2(\mathbb{R}),$$

which can be proved by Fourier transform. Thus, recalling the definition of the function $\check{u}_\delta^\tau(x,t) = (\widetilde{u}^\tau * \xi_\delta)(x,t)$ we obtain

$$A_{321}^\tau = \int_0^T \phi \left(\int_{\mathbb{R}} \left(\partial_x \check{u}_\delta^\tau\right)^2 dx - \int_{\mathbb{R}} \left(\partial_x \check{u}_\delta^\tau * \xi_\delta\right)^2 dx \right) dt.$$

Thanks to the nonnegativity of the function ϕ, Jensen's inequality, the fact that $\delta \xi_\delta^2(y) \leq \xi_\delta(y)$ for all $y \in \mathbb{R}$ we obtain

$$\begin{aligned} A_{321}^\tau &\geq \int_0^T \phi(t) \int_{\mathbb{R}} \left(\left(\partial_x \check{u}_\delta^\tau\right)^2 (x,t) - \delta \int_{-\delta/2}^{\delta/2} \left(\partial_x \check{u}_\delta^\tau\right)^2 (x-y,t)\, \xi_\delta^2(y)\, dy \right) dxdt \\ &\geq \int_0^T \phi(t) \int_{\mathbb{R}} \left(\left(\partial_x \check{u}_\delta^\tau\right)^2 (x,t) - \int_{-\delta/2}^{\delta/2} \left(\partial_x \check{u}_\delta^\tau\right)^2 (x-y,t)\, \xi_\delta(y)\, dy \right) dxdt \\ &\geq \int_0^T \phi(t) \int_{\mathbb{R}} \left(\left(\partial_x \check{u}_\delta^\tau\right)^2 - \left(\partial_x \check{u}_\delta^\tau\right)^2 * \xi_\delta \right) (x,t)\, dxdt. \end{aligned}$$

In particular it holds

lim.A321 (93) $$\lim_{\tau \downarrow 0} A_{321}^\tau \geq 0.$$

For A_{322}^τ we first observe that $\partial_x u_\delta^\tau(x,t) = 0$ for a.e. $(x,t) \in \mathbb{R} \setminus (-2\delta, X^0 + \delta)$. Therefore, applying the L^∞ bounds established on $\partial_x \widetilde{u}^\tau$ and $\partial_x u_\delta^\tau$ in Lemma 4.2, we get

$$|A_{322}^\tau| \leq 6\, C_1^2\, ||\phi||_{L^1(0,T)}\, \tau^{\vartheta' - 2\vartheta}.$$

Moreover the conditions (83) imply in particular that $\vartheta' > 2\vartheta$ and we conclude that

lim.A322 (94) $$\lim_{\tau \downarrow 0} A_{322}^\tau = 0.$$

Collecting (92)–(94) yield (86). It remains to prove that

lim.A6 (95) $$\lim_{\tau \downarrow 0} A_6^\tau = 0,$$

holds true. In this purpose, thanks to the proofs of Proposition 4.6 and Proposition 4.7, we obtain the existence of a constant C independent of τ such that

$$|A_6^\tau| \leq C\, ||\phi||_{L^\infty(0,T)} \left(\left|\left| \partial_x^2 (\chi\, u_\delta^\tau) \right|\right|_{L^\infty(\mathbb{R}\times(0,T))} + ||\partial_x (\chi u_\delta^\tau)||_{L^\infty(\mathbb{R}\times(0,T))} \right) \tau^{1-2\vartheta}.$$

Now, applying the estimates established in Lemma 4.2 and $||\chi||_{L^\infty([0,X^0))} \leq 1$, we have

$$\left|\left| \partial_x^2 (\chi\, u_\delta^\tau) \right|\right|_{L^\infty(\mathbb{R}\times(0,T))} \leq ||\chi''||_{L^\infty([0,X^0))}\, (2\exp(b) - \exp(a)) + 2C_1 ||\chi'||_{L^\infty([0,X^0))} \tau^{-\vartheta} + C_4 \tau^{-\vartheta-\vartheta'},$$

and

$$||\partial_x (\chi u_\delta^\tau)||_{L^\infty(\mathbb{R}\times(0,T))} \leq ||\chi'||_{L^\infty([0,X^0))}\, (2\exp(b) - \exp(a)) + C_1 \tau^{-\vartheta}.$$

Hence, we deduce that there exists a constant, still denoted C and independent of τ, such that

$$|A_6^\tau| \leq C\, \tau^{1-3\vartheta-\vartheta'}.$$

Since we assume that $1 - 3\vartheta - \vartheta' > 0$, see (83), we conclude that (95) holds. This completes the proof of Proposition 4.8. □

Now, passing to the limit $\tau \downarrow 0$ in (82) we conclude, thanks to Proposition 4.8, that the weak solution (ρ, M, X) of (1) satisfies the variational inequality (10). Therefore, the triplet (ρ, M, X) is a weak solution to (1)–(2).

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