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# SOLVABLE GROUPS AND AFFINE ACTIONS ON THE LINE

JOAQUÍN BRUM, NICOLÁS MATTE BON, CRISTÓBAL RIVAS, AND MICHELE TRIESTINO

**ABSTRACT.** We prove a structural result for orientation-preserving actions of finitely generated solvable groups on real intervals, considered up to semi-conjugacy. As applications we obtain new answers to a problem first considered by J. F. Plante, which asks under which conditions an action of a solvable group on a real interval is semi-conjugate to an action on the line by affine transformations. We show that this is always the case for actions by  $C^1$  diffeomorphisms on closed intervals. For arbitrary actions by homeomorphisms, for which this result is no longer true (as shown by Plante), we show that a semi-conjugacy to an affine action still exists in a local sense, at the level of germs near the endpoints. Finally for a vast class of solvable groups, including all solvable linear groups, we show that the family of affine actions on the line is robust, in the sense that any action by homeomorphisms on the line which is sufficiently close to an affine action must be semi-conjugate to an affine action. This robustness fails for general solvable groups, as illustrated by a counterexample.

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## 1. INTRODUCTION

**1.1. Background.** This work is about the study of group actions on one-manifold. This means, given a group  $G$ , to try to understand and classify the possible representations  $\varphi: G \rightarrow \text{Homeo}_0(X)$  into the groups of orientation-preserving homeomorphisms, or into the group  $\text{Diff}_0^r(X)$  of diffeomorphisms of regularity  $C^r$  where  $X$  is a connected one-manifold (that is, the circle or a real interval). With no real loss of generality we restrict to actions which have no global fixed points in the interior of  $X$ , which will be termed *irreducible* (indeed, an action with global fixed points can be decomposed as a union of irreducible actions on a countable family of invariant intervals). As usual in dynamics, actions can be considered up to (topological) conjugacy; however, in the one-dimensional setting it is always possible to obtain many non-conjugate actions from a given one by blowing up orbits as in the classical Denjoy's example. This redundancy can be avoided by considering a weaker equivalence relation, called *semi-conjugacy*, whose definition can be traced back to Ghys [11], although it has been unanimously formalized only recently (see for instance the monograph by Kim, Koberda, and Mj [14]). When  $X, Y$  are non-empty open intervals, two irreducible actions  $\varphi: G \rightarrow \text{Homeo}_0(X)$  and  $\psi: G \rightarrow \text{Homeo}_0(Y)$  are semi-conjugate if there is a monotone  $G$ -equivariant map  $h: X \rightarrow Y$ . We warn the reader that this semi-conjugacy notion is slightly different from the homonymous notion appearing in classical dynamical systems: notably, this is an equivalence relation. (We refer to §2.1 for more details.)

In this article, we are interested in the case where the group  $G$  is a finitely generated solvable group, a case which has attracted considerable attention, starting with the work of Plante [23, 24], whose motivation was from the theory of codimension-one foliations. A basic source

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of examples of solvable groups acting on one-manifolds are subgroups of the real affine group

$$\text{Aff}(\mathbb{R}) = \{x \mapsto ax + b : a > 0, b \in \mathbb{R}\}.$$

In a seminal article [24], Plante considered the following converse problem: under which conditions an action of a solvable group  $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$  must be semi-conjugate to an action by affine transformations  $\varphi: G \rightarrow \text{Aff}(\mathbb{R})$ ?

Plante provided an array of sufficient conditions for an affirmative answer. He deduced in particular that when  $G$  is a finitely generated polycyclic group, every irreducible action  $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$  is semi-conjugate to an affine action. In contrast, he showed that the result fails for general solvable groups: he constructed a faithful and minimal action of the group  $\mathbb{Z} \wr \mathbb{Z}$  on  $\mathbb{R}$  which is not semi-conjugate to an affine action (we recall this construction in Section 6). The idea behind the construction of Plante's action of  $\mathbb{Z} \wr \mathbb{Z}$  is quite robust and can be generalized in multiple ways (see for instance our previous work [5, Example 7.11]); as a consequence, it turns out the possible actions of general solvable groups on the real line can be quite wilder than expected. Since Plante's work, a number of papers have addressed the study of solvable group actions on one-manifolds, and the special role played by affine actions can be recognized as the main theme, see for instance works of Plante [25], Navas [20, 21] and references therein, Akhmedov [1, 2], Bonatti, Monteverde, Navas and the third author [3], Guelman and the third author [12] (see also work by Chiswell and Kropholler [6] and by the third author and Tessera [26] for tightly related results on left-orders on solvable groups).

**1.2. Results.** In this paper we develop a new approach to study general actions of solvable groups on the real line by homeomorphisms (we work under the slightly more general assumption that  $G$  is virtually solvable, i.e. contains a solvable subgroup of finite index). As application, we obtain three results which provide new answers to the problem originally considered by Plante, by relating arbitrary actions to affine actions.

Our first result deals with actions on intervals by diffeomorphisms of class  $C^1$ . Here and elsewhere, we say for short that an affine action  $\psi: G \rightarrow \text{Aff}(\mathbb{R})$  is *abelian* if its image is abelian.

**Theorem A** ( $C^1$  actions of solvable groups). *Let  $G$  be a finitely generated virtually solvable group and let  $\varphi: G \rightarrow \text{Diff}_0^1([0, 1])$  be an irreducible action. Then the restriction of  $\varphi$  to  $(0, 1)$  is semi-conjugate to an affine action  $\psi: G \rightarrow \text{Aff}(\mathbb{R})$ . Moreover, the semi-conjugacy is a topological conjugacy provided  $\psi$  is non-abelian.*

*Remark 1.1.* The main contribution of the theorem is the existence of the semi-conjugacy. When  $\psi$  is a non-abelian affine action, any  $C^1$  action semi-conjugate to  $\psi$  is automatically conjugate to  $\psi$  by a result of Bonatti, Monteverde, Navas, and the third author [3, §4.2].

*Remark 1.2.* Under the stronger assumption that actions be of class  $C^2$ , the result was proved by Navas in [20], who also obtained an algebraic description of solvable subgroups of  $\text{Diff}_0^2([0, 1])$ . Navas's approach is based essentially on the strong restriction on abelian groups of diffeomorphisms of class  $C^2$ . Rigidity results in  $C^2$  regularity (or even in  $C^r$  regularity for  $r > 1$ ) are relatively abundant. They are typically based on classical results of Kopell [15], Denjoy [8] and Hölder [13] together imply that an abelian subgroup of  $\text{Diff}_0^2([0, 1])$ , whose action on  $(0, 1)$  is irreducible, is conjugate to a group of translations of  $\mathbb{R}$ . Both Kopell's and Denjoy's results fail in  $C^1$  regularity, and this approach does not suffice even to exclude that the Plante's actions of  $\mathbb{Z} \wr \mathbb{Z}$  might be conjugate to a  $C^1$  action. In fact our proof of Theorem A proceeds by first reducing to this special case, and then ruling out this possibility (see Proposition 7.1, which is inspired by [3]).

In the  $C^0$  setting, our next result shows that the answer to Plante's problem is always affirmative in a *local* sense, namely a semi-conjugacy to an affine action can be found at the level of germs near fixed points.

**Theorem B** (Local semi-conjugacy near fixed points). *Let  $G$  be a finitely generated virtually solvable group, and let  $\varphi: G \rightarrow \mathbf{Homeo}_0(\mathbb{R})$  be an action without global fixed points on an interval  $I$  of the form  $(a, +\infty)$ .*

*Then there exists an irreducible affine action  $\psi: G \rightarrow \mathbf{Aff}(\mathbb{R})$  and a non-decreasing map  $h: I \rightarrow \mathbb{R}$ , with  $\lim_{x \rightarrow +\infty} h(x) = +\infty$  such that for every  $g \in G$  we have  $\psi(g)(h(x)) = h(\varphi(g)(x))$  for all  $x \in \mathbb{R}$  sufficiently large. This action  $\psi$  is unique up to affine conjugacy.*

*Remark 1.3.* Of course, equivalent results hold for irreducible actions on finite intervals, considering neighborhoods of the endpoints.

Theorem B should be compared with a result of Chiswell and Kropholler [6], stating that a solvable group is left-orderable if and only if it is locally indicable (this is true more generally for amenable groups, by a result of Witte Morris [18]). In an equivalent formulation, that result says that a finitely generated solvable group  $G$  has a non-trivial action  $\varphi: G \rightarrow \mathbf{Homeo}_0(\mathbb{R})$  if and only if it admits a non-trivial homomorphism to the group of translations  $(\mathbb{R}, +)$ . The reader may note that Theorem B easily recovers this result (using that every non-trivial subgroup of  $\mathbf{Aff}(\mathbb{R})$  has a homomorphism to  $(\mathbb{R}, +)$ ). An important difference is that Theorem B uniquely associates to every action of  $G$  on  $\mathbb{R}$ , a homomorphism to  $\mathbf{Aff}(\mathbb{R})$  with an explicit dynamical meaning; in contrast the proofs in [6] and [18] proceed through considerations on the set of all left-orders to deduce the existence of a homomorphism to  $\mathbb{R}$ , but do not to establish an explicit relation between a fixed action (or order) and such homomorphisms.

Our third main result addresses the question of rigidity of the family of affine actions under small deformations. To this end, given a group  $G$ , we denote by  $\mathbf{Hom}_{\text{irr}}(G, \mathbf{Homeo}_0(\mathbb{R}))$  the space of irreducible actions  $\varphi: G \rightarrow \mathbf{Homeo}_0(\mathbb{R})$ . Recall that this space has a natural topology, induced from the pointwise convergence topology of all maps from  $G \rightarrow \mathbf{Homeo}_0(\mathbb{R})$ , with respect to the compact-open topology on  $\mathbf{Homeo}_0(\mathbb{R})$ . In view of Plante's problem, it is then natural to ask whether an action of a solvable group which is sufficiently close to an affine action must be semi-conjugate to an affine action. The following result provides an affirmative answer for a vast class of solvable groups, namely those that are *virtually metanilpotent*, that is, admitting a nilpotent normal subgroup  $N$  such that  $G/N$  is virtually nilpotent. This class includes in particular all virtually solvable linear groups (that is, subgroups of  $\mathbf{GL}(n, \mathbb{K})$  for some field  $\mathbb{K}$ ), thanks to a well-known result of Mal'cev [17] (see Remark 8.1).

**Theorem C** (Perturbations of affine actions). *Let  $G$  be a finitely generated virtually metanilpotent group. Then, the subset of  $\mathbf{Hom}_{\text{irr}}(G, \mathbf{Homeo}_0(\mathbb{R}))$  of all irreducible actions which are semi-conjugate to a non-abelian affine action is open.*

*Remark 1.4.* Theorem C is false for general solvable group, already in the 3-step solvable case: we construct in §8.2 a 3-step finitely generated solvable group  $G$  and a sequence of irreducible actions  $(\varphi_n) \subset \mathbf{Hom}_{\text{irr}}(G, \mathbf{Homeo}_0(\mathbb{R}))$  which are not semi-conjugate to affine actions, yet converge to a non-abelian affine action.

*Remark 1.5.* Theorem C implies in particular that if  $\psi: G \rightarrow \mathbf{Aff}(\mathbb{R})$  is a non-abelian affine action, then every action  $\varphi$  sufficiently close to  $\psi$  remains semi-conjugate to a non-abelian affine action. It is however not true in general that  $\varphi$  is semi-conjugate to  $\psi$  itself: indeed even metabelian groups often admit continuous paths of representations into  $\mathbf{Aff}(\mathbb{R})$  (see e.g. Lemma 6.1).

**1.3. The structure of general actions.** Although affine actions are a central character in the results stated above, the essence of this work is about the understanding of actions of solvable groups on the line that are *not* semi-conjugate to any affine action. Our starting point is a result obtained in our previous work [5, Theorem 7.18], namely that every minimal action of a finitely generated solvable group  $G$  on the line is either conjugate to an affine action, or it is  $\mathbb{R}$ -focal in the terminology of [5]. We review this notion and its main properties in Section 3. An action  $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$  is  $\mathbb{R}$ -focal if it preserves a *lamination* on the real line, that is, a collection  $\mathcal{L}$  of bounded intervals without crossings, and the action of  $G$  on  $\mathcal{L}$  satisfies a hyperbolicity-like condition. As we recall in details in Section 3, from such an invariant lamination one may construct an action of  $G$  on a planar *directed (real) tree*, which roughly speaking is a space  $\mathbb{T}$  obtained by gluing together copies of the real line along intervals of the form  $[a, +\infty)$ . The resulting  $G$ -action on  $\mathbb{T}$  fixes an end  $\omega \in \partial\mathbb{T}$ , and the original action of  $G$  on the line can be reconstructed from the action on the remaining ends  $\partial\mathbb{T} \setminus \{\omega\}$ , endowed with a natural linear order. (One may intuitively think of  $\mathbb{T}$  as hanging over the line in the upper half plane, with one end in the vertical direction and the remaining ends on the line.)

This leads us to study actions of solvable group  $G$  on directed trees, which is the purpose of Sections 4 and 5. The outcome is Theorem 5.2, which is a central result of the article. That results says that if  $G$  is a finitely generated (virtually) solvable group, then every minimal  $\mathbb{R}$ -focal action  $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$  naturally comes with an associated action  $\varphi': G/N \rightarrow \text{Homeo}_0(\mathbb{R})$  of a proper quotient of  $G$ , where  $N \trianglelefteq G$  is such that  $G/N$  has solvable length strictly less than  $G$  (more precisely,  $N$  will be the *Fitting subgroup* of  $G$ ). This new action  $\varphi'$  retains information on the dynamics of  $\varphi$  at large scale. The precise relation is that  $\varphi$  is *horograded* by  $\varphi'$ , which means that the  $G$ -action on a directed tree  $\mathbb{T}$  associated to  $\varphi$  is governed by  $\varphi'$  in the vertical direction (see §3.3). A crucial feature of Theorem 5.2 is that it can be applied inductively: if the horograding action  $\varphi'$  is not semi-conjugate to an affine action, then it must be again  $\mathbb{R}$ -focal and horograded by an action  $\varphi''$  of a quotient of even smaller solvable length. In finitely many steps, we must reach an affine action  $\psi: G \rightarrow \text{Aff}(\mathbb{R})$ . This shows that actions of a solvable groups on the line are naturally organized in a tower of finitely many levels of complexity, where affine actions are the simplest.

