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ON THE ROLE OF ANISOTROPY IN THE WEAK STABILITY OF THE NAVIER-STOKES SYSTEM

HAJER BAHOURI, JEAN-YVES CHEMIN, AND ISABELLE GALLAGHER

ABSTRACT. In this article, we investigate the weak stability for the three dimensional incompressible Navier-Stokes system. Because of the invariances of this system, a positive answer in general to this question would imply global regularity for any data. Thus some restrictions have to be imposed to hope to prove such a weak openness result. The result we prove in this paper solves this issue under anisotropy assumption. To achieve our goal, we write a new kind of profile decomposition and establish global existence results for the Navier-Stokes system associated with new classes of arbitrarily large initial data generalizing the examples dealt with in [14, 15, 16].

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1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. **Setting of the problem.** We are interested in the Cauchy problem for the three dimensional, incompressible Navier-Stokes system

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0, \end{cases}$$

where $u(t, x)$ and $p(t, x)$ are respectively the velocity and the pressure of the fluid at time $t \geq 0$ and position $x \in \mathbb{R}^3$.

Key words and phrases. Navier-Stokes equations; anisotropy; Besov spaces; profile decomposition; weak stability.

An important point in the study of (NS) is its scale invariance, which reads as follows: defining the scaling operators, for any positive real number λ and any point x_0 of \mathbb{R}^3 ,

$$(1.1) \quad \Lambda_{\lambda, x_0} \phi(t, x) \stackrel{\text{def}}{=} \frac{1}{\lambda} \phi\left(\frac{t}{\lambda^2}, \frac{x - x_0}{\lambda}\right) \quad \text{and} \quad \Lambda_\lambda \phi(t, x) \stackrel{\text{def}}{=} \frac{1}{\lambda} \phi\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right),$$

if u solves (NS) with data u_0 , then $\Lambda_{\lambda, x_0} u$ solves (NS) with data $\Lambda_{\lambda, x_0} u_0$. Note in particular that in two space dimensions, $L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2))$ is scale invariant, while in three space dimensions that is the case for $L^\infty(\mathbb{R}^+; L^3(\mathbb{R}^3))$ or the family of spaces $L^\infty(\mathbb{R}^+; B_{p, q}^{-1+\frac{3}{p}}(\mathbb{R}^3))$ ¹, with $1 \leq p < \infty$ and $0 < q \leq \infty$.

Let us also emphasize that the (NS) system formally conserves the energy, in the sense that smooth enough solutions satisfy the following equality for all times $t \geq 0$:

$$(1.2) \quad \frac{1}{2} \|u(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla u(t')\|_{L^2(\mathbb{R}^3)}^2 dt' = \frac{1}{2} \|u_0\|_{L^2(\mathbb{R}^3)}^2.$$

The energy equality (1.2) can easily be derived observing that thanks to the divergence free condition, the nonlinear term is skew-symmetric in L^2 : one has indeed if u and p are smooth enough and decaying at infinity

$$(u(t) \cdot \nabla u(t) + \nabla p(t) | u(t))_{L^2} = 0.$$

The mathematical study of the Navier-Stokes system has a long history beginning with the founding paper [37] of J. Leray in 1933. In this article, J. Leray proved that any finite energy initial data (meaning square-integrable data) generates a (possibly non unique) global in time weak solution; and this in any dimension $d \geq 2$. He moreover proved in [38] the uniqueness of the solution in two space dimension, but in dimension three and more, the question of the uniqueness of Leray's solutions is still an open problem. Actually the difference between dimension 2 and higher dimensions is linked to the fact that $\|u(t)\|_{L^2(\mathbb{R}^2)}$ is both scale invariant and bounded globally in time thanks to the energy estimate, while it is not the case in dimension $d \geq 3$ (since $L^2(\mathbb{R}^3)$ is not scale invariant).

Recall that $u \in L^2_{\text{loc}}([0, T] \times \mathbb{R}^3)$ is said a *weak solution* of (NS) associated with the data u_0 if for any compactly supported, divergence free vector field ϕ in $C^\infty([0, T] \times \mathbb{R}^3)$ the following holds for all $t \leq T$:

$$\int_{\mathbb{R}^3} u \cdot \phi(t, x) dx = \int_{\mathbb{R}^3} u_0(x) \cdot \phi(0, x) dx + \int_0^t \int_{\mathbb{R}^3} (u \cdot \Delta \phi + u \otimes u : \nabla \phi + u \cdot \partial_t \phi) dx dt',$$

with

$$u \otimes u : \nabla \phi \stackrel{\text{def}}{=} \sum_{1 \leq j, k \leq 3} u^j u^k \partial_k \phi^j.$$

Weak solutions satisfying the energy inequality

$$(1.3) \quad \frac{1}{2} \|u(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla u(t')\|_{L^2(\mathbb{R}^3)}^2 dt' \leq \frac{1}{2} \|u_0\|_{L^2(\mathbb{R}^3)}^2$$

are said to be *turbulent solutions*, following the terminology of J. Leray [37].

In what follows, we say that a family $(X_T)_{T>0}$ of spaces of distributions over $[0, T] \times \mathbb{R}^3$ is *scaling invariant* if for all $T > 0$ one has, under Notation (1.1)

$$\forall \lambda > 0, \forall x_0 \in \mathbb{R}^3, u \in X_T \iff \Lambda_{\lambda, x_0} u \in X_{\lambda^{-2}T} \quad \text{with} \quad \|u\|_{X_T} = \|\Lambda_{\lambda, x_0} u\|_{X_{\lambda^{-2}T}}.$$

Similarly a space X_0 of distributions defined on \mathbb{R}^3 will be said scaling invariant if

$$\forall \lambda > 0, \forall x_0 \in \mathbb{R}^3, u_0 \in X_0 \iff \Lambda_{\lambda, x_0} u_0 \in X_0 \quad \text{with} \quad \|u_0\|_{X_0} = \|\Lambda_{\lambda, x_0} u_0\|_{X_0}.$$

¹Here $B_{p, q}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ denotes the usual homogeneous Besov space (see [2], [9] or [49] for a precise definition).

This leads to the definition of a scaled solution, which will be the notion of solution we consider throughout this article.

Definition 1.1. *A vector field u is said a scaled solution to (NS) associated with the data u_0 if it is a weak solution, belonging to a family of scaling invariant spaces.*

After Leray's results, the question of the global wellposedness of the Navier-Stokes system in dimension $d \geq 3$ was raised, and has been open ever since, although several partial answers to the construction of a global unique solution were established since (we refer for instance to [2] or [36] and the references therein for recent surveys on the subject). Let us simply recall the best result known to this day on the uniqueness of solutions to (NS), which is due to H. Koch and D. Tataru in [35] : if

$$\|u_0\|_{\text{BMO}^{-1}(\mathbb{R}^3)} \stackrel{\text{def}}{=} \|u_0\|_{B_{\infty,\infty}^{-1}(\mathbb{R}^3)} + \sup_{\substack{x \in \mathbb{R}^3 \\ R > 0}} \frac{1}{R^{\frac{3}{2}}} \left(\int_{[0,R^2] \times B(x,R)} |(e^{t\Delta} u_0)(t,y)|^2 dy dt \right)^{\frac{1}{2}}$$

is small enough, then there is a global, unique solution to (NS), lying in $\text{BMO}^{-1} \cap X$ for all times, with X a scale invariant space to be specified – we shall not be using that space in the sequel. Note that the space BMO^{-1} is invariant by the scaling operator Λ_{λ,x_0} and that the norm in $B_{\infty,\infty}^{-1}(\mathbb{R}^3)$ denotes a Besov norm. Actually, the Besov space $B_{\infty,\infty}^{-1}(\mathbb{R}^3)$ is the largest space in which any scale and translation invariant Banach space of tempered distributions embeds (see [41]). However, it was proved in [10] and [23] that (NS) is illposed for initial data in $B_{\infty,\infty}^{-1}(\mathbb{R}^3)$.

Our goal in this paper is to investigate the *stability* of global solutions. Let us recall that strong stability results were achieved. Namely, it was proved in [1] (see [20] for the Besov setting) that the set of initial data generating a global solution is open in BMO^{-1} . More precisely, denoting by VMO^{-1} the closure of smooth functions in BMO^{-1} , it was established in [1] that if u_0 belongs to VMO^{-1} and generates a global, smooth solution to (NS), then any sequence $(u_{0,n})_{n \in \mathbb{N}}$ converging to u_0 in the BMO^{-1} norm also generates a global smooth solution as soon as n is large enough.

In this paper we would like to address the question of *weak stability*:

If $(u_{0,n})_{n \in \mathbb{N}}$, bounded in some scale invariant space X_0 , converges to u_0 in the sense of distributions, with u_0 giving rise to a global smooth solution, is it the case for $u_{0,n}$ when n is large enough ?

Because of the invariances of (NS) system, a positive answer in general to this question would imply global regularity for any data and so would solve the question of the possible blow up in finite time of solutions to (NS) which is actually one of the Millenium Prize Problems in Mathematics. Indeed, consider for instance the sequence

$$(1.4) \quad u_{0,n} = \lambda_n \Phi_0(\lambda_n \cdot) = \Lambda_{\lambda_n} \Phi_0 \quad \text{with} \quad \lim_{n \rightarrow \infty} \left(\lambda_n + \frac{1}{\lambda_n} \right) = \infty,$$

with Φ_0 any smooth divergence-free vector field. If the weak stability result were true, then since the weak limit of $(u_{0,n})_{n \in \mathbb{N}}$ is zero (which gives rise to the unique, global solution which is identically zero) then for n large enough $u_{0,n}$ would give rise to a unique, global solution. By scale invariance then so would Φ_0 , and this for any Φ_0 , so that would solve the global regularity problem for (NS). Another natural example is the sequence

$$(1.5) \quad u_{0,n} = \Phi_0(\cdot - x_n) = \Lambda_{1,x_n} \Phi_0,$$

with $(x_n)_{n \in \mathbb{N}}$ a sequence of \mathbb{R}^3 whose norm goes to infinity. Thus sequences built by rescaling fixed divergence free vector fields according to the invariances of the equations have to be

excluded from our analysis, since solving (NS) system for any smooth initial data seems out of reach.

Thus clearly some restrictions have to be imposed to hope to prove such a weak openness result. Let us note that a first step in that direction was achieved in [4], under two additional assumptions to the weak convergence. The first one is an assumption on the asymptotic separation of the horizontal and vertical spectral supports of the sequence $(u_{0,n})_{n \in \mathbb{N}}$, while the second one requires that some of the profiles involved in the profile decomposition of $(u_{0,n})_{n \in \mathbb{N}}$ vanish at zero. In this paper, we remove the second assumption and give a positive answer to the question of weak stability, provided that the convergence of the sequence $(u_{0,n})_{n \in \mathbb{N}}$ towards u_0 holds "anisotropically" in frequency space (see Definition 1.4 below). The main ingredient which enables us to eliminate the second assumption required in [4] is a novel form of anisotropic profile decomposition. This new profile decomposition enables us to decompose the sequence of initial data $u_{0,n}$, up to a small remainder term, into a finite sum of orthogonal sequences of divergence-free vector fields ; these sequences are obtained from the classical anisotropic profile decompositions by grouping together all the profiles having the same horizontal scale. The price to pay is that the profiles are no longer fixed functions as in the classical case, but bounded sequences. To carry out the strategy of proof developed in [4] in this framework, we are led to establishing global existence results for (NS) associated to new classes of arbitrarily large initial data generalizing the examples dealt in [14, 15, 16], and where regularity is sharply estimated.

1.2. Statement of the main result. We prove in this article a weak stability result for the (NS) system under an anisotropy assumption. This leads us naturally to introducing anisotropic Besov spaces. These spaces generalize the more usual isotropic Besov spaces, which are studied for instance in [2, 9, 49].

Definition 1.2. Let $\widehat{\chi}$ (the Fourier transform of χ) be a radial function in $\mathcal{D}(\mathbb{R})$ such that $\widehat{\chi}(t) = 1$ for $|t| \leq 1$ and $\widehat{\chi}(t) = 0$ for $|t| > 2$. For $(j, k) \in \mathbb{Z}^2$, the horizontal truncations are defined by

$$\widehat{S}_k^h f(\xi) \stackrel{\text{def}}{=} \widehat{\chi}(2^{-k}|(\xi_1, \xi_2)|) \widehat{f}(\xi) \quad \text{and} \quad \Delta_k^h \stackrel{\text{def}}{=} S_{k+1}^h - S_k^h,$$

and the vertical truncations by

$$\widehat{S}_j^v f \stackrel{\text{def}}{=} \widehat{\chi}(2^{-j}|\xi_3|) \widehat{f}(\xi) \quad \text{and} \quad \Delta_j^v \stackrel{\text{def}}{=} S_{j+1}^v - S_j^v.$$

For all p in $[1, \infty]$ and q in $]0, \infty]$, and all (s, s') in \mathbb{R}^2 , with $s < 2/p, s' < 1/p$ (or $s \leq 2/p$ and $s' \leq 1/p$ if $q = 1$), the anisotropic homogeneous Besov space $B_{p,q}^{s,s'}$ is defined as the space of tempered distributions f such that

$$\|f\|_{B_{p,q}^{s,s'}} \stackrel{\text{def}}{=} \left\| 2^{ks+js'} \|\Delta_k^h \Delta_j^v f\|_{L^p} \right\|_{\ell^q} < \infty.$$

In all other cases of indexes s and s' , the Besov space is defined similarly, up to taking the quotient with polynomials.

Remark 1.3. The Besov spaces $B_{p,q}^{s,s'}$ (for $s \neq s'$) is anisotropic by essence, which as pointed out above, will be an important feature of our analysis. These spaces have properties which look very much like the ones of classical Besov spaces. We refer for instance to [2], [18], [25] and [43] for all necessary details. By construction, these spaces are defined using an anisotropic Littlewood-Paley decomposition. It is useful to point out that the horizontal and vertical truncations S_k^h , Δ_k^h , S_j^v and Δ_j^v introduced in Definition 1.2 map L^p into L^p with norms independent of k, j and p . For our purpose, it is crucial to recall the

following inequalities known as Bernstein inequalities: if $1 \leq p_1 \leq p_2 \leq \infty$, then for any $\alpha \in \mathbb{N}^2$ and $m \in \mathbb{N}$

$$(1.6) \quad \|\partial_{(x_1, x_2)}^\alpha \Delta_k^h f\|_{L^{p_2}(\mathbb{R}^2; L^r(\mathbb{R}))} \lesssim 2^{k(|\alpha| + 2(1/p_1 - 1/p_2))} \|\Delta_k^h f\|_{L^{p_1}(\mathbb{R}^2; L^r(\mathbb{R}))} \quad \text{and}$$

$$(1.7) \quad \|\partial_{x_3}^m \Delta_j^v f\|_{L^r(\mathbb{R}^2; L^{p_2}(\mathbb{R}))} \lesssim 2^{j(m+1/p_1 - 1/p_2)} \|\Delta_j^v f\|_{L^r(\mathbb{R}^2; L^{p_1}(\mathbb{R}))},$$

as well as the action of the heat flow on frequency localized distributions in an anisotropic context, namely for any p in $[1, \infty]$

$$(1.8) \quad \|e^{t\Delta} \Delta_k^h \Delta_j^v f\|_{L^p} \lesssim e^{-ct(2^{2k} + 2^{2j})} \|\Delta_k^h \Delta_j^v f\|_{L^p}$$

Notation. To avoid heaviness, in what follows we denote by $\mathcal{B}^{s, s'}$ the space $B_{2,1}^{s, s'}$, by \mathcal{B}^s the space $\mathcal{B}^{s, \frac{1}{2}}$ and by $\mathcal{B}_{p,q}$ the space $B_{p,q}^{-1 + \frac{2}{p}, \frac{1}{p}}$. In particular $\mathcal{B}_{2,1} = \mathcal{B}^0$.

Let us point out that the scaling operators (1.1) enjoy the following invariances:

$$\begin{aligned} \|\Lambda_{\lambda, x_0} \varphi\|_{\mathcal{B}_{p,q}} &= \|\varphi\|_{\mathcal{B}_{p,q}} \quad \text{and} \\ \forall r \in [1, \infty], \quad \|\Lambda_{\lambda, x_0} \Phi\|_{L^r(\mathbb{R}^+; B_{p,q}^{-1 + \frac{2}{p} + \frac{2}{r}, \frac{1}{p}})} &= \|\Phi\|_{L^r(\mathbb{R}^+; B_{p,q}^{-1 + \frac{2}{p} + \frac{2}{r}, \frac{1}{p}})}, \end{aligned}$$

and also the following scaling property:

$$(1.9) \quad \forall r \in [1, \infty], \quad \forall \sigma \in \mathbb{R}, \quad \|\Lambda_{\lambda, x_0} \Phi\|_{L^r(\mathbb{R}^+; B_{p,q}^{-1 + \frac{2}{p} + \frac{2}{r} - \sigma, \frac{1}{p}})} \sim \lambda^\sigma \|\Phi\|_{L^r(\mathbb{R}^+; B_{p,q}^{-1 + \frac{2}{p} + \frac{2}{r} - \sigma, \frac{1}{p}})}.$$

The Navier-Stokes system in anisotropic spaces has been studied in a number of frameworks. We refer for instance, among others, to [4], [18], [25], [27] and [43]. In particular in [4] it is proved that if u_0 belongs to \mathcal{B}^0 , then there is a unique solution (global in time if the data is small enough) in $L^2([0, T]; \mathcal{B}^1)$. That norm controls the equation, in the sense that as soon as the solution belongs to $L^2([0, T]; \mathcal{B}^1)$, then it lies in fact in $L^r([0, T]; \mathcal{B}^{\frac{2}{r}})$ for all $1 \leq r \leq \infty$. The space \mathcal{B}^1 is included in L^∞ and since the seminal work [37] of J. Leray, it is known that the $L^2([0, T]; L^\infty(\mathbb{R}^3))$ norm controls the propagation of regularity and also ensures weak uniqueness among turbulent solutions. Thus the space \mathcal{B}^0 is natural in this context.

As mentioned above the result we establish in this paper involves an anisotropy assumption on the sequence $(u_{0,n})_{n \in \mathbb{N}}$ of initial data. Let us introduce this assumption that we call notion of *anisotropically oscillating* sequences, and which is a natural adaptation to our setting of the vocabulary of P. Gérard in [22].

Definition 1.4. Let $0 < q \leq \infty$ be given. We say that a sequence $(f_n)_{n \in \mathbb{N}}$, bounded in $\mathcal{B}_{1,q}$, is *anisotropically oscillating* if the following property holds. There exists $p \geq 2$ such that for all sequences (k_n, j_n) in $\mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^{\mathbb{N}}$,

$$(1.10) \quad \liminf_{n \rightarrow \infty} 2^{k_n(-1 + \frac{2}{p}) + \frac{j_n}{p}} \|\Delta_{k_n}^h \Delta_{j_n}^v f_n\|_{L^p(\mathbb{R}^3)} = C > 0 \quad \implies \quad \lim_{n \rightarrow \infty} |j_n - k_n| = \infty.$$

Remark 1.5. In view of Bernstein inequalities (1.6) and (1.7), it is easy to see that any function f in $\mathcal{B}_{1,q}$ belongs also to $\mathcal{B}_{p,\infty}$ for any $p \geq 1$ hence

$$f \in \mathcal{B}_{1,q} \implies \sup_{(k,j) \in \mathbb{Z}^2} 2^{k(-1 + \frac{2}{p}) + \frac{j}{p}} \|\Delta_k^h \Delta_j^v f\|_{L^p} < \infty.$$

The left-hand side of (1.10) indicates which ranges of frequencies are predominant in the sequence (f_n) : if $\liminf_{n \rightarrow \infty} 2^{k_n(-1 + \frac{2}{p}) + \frac{j_n}{p}} \|\Delta_{k_n}^h \Delta_{j_n}^v f_n\|_{L^p}$ is zero for a couple of frequencies $(2^{k_n}, 2^{j_n})$, then the sequence $(f_n)_{n \in \mathbb{N}}$ is “unrelated” to those frequencies, with the vocabulary of P. Gérard in [22]. The right-hand side of (1.10) is then an *anisotropy* property. Indeed one sees easily that a sequence such as $(u_{0,n})_{n \in \mathbb{N}}$ defined in (1.4) is precisely not anisotropically oscillating: for the left-hand side of (1.10) to hold for this example one would need $j_n \sim k_n \sim n$,

which is precisely not the condition required on the right-hand side of (1.10). A typical sequence satisfying Assumption (1.10) is rather (for $a \in \mathbb{R}^3$)

$$f_n(x) := 2^{\alpha n} f(2^{\alpha n}(x_1 - a_1), 2^{\alpha n}(x_2 - a_2), 2^{\beta n}(x_3 - a_3)), \quad (\alpha, \beta) \in \mathbb{R}^2, \quad \alpha \neq \beta$$

with f smooth.

Our main result states as follows:

Theorem 1. *Let q be given in $]0, 1[$ and let u_0 in $\mathcal{B}_{1,q}$ generate a unique global solution to (NS) in $L^2(\mathbb{R}^+; \mathcal{B}^1)$. Let $(u_{0,n})_{n \in \mathbb{N}}$ be a sequence of divergence free vector fields converging towards u_0 in the sense of distributions, and such that $(u_{0,n} - u_0)_{n \in \mathbb{N}}$ is anisotropically oscillating. Then for n large enough, $u_{0,n}$ generates a unique, global solution to (NS) system in the space $L^2(\mathbb{R}^+; \mathcal{B}^1)$.*

Remark 1.6. One can see from the proof of Theorem 1 that the solution $u_n(t)$ associated with $u_{0,n}$ converges for all times, in the sense of distributions to the solution associated with u_0 . In this sense the Navier-Stokes system is stable by weak convergence.

The proof of Theorem 1 enables us to infer easily the following result, which generalizes the statement of Theorem 1 to the case when the solution to (NS) system generated by u_0 is assumed to blow up in finite time (for a strategy of proof, one can consult [4]).

Corollary 1.7. *Let $(u_{0,n})_{n \in \mathbb{N}}$ be a sequence of divergence free vector fields bounded in the space $\mathcal{B}_{1,q}$ for some $0 < q < 1$, converging towards some u_0 in $\mathcal{B}_{1,q}$ in the sense of distributions, with $u_0 - (u_{0,n})_{n \in \mathbb{N}}$ anisotropically oscillating. Let u be the solution to the Navier-Stokes system associated with u_0 and assume that the life span of u is $T^* < \infty$. Then for all positive times $T < T^*$, there is a subsequence such that the life span of the solution associated with $u_{0,n}$ is at least T .*

Remark 1.8. As explained above, the natural space in our context would be \mathcal{B}^0 . For technical reasons, we assume in our result more smoothness on the sequence of initial data, since obviously by Bernstein inequalities (1.6) and (1.7), we have $\mathcal{B}_{1,q} \hookrightarrow \mathcal{B}^0$.

1.3. Layout. The proof of Theorem 1 is addressed in Section 2. In Paragraph 2.2, we provide a new kind of “anisotropic profile decomposition” of the sequence of initial data, whose proof can be found in Section 3. This enables us to replace the sequence of Cauchy data, up to an arbitrarily small remainder term, by a finite (but large) sum of orthogonal sequences of divergence-free vector fields. In Paragraph 2.3, we state that each individual element involved in the decomposition derived in Paragraph 2.2 gives rise to a unique global solution to (NS) system (the proof is postponed to Section 4). Paragraph 2.4 is devoted to the proof of the fact that the sum of each individual profile does provide an approximate solution to the Navier-Stokes system, thanks to an orthogonality argument, which completes the proof of Theorem 1.

