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# An Essential, Hyperconnected, Local Geometric Morphism that is not Locally Connected

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## Abstract

Thomas Streicher asked on the category theory mailing list whether every essential, hyperconnected, local geometric morphism is automatically locally connected. We show that this is not the case, by providing a counterexample.

**Keywords** Grothendieck topos · Essential · Hyperconnected · Local · Locally connected · Geometric morphism

We show that an essential, hyperconnected, local geometric morphism is not necessarily locally connected, by constructing a counterexample. This answers a question asked by Thomas Streicher on the category theory mailing list. Our counterexample arises from our earlier work [1] and work-in-progress regarding properties of geometric morphisms  $\mathbf{PSh}(M) \rightarrow \mathbf{PSh}(N)$  for monoids  $M$  and  $N$ .

We thank Thomas Streicher for the interesting question and the subsequent discussion regarding this counterexample. Further, we would like to thank the anonymous reviewer for their suggestions, especially for the different proof that  $f$  is not locally connected (which replaces our previous, longer proof).

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Communicated by Thomas Streicher.

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### The counterexample

Let  $M$  be the monoid with presentation  $\langle e, x : e^2 = e, xe = x \rangle$ . Note that each element of  $M$  can be written as either  $x^n$  or  $ex^n$  for some  $n \in \{0, 1, 2, \dots\}$ . Further, let  $N$  be the free monoid on one variable  $a$ , so  $N = \{1, a, a^2, \dots\}$ . Consider the monoid morphism  $\phi : M \rightarrow N$  which on generators is given by  $\phi(e) = 1$  and  $\phi(x) = a$ . If we interpret  $M$  and  $N$  as categories, then  $\phi$  is a functor. There is an induced essential geometric morphism

$$\mathbf{PSh}(M) \xrightarrow{f} \mathbf{PSh}(N)$$

given by functors

$$\begin{array}{ccc} & \xrightarrow{f_!} & \\ \mathbf{PSh}(M) & \xleftarrow{f^*} & \mathbf{PSh}(N) \\ & \xrightarrow{f_*} & \end{array}$$

with the following description, for  $X$  in  $\mathbf{PSh}(M)$  and  $Y$  in  $\mathbf{PSh}(N)$ :

- $f_!(X) \simeq X \otimes_M N$  where  $N$  has left  $M$ -action defined by  $m \cdot n = \phi(m)n$  for  $m \in M$  and  $n \in N$ , and right  $N$ -action defined by multiplication;
- $f^*(Y) \simeq Y$  with right  $M$ -action defined as  $y \cdot m = y \cdot \phi(m)$  for  $y \in Y$  and  $m \in M$ ;
- $f_*(X) \simeq \mathcal{H}om_M(N, X)$ , where  $N$  has right  $M$ -action given by  $n \cdot m = n\phi(m)$  for  $n \in N$  and  $m \in M$ , and  $\mathcal{H}om_M(N, X)$  is the set of morphisms of right  $M$ -sets  $g : N \rightarrow X$ ; the right  $N$ -action on  $\mathcal{H}om_M(N, X)$  is defined as  $(g \cdot n)(n') = g(nn')$  for  $g \in \mathcal{H}om_M(N, X)$  and  $n, n' \in N$ .

For definitions and background regarding tensor products and Hom-sets, in the context of sets with a monoid action, we refer to [1, Subsection 1.2].

