



HAL
open science

Guarantees in Fair Division: General or Monotone Preferences

Anna Bogomolnaia, Hervé Moulin

► **To cite this version:**

Anna Bogomolnaia, Hervé Moulin. Guarantees in Fair Division: General or Monotone Preferences. Mathematics of Operations Research, 2022, 1, pp.1-602, C2. 10.1287/moor.2022.1255 . hal-03886828

HAL Id: hal-03886828

<https://hal.science/hal-03886828>

Submitted on 6 Dec 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Guarantees in Fair Division: general or monotone preferences

Anna Bogomolnaia[♠] and Hervé Moulin[♠]

September 2020

♠ University of Glasgow, UK, and Higher School of Economics, St Petersburg, Russia

Abstract

To divide a "manna" Ω of private items (commodities, workloads, land, time intervals) between n agents, the worst case measure of fairness is the welfare guaranteed to each agent, irrespective of others' preferences. If the manna is non atomic and utilities are continuous (not necessarily monotone or convex), we can guarantee the min-Max utility: that of our agent's best share in her worst partition of the manna; and implement it by Kuhn's generalisation of Divide and Choose. The larger Maxmin utility – of her worst share in her best partition – cannot be guaranteed, even for two agents.

If for all agents more manna is better than less (or less is better than more), our Bid & Choose rules implement guarantees between minMax and Maxmin by letting agents bid for the smallest (or largest) size of a share they find acceptable.

Acknowledgement 1 *We are grateful for the critical comments of Haris Aziz, Steve Brams, Eric Budish, Ariel Procaccia, Richard Stong, and participants in seminars and workshops at LUISS University, UNSW Sydney, Universities of Gothenburg, St Andrews, the London School of Economics, the 12th International Symposium on Algorithmic Game Theory, Athens, and webinars at at ETH Zurich and Caltech. Special thanks to Erel Segal-Halevi and Ron Hozman for their detailed comments on the first version of this paper.*

Bogomolnaia and Moulin acknowledge the support from the Basic Research Program of the National Research University Higher School of Economics.

1 Introduction and the punchlines

The fair division of a common property manna – resources privately consumed – is a complicated problem if its joint owners have heterogenous preferences over the manna. A coarse yet important benchmark is the welfare *Guarantee* a division rule offers to each participant: this is the highest welfare that a given agent can secure in this rule, irrespective of the preferences of other agents, even if our agent is clueless about the latter and assumes the worst. The more an agent is risk averse and the less she knows about others’ preferences, the more this worst case benchmark matters to her.

Our goal is to throw some light on the feasible Guarantees in the very general class of *non atomic* fair division problems: small changes in the size of a share result in small utility changes (a continuity property explained below). Our model places no other restrictions on the structure of preferences and corresponding utilities, or their direction: the manna may contain some desirable parts (money, tasty cake, valuable commodities), some not (unpleasant tasks, financial liabilities, burnt parts of the cake that must still be eaten [33]); agents may disagree over which parts are good or bad; utilities can be single-peaked over some parts (teaching loads, volunteering time, shares of a risky project), single-dipped on others, etc..

Assume that the manna Ω and the domain \mathcal{D} of potential preferences are common knowledge, and define a *Fair Guarantee* as a mapping $(u_i, n) \rightarrow \Gamma(u_i; n)$ selecting for each preference in \mathcal{D} , described for clarity as a utility function u_i , and each number n of joint owners, a utility level. The mapping is fair because it ignores agent i ’s identity, and it must be feasible: for any profile $(u_i)_{i=1}^n$ of utilities in \mathcal{D}^n there exists a partition $(S_i)_{i=1}^n$ of Ω such that $u_i(S_i) \geq \Gamma(u_i; n)$ for all i .

Given the division problem (Ω, \mathcal{D}) we ask what are the best (highest) Fair Guarantees? and what mechanism implements¹ them?

Observe first that any Fair Guarantee $\Gamma(u; n)$ is bounded above by the utility, denoted $Maxmin(u; n)$, of the worst share for u in the best n -partition of the manna. For all $u \in \mathcal{D}$ and n we have

$$\Gamma(u; n) \leq Maxmin(u; n) = \max_{\Pi=(S_i)_{i=1}^n} \min_{1 \leq i \leq n} u(S_i) \quad (1)$$

where the maximum (that may not be achieved exactly) bears on all n -

¹In the simple sense of implementation described in the last paragraph of this section.

partitions $\Pi = (S_i)_{i=1}^n$ of Ω . This follows by feasibility of $\Gamma(u; n)$: at the unanimous profile where $u_i = u$ for all i there is a partition Π such that $u(S_i) \geq \Gamma(u; n)$ for all i , hence $\Gamma(u; n) \leq \min_{1 \leq i \leq n} u(S_i) \leq \text{Maxmin}(u; n)$.

Therefore if $(u, n) \rightarrow \text{Maxmin}(u; n)$ is itself a Fair Guarantee (it is fair, but the issue is feasibility), it is the best possible one and answers the first of the two general questions above. This happens in two well known and much discussed families of fair division problems.

In the cake-cutting model due to Steinhaus ([35]) the manna Ω is a measurable space endowed with a non atomic measure, and utilities are additive measures, absolutely continuous with respect to the base measure. Additivity of u implies $\text{Maxmin}(u; n) \leq \frac{1}{n}u(\Omega)$; this is in fact an equality because the cake can be partitioned in n shares of equal utility. Agent i 's share S_i is *Proportionally Fair* if $u_i(S_i) \geq \frac{1}{n}u_i(\Omega)$: this is feasible for all agents at any preference profile $(u_i)_{i=1}^n$, therefore Proportional Fairness offers the best possible Guarantee in this model, and is the weakest and least controversial test of fairness throughout the cake-cutting literature ([14] and [32]).

In the microeconomic model of fair division the manna is a bundle $\omega \in \mathbb{R}_+^K$ of K divisible and non disposable items, and \mathcal{D} is the set of convex and continuous preferences over $[0, \omega]$ (not necessarily monotonic). It is feasible to give an equal share $\frac{1}{n}\omega$ to every agent, so that $\Gamma^{es}(u; n) = u(\frac{1}{n}\omega)$ is a feasible Guarantee. In \mathcal{D} the inequality $\text{Maxmin}(u; n) \leq u(\frac{1}{n}\omega)$ is also true.² Therefore the optimal Guarantee is Γ^{es} , aka the *Equal Split lower bound* $u_i(z_i) \geq u(\frac{1}{n}\omega)$ (where z_i is i 's share of ω). Here too it is the starting point of the discussion of fairness (see e. g., [38] and [28]).

As soon as we drop either additivity in the former model, or convexity in the latter one, the Maxmin benchmark is not a Fair Guarantee any more: already in some two person problems no division of the manna yields at least $\text{Maxmin}(u_i; 2)$ for both $i = 1, 2$. In a simple example Ann and Bob share 10 units of a single non disposable divisible item (e.g., time spent in a given activity). Ann's preferences are single-peaked (hence convex), while Bob's are single-dipped (*see Figure 1*):

$$u_A(x) = x(12 - x) \quad ; \quad u_B(x) = x(x - 6) \quad \text{for } 0 \leq x \leq 10$$

²Pick a hyperplane H supporting the upper contour of u at $\frac{1}{n}\omega$; the lower contour of u at $\frac{1}{n}\omega$ contains one closed half-space cut by H , and every division of ω as $\omega = \sum_1^n z_i$ includes at least one z_j in that half-space.

Compute

$$Maxmin(u_A) = 35 \text{ at } \Pi_1 = \{5, 5\} ; Maxmin(u_B) = 0 \text{ at } \Pi_2 = \{0, 10\}$$

If Bob's share is worth at least $Maxmin(u_B)$ then Ann gets either the whole manna or at most 4 units: so her utility is at most 32 and we see that $(Maxmin(u_A), Maxmin(u_B))$ is not feasible.

A second critical benchmark utility is $minMax(u; n)$, the utility of the best share for u in the worst possible n -partition of Ω :

$$minMax(u; n) = \min_{\Pi=(S_i)_{i=1}^n} \max_{1 \leq i \leq n} u(S_i)$$

where as before the minimum bears on all n -partitions of Ω .

Our first main result, Theorem 1 in Section 4, says that in any non atomic problem, the mapping $u \rightarrow minMax(u; n)$ is a Fair Guarantee; in particular $minMax(u; n) \leq Maxmin(u; n)$ for all $u \in \mathcal{D}$ and n (by (1)). Moreover the $minMax$ Guarantee is implemented by Kuhn's little known n -person generalisation of Divide and Choose ([20]), denoted here $D\&C_n$.

The result is clear in two person problems, where ordinary Divide and Choose clearly guarantees her $Maxmin$ to the Divider and his $Minmax$ to the Chooser. For instance in the example above Ann would Divide as $\Pi_1 = \{5, 5\}$ and Bob would get utility -5 , exactly his $minMax(u_B; 2)$; while Bob would Divide as $\Pi_2 = \{0, 10\}$, and Ann would Choose 10, thus achieving $minMax(u_A; 2) = 20$.

In three persons problems $D\&C_3$ works as follows. The Divider Ann offers a 3-partition $\Pi = \{S_1, S_2, S_3\}$ where all shares are of equal value to her; Bob *accepts all shares* worth at least $minMax(u_B; 3)$, and Charles all those worth at least $minMax(u_C; 3)$. If Bob and Charles can each be assigned a share they accept, we do so and Ann gets the last piece.³ If both accept a single share in Π , the same one, we give one of the remaining shares S_k to Ann (it does not matter which one) and then run $D\&C_2$ between Bob and Charles for $\Omega \setminus S_k$ (it does not matter who Divides or Chooses).