For all points $x = (x_1, x_2, x_3)$ in \mathbb{R}^3 and all vector fields $u = (u^1, u^2, u^3)$, we denote by

$$x_h \stackrel{\text{def}}{=} (x_1, x_2) \quad \text{and} \quad u^h \stackrel{\text{def}}{=} (u^1, u^2)$$

their horizontal parts. We also define horizontal differentiation operators $\nabla^h \stackrel{\text{def}}{=} (\partial_1, \partial_2)$ and $\text{div}_h \stackrel{\text{def}}{=} \nabla^h \cdot$, as well as $\Delta_h \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2$.

We also use the following shorthand notation: $X_h Y_v := X(\mathbb{R}^2; Y(\mathbb{R}))$ where X is a function space defined on \mathbb{R}^2 and Y is defined on \mathbb{R} .

As we shall be considering functions which have different types of variations in the x_3 variable and the x_h variable, the following notation will be used:

$$(1.11) \quad [f]_\beta(x) \stackrel{\text{def}}{=} f(x_h, \beta x_3).$$

Clearly, for any function f , we have the following identity which will be of constant use all along this paper:

$$(1.12) \quad \|[f]_\beta\|_{B_{p,1}^{s_1,s_2}} \sim \beta^{s_2 - \frac{1}{p}} \|f\|_{B_{p,1}^{s_1,s_2}}.$$

Finally we denote by C a constant which does not depend on the various parameters appearing in this paper, and which may change from line to line. We also denote sometimes $x \leq Cy$ by $x \lesssim y$.

2. PROOF OF THE MAIN THEOREM

2.1. General scheme of the proof. The main arguments leading to Theorem 1 are the following: by a profile decomposition argument, the sequence of initial data is decomposed into the weak limit u_0 and the sum of sequences of divergence-free vector fields, up to a small remainder term. Then to prove that each individual element of the decomposition generates a unique global solution to (NS), it is necessary to estimate sharply the regularity in scaling invariant (anisotropic) norms. The mutual orthogonality of each term in the decomposition of the initial data implies finally that the sum of the solutions associated to each element is itself an approximate solution to (NS), globally in time, which concludes the proof of the result.

2.2. Anisotropic profile decomposition. The study of the lack of compactness in critical Sobolev embeddings has attracted a lot of attention in the last decades, both for its interesting geometric features and for its applications to nonlinear partial differential equations. This study originates in the works of P.-L. Lions (see [39] and [40]) by means of defect measures, and earlier decompositions of bounded sequences into a sum of “profiles” can be found in the studies by H. Brézis and J.-M. Coron in [11] and M. Struwe in [48]. Our source of inspiration here is the work [22] of P. Gérard in which the defect of compactness of the critical Sobolev embeddings (for L^2 -based Sobolev spaces) in Lebesgue spaces is described by means of an asymptotic, orthogonal decomposition in terms of rescaled and translated profiles. This was generalized to L^p -based Sobolev spaces by S. Jaffard in [28], to Besov spaces by G. Koch [34], and finally to general critical embeddings by H. Bahouri, A. Cohen and G. Koch in [3] (see also [6, 7, 8] for the limiting case of Sobolev embeddings in Orlicz spaces and [47] for an abstract, functional analytic presentation of the concept in various settings).

In the pioneering works [5] (for the critical 3D wave equation) and [42] (for the critical 2D Schrödinger equation), it was highlighted that this type of decomposition provides applications to the study of nonlinear partial differential equations. The ideas of [5] were revisited in [33] and [19] in the context of the Schrödinger equations and Navier-Stokes system respectively, with an aim at describing the structure of bounded sequences of solutions to those equations. These profile decomposition techniques have since then been successfully used in order to study the possible blow-up of solutions to nonlinear partial differential equations, in various contexts; we refer for instance to [21], [26], [29], [30], [31], [32], [45], [46].

The first step in the proof of Theorem 1 consists in writing down an anisotropic profile decomposition of the sequence of initial data $(u_{0,n})_{n \in \mathbb{N}}$ (see Theorem 2). To state our result in a clear way, let us start by introducing some definitions and notations.

Definition 2.1. We say that two sequences of positive real numbers $(\lambda_n^1)_{n \in \mathbb{N}}$ and $(\lambda_n^2)_{n \in \mathbb{N}}$ are orthogonal if

$$\frac{\lambda_n^1}{\lambda_n^2} + \frac{\lambda_n^2}{\lambda_n^1} \rightarrow \infty, \quad n \rightarrow \infty.$$

A family of sequences $((\lambda_n^j)_{n \in \mathbb{N}})_j$ is said to be a family of scales if $\lambda_n^0 \equiv 1$ and if $(\lambda_n^j)_{n \in \mathbb{N}}$ and $(\lambda_n^k)_{n \in \mathbb{N}}$ are orthogonal when $j \neq k$.

Definition 2.2. Let μ be a positive real number less than $1/2$, fixed from now on.

We define $D_\mu \stackrel{\text{def}}{=} [-2 + \mu, 1 - \mu] \times [1/2, 7/2]$ and $\tilde{D}_\mu \stackrel{\text{def}}{=} [-1 + \mu, 1 - \mu] \times [1/2, 3/2]$. We denote by S_μ the space of functions a belonging to $\bigcap_{(s,s') \in D_\mu} \mathcal{B}^{s,s'}$ such that

$$\|a\|_{S_\mu} \stackrel{\text{def}}{=} \sup_{(s,s') \in D_\mu} \|a\|_{\mathcal{B}^{s,s'}} < \infty.$$

Notation. In all that follows, θ is a given function in $\mathcal{D}(B_{\mathbb{R}^3}(0,1))$ which has value 1 near $B_{\mathbb{R}^3}(0,1/2)$. For any positive real number η , we denote

$$(2.1) \quad \theta_\eta(x) \stackrel{\text{def}}{=} \theta(\eta x) \quad \text{and} \quad \theta_{h,\eta}(x_h) \stackrel{\text{def}}{=} \theta_\eta(x_h, 0).$$

In order to make notations as light as possible, the letter v (possibly with indices) will always denote a two-component divergence free vector field, which may depend on the vertical variable x_3 .

The following result, the proof of which is postponed to Section 3, is in the spirit of the *profile decomposition* theorem of P. Gérard in [22] concerning the critical Sobolev embedding in Lebesgue spaces.

Theorem 2. Under the assumptions of Theorem 1 and up to the extraction of a subsequence, the following holds. There is a family of scales $((\lambda_n^j)_{n \in \mathbb{N}})_{j \in \mathbb{N}}$ and for all $L \geq 1$ there is a family of sequences $((h_n^j)_{n \in \mathbb{N}})_{j \in \mathbb{N}}$ going to zero such that for any real number α in $]0, 1[$, there are families of sequences of divergence-free vector fields (for j ranging from 1 to L), $(v_{n,\alpha,L}^j)_{n \in \mathbb{N}}$, $(w_{n,\alpha,L}^j)_{n \in \mathbb{N}}$, $(v_{n,\alpha,L}^{0,\infty})_{n \in \mathbb{N}}$, $(w_{0,n,\alpha,L}^{0,\infty})_{n \in \mathbb{N}}$, $(v_{0,n,\alpha,L}^{0,\text{loc}})_{n \in \mathbb{N}}$ and $(w_{0,n,\alpha,L}^{0,\text{loc}})_{n \in \mathbb{N}}$ all belonging to S_μ , and a smooth, compactly supported function $u_{0,\alpha}$ such that the sequence $(u_{0,n})_{n \in \mathbb{N}}$ can be written under the form

$$\begin{aligned} u_{0,n} \equiv & u_{0,\alpha} + [(v_{0,n,\alpha,L}^{0,\text{loc}} + h_n^0 w_{0,n,\alpha,L}^{0,\text{loc,h}}, w_{0,n,\alpha,L}^{0,\text{loc},3})]_{h_n^0} + [(v_{0,n,\alpha,L}^{0,\infty} + h_n^0 w_{0,n,\alpha,L}^{0,\infty,h}, w_{0,n,\alpha,L}^{0,\infty,3})]_{h_n^0} \\ & + \sum_{j=1}^L \Lambda_{\lambda_n^j} [(v_{n,\alpha,L}^j + h_n^j w_{n,\alpha,L}^{j,h}, w_{n,\alpha,L}^{j,3})]_{h_n^j} + \rho_{n,\alpha,L} \end{aligned}$$

where $u_{0,\alpha}$ approximates u_0 in the sense that

$$(2.2) \quad \lim_{\alpha \rightarrow 0} \|u_{0,\alpha} - u_0\|_{\mathcal{B}_{1,q}} = 0,$$

where the remainder term satisfies

$$(2.3) \quad \lim_{L \rightarrow \infty} \lim_{\alpha \rightarrow 0} \limsup_{n \rightarrow \infty} \|e^{t\Delta} \rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} = 0,$$

while the following uniform bounds hold:

$$(2.4) \quad \mathcal{M} \stackrel{\text{def}}{=} \sup_{L \geq 1} \sup_{\alpha \in]0, 1[} \sup_{n \in \mathbb{N}} \left(\| (v_{0,n,\alpha,L}^{0,\infty}, w_{0,n,\alpha,L}^{0,\infty,3}) \|_{\mathcal{B}^0} + \| (v_{0,n,\alpha,L}^{0,\text{loc}}, w_{0,n,\alpha,L}^{0,\text{loc},3}) \|_{\mathcal{B}^0} \right. \\ \left. + \| u_{0,\alpha} \|_{\mathcal{B}^0} + \sum_{j=1}^L \| (v_{n,\alpha,L}^j, w_{n,\alpha,L}^{j,3}) \|_{\mathcal{B}^0} \right) < \infty$$

and for all α in $]0, 1[$,

$$(2.5) \quad \mathcal{M}_\alpha \stackrel{\text{def}}{=} \sup_{L \geq 1} \sup_{\substack{1 \leq j \leq L \\ n \in \mathbb{N}}} \left(\| (v_{0,n,\alpha,L}^{0,\infty}, w_{0,n,\alpha,L}^{0,\infty,3}) \|_{S_\mu} + \| (v_{0,n,\alpha,L}^{0,\text{loc}}, w_{0,n,\alpha,L}^{0,\text{loc},3}) \|_{S_\mu} \right. \\ \left. + \| u_{0,\alpha} \|_{S_\mu} + \| (v_{n,\alpha,L}^j, w_{n,\alpha,L}^{j,3}) \|_{S_\mu} \right)$$

is finite. Finally, we have

$$(2.6) \quad \lim_{L \rightarrow \infty} \lim_{\alpha \rightarrow 0} \limsup_{n \rightarrow \infty} \| (v_{0,n,\alpha,L}^{0,\text{loc}}, w_{0,n,\alpha,L}^{0,\text{loc},3})(\cdot, 0) \|_{B_{2,1}^0(\mathbb{R}^2)} = 0,$$

$$(2.7) \quad \forall (\alpha, L), \exists \eta(\alpha, L) / \forall \eta \leq \eta(\alpha, L), \forall n \in \mathbb{N}, (1 - \theta_{h,\eta})(v_{0,n,\alpha,L}^{0,\text{loc}}, w_{0,n,\alpha,L}^{0,\text{loc},3}) = 0 \text{ and}$$

$$(2.8) \quad \forall (\alpha, L, \eta), \exists n(\alpha, L, \eta) / \forall n \geq n(\alpha, L, \eta), \theta_{h,\eta}(v_{0,n,\alpha,L}^{0,\infty}, w_{0,n,\alpha,L}^{0,\infty,3}) = 0.$$

Theorem 2 states that the sequence $u_{0,n}$ is equal, up to a small remainder term, to a finite sum of orthogonal sequences of divergence-free vector fields. These sequences are obtained from the profile decomposition derived in [4] (see Proposition 2.4 in [4]) by grouping together all the profiles having the same horizontal scale λ_n , and the form they take depends on whether the scale λ_n is identically equal to one or not.

Note that in contrast with classical profile decompositions (see for instance [22]), cores of concentration do not appear in the profile decomposition given in Theorem 2 since all the profiles with the same horizontal scale are grouped together, and thus the decomposition is written in terms of scales only. The price to pay is that the profiles are no longer fixed functions, but bounded sequences. To carry out the strategy of proof developed in [4] in this framework, we have to establish that each element involved in the decomposition of Theorem 2 generates a global solution to (NS) system as soon as n is large enough. Since we deal with bounded sequences, it is necessary to estimate sharply the regularity.

Let us emphasize that in the case when λ_n goes to 0 or infinity, these sequences are of the type

$$(2.9) \quad \Lambda_{\lambda_n} [(v_{0,n}^h + h_n w_{0,n}^h, w_{0,n}^3)]_{h_n},$$

where we used Notation (1.11), and with h_n a sequence going to zero. It is essential (to establish our result) that the profiles that must be considered in that case are only profiles of type (2.9) with h_n tending to zero. Actually the divergence free assumption on $u_{0,n}$ allows to include the terms of type (2.9) with h_n tending to infinity into the remainder term and the anisotropically oscillating assumption for $(u_0 - u_{0,n})_{n \in \mathbb{N}}$ allows to exclude in the profile decomposition of $u_{0,n}$ sequences of type (2.9) with $h_n \equiv 1$.

In the case when λ_n is identically equal to one, we deal with three types of orthogonal sequences: the first one consists in $u_{0,\alpha}$, an approximation of the weak limit u_0 , the second one is of type (2.9) with $\lambda_n \equiv 1$ and h_n tending to zero, and is uniformly localized in the horizontal variable and vanishes at $x_3 = 0$, while the third one is also of type (2.9) with $\lambda_n \equiv 1$ and h_n converging to zero, and its support in the horizontal variable goes to infinity. Note that contrary to the case when the horizontal scale λ_n tends to 0 or infinity, all the profiles involved in the anisotropic decomposition of the sequence $(u_0 - u_{0,n})_{n \in \mathbb{N}}$ having the same

horizontal scale $\lambda_n \equiv 1$ are not grouped together: the sum of these profiles is divided into two parts depending on whether the horizontal cores of concentration escape to infinity or not. This splitting plays a key role in establishing our result under the only assumption of anisotropic oscillation, by removing the second assumption required in [4].

2.3. Propagation of profiles. The second step of the proof of Theorem 1 consists in proving that each individual profile involved in the decomposition of Theorem 2 generates a global solution to (NS) as soon as n is large enough. This is mainly based on the following results concerning respectively profiles of the type

$$\Lambda_{\lambda_n^j} [(v_{n,\alpha,L}^j + h_n^j w_{n,\alpha,L}^{j,h}, w_{n,\alpha,L}^{j,3})]_{h_n^j}$$

with λ_n^j going to 0 or infinity and h_n^j converging to zero, and the profiles of horizontal scale one, see respectively Theorems 3 and 4.

In order to state these theorems, let us begin by defining the function spaces we shall be working with.

Definition 2.3. – We define the space $\mathcal{A}^{s,s'} = L^\infty(\mathbb{R}^+; \mathcal{B}^{s,s'}) \cap L^2(\mathbb{R}^+; \mathcal{B}^{s+1,s'})$ equipped with the norm

$$\|a\|_{\mathcal{A}^{s,s'}} \stackrel{\text{def}}{=} \|a\|_{L^\infty(\mathbb{R}^+; \mathcal{B}^{s,s'})} + \|a\|_{L^2(\mathbb{R}^+; \mathcal{B}^{s+1,s'})},$$

and we denote $\mathcal{A}^s = \mathcal{A}^{s, \frac{1}{2}}$.

– We denote by $\mathcal{F}^{s,s'}$ any function space such that

$$\|L_0 f\|_{L^2(\mathbb{R}^+; \mathcal{B}^{s+1,s'})} \lesssim \|f\|_{\mathcal{F}^{s,s'}}$$

where, for any non negative real number τ , $L_\tau f$ denotes the solution of the heat equation

$$\begin{cases} \partial_t L_\tau f - \Delta L_\tau f = f \\ L_\tau f|_{t=\tau} = 0. \end{cases}$$

We denote $\mathcal{F}^s = \mathcal{F}^{s, \frac{1}{2}}$.

Examples. Using the smoothing effect of the heat flow, it is easy to prove that the spaces $\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{s-1,s'})$, $\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{s,s'-1})$ are $\mathcal{F}^{s,s'}$ spaces, as well as the spaces $L^1(\mathbb{R}^+; \mathcal{B}^{s,s'})$ and $L^1(\mathbb{R}^+; \mathcal{B}^{s+1,s'-1})$. Actually recalling that $L_0 f = \int_0^t e^{(t-t')\Delta} f(t') dt$ and taking advantage of (1.8), we get for any function in $\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{s-1,s'})$

$$\|\Delta_k^h \Delta_j^v L_0 f\|_{L^2} \lesssim \int_0^t e^{-ct'(2^{2k}+2^{2j})} \|\Delta_k^h \Delta_j^v f(t')\|_{L^2} dt',$$

where we make use of notations of Definition 1.2. We deduce that there is a sequence $d_{j,k}(t')$ in the sphere of $\ell^1(\mathbb{Z} \times \mathbb{Z}; L^2(\mathbb{R}^+))$ such that

$$\|\Delta_k^h \Delta_j^v L_0 f\|_{L^2} \lesssim \|f\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{s-1,s'})} 2^{-k(s-1)} 2^{-js'} \int_0^t e^{-ct'(2^{2k}+2^{2j})} d_{j,k}(t') dt'.$$

Young's inequality in time therefore gives

$$\|\Delta_k^h \Delta_j^v L_0 f\|_{L^2(\mathbb{R}^+; L^2)} \lesssim \|f\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{s-1,s'})} 2^{-k(s-1)-js'} d_{j,k},$$

where $d_{j,k}$ is a generic sequence in the sphere of $\ell^1(\mathbb{Z} \times \mathbb{Z})$, which ends the proof of the result in the case when f belongs to $\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{s-1,s'})$. The argument is similar in the other cases.

Notation. In the following we designate by $\mathcal{T}_0(A, B)$ a generic constant depending only on the quantities A and B . We denote by \mathcal{T}_1 a generic non decreasing function from \mathbb{R}^+ into \mathbb{R}^+ such that

$$(2.10) \quad \limsup_{r \rightarrow 0} \frac{\mathcal{T}_1(r)}{r} < \infty,$$

and by \mathcal{T}_2 a generic locally bounded function from \mathbb{R}^+ into \mathbb{R}^+ . All those functions may vary from line to line. Let us notice that for any positive sequence $(a_n)_{n \in \mathbb{N}}$ belonging to ℓ^1 , we have

$$(2.11) \quad \sum_n \mathcal{T}_1(a_n) \leq \mathcal{T}_2\left(\sum_n a_n\right).$$

As in the isotropic case, the following space-time (quasi)-norms, first introduced by J.-Y. Chemin and N. Lerner in [17]:

$$(2.12) \quad \|f\|_{\tilde{L}^r([0, T]; B_{p, q}^{s, s'})} \stackrel{\text{def}}{=} \|2^{ks+j s'} \|\Delta_k^h \Delta_j^v f\|_{L^r([0, T]; L^p)}\|_{\ell^q},$$

are very useful in the context of the Navier-Stokes system, and will be of constant use all along this paper. Notice that of course $\tilde{L}^r([0, T]; B_{p, r}^{s, s'}) = L^r([0, T]; B_{p, r}^{s, s'})$, and by Minkowski's inequality, we have the embedding $\tilde{L}^r([0, T]; B_{p, q}^{s, s'}) \subset L^r([0, T]; B_{p, q}^{s, s'})$ if $r \geq q$.

Our first theorem of global existence for the Navier-Stokes system, which concerns profiles with horizontal scales going to 0 or infinity, generalizes the example considered in [14].

Theorem 3. *A locally bounded function ε_1 from \mathbb{R}^+ into \mathbb{R}^+ exists which satisfies the following. For any (v_0, w_0^3) in S_μ (see Definition 2.2), for any positive real number β such that $\beta \leq \varepsilon_1(\|(v_0, w_0^3)\|_{S_\mu})$, the divergence free vector field*

$$\Phi_0 \stackrel{\text{def}}{=} [(v_0 - \beta \nabla^h \Delta_h^{-1} \partial_3 w_0^3, w_0^3)]_\beta$$

generates a global solution Φ_β to (NS) which satisfies

$$(2.13) \quad \|\Phi_\beta\|_{\mathcal{A}^0} \leq \mathcal{T}_1(\|(v_0, w_0^3)\|_{\mathcal{B}^0}) + \beta \mathcal{T}_2(\|(v_0, w_0^3)\|_{S_\mu}).$$

Moreover, for any (s, s') in $[-1 + \mu, 1 - \mu] \times [1/2, 7/2]$, we have, for any r in $[1, \infty]$,

$$(2.14) \quad \|\Phi_\beta\|_{L^r(\mathbb{R}^+; \mathcal{B}^{s+\frac{2}{r}})} + \frac{1}{\beta^{s'-\frac{1}{2}}} \|\Phi_\beta\|_{L^r(\mathbb{R}^+; \mathcal{B}_r^{s, s'})} \leq \mathcal{T}_2(\|(v_0, w_0^3)\|_{S_\mu}).$$

The proof of Theorem 3 is provided in Paragraph 4.1.

The existence of a global regular solution for the set of profiles associated with the horizontal scale 1 is ensured by the following theorem which can be viewed as a generalization of Theorem 3 of [15] and of Theorem 2 of [16].

Theorem 4. *With the notation of Theorem 2, let us consider the initial data:*

$$\Phi_{0, n, \alpha, L}^0 \stackrel{\text{def}}{=} u_{0, \alpha} + [(v_{0, n, \alpha, L}^{0, \infty} + h_n^0 w_{0, n, \alpha, L}^{0, \infty, h}, w_{0, n, \alpha, L}^{0, \infty, 3})]_{h_n^0} + [(v_{0, n, \alpha, L}^{0, \text{loc}} + h_n^0 w_{0, n, \alpha, L}^{0, \text{loc}, h}, w_{0, n, \alpha, L}^{0, \text{loc}, 3})]_{h_n^0}.$$

There is a constant ε_0 , depending only on u_0 and on \mathcal{M}_α , such that if $h_n^0 \leq \varepsilon_0$, then the initial data $\Phi_{0, n, \alpha, L}^0$ generates a global smooth solution $\Phi_{n, \alpha, L}^0$ which satisfies for all s in $[-1 + \mu, 1 - \mu]$ and all r in $[1, \infty]$,

$$(2.15) \quad \|\Phi_{n, \alpha, L}^0\|_{L^r(\mathbb{R}^+; \mathcal{B}^{s+\frac{2}{r}})} \leq \mathcal{T}_0(u_0, \mathcal{M}_\alpha).$$

The proof of Theorem 4 is provided in Paragraph 4.2.