**Proposition 1** *The geometric morphism  $f$  is hyperconnected and local.*

**Proof** Because  $\phi$  is surjective, it follows that  $f$  is hyperconnected, see [2, Example A.4.6.9]. We now show that  $f$  is local. Because  $f$  is connected (even hyperconnected), it follows from [3, Corollary 3.3] that  $f$  is local if and only if  $f_*$  has a further right adjoint  $f^!$ . Note that there is an isomorphism of right  $M$ -sets  $N \cong eM$ , so  $f_*(X) \simeq \mathcal{H}om_M(eM, X)$ . A map of right  $M$ -sets  $eM \rightarrow X$  is completely determined by the image of  $e$ , and this image can be any element of  $Xe$ . So we see that  $f_*(X) \simeq Xe$ , with the right  $N$ -action defined as  $b \cdot a = b \cdot x$  for  $b \in Xe$ . In other words,  $f_*(X) \simeq X \otimes_M Me$ , where  $Me$  has a left  $M$ -action by multiplication and a right  $N$ -action given by  $m \cdot a = m \cdot x$  for  $m \in Me$ . From the tensor–hom adjunction (see e.g. [1, Proposition 1.5]), we now know that  $f_*$  has a right adjoint given by  $f^!(Y) \simeq \mathcal{H}om_N(Me, Y)$  for  $Y$  in  $\mathbf{PSh}(N)$ . It follows that  $f$  is local.  $\square$

**Remark 2** As suggested by the reviewer, we can give an explicit description of the functor  $f^!$  as follows. We define  $f^!(Y) \simeq Y \times Y$ , with the right  $M$ -action defined as

$$(s, t) \cdot x = (t, t \cdot a), \quad (s, t) \cdot e = (s, s \cdot a).$$

for  $(s, t) \in Y \times Y$ . To see that this agrees with the description  $f^!(Y) \simeq \mathcal{H}om_N(Me, Y)$  as above, note that  $Me \cong N \sqcup N$  as right  $N$ -set. The right  $M$ -action on

$$f^!(Y) \simeq \mathcal{H}om_N(Me, Y)$$

is the one induced by the left  $M$ -action on  $Me$  (by multiplication).

If  $f$  were locally connected, then in particular  $f^*$  would preserve exponential objects. We will show that this is not the case, and as a result  $f$  is not locally connected.

**Proposition 3** *The functor  $f^*$  does not preserve exponential objects.*

**Proof** Let  $2 = 1 \sqcup 1$ . We claim that the comparison map

$$f^*(2^N) \longrightarrow f^*(2)^{f^*(N)}$$

is not an isomorphism. In our case, the comparison map is the map

$$\begin{aligned} \mathcal{H}om_N(N \times N, 2) &\longrightarrow \mathcal{H}om_M(M \times N, 2) \\ \gamma &\mapsto \gamma \circ g \end{aligned}$$

where  $g(m, n) = (\phi(m), n)$ . We can identify  $\mathcal{H}om_N(N \times N, 2)$  with the set of complemented sub- $N$ -sets of  $N \times N$ , and similarly we can identify  $\mathcal{H}om_M(M \times N, 2)$  with the set of complemented sub- $M$ -sets of  $M \times N$ . The comparison map then sends  $S \subseteq N \times N$  to  $g^{-1}(S) \subseteq M \times N$ . Now consider the (right) sub- $M$ -set

$$T = \{(ex^{n+1}, a^n) : n \geq 0\} \subseteq M \times N.$$

To verify that this is a complemented subset, note that there is a degree function

$$\begin{aligned} \text{deg} : M \times N &\rightarrow \mathbb{Z} \\ \text{deg}(x^i, a^j) &= \text{deg}(ex^i, a^j) = i - j \end{aligned}$$

such that  $\text{deg}(y \cdot m) = \text{deg}(y)$  for all  $y \in M \times N, m \in M$ . Further, if  $y \in M \times N$  is of the form  $y = (x^{n+1}, a^n)$ , then both  $y \cdot x = (x^{n+2}, a^{n+1})$  and  $y \cdot e = y$  are again of this form. So  $T$  is complemented, but it is not of the form  $g^{-1}(S)$  for some  $S \subseteq N \times N$ , because it contains the element  $(ex, 1)$  but not the element  $(x, 1)$ . So the comparison map is not surjective, and as a result  $f^*$  does not preserve exponential objects. □

**Corollary 4** *The geometric morphism  $f$  is not locally connected.*

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