The n -person division rule $D\&C_n$ proceeds similarly in at most $n - 1$ steps of Division and Acceptance between a shrinking set of agents sharing a shrinking manna. Its only subtlety is a simple combinatorial matching step (Lemma 2 in Section 4) after each partitioning of the remaining manna.

³Maybe more than one such assignment is feasible; any choice implements the target Guarantee, which is all we need.

The hard step in proving Theorem 1 is Lemma 1 in Subsection 3.2, stating that in each round of $D\&C_n$ the current Divider can find an *equipartition*: a partition of the remaining manna where all shares are equally valuable to this Divider. Because we only assume that the manna is measurable and endowed with a non atomic measure, and that utilities are continuous in that measure, the proof of Lemma 1 requires advanced tools in algebraic geometry: this the object of the companion paper [4], see the discussion in Subsection 3.2.

Our second main result, Theorem 2 in Subsection 5.2, focuses on non atomic problems where preferences are also *co-monotone*: that is, *increasing* if enlarging a share cannot make it worse and we speak of a *good manna*; or *decreasing* if the opposite holds and we have a *bad manna*. Either restriction on preferences opens the door to a new family of division rules significantly simpler than $D\&C_n$ and implementing a higher Guarantee than the *minMax*. These rules are inspired by the well known family of *Moving Knife* (MK_n) rules (Dubins and Spanier [19]) that we recall first.

Assume the manna is good: a knife cuts continuously an increasing share of the cake; agents can stop the knife at any time; the *first* agent who does gets the share cut so far. Repeat between the remaining agents and manna. For a bad manna, agents can drop at any time and the *last* one to drop gets the share cut so far.

A Moving Knife (MK) rule chooses a single arbitrary path for the knife, which tightly restricts the range of individual shares and partitions, hence can result in a very inefficient allocation. We introduce a large family of rules in the same spirit as MK but with all partitions in their range, that we call the *Bid & Choose* ($B\&C_n$) rules. Each rule is defined by fixing a benchmark *additive measure* of the shares, diversely interpreted as their size, their market price, etc.. If the manna is good a bid b_i by agent i is the smallest measure of a share that i finds acceptable: the smallest bidder i^* chooses freely a share of measure at most b_{i^*} , then we repeat between the remaining agents and manna. For a bad manna the bid b_i is the largest size of a share that i finds acceptable, and the largest bidder i^* picks any share of size at least b_{i^*} .

Theorem 2 in Section 5 shows that all $B\&C_n$ rules, as well as all MK_n rules implement a Guarantee between the *minMax* and *Maxmin* level.

A handful of examples in Subsection 5.3 show that the $B\&C_n$ Guarantee improves substantially the *minMax* Guarantee in the microeconomic model of fair division. There the Equal Split Guarantee is the *Maxmin* benchmark

(the best possible) for agents with convex preferences, while for agents with “concave” preferences (convex lower contours) Equal Split is the *minMax* Guarantee, which the $B\&C_n$ Guarantee improves significantly.

Throughout the paper we speak of implementation in the very simple sense adopted by most of the cake cutting literature (e. g., [14]), and formalized in the general collective decision context as implementation in “protective equilibrium” (Barbera and Dutta [7]). A rule implements (guarantees) a certain utility level γ means this: no matter what her preferences, each agent has a strategy that depends also upon Ω, n and \mathcal{D} , such that whatever other agents do the utility of her share is no less than γ . Moreover the “guaranteeing strategy” is essentially unique.

2 Relevant literature

The two welfare levels *Maxmin* and *minMax* are key to our results. In the *atomic* model where the manna is a set of indivisible items, they are introduced by Budish ([16]) and Bouveret and Lemaitre ([12]) respectively. If utilities are additive in that model, the basic inequality of our *non atomic* model is reversed:

$$Maxmin(u; n) \leq \frac{1}{n}u(\Omega) \leq minMax(u; n)$$

and *minMax*($u; n$) is obviously not a feasible Guarantee. It took a couple of years and many brain cells to check that the *Maxmin* lower bound may not be feasible either for three or more agents ([31]), though this happens in rare instances of the model ([22]).⁴ Our paper is the first systematic discussion of these two bounds in the non atomic model of cake division.

Kuhn’s 1967 n person generalisation of Divide and Choose ([20]) promptly implements the *minMax* guarantee in our model: Theorem 1. Except for a recent discussion in [1] for additive utilities, $D\&C_n$ has not received much attention, a situation which our paper may help to correct. In particular, unlike the Diminishing Share ([35]) Moving Knife ([19]), and Bid and Choose rules, it is very well suited to divide *mixed manna*, i. e., containing subjectively good and bad parts, as when we divide the assets and liabilities of a

⁴If the manna is atomic and utilities are not necessarily additive, it is easy to construct examples showing that all six orderings of *Maxmin*, *minMax*, and $\frac{1}{n}u(\Omega)$ are possible.

dissolving partnership. Introduced in [11] and [10] for the competitive fair division of commodities in the microeconomic model, the mixed manna model is discussed by [33] for a general cake, and by [5] for indivisible items.

Privacy preservation is a growing concern in a world of ever expanding information flows. The $D\&C_n$ rule stands out for its informational parsimony: each Divider only reports a partition with the understanding that she is indifferent between the two shares she just cut, and Choosers only accept a subset of these shares. If the manna is mixed, no one is asked to explain which parts they view as good or bad: for instance if we divide tasks, I may not want others to know which tasks I am actually happy to perform, or which ones are very painful to me.

The “cuts” selected by Dividers and “queries” answered by Choosers require only a modest cognitive effort: no one needs to form complete preference relations over all shares of the cake. Taking this feature to heart, a large literature in the cake cutting model evaluates the informational complexity of various mechanisms by the number of “cuts” and “queries” they involve: see [14] or [32], and more recently [17] and [18]. This line of research goes beyond the test of Proportional Guarantee, and find cuts and queries division rules more complex than $D\&C_n$ reaching an Envy-free division of the cake. The algorithms in Brams and Taylor ([13]), and more recently Aziz and McKenzie ([6]), do exactly this when utilities are additive and non atomic; but because they involve an astronomical number of cuts and queries they are of no practical interest and squarely contradict informational parsimony. See ([15], [21]) for some fine tuning of these general facts.

The “equipartition” Lemma (Subsection 3.2) is critical to the proof of Theorem 1, and proved in [4] by algebraic geometry techniques. These, or subtle variants of Sperner’s Lemma, demonstrate the existence of an Envy-free division under very general preferences, where which share I like best in a given partition can depend upon the partition itself, not just upon my own share: Stromquist’s ([36]) and Woodall’s ([39]) seminal insights are considerably strengthened by the recent results in [37], [33], [25] and [3]. However all these results assume that, either all agents (weakly) prefer any non empty share to the empty share, or all weakly prefer the empty share to any non empty one: this rules out a mixed manna.

We noted earlier that the concept of *unanimity utility* (the common efficient utility level in the economy where everyone has the same preferences) leads to the Equal Split Guarantee when we divide private goods and preferences are convex (see Footnote 2). When applied to fair division problems

involving production, it defines some compelling Fair Guarantees as well as some meaningful *upper bounds* on individual welfare: [27], [26].

3 Non atomic fair division

3.1 Basic definitions

The manna Ω is a bounded measurable set in an euclidian space, endowed with the Lebesgue measure $|\cdot|$, and such that $|\Omega| > 0$. A share S is a possibly empty measurable subset of Ω , and \mathcal{B} is the set of all shares. A n -partition of Ω is a n -tuple of shares $\Pi = (S_i)_{i=1}^n$ such that $\cup_{i=1}^n S_i = \Omega$ and $|S_i \cap S_j| = 0$ for all $i \neq j$; and $\mathcal{P}_n(\Omega)$ is the set of all partitions of Ω . We define similarly an n -partition of S for any share $S \in \mathcal{B}$, and write their set as $\mathcal{P}_n(S)$.

If $S \otimes T = (S \cup T) \setminus (S \cap T)$ is the symmetric difference of shares, recall that $\delta(S, T) = |S \otimes T|$ is a pseudo-metric on \mathcal{B} (a metric except that $\delta(S, T) = 0$ iff S and T differ by a set of measure zero).

A utility function u is a mapping from \mathcal{B} into \mathbb{R} such that $u(\emptyset) = 0$ and u is continuous for the pseudo-metric δ and bounded. So u does not distinguish between two shares at pseudo-distance zero (equal up to a set of measure zero): for instance $u(S) = 0$ if $|S| = 0$. Also if the sequence $|S^t|$ converges to zero in t , so does $u(S^t)$. We write $\mathcal{D}(\Omega)$ for this domain of utility functions.

So a non atomic division problem consists of $(\Omega, \mathcal{B}, (u_i)_{i=1}^n \in \mathcal{D}(\Omega)^n)$.

Several subdomains of $\mathcal{D}(\Omega)$ play a role below:

- additive utilities: $u \in \mathcal{Add}(\Omega)$ iff $u(S) = \int_S f(x)dx$ for all S , where f is bounded and measurable in Ω ;
- monotone increasing: $u \in \mathcal{M}^+(\Omega)$ iff $S \subset T \implies u(S) \leq u(T)$ for all S, T ;
- monotone decreasing: $u \in \mathcal{M}^-(\Omega)$ iff $S \subset T \implies u(S) \geq u(T)$ for all S, T ;
- separable: $u \in \mathcal{S}(\Omega)$ iff there is a finite set A , a partition $(C_a)_{a \in A} \in \mathcal{P}_{|A|}(\Omega)$ of Ω , and a continuous function v from \mathbb{R}_+^A into \mathbb{R} , such that $u(S) = v(|S \cap C_a|)_{a \in A}$ for all $S \in \mathcal{B}$.