Theorem B is a direct consequence of Theorem 5.2, the action  $\psi$  in its statement being exactly the affine action obtained from  $\varphi$  through the inductive procedure above. In Section 6 we specialize the discussion to the group  $G = \mathbb{Z} \wr \mathbb{Z}$ . We shall see that in this case Theorem 5.2 readily translates to a classification of actions of  $G$  on the line up to semi-conjugacy: these are either affine or one of the Plante's action from [24]. In addition, for every  $\mathbb{R}$ -focal action of a solvable group on the line, the group  $G$  must have a subgroup isomorphic to  $\mathbb{Z} \wr \mathbb{Z}$  whose action is semi-conjugate to a Plante action. To prove Theorem A, we are reduced to show that this action cannot be semi-conjugate to a  $C^1$  action on the interval, which is done in Section 7. Finally, in Section 8 we consider the class of metanilpotent groups. We shall see that for these groups, Theorem 5.2 takes a much stronger form, and deduce Theorem C.

In a companion paper in preparation [4], we will further use Theorem 5.2 and the inductive approach developed here to provide more detail results on the structure of semi-conjugacy classes in the space  $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$  for a finitely generated solvable group, through an analysis of the space  $\text{Der}_\mu(G)$  of normalized  $\mu$ -harmonic actions (in the sense of [9]). We shall see in particular that the quotient of  $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$  by the semi-conjugacy relation has a particularly nice structure (contrary to arbitrary finitely generated groups, for which it might not even be a standard Borel space).

**Note to the reader.** While this paper builds on concepts introduced in our previous work [5], we made an effort to make the presentation as self-contained as possible. The dependence on [5] is limited to a few statements, all contained in Sections 7 and 8 there, which are largely independent from the rest of [5]. Some other statements from [5, §§7-8] had to be strengthened and thus a complete proof can be found here, or a sketch of proof is repeated for completeness.

## 2. NOTATION AND PRELIMINARIES

**2.1. Actions on the line.** Let us start with some notation and terminology. When a group  $G$  acts on a set  $X$ , we simply write  $g.x$  for the action, unless there is risk of confusion. If  $\varphi : G \rightarrow \text{Sym}(X)$  is an action, we write  $\text{Stab}^\varphi(x)$  for the stabilizer of  $x \in X$  in  $G$  under the action, and  $\text{Fix}^\varphi(g)$  for the subset of fixed points in  $X$  of an element  $g \in G$ , and we write  $\text{Supp}^\varphi(g) = X \setminus \text{Fix}^\varphi(g)$  for its support.

Let  $X$  be a real interval with non-empty interior (that is, up to diffeomorphism, one of the spaces  $\mathbb{R}, [0, 1], [0, 1)$ ). The set of interior points of  $X$  is denoted by  $\text{Int}(X)$ . The group of orientation-preserving homeomorphisms of  $X$  is denoted by  $\text{Homeo}_0(X)$ , and similarly we denote the group of orientation-preserving  $C^r$  diffeomorphisms  $\text{Diff}_0^r(X)$ , for  $r \geq 1$ . We will say that a group action  $\varphi : G \rightarrow \text{Homeo}_0(X)$  is *irreducible* if it has no fixed point in  $\text{Int}(X)$ . We also recall the notion of semi-conjugacy for actions on intervals (see Kim, Koberda, and Mj [14]).

**Definition 2.1.** Let  $X$  and  $Y$  be real intervals with non-empty interior, and let  $\varphi : G \rightarrow \text{Homeo}_0(X)$  and  $\psi : G \rightarrow \text{Homeo}_0(Y)$  be two irreducible actions of a group  $G$ . We say that  $\varphi$  and  $\psi$  are *semi-conjugate* if there exists a monotone map  $h : \text{Int}(X) \rightarrow \text{Int}(Y)$  such that  $h \circ \varphi(g) = \psi(g) \circ h$  for any  $g \in G$ . When  $h$  is non-decreasing, we say that  $\varphi$  and  $\psi$  are *positively semi-conjugate*. When  $h$  is an (orientation-preserving) homeomorphism, we say that  $\varphi$  and  $\psi$  are *(positively) conjugate*.

A well-known argument (see [22, Section 2.1]) shows that if the group  $G$  is finitely generated, then every irreducible group action  $\varphi : G \rightarrow \text{Homeo}_0(X)$  admits a non-empty closed minimal invariant subset, which is either  $\mathbb{R}$ , a perfect set of empty interior, or a closed (and thus discrete) orbit. From the perspective of semi-conjugacy, this implies that  $\rho$  is always semi-conjugate to an action  $\psi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  which is either *minimal* (that is, every orbit is dense), or which takes values in the group  $(\mathbb{Z}, +)$  of integer translations, in which case we will say that  $\psi$  is *cyclic*. Moreover, such a minimal or cyclic representative of each semi-conjugacy class is unique up to conjugacy. Thus when studying continuous actions of  $G$  up to semi-conjugacy it is enough to restrict to minimal actions (the cyclic case being rather trivial), and we will often do so.

We will often build actions of groups on real intervals starting from actions on totally ordered sets. This strategy is quite general, but for the purpose of this note we will restrict our attention to actions on ordered sets  $(\Omega, <)$  which are countable, have no minimal element neither maximal element, and which are *densely ordered* in the sense that for every  $\omega_1 < \omega_2$  in  $\Omega$  there is  $\omega' \in \Omega$  such that  $\omega_1 < \omega' < \omega_2$ . The good thing about these sets is that Cantor's "back and forth argument" ensures that there is an order-preserving bijection  $t : (\Omega, <) \rightarrow (\mathbb{Q}, <)$  where rationals are equipped with its natural order (see for instance Clay and Rolfsen [7, Theorem 2.22]).

**Definition 2.2.** Let  $(\Omega, <)$  be a countable densely ordered set having no maximal neither minimal element,  $t : \Omega \rightarrow \mathbb{Q}$  an order-preserving bijection, and let  $\varphi : G \rightarrow \text{Aut}(\Omega, <)$  be a non-trivial order-preserving action. A *dynamical realization* of  $\varphi$  is an irreducible action

$\bar{\varphi} : G \rightarrow \text{Homeo}_0(\mathbb{R})$  with the property that  $\bar{\varphi}(g)(t(\omega)) = t(\varphi(g)(\omega))$  for every  $g \in G$  and every  $\omega \in \Omega$ .

A dynamical realization always exists, see [7, Section 2.4] for details. Moreover, if  $\bar{\varphi}_1$  and  $\bar{\varphi}_2$  are two dynamical realization of  $\varphi : G \rightarrow \text{Aut}(\Omega, <)$ , then they are positively conjugate. Indeed, if  $t_1$  and  $t_2$  are the corresponding bijections from  $\Omega$  to  $\mathbb{Q}$ , then  $h = t_1 \circ t_2^{-1}$  can be extended to the whole real line so that it conjugates  $\bar{\varphi}_1$  to  $\bar{\varphi}_2$ . This is why we usually refer to *the* dynamical realization of  $\varphi$ . The following result will be useful later.

**Lemma 2.3.** *Let  $(\Omega, <)$  be a countable, densely ordered set without maximal or minimal element. Let  $\varphi : G \rightarrow \text{Aut}(\Omega, <)$  be an order-preserving action of a group  $G$ , whose dynamical realization  $\bar{\varphi} : G \rightarrow \text{Homeo}_0(\mathbb{R})$  is minimal. Let  $\psi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  be an irreducible action, and suppose that  $\sigma : \Omega \rightarrow \mathbb{R}$  is a map which is non-decreasing and  $G$ -equivariant:*

$$\sigma(\varphi(g)\omega) = \psi(g)\sigma(\omega) \quad \text{for every } g \in G, \omega \in \Omega.$$

*Then,  $\sigma$  is injective, and  $\psi$  and  $\bar{\varphi}$  are positively semi-conjugate. In particular if  $\psi$  is minimal, then it is conjugate to  $\bar{\varphi}$ .*

*Proof.* Let  $t : \Omega \rightarrow \mathbb{Q}$  be the injection used to define the dynamical realization  $\bar{\varphi}$ . We define a map  $\tilde{\sigma} : \mathbb{R} \rightarrow \mathbb{R}$  by setting  $\tilde{\sigma}(x) = \sup\{\sigma(\omega) : t(\omega) \leq x\}$ . Then  $\tilde{\sigma}$  is non-decreasing and equivariant with respect to the  $\bar{\varphi}$ -action on the source  $\mathbb{R}$  and the  $\psi$ -action on the target line, showing that  $\psi$  is semi-conjugate to  $\bar{\varphi}$ . We claim that  $\tilde{\sigma}$  is injective. To see this, consider the subset

$$S = \{x \in \mathbb{R} : \tilde{\sigma} \text{ is constant on a neighborhood of } x\}.$$

It is direct to see that  $S$  is open and  $\bar{\varphi}$ -invariant, and therefore it must be empty by minimality of  $\bar{\varphi}$ . To conclude the proof of the claim notice that if  $\tilde{\sigma}(x) \neq \tilde{\sigma}(y)$  for  $x < y$  then, monotonicity of  $\tilde{\sigma}$  would imply  $(x, y) \subseteq S$ , which is a contradiction. In particular, since  $t$  is injective, so is  $\sigma = \tilde{\sigma} \circ t$ .

Assume now that  $\Psi$  is minimal. In this case, the image of  $\tilde{\sigma}$  must be dense, or otherwise its closure would be a proper closed  $\Psi$ -invariant subset. Therefore, since  $\tilde{\sigma}$  is monotone, injective, and has dense image, it must be a homeomorphism. This shows that  $\bar{\varphi}$  and  $\psi$  are conjugate.  $\square$

**2.2. Virtually solvable groups and Fitting series.** In this paper, we will mostly be interested in groups  $G$  which are *virtually solvable*, that is, contain a (normal) solvable subgroup of finite index. We refer the reader to the book by Lennox and Robinson [16] for a general reference on the theory of solvable groups.

Recall that the *Fitting subgroup*  $\text{Fit}(G)$  of a group  $G$  is the subgroup generated by all nilpotent normal subgroups of  $G$ . Fitting's theorem states that the group generated by two nilpotent normal subgroups of  $G$  is again nilpotent. In particular,  $\text{Fit}(G)$  is in fact the union of all nilpotent normal subgroups. The following basic lemma will be used without mention.

**Lemma 2.4.** *Let  $G$  be a group, and  $G_0 \trianglelefteq G$  be a normal solvable subgroup of finite index. Then  $\text{Fit}(G)$  contains every nilpotent normal subgroup of  $G_0$ .*

*Proof.* Let  $N \trianglelefteq G_0$  be nilpotent. Then  $N$  has finitely many conjugate subgroups in  $G$ , all contained in  $G_0$ . By Fitting's theorem the subgroup  $M$  generated by such conjugates is nilpotent, and it is normal in  $G$ . Hence  $N \leq M \leq \text{Fit}(G)$ .  $\square$

Note that in the previous situation  $\text{Fit}(G)$  contains in particular the last non-trivial term of the derived series of  $G_0$ .

**Definition 2.5.** Let  $G$  be a virtually solvable group. The *upper Fitting series* of  $G$  is defined by setting  $F_1 = \text{Fit}(G)$ , and  $F_{i+1}$  to be the preimage of  $\text{Fit}(G/F_i)$  in  $G$ . Since  $G$  is virtually solvable, there must exist some  $k \in \mathbb{N}$  such that  $G/F_k$  is finite. We call the smallest such  $k$  the *virtual Fitting length* of  $G$  and denote it by  $\text{vf}(G)$ .

### 3. GENERALITIES ON $\mathbb{R}$ -FOCAL ACTIONS

The notion of  $\mathbb{R}$ -focal action was introduced in our previous work [5]. We recall and further study this notion and its consequences. We follow [5, Sections 7, 8] with some slight differences in the terminology (which will be pointed out).

**3.1. Minimal laminations.** We identify the space of bounded open intervals of the real line with the set  $\mathbb{R}^{(2)} = \{(a, b) \in \mathbb{R}^2 : a < b\}$ , and endow it with the topology induced from  $\mathbb{R}^2$ . We say that two intervals are *non-crossed* if they are either disjoint or nested.

**Definition 3.1.** A *lamination* of the real line is a non-empty closed subset  $\mathcal{L} \subset \mathbb{R}^{(2)}$  consisting of pairwise non-crossed intervals.

**Definition 3.2.** An action  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  is  *$\mathbb{R}$ -focal* if it preserves a lamination  $\mathcal{L}$  such that for every  $l \in \mathcal{L}$  there exists a sequence  $(g_n) \subset G$  such that  $(g_n.l)$  is an increasing exhaustion of the line. When we want to emphasize the role of the lamination, we will say that  $\varphi$  is  *$\mathbb{R}$ -focal with respect to  $\mathcal{L}$* .

Definition 3.2 is slightly different from the definition of  $\mathbb{R}$ -focal action introduced in [5], where we were not assuming that  $\mathcal{L}$  be closed (the terminology being *cross-free cover* instead of lamination). However, the closure of a  $G$ -invariant cross-free cover is a lamination, and therefore the two definitions are equivalent.

**Definition 3.3.** Let  $\mathcal{L} \subset \mathbb{R}^{(2)}$  be a lamination which is invariant under an action  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ . We say that  $\mathcal{L}$  is a *minimal  $\varphi$ -invariant lamination* if the subset  $\mathcal{L}$  is minimal with respect to inclusion, among the family of  $\varphi$ -invariant laminations.

In other terms,  $\mathcal{L}$  is a minimal  $\varphi$ -invariant lamination if and only if the subset  $\mathcal{L} \subset \mathbb{R}^{(2)}$  is a minimal set (in the classical dynamical sense) for the diagonal action induced by  $\varphi$  on  $\mathbb{R}^{(2)}$ . When there is no risk of confusion, we will simply adopt the terminology “minimal lamination” instead of “minimal  $\varphi$ -invariant lamination”.