2.4. End of the proof of the main theorem. To end the proof of Theorem 1, we need to check that the sum of the propagation of the remainder term through the transport-diffusion equation and the solutions to (NS) associated to each individual profile (provided by Theorems 3 and 4) is an approximate solution to the Navier-Stokes system. This can be achieved by proving that the nonlinear interactions of all the solutions are negligible, thanks to the orthogonality between the scales. For that purpose, let us look at the profile decomposition given by Theorem 2. For a given positive and small ε , Assertion (2.3) allows to choose α , L and N_0 (depending of course on ε) such that

$$(2.16) \quad \forall n \geq N_0, \quad \|e^{t\Delta} \rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \leq \varepsilon.$$

The parameters α and L are fixed so that (2.16) holds, let us consider the two functions ε_1 , \mathcal{T}_1 and \mathcal{T}_2 (resp. ε_0 and \mathcal{T}_0) which appear in the statement of Theorem 3 (resp. Theorem 4). Since each sequence $(h_n^j)_{n \in \mathbb{N}}$, for $0 \leq j \leq L$, goes to zero as n goes to infinity, one can choose an integer N_1 greater than or equal to N_0 such that

$$(2.17) \quad \forall n \geq N_1, \quad \forall j \in \{0, \dots, L\}, \quad h_n^j \leq \min \left\{ \varepsilon_1(\mathcal{M}_\alpha), \varepsilon_0, \frac{\varepsilon}{L\mathcal{T}_2(\mathcal{M}_\alpha)} \right\}.$$

Now for $1 \leq j \leq L$ (resp. $j = 0$), let us denote by $\Phi_{n,\varepsilon}^j$ (resp. $\Phi_{n,\varepsilon}^0$) the global solution of (NS) associated with the initial data

$$\begin{aligned} & [(v_{n,\alpha,L}^j + h_n^j w_{n,\alpha,L}^{j,h}, w_{n,\alpha,L}^{j,3})]_{h_n^j} \\ & \left(\text{resp. } u_{0,\alpha} + [(v_{0,n,\alpha,L}^{0,\infty} + h_n^0 w_{0,n,\alpha,L}^{0,\infty,h}, w_{0,n,\alpha,L}^{0,\infty,3})]_{h_n^0} + [(v_{0,n,\alpha,L}^{0,\text{loc}} + h_n^0 w_{0,n,\alpha,L}^{0,\text{loc},h}, w_{0,n,\alpha,L}^{0,\text{loc},3})]_{h_n^0} \right) \end{aligned}$$

given by Theorem 3 (resp. Theorem 4). We look for the global solution associated with $u_{0,n}$ under the form

$$(2.18) \quad u_n = u_{n,\varepsilon}^{\text{app}} + R_{n,\varepsilon} \quad \text{with} \quad u_{n,\varepsilon}^{\text{app}} \stackrel{\text{def}}{=} \sum_{j=0}^L \Lambda_{\lambda_n^j} \Phi_{n,\varepsilon}^j + e^{t\Delta} \rho_{n,\alpha,L}.$$

In view of the scaling invariance of the Navier-Stokes system, $\Lambda_{\lambda_n^j} \Phi_{n,\varepsilon}^j$ solves (NS) with the initial data $\Lambda_{\lambda_n^j} [(v_{n,\alpha,L}^j + h_n^j w_{n,\alpha,L}^{j,h}, w_{n,\alpha,L}^{j,3})]_{h_n^j}$. This gives the following equation on $R_{n,\varepsilon}$:

$$\begin{aligned} & \partial_t R_{n,\varepsilon} - \Delta R_{n,\varepsilon} + \text{div}(R_{n,\varepsilon} \otimes R_{n,\varepsilon} + R_{n,\varepsilon} \otimes u_{n,\varepsilon}^{\text{app}} + u_{n,\varepsilon}^{\text{app}} \otimes R_{n,\varepsilon}) + \nabla p_{n,\varepsilon} \\ & = F_{n,\varepsilon} \stackrel{\text{def}}{=} F_{n,\varepsilon}^1 + F_{n,\varepsilon}^2 + F_{n,\varepsilon}^3 \quad \text{with} \\ & F_{n,\varepsilon}^1 \stackrel{\text{def}}{=} -\text{div}(e^{t\Delta} \rho_{n,\alpha,L} \otimes e^{t\Delta} \rho_{n,\alpha,L}) \\ & F_{n,\varepsilon}^2 \stackrel{\text{def}}{=} -\sum_{j=0}^L \text{div}(\Lambda_{\lambda_n^j} \Phi_{n,\varepsilon}^j \otimes e^{t\Delta} \rho_{n,\alpha,L} + e^{t\Delta} \rho_{n,\alpha,L} \otimes \Lambda_{\lambda_n^j} \Phi_{n,\varepsilon}^j) \quad \text{and} \\ & F_{n,\varepsilon}^3 \stackrel{\text{def}}{=} -\sum_{\substack{0 \leq j,k \leq L \\ j \neq k}} \text{div}(\Lambda_{\lambda_n^j} \Phi_{n,\varepsilon}^j \otimes \Lambda_{\lambda_n^k} \Phi_{n,\varepsilon}^k), \end{aligned} \tag{2.19}$$

and where $(\text{div}(u \otimes v))^j = \sum_{k=1}^3 \partial_k(u^j v^k)$.

In order to establish that the function u_n defined by (2.18) provides a global solution to (NS) system, it suffices to prove that there exist some space \mathcal{F}^0 as in Definition 2.3 and an integer $N \geq N_1$ such that

$$(2.20) \quad \forall n \geq N, \quad \|F_{n,\varepsilon}\|_{\mathcal{F}^0} \leq C\varepsilon,$$

where C only depends on L and \mathcal{M}_α . In the next estimates we omit the dependence of all constants on α and L , which are fixed. Indeed if (2.20) holds, then $R_{n,\varepsilon}$ exists globally thanks to strong stability in \mathcal{B}^0 (see [4] for the setting of $\mathcal{B}_{1,1}$).

Let us start with the estimate of $F_{n,\varepsilon}^1$. Using the fact that \mathcal{B}^1 is an algebra, we have

$$\|e^{t\Delta}\rho_{n,\alpha,L}^h \otimes e^{t\Delta}\rho_{n,\alpha,L}\|_{L^1(\mathbb{R}^+;\mathcal{B}^1)} \lesssim \|e^{t\Delta}\rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}^2,$$

so

$$\|\operatorname{div}_h(e^{t\Delta}\rho_{n,\alpha,L}^h \otimes e^{t\Delta}\rho_{n,\alpha,L})\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)} \lesssim \|e^{t\Delta}\rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}^2$$

and

$$\|\partial_3(e^{t\Delta}\rho_{n,\alpha,L}^3 e^{t\Delta}\rho_{n,\alpha,L})\|_{L^1(\mathbb{R}^+;\mathcal{B}^{1,-\frac{1}{2}})} \lesssim \|e^{t\Delta}\rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}^2.$$

According to Inequality (2.16), this gives rise to

$$(2.21) \quad \forall n \geq N_1, \quad \|F_{n,\varepsilon}^1\|_{\mathcal{F}^0} \lesssim \varepsilon^2.$$

Now let us consider $F_{n,\varepsilon}^2$. By the scaling invariance of the operators $\Lambda_{\lambda_n^j}$ in $L^2(\mathbb{R}^+;\mathcal{B}^1)$ and again the fact that \mathcal{B}^1 is an algebra, we get

$$(2.22) \quad \begin{aligned} \|\Lambda_{\lambda_n^j}\Phi_{n,\varepsilon}^j \otimes e^{t\Delta}\rho_{n,\alpha,L} + e^{t\Delta}\rho_{n,\alpha,L} \otimes \Lambda_{\lambda_n^j}\Phi_{n,\varepsilon}^j\|_{L^1(\mathbb{R}^+;\mathcal{B}^1)} \\ \lesssim \|\Phi_{n,\varepsilon}^j\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \|e^{t\Delta}\rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}. \end{aligned}$$

Making use of Estimates (2.13) and (2.15), we infer that

$$\sum_{j=0}^L \|\Phi_{n,\varepsilon}^j\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \leq \mathcal{T}_0(u_0, \mathcal{M}_\alpha) + \mathcal{T}_2(\mathcal{M}) + \sum_{j=1}^L h_n^j \mathcal{T}_2(\mathcal{M}_\alpha),$$

which in view of Condition (2.17) on the sequences $(h_n^j)_{n \in \mathbb{N}}$ implies that

$$\left\| \sum_{j=0}^L \Phi_{n,\varepsilon}^j \right\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \leq \mathcal{T}_0(u_0, \mathcal{M}_\alpha) + \mathcal{T}_2(\mathcal{M}) + \varepsilon.$$

It follows (of course up to a change of \mathcal{T}_2) that for small enough ε

$$(2.23) \quad \left\| \sum_{j=0}^L \Phi_{n,\varepsilon}^j \right\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \leq \mathcal{T}_0(u_0, \mathcal{M}_\alpha) + \mathcal{T}_2(\mathcal{M}).$$

Thanks to (2.16) and (2.22), this gives rise to

$$(2.24) \quad \forall n \geq N_1, \quad \|F_{n,\varepsilon}^2\|_{\mathcal{F}^0} \leq \varepsilon (\mathcal{T}_0(u_0, \mathcal{M}_\alpha) + \mathcal{T}_2(\mathcal{M})).$$

Finally let us consider $F_{n,\varepsilon}^3$. Using the fact that \mathcal{B}^1 is an algebra along with the Hölder inequality, we infer that for a small enough γ in $]0, 1[$,

$$\|\Lambda_{\lambda_n^j}\Phi_{n,\varepsilon}^j \otimes \Lambda_{\lambda_n^k}\Phi_{n,\varepsilon}^k\|_{L^1(\mathbb{R}^+;\mathcal{B}^1)} \leq \|\Lambda_{\lambda_n^j}\Phi_{n,\varepsilon}^j\|_{L^{\frac{2}{1+\gamma}}(\mathbb{R}^+;\mathcal{B}^1)} \|\Lambda_{\lambda_n^k}\Phi_{n,\varepsilon}^k\|_{L^{\frac{2}{1-\gamma}}(\mathbb{R}^+;\mathcal{B}^1)}.$$

The scaling invariance (1.9) gives

$$\begin{aligned} \|\Lambda_{\lambda_n^j}\Phi_{n,\varepsilon}^j\|_{L^{\frac{2}{1+\gamma}}(\mathbb{R}^+;\mathcal{B}^1)} &\sim (\lambda_n^j)^\gamma \|\Phi_{n,\varepsilon}^j\|_{L^{\frac{2}{1+\gamma}}(\mathbb{R}^+;\mathcal{B}^1)} \quad \text{and} \\ \|\Lambda_{\lambda_n^k}\Phi_{n,\varepsilon}^k\|_{L^{\frac{2}{1-\gamma}}(\mathbb{R}^+;\mathcal{B}^1)} &\sim \frac{1}{(\lambda_n^k)^\gamma} \|\Phi_{n,\varepsilon}^k\|_{L^{\frac{2}{1-\gamma}}(\mathbb{R}^+;\mathcal{B}^1)}. \end{aligned}$$

For small enough γ , Theorems 3 and 4 imply that

$$\|\Lambda_{\lambda_n^j}\Phi_{n,\varepsilon}^j \otimes \Lambda_{\lambda_n^k}\Phi_{n,\varepsilon}^k\|_{L^1(\mathbb{R}^+;\mathcal{B}^1)} \lesssim \left(\frac{\lambda_n^j}{\lambda_n^k}\right)^\gamma.$$

We deduce that

$$\|F_{n,\varepsilon}^3\|_{\mathcal{F}^0} \lesssim \sum_{\substack{0 \leq j, k \leq L \\ j \neq k}} \min \left\{ \frac{\lambda_n^j}{\lambda_n^k}, \frac{\lambda_n^k}{\lambda_n^j} \right\}^\gamma.$$

As the sequences $(\lambda_n^j)_{n \in \mathbb{N}}$ and $(\lambda_n^k)_{n \in \mathbb{N}}$ are orthogonal (see Definition 2.1), we have for any j and k such that $j \neq k$

$$\lim_{n \rightarrow \infty} \min \left\{ \frac{\lambda_n^j}{\lambda_n^k}, \frac{\lambda_n^k}{\lambda_n^j} \right\} = 0.$$

Thus an integer N_2 greater than or equal to N_1 exists such that

$$\forall n \geq N_2, \quad \|F_{n,\varepsilon}^3\|_{\mathcal{F}^0} \lesssim \varepsilon.$$

Together with (2.21) and (2.24), this implies that

$$n \geq N_2 \implies \|F_{n,\varepsilon}\|_{\mathcal{F}^0} \lesssim \varepsilon,$$

which proves (2.20) and thus concludes the proof of Theorem 1. \square

3. PROFILE DECOMPOSITION OF THE SEQUENCE OF INITIAL DATA: PROOF OF THEOREM 2

The proof of Theorem 2 is structured as follows. First, in Section 3.1 we write down the profile decomposition of any bounded sequence of *anisotropically oscillating* divergence free vector fields, following the results of [4]. Next we reorganize the profile decomposition by grouping together all profiles having the same horizontal scale and we check that all the conclusions of Theorem 2 hold: that is performed in Section 3.2.

3.1. Profile decomposition of anisotropically oscillating, divergence free vector fields. In this section we start by recalling the result of [4], where an anisotropic profile decomposition of sequences of $\mathcal{B}_{1,q}$ anisotropically oscillating is achieved. Let us first define anisotropic scaling operators, similar to the operators defined in (1.1): for any two sequences of positive real numbers $(\varepsilon_n)_{n \in \mathbb{N}}$ and $(\gamma_n)_{n \in \mathbb{N}}$, and for any sequence $(x_n)_{n \in \mathbb{N}}$ of points in \mathbb{R}^3 , we denote

$$\Lambda_{\varepsilon_n, \gamma_n, x_n} \phi(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon_n} \phi \left(\frac{x_h - x_{n,h}}{\varepsilon_n}, \frac{x_3 - x_{n,3}}{\gamma_n} \right).$$

Let us also introduce the definition of orthogonal triplets of sequences, analogous to Definition 2.1.

Definition 3.1. We say that two triplets of sequences $(\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell)_{n \in \mathbb{N}}$ with ℓ belonging to $\{1, 2\}$, where $(\varepsilon_n^\ell, \gamma_n^\ell)_{n \in \mathbb{N}}$ are two sequences of positive real numbers and x_n^ℓ are sequences in \mathbb{R}^3 , are orthogonal if, when n tends to infinity,

$$\begin{aligned} & \text{either } \frac{\varepsilon_n^1}{\varepsilon_n^2} + \frac{\varepsilon_n^2}{\varepsilon_n^1} + \frac{\gamma_n^1}{\gamma_n^2} + \frac{\gamma_n^2}{\gamma_n^1} \rightarrow \infty \\ & \text{or } (\varepsilon_n^1, \gamma_n^1) \equiv (\varepsilon_n^2, \gamma_n^2) \quad \text{and} \quad |(x_n^1)^{\varepsilon_n^1, \gamma_n^1} - (x_n^2)^{\varepsilon_n^2, \gamma_n^2}| \rightarrow \infty, \end{aligned}$$

where we have denoted $(x_n^\ell)^{\varepsilon_n^\ell, \gamma_n^\ell} \stackrel{\text{def}}{=} \left(\frac{x_{n,h}^\ell}{\varepsilon_n^\ell}, \frac{x_{n,3}^\ell}{\gamma_n^\ell} \right)$. A family of sequences $((\varepsilon_n^j, \gamma_n^j, x_n^j)_{n \in \mathbb{N}})_{j \geq 0}$ is said to be a family of scales and cores if $\varepsilon_n^0 \equiv \gamma_n^0 \equiv 1$, $x_n^0 \equiv 0$, and if $(\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell)_{n \in \mathbb{N}}$ and $(\varepsilon_n^k, \gamma_n^k, x_n^k)_{n \in \mathbb{N}}$ are orthogonal when $\ell \neq k$.

Now, let us recall without proof the following result.

Proposition 3.2 ([4]). *Under the assumptions of Theorem 1, the following holds. For all integers $\ell \geq 0$ there is a triplet of scales and cores in the sense of Definition 3.1, denoted by $(\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell)_{n \in \mathbb{N}}$ and for all α in $]0, 1[$ there are arbitrarily smooth divergence free vector fields $(\tilde{\phi}_\alpha^{\text{h},\ell}, 0)$ and $(-\nabla^{\text{h}} \Delta_{\text{h}}^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell)$ with $\tilde{\phi}_\alpha^{\text{h},\ell}$ and ϕ_α^ℓ compactly supported, and such that up to extracting a subsequence, one can write the sequence $(u_{0,n})_{n \in \mathbb{N}}$ under the following form, for each $L \geq 1$:*

$$(3.1) \quad u_{0,n} = u_0 + \sum_{\ell=1}^L \Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell} \left(\tilde{\phi}_\alpha^{\text{h},\ell} + \tilde{r}_\alpha^{\text{h},\ell} - \frac{\varepsilon_n^\ell}{\gamma_n^\ell} \nabla^{\text{h}} \Delta_{\text{h}}^{-1} \partial_3 (\phi_\alpha^\ell + r_\alpha^\ell), \phi_\alpha^\ell + r_\alpha^\ell \right) + (\tilde{\psi}_n^{\text{h},L} - \nabla^{\text{h}} \Delta_{\text{h}}^{-1} \partial_3 \psi_n^L, \psi_n^L),$$

where $\tilde{\psi}_n^{\text{h},L}$ and ψ_n^L are independent of α and satisfy

$$(3.2) \quad \limsup_{n \rightarrow \infty} \left(\|\tilde{\psi}_n^{\text{h},L}\|_{\mathcal{B}^0} + \|\psi_n^L\|_{\mathcal{B}^0} \right) \rightarrow 0, \quad L \rightarrow \infty,$$

while $\tilde{r}_\alpha^{\text{h},\ell}$ and r_α^ℓ are independent of n and L and satisfy for each $\ell \in \mathbb{N}$

$$(3.3) \quad \|\tilde{r}_\alpha^{\text{h},\ell}\|_{\mathcal{B}_{1,q}} + \|r_\alpha^\ell\|_{\mathcal{B}_{1,q}} \leq \alpha.$$

Moreover the following properties hold:

$$(3.4) \quad \forall \ell \geq 1, \quad \lim_{n \rightarrow \infty} (\gamma_n^\ell)^{-1} \varepsilon_n^\ell \in \{0, \infty\},$$

and $\lim_{n \rightarrow \infty} (\gamma_n^\ell)^{-1} \varepsilon_n^\ell = \infty \implies \phi_\alpha^\ell \equiv r_\alpha^\ell \equiv 0,$

as well as the following stability result, which is uniform in α :

$$(3.5) \quad \sum_{\ell \geq 1} \left(\|\tilde{\phi}_\alpha^{\text{h},\ell}\|_{\mathcal{B}_{1,q}} + \|\tilde{r}_\alpha^{\text{h},\ell}\|_{\mathcal{B}_{1,q}} + \|\phi_\alpha^\ell\|_{\mathcal{B}_{1,q}} + \|r_\alpha^\ell\|_{\mathcal{B}_{1,q}} \right) \lesssim \sup_n \|u_{0,n}\|_{\mathcal{B}_{1,q}} + \|u_0\|_{\mathcal{B}_{1,q}}.$$

Remark 3.3. *As pointed out in [4, Section 2], if two scales appearing in the above decomposition are not orthogonal, then they can be chosen to be equal. We shall therefore assume from now on that is the case: two sequences of scales are either orthogonal, or equal.*

3.2. Regrouping of profiles according to horizontal scales. In order to proceed with the re-organization of the profile decomposition provided in Proposition 3.2, we introduce some more definitions, keeping the notation of Proposition 3.2. For a given $L \geq 1$ we define recursively an increasing (finite) sequence of indexes $\ell_k \in \{1, \dots, L\}$ by

$$(3.6) \quad \ell_0 \stackrel{\text{def}}{=} 0, \quad \ell_{k+1} \stackrel{\text{def}}{=} \min \left\{ \ell \in \{\ell_k + 1, \dots, L\} / \frac{\varepsilon_n^\ell}{\gamma_n^\ell} \rightarrow 0 \quad \text{and} \quad \ell \notin \bigcup_{k'=0}^k \Gamma^L(\varepsilon_n^{\ell_{k'}}) \right\},$$

where for $0 \leq \ell \leq L$, we define (recalling that by Remark 3.3 if two scales are not orthogonal, then they are equal),

$$(3.7) \quad \Gamma^L(\varepsilon_n^\ell) \stackrel{\text{def}}{=} \left\{ \ell' \in \{1, \dots, L\} / \varepsilon_n^{\ell'} \equiv \varepsilon_n^\ell \quad \text{and} \quad \varepsilon_n^{\ell'} (\gamma_n^{\ell'})^{-1} \rightarrow 0, \quad n \rightarrow \infty \right\}.$$

We call $\mathcal{L}(L)$ the largest index of the sequence (ℓ_k) and we may then introduce the following partition:

$$(3.8) \quad \left\{ \ell \in \{1, \dots, L\} / \varepsilon_n^\ell (\gamma_n^\ell)^{-1} \rightarrow 0 \right\} = \bigcup_{k=0}^{\mathcal{L}(L)} \Gamma^L(\varepsilon_n^{\ell_k}).$$

We shall now regroup profiles in the decomposition (3.1) of $u_{0,n}$ according to the value of their horizontal scale. We fix from now on an integer $L \geq 1$.

3.2.1. *Construction of the profiles for $\ell = 0$.* Before going into the technical details of the construction, let us discuss an example explaining the computations of this paragraph. Consider the particular case when $u_{0,n}$ is given by

$$u_{0,n}(x) = u_0(x) + (v_0^0(x_h, 2^{-n}x_3) + w_0^{0,h}(x_h, 2^{-2n}x_3), 0) + (v_0^0(x_1 + 2^n, x_2, 2^{-n}x_3), 0),$$

with v_0^0 and $w_0^{0,h}$ smooth (say in $B_{1,q}^{s,s'}$ for all s, s' in \mathbb{R}) and compactly supported. Let us assume that $u_{0,n}$ converges towards u_0 in the sense of distributions, and that $(u_{0,n} - u_0)_{n \in \mathbb{N}}$ is anisotropically oscillating. Then we can write

$$u_{0,n}(x) = u_0(x) + (v_{0,n}^{0,\text{loc}}(x_h, 2^{-n}x_3), 0) + (v_{0,n}^{0,\infty}(x_h, 2^{-n}x_3), 0),$$

with $v_{0,n}^{0,\text{loc}}(y) := v_0^0(y) + w_0^{0,h}(y_h, 2^{-n}y_3)$ and $v_{0,n}^{0,\infty}(y) = v_0^0(y_1 + 2^n, y_2, y_3)$. Now since $u_{0,n} \rightharpoonup u_0$ as n goes to infinity, we have that $v_0^0(x_h, 0) + w_0^h(x_h, 0) \equiv 0$, hence $v_{0,n}^{0,\text{loc}}(x_h, 0) = 0$. The initial data $u_{0,n}$ has therefore been re-written as

$$u_{0,n}(x) = u_0(x) + (v_{0,n}^{0,\text{loc}}(x_h, 2^{-n}x_3), 0) + (v_{0,n}^{0,\infty}(x_h, 2^{-n}x_3), 0) \quad \text{with} \quad v_{0,n}^{0,\text{loc}}(x_h, 0) = 0$$

and where the support in x_h of $v_{0,n}^{0,\text{loc}}(x_h, 2^{-n}x_3)$ is in a fixed compact set whereas the support in x_h of $v_{0,n}^{0,\infty}(x_h, 2^{-n}x_3)$ escapes to infinity. This is of the same form as in the statement of Theorem 2.

When considering all the profiles having the same horizontal scale (1 here), the point is therefore to choose the smallest vertical scale (2^n here) and to write the decomposition in terms of that scale only. Of course that implies that contrary to usual profile decompositions, the profiles are no longer fixed functions in $\mathcal{B}_{1,q}$, but sequences of functions, bounded in $\mathcal{B}_{1,q}$.