The separable domain $\mathcal{S}(\Omega)$ captures the standard microeconomic fair division model: A is a set of divisible commodities, the manna is the bundle $\omega \in \mathbb{R}_+^A$ such that $\omega_a = |C_a|$ for all a , a share S_i gives to agent i the amount $z_{ia} = |S_i \cap C_a|$ of commodity a , and the partition $\Pi = (S_i)_{i=1}^n$ corresponds to the division of the manna as $\omega = \sum_1^n z_i$.

In the general non atomic division problem, the set of shares \mathcal{B} is not compact for the pseudo-metric δ . It follows that when we maximize or minimize utilities over shares, or look for a partition achieving a benchmark utility *minMax* or *Maxmin*, we cannot claim the existence of an exact solution to the program: the *minMax* is not a true minimum, only an infimum, and *Maxmin* is only a supremum, not a true maximum. As this will cause no confusion, we stick to the *min* and *Max* notation throughout.

However in the microeconomic model, the set of shares and of partitions are both compact so for this important set of problems (where all our examples live) the *min* and *Max* notation are strictly justified.

One can also specialise the general model by imposing constraints on the set of feasible shares. The most important instance is the familiar *interval model*, where the manna is $\Omega = [0, 1]$ and a share must be an interval, so an n -partition is made of n adjacent intervals. Other instances assume Ω is a polytope, and shares are polytopes of a certain type: e.g. triangles or tetrahedrons ([34]). And sometimes shares must be connected subsets of Ω ([8], [2]).

The Divide and Choose $_n$ rules, as well as our Bid and Choose $_n$ rules, do not work in these models⁵, so our Theorems 1 and 2 do not apply. But the interval model is still useful here in a technical sense: the proof of the critical Lemma 1 in Subsection 3.2 starts by projecting the general problem onto an interval model and proving existence of an equipartition there.

3.2 Equipartitions

Definition 1 *An n -equipartition of the share $T \in \mathcal{B}$ for utility $u \in \mathcal{D}(T)$ is a partition $\Pi^e = (S_i)_{i=1}^n \in \mathcal{P}_n(T)$ such that $u(S_i) = u(S_j)$ for all $i, j \in \{1, \dots, n\}$; we write $u(\Pi^e)$ for this common value, and $\mathcal{EP}_n(T; u)$ for the set of these n -equipartitions.*

⁵For instance in the interval model, the first divider can find an equipartition made of adjacent intervals (by our Lemma 1), but the next agent called to divide is typically left with disconnected intervals.

It is clear that $\mathcal{EP}_n(S; u)$ is non empty if u is additive: if $\mathcal{B}[S]$ is the subset of shares included in S , Lyapunov Theorem implies that the range $u(\mathcal{B}[S])$ is convex, so it contains $\frac{1}{n}u(S)$; then we replace n by $n - 1$ and repeat the argument on the remaining share.

The same is true if u is monotone ($u \in \mathcal{M}^\pm(\Omega)$), and the proof, outlined in Remark 1 below, is fairly simple. That of our next statement is much harder.

Lemma 1 ([4])

Fix a share $S \in \mathcal{B}$ and a utility $u \in \mathcal{D}(\Omega)$. The set $\mathcal{EP}_n(S; u)$ of n -equipartitions of S at u is non empty.

Proof. The Theorem in [4] proves Lemma 1 for the interval model (which, as mentioned above, is not a special case of our model). Fix a real valued function f on the set of intervals $[a, b] \subset [0, 1]$, continuous in the standard topology and such that $f(a, a) = 0$ for all $a \in [0, 1]$. Then there exist n subintervals $[0 = x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n = 1]$ of $[0, 1]$ forming an equipartition of f : $f(x_{i-1}, x_i)$ is constant for $i = 1, \dots, n$.

Start now from a share S in the statement of Lemma 1 and pick a *moving knife* through S , i. e., a path $\kappa : [0, 1] \ni t \rightarrow K(t) \in \mathcal{B}$ from $K(0) = \emptyset$ to $K(1) = S$, continuous for the pseudo-metric δ on \mathcal{B} and weakly inclusion increasing:

$$0 \leq t < t' \leq 1 \implies K(t) \subseteq K(t')$$

(in Subsection 5.1 moving knives must be strictly inclusion increasing). Then the function

$$f(a, b) = u(K(b) \setminus K(a))$$

is as in the previous paragraph, and an f -equipartition $([x_{i-1}, x_i])_{i=1}^n$ of $[0, 1]$ yields the desired u -equipartition $(K(x_i) \setminus K(x_{i-1}))_{i=1}^n$ of S . ■

Remark 1 It is easy to prove Lemma 1 if we assume that the sign of u is constant: all shares are weakly preferred to the empty share, or all are weakly worse. Assume the former and use as above a moving knife to project S onto $[0, 1]$, where a n -partition is identified with a point in the simplex of dimension $n - 1$. Then apply the Knaster–Kuratowski–Mazurkiewicz Lemma to the sets Q_i of partitions of the interval where the i -th interval gives the lowest utility: each Q_i is closed, contains the i -th face of the simplex, and their union covers it entirely. Thus these sets intersect.

One can also invoke the stronger results in [36] and [37] showing the existence of an Envy-free partition under this assumption. But recall that a

key feature in the division of a mixed manna is that the sign of u is **not** constant across shares.

3.3 Two utility benchmarks

Definition 2 Fix n , the manna (Ω, \mathcal{B}) and $u \in \mathcal{D}(\Omega)$:

$$\minMax(u; n) = \min_{\Pi \in \mathcal{P}_n(\Omega)} \max_{1 \leq i \leq n} u(S_i) ; \text{Maxmin}(u; n) = \max_{\Pi \in \mathcal{P}_n(\Omega)} \min_{1 \leq i \leq n} u(S_i) \quad (2)$$

Recall that \minMax is the utility agent u can achieve by having first pick in the worst possible n -partition of Ω , and Maxmin by having last pick in the best possible n -partition of Ω .

Proposition 1

- i) If $u \in \text{Add}(\Omega)$ then $\minMax(u; n) = \text{Maxmin}(u; n) = \frac{1}{n}u(\Omega)$
- ii) If $u \in \mathcal{M}^\pm(\Omega)$

$$\minMax(u; n) = \min_{\Pi^e \in \mathcal{EP}_n(\Omega; u)} u(\Pi^e) ; \text{Maxmin}(u; n) = \max_{\Pi^e \in \mathcal{EP}_n(\Omega; u)} u(\Pi^e) \quad (3)$$

- iii) If $u \in \mathcal{D}(\Omega)$

$$\minMax(u; n) \leq \min_{\Pi^e \in \mathcal{EP}_n(\Omega; u)} u(\Pi^e) \leq \max_{\Pi^e \in \mathcal{EP}_n(\Omega; u)} u(\Pi^e) \leq \text{Maxmin}(u; n) \quad (4)$$

Proof

Statement iii) If Π^e is an n -equipartition, $u(\Pi^e)$ is the utility of its best share, hence $\minMax(u; n) \leq u(\Pi^e)$; proving the other inequality in (4) is just as easy.

Statement i) By additivity of u , for any n -partition Π we have $\max_i u(P_i) \geq \frac{1}{n}u(\Omega)$ implying $\minMax(u; n) \geq \frac{1}{n}u(\Omega)$; we check symmetrically $\frac{1}{n}u(\Omega) \geq \text{Maxmin}(u; n)$, and the conclusion follows by comparing these inequalities to those in (4).

Statement ii) Assume $u \in \mathcal{M}^+(\Omega)$; the proof for $\mathcal{M}^-(\Omega)$ is identical. The continuity and monotonicity of u imply: if S, T are two disjoint shares such that $u(S) > u(T)$, we can trim part of S and add it to T to get two disjoint shares with equal utility in between $u(S)$ and $u(T)$. Expanding this argument, if S_1, \dots, S_k and T are disjoint shares such that

$$u(S_1) = u(S_2) = \dots = u(S_k) > u(T)$$

we can simultaneously trim S_1, \dots, S_k keeping them of equal utility and add the trimming to T , so that the resulting $k + 1$ shares are all equally good and their common utility is between the two utilities above. Iterating this process, we see that if $\Pi = (S_i)_{i=1}^n \in \mathcal{P}_n(\Omega)$ is such that $\max_{1 \leq i \leq n} u(S_i) > \min_{1 \leq j \leq n} u(S_j)$, we can construct an equipartition $\Pi^e \in \mathcal{EP}_n(\Omega; u)$ such that

$$\max_{1 \leq i \leq n} u(S_i) > u(\Pi^e) > \min_{1 \leq j \leq n} u(S_j)$$

Now fix $\varepsilon > 0$, arbitrarily small, pick $\Pi = (S_i)_{i=1}^n \in \mathcal{P}_n(\Omega)$ such that $\min_{1 \leq j \leq n} u(S_j) \geq \text{Maxmin}(u; n) - \varepsilon$, and assume that Π is not an equipartition. By the argument above we can find $\Pi^e \in \mathcal{EP}_n(\Omega; u)$ such that $u(\Pi^e) > \min_{1 \leq j \leq n} u(S_j)$, therefore Π^e too is an ε -approximation of $\text{Maxmin}(u; n)$, and the right-hand inequality in (3) follows. The proof of the left-hand inequality is similar. ■

In the general domain $\mathcal{D}(\Omega)$, the partitions achieving the *Maxmin* and *minMax* utilities are not necessarily equipartitions. In the microeconomic example of Section 1, Ann has single-peaked preferences and her *minMax* is achieved by the all-or-nothing partition $\{\emptyset, \Omega\}$; Bob has single-dipped preferences and the same partition delivers his *Maxmin*; but $\{\emptyset, \Omega\}$ is not an equipartition for either utility.