*Remark 3.4.* The property of being an  $\mathbb{R}$ -focal action is invariant under semi-conjugacy, and every  $\mathbb{R}$ -focal action is semi-conjugate to a minimal  $\mathbb{R}$ -focal action. See [5, Proposition 7.6].

*Remark 3.5.* Assume that  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  is a *minimal* action. If  $\mathcal{L}$  is any  $\varphi$ -invariant lamination, then  $\varphi$  is automatically  $\mathbb{R}$ -focal with respect to  $\mathcal{L}$ . See [5, Proposition 7.7].

Our first general result is that, in analogy to minimal invariant sets for actions on the real line, minimal laminations always exist for  $\mathbb{R}$ -focal actions of finitely generated groups.

**Proposition 3.6.** *Let  $G$  be a finitely generated group, and let  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  be an  $\mathbb{R}$ -focal with respect to a lamination  $\mathcal{L}$ . Then,  $\mathcal{L}$  contains a minimal  $\varphi$ -invariant lamination.*

*Proof.* Consider a finite symmetric generating system  $S = \{g_1, \dots, g_n\}$  for  $G$ , and let  $\|\cdot\|_S$  be the associated word length. As  $\mathcal{L}$  exhausts the real line, we can take an interval  $l \in \mathcal{L}$  so that  $h.l \cap l \neq \emptyset$  for every  $h \in S$ . The non-crossing condition gives that for every  $h \in S$  either  $h.l$  or  $h^{-1}.l$  contains  $l$ . This implies that the subset  $L := \bigcup_{h \in S} h.l$  is an interval which belongs to  $\mathcal{L}$ . We can then consider the subset  $\mathcal{K} := \{k \in \mathcal{L} : l \subseteq k \subseteq L\}$ , which is a compact subset of  $\mathcal{L}$ .



We want to show that any orbit in  $\mathcal{L}$  intersects  $\mathcal{K}$ . Take  $k_0 \in \mathcal{L}$ . If  $k_0 \in \mathcal{K}$ , there is nothing to prove. Otherwise, by definition of  $\mathbb{R}$ -focal action we can find an element  $h \in G$  such that  $L \subseteq h.k_0$ , and such that  $n := \|h\|_S$  is minimal among lengths of elements with this property. Write  $h = f_n \cdots f_1$ , with  $f_i \in S$  for every  $i \in \{1, \dots, n\}$ . By the minimality assumption on  $h$ , the interval  $k_* = f_n^{-1}h.k_0$  does not contain  $L$ . On the other hand, by definition of  $L$ , we have  $f_n.l \subseteq L$ , and since  $L \subseteq h.k_0 = f_n.k_*$ , we must have  $l \subseteq k_*$ . As  $l \subseteq L$ , this gives  $k_* \cap L \neq \emptyset$ , so that by the non-crossing condition we must have  $l \subseteq k_* \subseteq L$ , or equivalently  $k_* \in \mathcal{K}$ , as desired.

Consider now the families

$$\mathcal{C} = \{C \subset \mathcal{L} : C \text{ is } \varphi\text{-invariant and closed}\}, \quad \mathcal{C}_{\mathcal{K}} = \{C \cap \mathcal{K} : C \in \mathcal{C}\}.$$

Denote by  $i : \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{K}}$  the map given by  $C \mapsto C \cap \mathcal{K}$ . Since every  $\varphi$ -orbit meets  $\mathcal{K}$ , we have that  $i$  is a partial-order-isomorphism with respect to inclusion. On the other hand, since  $\mathcal{K}$  is compact, every decreasing chain in  $\mathcal{C}_{\mathcal{K}}$  has an infimum, so that after Zorn's lemma we can find a  $\varphi$ -invariant lamination which is minimal with respect to inclusion in  $\mathcal{C}$ .  $\square$

**3.2. Minimal actions on planar trees.** Let us recall some further terminology from [5, Section 8]. To study  $\mathbb{R}$ -focal actions it is helpful to study actions on a certain type of (real) trees with a preferred direction to infinity. The precise setting is the following.

**Definition 3.7.** A *directed tree* is a partially ordered set (*poset* for short)  $(\mathbb{T}, \triangleleft)$  with the following properties.

- (T1) For every  $v \in \mathbb{T}$  the subset  $\{u \in \mathbb{T} : v \trianglelefteq u\}$  is totally ordered and order-isomorphic to the half-line  $[0, +\infty)$ .
- (T2) Every pair of points  $u, v \in \mathbb{T}$  has a smallest common upper bound, denoted  $u \wedge v$ .
- (T3) There exists a countable subset  $\Sigma \subset \mathbb{T}$  such that for every distinct  $u, v \in \mathbb{T}$  with  $u \trianglelefteq v$  there exists  $z \in \Sigma$  such that  $u \trianglelefteq z \trianglelefteq v$ .

We say that a point  $u \in \mathbb{T}$  is *below*  $v \in \mathbb{T}$  (or that  $v$  is *above*  $u$ ) if  $u \neq v$  and  $u \trianglelefteq v$ , and write  $u \triangleleft v$ . When  $u$  is below  $v$ , we write  $[u, v] = [v, u] = \{w \in \mathbb{T} : u \trianglelefteq w \trianglelefteq v\}$  for the arc between  $u$  and  $v$ , and for general  $u$  and  $v$  we set  $[u, v] = [u, u \wedge v] \cup [v, u \wedge v]$ . We also write  $]u, v[ = [u, v] \setminus \{u, v\}$ , and similarly we define  $]u, v]$  and  $[u, v[$ . Arcs allow to define path-components and introduce the notion of *branching points*, that is, points  $v \in \mathbb{T}$  such that  $\mathbb{T} \setminus \{v\}$  has at least three path-components. We write  $\text{Br}(\mathbb{T})$  for the collection of branching points. We say that a directed tree  $(\mathbb{T}, \triangleleft)$  is *simplicial* if  $\text{Br}(\mathbb{T}) \cap [v, w]$  is finite for every  $v, w \in \mathbb{T}$ . A point  $v \in \mathbb{T}$  is called a *leaf* if  $\mathbb{T} \setminus \{v\}$  has only one path-component, or equivalently, if  $v$  is minimal with respect to  $\triangleleft$ . We will only be interested in trees which have no leaves (as justified by Remark 3.13 below).

The *end completion* of a directed tree  $(\mathbb{T}, \triangleleft)$  is the poset  $(\bar{\mathbb{T}}, \triangleleft)$  obtained by adding a unique  $\triangleleft$ -maximal point  $\omega$ , called the *focus*, and completing all unbounded maximal totally  $\triangleleft$ -ordered subsets by adding a least point. The set  $\partial\mathbb{T} := \bar{\mathbb{T}} \setminus \mathbb{T}$  is called the *boundary* of  $\mathbb{T}$ . We write  $\partial^*\mathbb{T} := \partial\mathbb{T} \setminus \{\omega\}$ . Given a point  $v \in \mathbb{T}$ , we denote by  $\partial U_v \subset \partial\mathbb{T} \setminus \{\omega\}$  the collection of boundary points corresponding to maximal totally  $\triangleleft$ -ordered subsets containing the point  $v$ , and we call it the *shadow* of  $v$ .

Next we introduce the notion of *planar tree*. Roughly speaking, this means a directed tree together with an order on branches which corresponds to a preferred way to draw it in the plane. The definition we give is slightly different from that appearing in [5], but equivalent (see Remark 3.9 below). To state it, when  $X$  is a set and  $\triangleleft, \prec$  are two partial orders on  $X$ , we

say that  $\triangleleft$  and  $\prec$  are *transverse* if for every  $u, v \in X$  it holds that  $u$  and  $v$  are  $\triangleleft$ -related if and only if they are not  $\prec$ -related.

**Definition 3.8.** Let  $(\mathbb{T}, \triangleleft)$  be a directed tree. A *planar structure* on  $(\mathbb{T}, \triangleleft)$  is a partial order  $\prec$  on  $\mathbb{T}$  which is transverse to  $\triangleleft$  and satisfies the *compatibility condition*:

(CC) for every  $u, u_0, v, v_0 \in \mathbb{T}$  with  $u \prec v$ ,  $u_0 \trianglelefteq u$  and  $v_0 \trianglelefteq v$ , it holds that  $u_0 \prec v_0$ .

We say that  $(\mathbb{T}, \triangleleft, \prec)$  is a *planar directed tree* (or simply *planar tree*) if  $(\mathbb{T}, \triangleleft)$  is a directed tree and  $\prec$  is a planar structure on  $(\mathbb{T}, \triangleleft)$ .

*Remark 3.9.* In [5, Definition 8.2], the definition of planar tree was given instead by fixing a linear order  $\prec_v$  on the set of path-components of  $\mathbb{T} \setminus \{v\}$  below any point  $v \in \text{Br}(\mathbb{T})$  (also called *directions* below  $v$ ). The two definitions are equivalent, and it is immediate to pass from one to the other. We prefer to adopt this alternative definition because in this way we can avoid introducing the related terminology, which is unnecessary for this work.

*Remark 3.10.* Any lamination  $\mathcal{L}$  admits two distinct partial orders, defined by the inclusion relation  $\subset$  and the order of the real line  $<$  (for  $l, k \in \mathcal{L}$ , we write  $l < k$  when  $\sup l \leq \inf k$ ). These two partial orders are transverse and satisfy the compatibility condition (CC).

Note that whenever  $(\mathbb{T}, \triangleleft, \prec)$  is a planar tree, the partial order  $\prec$  naturally induces a linear order on  $\partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\omega\}$ , by declaring  $\xi \prec \eta$  if there exist  $v \in \mathbb{T}$  and  $w \in \mathbb{T}$  satisfying  $\xi \triangleleft v, \eta \triangleleft w$ , and  $v \prec w$ .

We next want to describe the dynamics of group actions on (planar) directed trees.

**Definition 3.11.** Let  $G \subseteq \text{Aut}(\mathbb{T}, \triangleleft)$  be a group of automorphisms of a directed tree.

- $G$  is *focal* if for any  $u, v \in \mathbb{T}$  there exists  $g \in G$  such that  $v \triangleleft g.u$ .
- The action of  $G$  is *d-minimal* if for every  $v_1, v_2, w_-, w \in \mathbb{T}$  with  $v_1 \triangleleft v_2$  and  $w_- \triangleleft w$ , there exist  $g \in G$  and  $w' \in [w_-, w[$  such that  $g.[w', w] \subset ]v_1, v_2[$ .
- The action of  $G$  is *simplicial* if  $(\mathbb{T}, \triangleleft)$  is simplicial and for every  $v, w \in \text{Br}(\mathbb{T})$  the stabilizer  $\text{Stab}^\Phi(\{v, w\})$  acts trivially on  $[v, w]$ .

When  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  is an action on a directed tree, we say that  $\Phi$  is *focal*, *d-minimal*, or *simplicial*, according to what  $\Phi(G)$  is.

*Remark 3.12.* Any d-minimal action is focal. The letter “d” stands for *directionally*. Indeed the definition says that it is possible to send every point  $w$  inside any interval  $[v_1, v_2[$ , and moreover this can be done so that any given direction below  $w$  (defined by  $w_-$ ) is mapped to the direction defined by the point  $v_1$ .

*Remark 3.13.* If  $(\mathbb{T}, \triangleleft)$  admits a focal group of automorphisms, then it has no leaves.

Any group action  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  extends to an action on  $\overline{\mathbb{T}} = \mathbb{T} \cup \partial\mathbb{T}$ , which fixes the focus  $\omega$ . As a consequence, we obtain an action on the subset  $\partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\omega\}$ . In the case the focal action preserves a planar structure  $\prec$  on  $\mathbb{T}$ , we get that the linear order on  $\partial^*\mathbb{T}$  is preserved by the extension of the focal action. Fixing an end  $\xi \in \partial^*\mathbb{T}$  we get an order-preserving action on the totally ordered set  $(G.\xi, \prec)$  where  $\prec$  is the order induced by the linear order on  $\partial^*\mathbb{T}$ . We proved in [5, Proposition 8.19] that, for a given focal action  $\Phi$ , the dynamical realization of the action on  $(G.\xi, \prec)$  gives a minimal  $\mathbb{R}$ -focal action that does not depend on the choice of  $\xi \in \partial^*\mathbb{T}$  (up to positive conjugacy). We call this action the *dynamical realization* of  $\Phi$ . Conversely, by [5, Proposition 8.20], any minimal  $\mathbb{R}$ -focal action of a countable group  $G$  can be reconstructed (up to positive semi-conjugacy) as the dynamical realization

of a focal action  $\Phi$  of  $G$  on a planar directed tree. In this case we say that  $\Phi$  *represents* the  $\mathbb{R}$ -focal action.

The following lemma gives an alternative characterization of this notion.

**Lemma 3.14.** *Let  $G$  be a countable group and  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  a minimal  $\mathbb{R}$ -focal action. Let  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$  be a focal action on a planar tree. Then  $\Phi$  represents  $\varphi$  if and only if there exists a non-decreasing  $G$ -equivariant map  $\sigma : (\partial^*\mathbb{T}, \prec) \rightarrow (\mathbb{R}, <)$ . Moreover such a map  $\sigma$  is necessarily injective on each  $G$ -orbit.*

*Proof.* Assume first that  $\Phi$  represents  $\varphi$ . This means that there exists  $\xi \in \partial^*\mathbb{T}$  and an equivariant order preserving bijection  $t : G \cdot \xi \rightarrow \mathbb{Q}$ , which can be extended to a non-decreasing  $G$ -equivariant map  $\sigma : \partial^*\mathbb{T} \rightarrow \mathbb{R}$  given by  $\sigma(\eta) = \sup\{t(\xi') : \xi' \in G \cdot \xi, \xi' \preceq \eta\}$ . Conversely, assume that there is such a map  $\sigma$ . Then for every  $\xi \in \partial^*\mathbb{T}$ , the restriction of  $\sigma$  to  $G \cdot \xi$  satisfies the assumption of Lemma 2.3. We deduce both that  $\sigma$  is injective on  $G \cdot \xi$  and that  $\varphi$  is the dynamical realization of  $\Phi$ .  $\square$

The next result is a more precise version of [5, Proposition 8.20].