In view of the above example, let ℓ_0^- be an integer such that $\gamma_n^{\ell_0^-}$ is the smallest vertical scale going to infinity, associated with profiles for $1 \leq \ell \leq L$, having 1 for horizontal scale. More precisely we ask that

$$\gamma_n^{\ell_0^-} = \min_{\ell \in \Gamma^L(1)} \gamma_n^\ell,$$

where according to (3.7),

$$\Gamma^L(1) = \left\{ \ell' \in \{1, \dots, L\} / \varepsilon_n^{\ell'} \equiv 1 \quad \text{and} \quad \gamma_n^{\ell'} \rightarrow \infty, n \rightarrow \infty \right\}.$$

Notice that the minimum of the sequences γ_n^ℓ is well defined in our context thanks to the fact that due to Remark 3.3, either two sequences are orthogonal in the sense of Definition 3.1, or they are equal. Observe also that ℓ_0^- is by no means unique, as several profiles may have the same horizontal scale as well as the same vertical scale (in which case the concentration cores must be orthogonal). Now we denote

$$(3.9) \quad h_n^0 \stackrel{\text{def}}{=} (\gamma_n^{\ell_0^-})^{-1},$$

and we notice that h_n^0 goes to zero as n goes to infinity for each L . Note also that h_n^0 depends on L through the choice of ℓ_0^- , since if L increases then ℓ_0^- may also increase; this dependence is omitted in the notation for simplicity. Let us define (up to a subsequence extraction)

$$(3.10) \quad a^\ell \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \left(x_{n,h}^\ell, \frac{x_{n,3}^\ell}{\gamma_n^\ell} \right).$$

We then define the divergence-free vector fields

$$(3.11) \quad v_{0,n,\alpha,L}^{0,\text{loc}}(y) \stackrel{\text{def}}{=} \sum_{\substack{\ell \in \Gamma^L(1) \\ a_h^\ell \in \mathbb{R}^2}} \tilde{\phi}_\alpha^{h,\ell} \left(y_h - x_{n,h}^\ell, \frac{y_3}{h_n^0 \gamma_n^\ell} - \frac{x_{n,3}^\ell}{\gamma_n^\ell} \right)$$

and

$$(3.12) \quad w_{0,n,\alpha,L}^{0,\text{loc}}(y) \stackrel{\text{def}}{=} \sum_{\substack{\ell \in \Gamma^L(1) \\ a_h^\ell \in \mathbb{R}^2}} \left(-\frac{1}{h_n^0 \gamma_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \right) \left(y_h - x_{n,h}^\ell, \frac{y_3}{h_n^0 \gamma_n^\ell} - \frac{x_{n,3}^\ell}{\gamma_n^\ell} \right).$$

By construction we have

$$w_{0,n,\alpha,L}^{0,\text{loc},h} = -\nabla^h \Delta_h^{-1} \partial_3 w_{0,n,\alpha,L}^{0,\text{loc},3}.$$

Similarly we define

$$(3.13) \quad v_{0,n,\alpha,L}^{0,\infty}(y) \stackrel{\text{def}}{=} \sum_{\substack{\ell \in \Gamma^L(1) \\ |a_h^\ell| = \infty}} \tilde{\phi}_\alpha^{h,\ell} \left(y_h - x_{n,h}^\ell, \frac{y_3}{h_n^0 \gamma_n^\ell} - \frac{x_{n,3}^\ell}{\gamma_n^\ell} \right)$$

and

$$(3.14) \quad w_{0,n,\alpha,L}^{0,\infty}(y) \stackrel{\text{def}}{=} \sum_{\substack{\ell \in \Gamma^L(1) \\ |a_h^\ell| = \infty}} \left(-\frac{1}{h_n^0 \gamma_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \right) \left(y_h - x_{n,h}^\ell, \frac{y_3}{h_n^0 \gamma_n^\ell} - \frac{x_{n,3}^\ell}{\gamma_n^\ell} \right).$$

By construction we have again

$$w_{0,n,\alpha,L}^{0,\infty,h} = -\nabla^h \Delta_h^{-1} \partial_3 w_{0,n,\alpha,L}^{0,\infty,3}.$$

Moreover recalling the notation

$$[f]_{h_n^0}(x) \stackrel{\text{def}}{=} f(x_h, h_n^0 x_3)$$

and

$$\Lambda_{\varepsilon_n, \gamma_n, x_n} \phi(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon_n} \phi \left(\frac{x_h - x_{n,h}}{\varepsilon_n}, \frac{x_3 - x_{n,3}}{\gamma_n} \right),$$

one can compute that

$$(3.15) \quad \sum_{\substack{\ell \in \Gamma^L(1) \\ a_h^\ell \in \mathbb{R}^2}} \Lambda_{1, \gamma_n^\ell, x_n^\ell} \left(\tilde{\phi}_\alpha^{h,\ell} - \frac{1}{\gamma_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \right) = [(v_{0,n,\alpha,L}^{0,\text{loc}} + h_n^0 w_{0,n,\alpha,L}^{0,\text{loc},h}, w_{0,n,\alpha,L}^{0,\text{loc},3})]_{h_n^0}$$

and

$$(3.16) \quad \sum_{\substack{\ell \in \Gamma^L(1) \\ |a_h^\ell| = \infty}} \Lambda_{1, \gamma_n^\ell, x_n^\ell} \left(\tilde{\phi}_\alpha^{h,\ell} - \frac{1}{\gamma_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \right) = [(v_{0,n,\alpha,L}^{0,\infty} + h_n^0 w_{0,n,\alpha,L}^{0,\infty,h}, w_{0,n,\alpha,L}^{0,\infty,3})]_{h_n^0}.$$

Let us now check that $v_{0,n,\alpha,L}^{0,\text{loc}}$, $w_{0,n,\alpha,L}^{0,\text{loc}}$, $v_{0,n,\alpha,L}^{0,\infty}$ and $w_{0,n,\alpha,L}^{0,\infty}$ satisfy the bounds given in the statement of Theorem 2. We shall only study $v_{0,n,\alpha,L}^{0,\text{loc}}$ and $w_{0,n,\alpha,L}^{0,\text{loc}}$ as the other study is very similar. On the one hand, by translation and scale invariance of \mathcal{B}^0 and using definitions (3.11) and (3.12), we get

$$(3.17) \quad \|v_{0,n,\alpha,L}^{0,\text{loc},h}\|_{\mathcal{B}^0} \leq \sum_{\ell \geq 1} \|\tilde{\phi}_\alpha^{h,\ell}\|_{\mathcal{B}^0} \quad \text{and} \quad \|w_{0,n,\alpha,L}^{0,\text{loc},3}\|_{\mathcal{B}^0} \leq \sum_{\ell \geq 1} \|\phi_\alpha^\ell\|_{\mathcal{B}^0}.$$

According to (3.5) and the Sobolev embedding $\mathcal{B}_{1,q} \hookrightarrow \mathcal{B}^0$, this gives rise to

$$(3.18) \quad \|v_{0,n,\alpha,L}^0\|_{\mathcal{B}^0} + \|w_{0,n,\alpha,L}^{0,\text{loc},3}\|_{\mathcal{B}^0} \leq C \quad \text{uniformly in } \alpha, L, n.$$

Moreover for each given α , the profiles are as smooth as needed, and since in the above sums by construction $\gamma_{n,L}^{\bar{0}} \leq \gamma_n^\ell$, one gets also after an easy computation

$$(3.19) \quad \forall s \in \mathbb{R}, \forall s' \geq 1/2, \quad \|v_{0,n,\alpha,L}^{0,\text{loc}}\|_{\mathcal{B}^{s,s'}} + \|w_{0,n,\alpha,L}^{0,\text{loc},3}\|_{\mathcal{B}^{s,s'}} \leq C(\alpha) \quad \text{uniformly in } n, L.$$

Estimates (3.18) and (3.19) give easily (2.4) and (2.5).

Finally let us estimate $v_{0,n,\alpha,L}^{0,\text{loc},h}(\cdot, 0)$ and $w_{0,n,\alpha,L}^{0,\text{loc},3}(\cdot, 0)$ in $B_{2,1}^0(\mathbb{R}^2)$ and prove (2.6). On the one hand by assumption we know that $u_{0,n} \rightarrow u_0$ in the sense of distributions. On the other hand we can take weak limits in the decomposition of $u_{0,n}$ provided by Proposition 3.2. We recall that by (3.4), if $\varepsilon_n^\ell/\gamma_n^\ell \rightarrow \infty$ then $\phi_\alpha^\ell \equiv r_\alpha^\ell \equiv 0$. Then we notice that clearly

$$\varepsilon_n^\ell \rightarrow 0 \text{ or } \varepsilon_n^\ell \rightarrow \infty \implies \Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell} f \rightharpoonup 0$$

for any value of the sequences γ_n^ℓ, x_n^ℓ and any function f . Moreover

$$\gamma_n^\ell \rightarrow 0 \implies \Lambda_{1, \gamma_n^\ell, x_n^\ell} f \rightharpoonup 0$$

for any sequence of cores x_n^ℓ and any function f , so we are left with the study of profiles such that $\varepsilon_n^\ell \equiv 1$ and $\gamma_n^\ell \rightarrow \infty$. Then we also notice that if $\gamma_n^\ell \rightarrow \infty$, then with Notation (3.10),

$$(3.20) \quad |a_h^\ell| = \infty \implies \Lambda_{1, \gamma_n^\ell, x_n^\ell} f \rightharpoonup 0.$$

In that case, in view of (3.2) and (3.5)

$$\sum_{\ell=1}^L \Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell} \frac{\varepsilon_n^\ell}{\gamma_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 (\phi_\alpha^\ell + r_\alpha^\ell) + \nabla^h \Delta_h^{-1} \partial_3 \psi_n^L \rightharpoonup 0.$$

Consequently for each $L \geq 1$ and each α in $]0, 1[$, we have in view of (3.1), as n goes to infinity

$$(3.21) \quad \begin{aligned} -\psi_n^L - \sum_{\ell \in \Gamma^L(1)} r_\alpha^\ell \left(\cdot - x_{n,h}^\ell, \frac{\cdot - x_{n,3}^\ell}{\gamma_n^\ell} \right) &\rightharpoonup \sum_{\substack{\ell \in \Gamma^L(1) \\ \text{s.t. } a_h^\ell \in \mathbb{R}^2}} \phi_\alpha^\ell (\cdot - a_h^\ell, 0) \\ -\tilde{\psi}_n^{h,L} - \sum_{\ell \in \Gamma^L(1)} \tilde{r}_\alpha^{h,\ell} \left(\cdot - x_{n,h}^\ell, \frac{\cdot - x_{n,3}^\ell}{\gamma_n^\ell} \right) &\rightharpoonup \sum_{\substack{\ell \in \Gamma^L(1) \\ \text{s.t. } a_h^\ell \in \mathbb{R}^2}} \tilde{\phi}_\alpha^{h,\ell} (\cdot - a_h^\ell, 0). \end{aligned}$$

Now let $\eta > 0$ be given. Then thanks to (3.2) and (3.3), there is $L_0 \geq 1$ such that for all $L \geq L_0$ there is $\alpha_0 \leq 1$ (depending on L) such that for all $L \geq L_0$ and $\alpha \leq \alpha_0$, uniformly in $n \geq n(L_0, \eta)$

$$\left\| (\tilde{\psi}_n^{h,L}, \psi_n^L) \right\|_{\mathcal{B}^0} + \left\| \sum_{\ell \in \Gamma^L(1)} (\tilde{r}_\alpha^{h,\ell}, r_\alpha^\ell) \left(\cdot - x_{n,h}^\ell, \frac{\cdot - x_{n,3}^\ell}{\gamma_n^\ell} \right) \right\|_{\mathcal{B}^0} \leq \eta.$$

Using the fact that \mathcal{B}^0 is embedded in $L^\infty(\mathbb{R}; B_{2,1}^0(\mathbb{R}^2))$, we infer from (3.21) that for $L \geq L_0$ and $\alpha \leq \alpha_0$

$$(3.22) \quad \left\| \sum_{\substack{\ell \in \Gamma^L(1) \\ \text{s.t. } a_h^\ell \in \mathbb{R}^2}} \tilde{\phi}_\alpha^{h,\ell} (\cdot - a_h^\ell, 0) \right\|_{B_{2,1}^0(\mathbb{R}^2)} \leq \eta$$

and

$$(3.23) \quad \left\| \sum_{\substack{\ell \in \Gamma^L(1) \\ \text{s.t. } a_h^\ell \in \mathbb{R}^2}} \phi_\alpha^\ell(\cdot - a_h^\ell, 0) \right\|_{B_{2,1}^0(\mathbb{R}^2)} \leq \eta.$$

But by (3.11), we have

$$w_{0,n,\alpha,L}^{0,\text{loc,h}}(\cdot, 0) = \sum_{\substack{\ell \in \Gamma^L(1) \\ a_h^\ell \in \mathbb{R}^2}} \tilde{\phi}_\alpha^{h,\ell} \left(\cdot - x_{n,h}^\ell, -\frac{x_{n,3}^\ell}{\gamma_n^\ell} \right)$$

and by (3.12) we have also

$$w_{0,n,\alpha,L}^{0,\text{loc,3}}(\cdot, 0) = \sum_{\substack{\ell \in \Gamma^L(1) \\ a_h^\ell \in \mathbb{R}^2}} \phi_\alpha^\ell \left(\cdot - x_{n,h}^\ell, -\frac{x_{n,3}^\ell}{\gamma_n^\ell} \right).$$

It follows that we can write for all $L \geq L_0$ and $\alpha \leq \alpha_0$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|v_{0,n,\alpha,L}^{0,\text{loc,h}}(\cdot, 0)\|_{B_{2,1}^0(\mathbb{R}^2)} &\leq \left\| \sum_{\substack{\ell \in \Gamma^L(1) \\ a_h^\ell \in \mathbb{R}^2}} \tilde{\phi}_\alpha^{h,\ell}(\cdot - a_h^\ell, 0) \right\|_{B_{2,1}^0(\mathbb{R}^2)} \\ &\leq \eta \end{aligned}$$

thanks to (3.22). A similar estimate for $w_{0,n,\alpha,L}^{0,\text{loc,3}}(\cdot, 0)$ using (3.23) gives finally

$$(3.24) \quad \lim_{L \rightarrow \infty} \lim_{\alpha \rightarrow 0} \limsup_{n \rightarrow \infty} \left(\|v_{0,n,\alpha,L}^{0,\text{loc,h}}(\cdot, 0)\|_{B_{2,1}^0(\mathbb{R}^2)} + \|w_{0,n,\alpha,L}^{0,\text{loc,3}}(\cdot, 0)\|_{B_{2,1}^0(\mathbb{R}^2)} \right) = 0.$$

The results (2.7) and (2.8) involving the cut-off function θ are simply due to the fact that the profiles are compactly supported.

3.2.2. Construction of the profiles for $\ell \geq 1$. The construction is very similar to the previous one. We start by considering a fixed integer $j \in \{1, \dots, \mathcal{L}(L)\}$. Then we define an integer ℓ_j^- so that, up to a sequence extraction,

$$\gamma_n^{\ell_j^-} = \min_{\ell \in \Gamma^L(\varepsilon_n^{\ell_j^-})} \gamma_n^\ell,$$

where as in (3.7)

$$\Gamma^L(\varepsilon_n^\ell) \stackrel{\text{def}}{=} \left\{ \ell' \in \{1, \dots, L\} / \varepsilon_n^{\ell'} \equiv \varepsilon_n^\ell \quad \text{and} \quad \varepsilon_n^{\ell'} (\gamma_n^{\ell'})^{-1} \rightarrow 0, n \rightarrow \infty \right\}.$$

Notice that necessarily $\varepsilon_n^{\ell_j^-} \not\equiv 1$. Finally we define

$$h_n^j \stackrel{\text{def}}{=} \varepsilon_n^{\ell_j^-} (\gamma_n^{\ell_j^-})^{-1}.$$

By construction we have that $h_n^j \rightarrow 0$ as $n \rightarrow \infty$ (recall that $\varepsilon_n^{\ell_j} \equiv \varepsilon_n^{\ell_j^-}$). Then we define for $j \leq \mathcal{L}(L)$

$$(3.25) \quad v_{n,\alpha,L}^{j,h}(y) \stackrel{\text{def}}{=} \sum_{\ell \in \Gamma^L(\varepsilon_n^{\ell_j^-})} \tilde{\phi}_\alpha^{h,\ell} \left(y_h - \frac{x_{n,h}^\ell}{\varepsilon_n^{\ell_j^-}}, \frac{\varepsilon_n^{\ell_j^-}}{h_n^j \gamma_n^{\ell_j^-}} y_3 - \frac{x_{n,3}^\ell}{\gamma_n^{\ell_j^-}} \right)$$

and

$$w_{n,\alpha,L}^j(y) \stackrel{\text{def}}{=} \sum_{\ell \in \Gamma^L(\varepsilon_n^{\ell_j})} \left(-\frac{\varepsilon_n^{\ell_j}}{h_n^j \gamma_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \right) \left(y_h - \frac{x_{n,h}^\ell}{\varepsilon_n^{\ell_j}}, \frac{\varepsilon_n^{\ell_j}}{h_n^j \gamma_n^\ell} y_3 - \frac{x_{n,3}^\ell}{\gamma_n^\ell} \right)$$

and we choose

$$(3.26) \quad \mathcal{L}(L) < j \leq L \quad \Rightarrow \quad v_{n,\alpha,L}^{j,h} \equiv 0 \quad \text{and} \quad w_{n,\alpha,L}^j \equiv 0.$$

We notice that

$$w_{n,\alpha,L}^{j,h} = -\nabla^h \Delta_h^{-1} \partial_3 w_{n,\alpha,L}^{j,3}.$$

Defining

$$\lambda_n^j \stackrel{\text{def}}{=} \varepsilon_n^{\ell_j},$$

a computation, similar to that giving (3.15) implies directly that

$$(3.27) \quad \sum_{\ell \in \Gamma^L(\varepsilon_n^{\ell_j})} \Lambda_{\varepsilon_n^{\ell_j}, \gamma_n^\ell, x_n^\ell} \left(\tilde{\phi}_\alpha^{h,\ell} - \frac{\lambda_n^j}{\gamma_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \right) \\ = \Lambda_{\lambda_n^j} \left[(v_{n,\alpha,L}^{j,h} + h_n^j w_{n,\alpha,L}^{j,h}, w_{n,\alpha,L}^{j,3}) \right]_{h_n^j}.$$

Notice that since $\varepsilon_n^{\ell_j} \not\equiv 1$ as recalled above, we have that $\lambda_n^j \rightarrow 0$ or ∞ as $n \rightarrow \infty$.

The a priori bounds for the profiles $(v_{n,\alpha,L}^{j,h}, w_{n,\alpha,L}^{j,3})_{1 \leq j \leq L}$ are obtained exactly as in the previous paragraph: let us prove that

$$(3.28) \quad \sum_{j \geq 1} (\|v_{n,\alpha,L}^{j,h}\|_{\mathcal{B}^0} + \|w_{n,\alpha,L}^{j,3}\|_{\mathcal{B}^0}) \leq C, \quad \text{and} \\ \forall s \in \mathbb{R}, \quad \forall s' \geq 1/2, \quad \sum_{j \geq 1} (\|v_{n,\alpha,L}^{j,h}\|_{\mathcal{B}^{s,s'}} + \|w_{n,\alpha,L}^{j,3}\|_{\mathcal{B}^{s,s'}}) \leq C(\alpha).$$

We shall detail the argument for the first inequality only, and in the case of $v_{n,\alpha,L}^{j,h}$ as the study of $w_{n,\alpha,L}^{j,3}$ is similar. We write, using the definition of $v_{n,\alpha,L}^{j,h}$ in (3.25),

$$\sum_{j=1}^L \|v_{n,\alpha,L}^{j,h}\|_{\mathcal{B}^0} = \sum_{j=1}^L \left\| \sum_{\ell \in \Gamma^L(\varepsilon_n^{\ell_j})} \tilde{\phi}_\alpha^{h,\ell} \left(y_h - \frac{x_{n,h}^\ell}{\varepsilon_n^{\ell_j}}, \frac{\varepsilon_n^{\ell_j}}{h_n^j \gamma_n^\ell} y_3 - \frac{x_{n,3}^\ell}{\gamma_n^\ell} \right) \right\|_{\mathcal{B}^0},$$

so by definition of the partition (3.8) and by scale and translation invariance of \mathcal{B}^0 we find thanks to (3.5), that there is a constant C independent of L such that

$$\sum_{j=1}^L \|v_{n,\alpha,L}^{j,h}\|_{\mathcal{B}^0} \leq \sum_{\ell=1}^L \|\tilde{\phi}_\alpha^{h,\ell}\|_{\mathcal{B}^0} \leq C.$$

The result is proved.