Remark 2: In the interval model with a monotone utility u , it is easy to check that any two n -equipartitions have the same utility and in turn this implies $\text{minMax}(u; n) = \text{Maxmin}(u; n)$: hence this is the best Fair Guarantee. The numerical example above can be viewed as an instance of the interval model where the two agents are indifferent between $[0, x]$ and $[1 - x, 1]$ for all x : so only the inequality (4) holds true in the general (non monotone) interval model.

3.4 Fair Guarantees

Definition 3 Fix the manna (Ω, \mathcal{B}) and a subdomain \mathcal{D}^* , $\mathcal{D}^* \subseteq \mathcal{D}(\Omega)$. A Fair Guarantee in \mathcal{D}^* is a mapping $\Gamma : u \rightarrow \Gamma(u; n)$ such that for any profile $(u_i)_{i=1}^n \in (\mathcal{D}^*)^n$ there exists $\Pi = (S_i)_{i=1}^n \in \mathcal{P}_n(\Omega)$ such that $u_i(S_i) \geq \Gamma(u; n)$ for all i .

In Section 1 we observed, by looking at unanimity profiles, that $\text{Maxmin}(\cdot; n)$ is an upper bound for any Fair Guarantee: inequality (1). We also mentioned two subdomains where $\text{Maxmin}(\cdot; n)$ itself is a (hence the optimal)

Fair Guarantee: the additive domain $\mathcal{Add}(\Omega)$ and the subdomain of the separable one $\mathcal{S}(\Omega)$ where preferences are also convex. Finally we used the Ann and Bob microeconomic example with a single commodity to show that $Maxmin(\cdot; n)$ is not a Fair Guarantee in $\mathcal{D}(\Omega)$, even for $n = 2$ and a one dimensional manna.

Before proving in the next Section that $minMax(\cdot; n)$ is a Fair Guarantee in the whole domain $\mathcal{D}(\Omega)$ we construct a microeconomic problem with two divisible items and two agents u_1 and u_2 where

$$\begin{aligned} minMax(u_i; 2) &= 0 < 1 = Maxmin(u_i; 2) \text{ for } i = 1, 2 \\ \text{and } (minMax(u_1), minMax(u_2)) &\text{ is weakly Pareto optimal} \end{aligned}$$

This implies that for *any* Fair Guarantee Γ , at least one of $\Gamma(u_1; 2) = 0$ and $\Gamma(u_2; 2) = 0$ must hold. In words, for some problems, no Fair Guarantee can reduce the gap from $minMax$ to $Maxmin$ for both agents.⁶

The manna is $\omega = (1, 1)$ and we a share as $z = (x, y)$. Both utilities are symmetric in x, y : $u_i(x, y) = u_i(y, x)$ so it is enough to define them for $x \leq y$:

$$\begin{aligned} u_1(z) &= 0 && \text{if } x \leq \frac{1}{2} \leq y \\ u_1(z) &= 1 - 2y && \text{if } x \leq y \leq \frac{1}{2} \\ u_1(z) &= 2x - 1 && \text{if } \frac{1}{2} \leq x \leq y \end{aligned}$$

$$\begin{aligned} u_2(z) &= 0 && \text{if } x \leq y \leq \frac{1}{2} \text{ or } \frac{1}{2} \leq x \leq y \\ u_2(z) &= 2y - 1 && \text{if } \frac{1}{2} \leq y \leq 1 - x \\ u_2(z) &= 1 - 2x && \text{if } \frac{1}{2} \leq 1 - x \leq y \end{aligned}$$

The range of both utilities is $[0, 1]$. Agent 1's utility $u_1(z_1)$ is null in the NW and SE quadrant of the box $[0, 1]^2$ with center at $(\frac{1}{2}, \frac{1}{2})$; it is strictly positive in the SW and NE quadrants except on the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$. Agent 2's utility $u_2(z_2)$ is symmetrically null in the SW and NE quadrants, and strictly positive in the NW and SE quadrants except on the same two lines. Therefore for any division $(1, 1) = z_1 + z_2$ of the manna we have $u_1(z_1) \cdot u_2(z_2) = 0$: there is no feasible division s. t. $u_i(z_i) > 0$ for $i = 1, 2$.

The partition $\{(0, 0), (1, 1)\}$ achieves $Maxmin(u_1) = 1$ and $minMax(u_2) = 0$; the partition $\{(0, 1), (1, 0)\}$ achieves $Maxmin(u_2) = 1$ and $minMax(u_1) = 0$.

⁶Divide and Choose implements the utility profile $(minMax(u_i; 2), Maxmin(u_j; 2))$: this gap can be closed for one agent.

4 The Divide & Choose_n rule

Start by a combinatorial observation. Let G be a bilateral graph between the sets M of agents and R of shares: interpret $(m, r) \in G$ as agent m likes share r . We say that the subset \widetilde{M} of agents are *properly matched* to the subset \widetilde{R} of shares if $|\widetilde{M}| = |\widetilde{R}|$, agents in \widetilde{M} are each matched (one-to-one) to a share they like in \widetilde{R} , and no one outside \widetilde{M} likes any share in \widetilde{R} .

Lemma 2. *Assume $|M| = |R|$, each agent in M likes at least one object in R and some agent i^* likes all objects in R . Then there is a (non empty) largest set M^* of properly matchable agents containing i^* : if \widetilde{M} is properly matched to \widetilde{R} , then $\widetilde{M} \subseteq M^*$.*

Proof. We apply the Gallai-Edmonds decomposition of a bipartite graph: see e.g. [23] Chap 3 (or Lemma 1 in [9]). If M can be matched with R this is a proper match and the statement holds true. If M and R cannot be matched, then we can uniquely partition M as (M^+, M^*) and R as (R^+, R^*) such that:

1. $|M^+| > |R^+|$, the agents in M^+ do not like any object in R^* , and they compete for the over-demanded objects in R^+ : every subset of R^+ is liked by a strictly larger subset of agents in M^+ ;
2. $|M^*| < |R^*|$ and the agents in M^* can be matched with some subset of R^* .

By the general Gallai-Edmonds result, M^+ and R^* are non empty. Here M^* is non empty as well because it contains the special agent i^* . Every match of M^* to a subset of R^* is proper. Finally suppose \widetilde{M} is properly matched to \widetilde{R} and $\widehat{M} = \widetilde{M} \cap M^+$ is non empty. Then \widehat{M} is matched to some subset \widehat{R} of R^+ but \widehat{R} is liked by more agents in M^+ than there are in \widehat{M} , therefore the match is not proper: contradiction. So \widetilde{M} does not intersect M^+ as was to be proved. ■

Definition 4: *the D&C_n rule.*

Fix the manna (Ω, \mathcal{B}) and the ordered set of agents $N = \{1, \dots, n\}$, each with a utility in $\mathcal{D}(\Omega)$.

Step 1. Agent 1 proposes a partition $\Pi^1 \in \mathcal{P}_n(\Omega)$; all other agents report which shares in Π^1 they like (at least one). In the resulting bipartite graph between N and the shares in Π^1 , where agent 1 likes all the shares, we use Lemma 2 to match properly the largest possible set of agents N^1 (it contains agent 1) with some set of shares R ; if $N^1 = N$ we are done, otherwise we go to

Step 2. Repeat with the remaining manna Ω^2 and agents in $N \setminus N^1$. Ask the first agent in the exogenous ordering to propose a partition $\Pi^2 \in \mathcal{P}_{n-|N_1|}(\Omega^2)$, while others report which of these new shares they like. And so on.

At least one agent, the Divider, is served in each step, thus the algorithm just described takes at most $n - 1$ steps. But the algorithm matches as many agents as possible, so as to minimize information disclosure, and typically takes fewer steps.

There is some flexibility in the Definition of the rule: although the set of agents matched in each step is unambiguous, we have typically several choices for the set R of shares to assign in each step, and multiple ways to assign these to the agents.

Our first main result is that *minMax* is a Fair Guarantee, implemented by the $D\&C_n$ rule in the full domain $\mathcal{D}(\Omega)$.

Theorem 1

Fix the manna (Ω, \mathcal{B}) and n .

- i) In the $D\&C_n$ rule, an agent with utility $u \in \mathcal{D}(\Omega)$ guarantees the $\text{minMax}(u; n)$ utility level by 1) when called to divide, proposing an equipartition $\Pi^e \in \mathcal{EP}_m(S; u)$ of the remaining share S of manna among the m remaining agents, and 2) when reporting shares he likes, accepting **all** shares, and only those, not worse than $\text{minMax}(u; n)$ (the *minMax* level in the **initial** problem).*
- ii) Moreover the first Divider (and no one else) guarantees her *Maxmin* utility by proposing her *Maxmin* partition in Step 1. Other agents cannot guarantee more than their *minMax* utility.*

Proof. *Statement i).* Consider agent u using the strategy in the statement. At a step where he must report which shares he likes among those offered at that step, he can for sure find one worth at least $\text{minMax}(u; n)$: all shares previously assigned are worth to him strictly less than $\text{minMax}(u; n)$, and together with the freshly cut shares they form a partition in $\mathcal{P}_n(\Omega)$; in any partition at least one share is worth $\text{minMax}(u; n)$ or more.