**Proposition 3.15.** *Let  $G$  be a finitely generated group and let  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  be a minimal action which is  $\mathbb{R}$ -focal. Then,  $\varphi$  can be represented by a focal action  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$  which is either  $d$ -minimal, or simplicial and transitive on branching points.*

*Proof.* As we are assuming that  $G$  is finitely generated, after Proposition 3.6, we can take a minimal  $\varphi$ -invariant lamination  $\mathcal{L}$ . Since the subset of isolated leaves is both  $\varphi$ -invariant and open in  $\mathcal{L}$ , minimality of  $\mathcal{L}$  implies that either  $\mathcal{L}$  is discrete or  $\mathcal{L}$  is perfect.

We first treat the discrete case. In this case, each  $\varphi$ -orbit in  $\mathcal{L}$  is open, so that by minimality  $\mathcal{L}$  consists of a single  $\varphi$ -orbit. We consider the simplicial tree  $\mathbb{T}$  associated with the partial order on  $\mathcal{L}$  defined by the inclusion relation  $\subset$ . It is straightforward to check that  $(\mathbb{T}, \subset)$  is a simplicial directed tree, where  $\text{Br}(\mathbb{T})$  is naturally identified with  $\mathcal{L}$ . Moreover, the partial order relation  $<$  induced by the order on the real line makes  $(\mathbb{T}, \subset, <)$  a planar directed tree (see Remark 3.10). The action of  $\varphi$  on  $\mathcal{L}$  naturally extends to a simplicial action  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \subset, <)$ . Finally, consider the map  $\iota : \partial^*\mathbb{T} \rightarrow \mathbb{R}$  that sends the boundary point  $\xi$ , represented by a  $\subset$ -monotone ray  $r_\xi : [0, +\infty) \rightarrow \mathbb{T}$ , to the intersection of nested closed intervals

$$\iota(\xi) := \bigcap_{l \in \text{Im}(r_\xi) \cap \mathcal{L}} \bar{l}.$$

It is clear that  $\iota(\xi)$  does not depend on the choice of  $r_\xi$ . To see that  $\iota$  is well defined (that is, it takes value in  $\mathbb{R}$ ) we need to check that  $\iota(\xi)$  has empty interior. To see this, first notice that  $\text{Int}(\iota(\xi_1)) \cap \text{Int}(\iota(\xi_2)) = \emptyset$  for any  $\xi_1 \neq \xi_2$ . Thus, if some  $\xi \in \partial^*\mathbb{T}$  had non-empty interior we would obtain a proper  $\varphi$ -invariant open subset, contradicting minimality of the action  $\varphi$ . It is direct to check that  $\iota$  is  $<$ -monotone and equivariant with respect to  $\Phi$  and  $\varphi$ . We conclude that  $\varphi$  is represented by  $\Phi$ .

Assume now that  $\mathcal{L}$  is perfect. Say that a leaf  $l \in \mathcal{L}$  is *accumulated from above* if the subset  $\{k \in \mathcal{L} : k \supsetneq l\}$  accumulates on  $l$ . Analogously we define when a leaf is *accumulated from below*. Given  $l \in \mathcal{L}$  write

$$l^* = \text{Int} \left( \bigcap_{k \in \mathcal{L}, k \supsetneq l} k \right),$$

and note that  $l^* \in \mathcal{L}$  as  $\mathcal{L}$  is closed in  $\mathbb{R}^{(2)}$ .

**Claim.** *The subset  $\mathcal{L}^* := \{l^* : l \in \mathcal{L}\}$  coincides with the subset of leaves in  $\mathcal{L}$  accumulated from above, and the equality  $(l^*)^* = l^*$  holds for every  $l \in \mathcal{L}$ .*

*Proof of claim.* First notice that, by definition, if  $l \in \mathcal{L}$  is accumulated from above, then  $l = l^*$  and therefore  $l \in \mathcal{L}^*$ . Take now  $k = l^*$  in  $\mathcal{L}^*$ . If  $l = k$  then, by definition again,  $k$  is accumulated from above. On the other hand, when  $l \neq k$ , the leaf  $k$  is not accumulated from below. Thus, since  $\mathcal{L}$  is perfect,  $k$  must be accumulated from above. Also notice that  $k^* = k$  holds in both cases.  $\square$

We want to show that the action induced by  $\varphi$  on  $(\mathcal{L}^*, \subset, <)$  is the desired focal action. We first need to verify that  $(\mathcal{L}^*, \subset)$  is a directed tree. We start with condition (T1): for fixed  $l \in \mathcal{L}^*$ , we need to show that the subset  $R = \{k \in \mathcal{L}^* : l \subseteq k\}$  is order-isomorphic to  $[0, +\infty)$ . Consider the function  $j : (R, \subset) \rightarrow [0, +\infty)$ , defined by  $j(k) = |k| - |l|$ , where  $|\cdot|$  denotes the size of an interval. After the claim, the function  $j$  is order preserving and unbounded, and every point in its image is accumulated from the right. We can then find monotone surjective map  $h : [0, +\infty) \rightarrow [0, +\infty)$ , which is bijective on  $j(R)$ . The composition  $h \circ j$  gives the desired order-isomorphism. We next focus on condition (T2), and for this fix  $l_1, l_2 \in \mathcal{L}^*$ . Consider the subset  $S = \{k \in \mathcal{L}^* : (l_1 \cup l_2) \subset k\}$ , and define  $L := \text{Int}(\bigcap_{k \in S} k)$ . We need to show that  $L \in \mathcal{L}^*$ , or equivalently, by the claim, that  $L$  is accumulated from above. If not, since  $\mathcal{L}$  is perfect,  $L$  must be accumulated from below by leaves in  $\mathcal{L}^*$ . Taking a leaf  $L' \in \mathcal{L}^*$  strictly contained in  $L$  and sufficiently close to  $L$ , we must have  $L' \cap l_i \neq \emptyset$  for  $i \in \{1, 2\}$ , and by the non-crossing condition, this implies  $L' \supset (l_1 \cup l_2)$ , contradicting the choice of  $L$ . For condition (T3), take any  $l_0 \in \mathcal{L}^*$  and consider the countable subset  $\Sigma := \{g.l_0 : g \in G\}$ . Using the claim and the fact that  $\mathcal{L}$  is a minimal lamination, for every distinct  $l, k \in \mathcal{L}^*$  we can find an element  $g \in G$  such that  $l \subseteq g.l_0 \subseteq k$ , as desired.

As for the discrete case, the linear order  $<$  on the real line makes  $(\mathcal{L}^*, \subset, <)$  a planar directed tree, so that the action induced by  $\varphi$  on  $\mathcal{L}^*$  preserves this structure, and represents the  $\mathbb{R}$ -focal action  $\varphi$ . It remains to prove that this action, call it  $\Phi : G \rightarrow \text{Aut}(\mathcal{L}^*, \subset, <)$ , is d-minimal (so that it is automatically focal, see Remark 3.12). For this, take leaves  $l_1 \subset l_2$  and  $k_1 \subset k_2$  in  $\mathcal{L}^*$ , and we want to find an element  $g \in G$  and a leaf  $k \in [k_1, k_2]$  such that  $g.[k, k_2] \subset ]l_1, l_2[$ . Taking a smaller  $l_2$  if necessary, we can assume that  $l_2$  is also accumulated from below. Consider the leaf  $\bar{k}$  in  $\mathcal{L}$  given by

$$\bar{k} := \bigcup_{k \in \mathcal{L}; k_1 \subset k \subsetneq k_2} k,$$

and note that  $\bar{k}$  is accumulated from below. Again, by the claim and minimality of  $\mathcal{L}$ , there exists an element  $g \in G$  such that  $l_1 \subsetneq g.\bar{k} \subsetneq l_2$ . As  $l_2$  is accumulated from below, and  $k_2$  is either  $\bar{k}$  or its successor, we must have  $l_1 \subsetneq g.k_2 \subsetneq l_2$ . As  $\bar{k}$  is accumulated from below, we can also find a leaf  $k \in \mathcal{L}^*$  such that  $l_1 \subsetneq g.k \subsetneq g.\bar{k}$ . This gives the desired conclusion.  $\square$

**3.3. Horogradings actions.** One of the advantages when passing from an  $\mathbb{R}$ -focal action to a focal representation on a directed tree, is that some new information about the dynamics might become visible at the level of the tree. This happens when there is a consistent way of measuring how the distance from the focus changes under the action. Let us recall from [5, §8.1.3] that an (*increasing*) *horograding* of a directed tree  $(\mathbb{T}, \triangleleft)$  is an increasing map  $\pi : \mathbb{T} \rightarrow \mathbb{R}$  such that for every  $u, v \in \mathbb{T}$  verifying  $u \triangleleft v$ , the restriction of  $\pi$  to the arc  $[u, v] \subset \mathbb{T}$  is an order-preserving bijection onto the interval  $[\pi(u), \pi(v)] \subset \mathbb{R}$ . Every directed tree admits horogradings, but they are not always adapted to study focal actions. For this, given an action  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$ , we say that  $\Phi$  is *horograded* by an action  $j : G \rightarrow \text{Homeo}_0(\mathbb{R})$  if there

exists a  $G$ -equivariant surjective horograding  $\pi : \mathbb{T} \rightarrow \mathbb{R}$ . We also say that an  $\mathbb{R}$ -focal action  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  is horograded by an action  $j : G \rightarrow \text{Homeo}_0(\mathbb{R})$ , if  $j$  is a horograding for a focal tree representation of  $\varphi$ .

Horogradings are good to detect the dynamical behavior of single elements, and for this we recall the following terminology.

**Definition 3.16.** A homeomorphism  $h \in \text{Homeo}_0(\mathbb{R})$  is an *expanding pseudohomothety* if there exists a compact subset  $K \subset \mathbb{R}$  such that  $h(U) \supset \overline{U}$  for every open subset  $U \supset K$ . When,  $h^{-1}$  is an expanding pseudohomothety we say that  $h$  is a *contracting pseudohomothety*. In the case where  $K$  can be taken to be a point, we simply say  $h$  is an (expanding or contracting) *homothety*. In contrast, we say that  $h$  is *totally bounded* if  $\text{Fix}(h)$  accumulates on both  $+\infty$  and  $-\infty$ .

It was shown in [5, Corollary 8.28] that, in an  $\mathbb{R}$ -focal action  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$ , every element is either a pseudohomothety or totally bounded. In the case of a horograded  $\mathbb{R}$ -focal action, this can be determined by the horograding action:

**Proposition 3.17** (Proposition 8.33 in [5]). *Let  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  be a minimal  $\mathbb{R}$ -focal action positively horograded by an action  $j : G \rightarrow \text{Homeo}_0(\mathbb{R})$ . For any element  $g \in G$ , we have the following.*

- (1) *If  $\text{Fix}(j(g))$  does not accumulate on  $+\infty$ , then  $\varphi(g)$  is a pseudohomothety, which is expanding if  $j(g)(x) > x$  for every large enough  $x$ , and contracting otherwise.*
- (2) *Otherwise,  $\varphi(g)$  is totally bounded.*

Moreover, in the former situation, if  $\text{Fix}(j(g)) = \emptyset$  then  $\varphi(g)$  is a homothety.

The following result establishes a more explicit relation between an  $\mathbb{R}$ -focal action and an action that horogrades it. Roughly speaking it says that if  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  is horograded by  $j : G \rightarrow \text{Homeo}_0(\mathbb{R})$ , then there are locally defined semi-conjugacies at the level of germs near  $\infty$ .

**Proposition 3.18.** *Let  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  be a minimal  $\mathbb{R}$ -focal action, and assume that  $\varphi$  can be horograded by  $j : G \rightarrow \text{Homeo}_0(\mathbb{R})$ . Then, there exist two maps  $h_+, h_- : \mathbb{R} \rightarrow \mathbb{R}$  satisfying the following conditions.*

- (1)  *$h_+$  is monotone non-decreasing,*
- (2)  *$h_-$  is monotone non-increasing,*
- (3) *we have  $\lim_{x \rightarrow +\infty} h_+(x) = \lim_{x \rightarrow -\infty} h_-(x) = +\infty$ ,*
- (4) *for every  $g \in G$  there exists  $M > 0$  such that  $h_+(\varphi(g)(x)) = j(g)(h_+(x))$  for every  $x > M$ , and  $h_-(\varphi(g)(x)) = j(g)(h_-(x))$  for every  $x < -M$ .*

*Proof.* Let  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$  be a focal action representing  $\varphi$  together with a horograding  $\pi : \mathbb{T} \rightarrow \mathbb{R}$  that is  $G$ -equivariant with respect to  $\Phi$  and  $j$ . Given a boundary point  $\xi \in \partial^* \mathbb{T}$ , let  $\iota : G \cdot \xi \rightarrow \mathbb{R}$  be an order-preserving map which is  $G$ -equivariant with respect to  $\Phi$  and  $\varphi$ . For  $v \in \mathbb{T}$ , recall that we denote  $\partial U_v \subset \partial^* \mathbb{T}$  the shadow of  $v$ , consisting of points  $\xi \in \partial^* \mathbb{T}$  such that  $\xi \triangleleft v$ . Set  $I_v = \overline{\iota(G \cdot \xi \cap \partial U_v)}$ , which is a compact interval. Choose a point  $v_0 \in \mathbb{T}$ . The collection  $\{I_v : v_0 \triangleleft v\}$  is a totally ordered family of intervals which exhausts  $\mathbb{R}$ . For  $x \in \mathbb{R}$ , set  $v_x = \inf_{\triangleleft} \{v \triangleright v_0 : x \in I_v\}$ . Let  $I_{v_0} = [a, b]$  and note that if  $x \leq y$  both lie in  $[b, +\infty)$  then we have  $v_x \triangleleft v_y$ , and if they lie in  $(-\infty, a]$  we have  $v_y \triangleleft v_x$ . Now define  $h_+$  and  $h_-$  by letting  $h_+(x) = \pi(v_x)$  for  $x \in [b, +\infty)$  and  $h_-(x) = \pi(v_x) = (-\infty, a]$  (extend them arbitrarily to the rest of  $\mathbb{R}$  by ensuring that  $h_+$  is non-decreasing and  $h_-$  is non-increasing). Then  $h_+$  and  $h_-$  satisfy the desired properties.  $\square$

**3.4. Abelian horogradings and invariant  $\mathbb{R}$ -tree metrics.** Recall that a *metric  $\mathbb{R}$ -tree* is a metric space  $(X, d)$  where every pair of points  $v, w \in X$  can be joined by a unique arc, and this arc can be chosen to be a geodesic. We say that a directed tree  $(\mathbb{T}, \triangleleft)$  has a *compatible metric* if  $\mathbb{T}$  is endowed with an  $\mathbb{R}$ -tree metric where the subsets of the form  $[v, w]$  associated to  $\triangleleft$ , coincide with the geodesic segments associated to the  $\mathbb{R}$ -tree metric. In [5, Section 8.1.4] it is shown that every directed tree can be endowed with such a compatible  $\mathbb{R}$ -tree metric. However, in the presence of an action  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  it is not always possible to find  $\Phi$ -invariant compatible metrics. The following proposition describes when it is the case in terms of the existence of abelian horogradings. Its proof is roughly contained in [5, Proposition 8.34] and [5, Remark 8.30] up to minor terminological differences. This point of view will be helpful in Section 8.