3.2.3. Construction of the remainder term. With the notation of Proposition 3.2, let us first define the remainder terms

$$(3.29) \quad \tilde{\rho}_{n,\alpha,L}^{(1),h} \stackrel{\text{def}}{=} -\sum_{\ell=1}^L \frac{\varepsilon_n^\ell}{\gamma_n^\ell} \Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 r_\alpha^\ell - \nabla^h \Delta_h^{-1} \partial_3 \psi_n^L$$

and

$$(3.30) \quad \rho_{n,\alpha,L}^{(2)} \stackrel{\text{def}}{=} \sum_{\ell=1}^L \Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell} (\tilde{r}_\alpha^{h,\ell}, 0) + \sum_{\ell=1}^L \Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell} (0, r_\alpha^\ell) + (\tilde{\psi}_n^{h,L}, \psi_n^L).$$

Observe that by construction, thanks to (3.2) and (3.3) and to the fact that if $r_\alpha^\ell \neq 0$, then $\varepsilon_n^\ell/\gamma_n^\ell$ goes to zero as n goes to infinity, we have

$$(3.31) \quad \begin{aligned} \lim_{L \rightarrow \infty} \lim_{\alpha \rightarrow 0} \limsup_{n \rightarrow \infty} \|\tilde{\rho}_{\alpha,n,L}^{(1),h}\|_{\mathcal{B}^{1,-\frac{1}{2}}} &= 0, \\ \text{and } \lim_{L \rightarrow \infty} \lim_{\alpha \rightarrow 0} \limsup_{n \rightarrow \infty} \|\rho_{\alpha,n,L}^{(2)}\|_{\mathcal{B}^0} &= 0. \end{aligned}$$

Then we notice that for each $\ell \in \mathbb{N}$ and each $\alpha \in]0, 1[$, we have by a direct computation

$$\left\| \Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell}(\tilde{\phi}_\alpha^{h,\ell}, 0) \right\|_{\mathcal{B}^{1,-\frac{1}{2}}} \sim \frac{\gamma_n^\ell}{\varepsilon_n^\ell} \left\| \tilde{\phi}_\alpha^{h,\ell} \right\|_{\mathcal{B}^{1,-\frac{1}{2}}}.$$

We deduce that if $\varepsilon_n^\ell/\gamma_n^\ell \rightarrow \infty$, then $\Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell}(\tilde{\phi}_\alpha^{h,\ell}, 0)$ goes to zero in $\mathcal{B}^{1,-\frac{1}{2}}$ as n goes to infinity, hence so does the sum over $\ell \in \{1, \dots, L\}$. It follows that for each given α in $]0, 1[$ and $L \geq 1$ we may define

$$\rho_{n,\alpha,L}^{(1)} \stackrel{\text{def}}{=} \tilde{\rho}_{n,\alpha,L}^{(1),h} + \sum_{\substack{\ell=1 \\ \varepsilon_n^\ell/\gamma_n^\ell \rightarrow \infty}}^L \Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell}(\tilde{\phi}_\alpha^{h,\ell}, 0)$$

and we have

$$(3.32) \quad \lim_{L \rightarrow \infty} \lim_{\alpha \rightarrow 0} \limsup_{n \rightarrow \infty} \|\rho_{n,\alpha,L}^{(1)}\|_{\mathcal{B}^{1,-\frac{1}{2}}} = 0.$$

Finally, as $\mathcal{D}(\mathbb{R}^3)$ is dense in $\mathcal{B}_{1,q}$, let us choose a family $(u_{0,\alpha})_\alpha$ of functions in $\mathcal{D}(\mathbb{R}^3)$ such that $\|u_0 - u_{0,\alpha}\|_{\mathcal{B}_{1,q}} \leq \alpha$ and let us define

$$(3.33) \quad \rho_{n,\alpha,L} \stackrel{\text{def}}{=} \rho_{n,\alpha,L}^{(1)} + \rho_{n,\alpha,L}^{(2)} + u_0 - u_{0,\alpha}.$$

Inequalities (3.31) and (3.32) give

$$(3.34) \quad \lim_{L \rightarrow \infty} \lim_{\alpha \rightarrow 0} \limsup_{n \rightarrow \infty} \|e^{t\Delta} \rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} = 0.$$

3.2.4. End of the proof of Theorem 2. Let us return to the decomposition given in Proposition 3.2, and use definitions (3.29), (3.30) and (3.33) which imply that

$$u_{0,n} = u_{0,\alpha} + \sum_{\substack{\ell=1 \\ \varepsilon_n^\ell/\gamma_n^\ell \rightarrow 0}}^L \Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell} \left(\tilde{\phi}_\alpha^{h,\ell} - \frac{\varepsilon_n^\ell}{\gamma_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \right) + \rho_{n,\alpha,L}.$$

We recall that for all ℓ in \mathbb{N} , we have $\lim_{n \rightarrow \infty} (\gamma_n^\ell)^{-1} \varepsilon_n^\ell \in \{0, \infty\}$ and in the case where the ratio $\varepsilon_n^\ell/\gamma_n^\ell$ goes to infinity then $\phi_\alpha^\ell \equiv 0$. Next we separate the case when the horizontal scale is one, from the others: with the notation (3.7) we write

$$\begin{aligned} u_{0,n} &= u_{0,\alpha} + \sum_{\ell \in \Gamma^L(1)} \Lambda_{1, \gamma_n^\ell, x_n^\ell} \left(\tilde{\phi}_\alpha^{h,\ell} - \frac{1}{\gamma_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \right) \\ &+ \sum_{\substack{\ell=1 \\ \varepsilon_n^\ell \neq 1 \\ \varepsilon_n^\ell/\gamma_n^\ell \rightarrow 0}}^L \Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell} \left(\tilde{\phi}_\alpha^{h,\ell} - \frac{\varepsilon_n^\ell}{\gamma_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \right) + \rho_{n,\alpha,L}. \end{aligned}$$

With (3.15) this can be written

$$\begin{aligned} u_{0,n} &= u_{0,\alpha} + [(v_{0,n,\alpha,L}^{0,\text{loc,h}} + h_n^0 w_{0,n,\alpha,L}^{0,\text{loc,h}}, w_{0,n,\alpha,L}^{0,\text{loc,3}})]_{h_n^0} + [(v_{0,n,\alpha,L}^{0,\infty,h} + h_n^0 w_{0,n,\alpha,L}^{0,\infty,h}, w_{0,n,\alpha,L}^{0,\infty,3})]_{h_n^0} \\ &+ \sum_{\substack{\ell=1 \\ \varepsilon_n^\ell \neq 1 \\ \varepsilon_n^\ell / \gamma_n^\ell \rightarrow 0}} \Lambda_{\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell} \left(\tilde{\phi}_\alpha^{h,\ell} - \frac{\varepsilon_n^\ell}{\gamma_n^\ell} \nabla^h \Delta_h^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \right) + \rho_{n,\alpha,L}. \end{aligned}$$

Next we use the partition (3.8), so that with notation (3.6) and (3.7),

$$\begin{aligned} u_{0,n} &= u_{0,\alpha} + [(v_{0,n,\alpha,L}^{0,\text{loc,h}} + h_n^0 w_{0,n,\alpha,L}^{0,\text{loc,h}}, w_{0,n,\alpha,L}^{0,\text{loc,3}})]_{h_n^0} + [(v_{0,n,\alpha,L}^{0,\infty,h} + h_n^0 w_{0,n,\alpha,L}^{0,\infty,h}, w_{0,n,\alpha,L}^{0,\infty,3})]_{h_n^0} \\ &+ \sum_{j=1}^{\mathcal{L}(L)} \sum_{\substack{\ell \in \Gamma^L(\varepsilon_n^{\ell_j}) \\ \varepsilon_n^{\ell_j} \neq 1}} \Lambda_{\varepsilon_n^{\ell_j}, \gamma_n^{\ell_j}, x_n^{\ell_j}} \left(\tilde{\phi}_\alpha^{h,\ell_j} - \frac{\varepsilon_n^{\ell_j}}{\gamma_n^{\ell_j}} \nabla^h \Delta_h^{-1} \partial_3 \phi_\alpha^{\ell_j}, \phi_\alpha^{\ell_j} \right) + \rho_{n,\alpha,L}. \end{aligned}$$

Then we finally use the identity (3.27) which gives

$$\begin{aligned} u_{0,n} &= u_{0,\alpha} + [(v_{0,n,\alpha,L}^{0,\text{loc,h}} + h_n^0 w_{0,n,\alpha,L}^{0,\text{loc,h}}, w_{0,n,\alpha,L}^{0,\text{loc,3}})]_{h_n^0} + [(v_{0,n,\alpha,L}^{0,\infty,h} + h_n^0 w_{0,n,\alpha,L}^{0,\infty,h}, w_{0,n,\alpha,L}^{0,\infty,3})]_{h_n^0} \\ &+ \sum_{j=1}^L \Lambda_{\lambda_n^j} [(v_{n,\alpha,L}^{j,h} + h_n^j w_{n,\alpha,L}^{j,h}, w_{n,\alpha,L}^{j,3})]_{h_n^j} + \rho_{n,\alpha,L}. \end{aligned}$$

The end of the proof follows from the estimates (3.18), (3.19), (3.24), (3.28), along with (3.34). Theorem 2 is proved.

4. PROOF OF THEOREMS 3 AND 4

4.1. Proof of Theorem 3. In order to prove that the initial data defined by

$$\Phi_0 \stackrel{\text{def}}{=} [(v_0 - \beta \nabla^h \Delta_h^{-1} \partial_3 w_0^3, w_0^3)]_\beta,$$

with (v_0, w_0^3) satisfying the assumptions of Theorem 3, gives rise to a global smooth solution for small enough β , we look for the solution under the form

$$(4.1) \quad \Phi_\beta = \Phi^{\text{app}} + \psi \quad \text{with} \quad \Phi^{\text{app}} \stackrel{\text{def}}{=} [(v + \beta w^h, w^3)]_\beta$$

where v solves the two-dimensional Navier-Stokes equations

$$(NS2D)_{x_3} \quad \begin{cases} \partial_t v + v \cdot \nabla^h v - \Delta_h v = -\nabla^h p & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ \text{div}_h v = 0 \\ v|_{t=0} = v_0(\cdot, x_3), \end{cases}$$

while w^3 solves the transport-diffusion equation

$$(T_\beta) \quad \begin{cases} \partial_t w^3 + v \cdot \nabla^h w^3 - \Delta_h w^3 - \beta^2 \partial_3^2 w^3 = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ w^3|_{t=0} = w_0^3 \end{cases}$$

and w^h is determined by the divergence free condition on w .

In Paragraph 4.1.1 (resp. 4.1.2), we establish a priori estimates on v (resp. w), and in Paragraph 4.1.3, we achieve the proof of Theorem 3 by studying the perturbed Navier-Stokes equation satisfied by ψ .

4.1.1. *Two dimensional flows with parameter.* The goal of this section is to prove the following proposition on v , the solution of $(\text{NS2D})_{x_3}$. It is a general result on the regularity of the solution of (NS2D) when the initial data depends on a real parameter x_3 , measured in terms of Besov spaces with respect to the variable x_3 .

Proposition 4.1. *Let v_0 be a two-component divergence free vector field depending on the vertical variable x_3 , and belonging to S_μ . Then the unique, global solution v to $(\text{NS2D})_{x_3}$ belongs to \mathcal{A}^0 and satisfies the following estimate:*

$$(4.2) \quad \|v\|_{\mathcal{A}^0} \leq \mathcal{T}_1(\|v_0\|_{\mathcal{B}^0}).$$

Moreover, for all (s, s') in D_μ , we have

$$(4.3) \quad \forall r \in [1, \infty], \quad \|v\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}^{s+\frac{2}{r}, s'})} \leq \mathcal{T}_2(\|v_0\|_{S_\mu}).$$

Proof. The proof of Proposition 4.1 is done in three steps. First, we deduce from the classical energy estimate for the two dimensional Navier-Stokes system, a stability result in the spaces $L^r(\mathbb{R}^+; H^{s+\frac{2}{r}}(\mathbb{R}^2))^2$ with r in $[2, \infty]$ and s in $] -1, 1[$. This is the purpose of Lemma 4.2, the proof of which uses essentially energy estimates together with paraproduct laws. Then we have to translate the stability result of Lemma 4.2 in terms of Besov spaces with respect to the third variable, seen before simply as a parameter. This is the object of Lemma 4.3, the proof of which relies on the equivalence of two definitions of Besov spaces with regularity index in $]0, 1[$: the first one involving the dyadic decomposition of the frequency space, and the other one consisting in estimating integrals in physical space. Finally, invoking the Gronwall lemma and product laws we conclude the proof of the proposition.

Step 1: 2D-stability result. Let us start by proving the following lemma.

Lemma 4.2. *For any compact set I included in $] -1, 1[$, a constant C exists such that, for any r in $[2, \infty]$ and any s in I , we have for any two solutions v_1 and v_2 of the two-dimensional Navier-Stokes equations*

$$(4.4) \quad \|v_1 - v_2\|_{L^r(\mathbb{R}^+; H^{s+\frac{2}{r}}(\mathbb{R}^2))} \lesssim \|v_1(0) - v_2(0)\|_{H^s(\mathbb{R}^2)} E_{12}(0),$$

where we define

$$E_{12}(0) \stackrel{\text{def}}{=} \exp C(\|v_1(0)\|_{L^2}^2 + \|v_2(0)\|_{L^2}^2).$$

Proof. Defining $v_{12}(t) \stackrel{\text{def}}{=} v_1(t) - v_2(t)$, we find that

$$(4.5) \quad \partial_t v_{12} + v_2 \cdot \nabla^h v_{12} - \Delta_h v_{12} = -v_{12} \cdot \nabla^h v_1 - \nabla^h p.$$

Thus taking the H^s scalar product with v_{12} , we get thanks to the divergence free condition

$$\frac{1}{2} \frac{d}{dt} \|v_{12}(t)\|_{H^s}^2 + \|\nabla^h v_{12}(t)\|_{H^s}^2 = -(v_2(t) \cdot \nabla^h v_{12}(t) | v_{12}(t))_{H^s} - (v_{12}(t) \cdot \nabla^h v_1(t) | v_{12}(t))_{H^s}.$$

Whence, by time integration we get

$$\begin{aligned} \|v_{12}(t)\|_{H^s}^2 + 2 \int_0^t \|\nabla^h v_{12}(t')\|_{H^s}^2 dt' &= \|v_{12}(0)\|_{H^s}^2 - 2 \int_0^t (v_2(t') \cdot \nabla^h v_{12}(t') | v_{12}(t'))_{H^s} dt' \\ &\quad - 2 \int_0^t (v_{12}(t') \cdot \nabla^h v_1(t') | v_{12}(t'))_{H^s} dt'. \end{aligned}$$

Now making use of the following estimate proved in [12, Lemma 1.1]:

$$(4.6) \quad (v \cdot \nabla^h a | a)_{H^s} \lesssim \|\nabla^h v\|_{L^2} \|a\|_{H^s} \|\nabla^h a\|_{H^s},$$

²Here $H^{s+\frac{2}{r}}(\mathbb{R}^2)$ denotes the usual homogeneous Sobolev space.

available uniformly with respect to s in any compact set of $] - 2, 1[$, we deduce that there is a positive constant C such that for any s in I , we have

$$(4.7) \quad 2 \left| \int_0^t (v_{12}(t') \cdot \nabla^h v_{12}(t') |v_{12}(t')|)_{H^s} dt' \right| \\ \leq \frac{1}{2} \int_0^t \|\nabla^h v_{12}(t')\|_{H^s}^2 dt' + \frac{C^2}{2} \int_0^t \|v_{12}(t')\|_{H^s}^2 \|\nabla^h v_{12}(t')\|_{L^2}^2 dt'.$$

Noticing that

$$\int_0^t (v_{12}(t') \cdot \nabla^h v_1(t') |v_{12}(t')|)_{H^s} dt' \leq \int_0^t \|\nabla^h v_{12}(t')\|_{H^s} \|v_{12}(t') \cdot \nabla^h v_1(t')\|_{H^{s-1}} dt',$$

we deduce by Cauchy-Schwarz inequality and product laws in Sobolev spaces on \mathbb{R}^2 that for s in I ,

$$(4.8) \quad 2 \left| \int_0^t (v_{12}(t') \cdot \nabla^h v_1(t') |v_{12}(t')|)_{H^s} dt' \right| \\ \leq \frac{1}{2} \int_0^t \|\nabla^h v_{12}(t')\|_{H^s}^2 dt' + \frac{C^2}{2} \int_0^t \|v_{12}(t')\|_{H^s}^2 \|\nabla^h v_1(t')\|_{L^2}^2 dt'.$$

Combining (4.7) and (4.8), we get for s in I ,

$$\|v_{12}(t)\|_{H^s}^2 + \int_0^t \|\nabla^h v_{12}(t')\|_{H^s}^2 dt' \lesssim \|v_{12}(0)\|_{H^s}^2 \\ + \int_0^t \|v_{12}(t')\|_{H^s}^2 (\|\nabla^h v_1(t')\|_{L^2}^2 + \|\nabla^h v_2(t')\|_{L^2}^2) dt'.$$

Gronwall's lemma implies that there exists a positive constant C such that

$$\|v_{12}(t)\|_{H^s}^2 + \int_0^t \|\nabla^h v_{12}(t')\|_{H^s}^2 dt' \lesssim \|v_{12}(0)\|_{H^s}^2 \exp C \int_0^t (\|\nabla^h v_1(t')\|_{L^2}^2 + \|\nabla^h v_2(t')\|_{L^2}^2) dt'.$$

But for any i in $\{1, 2\}$, we have by the L^2 energy estimate (1.2)

$$(4.9) \quad \int_0^t \|\nabla^h v_i(t')\|_{L^2}^2 dt' \leq \frac{1}{2} \|v_i(0)\|_{L^2}^2.$$

Consequently for s in I ,

$$\|v_{12}(t)\|_{H^s}^2 + \int_0^t \|\nabla^h v_{12}(t')\|_{H^s}^2 dt' \lesssim \|v_{12}(0)\|_{H^s}^2 E_{12}(0),$$

which leads to the result by interpolation. \square

Step 2: propagation of vertical regularity. Thanks to Lemma 4.2, we can propagate vertical regularity as stated in the following result.

Lemma 4.3. *For any compact set I included in $] - 1, 1[$, a constant C exists such that, for any r in $[2, \infty]$ and any s in I , we have for any solution v to $(\text{NS2D})_{x_3}$,*

$$\|v\|_{L^r(\mathbb{R}^+; L_v^\infty(H_h^{s+\frac{2}{r}}))} \lesssim \|v_0\|_{B^s} E(0) \quad \text{with} \quad E(0) \stackrel{\text{def}}{=} \exp(C \|v(0)\|_{L_v^\infty L_h^2}^2).$$

Proof. As mentioned above, the proof of Lemma 4.3 uses crucially the characterization of Besov spaces via differences in physical space, namely that for any Banach space X of distributions one has (see for instance Theorem 2.36 of [2])

$$(4.10) \quad \|(2^{\frac{j}{2}} \|\Delta_j^v u\|_{L_v^2(X)})_j\|_{\ell^1(\mathbb{Z})} \sim \int_{\mathbb{R}} \frac{\|u - (\tau_{-z} u)\|_{L_v^2(X)} dz}{|z|^{\frac{1}{2}} |z|}$$

where the translation operator τ_{-z} is defined by

$$(\tau_{-z}u)(t, x_h, x_3) \stackrel{\text{def}}{=} u(t, x_h, x_3 + z).$$

Lemma 4.2 asserts that, for any r in $[2, \infty]$, any s in I and any couple (x_3, z) in \mathbb{R}^2 , the solution v to $(\text{NS2D})_{x_3}$ satisfies

$$\|v - \tau_{-z}v\|_{Y_r^s} \lesssim \|v_0 - \tau_{-z}v_0\|_{H_h^s} E(0) \quad \text{with} \quad Y_r^s \stackrel{\text{def}}{=} L^r(\mathbb{R}^+; H_h^{s+\frac{2}{r}}).$$

Taking the L^2 norm of the above inequality with respect to the x_3 variable and then the L^1 norm with respect to the measure $|z|^{-\frac{3}{2}} dz$ gives

$$(4.11) \quad \int_{\mathbb{R}} \frac{\|v - \tau_{-z}v\|_{L_v^2(Y_r^s)} dz}{|z|^{\frac{1}{2}} |z|} \lesssim \int_{\mathbb{R}} \frac{\|v_0 - \tau_{-z}v_0\|_{L_v^2(H_h^s)} dz}{|z|^{\frac{1}{2}} |z|} E(0).$$

Now, making use of the characterization (4.10) with $X = Y_r^s$, we find that

$$\int_{\mathbb{R}} \frac{\|v - \tau_{-z}v\|_{L_v^2(Y_r^s)} dz}{|z|^{\frac{1}{2}} |z|} \sim \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \left\| \left\| (2^{k(s+\frac{2}{r})} \Delta_j^v \Delta_k^h v(t, \cdot, z))_k \right\|_{L^r(\mathbb{R}^+; \ell^2(\mathbb{Z}; L_h^2))} \right\|_{L_v^2}.$$

Similarly we have

$$\int_{\mathbb{R}} \frac{\|v_0 - \tau_{-z}v_0\|_{L_v^2(H_h^s)} dz}{|z|^{\frac{1}{2}} |z|} \sim \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \left\| \left\| (2^{ks} \Delta_j^v \Delta_k^h v_0)_k \right\|_{\ell^2(\mathbb{Z}; L_h^2)} \right\|_{\ell^2(\mathbb{Z}; L_v^2)}.$$

Thus by the embedding from $\ell^1(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$, we get

$$\int_{\mathbb{R}} \frac{\|v_0 - \tau_{-z}v_0\|_{L_v^2(H_h^s)} dz}{|z|^{\frac{1}{2}} |z|} \lesssim \sum_{(j,k) \in \mathbb{Z}^2} 2^{\frac{j}{2}} 2^{ks} \|\Delta_j^v \Delta_k^h v_0\|_{L^2(\mathbb{R}^3)}.$$

This implies that Estimate (4.11) also writes

$$\sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \left\| \left\| (2^{k(s+\frac{2}{r})} \Delta_j^v \Delta_k^h v(t, \cdot, z))_k \right\|_{L^r(\mathbb{R}^+; \ell^2(\mathbb{Z}; L_h^2))} \right\|_{L_v^2} \lesssim \|v_0\|_{\mathcal{B}^s} E(0).$$

As $r \geq 2$, Minkowski's inequality implies that

$$\sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \left\| \left\| (2^{k(s+\frac{2}{r})} \Delta_j^v \Delta_k^h v(t, \cdot))_k \right\|_{\ell^2(\mathbb{Z}; L^2(\mathbb{R}^3))} \right\|_{L^r(\mathbb{R}^+)} \lesssim \|v_0\|_{\mathcal{B}^s} E(0).$$

Bernstein inequalities (1.6) and (1.7) ensure that

$$\|\Delta_j^v \Delta_k^h v(t, \cdot)\|_{L_v^\infty(L_h^2)} \lesssim 2^{\frac{j}{2}} \|\Delta_k^h v(t, \cdot)\|_{L^2(\mathbb{R}^3)},$$

which gives rise to

$$\left\| \left\| (2^{k(s+\frac{2}{r})} \|\Delta_k^h v\|_{L_v^\infty(L_h^2)})_k \right\|_{\ell^2(\mathbb{Z})} \right\|_{L^r(\mathbb{R}^+)} \lesssim \|v_0\|_{\mathcal{B}^s} E(0).$$

Permuting the ℓ^2 norm and the L_v^∞ norm thanks to Minkowski's inequality again, achieves the proof of the lemma. \square

Step 3: end of the proof of Proposition 4.1. Our aim is to establish (4.3) for all (s, s') in D_μ . Let us start by proving the following inequality: for any v solving $(\text{NS2D})_{x_3}$, for any r in $[4, \infty]$, any s in $]-\frac{1}{2}, \frac{1}{2}[$ and any positive s' ,

$$(4.12) \quad \|v\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}^{s+\frac{2}{r}, s'})} \lesssim \|v_0\|_{\mathcal{B}^{s, s'}} \exp\left(\int_0^\infty C(\|v(t)\|_{L_v^\infty(L_h^4)}^4 + \|v(t)\|_{L_v^\infty(H_h^1)}^2) dt\right).$$

For that purpose, let us introduce, for any nonnegative λ , the following notation: for any function F we define

$$F_\lambda(t) \stackrel{\text{def}}{=} F(t) \exp\left(-\lambda \int_0^t \phi(t') dt'\right) \quad \text{with} \quad \phi(t) \stackrel{\text{def}}{=} \|v(t)\|_{L^\infty(L_h^4)}^4 + \|v(t)\|_{L^\infty(H_h^1)}^2.$$

Combining Lemma 4.3 with the Sobolev embedding of $H^{\frac{1}{2}}(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$, we find that

$$(4.13) \quad \int_0^t \phi(t') dt' \lesssim E(0)(\|v_0\|_{\mathcal{B}^0}^2 + \|v_0\|_{\mathcal{B}^0}^4).$$

Now making use of the Duhamel formula and the action of the heat flow (see for instance Proposition B.2 in [4]), we infer that

$$(4.14) \quad \begin{aligned} \|\Delta_j^v \Delta_k^h v_\lambda(t)\|_{L^2} &\leq C e^{-c2^{2k}t} \|\Delta_j^v \Delta_k^h v_0\|_{L^2} \\ &+ C 2^k \int_0^t \exp\left(-c(t-t')2^{2k} - \lambda \int_{t'}^t \phi(t'') dt''\right) \|\Delta_j^v \Delta_k^h (v \otimes v)_\lambda(t')\|_{L^2} dt'. \end{aligned}$$

Recall that $(v \otimes v)_\lambda = v \otimes v_\lambda$. Now to estimate the term $\|\Delta_j^v \Delta_k^h (v \otimes v)_\lambda(t')\|_{L^2}$, we make use of the anisotropic version of Bony's paraproduct decomposition (one can consult [2] and [43] for an introduction to anisotropic Littlewood-Paley theory), writing

$$(4.15) \quad \begin{aligned} ab &= \sum_{\ell=1}^4 T^\ell(a, b) \quad \text{with} \\ T^1(a, b) &= \sum_{j,k} S_j^v S_k^h a \Delta_j^v \Delta_k^h b, \\ T^2(a, b) &= \sum_{j,k} S_j^v \Delta_k^h a \Delta_j^v S_{k+1}^h b, \\ T^3(a, b) &= \sum_{j,k} \Delta_j^v S_k^h a S_{j+1}^v \Delta_k^h b, \\ T^4(a, b) &= \sum_{j,k} \Delta_j^v \Delta_k^h a S_{j+1}^v S_{k+1}^h b. \end{aligned}$$

In light of Bernstein inequality (1.6), we have

$$\|\Delta_j^v \Delta_k^h T^1(v(t), v_\lambda(t))\|_{L^2} \lesssim 2^{\frac{k}{2}} \|\Delta_j^v \Delta_k^h T^1(v(t), v_\lambda(t))\|_{L^{\frac{4}{3}}(L_h^{\frac{4}{3}})},$$

which, in view of (4.15), Hölder's inequalities and the action of the horizontal and vertical truncations on Lebesgue spaces, ensures the existence of some fixed nonzero integer N_0 such that

$$\begin{aligned} \|\Delta_j^v \Delta_k^h T^1(v(t), v_\lambda(t))\|_{L^2} &\lesssim 2^{\frac{k}{2}} \sum_{\substack{j' \geq j - N_0 \\ k' \geq k - N_0}} \|S_{j'}^v S_{k'}^h v(t)\|_{L^\infty(L_h^4)} \|\Delta_{j'}^v \Delta_{k'}^h v_\lambda(t)\|_{L^2} \\ &\lesssim 2^{\frac{k}{2}} \|v(t)\|_{L^\infty(L_h^4)} \sum_{\substack{j' \geq j - N_0 \\ k' \geq k - N_0}} \|\Delta_{j'}^v \Delta_{k'}^h v_\lambda(t)\|_{L^2}. \end{aligned}$$

According to the definition of $\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})$, we get

$$\begin{aligned} 2^{js'} 2^{ks} \|\Delta_j^v \Delta_k^h T^1(v(t), v_\lambda(t))\|_{L^2} \\ \lesssim \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})} \|v(t)\|_{L^\infty(L_h^4)} \sum_{\substack{j' \geq j - N_0 \\ k' \geq k - N_0}} 2^{-(j'-j)s'} 2^{-(k'-k)(s+\frac{1}{2})} \tilde{f}_{j', k'}(t). \end{aligned}$$

where $\tilde{f}_{j',k'}(t)$, defined by

$$\tilde{f}_{j',k'}(t) \stackrel{\text{def}}{=} \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})}^{-1} 2^{k'(s+\frac{1}{2})} 2^{j's'} \|\Delta_{j'}^v \Delta_{k'}^h v_\lambda(t)\|_{L^2},$$

is on the sphere of $\ell^1(\mathbb{Z}^2; L^4(\mathbb{R}^+))$.