At a step where our agent is called to cut, he proposes to the remaining agents an m -equipartition $\Pi^e \in \mathcal{EP}_m(S; u)$ of the remaining manna S . To check the inequality $u(\Pi^e) \geq \text{minMax}(u; n)$ note that Π^e together with the previously assigned shares is a partition of Ω in which the old shares are worth strictly less than $\text{minMax}(u; n)$.⁷

⁷After Step 1 an agent can secure his *Maxmin* utility for the smaller manna S among m agents, but this may be below the *Maxmin* utility in the initial problem.

Statement ii). This is clear for the first Divider. Fix now an agent i with utility u and check that if he is not the first Divider, for certain moves of the other agents, agent u gets exactly his *minMax* utility. Pick a partition $\Pi \in \mathcal{P}_n(\Omega)$ achieving $\text{minMax}(u; n)$ (as usual, the existence assumption is without loss). Suppose that the first Divider, who is not agent i , offers Π , and all agents other than i (including the Divider) find all shares acceptable: then a full match is feasible (i must accept at least one share) so i 's share cannot be worth more than $\text{minMax}(u; n)$. ■

5 Bid and Choose and Moving Knives for good or bad manna

We now assume that the manna is unanimously good, $u \in \mathcal{M}^+(\Omega)$, or unanimously bad, $u \in \mathcal{M}^-(\Omega)$. Because $u(\emptyset) = 0$, for all S we have $u(S) \geq 0$ in the former case and $u(S) \leq 0$ in the latter. Recall that in these two domains, the *minMax* (resp. *Maxmin*) utility is the smallest (resp. largest) equipartition utility: property (3) in Proposition 1.

We check first that the profile of *Maxmin* utility levels still may not be feasible, even in the simple microeconomic model (corresponding to the separable domain $\mathcal{S}(\Omega)$ in Subsection 3.1). We have one unit each of two divisible goods, $\omega = (1, 1)$, to be shared between two agents. The first agent has Leontief preferences $u_1(z) = \min\{x, y\}$ so his worst case partition is $\Pi = \{(1, 0), (0, 1)\}$ and his best one is the equal split partition $\Pi' = \{\frac{1}{2}\omega, \frac{1}{2}\omega\}$: $\text{minMax}(u_1; 2) = 0 < \frac{1}{2} = \text{Maxmin}(u_1; 2)$. Agent 2 has anti-Leontief preferences: $u_2(z) = \max\{x, y\}$. For her the equal split partition Π' is the worst and the best one is Π : $\text{minMax}(u_2; 2) = \frac{1}{2} < 1 = \text{Maxmin}(u_2; 2)$. Clearly the profile of *Maxmin* utilities $(\frac{1}{2}, 1)$ is not feasible, while D&C₂ implements $(\frac{1}{2}, \frac{1}{2})$ and $(0, 1)$, depending on who is the Divider.

We show that the *minMax* guarantee is always improved, at least weakly, by the large family of Bid and Choose (B&C _{n}) rules, inspired by the familiar Moving Knives (MK _{n}) rules ([19]).

5.1 MK _{n} ^{κ} and B&C _{n} ^{θ} rules

A *moving knife* through the manna $(\Omega, \mathcal{B}, |\cdot|)$ is a path $\kappa : [0, 1] \ni t \rightarrow K(t) \in \mathcal{B}$ from $K(0) = \emptyset$ to $K(1) = \Omega$, continuous for the pseudo-metric δ

on \mathcal{B} and strictly inclusion increasing:

$$0 \leq t < t' \leq 1 \implies K(t) \subset K(t') \text{ and } |K(t') \setminus K(t)| > 0$$

The moving knife κ arranges shares of increasing value to all participants along the specific path of the knife. An example is $K(t) = B(t) \cap \Omega$, where $t \rightarrow B(t)$ is a path of balls with a fixed center and radius growing from 0 to 1, so that $B(1)$ contains Ω . Moving knives can take many other shapes, for instance hyperplanes.

Our Bid and Choose rules offer more choices than Moving Knives to the agents, with the help of a benchmark *measure* θ of the shares, chosen by the rule designer: θ is a positive σ -additive measure on (Ω, \mathcal{B}) , normalised to $\theta(\Omega) = 1$. It is absolutely continuous w.r.t. the Lebesgue measure $|\cdot|$ and vice versa: the density of θ w.r.t. $|\cdot|$ is strictly positive. In particular θ is strictly inclusion increasing:

$$\forall S, T \in \mathcal{B} : S \subset T \text{ and } |T \setminus S| > 0 \implies \theta(S) < \theta(T)$$

In applications θ can evaluate for instance the market value, physical size, or weight of a share.

Fixing a moving knife κ and a measure θ , we define in parallel the Moving Knife (MK_n^κ) and the Bid and Choose ($B\&C_n^\theta$) rules. In both cases a clock t runs from $t = 0$ to $t = 1$.

Definition 5 *the MK_n^κ and $B\&C_n^\theta$ rules with increasing utilities*

Step 1. The first agent i_1 to stop the clock, at t^1 , gets the share $K(t^1)$ in MK_n^κ , or in $B\&C_n^\theta$ chooses any share in Ω s.t. $\theta(S) = t^1$, say S_{i_1} , and leaves;
Step k : Whoever stops the clock first at t^k gets the share $K(t^k) \setminus K(t^{k-1})$ in MK_n^κ , or in $B\&C_n^\theta$ chooses any share in $\Omega \setminus \cup_1^{k-1} S_{i_\ell}$ s.t. $\theta(S) = t^k - t^{k-1}$, say S_{i_k} , and leaves;

In Step $n - 1$ the single remaining agent who did not stop the clock takes the remaining share $\Omega \setminus K(t^{n-1})$ or $\Omega \setminus \cup_1^{n-1} S_{i_\ell}$.

Definition 5* *with decreasing utilities*

In each step all agents must choose a time to “drop”, and the last agent i_1 who drops, at t^1 , gets $K(t^1)$ in MK_n^κ , or in $B\&C_n^\theta$ chooses S_{i_1} s.t. $\theta(S_{i_1}) = t^1$. The other steps are similarly adjusted.

Breaking ties between agents stopping the clock (or dropping) at the same time is the only indeterminacy in these rules, much less severe than in $D\&C_n$, where we serve at each step an unambiguous set of agents, but there are typically several ways to match them properly.

Up to tie-breaking, $B\&C_n^\theta$ and MK_n^κ are anonymous (do not discriminate between agents) but not neutral (do discriminate between shares), while $D\&C_n$ is neutral but not anonymous.

In MK_n^κ the share of an agent takes the form $K(t) \setminus K(t')$ so it covers a set of dimension 2 (and feasible partitions move in a set of dimension $n - 1$). By contrast every partition in $\mathcal{P}_m(\Omega)$ is feasible under the $B\&C_n^\theta$ rule.

To check this fix $\Pi = (S_i)_{i=1}^n$ and assume first $|S_i| > 0$ for all i . Consider n agents deciding (cooperatively) to achieve Π . By the strict monotonicity of θ the sequence $t^i = \theta(\cup_{j=1}^i S_j)$ increases strictly therefore they can stop the clock (or drop) at these successive times and choose the corresponding shares in Π . If there are shares of measure zero they can all be distributed at time 0.

On the other hand in $B\&C_n^\theta$ all but one agent must pick a share under constraints, thus revealing more information than in MK_n^κ . Loosely speaking, $B\&C_n^\theta$ is informationally comparable to $D\&C_n$.

Remark 3. We can also implement the Guarantees described in the next Subsection by alternative static versions of MK_n^κ and $B\&C_n^\theta$ where agents bid all at once for potential stopping times; we do not discuss these rules for the sake of brevity.

5.2 $B\&C^\theta$ and MK^κ Guarantees

We fix an increasing utility $u \in \mathcal{M}^+(\Omega)$. The results are identical, and identically phrased, for a bad manna $u \in \mathcal{M}^-(\Omega)$. See also Remark 4 at the end of this Subsection.

Define the triangle $\mathcal{T} = \{(t^1, t^2) | 0 \leq t^1 \leq t^2 \leq 1\}$ in \mathbb{R}_+^2 and the set $\Upsilon(n)$ of increasing sequences $\tau = (t^k)_{0 \leq k \leq n}$ in $[0, 1]$ s.t.

$$t^0 = 0 \leq t^1 \leq \dots \leq t^{n-1} \leq 1 = t^n$$

For a moving knife κ , utilities of the shares in MK_n^κ are described by the function u^κ on \mathcal{T} :

$$u^\kappa(t^1, t^2) = u(K(t^2) \setminus K(t^1)) \text{ for all } (t^1, t^2) \in \mathcal{T}$$

For a measure θ , the corresponding definition in $B\&C^\theta$ is the *indirect utility* u^θ :

$$u^\theta(t^1, t^2) = \min_{T: \theta(T)=t^1} \max_{S: S \cap T = \emptyset; \theta(S)=t^2-t^1} u(S) \text{ for all } (t^1, t^2) \in \mathcal{T} \quad (5)$$

Both u^κ and u^θ decrease (weakly) in t^1 and increase (weakly) in t^2 .