**Proposition 3.19.** *Let  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  be a focal action on a directed tree. Then,  $\Phi$  preserves a compatible metric on  $(\mathbb{T}, \triangleleft)$  if and only if  $\Phi$  can be horograded by an action by translations  $j : G \rightarrow (\mathbb{R}, +)$ . Moreover, if  $\Phi$  is simplicial then  $\Phi$  is horograded by a cyclic action.*

*Sketch of proof.* First suppose that  $\Phi$  is horograded by an action  $j : G \rightarrow \text{Homeo}_0(\mathbb{R})$  whose image consists on translations, and denote  $\pi : \mathbb{T} \rightarrow \mathbb{R}$  its associated horograding. Then, define the distance  $d$  on  $\mathbb{T}$  so that

$$d(w_1, w_2) = \left( \pi(w_1 \wedge w_2) - \pi(w_1) \right) + \left( \pi(w_1 \wedge w_2) - \pi(w_2) \right).$$

It is direct to check that  $d$  is a compatible metric on  $\mathbb{T}$ . Also notice that, since  $j$  is an action by translations,  $d$  is  $\Phi$ -invariant.

For the converse, suppose  $\Phi$  preserves a compatible metric  $d$  on  $(\mathbb{T}, \triangleleft)$ . Denote by  $l_d(w_1, w_2)$  the length induced by  $d$  on the geodesic  $[w_1, w_2]$ . Fix  $v \in \mathbb{T}$ ,  $x \in \mathbb{R}$  and define  $\pi : \mathbb{T} \rightarrow \mathbb{R}$  as

$$\pi(w) = x + l_d([v, v \wedge w]) - l_d(w, v \wedge w).$$

Notice that  $\pi$  is the unique horograding satisfying:

- (1)  $\pi(v) = x$
- (2) the restriction of  $\pi$  to any ray of the form  $[w, \omega[$  is an order-preserving isometry.

By considering all possible values of  $x \in \mathbb{R}$  we deduce that there exists exactly a one-parameter family of horogradings that satisfy condition (2), and that all these horogradings can be obtained as  $T_y \circ \pi$  where  $T_y : \mathbb{R} \rightarrow \mathbb{R}$  is the translation by  $y \in \mathbb{R}$ . Since  $\Phi$  preserves  $d$ ,  $\pi \circ \Phi(g)$  satisfies condition (2) for every  $g \in G$ . Putting all together we deduce that, for every  $g \in G$ , there exist a unique  $t_g \in \mathbb{R}$  such that  $\pi \circ \Phi(g) = T_{t_g} \circ \pi$ . Thus, the map  $g \mapsto t_g$  is a morphism and  $\Phi$  is horograded by the action  $j : G \rightarrow \text{Homeo}_0(\mathbb{R})$  defined as  $j(g) = T_{t_g}$ .

In the particular case where  $\Phi$  is simplicial, one can consider the compatible metric such that  $d(v, w) = 1$  for every pair of adjacent branching points, that is for every  $v, w \in \text{Br}(\mathbb{T})$  so that  $]v, w[ \cap \text{Br}(\mathbb{T}) = \emptyset$ . Since  $\Phi$  is simplicial this metric is  $\Phi$  invariant. Finally, it is direct to check that

$$t_g = l_d([v, v \wedge g.v]) - l_d([g.v, v \wedge g.v]),$$

which belongs to  $\mathbb{Z}$ . This implies that  $\Phi$  is horograded by a cyclic action as desired.  $\square$

#### 4. ELLIPTIC SUBGROUPS IN FOCAL ACTIONS

In this section we explain a technical mechanism that allows, given a d-minimal action  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  on a directed tree and a suitable normal subgroup  $N$  of  $G$ , to “mod out

$N$ ” and construct an action of  $G/N$  on a smaller directed tree. This mechanism will be an important tool to study  $\mathbb{R}$ -focal actions of solvable groups, by successively simplifying them.

#### 4.1. Elliptic subgroups.

**Definition 4.1.** An automorphism  $g \in \text{Aut}(\mathbb{T}, \triangleleft)$  of a directed tree is *elliptic* if for every  $v \in \mathbb{T}$  verifying  $g.v \triangleleft v$ , one has  $g.v = v$ . A subgroup  $G \subseteq \text{Aut}(\mathbb{T}, \triangleleft)$  of automorphisms is elliptic, if it contains only elliptic elements. Similarly, when  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  is an action such that  $\Phi(G)$  is elliptic, we say that  $\Phi$  is elliptic.

*Remark 4.2.* When the directed tree  $(\mathbb{T}, \triangleleft)$  is simplicial (or more generally, when an action preserves an  $\mathbb{R}$ -tree metric), the definition above corresponds to the classical notion of elliptic isometry. Indeed, for isometric actions on simplicial trees, every automorphism is either elliptic (which is equivalent to having a fixed point in  $\mathbb{T}$ ) or *hyperbolic*. Recall that hyperbolic automorphisms admit a unique translation axis (an embedded line on which the automorphism acts by translation), and in the case of directed trees, the axis contains the focus in its boundary.

**Proposition 4.3.** *Let  $G \subseteq \text{Aut}(\mathbb{T}, \triangleleft)$  be a focal subgroup of automorphisms of a directed tree, whose action is either simplicial or d-minimal. Then any normal subgroup of  $G$  is either focal or elliptic.*

*Proof.* Let  $N \triangleleft G$  be a normal subgroup. Assume first that the action of  $G$  is simplicial. We distinguish two cases according to whether the image of  $N$  contains or not hyperbolic automorphisms of  $\mathbb{T}$  (see Remark 4.2). If  $N$  does not contain any hyperbolic element, then it is elliptic. If  $N$  does contain a hyperbolic element, let  $\mathbb{S}$  be the union of the axes of its hyperbolic elements. Note that  $\mathbb{S}$  is a directed subtree, and the action of  $N$  on  $\mathbb{S}$  is focal. As  $N$  is normal,  $\mathbb{S}$  is also  $G$ -invariant. Finally, take a point  $v \in \mathbb{S}$  and note that as  $G$  is focal, for any  $u \in \mathbb{T}$  there exists  $g \in G$  such that  $g.u \in [v, \omega[ \subset \mathbb{S}$ . This shows that  $\mathbb{T} = \mathbb{S}$ , and we have already observed that the action of  $N$  on  $\mathbb{T} = \mathbb{S}$  is focal, as desired.

Consider next the case where the action of  $G$  is d-minimal and assume by contradiction that  $N$  is neither elliptic nor focal.

**Claim.** *For any  $v \in \mathbb{T}$  and  $v' \triangleleft v$ , there exist  $k \in N$  and  $w \in [v', v[$  such that  $v \triangleleft k.w$ .*

*Proof.* Since  $N$  is not elliptic, we can choose a point  $v_0 \in \mathbb{T}$  and  $h \in N$  such that  $v_0 \triangleleft h.v_0$ . By d-minimality, for given  $v' \triangleleft v$ , we can find  $w \in [v', v[$  and  $g \in G$  such that  $g.[w, v[ \subset ]v_0, h.v_0[$ . By the choice of  $h$ , we must then have  $g.v \triangleleft h.v_0 \triangleleft hg.w$ . Applying  $g^{-1}$ , we get  $v \triangleleft g^{-1}hg.w$ . As  $N$  is normal and  $h \in N$ , the element  $k := g^{-1}hg \in N$  is the element we were looking for.  $\square$

Now since  $N$  is not focal, there exists  $u \in \mathbb{T}$  such that  $N.u \cap [u, \omega[$  is upper bounded, and set  $v = \sup_{\triangleleft}(N.u \cap [u, \omega[)$ . After the claim, we can take  $k \in N$  and  $w \in [u, v[$  such that  $v \triangleleft k.w$ . Note that the above relation holds for any  $w' \in [w, v[$ . In particular, for any  $w' \in N.u \cap [w, \omega[$ , we will have  $v \triangleleft k.w'$ , and as  $k.w' \in N.u \cap [u, \omega[$ , this contradicts the choice of  $v$ .  $\square$

From the previous proposition, we obtain the following.

**Corollary 4.4.** *Let  $G \subseteq \text{Aut}(\mathbb{T}, \triangleleft)$  be a focal subgroup of automorphisms of a directed tree, and let  $N$  be a normal subgroup of  $G$  whose centralizer in  $G$  is non-trivial. Assume further that the action of  $G$  is either simplicial or d-minimal, and that  $\mathbb{T} \neq \mathbb{R}$ . Then,  $N$  is elliptic.*

The proof of the corollary is also based on the next general result for focal subgroups.

**Lemma 4.5.** *Let  $G \subseteq \text{Aut}(\mathbb{T}, \triangleleft)$  be a focal subgroup of automorphisms of a directed tree  $\mathbb{T} \neq \mathbb{R}$ . Assume that the action of  $G$  is either  $d$ -minimal or simplicial. Let  $H < G$  be a focal subgroup of  $G$ . Then the centralizer of  $H$  in  $G$  is trivial.*

*Proof.* By contradiction, we assume there exists a non-trivial automorphism  $k \in G$  centralizing  $G$ . We first claim that there exists a point  $v \in \mathbb{T}$  with  $v$  and  $k.v$  that are not  $\triangleleft$ -related. To see this, assume first that the action of  $G$  is  $d$ -minimal. This assumption, together with the fact that  $\mathbb{T} \neq \mathbb{R}$ , implies that every non-empty open path  $]v, w[ \subset \mathbb{T}$  contains branching points. Choose  $w \in \mathbb{T}$  such that  $k.w \neq w$ . If  $w, k.w$  are not  $\triangleleft$ -related, we choose  $v = w$ . Else assume, say, that  $w \triangleleft k.w$ , and let  $v_0 \in ]w, k.w[$  be a branching point. Let  $v \triangleleft v_0$  lie in a different path-component of  $\mathbb{T} \setminus \{v_0\}$  as  $w$ . Then  $v, k.v$  are not  $\triangleleft$ -related. If the action is simplicial, choose  $w \in \text{Br}(\mathbb{T})$  such that  $k.w \neq w$ , and let  $v_1, v_2$  be two distinct branching points below  $w$ . Then at least one between  $v = v_1$  or  $v = v_2$  must be such that  $k.v$  and  $v$  are not  $\triangleleft$ -related.

Let now  $v$  be as in the claim. Write  $w := v \wedge k.v$ . Since  $H$  is focal we can take  $g \in H$  so that  $g.v \triangleright w$ . On the one hand, since  $v$  and  $k.v$  are not  $\triangleleft$ -related, neither are  $g.v$  and  $gk.v$ . Thus, since  $k.v \triangleleft w \triangleleft g.v$ , we conclude that  $k.v$  and  $gk.v$  are non  $\triangleleft$ -related. On the other hand, since  $v \triangleleft w \triangleleft g.v$  we have that  $k.v \triangleleft kg.v = gk.v$  which contradicts the previous conclusion.  $\square$

*Proof of Corollary 4.4.* Combine Lemma 4.5 with Proposition 4.3.  $\square$

**4.2. Modding out elliptic normal subgroups.** The aim of this section is to explain that if  $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  is an action and  $N$  is a normal elliptic subgroup, then  $N$  can be mod out to obtain a new action of  $G/N$  on a directed tree (Proposition 4.10 below). This is quite intuitive: the idea is to consider the action on the quotient  $\mathbb{T}/N$ . The issue however, is that the space  $\mathbb{T}/N$  might not be a directed tree. Fortunately, this can be readily solved by passing to a slightly smaller quotient. This the goal of the discussion below.

**Definition 4.6.** We say that a poset  $(\mathbb{T}, \triangleleft)$  is a *pre-directed tree* if it satisfies conditions (T1) and (T3) in the definition of directed tree and

(T2') for every pair of points  $u, v \in \mathbb{T}$  there is  $w \in \mathbb{T}$  such that  $u \triangleleft w$  and  $v \triangleleft w$ .

We define  $d$ -minimality for actions on pre-directed tree in the same way as in the case of directed trees.

*Remark 4.7.* As an example, think of the space obtained by gluing two copies of  $\mathbb{R}$  along the open ray  $(0, +\infty)$ . Although we did not consider any topology on a directed tree, one may think of pre-directed tree as a directed tree which is possibly not Hausdorff.

**Definition 4.8.** Let  $(\mathbb{T}_1, \triangleleft)$  and  $(\mathbb{T}_2, \triangleleft)$  be pre-directed trees. We say that a map  $\pi: \mathbb{T}_1 \rightarrow \mathbb{T}_2$  is a *grading* if it is surjective and for every  $v, w \in \mathbb{T}_1$  such that  $v \triangleleft w$ , we have that  $\pi(v) \triangleleft \pi(w)$  and the restriction of  $\pi$  to  $[v, w]$  is an order-preserving isomorphism onto  $[\pi(v), \pi(w)]$ . If in addition, for  $i \in \{1, 2\}$ , one has that  $\Phi_i: G \rightarrow \text{Aut}(\mathbb{T}_i, \triangleleft)$  are order-preserving actions admitting a  $G$ -equivariant grading from  $\mathbb{T}_1$  onto  $\mathbb{T}_2$ , we say that  $\Phi_1$  can be *graded* by  $\Phi_2$ .