Since $s > -\frac{1}{2}$ and $s' > 0$, it follows by Young's inequality on series, that

$$2^{j's'} 2^{ks} \|\Delta_{j'}^v \Delta_k^h T^1(v(t), v_\lambda(t))\|_{L^2} \lesssim \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})} \|v(t)\|_{L^\infty(L_h^4)} f_{j,k}(t)$$

where $f_{j,k}(t)$ is on the sphere of $\ell^1(\mathbb{Z}^2; L^4(\mathbb{R}^+))$.

As by definition $\phi(t)$ is greater than $\|v(t)\|_{L^\infty(L_h^4)}^4$, we infer that

$$\begin{aligned} \mathcal{T}_{j,k,\lambda}^1(t) &\stackrel{\text{def}}{=} 2^k 2^{j's'} 2^{ks} \int_0^t \exp\left(-c(t-t')2^{2k} - \lambda \int_{t'}^t \phi(t'') dt''\right) \\ &\quad \times \|\Delta_{j'}^v \Delta_k^h T^1(v(t'), v_\lambda(t'))\|_{L^2} dt' \\ (4.16) \quad &\lesssim \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})} \\ &\quad \times 2^k \int_0^t \exp\left(-c(t-t')2^{2k} - \lambda \int_{t'}^t \phi(t'') dt''\right) \phi^{\frac{1}{4}}(t') f_{j,k}(t') dt'. \end{aligned}$$

By Hölder's inequality, this leads to

$$\begin{aligned} \mathcal{T}_{j,k,\lambda}^1(t) &\lesssim \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})} \left(\int_0^t e^{-c(t-t')2^{2k}} f_{j,k}^4(t') dt' \right)^{\frac{1}{4}} \\ &\quad \times 2^k \left(\int_0^t \exp\left(-c(t-t')2^{2k} - \frac{4}{3}\lambda \int_{t'}^t \phi(t'') dt''\right) \phi(t')^{\frac{1}{3}} dt' \right)^{\frac{3}{4}}. \end{aligned}$$

Finally applying Hölder's inequality in the last term of the above inequality, we get

$$(4.17) \quad \mathcal{T}_{j,k,\lambda}^1(t) \lesssim \frac{1}{\lambda^{\frac{1}{4}}} \left(\int_0^t e^{-c(t-t')2^{2k}} f_{j,k}^4(t') dt' \right)^{\frac{1}{4}} \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})}.$$

Now let us study the term with T^2 . Using again that the support of the Fourier transform of the product of two functions is included in the sum of the two supports, let us write that

$$\|\Delta_j^v \Delta_k^h T^2(v(t), v_\lambda(t))\|_{L^2} \lesssim \sum_{\substack{j' \geq j - N_0 \\ k' \geq k - N_0}} \|S_{j'}^v \Delta_{k'}^h v(t)\|_{L^\infty(L_h^2)} \|\Delta_{j'}^v S_{k'+1}^h v_\lambda(t)\|_{L^2(L_h^\infty)}.$$

Combining Bernstein inequality (1.6) with the definition of the function ϕ , we get

$$(4.18) \quad \|S_{j'}^v \Delta_{k'}^h v(t)\|_{L^\infty(L_h^2)} \lesssim 2^{-k'} \|v(t)\|_{L^\infty(H_h^1)} \lesssim 2^{-k'} \phi^{\frac{1}{2}}(t).$$

Now let us observe that using again the Bernstein inequality, we have

$$\begin{aligned} \|\Delta_{j'}^v S_{k'+1}^h v_\lambda(t)\|_{L^2(L_h^\infty)} &\lesssim \sum_{k'' \leq k'} \|\Delta_{j'}^v \Delta_{k''}^h v_\lambda(t)\|_{L^2(L_h^\infty)} \\ &\lesssim \sum_{k'' \leq k'} 2^{k''} \|\Delta_{j'}^v \Delta_{k''}^h v_\lambda(t)\|_{L^2}. \end{aligned}$$

By definition of the $\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})$ norm, we have

$$2^{j's'} 2^{k'(s-\frac{1}{2})} \|\Delta_{j'}^v S_{k'+1}^h v_\lambda(t)\|_{L^2(L_h^\infty)} \lesssim \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})} \sum_{k'' \leq k'} 2^{(k'-k'')(s-\frac{1}{2})} \tilde{f}_{j',k''}(t)$$

where $\underline{f}_{j',k''}(t)$, on the sphere of $\ell^1(\mathbb{Z}^2; L^4(\mathbb{R}^+))$, is defined by

$$\underline{f}_{j',k''}(t) \stackrel{\text{def}}{=} \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})}^{-1} 2^{j's'} 2^{k''(s+\frac{1}{2})} \|\Delta_{j'}^v \Delta_{k''}^h v_\lambda(t)\|_{L^2}.$$

Since $s < \frac{1}{2}$, this ensures by Young's inequality that

$$\|\Delta_{j'}^v S_{k'+1}^h v_\lambda(t)\|_{L_v^2(L_h^\infty)} \lesssim 2^{-j's'} 2^{-k'(s-\frac{1}{2})} \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})} \tilde{f}_{j',k'}(t)$$

where $\tilde{f}_{j',k'}(t)$ is on the sphere of $\ell^1(\mathbb{Z}^2; L^4(\mathbb{R}^+))$. Together with Inequality (4.18), this gives

$$2^{j's'} 2^{k(s+\frac{1}{2})} \|\Delta_j^v \Delta_k^h T^2(v(t), v_\lambda(t))\|_{L^2} \lesssim \phi(t)^{\frac{1}{2}} \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})} f_{j,k}(t),$$

where $f_{j,k}(t)$ is on the sphere of $\ell^1(\mathbb{Z}^2; L^4(\mathbb{R}^+))$. We deduce that

$$\begin{aligned} \mathcal{T}_{j,k,\lambda}^2(t) &\stackrel{\text{def}}{=} 2^k 2^{j's'} 2^{ks} \int_0^t \exp(-c(t-t')2^{2k} - \lambda \int_{t'}^t \phi(t'') dt'') \\ &\quad \times \|\Delta_j^v \Delta_k^h T^2(v(t'), v_\lambda(t'))\|_{L^2} dt' \\ (4.19) \quad &\lesssim \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})} \\ &\quad \times 2^{\frac{k}{2}} \int_0^t \exp(-c(t-t')2^{2k} - \lambda \int_{t'}^t \phi(t'') dt'') \phi(t')^{\frac{1}{2}} f_{j,k}(t') dt'. \end{aligned}$$

Using Hölder's inequality twice, we get

$$\begin{aligned} \mathcal{T}_{j,k,\lambda}^2(t) &\lesssim \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})} \left(\int_0^t e^{-c(t-t')2^{2k}} f_{j,k}^4(t') dt' \right)^{\frac{1}{4}} \\ &\quad \times 2^{\frac{k}{2}} \left(\int_0^t \exp(-c(t-t')2^{2k} - \lambda \int_{t'}^t \phi(t'') dt'') \phi(t')^{\frac{2}{3}} dt' \right)^{\frac{3}{4}} \\ (4.20) \quad &\lesssim \frac{1}{\lambda^{\frac{1}{2}}} \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})} \left(\int_0^t e^{-c(t-t')2^{2k}} f_{j,k}^4(t') dt' \right)^{\frac{1}{4}}. \end{aligned}$$

As T^3 is estimated like T^1 and T^4 is estimated like T^2 , this implies finally that

$$\begin{aligned} 2^{j's'} 2^{ks} \|\Delta_j^v \Delta_k^h v_\lambda(t)\|_{L^2} &\lesssim 2^{j's'} 2^{ks} e^{-c2^{2k}t} \|\Delta_j^v \Delta_k^h v_0\|_{L^2} \\ &\quad + \left(\int_0^t e^{-c(t-t')2^{2k}} f_{j,k}^4(t') dt' \right)^{\frac{1}{4}} \left(\frac{1}{\lambda^{\frac{1}{4}}} + \frac{1}{\lambda^{\frac{1}{2}}} \right) \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})}. \end{aligned}$$

As we have

$$\begin{aligned} \left(\int_0^\infty \left(\int_0^t e^{-c(t-t')2^{2k}} f_{j,k}^4(t') dt' \right)^{\frac{1}{4} \times 4} dt \right)^{\frac{1}{4}} &= c^{-1} d_{j,k} 2^{-\frac{k}{2}} \\ \text{and } \sup_{t \in \mathbb{R}^+} \left(\int_0^t e^{-c(t-t')2^{2k}} f_{j,k}^4(t') dt' \right)^{\frac{1}{4}} &= d_{j,k}, \quad \text{with } d_{j,k} \in \ell^1(\mathbb{Z}^2), \end{aligned}$$

we infer that

$$\begin{aligned} 2^{j's'} 2^{ks} (\|\Delta_j^v \Delta_k^h v_\lambda\|_{L^\infty(\mathbb{R}^+; L^2)} + 2^{\frac{k}{2}} \|\Delta_j^v \Delta_k^h v_\lambda\|_{L^4(\mathbb{R}^+; L^2)}) \\ \lesssim 2^{j's'} 2^{ks} \|\Delta_j^v \Delta_k^h v_0\|_{L^2} + d_{j,k} \left(\frac{1}{\lambda^{\frac{1}{4}}} + \frac{1}{\lambda^{\frac{1}{2}}} \right) \|v_\lambda\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})}. \end{aligned}$$

This ends the proof of (4.12) by taking the sum over j and k and choosing λ large enough.

Now to show that Estimate (4.12) remains available for $r = 2$, we start from Formula (4.14) with $\lambda = 0$. Applying again anisotropic paraproduct decomposition, we find by arguments similar to those conducted above

$$\begin{aligned} 2^{js'} 2^{k(s+1)} \|\Delta_j^v \Delta_k^h v(t)\|_{L^2} &\lesssim 2^{js'} 2^{k(s+1)} e^{-c2^{2k}t} \|\Delta_j^v \Delta_k^h v_0\|_{L^2} \\ &\quad + 2^{2k} \|v\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{s+\frac{1}{2}, s'})} \int_0^t e^{-c(t-t')2^{2k}} ((g_{j,k}(t') + 2^{-\frac{k}{2}} h_{j,k}(t')) dt', \end{aligned}$$

where $g_{j,k}$ (resp. $h_{j,k}$) are in $\ell^1(\mathbb{Z}^2; L^2(\mathbb{R}^+))$ (resp. $\ell^1(\mathbb{Z}^2; L^{\frac{4}{3}}(\mathbb{R}^+))$), with

$$\sum_{(j,k) \in \mathbb{Z}^2} \|g_{j,k}\|_{L^2(\mathbb{R}^+)} \lesssim \|\phi\|_{L^1}^{\frac{1}{4}} \quad \text{and} \quad \sum_{(j,k) \in \mathbb{Z}^2} \|h_{j,k}\|_{L^{\frac{4}{3}}(\mathbb{R}^+)} \lesssim \|\phi\|_{L^1}^{\frac{1}{2}}.$$

Laws of convolution in the time variable, summation over j and k and (4.12) imply that

$$\|v\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{s+1, s'})} \lesssim \|v_0\|_{\mathcal{B}^{s, s'}} \exp\left(C \int_0^\infty \phi(t) dt\right).$$

This implies by interpolation in view of (4.12) that for all r in $[2, \infty]$, all s in $] -\frac{1}{2}, \frac{1}{2}[$ and all positive s'

$$(4.21) \quad \|v\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}^{s+\frac{2}{r}, s'})} \lesssim \|v_0\|_{\mathcal{B}^{s, s'}} \exp\left(C \int_0^\infty \phi(t) dt\right),$$

which in view of (4.13) ensures Inequality (4.2) and achieves the proof of Estimate (4.3) in the case when s belongs to $] -\frac{1}{2}, \frac{1}{2}[$.

To conclude the proof of the proposition, it remains to complete the range of indexes. Let us first double the interval on the index s , by proving that for any s in $] -1, 1[$, any $s' \geq 1/2$ and any r in $[2, \infty]$ we have

$$(4.22) \quad \|v\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}^{s+\frac{2}{r}, s'})} \lesssim \|v_0\|_{\mathcal{B}^{s, s'}} + \|v_0\|_{\mathcal{B}^{\frac{s}{2}, s'}} \|v_0\|_{\mathcal{B}^{\frac{s}{2}}} \exp(C \|v_0\|_{\mathcal{B}^0} E_0).$$

Anisotropic product laws (see for instance Appendix B in [4]) ensure that for any s in $] -1, 1[$ and any $s' \geq 1/2$, we have

$$\|v(t) \otimes v(t)\|_{\mathcal{B}^{s, s'}} \lesssim \|v(t)\|_{\mathcal{B}^{\frac{s+1}{2}}} \|v(t)\|_{\mathcal{B}^{\frac{s+1}{2}, s'}}.$$

According to Formula (4.14) and the smoothing effect of the horizontal heat flow, we find that, for any s belonging to $] -1, 1[$, any $s' \geq 1/2$ and any r in $[2, \infty]$,

$$\begin{aligned} \|v\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}^{s+\frac{2}{r}, s'})} &\lesssim \|v_0\|_{\mathcal{B}^{s, s'}} + \|v \otimes v\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{s, s'})} \\ &\lesssim \|v_0\|_{\mathcal{B}^{s, s'}} + \|v\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{\frac{s+1}{2}})} \|v\|_{\tilde{L}^4(\mathbb{R}^+; \mathcal{B}^{\frac{s+1}{2}, s'})}. \end{aligned}$$

Finally Inequality (4.12) ensures that for any s in $] -1, 1[$, any $s' \geq 1/2$ and any r in $[2, \infty]$,

$$(4.23) \quad \|v\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}^{s+\frac{2}{r}, s'})} \lesssim \|v_0\|_{\mathcal{B}^{s, s'}} + \|v_0\|_{\mathcal{B}^{\frac{s}{2}}} \|v_0\|_{\mathcal{B}^{\frac{s}{2}, s'}} \exp(C \|v_0\|_{\mathcal{B}^0} E(0)),$$

which ends the proof of Inequality (4.22), and thus for (4.3) when r in $[2, \infty]$ and s in $] -1, 1[$.

Let us now treat the case when s belongs $] -2, 0]$ and $s' \geq 1/2$. Again by anisotropic product laws, we have

$$\|v(t) \otimes v(t)\|_{\mathcal{B}^{s+1, s'}} \lesssim \|v(t)\|_{\mathcal{B}^{\frac{s}{2}+1}} \|v(t)\|_{\mathcal{B}^{\frac{s}{2}+1, s'}},$$

which implies that

$$\|v \otimes v\|_{L^1(\mathbb{R}^+; \mathcal{B}^{s+1, s'})} \lesssim \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^{\frac{s}{2}+1})} \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^{\frac{s}{2}+1, s'})}.$$

The smoothing effect of the heat flow gives then, for any r in $[1, \infty]$, any s in $] - 2, 0]$ and any $s' \geq 1/2$

$$\|v\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}^{s+\frac{2}{r}, s'})} \lesssim \|v_0\|_{\mathcal{B}^{s, s'}} + \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^{\frac{s}{2}+1})} \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^{\frac{s}{2}+1, s'})}.$$

Inequality (4.23) implies that, for any r in $[1, \infty]$, any s in $] - 2, 0]$ and any $s' \geq 1/2$

$$\begin{aligned} \|v\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}^{s+\frac{2}{r}, s'})} &\lesssim \|v_0\|_{\mathcal{B}^{s, s'}} + \|v_0\|_{\mathcal{B}^{\frac{s}{2}}} (\|v_0\|_{\mathcal{B}^{\frac{s}{2}, s'}} + \|v_0\|_{\mathcal{B}^{\frac{s}{4}}} \|v_0\|_{\mathcal{B}^{\frac{s}{4}, s'}} \exp(C\|v_0\|_{\mathcal{B}^0} E_0)) \\ &\quad + \|v_0\|_{\mathcal{B}^{\frac{s}{4}}}^2 (\|v_0\|_{\mathcal{B}^{\frac{s}{2}, s'}} + \|v_0\|_{\mathcal{B}^{\frac{s}{4}}} \|v_0\|_{\mathcal{B}^{\frac{s}{4}, s'}} \exp(C\|v_0\|_{\mathcal{B}^0} E_0)). \end{aligned}$$

This concludes the proof of Estimate (4.3), and thus achieves the proof of Proposition 4.1. \square

4.1.2. *Propagation of regularity by a transport diffusion equation.* Now let us estimate the norm of the function w^3 defined as the solution of (T_β) defined page 22. This is described in the following proposition.

Proposition 4.4. *Let v_0 and v be as in Proposition 4.1. For any non negative real number β , let us consider w^3 the solution of*

$$(T_\beta) \quad \begin{cases} \partial_t w^3 + v \cdot \nabla^h w^3 - \Delta_h w^3 - \beta^2 \partial_z^2 w^3 = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ w^3|_{t=0} = w_0^3. \end{cases}$$

Then w^3 satisfies the following estimates where all the constants are independent of β :

$$(4.24) \quad \|w^3\|_{\mathcal{A}^0} \lesssim \|w_0^3\|_{\mathcal{B}^0} \exp(\mathcal{T}_1(\|v_0\|_{\mathcal{B}^0})),$$

and for any s in $[-2 + \mu, 0]$ and any $s' \geq 1/2$, we have

$$(4.25) \quad \|w^3\|_{\mathcal{A}^{s, s'}} \lesssim (\|w_0^3\|_{\mathcal{B}^{s, s'}} + \|w_0^3\|_{\mathcal{B}^0} \mathcal{T}_2(\|v_0\|_{S_\mu})) \exp(\mathcal{T}_1(\|v_0\|_{\mathcal{B}^0})).$$

Proof. Proposition 4.4 follows easily from the following lemma which is a general result about the propagation of anisotropic regularity by a transport-diffusion equation.

Lemma 4.5. *Let us consider (s, s') a couple of real numbers, and \mathcal{Q} a bilinear operator which maps continuously $\mathcal{B}^1 \times \mathcal{B}^{s+1, s'}$ into $\mathcal{B}^{s, s'}$. A constant C exists such that for any two-component vector field v in $L^2(\mathbb{R}^+; \mathcal{B}^1)$, any f in $L^1(\mathbb{R}^+; \mathcal{B}^{s, s'})$, any a_0 in $\mathcal{B}^{s, s'}$ and for any non negative β , if $\Delta_\beta \stackrel{\text{def}}{=} \Delta_h + \beta^2 \partial_z^2$ and a is the solution of*

$$\partial_t a - \Delta_\beta a + \mathcal{Q}(v, a) = f \quad \text{and} \quad a|_{t=0} = a_0,$$

then a satisfies

$$\forall r \in [1, \infty], \|a\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}^{s+\frac{2}{r}, s'})} \leq C (\|a_0\|_{\mathcal{B}^{s, s'}} + \|f\|_{L^1(\mathbb{R}^+; \mathcal{B}^{s, s'})}) \exp\left(C \int_0^\infty \|v(t)\|_{\mathcal{B}^1}^2 dt\right).$$

The fact that the third index of the Besov spaces is one, induces some technical difficulties which lead us to work first on subintervals I of \mathbb{R}^+ on which $\|v\|_{L^2(I; \mathcal{B}^1)}$ is small.