We show below that the Guarantees Γ^κ and Γ^θ implemented by MK_n^κ and $B\&C_n^\theta$ respectively are computed as

$$\Gamma^\alpha(u; n) = \max_{\tau \in \Upsilon(n)} \min_{0 \leq k \leq n-1} u^\alpha(t^k; t^{k+1}) \text{ where } \alpha \text{ is } \kappa \text{ or } \theta \quad (6)$$

For instance in MK_2^κ with two agents, write τ_κ for the (not necessarily unique) position of the knife making our agent indifferent between the share $K(\tau_\kappa)$ and its complement. Then

$$\Gamma^\kappa(u; 2) = \max_{0 \leq t^1 \leq 1} \min\{u(K(t^1)), u(\Omega \setminus K(t^1))\} = u(K(\tau_\kappa)) = u(\Omega \setminus K(\tau_\kappa))$$

In $B\&C_2^\theta$ the bid τ_θ makes the best share of size τ_θ as good as the worst share of size $1 - \tau_\theta$:

$$\Gamma^\theta(u; 2) = \max_{0 \leq t^1 \leq 1} \min\left\{\max_{\theta(S)=t^1} u(S), \min_{\theta(S)=t^1} u(\Omega \setminus S)\right\} = \max_{\theta(S)=\tau_\theta} u(S) = \min_{\theta(S)=\tau_\theta} u(\Omega \setminus S) \quad (7)$$

Lemma 4

- i) The utility u^κ and the indirect utility u^θ are continuous. Both the minimum and maximum in (5) are achieved.*
- ii) The maximum of problem (6) (for both rules) is achieved at some $\tau \in \Upsilon(n)$ where the sequence t^k increases in k , all the $u^\alpha(t^k; t^{k+1})$ are equal, and this common utility is the optimal value of (6).*

Proof in the Appendix.

Theorem 2

Fix the manna (Ω, \mathcal{B}) , the number of agents n , and a utility $u \in \mathcal{M}^+(\Omega)$.

- i) With the MK_n^κ rule, an agent guarantees the utility $\Gamma^\kappa(u; n)$ by committing to stop the knife at t_κ^k if exactly $k - 1$ other agents have been served before;*
- ii) With the $B\&C_n^\theta$ rule, she guarantees $\Gamma^\theta(u; n)$ by stopping the clock at t_θ^k if exactly $k - 1$ other agents have been served before; and choosing then the best available share of size $t^k - t^{k-1}$.*
- iii) $\min \text{Max}(u; n) \leq \Gamma^\alpha(u; n) \leq \text{Max} \min(u; n)$ where α is κ or θ .*

Proof.

Statement i) and iii) for MK_n^κ . Recall the equipartition $\Pi = (K(t_\kappa^k) \setminus K(t_\kappa^{k-1}))_1^n$ has $u(\Pi) = \Gamma^\kappa(u; n)$. Thus (3) in Proposition 1 implies the inequalities *iii)*. Next if the knife has been stopped $k - 1$ times before our agent is served, the last stop occurred at or before t_κ^{k-1} therefore if she does stop the knife at t_κ^k

(and wins the possible tie break) her share is at least $K(t_\kappa^k) \setminus K(t_\kappa^{k-1})$. If she never gets to stop the knife, the last stop is at or before t_κ^{n-1} and she gets at least $\Omega \setminus K(t_\kappa^{n-1})$.

Statement ii). If she is the first to stop the clock (perhaps also winning the tie break) at step k , in step $k-1$ the clock stopped at $t^{k-1} \leq t_\theta^{k-1}$ and the share T already distributed at that time has $\theta(T) = t^{k-1}$: therefore she can choose a share with utility $u^\theta(t^{k-1}; t_\theta^k) \geq u^\theta(t_\theta^{k-1}; t_\theta^k) = \Gamma^\theta(u; n)$. If she is the last to be served, having never stopped the clock (or lost some tie breaks) the share assigned to all other agents has $\theta(T) = t^{n-1} \leq t_\theta^{n-1}$ therefore her share is worth $u^\theta(t^{n-1}; 1) \geq u^\theta(t_\theta^{n-1}; 1) = \Gamma^\theta(u; n)$.

Statement iii) for $B\mathcal{E}C_n^\theta$.

Right hand inequality. It is enough to construct a partition $\Pi = (S_k)_1^n$ in which the utility of every share $S_k, 0 \leq k \leq n-1$ is at least $u^\theta(t_\theta^{k-1}, t_\theta^k)$, implying $\min_k u(S_k) \geq \Gamma^\theta(u; n)$. We proceed by induction on the steps of $B\mathcal{E}C_n^\theta$. First S_1 maximizes $u(S)$ s.t. $\theta(S) = t_\theta^1$ so $u(S_1) = u^\theta(0; t_\theta^1) = \Gamma^\theta(u; n)$ and $\theta(S_1) = t_\theta^1$. Assume the sets S_ℓ are constructed for $1 \leq \ell \leq k$, mutually disjoint, s.t. $\theta(S_\ell) = t_\theta^\ell - t_\theta^{\ell-1}$ and $u(S_\ell) \geq u^\theta(t_\theta^\ell, t_\theta^{\ell-1})$: then the set $T = \cup_1^k S_\ell$ is of size t_θ^k and we pick S_{k+1} maximizing $u(S)$ s.t. $S \cap T = \emptyset$ and $\theta(S) = t_\theta^{k+1} - t_\theta^k$. By definition (5) we have $u(S_k) \geq u^\theta(t_\theta^k; t_\theta^{k+1})$ and the induction proceeds. Note that in fact $\min_k u(S_k) = \Gamma^\theta(u; n)$.

Left hand inequality. We need now construct a partition $\Pi = (R_k)_1^n$ s. t. $u(R_k) \leq u^\theta(t_\theta^{k-1}; t_\theta^k)$ for $1 \leq k \leq n$. We do this by a decreasing induction in n . In (the first) step n of the induction we define the 2-partition $\Pi^n = (T_{n-1}, R_n)$ of Ω where T_{n-1} is any solution of the program $\min_{T: \theta(T) = t_\theta^{n-1}} u(\Omega \setminus T)$, and $R_n = \Omega \setminus T_{n-1}$. Thus $u(R_n) = u^\theta(t_\theta^{n-1}; 1)$ and $\theta(T_{n-1}) = t_\theta^{n-1}$.

Assume that in step k we constructed the $(n-k+2)$ -partition $\Pi^k = (T_{k-1}, R_k, R_{k+1}, \dots, R_n)$ s.t. $\theta(T_{k-1}) = t_\theta^{k-1}$ and $u(R_\ell) \leq u^\theta(t_\theta^{\ell-1}; t_\theta^\ell)$ for $k \leq \ell \leq n$. Pick \tilde{T} a solution of

$$\min_{T: \theta(T) = t_\theta^{k-2}} \max_{S: S \cap T = \emptyset; \theta(S) = t_\theta^{k-1} - t_\theta^{k-2}} u(S) = u^\theta(t_\theta^{k-2}; t_\theta^{k-1})$$

As $\theta(\tilde{T} \cap T_{k-1}) \leq t_\theta^{k-2}$ and $\theta(T_{k-1}) = t_\theta^{k-1}$ we can choose T_{k-2} s.t. $\tilde{T} \cap T_{k-1} \subseteq T_{k-2} \subseteq T_{k-1}$ and $\theta(T_{k-2}) = t_\theta^{k-2}$. Then we set $R_{k-1} = T_{k-1} \setminus T_{k-2}$ so that $u(R_{k-1}) \leq u^\theta(t_\theta^{k-2}; t_\theta^{k-1})$ follows from $R_{k-1} \cap \tilde{T} = \emptyset$ and the definition of \tilde{T} . This completes the induction step. We note finally that each set R^k thus constructed is of θ -size $t_\theta^k - t_\theta^{k-1}$, and that $\max_k u(S_k) = \Gamma^\theta(u; n)$. ■

It is easy to check that no agent can secure more utility than Γ_n^κ in MK_n^κ or Γ_n^θ in B\&C_n^θ .

Remark 4. The *minMax Guarantee* and *Maxmin upper bound* for $u \in \mathcal{M}^\varepsilon(\Omega)$ and $-u \in \mathcal{M}^{-\varepsilon}(\Omega)$, where $\varepsilon = \pm$, are related: $\text{minMax}(-u; n) = -\text{Maxmin}(u; n)$. With two agents the Guarantees $\Gamma^\kappa(u; 2)$ and $\Gamma^\theta(u; 2)$ are similarly antisymmetric:

$$\Gamma^\alpha(-u; 2) = -\Gamma^\alpha(u; 2) \text{ where } \alpha \text{ is } \kappa \text{ or } \theta \quad (8)$$

This is clear for Γ^κ and we check it for Γ^θ by means of the change of variable $S \rightarrow S' = \Omega \setminus S$:

$$\begin{aligned} \Gamma^\theta(-u; 2) &= - \min_{0 \leq t^1 \leq 1} \max \left\{ \min_{\theta(S)=t^1} u(S), \max_{\theta(S)=t^1} u(\Omega \setminus S) \right\} = \\ &= - \min_{0 \leq t^1 \leq 1} \max \left\{ \min_{\theta(S')=1-t^1} u(\Omega \setminus S'), \max_{\theta(S')=1-t^1} u(S') \right\} \\ &= - \min_{0 \leq t^1 \leq 1} \max \left\{ \max_{\theta(S')=t^1} u(S'), \min_{\theta(S')=t^1} u(\Omega \setminus S') \right\} \end{aligned}$$

and the claim follows because if two continuous functions $t \rightarrow f(t)$ and $t \rightarrow g(t)$ intersect in $[0, 1]$ and one increases while the other decreases, then $\min_{0 \leq t \leq 1} \max \{f(t), g(t)\} = \max_{0 \leq t \leq 1} \min \{f(t), g(t)\}$.