**Lemma 4.9.** *Let  $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  be an action on a pre-directed tree. Then, there exists an action  $\widehat{\Phi}: G \rightarrow \text{Aut}(\widehat{\mathbb{T}}, \triangleleft)$  on a directed tree, grading  $\Phi$ . Moreover, if  $\Phi$  is  $d$ -minimal so is  $\widehat{\Phi}$ .*

*Proof.* Given  $v \in \mathbb{T}$ , write  $\widehat{v} := \{w \in \mathbb{T} : v \triangleleft w\}$  and  $\widehat{\mathbb{T}} := \{\widehat{v} : v \in \mathbb{T}\}$ . We set  $\widehat{v} \triangleleft \widehat{w}$  if and only if  $\widehat{v} \supsetneq \widehat{w}$ . Equivalently,  $\widehat{w} \triangleleft \widehat{v}$  if and only if  $w \in \widehat{v}$ . It is direct to check that if  $\Sigma \subset \mathbb{T}$  is a countable subset given by condition (T3) for  $(\mathbb{T}, \triangleleft)$ , then the subset  $\widehat{\Sigma} := \{\widehat{v} \in \widehat{\mathbb{T}} : v \in \Sigma\}$  also satisfies condition (T3). In order to check condition (T1), for any  $v \in \mathbb{T}$  write

$$(4.1) \quad L_v = \{\widehat{w} \in \widehat{\mathbb{T}} : \widehat{v} \triangleleft \widehat{w}\} = \{\widehat{w} \in \widehat{\mathbb{T}} : w \in \widehat{v}\}.$$



From this, it follows that the poset  $(\widehat{\mathbb{T}}, \triangleleft)$  satisfies condition (T1) in the definition of directed tree and that the map  $\pi : \mathbb{T} \rightarrow \widehat{\mathbb{T}}$  defined by  $\pi(v) = \widehat{v}$  is a grading. Notice that the action  $\Phi$  projects to an action  $\widehat{\Phi} : G \rightarrow \text{Aut}(\widehat{\mathbb{T}}, \triangleleft)$  that grades  $\widehat{\Phi}$  through the projection  $\pi$ . It also follows from (4.1) that whenever  $\Phi$  is d-minimal, so is  $\widehat{\Phi}$ . It remains to show that  $(\widehat{\mathbb{T}}, \triangleleft)$  satisfies condition (T2) in the definition of directed tree. For this, note that by condition (T2'), given  $v_1, v_2 \in \mathbb{T}$ , there exists  $w \in \mathbb{T}$  such that either  $\widehat{v}_1 \cap \widehat{v}_2 = \widehat{w}$  or  $\widehat{v}_1 \cap \widehat{v}_2 = \widehat{w} \cup \{w\}$ . In both cases, the set of common upper bounds of  $\widehat{v}_1$  and  $\widehat{v}_2$  equals  $L_{v_1} \cap L_{v_2} = L_w \cup \{\widehat{w}\}$  and therefore  $\widehat{w}$  is the least common upper bound of  $\widehat{v}_1$  and  $\widehat{v}_2$ , as wanted.  $\square$

**Proposition 4.10.** *Let  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  be an action on a directed tree. Assume that  $N \trianglelefteq G$  is a normal subgroup whose action is elliptic. Then,  $\Phi$  can be graded by an action  $\widehat{\Phi} : G \rightarrow \text{Aut}(\widehat{\mathbb{T}}, \triangleleft)$  on a directed tree where  $N$  acts trivially. Moreover, if  $\Phi$  is d-minimal so is  $\widehat{\Phi}$ .*

*Proof.* We will first show that  $\Phi$  is graded by an action on a pre-directed tree where  $N$  acts trivially. For this, consider  $\widehat{\mathbb{T}}_0 = \mathbb{T}/\Phi(N)$ , namely the quotient space of  $\mathbb{T}$  under the action of  $N$ , and denote by  $\pi : \mathbb{T} \rightarrow \widehat{\mathbb{T}}_0$  the quotient projection. We define a partial order on  $\widehat{\mathbb{T}}_0$  by declaring  $x_1 \trianglelefteq x_2$  if there exists  $v_i \in \pi^{-1}(x_i)$  for  $i \in \{1, 2\}$ , with  $v_1 \trianglelefteq v_2$ . In order to check that the relation is transitive, take  $x_1, x_2, x_3 \in \widehat{\mathbb{T}}_0$  satisfying  $x_1 \trianglelefteq x_2 \trianglelefteq x_3$ . By definition of the partial order there are points  $v_i \in \pi^{-1}(x_i)$  for  $i \in \{1, 2\}$  and  $w_i \in \pi^{-1}(x_i)$  for  $i \in \{2, 3\}$  such that  $v_1 \trianglelefteq v_2$  and  $w_2 \trianglelefteq w_3$ . On the other hand, since the points  $v_2$  and  $w_2$  are in the same  $N$ -orbit, there exists  $h \in N$  such that  $h.w_2 = v_2$ . Thus, we have  $v_1 \trianglelefteq h.w_2 \trianglelefteq h.w_3$  and, since  $h.w_3$  projects to  $x_3$ , we get  $x_1 \trianglelefteq x_3$ , as desired. To check antisymmetry of  $\triangleleft$ , suppose that for  $x_1, x_2 \in \widehat{\mathbb{T}}_0$  it holds  $x_1 \trianglelefteq x_2$  and  $x_2 \trianglelefteq x_1$ . In this case, again by the definition of the partial order, there exist  $v_i, w_i \in \pi^{-1}(x_i)$  for  $i \in \{1, 2\}$ , such that  $v_1 \trianglelefteq v_2$  and  $w_2 \trianglelefteq w_1$ . Take  $h_1, h_2 \in N$  so that  $w_i = h_i.v_i$  for  $i \in \{1, 2\}$ . Then, applying  $h_2^{-1}$ , we get  $v_1 \trianglelefteq v_2 \trianglelefteq h_2^{-1}h_1.v_1$ . Since  $h_2^{-1}h_1$  is elliptic, these must be equalities, implying that  $v_1 = v_2$  and therefore  $x_1 = x_2$ . This finishes the proof that  $(\widehat{\mathbb{T}}_0, \triangleleft)$  is a poset.

We proceed to check that  $(\widehat{\mathbb{T}}_0, \triangleleft)$  satisfies condition (T1) in the definition of pre-directed tree. Recall that for given  $x \in \widehat{\mathbb{T}}_0$ , we write  $[x, \widehat{\omega}[ = \{y \in \widehat{\mathbb{T}}_0 : x \trianglelefteq y\}$ . We want to show that  $[x, \widehat{\omega}[$  is order-isomorphic to the half-line  $[0, +\infty[$ . To see this, take  $v \in \pi^{-1}(x)$  and denote by  $\pi_0 : [v, \omega[ \rightarrow \widehat{\mathbb{T}}$  the restriction of the projection  $\pi$  to the line  $[v, \omega[$ . In order to prove the claim we will show that  $\pi_0$  is an order-preserving bijection onto  $[x, \widehat{\omega}[$ . Note that, by construction,  $\pi_0$  is order-preserving and takes values in  $[x, \widehat{\omega}[$ . From the assumption that the action of  $N$  is elliptic, we get that  $\pi_0$  is injective. Let us prove that  $\pi_0$  is surjective. For this, take  $x' \in [x, \widehat{\omega}[$ . Then, there exist  $w' \in \pi^{-1}(x')$ ,  $w \in \pi^{-1}(x)$  with  $w \trianglelefteq w'$ , and  $h \in N$  with  $h.w = v$ . Since  $v = h.w \trianglelefteq h.w'$  we deduce that  $h.w' \in [v, \omega[$ , and  $\pi_0(h.w') = \pi_0(w') = x'$ , as desired. This proves condition (T1). Condition (T3) in the definition of pre-directed tree follows taking a countable subset  $\Sigma_0 \subset \mathbb{T}$  and considering the subset  $\widehat{\Sigma}_0 := \pi(\Sigma_0)$ . Finally, for condition (T2'), take  $x_1, x_2 \in \widehat{\mathbb{T}}_0$ , and for  $i \in \{1, 2\}$  take  $v_i \in \pi^{-1}(x_i)$ . As  $\mathbb{T}$  is a directed tree, the least common upper bound  $v_1 \wedge v_2 \in \mathbb{T}$  exists, and set  $y = \pi(v_1 \wedge v_2) \in \widehat{\mathbb{T}}_0$ . By the definition of the partial order on  $\widehat{\mathbb{T}}_0$ , we see that  $y$  is a common upper bound for  $x_1$  and  $x_2$ . Thus, we have proved that  $(\widehat{\mathbb{T}}_0, \triangleleft)$  is a pre-directed tree. Notice that the action  $\Phi$  projects to an action on  $\widehat{\mathbb{T}}_0$  which clearly preserves  $\triangleleft$ . Denote by  $\widehat{\Phi}_0 : G \rightarrow \text{Aut}(\widehat{\mathbb{T}}_0, \triangleleft)$  this action and notice that  $N$  is contained in its kernel. It follows from the proof that  $(\widehat{\mathbb{T}}_0, \triangleleft)$  is a pre-directed tree, that the projection map  $\pi : \mathbb{T} \rightarrow \widehat{\mathbb{T}}_0$  is a  $G$ -equivariant and surjective grading. Moreover, it follows directly from the definitions that in the case where  $\Phi$  is d-minimal, so is  $\widehat{\Phi}_0$ .

Finally, by applying Lemma 4.9 we get a directed tree  $(\widehat{\mathbb{T}}, \triangleleft)$  together with an action  $\widehat{\Phi} : G \rightarrow \text{Aut}(\widehat{\mathbb{T}}, \triangleleft)$  which grades  $\widehat{\Phi}_0$ . Moreover, Lemma 4.9 also implies that if  $\widehat{\Phi}_0$  is d-minimal so is  $\widehat{\Phi}$ . Notice that, since  $N$  is contained in the kernel of  $\widehat{\Phi}_0$  and gradings are surjective,  $N$  is also contained in the kernel of  $\widehat{\Phi}$ . Finally, since the composition of gradings is a grading, the result follows.  $\square$

## 5. $\mathbb{R}$ -FOCAL ACTIONS OF SOLVABLE GROUPS

**5.1. Actions of solvable groups on directed trees.** The results in the previous sections imply the following for actions of solvable groups on directed trees. Recall that given a virtually solvable group, we denote by  $\text{Fit}(G)$  its Fitting subgroup and by  $\text{vf}(G)$  its virtual Fitting length (see §2.2).

**Theorem 5.1.** *Let  $G$  be a virtually solvable group, and let  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  be a d-minimal action on a directed tree  $\mathbb{T} \neq \mathbb{R}$ . Then  $\Phi$  can be horograded by a minimal action  $j : G \rightarrow \text{Homeo}_0(\mathbb{R})$  such that  $\text{Fit}(G) \subset \ker j$ .*

*Proof.* Since  $\text{Fit}(G)$  is the union of nilpotent normal subgroups, and the action of nilpotent normal subgroups is elliptic by Corollary 4.4, we deduce that the action of  $\text{Fit}(G)$  must be elliptic. Thus, applying Proposition 4.10 to  $N = \text{Fit}(G)$  we obtain that  $\Phi$  is graded by a d-minimal action  $\Phi_0 : G \rightarrow \text{Aut}(\mathbb{T}_0, \triangleleft)$  where  $\text{Fit}(G)$  acts trivially. Since  $\Phi_0$  is d-minimal,  $G/\text{Fit}(G)$  must be infinite and therefore  $\text{vf}(G/\text{Fit}(G)) = \text{vf}(G) - 1 \geq 1$ . In particular  $\text{vf}(G) \geq 2$ . We proceed with the proof by induction on  $\text{vf}(G)$ . Assume that  $\text{vf}(G) = 2$  and consider the d-minimal action  $\Phi_0 : G \rightarrow \text{Aut}(\mathbb{T}_0, \triangleleft)$  with  $\text{Fit}(G) \subset \text{Ker}(\Phi_0)$  grading  $\Phi$ . Since  $\text{vf}(G/\text{Fit}(G)) = 1$  it must hold that  $\mathbb{T}_0 = \mathbb{R}$  and therefore the conclusion follows by setting  $j = \Phi_0$ . For the inductive step, consider a d-minimal action  $\Phi : G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  and its corresponding grading action  $\Phi_0 : G \rightarrow \text{Aut}(\mathbb{T}_0, \triangleleft)$ . We distinguish two cases according to whether  $\mathbb{T}_0 = \mathbb{R}$  or not. In the former case the conclusion follows by setting  $j = \Phi_0$ . In the latter case, we can apply the induction hypothesis to  $\Phi_0$  since it factors through  $G/\text{Fit}(G)$  which has strictly smaller virtual Fitting length. Therefore  $\Phi_0$  is horograded by an action  $j : G \rightarrow \text{Homeo}_0(\mathbb{R})$  such that  $\text{Fit}(G) \subset \ker j$  and the same holds for  $\Phi$ .  $\square$

**5.2. Actions of virtually solvable groups on the line.** In [5, Proposition 7.18] we observed that if  $G$  is a finitely generated solvable group, then its minimal actions on the line satisfy a dichotomy: they are either conjugate to an affine action, or  $\mathbb{R}$ -focal. Combining this result with Theorem 5.1, we obtain the following, which is the pivotal result of this paper.

**Theorem 5.2.** *Let  $G$  be a finitely generated virtually solvable group, and let  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  be a minimal action. Then either*

- (1)  $\varphi$  is conjugate to an affine action, or
- (2)  $\varphi$  is  $\mathbb{R}$ -focal and can be horograded by an action  $j : G \rightarrow \text{Homeo}_0(\mathbb{R})$  which factors through  $G/\text{Fit}(G)$ . Moreover this action  $j$  is either minimal or cyclic.