Let us then start by considering any subinterval $I = [\tau_0, \tau_1]$ of \mathbb{R}^+ . The Duhamel formula and the smoothing effect of the heat flow imply that

$$\begin{aligned} \|\Delta_k^h \Delta_j^v a(t)\|_{L^2} &\leq e^{-c2^{2k}(t-\tau_0)} \|\Delta_k^h \Delta_j^v a(\tau_0)\|_{L^2} \\ &\quad + C \int_{\tau_0}^t e^{-c2^{2k}(t-t')} \|\Delta_k^h \Delta_j^v (\mathcal{Q}(v(t'), a(t')) + f(t'))\|_{L^2} dt'. \end{aligned}$$

After multiplication by $2^{ks+j's'}$ and using Young's inequality in the time integral, we deduce that

$$\begin{aligned} 2^{ks+j's'} (\|\Delta_k^h \Delta_j^v a\|_{L^\infty(I; L^2)} + 2^{2k} \|\Delta_k^h \Delta_j^v a\|_{L^1(I; L^2)}) &\leq C 2^{ks+j's'} \|\Delta_k^h \Delta_j^v a(\tau_0)\|_{L^2} \\ &\quad + C \int_I d_{k,j}(t') (\|v(t')\|_{\mathcal{B}^1} \|a(t')\|_{\mathcal{B}^{s+1, s'}} + \|f(t')\|_{\mathcal{B}^{s, s'}}) dt', \end{aligned}$$

where for any t , $d_{k,j}(t)$ is an element of the sphere of $\ell^1(\mathbb{Z}^2)$. By summation over (k, j) and using the Cauchy-Schwarz inequality, we infer that

$$(4.26) \quad \begin{aligned} \|a\|_{\tilde{L}^\infty(I; \mathcal{B}^{s, s'})} + \|a\|_{L^1(I; \mathcal{B}^{s+2, s'})} &\leq C \|a(\tau_0)\|_{\mathcal{B}^{s, s'}} + C \|f\|_{L^1(I; \mathcal{B}^{s, s'})} \\ &\quad + C \|v\|_{L^2(I; \mathcal{B}^1)} \|a\|_{L^2(I; \mathcal{B}^{s+1, s'})}. \end{aligned}$$

Let us define an increasing sequence $(T_m)_{0 \leq m \leq M+1}$ by induction such that $T_0 = 0$, $T_{M+1} = \infty$ and

$$\forall m < M, \quad \int_{T_m}^{T_{m+1}} \|v(t)\|_{\mathcal{B}^1}^2 dt = c_0 \quad \text{and} \quad \int_{T_M}^{\infty} \|v(t)\|_{\mathcal{B}^1}^2 dt \leq c_0,$$

for some given c_0 which will be chosen later on. Obviously, we have

$$(4.27) \quad \int_0^{\infty} \|v(t)\|_{\mathcal{B}^1}^2 dt \geq \int_0^{T_M} \|v(t)\|_{\mathcal{B}^1}^2 dt = M c_0.$$

Thus the number M of T'_m 's such that T_m is finite is less than $c_0^{-1} \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)}^2$.

Applying Estimate (4.26) to the interval $[T_m, T_{m+1}]$, we get

$$\begin{aligned} \|a\|_{L^\infty([T_m, T_{m+1}]; \mathcal{B}^{s, s'})} + \|a\|_{L^1([T_m, T_{m+1}]; \mathcal{B}^{s+2, s'})} &\leq \|a\|_{L^2([T_m, T_{m+1}]; \mathcal{B}^{s+1, s'})} \\ &\quad + C (\|a(T_m)\|_{\mathcal{B}^{s, s'}} + C \|f\|_{L^1([T_m, T_{m+1}]; \mathcal{B}^{s, s'})}), \end{aligned}$$

if c_0 is chosen such that $C\sqrt{c_0} \leq 1$.

Since

$$\|a\|_{L^2([T_m, T_{m+1}]; \mathcal{B}^{s+1, s'})} \leq \|a\|_{L^\infty([T_m, T_{m+1}]; \mathcal{B}^{s, s'})}^{\frac{1}{2}} \|a\|_{L^1([T_m, T_{m+1}]; \mathcal{B}^{s+2, s'})}^{\frac{1}{2}},$$

we infer that

$$(4.28) \quad \begin{aligned} \|a\|_{L^\infty([T_m, T_{m+1}]; \mathcal{B}^{s, s'})} + \|a\|_{L^1([T_m, T_{m+1}]; \mathcal{B}^{s+2, s'})} \\ \leq 2C (\|a(T_m)\|_{\mathcal{B}^{s, s'}} + \|f\|_{L^1([T_m, T_{m+1}]; \mathcal{B}^{s, s'})}). \end{aligned}$$

Now let us us prove by induction that

$$\|a\|_{L^\infty([0, T_m]; \mathcal{B}^{s, s'})} \leq (2C)^m (\|a_0\|_{\mathcal{B}^{s, s'}} + \|f\|_{L^1([0, T_m]; \mathcal{B}^{s, s'})}).$$

Using (4.28) and the induction hypothesis we get

$$\begin{aligned} \|a\|_{L^\infty([T_m, T_{m+1}]; \mathcal{B}^{s, s'})} &\leq 2C (\|a\|_{L^\infty([0, T_m]; \mathcal{B}^{s, s'})} + \|f\|_{L^1([T_m, T_{m+1}]; \mathcal{B}^{s, s'})}) \\ &\leq (2C)^{m+1} (\|a_0\|_{\mathcal{B}^{s, s'}} + \|f\|_{L^1([0, T_{m+1}]; \mathcal{B}^{s, s'})}), \end{aligned}$$

provided that $2C \geq 1$, which ensures in view of (4.27) that

$$\|a\|_{L^\infty(\mathbb{R}^+; \mathcal{B}^{s, s'})} \leq C (\|a_0\|_{\mathcal{B}^{s, s'}} + \|f\|_{L^1(\mathbb{R}^+; \mathcal{B}^{s, s'})}) \exp\left(C \int_0^{\infty} \|v(t)\|_{\mathcal{B}^1}^2 dt\right).$$

We deduce from (4.28) that

$$\begin{aligned} \|a\|_{L^1([T_m, T_{m+1}]; \mathcal{B}^{s+2, s'})} &\leq C (\|a_0\|_{\mathcal{B}^{s, s'}} + \|f\|_{L^1(\mathbb{R}^+; \mathcal{B}^{s, s'})}) \exp\left(C \int_0^{\infty} \|v(t)\|_{\mathcal{B}^1}^2 dt\right) \\ &\quad + C \|f\|_{L^1([T_m, T_{m+1}]; \mathcal{B}^{s, s'})}. \end{aligned}$$

Once noticed that $x e^{C x^2} \leq e^{C' x^2}$, the result comes by summation over m and the fact that the total number of m 's is less than or equal to $c_0^{-1} \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)}^2$. This ends the proof of the lemma and thus of Proposition 4.4. \square

As w^h is defined by $w^h = -\nabla_h \Delta_h^{-1} \partial_3 w^3$, we deduce from Proposition 4.4 the following corollary.

Corollary 4.6. *For any s in $[-2 + \mu, 0]$ and any $s' \geq 1/2$,*

$$\|w^h\|_{\mathcal{A}^{s+1, s'-1}} \lesssim (\|w_0^3\|_{\mathcal{B}^{s, s'}} + \|w_0^3\|_{\mathcal{B}^0} \mathcal{T}_2(\|v_0\|_{S_\mu})) \exp(\mathcal{T}_1(\|v_0\|_{\mathcal{B}^0})).$$

4.1.3. *Conclusion of the proof of Theorem 3.* Using the definition of the approximate solution Φ^{app} given in (4.1), we infer from Propositions 4.1 and 4.4 and Corollary 4.6 that

$$(4.29) \quad \|\Phi^{\text{app}}\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \leq \mathcal{T}_1(\|(v_0, w_0^3)\|_{\mathcal{B}^0}) + \beta \mathcal{T}_2(\|(v_0, w_0^3)\|_{S_\mu}).$$

Moreover, the error term ψ satisfies the following modified Navier-Stokes system, with null Cauchy data:

$$(4.30) \quad \begin{aligned} \partial_t \psi + \operatorname{div}(\psi \otimes \psi + \Phi^{\text{app}} \otimes \psi + \psi \otimes \Phi^{\text{app}}) - \Delta \psi &= -\nabla q_\beta + \sum_{\ell=1}^4 E_\beta^\ell \quad \text{with} \\ E_\beta^1 &\stackrel{\text{def}}{=} \partial_3^2 [(v, 0)]_\beta + \beta(0, [\partial_3 p]_\beta), \\ E_\beta^2 &\stackrel{\text{def}}{=} \beta \left[\left(w^3 \partial_3 (v, w^3) + (\nabla^h \Delta_h^{-1} \operatorname{div}_h \partial_3 (vw^3), 0) \right) \right]_\beta, \\ E_\beta^3 &\stackrel{\text{def}}{=} \beta \left[\left(w^h \cdot \nabla_h (v, w^3) + v \cdot \nabla^h (w^h, 0) \right) \right]_\beta \quad \text{and} \\ E_\beta^4 &\stackrel{\text{def}}{=} \beta^2 \left[\left(w^h \cdot \nabla^h (w^h, 0) + w^3 \partial_3 (w^h, 0) \right) \right]_\beta. \end{aligned}$$

If we prove that

$$(4.31) \quad \left\| \sum_{\ell=1}^4 E_\beta^\ell \right\|_{\mathcal{F}^0} \leq \beta \mathcal{T}_2(\|(v_0, w_0^3)\|_{S_\mu}),$$

then according to the fact $\psi|_{t=0} = 0$, ψ exists globally and satisfies

$$(4.32) \quad \|\psi\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \lesssim \beta \mathcal{T}_2(\|(v_0, w_0^3)\|_{S_\mu}).$$

This in turn implies that Φ_0 generates a global regular solution Φ_β in $L^2(\mathbb{R}^+; \mathcal{B}^1)$ which satisfies

$$(4.33) \quad \|\Phi_\beta\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \leq \mathcal{T}_1(\|(v_0, w_0^3)\|_{\mathcal{B}^0}) + \beta \mathcal{T}_2(\|(v_0, w_0^3)\|_{S_\mu}).$$

Once this bound in $L^2(\mathbb{R}^+; \mathcal{B}^1)$ is obtained, the bound in \mathcal{A}^0 follows by heat flow estimates, and in $\mathcal{A}^{s, s'}$ by propagation of regularity for the Navier-Stokes system.

So all we need to do is to prove Inequality (4.31). Let us first estimate the term $\partial_3^2 [(v, 0)]_\beta$. This requires the use of some $\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{s, s'})$ norms. Clearly, we have

$$\|\partial_3^2 [v]_\beta\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{0, -\frac{1}{2}})} \lesssim \|[v]_\beta\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{0, \frac{3}{2}})},$$

which implies in view of the vertical scaling property (1.12) of the space $\mathcal{B}^{0, \frac{3}{2}}$

$$\|\partial_3^2 [v]_\beta\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{0, -\frac{1}{2}})} \lesssim \beta \|v\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{0, \frac{3}{2}})}.$$

Therefore Proposition 4.1 ensures that

$$(4.34) \quad \|\partial_3^2 [v]_\beta\|_{\tilde{L}^2(\mathbb{R}^+; \mathcal{B}^{0, -\frac{1}{2}})} \leq \beta \mathcal{T}_2(\|v_0\|_{S_\mu}).$$

Now let us study the pressure term. By applying the horizontal divergence to the equation satisfied by v we get, thanks to the fact that $\operatorname{div}_h v = 0$,

$$\partial_3 p = -\partial_3 \Delta_h^{-1} \sum_{\ell, m=1}^2 \partial_\ell \partial_m (v^\ell v^m).$$

Since ℓ and m belong to $\{1, 2\}$, the operator $\Delta_h^{-1} \partial_\ell \partial_m$ is a zero-order horizontal Fourier multiplier, which implies that

$$\begin{aligned} \|[\partial_3 p]_\beta\|_{L^1(\mathbb{R}^+; \mathcal{B}^0)} &= \|\partial_3 p\|_{L^1(\mathbb{R}^+; \mathcal{B}^0)} \\ &\lesssim \|v \partial_3 v\|_{L^1(\mathbb{R}^+; \mathcal{B}^0)}. \end{aligned}$$

According to laws of product in anisotropic Besov spaces, we obtain

$$\|v(t) \partial_3 v(t)\|_{\mathcal{B}^0} \lesssim \|v(t)\|_{\mathcal{B}^1} \|\partial_3 v(t)\|_{\mathcal{B}^0},$$

which gives rise to

$$(4.35) \quad \begin{aligned} \|[\partial_3 p]_\beta\|_{L^1(\mathbb{R}^+; \mathcal{B}^0)} &\lesssim \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \|\partial_3 v\|_{L^2(\mathbb{R}^+; \mathcal{B}^0)} \\ &\lesssim \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^{0, \frac{3}{2}})}. \end{aligned}$$

Combining (4.34) and (4.35), we get by virtue of Proposition 4.1

$$(4.36) \quad \|E_\beta^1\|_{\mathcal{F}^0} \leq \beta \mathcal{T}_2(\|v_0\|_{S_\mu}).$$

In the same way, we treat the terms E_β^2 , E_β^3 and E_β^4 , achieving the proof of Estimate (4.31). This ends the proof of the fact that the solution Φ_β of (NS) with initial data

$$\Phi_0 = [(v_0 - \beta \nabla^h \Delta_h^{-1} \partial_3 w_0^3, w_0^3)]_\beta$$

is global and belongs to $L^2(\mathbb{R}^+; \mathcal{B}^1)$.

The proof of the whole Theorem 3 is then achieved. \square

4.2. Proof of Theorem 4. The proof of Theorem 4 is done in three steps. First we define an approximate solution, using results proved in the previous section, and then we prove useful localization results on the different parts entering in the definition of the approximate solution. In the last step, we conclude the proof of the theorem, using those localization results.

4.2.1. The approximate solution. With the notation of Theorem 2, let us consider the divergence free vector field:

$$\Phi_{0,n,\alpha,L}^0 \stackrel{\text{def}}{=} u_{0,\alpha} + [(v_{0,n,\alpha,L}^{0,\infty} + h_n^0 w_{0,n,\alpha,L}^{0,\infty,h}, w_{0,n,\alpha,L}^{0,\infty,3})]_{h_n^0} + [(v_{0,n,\alpha,L}^{0,\text{loc}} + h_n^0 w_{0,n,\alpha,L}^{0,\text{loc},h}, w_{0,n,\alpha,L}^{0,\text{loc},3})]_{h_n^0}.$$

Our purpose is to establish that for h_n^0 small enough, depending only on the weak limit u_0 and on $\|(v_{0,n,\alpha,L}^{0,\infty}, w_{0,n,\alpha,L}^{0,\infty,3})\|_{S_\mu}$ as well as $\|(v_{0,n,\alpha,L}^{0,\text{loc}}, w_{0,n,\alpha,L}^{0,\text{loc},3})\|_{S_\mu}$, there is a unique, global smooth solution to (NS) with data $\Phi_{0,n,\alpha,L}^0$.

Let us start by solving globally (NS) with the data $u_{0,\alpha}$. By using the global strong stability of (NS) in $\mathcal{B}_{1,1}$ (see [4], Corollary 3) and the convergence result (2.2), we deduce that, for α small enough, $u_{0,\alpha}$ generates a unique, global solution u_α to (NS) system belonging to $L^2(\mathbb{R}^+; \mathcal{B}_{1,1}^{2, \frac{1}{2}})$. Actually in view of the Sobolev embedding of $\mathcal{B}_{1,1}^{2, \frac{1}{2}}$ into \mathcal{B}^1 , $u_\alpha \in L^2(\mathbb{R}^+; \mathcal{B}^1)$.

Next let us define

$$\Phi_{0,n,\alpha,L}^{0,\infty} \stackrel{\text{def}}{=} [(v_{0,n,\alpha,L}^{0,\infty} + h_n^0 w_{0,n,\alpha,L}^{0,\infty,h}, w_{0,n,\alpha,L}^{0,\infty,3})]_{h_n^0}.$$

Thanks to Theorem 3, we know that for h_n^0 smaller than $\varepsilon_1(\|(v_{0,n,\alpha,L}^{0,\infty}, w_{0,n,\alpha,L}^{0,\infty,3})\|_{S_\mu})$ there is a unique global smooth solution $\Phi_{n,\alpha,L}^{0,\infty}$ associated with $\Phi_{0,n,\alpha,L}^{0,\infty}$, which belongs to \mathcal{A}_0 , and

using the notation and results of Section 2.3, in particular (4.1) and (4.32), we can write

$$(4.37) \quad \begin{aligned} \Phi_{n,\alpha,L}^{0,\infty} &\stackrel{\text{def}}{=} \Phi_{n,\alpha,L}^{0,\infty,\text{app}} + \psi_{n,\alpha,L}^{0,\infty} \quad \text{with} \\ \Phi_{n,\alpha,L}^{0,\infty,\text{app}} &\stackrel{\text{def}}{=} [v_{n,\alpha,L}^{0,\infty} + h_n^0 w_{n,\alpha,L}^{0,\infty,\text{h}}, w_{n,\alpha,L}^{0,\infty,3}]_{h_n^0} \text{ and for all } r \text{ in } [2, \infty] \\ \|\psi_{n,\alpha,L}^{0,\infty}\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}_{\frac{2}{r}})} &\lesssim h_n^0 \mathcal{T}_2(\|(v_{0,n,\alpha,L}^{0,\infty}, w_{0,n,\alpha,L}^{0,\infty,3})\|_{S_\mu}), \end{aligned}$$

for all r in $[2, \infty]$,

$$(4.38) \quad \lim_{L \rightarrow \infty} \lim_{\alpha \rightarrow 0} \limsup_{n \rightarrow \infty} \|\Phi_{n,\alpha,L}^{0,\text{loc}}(\cdot, 0)\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}_{2,1}^{\frac{2}{r}}(\mathbb{R}^2))} = 0,$$

where $v_{n,\alpha,L}^{0,\infty}$ solves $(\text{NS2D})_{x_3}$ with data $v_{0,n,\alpha,L}^{0,\infty}$, $w_{n,\alpha,L}^{0,\infty,3}$ solves the transport-diffusion equation $(T_{h_n^0})$ defined page 22 with data $w_{0,n,\alpha,L}^{0,\infty,3}$ and where

$$w_{n,\alpha,L}^{0,\infty,\text{h}} = -\nabla^{\text{h}} \Delta_{\text{h}}^{-1} \partial_3 w_{n,\alpha,L}^{0,\infty,3}.$$

Similarly defining

$$\Phi_{0,n,\alpha,L}^{0,\text{loc}} \stackrel{\text{def}}{=} [(v_{0,n,\alpha,L}^{0,\text{loc}} + h_n^0 w_{0,n,\alpha,L}^{0,\text{loc},\text{h}}, w_{0,n,\alpha,L}^{0,\text{loc},3})]_{h_n^0},$$

then for h_n^0 smaller than $\varepsilon_1(\|(v_{0,n,\alpha,L}^{0,\text{loc}}, w_{0,n,\alpha,L}^{0,\text{loc},3})\|_{S_\mu})$ there is a unique global smooth solution $\Phi_{n,\alpha,L}^{0,\text{loc}}$ associated with $\Phi_{0,n,\alpha,L}^{0,\text{loc}}$, which belongs to \mathcal{A}_0 , and

$$(4.39) \quad \begin{aligned} \Phi_{n,\alpha,L}^{0,\text{loc}} &\stackrel{\text{def}}{=} \Phi_{n,\alpha,L}^{0,\text{loc},\text{app}} + \psi_{n,\alpha,L}^{0,\text{loc}} \quad \text{with} \\ \Phi_{n,\alpha,L}^{0,\text{loc},\text{app}} &\stackrel{\text{def}}{=} [v_{n,\alpha,L}^{0,\text{loc}} + h_n^0 w_{n,\alpha,L}^{0,\text{loc},\text{h}}, w_{n,\alpha,L}^{0,\text{loc},3}]_{h_n^0} \text{ and for all } r \text{ in } [2, \infty] \\ \|\psi_{n,\alpha,L}^{0,\text{loc}}\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}_{\frac{2}{r}})} &\lesssim h_n^0 \mathcal{T}_2(\|(v_{0,n,\alpha,L}^{0,\text{loc}}, w_{0,n,\alpha,L}^{0,\text{loc},3})\|_{S_\mu}), \end{aligned}$$

where $v_{n,\alpha,L}^{0,\text{loc}}$ solves $(\text{NS2D})_{x_3}$ with data $v_{0,n,\alpha,L}^{0,\text{loc}}$ and $w_{n,\alpha,L}^{0,\text{loc},3}$ solves $(T_{h_n^0})$ with data $w_{0,n,\alpha,L}^{0,\text{loc},3}$. Finally we recall that $w_{n,\alpha,L}^{0,\text{loc},\text{h}} = -\nabla^{\text{h}} \Delta_{\text{h}}^{-1} \partial_3 w_{n,\alpha,L}^{0,\text{loc},3}$.

In the next step, we establish localization properties on $\Phi_{n,\alpha,L}^{0,\infty}$ and $\Phi_{n,\alpha,L}^{0,\text{loc}}$. Those localization properties will enable us to prove that the function $u_\alpha + \Phi_{n,\alpha,L}^{0,\infty} + \Phi_{n,\alpha,L}^{0,\text{loc}}$ approximate the solution to (NS) system associated to the Cauchy data $\Phi_{0,n,\alpha,L}^0$.

4.2.2. Localization properties of the approximate solution. In this step, we prove localization properties on $\Phi_{n,\alpha,L}^{0,\infty}$ and $\Phi_{n,\alpha,L}^{0,\text{loc}}$, namely the fact that $\Phi_{n,\alpha,L}^{0,\infty,\text{app}}$ escapes to infinity in the space variable, while $\Phi_{n,\alpha,L}^{0,\text{loc},\text{app}}$ remains localized (approximately), and we also prove that $\Phi_{n,\alpha,L}^{0,\text{loc},\text{app}}$ remains small near $x_3 = 0$. Let us recall that as claimed by (2.6), (2.7) and (2.8), those properties are true for their respective initial data. A first part of these localization properties derives from the following result.

Proposition 4.7. *Under the assumptions of Proposition 4.1, the control of the value of v at the point $x_3 = 0$ is given by*

$$(4.40) \quad \forall r \in [1, \infty], \quad \|v(\cdot, 0)\|_{\tilde{L}^r(\mathbb{R}^+; \mathcal{B}_{2,1}^{\frac{2}{r}}(\mathbb{R}^2))} \lesssim \|v_0(\cdot, 0)\|_{B_{2,1}^0(\mathbb{R}^2)} + \|v_0(\cdot, 0)\|_{L^2(\mathbb{R}^2)}^2.$$

Moreover we have for all η in $]0, 1[$ and γ in $\{0, 1\}$,

$$(4.41) \quad \|(\gamma - \theta_{h,\eta})v\|_{\mathcal{A}^0} \leq \|(\gamma - \theta_{h,\eta})v_0\|_{\mathcal{B}^0} \exp \mathcal{T}_1(\|v_0\|_{\mathcal{B}^0}) + \eta \mathcal{T}_2(\|v_0\|_{S_\mu}),$$

with $\theta_{h,\eta}$ is the truncation function defined by (2.1).

Proof. In order to establish Proposition 4.7, let us start by pointing out that the proof of Lemma 1.1 of [12] claims that for all x_3 in \mathbb{R} ,

$$(4.42) \quad \begin{aligned} & (\Delta_k^h(v(t, \cdot, x_3) \cdot \nabla^h v(t, \cdot, x_3)) | \Delta_k^h v(t, \cdot, x_3))_{L^2} \\ & \lesssim d_k(t, x_3) \|\nabla^h v(t, \cdot, x_3)\|_{L^2}^2 \|\Delta_k^h v(t, \cdot, x_3)\|_{L^2}, \end{aligned}$$

where $(d_k(t, x_3))_{k \in \mathbb{Z}}$ is a generic element of the sphere of $\ell^1(\mathbb{Z})$.

Taking $x_3 = 0$, we deduce by an L^2 energy estimate in \mathbb{R}^2

$$\frac{1}{2} \frac{d}{dt} \|\Delta_k^h v(t, \cdot, 0)\|_{L^2}^2 + c2^{2k} \|\Delta_k^h v(t, \cdot, 0)\|_{L^2}^2 \lesssim d_k(t) \|\nabla^h v(t, \cdot, 0)\|_{L^2}^2 \|\Delta_k^h v(t, \cdot, 0)\|_{L^2},$$

where $(d_k(t))_{k \in \mathbb{Z}}$ belongs to the sphere of $\ell^1(\mathbb{Z})$, which after division by $\|\Delta_k^h v(t, \cdot, 0)\|_{L^2}$ and time integration leads to

$$(4.43) \quad \begin{aligned} & \|\Delta_k^h v(\cdot, 0)\|_{L^\infty(\mathbb{R}^+; L^2)} + c2^{2k} \|\Delta_k^h v(\cdot, 0)\|_{L^1(\mathbb{R}^+; L^2)} \\ & \leq \|\Delta_k v_0(\cdot, 0)\|_{L^2} + C \int_0^\infty d_k(t) \|\nabla^h v(t, \cdot, 0)\|_{L^2}^2 dt. \end{aligned}$$

By summation over k and in view of (4.9), we obtain Inequality (4.40) of Proposition 4.7.