The identity (8) generalises to $n \geq 3$ for the MK^κ Guarantee, but not for the B\&C^θ one.

5.3 Microeconomic fair division

We must divide a good manna $\omega \in \mathbb{R}_+^K$ in n shares $z_i \in \mathbb{R}_+^K$. Utilities $u \in \mathcal{M}^+(\omega)$ are continuous and weakly increasing on $[0, \omega]$.

A Moving Knife is a continuous increasing path $t \rightarrow K(t)$ from 0 to ω : a natural choice is $K(t) = t\omega, 0 \leq t \leq 1$: the corresponding Guarantee $\Gamma^\kappa(u; n) = u(\frac{1}{n}\omega)$ is the *Equal Split* utility $\Gamma^{es}(u; n) = u(\frac{1}{n}\omega)$. A positive, additive measure θ defining B\&C^θ is a “price” $\theta(z) = p \cdot z, p \in \mathbb{R}_+^K \setminus \{0\}$, so we write the corresponding Guarantee as Γ^p .

Recall from Section 1 that if an agent’s preferences are convex her Equal Split utility equals her *Maxmin* utility, the upper bound on all Fair Guarantees ((1)), in particular it is weakly larger than the B\&C^p guarantee for any p . The converse inequality holds for “concave preferences”.

Lemma 5

- i)* If the upper contours of the utility $u \in \mathcal{M}^+(\omega)$ are convex, then $\Gamma^p(u; n) \leq u(\frac{1}{n}\omega) = \text{Maxmin}(u; n)$.
- ii)* If the lower contours of the utility $u \in \mathcal{M}^+(\omega)$ are convex, then $\text{minMax}(u; n) = u(\frac{1}{n}\omega) \leq \Gamma^p(u; n)$.

The equality in statement *i)* was proven in Section 1. A symmetrical argument gives statement *ii)*.

We turn to a handful of numerical examples where $K = 2$, $\omega = (1, 1)$, and $p \cdot z = \frac{1}{2}(x + y)$. Shares are $z = (x, y)$, utilities are 1-homogenous and normalised so that $u(\omega) = 10$. We compute our three Guarantees: Bid and Choose Γ^p , Equal Split, and *minMax*, and compare them to the *Maxmin* upper bound.

The first three utilities (Leontief, Cobb Douglas and CES) define convex preferences, the last two define ‘‘concave preferences’’ (represented by quadratic and ‘‘anti-Leontief’’ utilities).

Our first table assumes two agents, $n = 2$, and illustrates Lemma 5. An agent with convex (resp. concave) preferences gets a better Guarantee under Equal Split (resp. Bid and Choose):

$u(x, y)$	$\text{minMax}(u; 2)$	$\Gamma^p(u; 2)$	$u(\frac{1}{2}\omega)$	$\text{Maxmin}(u; 2)$
$10 \min\{x, y\}$	0	3.3	5	5
$10\sqrt{x \cdot y}$	0	4.1	5	5
$\frac{5}{2}(\sqrt{x} + \sqrt{y})^2$	2.5	4.4	5	5
$5(x + y)$	5	5	5	5
$5\sqrt{2(x^2 + y^2)}$	5	5.9	5	7.1
$10 \max\{x, y\}$	5	6.7	5	10

The equal split partition delivers the *Maxmin* utility for the first four preferences, and the *minMax* utilities for the last three. The equipartition $\Pi = \{(1, 0), (0, 1)\}$ gives similarly the *minMax* utilities of the first four, and the *Maxmin* ones for the last three.

To compute $\Gamma^p(u; 2)$ we know from (7) that the optimal bid t^1 (denoted t for simplicity) solves

$$\max_{\frac{1}{2}(x+y) \leq t} u(x, y) = \min_{\frac{1}{2}(x+y) \leq t} u(1-x, 1-y) = \min_{\frac{1}{2}(x+y) \geq 1-t} u(x, y)$$

This equality implies $0 \leq t \leq \frac{1}{2}$. If u represents convex preferences symmetric in the two goods, $u(x, y)$ is maximal under $\frac{1}{2}(x + y) \leq t$ at $x = y = t$, and

minimal under $x + y \geq 2(1 - t)$ at $x = 1, y = 1 - 2t$. So we must solve $u(t, t) = u(1, 1 - 2t)$: see Figure 2.

If u represents concave symmetric preferences its maximum under $\frac{1}{2}(x + y) \leq t$ is at $x = 0, y = 2t$, and its minimum under $x + y \geq 2(1 - t)$ at $x = y = 1 - t$, so we solve $u(0, 2t) = u(1 - t, 1 - t)$: see Figure 3.

We compute finally the same Guarantees with three agents:

$u(x, y)$	$minMax(u; 3)$	$\Gamma^p(u; 3)$	$u(\frac{1}{3}\omega)$	$Maxmin(u; 3)$
$10 \min\{x, y\}$	0	2	3.3	3.3
$10\sqrt{x \cdot y}$	0	2.4	3.3	3.3
$\frac{5}{2}(\sqrt{x} + \sqrt{y})^2$	2	2.5	3.3	3.3
$5(x + y)$	3.3	3.3	3.3	3.3
$5\sqrt{2(x^2 + y^2)}$	3.3	4.1	3.3	4.1
$10 \max\{x, y\}$	3.3	5	3.3	5

The $minMax$ equipartition for $u = \frac{5}{2}(\sqrt{x} + \sqrt{y})^2$ and the $Maxmin$ equipartition for $u' = 5\sqrt{2(x^2 + y^2)}$ have the same form $\Pi = \{(x, 0), (0, x), (1 - x, 1 - x)\}$: in the former case we find $x = \frac{4}{5}$ and $minMax(u; 3) = 2$, in the latter we get $x = 2 - \sqrt{2}$ and $Maxmin(u'; 3) = 10(\sqrt{2} - 1)$. Lemma 5 and the partition $\Pi' = \{(1, 0), (0, \frac{1}{2}), (0, \frac{1}{2})\}$ fill the remaining values of $minMax$ and $Maxmin$.

To compute $\Gamma^p(u; 3)$ we know by Lemma 4 that the three terms in (6) are equal. They are

$$u^p(0, t^1) = \max_{\frac{1}{2}(x+y) \leq t^1} u(x, y)$$

$$u^p(t^1, t^2) = \min_{\frac{1}{2}(x^1+y^1) \leq t^1} \max_{\frac{1}{2}(x+y) \leq t^2 - t^1 \text{ and } (x^1+x, y^1+y) \leq (1,1)} u(x, y)$$

$$u^p(t^2, 1) = \min_{\frac{1}{2}(x^2+y^2) \leq t^2} u(1 - x^2, 1 - y^2)$$

Clearly $t^1 \leq \frac{1}{3}$ (as $t^2 - t^1 < \frac{1}{3} < t^1$ and $1 - t^2 < \frac{1}{3} < t^1$ are both impossible). Therefore $u^p(0, t^1) = u^p(t^1, t^2)$ is achieved by $t^2 = 2t^1$ (the constraint $(x^1 + x, y^1 + y) \leq (1, 1)$ does not bind). Writing $t = t^1 = t^2 - t^1$ it remains to solve

$$\max_{\frac{1}{2}(x+y) \leq t} u(x, y) = \min_{\frac{1}{2}(x^2+y^2) \leq 2t} u(1 - x^2, 1 - y^2) = \min_{\frac{1}{2}(x+y) \geq 1-2t} u(x, y)$$

When u represents convex preferences symmetric in the two goods, the minimum on the right-hand side is achieved by $(x, y) = (1 - 4t, 1)$ so we solve $u(t, t) = u(1 - 4t, 1)$. See Figure 4.

If u represents concave symmetric preferences, the minimum on the right-hand side is achieved by $(x, y) = (1 - 2t, 1 - 2t)$ so we solve $u(2t, 0) = u(1 - 2t, 1 - 2t)$. See Figure 5.

6 Concluding comments

Comparing B&C_n versus D&C_n rules The exogenous ordering of the agents greatly affects the outcome of D&C_n, whereas B&C_n treats the agents symmetrically. On the other hand the choice of the benchmark measure in B&C_n is exogenous, which allows much, perhaps too much flexibility to the designer.

In D&C_n the dividing agent may have many different strategies guaranteeing her *minMax* utility. By contrast in B&C_n the solution to programs (7) and (6) is often unique. Multiple choices and the resulting indeterminacy of the outcome may be appealing for the sake of privacy preservation, less so from the implementation viewpoint.

Two challenging open questions 1). Fix the manna (Ω, \mathcal{B}) as in Theorem 1, and each of the n agents with his own utility in $\mathcal{D}(\Omega)$. As mentioned in Section 2 and Subsection 3.2, Stromquist ([36]) showed that an Envy-free partition of Ω exists *if all utilities are non negative for all shares*. Without the sign assumption on utilities, Avvakumov and Karasev ([3]) prove existence of an Envy-free partition if n is a power of a prime number. Whether this remains true for all n is still an open question.

2) If the utilities vary in a domain $\mathcal{U}(\Omega)$ where the *Maxmin* utility is not feasible, we would like to describe the family of *undominated* Fair Guarantees $u \rightarrow \Gamma(u; n)$. For instance in the microeconomic domain $\mathcal{M}^+(\omega)$ of Subsection 5.3, the Equal Split Guarantee is clearly undominated. We conjecture that in the domains $\mathcal{M}^\pm(\Omega)$ the B&C Guarantees Γ^θ (Subsection 5.2) are undominated as well.