*Proof.* Let  $\varphi$  be a minimal action. It was already shown in [5, Proposition 7.18] that  $\varphi$  conjugate to an affine action or  $\mathbb{R}$ -focal ( $G$  is supposed there to be solvable but the proof works with minimal changes if  $G$  is virtually solvable). Let us repeat a sketch for completeness, and refer to [5, Proposition 7.18] for more details. Upon replacing  $G$  by a quotient (which remains virtually solvable), we can suppose that  $\varphi$  is faithful. Let  $A$  be an abelian normal subgroup of  $G$ . Then  $\text{Fix}^\varphi(A) = \emptyset$ , by minimality. We distinguish two cases according to whether  $A$  acts freely or not. If  $A$  acts freely, then by Hölder's theorem its action is semi-conjugate to an

action by translations associated to an embedding  $\iota: A \rightarrow (\mathbb{R}, +)$  (see [10, Example 3.1.6]). If  $\iota(A)$  is dense in  $(\mathbb{R}, +)$ , then  $\varphi(A)$  has a unique non-empty closed minimal invariant subset  $\Lambda \subseteq \mathbb{R}$ . As  $A$  is normal,  $\Lambda$  is preserved by  $\varphi(G)$ , and thus by minimality  $\Lambda = \mathbb{R}$  and  $\varphi(A)$  is actually conjugate to  $\iota(A)$ , and after conjugating we can assume  $\varphi(A) = \iota(A)$ . Then the Lebesgue measure is the unique  $A$ -invariant Radon measure up to a positive constant. It follows that every element of  $\varphi(G)$  sends the Lebesgue measure to a multiple of itself, and thus is an affine map. In the case where  $\iota(A)$  is cyclic, we have that  $\varphi(A)$  is conjugate to a cyclic group of translations centralized by  $\varphi(G)$ , and  $\varphi$  descends to a minimal action on the topological circle  $\mathbb{R}/\varphi(A)$ . Since  $G$  is amenable, this action preserves a probability measure, and thus  $\varphi$  preserves a non-zero Radon measure on  $\mathbb{R}$ . In this case it follows that  $\varphi$  is conjugate to an action by translations, thus also affine. Assume now that  $A$  acts non-freely; then there exists  $a \in A \setminus \{\text{id}\}$  such that  $\text{Fix}^\varphi(a) \neq \emptyset$ . But  $\text{Fix}^\varphi(a)$  is an  $A$ -invariant closed subset (as all elements of  $A$  commute with  $a$ ), and thus it must accumulate to both  $\pm\infty$ . Let  $I$  be a bounded connected component of  $\text{Supp}^\varphi(a)$ . Then for every  $g \in G$ ,  $g.I$  is a connected component of  $\text{Supp}^\varphi(gag^{-1})$ , and since  $gag^{-1} \in A$  commutes with  $a$ , we have that  $g.I$  and  $I$  cannot cross. Thus the closure of the orbit of  $I$  in  $\mathbb{R}^{(2)}$  is a  $G$ -invariant lamination  $\mathcal{L}$ , and since  $\varphi$  is minimal it is automatically  $\mathbb{R}$ -focal with respect to  $\mathcal{L}$  [5, Proposition 7.7]. This shows that  $\varphi$  is either conjugate to an affine action, or it is  $\mathbb{R}$ -focal.

Now in the  $\mathbb{R}$ -focal case, by Proposition 3.15 we have that  $\varphi$  can be represented by a focal action  $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \triangleleft)$  on a planar directed tree, which is either d-minimal or simplicial and transitive on branching points. If  $\Phi$  is d-minimal, then by Theorem 5.1,  $\Phi$  can be horograded by an action  $j: G \rightarrow \text{Homeo}_0(\mathbb{R})$  which satisfies the desired conclusion. If it is simplicial, we have from Proposition 3.19 that  $\Phi$  can be horograded by a cyclic action  $j: G \rightarrow \text{Homeo}_0(\mathbb{R})$ . Then every element  $g \in G$  with  $g \notin \ker j$  is a hyperbolic isometry. As argued at the beginning of the proof of Theorem 5.1, Corollary 4.4 gives that  $\Phi(\text{Fit}(G))$  is elliptic, so that  $\text{Fit}(G) \subset \ker j$ .  $\square$

A crucial aspect of Theorem 5.2 is that if  $\varphi$  falls in case (2), then we may apply again the theorem to the action  $j$  which horogrades  $\varphi$ . Since this action factors through a quotient of smaller virtual Fitting length, in finitely many steps we must reach an affine action, which remembers some of the dynamics of the original action. This shows that actions of a virtually solvable group on the line are naturally organized into finitely many layers of complexity, the affine actions being the simplest type, and one might study them inductively.

As a first simple illustration of this philosophy we prove Theorem B, which is its most direct consequence. Indeed the combination of Theorem 5.2 and Proposition 3.18 gives the following.

**Proposition 5.3.** *Let  $G$  be a finitely generated virtually solvable group and  $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$  a minimal  $\mathbb{R}$ -focal action. Then, there exists an irreducible affine action  $\psi: G \rightarrow \text{Aff}(\mathbb{R})$  and two monotone maps  $h_+, h_-: \mathbb{R} \rightarrow \mathbb{R}$  satisfying the following conditions.*

- (1)  $h_+$  is monotone non-decreasing,
- (2)  $h_-$  is monotone non-increasing,
- (3) we have  $\lim_{x \rightarrow +\infty} h_+(x) = \lim_{x \rightarrow -\infty} h_-(x) = +\infty$ ,
- (4) for every  $g \in G$  there exists  $M > 0$  such that  $h_+(\varphi(g)(x)) = \psi(g)(h_+(x))$  for every  $x > M$ , and  $h_-(\varphi(g)(x)) = \psi(g)(h_-(x))$  for every  $x < -M$ .

*Proof.* We only discuss the existence of  $h_+$ , the other case being analogous. We argue by induction on the virtual Fitting length  $\text{vf}(G)$  of  $G$ . Let  $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$  be a minimal  $\mathbb{R}$ -focal action. Apply Theorem 5.2 to obtain an action  $j: G \rightarrow \text{Homeo}_0(\mathbb{R})$  as in the statement that horogrades  $\varphi$ . We apply Proposition 3.18 to obtain a map  $h'_+$  as in its statement. If

$j$  is an affine action (which includes the possibility that it is cyclic), then we have already reached the conclusion. Else,  $j$  is  $\mathbb{R}$ -focal, and by the inductive hypothesis for  $G/\text{Fit}(G)$ , there exists an irreducible affine action  $\psi$  and a non-decreasing map  $h'_+ : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $\lim_{x \rightarrow +\infty} h'_+(x) = +\infty$  and  $h'_+(j(g)(x)) = \psi(g)(h'_+(x))$  for any sufficiently large  $x \in \mathbb{R}$ . We then set  $h_+ = h''_+ \circ h'_+$ .  $\square$

*Proof of Theorem B.* If the action  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  is semi-conjugate to an affine action, we simply have to take this action as the desired  $\psi$ . Otherwise, by Theorem 5.2, we have that  $\varphi$  is semi-conjugate to a minimal  $\mathbb{R}$ -focal action, so that we can apply Proposition 5.3, which gives the desired irreducible affine action  $\psi$ .

We next prove that  $\psi$  is unique up to affine conjugacy. For this, assume there exist two such irreducible affine actions  $\psi_1$  and  $\psi_2$ , with corresponding maps  $h_1 : I_1 \rightarrow \mathbb{R}$  and  $h_2 : I_2 \rightarrow \mathbb{R}$ . Then, for any  $g \in G$  and sufficiently large  $x \in \mathbb{R}$ , we have that

$$\psi_2(g)h_2h_1^{-1}(x) = h_2\varphi(g)h_1^{-1}(x) = h_2h_1^{-1}\psi_1(g)(x),$$

so that the actions  $\psi_1$  and  $\psi_2$  are “semi-conjugate near  $+\infty$ ” by the map  $h := h_2 \circ h_1^{-1}$ , defined on some interval of the form  $(c, +\infty)$ . For  $i \in \{1, 2\}$  we write  $A_i = \psi_i(G)$ . Observe first that if  $\psi_1(g)$  is trivial for some  $g$ , then  $\psi_2(g)$  is an affine map fixing the image of  $h$ , and thus is trivial as well. Thus  $\ker \psi_1 = \ker \psi_2$ . It follows that if one of the groups  $A_i$  is a cyclic group (necessarily generated by a translation), then the other is as well, and this case is readily solved as any two positive translations are conjugate by an affine map. We can thus assume that  $A_i$  is not a cyclic group of translations (and thus acts minimally on  $\mathbb{R}$ ).

Given a finite subset  $S$  of homeomorphisms of the line and a point  $x \in \mathbb{R}$ , denote by  $\mathcal{R}(S, x)$  the equivalence relation on the interval  $(x, +\infty)$  defined by  $(y, z) \in \mathcal{R}(S, x)$  if there exist  $s_1, \dots, s_n \in S$  such that  $s_n \cdots s_1(y) = z$  and  $s_i \cdots s_1(y) > x$  for every  $i \in \{1, \dots, n\}$ .

**Claim.** *For  $i \in \{1, 2\}$ , there exists  $S_i \subset A_i$  such that for every sufficiently large  $x \in \mathbb{R}$ , the relation  $\mathcal{R}(S_i, x)$  is minimal (i.e. its equivalence classes are dense in  $(x, +\infty)$ ).*

*Proof of claim.* Assume first that  $A_i$  is contained in the group of translations. Since it is finitely generated and not cyclic, it contains two rationally independent translations  $a, b$ , and one readily checks that  $\mathcal{R}(\{a^{\pm 1}, b^{\pm 1}\}, x)$  is minimal for every  $x \in \mathbb{R}$ . Otherwise  $A_i$  contains a contracting homothety  $a$  and a positive translation  $b$ . Up to conjugating by an affine map, assume that  $a(x) = \lambda x$  with  $\lambda < 1$  and  $b(x) = x + 1$ . For  $x > 0$  and  $n, m > 0$  we have  $a^m b^n a^{-m}(x) = x + n\lambda^m$  and one readily checks that the successive applications of  $a^{\pm 1}, b$  in the word  $a^m b^n a^{-m}$  never move  $x$  to the left of itself. Since  $\lambda^m$  can be made arbitrarily small, we deduce that  $\mathcal{R}(\{a^{\pm 1}, b\}, x)$  is minimal for every  $x > 0$ .  $\square$

Let now  $\tilde{S}_i \subset G$  be finite subsets such that  $\psi_i(\tilde{S}_i) = S_i$ . Choose  $x$  large enough so that  $\mathcal{R}(S_1, x)$  and  $\mathcal{R}(S_2, h(x))$  are both minimal, and such that the equivariance condition  $h \circ \psi_1(s)(y) = \psi_2(s) \circ h(y)$  holds for all  $y \geq x$  and all  $s \in \tilde{S}_1 \cup \tilde{S}_2$ . Then the image of  $h|_{(x, +\infty)}$  is invariant under the relation  $\mathcal{R}(S_2, h(x))$ . Thus, arguments similar to those in Lemma 2.3 show that this image must be dense, implying that  $h$  is continuous on  $(x, +\infty)$ . Again as in the proof of Lemma 2.3, the minimality of  $\mathcal{R}(S_1, x)$  shows that  $h|_{(x, +\infty)}$  must be injective and thus a homeomorphism onto its image.

Now note that since  $K := \ker \psi_1 = \ker \psi_2$  we have that  $A_1$  is abelian (and hence a group of translations) if and only if  $A_2$  is. Otherwise, both groups are non-abelian and their derived subgroup consists of translations. In the first case, set  $H = G$  and in the second case set  $H = [G, G]$ . In either case  $\psi_i(H)$  is a minimal group of translations for  $i \in \{1, 2\}$ . Let  $g \in H$  be such that  $\psi_1(g)$  is a non-trivial positive translation. Then so is  $\psi_2(g)$ , by

the equivariance near  $+\infty$  of the map  $h$ . Up to conjugating  $\psi_2$  by an affine map we can assume  $\psi_1(g) = \psi_2(g)$ . Let  $J_1 \subset (x, +\infty)$  be a fundamental interval for the translation  $\psi_1(g)$  and set  $J_2 = h(J_1)$ , which is also a fundamental interval for the translation  $\psi_1(g) = \psi_2(g)$ . Therefore, up to conjugating the action  $\psi_2$  by a translation, we can assume that  $J_1 = J_2$ . Then for any  $i \in \{1, 2\}$ ,  $\psi_i|_H$  descends to a minimal action by rotations on the circle  $J_i/\psi_i(g)$ . Moreover, the homeomorphism  $h : (c, +\infty) \rightarrow (h(c), +\infty)$  induces a homeomorphism  $\bar{h}$  of  $J_1/\psi_1(g) = J_2/\psi_2(g)$  which conjugates the two actions. This implies that  $\bar{h}$  is a rotation, and thus  $h$  is a translation. As  $h(J_1) = J_1$ , we deduce that  $h$  is trivial. It follows that for every  $g \in G$ , the affine maps  $\psi_1(g)$  and  $\psi_2(g)$  coincide on a neighborhood of  $+\infty$ , and thus are equal.  $\square$

For later use, let us record the following consequence of Proposition 5.3.

**Lemma 5.4.** *Let  $G$  be a finitely generated virtually solvable group and let  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  be a minimal  $\mathbb{R}$ -focal action. Then there exists an element  $h \in G$  whose image is a pseudohomothety.*

*Proof.* Let  $\psi : G \rightarrow \text{Aff}(\mathbb{R})$  be the affine action given by Proposition 5.3. Choose  $g \in G$  with  $\psi(g) \neq \text{id}$ . Assume, say, that  $\psi(g)(x) > x$  for every sufficiently large  $x$ . Then  $\varphi(g)(x) > x$  for all  $x$  close enough to  $+\infty$  and similarly  $\varphi(g)(x) < x$  for all  $x$  close enough to  $-\infty$ . Thus  $\varphi(g)$  is a pseudohomothety.  $\square$

## 6. THE CASE OF $\mathbb{Z} \wr \mathbb{Z}$

The goal of this section is to apply the previous discussion to the lamplighter group  $\mathbb{Z} \wr \mathbb{Z}$ . This gives an illustration of how Theorem 5.2 can be applied in the simplest special case, and at the same time the results of this section will be invoked in the proof of Theorem A.