Now to go to the proof of Inequality (4.41), let us define $v_{\gamma, \eta} \stackrel{\text{def}}{=} (\gamma - \theta_{h, \eta})v$ and write that

$$(4.44) \quad \begin{aligned} & \partial_t v_{\gamma, \eta} - \Delta_h v_{\gamma, \eta} + \text{div}_h(v \otimes v_{\gamma, \eta}) = E_\eta(v) = \sum_{i=1}^3 E_\eta^i(v) \quad \text{with} \\ & E_\eta^1(v) \stackrel{\text{def}}{=} -2\eta(\nabla^h \theta)_{h, \eta} \nabla^h v - \eta^2(\Delta_h \theta)_{h, \eta} v, \\ & E_\eta^2(v) \stackrel{\text{def}}{=} \eta v \cdot (\nabla^h \theta)_{h, \eta} v \quad \text{and} \\ & E_\eta^3(v) \stackrel{\text{def}}{=} -(\gamma - \theta_{h, \eta}) \nabla^h \Delta_h^{-1} \sum_{1 \leq \ell, m \leq 2} \partial_\ell \partial_m (v^\ell v^m). \end{aligned}$$

Lemma 4.5 applied with $s = 0$, $s' = 1/2$, $a = v_{\gamma, \eta}$, $\mathcal{Q}(v, a) = \text{div}_h(v \otimes a)$, $f = E_\eta(v)$ and $\beta = 0$ reduces the problem to the proof of the following estimate

$$(4.45) \quad \|E_\eta(v)\|_{L^1(\mathbb{R}^+; \mathcal{B}^0)} \lesssim \eta \mathcal{T}_2(\|v_0\|_{S_\mu}).$$

Actually, in view of Inequality (4.3) applied with $r = 1$ and $s = -1$ (resp. with $r = 2$ and $s = -1/2$) this will follow from

$$(4.46) \quad \|E_\eta(v)\|_{L^1(\mathbb{R}^+; \mathcal{B}^0)} \lesssim \eta (\|v\|_{L^1(\mathbb{R}^+; \mathcal{B}^1)} + \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})}^2).$$

Product laws in anisotropic Besov spaces and the scaling properties of homogeneous Besov spaces give

$$\begin{aligned} \|(\nabla^h \theta)_{h, \eta} \nabla^h v(t)\|_{\mathcal{B}^0} & \lesssim \|(\nabla^h \theta)_{h, \eta}\|_{B_{2,1}^1(\mathbb{R}^2)} \|\nabla^h v(t)\|_{\mathcal{B}^0} \\ & \lesssim \|\nabla^h \theta\|_{B_{2,1}^1(\mathbb{R}^2)} \|v(t)\|_{\mathcal{B}^1}. \end{aligned}$$

Along the same lines, we get

$$\begin{aligned} \|(\Delta_h \theta)_{h, \eta} v(t)\|_{\mathcal{B}^0} & \lesssim \|(\Delta_h \theta)_{h, \eta}\|_{B_{2,1}^0(\mathbb{R}^2)} \|v(t)\|_{\mathcal{B}^1} \\ & \lesssim \frac{1}{\eta} \|\Delta_h \theta\|_{B_{2,1}^0(\mathbb{R}^2)} \|v(t)\|_{\mathcal{B}^1}. \end{aligned}$$

Consequently

$$(4.47) \quad \|E_\eta^1(v)\|_{L^1(\mathbb{R}^+; \mathcal{B}^0)} \lesssim \eta \|v\|_{L^1(\mathbb{R}^+; \mathcal{B}^1)}.$$

The same arguments enable us to deal with the term $E_\eta^2(v)$ and to prove that

$$(4.48) \quad \|E_\eta^2(v)\|_{L^1(\mathbb{R}^+; \mathcal{B}^0)} \lesssim \eta \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})}^2.$$

Let us finally study the term $E_\eta^3(v)$ which is most challenging. To this end, we make use of the horizontal paraproduct decomposition:

$$av = T_v^h a + T_a^h v + R^h(a, b) \quad \text{with} \quad T_a^h b \stackrel{\text{def}}{=} \sum_k S_{k-1}^h a \Delta_k^h b \quad \text{and} \quad R^h(a, b) \stackrel{\text{def}}{=} \sum_k \tilde{\Delta}_k^h a \Delta_k^h b,$$

where $\tilde{\Delta}_k^h \stackrel{\text{def}}{=} \tilde{\varphi}(2^{-k} \xi_h)$ with $\tilde{\varphi}$ is a smooth compactly supported (in $\mathbb{R}^2 \setminus \{0\}$) function which has value 1 near $B(0, 2^{-N_0}) + \mathcal{C}$, where \mathcal{C} is an adequate annulus. This allows us to write

$$(4.49) \quad \begin{aligned} E_\eta^3(v) &= \sum_{\ell=1}^3 E_\eta^{3,\ell}(v) \quad \text{with} \\ E_\eta^{3,1}(v) &\stackrel{\text{def}}{=} \tilde{T}_{\nabla^h p}^h \theta_{h,\eta} \quad \text{with} \quad \nabla^h p = \nabla^h \Delta_h^{-1} \sum_{1 \leq \ell, m \leq 2} \partial_\ell \partial_m (v^\ell v^m), \\ E_\eta^{3,2}(v) &\stackrel{\text{def}}{=} - \sum_{1 \leq \ell, m \leq 2} [T_{\gamma - \theta_{h,\eta}}^h, \nabla^h \Delta_h^{-1} \partial_\ell \partial_m] v^\ell v^m \quad \text{and} \\ E_\eta^{3,3}(v) &\stackrel{\text{def}}{=} \sum_{1 \leq \ell, m \leq 2} \nabla^h \Delta_h^{-1} \partial_\ell \partial_m \tilde{T}_{v^\ell v^m}^h \theta_{h,\eta}, \end{aligned}$$

where $\tilde{T}_a^h b = T_a^h b + R^h(a, b)$. Combining laws of product with scaling properties of Besov spaces, we obtain

$$\begin{aligned} \|\tilde{T}_{\nabla^h p(t)}^h \theta_{h,\eta}\|_{\mathcal{B}^0} &\lesssim \|\nabla^h p(t)\|_{\mathcal{B}^{-1}} \|\theta_{h,\eta}\|_{B_{2,1}^2(\mathbb{R}^2)} \\ &\lesssim \eta \sup_{1 \leq \ell, m \leq 2} \|v^\ell(t) v^m(t)\|_{\mathcal{B}^0} \|\theta\|_{B_{2,1}^2(\mathbb{R}^2)} \\ &\lesssim \eta \|v(t)\|_{\mathcal{B}^{\frac{1}{2}}}^2 \|\theta\|_{B_{2,1}^2(\mathbb{R}^2)}. \end{aligned}$$

Along the same lines we get

$$\begin{aligned} \|\nabla^h \Delta_h^{-1} \partial_\ell \partial_m \tilde{T}_{v^\ell(t) v^m(t)}^h \theta_{h,\eta}\|_{\mathcal{B}^0} &\lesssim \|\tilde{T}_{v^\ell(t) v^m(t)}^h \theta_{h,\eta}\|_{\mathcal{B}^1} \\ &\lesssim \|v^\ell(t) v^m(t)\|_{\mathcal{B}^0} \|\theta_{h,\eta}\|_{B_{2,1}^2(\mathbb{R}^2)} \\ &\lesssim \eta \|v(t)\|_{\mathcal{B}^{\frac{1}{2}}}^2 \|\theta\|_{B_{2,1}^2(\mathbb{R}^2)}. \end{aligned}$$

We deduce that

$$(4.50) \quad \|E_\eta^{3,1}(v) + E_\eta^{3,3}(v)\|_{L^1(\mathbb{R}^+; \mathcal{B}^0)} \lesssim \eta \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})}^2.$$

Now let us estimate $E_\eta^{3,2}(v)$. By definition, we have

$$[T_{\gamma - \theta_{h,\eta}}^h, \nabla^h \Delta_h^{-1} \partial_\ell \partial_m] v^\ell v^m = \sum_k \mathcal{E}_{k,\eta}(v) \quad \text{with}$$

$$\mathcal{E}_{k,\eta}(v) \stackrel{\text{def}}{=} [S_{k-N_0}^h(\gamma - \theta_{h,\eta}), \tilde{\Delta}_k^h \nabla^h \Delta_h^{-1} \partial_\ell \partial_m] \Delta_k^h (v^\ell v^m).$$

Then by commutator estimates (see for instance Lemma 2.97 in [2])

$$\|\Delta_j^v \mathcal{E}_{k,\eta}(v(t))\|_{L^2} \lesssim \|\nabla \theta_{h,\eta}\|_{L^\infty} \|\Delta_k^h \Delta_j^v (v^\ell(t) v^m(t))\|_{L^2}.$$

Noticing that $\|\nabla \theta_{h,\eta}\|_{L^\infty} = \eta \|\nabla \theta\|_{L^\infty}$, we get by virtue of laws of product

$$\|E_\eta^{3,2}(v)\|_{L^1(\mathbb{R}^+; \mathcal{B}^0)} \lesssim \eta \|v\|_{L^2(\mathbb{R}^+; \mathcal{B}^{\frac{1}{2}})}^2,$$

which ends the proof of Estimate (4.45) and thus of Proposition 4.7. \square

A similar result holds for the solution w^3 of

$$(T_\beta) \quad \partial_t w^3 + v \cdot \nabla^h w^3 - \Delta_h w^3 - \beta^2 \partial_3^2 w^3 = 0 \quad \text{and} \quad w^3|_{t=0} = w_0^3,$$

where β is any non negative real number. In the following statement, all the constants are independent of β .

Proposition 4.8. *Let v and w_3 be as in Proposition 4.4. The control of the value of w^3 at the point $x_3 = 0$ states as follows: For any r in $[2, \infty]$,*

$$(4.51) \quad \|w^3(\cdot, 0)\|_{\tilde{L}^r(\mathbb{R}^+; B_{2,1}^{\frac{2}{r}}(\mathbb{R}^2))} \leq \mathcal{T}_2(\|(v_0, w_0^3)\|_{S_\mu}) \left(\|w_0^3(\cdot, 0)\|_{B_{2,1}^{\frac{1-2\mu}{4(1-\mu)}(\mathbb{R}^2)}} + \beta \right).$$

Moreover, with the notations of Theorem 3, we have for all η in $]0, 1[$ and γ in $\{0, 1\}$,

$$(4.52) \quad \|(\gamma - \theta_{h,\eta})w^3\|_{\mathcal{A}^0} \leq \|(\gamma - \theta_{h,\eta})w_0^3\|_{\mathcal{B}^0} \exp \mathcal{T}_1(\|v_0\|_{\mathcal{B}^0}) + \eta \mathcal{T}_2(\|(v_0, w_0^3)\|_{S_\mu}).$$

The proof of Proposition 4.8 is very similar to that of Proposition 4.7 and is left to the reader.

Propositions 4.7 and 4.8 imply easily the following result, using the special form of $\Phi_{n,\alpha,L}^{0,\infty}$ and $\Phi_{n,\alpha,L}^{0,\text{loc}}$ recalled in (4.37) and (4.39), and thanks to (2.6), (2.7) and (2.8).

Corollary 4.9. *The vector fields $\Phi_{n,\alpha,L}^{0,\text{loc}}$ and $\Phi_{n,\alpha,L}^{0,\infty}$ satisfy the following: $\Phi_{n,\alpha,L}^{0,\text{loc}}$ vanishes at $x_3 = 0$, in the sense that for all r in $[2, \infty]$,*

$$(4.53) \quad \lim_{L \rightarrow \infty} \lim_{\alpha \rightarrow 0} \limsup_{n \rightarrow \infty} \|\Phi_{n,\alpha,L}^{0,\text{loc}}(\cdot, 0)\|_{\tilde{L}^r(\mathbb{R}^+; B_{2,1}^{\frac{2}{r}}(\mathbb{R}^2))} = 0,$$

and there is a constant $C(\alpha, L)$ such that for all η in $]0, 1[$,

$$\limsup_{n \rightarrow \infty} \left(\|(1 - \theta_{h,\eta})\Phi_{n,\alpha,L}^{0,\text{loc}}\|_{\mathcal{A}^0} + \|\theta_{h,\eta}\Phi_{n,\alpha,L}^{0,\infty}\|_{\mathcal{A}^0} \right) \leq C(\alpha, L)\eta.$$

Proof. In view of (4.39) and under Notation (1.11)

$$\begin{aligned} \Phi_{n,\alpha,L}^{0,\text{loc}} &= \Phi_{n,\alpha,L}^{0,\text{loc,app}} + \psi_{n,\alpha,L}^{0,\text{loc}} \quad \text{with} \\ \Phi_{n,\alpha,L}^{0,\text{loc,app}} &= [v_{n,\alpha,L}^{0,\text{loc}} + h_n^0 w_{n,\alpha,L}^{0,\text{loc,h}}, w_{n,\alpha,L}^{0,\text{loc,3}}]_{h_n^0}, \end{aligned}$$

where $v_{n,\alpha,L}^{0,\text{loc}}$ solves (NS2D) $_{x_3}$ with data $v_{0,n,\alpha,L}^{0,\text{loc}}$, $w_{n,\alpha,L}^{0,\text{loc,3}}$ solves the transport-diffusion equation $(T_{h_n^0})$ defined page 22 with data $w_{0,n,\alpha,L}^{0,\text{loc,3}}$ and $w_{n,\alpha,L}^{0,\text{loc,h}} = -\nabla^h \Delta_h^{-1} \partial_3 w_{n,\alpha,L}^{0,\text{loc,3}}$. Combining Property (2.6) together with Propositions 4.7 and 4.8, we infer that

$$\lim_{L \rightarrow \infty} \lim_{\alpha \rightarrow 0} \limsup_{n \rightarrow \infty} \|\Phi_{n,\alpha,L}^{0,\text{loc,app}}(\cdot, 0)\|_{\tilde{L}^r(\mathbb{R}^+; B_{2,1}^{\frac{2}{r}}(\mathbb{R}^2))} = 0,$$

which ends the proof of (4.53) invoking (2.5) and (4.39). The argument is similar for the other estimates. \square

4.2.3. *Conclusion of the proof of Theorem 4.* Now, with the above notations, we look for the solution to (NS) system associated to the Cauchy data $\Phi_{0,n,\alpha,L}^0$ under the form:

$$\Phi_{n,\alpha,L}^0 \stackrel{\text{def}}{=} u_\alpha + \Phi_{n,\alpha,L}^{0,\infty} + \Phi_{n,\alpha,L}^{0,\text{loc}} + \psi_{n,\alpha,L}.$$

In particular the two vector fields $\Phi_{n,\alpha,L}^{0,\text{loc}}$ and $\Phi_{n,\alpha,L}^{0,\infty}$ satisfy Corollary 4.9, and furthermore thanks to the Lebesgue theorem,

$$(4.54) \quad \lim_{\eta \rightarrow 0} \|(1 - \theta_\eta)u_\alpha\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} = 0.$$

Given a small number $\varepsilon > 0$, to be selected later on, we choose L , α and $\eta = \eta(\alpha, L, u_0)$ so that thanks to Corollary 4.9 and (4.54), for all r in $[2, \infty]$, and for n large enough,

$$(4.55) \quad \begin{aligned} & \|\Phi_{n,\alpha,L}^{0,\text{loc}}(\cdot, 0)\|_{L^r(\mathbb{R}^+; B_{2,1}^{\frac{2}{r}}(\mathbb{R}^2))} + \|(1 - \theta_{h,\eta})\Phi_{n,\alpha,L}^{0,\text{loc}}\|_{\mathcal{A}^0} + \|(1 - \theta_\eta)u_\alpha\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \\ & + \|\theta_{h,\eta}\Phi_{n,\alpha,L}^{0,\infty}\|_{\mathcal{A}^0} \leq \varepsilon. \end{aligned}$$

For sake of simplicity, denote in the sequel

$$(\Phi_\varepsilon^{0,\infty}, \Phi_\varepsilon^{0,\text{loc}}, \psi_\varepsilon) \stackrel{\text{def}}{=} (\Phi_{n,\alpha,L}^{0,\infty}, \Phi_{n,\alpha,L}^{0,\text{loc}}, \psi_{n,\alpha,L}) \quad \text{and} \quad \Phi_\varepsilon^{\text{app}} \stackrel{\text{def}}{=} u_\alpha + \Phi_\varepsilon^{0,\infty} + \Phi_\varepsilon^{0,\text{loc}}.$$

By straightforward computations, one can verify that the vector field ψ_ε satisfies the following equation, with null Cauchy data:

$$(4.56) \quad \begin{aligned} & \partial_t \psi_\varepsilon - \Delta \psi_\varepsilon + \text{div}(\psi_\varepsilon \otimes \psi_\varepsilon + \Phi_\varepsilon^{\text{app}} \otimes \psi_\varepsilon + \psi_\varepsilon \otimes \Phi_\varepsilon^{\text{app}}) = -\nabla q_\varepsilon + E_\varepsilon, \\ & \text{with } E_\varepsilon = E_\varepsilon^1 + E_\varepsilon^2 \quad \text{and} \\ & E_\varepsilon^1 \stackrel{\text{def}}{=} \text{div} \left(\Phi_\varepsilon^{0,\infty} \otimes (\Phi_\varepsilon^{0,\text{loc}} + u_\alpha) + (\Phi_\varepsilon^{0,\text{loc}} + u_\alpha) \otimes \Phi_\varepsilon^{0,\infty} \right. \\ & \quad \left. + \Phi_\varepsilon^{0,\text{loc}} \otimes (1 - \theta_\eta)u_\alpha + (1 - \theta_\eta)u_\alpha \otimes \Phi_\varepsilon^{0,\text{loc}} \right), \\ & E_\varepsilon^2 \stackrel{\text{def}}{=} \text{div} \left(\Phi_\varepsilon^{0,\text{loc}} \otimes \theta_\eta u_\alpha + \theta_\eta u_\alpha \otimes \Phi_\varepsilon^{0,\text{loc}} \right). \end{aligned}$$

The heart of the matter consists in proving that

$$(4.57) \quad \lim_{\varepsilon \rightarrow 0} \|E_\varepsilon\|_{\mathcal{F}^0} = 0.$$

Indeed, exactly as in the proof of Theorem 3, this ensures that ψ_ε belongs to $L^2(\mathbb{R}^+; \mathcal{B}^1)$, with

$$\lim_{\varepsilon \rightarrow 0} \|\psi_\varepsilon\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} = 0,$$

and allows to conclude the proof of Theorem 4.

So let us focus on (4.57). The term E_ε^1 is the easiest, thanks to the separation of the spatial supports. Let us first write $E_\varepsilon^1 = E_{\varepsilon,h}^1 + E_{\varepsilon,3}^1$ with

$$(4.58) \quad \begin{aligned} E_{\varepsilon,h}^1 & \stackrel{\text{def}}{=} \text{div}_h \left((\Phi_\varepsilon^{0,\text{loc}} + u_\alpha) \otimes \Phi_\varepsilon^{0,\infty,h} + \Phi_\varepsilon^{0,\infty} \otimes (\Phi_\varepsilon^{0,\text{loc},h} + u_\alpha^h) \right. \\ & \quad \left. + (1 - \theta_\eta)u_\alpha \otimes \Phi_\varepsilon^{0,\text{loc},h} + \Phi_\varepsilon^{0,\text{loc}} \otimes (1 - \theta_\eta)u_\alpha^h \right) \quad \text{and} \\ E_{\varepsilon,3}^1 & \stackrel{\text{def}}{=} \partial_3 \left((\Phi_\varepsilon^{0,\text{loc}} + u_\alpha) \Phi_\varepsilon^{0,\infty,3} + \Phi_\varepsilon^{0,\infty} (\Phi_\varepsilon^{0,\text{loc},3} + u_\alpha^3) \right. \\ & \quad \left. + (1 - \theta_\eta)u_\alpha \Phi_\varepsilon^{0,\text{loc},3} + \Phi_\varepsilon^{0,\text{loc}} (1 - \theta_\eta)u_\alpha^3 \right). \end{aligned}$$

Now using as usual the action of derivatives and the fact that \mathcal{B}^1 is an algebra, we infer that

$$(4.59) \quad \begin{aligned} & \|E_{\varepsilon,h}^1\|_{L^1(\mathbb{R}^+; \mathcal{B}^0)} + \|E_{\varepsilon,3}^1\|_{L^1(\mathbb{R}^+; B_{2,1}^{1,-\frac{1}{2}})} \leq \|\theta_{h,\eta}\Phi_\varepsilon^{0,\infty}\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \|\Phi_\varepsilon^{0,\text{loc}} + u_\alpha\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \\ & \quad + \|(1 - \theta_{h,\eta})(\Phi_\varepsilon^{0,\text{loc}} + u_\alpha)\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \|\Phi_\varepsilon^{0,\infty}\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \\ & \quad + \|\Phi_\varepsilon^{0,\text{loc}}\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \|u_\varepsilon^\infty\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)}, \end{aligned}$$

where we denote by u_ε^∞ the function $(1 - \theta_\eta)u_\alpha$. Thanks to (4.55) and to the a priori bounds on $\Phi_\varepsilon^{0,\infty}$, $\Phi_\varepsilon^{0,\text{loc}}$ and u_α , we easily get

$$(4.60) \quad \lim_{\varepsilon \rightarrow 0} \|E_\varepsilon^1\|_{\mathcal{F}^0} = 0.$$

Next let us turn to E_ε^2 . To this end, we use the following estimate (see for instance, Lemma 3.3 of [16]):

$$(4.58) \quad \|ab\|_{\mathcal{B}^1} \lesssim \|a\|_{\mathcal{B}^1} \|b(\cdot, 0)\|_{B_{2,1}^1(\mathbb{R}^2)} + \|x_3 a\|_{\mathcal{B}^1} \|\partial_3 b\|_{\mathcal{B}^1}.$$

Defining $u_\varepsilon^{\text{loc}} \stackrel{\text{def}}{=} \theta_\eta u_\alpha$, we get applying Estimate (4.58)

$$\begin{aligned} \|E_\varepsilon^2\|_{\mathcal{F}^0} &\lesssim \|u_\varepsilon^{\text{loc}}\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \|\Phi_\varepsilon^{0, \text{loc}}(\cdot, 0)\|_{L^2(\mathbb{R}^+; B_{2,1}^1(\mathbb{R}^2))} \\ &\quad + \|x_3 u_\varepsilon^{\text{loc}}\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)} \|\partial_3 \Phi_\varepsilon^{0, \text{loc}}\|_{L^2(\mathbb{R}^+; \mathcal{B}^1)}. \end{aligned}$$

Thanks to (4.55) as well as Inequality (2.14) of Theorem 3, we obtain

$$\lim_{\varepsilon \rightarrow 0} \|E_\varepsilon^2\|_{\mathcal{F}^0} = 0.$$

This proves (4.57), hence Theorem 4. \square

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