7 Appendix: proof of Lemma 4

1). *First statement.* Recall that we can replace in definition (5) the equalities like $\theta(T) = t^1$ with inequalities $\theta(T) \leq t^1$. We check first that the correspondence $t \rightarrow \{S \in \mathcal{B} | \theta(S) \leq t\}$ is continuous. Upper hemi continuity follows by the continuity of θ . For lower hemi continuity pick a sequence t_n converging to t and $S \in \mathcal{B}$ s.t. $\theta(S) \leq t$. If t_n has a decreasing subsequence, we set $S_n = S$ so that $\theta(S_n) \leq t_n$ and S_n converges to S . If t_n has an increasing subsequence we construct an inclusion increasing sequence S_m converging to S and s.t. $|S_m| < |S|$ for all m : because θ increases strictly, so does the

sequence $\theta(S_m)$ converging to $\theta(S)$, therefore we can pick subsequences S_p of S_m and t_p of t_n s.t. $\theta(S_p) \leq t_p$, as desired.

Next we apply the Maximum Theorem twice. The first one to show that the function $(T, t^1, t^2) \rightarrow C(T, t^1, t^2) = \max\{u(S) | S \subset \Omega \setminus T; \theta(T \cup S) \leq t^1 + t^2\}$ is continuous because the correspondence $(T, t^1, t^2) \rightarrow \{S | S \subset \Omega \setminus T; \theta(T \cup S) \leq t^1 + t^2\}$ is continuous. The second one to deduce that the function $\min_{T: \theta(T) \leq t^1} C(T, t^1, t^2)$ is continuous.

2). *Second statement.* For simplicity we assume $n = 3$, the general proof is entirely similar. Fixing u and t^1 there is some t^2 such that $u^\theta(t^1; t^2) = u^\theta(t^2; 1)$. This is because of the monotonicity properties of u^θ and of the inequalities $u^\theta(t^1; t^1) = 0 \leq u^\theta(t^1; 1)$ and $u^\theta(t^1; 1) \geq 0 = u^\theta(1; 1)$. This common value is unique (though t^2 may not be) and defines a function $g(t^1) = u^\theta(t^1; t^2) = u^\theta(t^2; 1)$. It is easy to check from the continuity and monotonicity properties of u^θ that g is weakly decreasing and continuous. Then we find in the same way t^1 s.t. $g(t^1) = u^\theta(0; t^1)$.

Check finally that if $\tau_* \in \Upsilon(n)$ is such that all terms $u^\theta(t_*^k; t_*^{k+1})$, $0 \leq k \leq n - 1$, equal a common value λ , then τ_* solves program (6). If it does not there is a τ such that $u^\theta(t^k; t^{l+1}) > \lambda$ for $0 \leq k \leq n - 1$. Applying this inequality at $k = 0$ gives $t^1 > t_*^1$; next at $k = 1$ we get $u^\theta(t^1, t^2) > u^\theta(t_*^1, t_*^2)$ implying $t^2 > t_*^2$; and so on until we reach a contradiction with the fact that both τ and τ_* are in $\Upsilon(n)$.

Finally, the optimal sequence t^k increases in k , strictly if u is not everywhere zero because $u(t, t) = 0$ for all t .

References

- [1] Aigner-Horev E, Segal-Halevi E. 2019. Envy-free Matchings in Bipartite Graphs and their Applications to Fair Division, [arXiv.org/cs/arXiv:1901.09527](https://arxiv.org/abs/1901.09527)
- [2] Aumann Y, Dombb Y. 2010. The Efficiency of Fair Division with Connected Pieces, *International Workshop on Internet and Network Economics WINE 2010: Internet and Network Economics*, 26-37
- [3] Avvakumov S., Karasev R. 2019. Envy-free Division under Mapping Degree, arxiv.org/pdf/1907.11183.pdf

- [4] Avvakumov S., Karasev R. 2020. Equipartition of an interval, *in preparation*.
- [5] Aziz H, Caragianis I, Igarashi A, Walsh T. 2018. Fair allocation of combinations of indivisible goods and chores, arXiv.org > cs > arXiv:1807.10684
- [6] Aziz A, McKenzie S. 2016. A Discrete and Bounded Envy-free Cake Cutting Protocol for Any Number of Agents, *IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, New Brunswick, NJ, USA, 9-11 Oct. 2016
- [7] Barbera S, Dutta B. 1982. Implementability via protective equilibria. *Journal of Mathematical Economics*, 10 (1), p. 49-65
- [8] Berliant M, Dunz K, Thomson W. 1992. On the fair division of a heterogeneous commodity, *J. Math. Econ.*, 21, 201–16
- [9] Bogomolnaia A, Moulin H. 2004. Random matching under divchotomous preferences, *Econometrica*, 72, 1, 257-279
- [10] Bogomolnaia A, Moulin H, Sandomirskiy F, Yanovskaia E. 2019. Dividing Bads under Additive Utilities, *Soc.Choice Welfare*, 52 (3), 395-417
- [11] Bogomolnaia A, Moulin H, Sandomirskiy F, Yanovskaia E. 2017. Competitive division of a mixed manna, *Econometrica*, 85(6), 1847-71
- [12] Bouveret S, Lemaitre M. 2016. Characterizing conflicts in fair division of indivisible goods using a scale of criteria. *Auton Agent Multi-Agent Syst*, 30:259–290
- [13] Brams SJ, Taylor AD. 1995. An envy-free cake division protocol, *American Math. Monthly*, 102(1), 9-18
- [14] Brams SJ, Taylor AD. 1996. *Fair Division: From Cake-Cutting to Dispute Resolution*, Cambridge University Press, Cambridge, Mass, USA
- [15] Branzei S. 2015. A note on envy-free cake cutting with polynomial valuations, *Information Processing Letters* 115, 2, 93–95

- [16] Budish E. 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *J. Polit. Econ.* 119, 6, 1061–103
- [17] Cseh A, Fleiner T. 2018. The complexity of cake cutting with unequal shares, [arxiv.org/cs/1709.03152](https://arxiv.org/abs/1709.03152)
- [18] Crew L, Narayanan B, Spirkl S. 2019. Disproportionate division, [arXiv.org > math > arXiv:1909.07141](https://arxiv.org/abs/1909.07141)
- [19] Dubins LE, Spanier EH. 1961. How to cut a cake fairly. *American Math. Monthly*, 68, 1–17
- [20] Kuhn, H.1967. On games of fair division, *Essays in Mathematical Economics in Honour of Oskar Morgenstern*, Princeton University Press, pp. 29–37
- [21] Kurokawa D, Lai JK., Procaccia AD. 2013. How to Cut a Cake Before the Party Ends, Proceedings of the 27th AAAI Conference on Artificial Intelligence
- [22] Kurokawa D, Procaccia AD, Wang J. 2016. When can the maximin share guarantee be guaranteed?, *Proceedings of the 30th AAAI Conference on Artificial Intelligence (AAAI)*,523-9
- [23] Lovasz L, Plummer MD. 1986. *Matching Theory*, Annals of Discrete Mathematics, Vol. 29. Amsterdam: North-Holland.
- [24] Massey WS. 1977. *Algebraic Topology: An Introduction*, Springer.
- [25] Meunier F, Zerbib S. 2018. Envy-free cake division without assuming the players prefer nonempty pieces, forthcoming, *Israel Journal of Mathematics*, [arxiv.org/pdf/1804.00449.pdf](https://arxiv.org/abs/1804.00449)
- [26] Moulin H. 1992. All Sorry to Disagree: a General Principle for the Provision of Non-rival Goods, *Scandinavian Journal of Economics*, 94, 37–51
- [27] Moulin H. 1992. Welfare Bounds in the Cooperative Production Problem, *Games and Economic Behavior*, 4, 373–401
- [28] Moulin H. 2019. Fair Division in the Internet Age, *Annual Review of Economics*,11:407-41.

- [29] Munkres JR. 1996. *Elements of Algebraic Topology*, Avalon.
- [30] Procaccia AD. 2013. Cake cutting: not just child's play, *Communications of the ACM (CACM Homepage archive)*, 56(7), 78-87
- [31] Procaccia AD, Wang J. 2014. Fair Enough: Guaranteeing Approximate Maximin Shares, *Proceedings of the 14th ACM Conference on Economics and Computation EC'14*, 675–692
- [32] Robertson JM, Webb WA.1998. *Cake Cutting Algorithms: Be Fair If You Can*, A. K. Peters
- [33] Segal-Halevi E. 2018. Fairly Dividing a Cake after Some Parts Were Burnt in the Oven, *Proceeding of AAMAS '18*, 1276-1284
- [34] Segal-Halevi E, Nitzan S, Hassidim A, Aumann Y. 2017. Fair and square: Cake-cutting in two dimensions, *J. Math. Econ.*, 70, 1-289
- [35] Steinhaus H. 1949. Sur la division pragmatique *Econometrica* (supplement), 17, 315-319
- [36] Stromquist W. 1980. How to cut a cake fairly, *Amer. Math. Monthly* 87, 640-644
- [37] Su F. E. 1999. Rental harmony: Sperner's lemma in fair division. *American Math. Monthly*, 10, 930–42
- [38] Thomson W. 2005. *The theory of fair allocation*. Princeton University Press, Princeton NJ, USA
- [39] Woodall DR. 1980. Dividing a cake fairly, *Journal of Mathematical Analysis and Applications*, 78(1):233-247