We set  $G := \mathbb{Z} \wr \mathbb{Z} = L \rtimes \mathbb{Z}$ , where  $L = \bigoplus_{\mathbb{Z}} \mathbb{Z}$  is the group of finitely supported configurations  $f : \mathbb{Z} \rightarrow \mathbb{Z}$ , and  $\mathbb{Z}$  acts on  $L$  by shifting the indices. We consider the generating set  $\mathbb{Z} \wr \mathbb{Z} = \langle g, h_0 \rangle$ , where  $g$  generates the factor  $\mathbb{Z}$ , and  $h_0 \in L$  is the configuration given by  $h_0(0) = 1$  and  $h_0(n) = 0$  for  $n \neq 0$ . We further denote by  $h_n = g^n h_0 g^{-n}$  ( $n \in \mathbb{Z}$ ) the elements of the canonical basis of  $L$ . The group  $G$  admits the infinite presentation:

$$(6.1) \quad G = \langle g, h_0 \mid h_n = g^n h_0 g^{-n}, [h_n, h_m] = \text{id} \quad (n, m \in \mathbb{Z}) \rangle.$$

The group  $G$  is solvable (more precisely, metabelian) and it is clear that  $L = \text{Fit}(G)$ .

**6.1. Affine actions.** For completeness, let us first recall that the group  $G$  admits a continuous family of affine actions on  $\mathbb{R}$ , which are readily classified:

**Lemma 6.1.** *Let  $G = \mathbb{Z} \wr \mathbb{Z}$  with the standard generating pair  $h_0, g$  defined above. Then every irreducible affine action  $\varphi : G \rightarrow \text{Aff}(\mathbb{R})$  is positively conjugate by an affine map to an action in one of the following families:*

- actions by translations obtained by setting

$$\varphi(h_0) : x \mapsto x + \alpha, \quad \varphi(g) : x \mapsto x + \beta \quad (\alpha^2 + \beta^2 = 1);$$

- non-abelian affine actions obtained by setting

$$\varphi(h_0) : x \mapsto x \pm 1, \quad \varphi(g) : x \mapsto \lambda x \quad (\lambda > 0, \lambda \neq 1).$$

*Proof.* First observe that the images of the generators under the actions above satisfy the relations in (6.1), so that these actions are well-defined. Conversely, assume  $\varphi$  is an affine action of  $G$ . Assume by contradiction that  $\varphi(h_0)$  is a homothety, with fixed point  $p$ . Then  $p$  cannot be fixed by  $\varphi(g)$  or it would be a global fixed point. Thus  $\varphi(h_1)$  is a homothety with fixed point  $g.p \neq p$ . This contradicts that  $\varphi(h_0)$  and  $\varphi(h_1)$  commute. Thus  $\varphi(h_0)$  is a translation (possibly trivial). If  $\varphi(g)$  is also a translation, up to conjugation by an affine map we are in the first case. If  $\varphi(g)$  is a homothety, then  $\varphi(h_0)$  cannot be trivial, and we can conjugate by an affine map to suppose that  $\varphi(h)$  is a translation by  $\pm 1$  and that  $\varphi(g)$  fixes 0.  $\square$

*Remark 6.2.* In the previous classification, actions by translations are obviously never faithful, while non-abelian affine actions are faithful if and only if  $\lambda$  is a trascendental number.

**6.2. Plante's actions and classification.** We now recall Plante's construction of actions on the line  $G$  that are not semi-conjugate to any affine action [24]. One can identify the subgroup  $L$  with the ring of Laurent polynomials  $\mathbb{Z}[X, X^{-1}]$ , in which case  $h_0$  represents the constant polynomial  $P_0(X) \equiv 1$ . With this point of view, the group  $G = L \rtimes \mathbb{Z}$  admits a natural action on  $\mathbb{Z}[X, X^{-1}]$ , where  $L \cong \mathbb{Z}[X, X^{-1}]$  acts additively, and  $g$  acts by multiplication by  $X$ . This action of  $G$  on  $\mathbb{Z}[X, X^{-1}]$  preserve two natural orders  $\prec_+$  and  $\prec_-$  of lexicographic type, obtained by looking at the sign of the coefficient of the monomial of maximal or minimal power, respectively. More precisely, we declare  $P \succ_+ 0$  if the coefficient of the highest power of  $X$  in  $P(X)$  is positive. Similarly, we declare  $P \succ_- 0$  if the coefficient of the lowest power of  $X$  in  $P(X)$  is positive. It is routine to check that these orders are invariant under the action of  $G$  defined above. Taking the dynamical realization of such order-preserving actions (see §2.1), one obtains two non-conjugate minimal  $\mathbb{R}$ -focal actions of  $G$  on the real line, that we call the *Plante's actions* of  $\mathbb{Z} \wr \mathbb{Z}$ . Note that in the dynamical realization of  $\prec_+$ , the generator  $g$  acts as an expanding homothety, while in case of  $\prec_-$  it acts as a contracting homothety. See [5, Example 7.11] for more details, and for a generalization of this construction to other wreath products. The following result (together with Lemma 6.1) classifies actions of  $\mathbb{Z} \wr \mathbb{Z}$  on the line up to semi-conjugacy.

**Proposition 6.3.** *Every irreducible action  $\varphi: \mathbb{Z} \wr \mathbb{Z} \rightarrow \text{Homeo}_0(\mathbb{R})$  is semi-conjugate either to an affine action or to a Plante's action.*

*Proof.* Set  $G = \mathbb{Z} \wr \mathbb{Z}$  and  $L = \bigoplus_{\mathbb{Z}} \mathbb{Z}$ , as before. By Theorem 5.2, it is enough to show that every minimal  $\mathbb{R}$ -focal action  $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$  is conjugate to one of the two Plante's actions. So assume that  $\varphi$  is  $\mathbb{R}$ -focal and minimal. By Theorem 5.2,  $\varphi$  can be horograded by an action  $j: G \rightarrow \text{Homeo}_0(\mathbb{R})$  which is either cyclic or minimal, and where  $\text{Fit}(G)$  acts trivially. Since  $\text{Fit}(G) = L$  and  $G/L \cong \mathbb{Z}$ , we deduce that  $j$  is a cyclic action. Moreover, since  $j(g)$  generates the image of  $g$ , it has no fixed points, and thus  $\varphi(g)$  is a homothety by Proposition 3.17. Also notice that, again by Proposition 3.17, the elements  $\varphi(h_n)$  ( $n \in \mathbb{Z}$ ) are totally bounded. Assume first that  $\varphi(g)$  is expanding, and let  $p$  be its unique fixed point. Note that no element  $\varphi(h_n)$  ( $n \in \mathbb{Z}$ ) can fix  $p$ , as otherwise  $p$  would be a global fixed point for the action. Thus, upon conjugating  $\varphi$  by an orientation-reversing homeomorphism, we can assume that  $h_0.p > p$  and thus  $h_n.p > p$  for every  $n \in \mathbb{Z}$ . For fixed  $n \in \mathbb{Z}$ , let  $I_n$  be the connected component of  $\text{Supp}^\varphi(h_n)$  containing  $p$ . Then, for every  $n \in \mathbb{Z}$ , the homothety  $\varphi(g)$  sends  $I_n$  to  $I_{n+1}$  and thus  $\overline{I_n} \subset I_{n+1}$ ; moreover for each  $l < n$  the element  $h_n$  must send  $I_l$  entirely to the right of itself, as it must permute the connected components of  $\text{Supp}^\varphi(h_l)$ . From this, it easily follows that for every element  $f = h_{n_1}^{k_1} \cdots h_{n_r}^{k_r}$  with  $n_1 < \cdots < n_r$  and  $k_r \neq 0$ , we have  $f.p > p$  if  $k_r > 0$ , and  $f.p < p$  otherwise. Since  $g.p = p$ , it follows that the the map that to

any element  $f \in L \cong \mathbb{Z}[X, X^{-1}]$  associates  $f.p$  is a  $G$ -equivariant order-preserving embedding of  $(\mathbb{Z}[X, X^{-1}], \prec_+)$  into  $\mathbb{R}$ , and thus  $\varphi$  must be conjugate with the Plante's action associated with  $\prec_+$  by Lemma 2.3. The case where  $\varphi(g)$  is a contracting homothety is treated similarly and gives rise to the Plante's action associated with  $\prec_-$ . Summarizing we get that, up to conjugating by an order-reversing homeomorphism,  $\varphi$  is either semi-conjugate to the Plante's action associated with  $\prec_+$  or to that associated with  $\prec_-$ .  $\square$

**6.3. Plante subactions in  $\mathbb{R}$ -focal actions.** Plante's actions of  $\mathbb{Z} \wr \mathbb{Z}$  are in a precise sense the "smallest"  $\mathbb{R}$ -focal actions for solvable groups:

**Proposition 6.4.** *Let  $G$  be a virtually solvable group and let  $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$  be a minimal and faithful  $\mathbb{R}$ -focal action. Then  $G$  contains a subgroup isomorphic to  $\mathbb{Z} \wr \mathbb{Z}$  whose action is irreducible and semi-conjugate to a Plante's action.*

*Proof.* By Lemma 5.4 there exists an element  $s \in G$  which acts as an expanding pseudohomothety. Let  $[\xi_1, \xi_2]$  be an interval containing  $\text{Fix}^\varphi(s)$ . Let  $A$  be a non-trivial abelian normal subgroup of  $G$ . Since the action of  $G$  is minimal and faithful, we have  $\text{Fix}^\varphi(A) = \emptyset$ , and hence there exists  $a \in A$  such that  $a.\xi_0 > \xi_1$ . Then  $\text{Fix}^\varphi(a) \cap \text{Fix}^\varphi(s) = \emptyset$  and thus  $H := \langle a, s \rangle$  acts irreducibly. The conjugates of  $a$  by powers of  $s^n$  all belong to  $A$  and thus commute. It follows that there is well-defined homomorphism  $\rho: \mathbb{Z} \wr \mathbb{Z} \rightarrow H$  sending  $h_0$  to  $a$  and  $g$  to  $s$ . Now  $\varphi \circ \rho$  cannot be semi-conjugate to an affine action (for instance because  $\varphi(G)$  does not contain any element conjugate to a translation, by Proposition 3.17), and thus it is semi-conjugate to a Plante's action by Proposition 6.3; we have *a fortiori* that  $\rho$  is an isomorphism, as Plante's actions are faithful.  $\square$

## 7. $C^1$ ACTIONS ON INTERVALS

In this section we prove Theorem A. After Proposition 6.4, we are reduced to consider actions which are semi-conjugate to Plante's actions of  $\mathbb{Z} \wr \mathbb{Z}$ . Let us outline the main idea, which comes from the work of Bonatti, Monteverde, Navas, and the third author [3]. It is proven there that for any  $C^1$  action of the Baumslag–Solitar group  $\text{BS}(1, n) = \langle a, b \mid aba^{-1} = b^n \rangle$  on the closed interval, which is topologically conjugate to its standard affine action, the derivative of the image of  $a$  at its unique fixed point must be equal to  $n$ . Observe that the affine action of  $\text{BS}(1, n)$  can also be seen as an action of  $\mathbb{Z} \wr \mathbb{Z}$ , by precomposing it with the epimorphism  $\mathbb{Z} \wr \mathbb{Z} \rightarrow \text{BS}(1, n)$  defined on the generators as  $h_0 \mapsto b$  and  $g \mapsto a$ . Letting  $n$  increase we obtain a sequence  $(\varphi_n)$  of actions of  $\mathbb{Z} \wr \mathbb{Z}$ . One can show that the sequence  $(\varphi_n)$  can be conjugated to make converge to a Plante's action of  $\mathbb{Z} \wr \mathbb{Z}$ . This strongly suggests that if Plante's action of  $\mathbb{Z} \wr \mathbb{Z}$  were conjugate to a  $C^1$  action, the derivative of  $g$  at its unique fixed point should be infinite, thus getting a contradiction. While this limiting argument is not an actual proof (as the conjugacy and the convergence are only  $C^0$ ), it turns out that the argument of in the proof of [3, Proposition 4.13] can be adapted to Plante's actions to obtain the following.

**Proposition 7.1.** *Let  $\rho: H \rightarrow \mathbb{Z} \wr \mathbb{Z}$  be an epimorphism from a finitely generated group  $H$ , and let  $\psi: H \rightarrow \text{Homeo}_0(\mathbb{R})$  be the action obtained by postcomposing  $\rho$  with a Plante's action. Let  $\varphi: H \rightarrow \text{Homeo}_0([0, 1])$  be an action whose restriction to  $(0, 1)$  is semi-conjugate to  $\psi$ . Then,  $\varphi$  cannot be of class  $C^1$  on  $[0, 1]$ .*

*Proof of Proposition 7.1.* Assume by contradiction that  $\varphi$  is of class  $C^1$ . As  $H$  is finitely generated, using a trick attributed to Muller [19] and Tsuboi [27] (see also [3]), we can assume, up to considering a  $C^0$  conjugate action of class  $C^1$ , that  $\varphi(k)'(0) = \varphi(k)'(1) = 1$  for every  $k \in H$ . Recall that we denote by  $h_0$  and  $g$  the standard generators of  $\mathbb{Z} \wr \mathbb{Z}$ , and  $h_n = g^n h_0 g^{-n}$ .

For  $n \in \mathbb{Z}$  we consider the subgroups  $L_n := \langle h_j, j \geq n \rangle$  of  $\mathbb{Z} \wr \mathbb{Z}$ . Choose corresponding preimages  $\tilde{h}_0 \in \rho^{-1}(h_0)$  and  $\tilde{g} \in \rho^{-1}(g)$  in  $H$ , and set  $\tilde{h}_n = \tilde{g}^n \tilde{h}_0 \tilde{g}^{-n}$ . Assume that  $\psi$  is the Plante's action from the lexicographic order  $\prec_+$ , so that  $\psi(\tilde{g})$  is an expanding homothety and  $\psi(\tilde{h}_0)$  moves to the right (the other case can be treated identically). Let  $\tau: (0, 1) \rightarrow \mathbb{R}$  be a non-decreasing map that semi-conjugates  $\varphi$  to  $\psi$ , in the sense that  $\tau\varphi(k) = \psi(k)\tau$  for every  $k \in G$ . As Plante's actions are minimal (see §6), the semi-conjugacy  $\tau$  is continuous. To avoid confusion, we will use Latin letters to denote points of  $\mathbb{R}$  (where the Plante's action lives), and Greek letters for points of  $[0, 1]$  (where the action  $\varphi$  lives). Let  $x_0 \in \mathbb{R}$  be the unique fixed point of  $\psi(g)$ . Its preimage  $\tau^{-1}(x_0)$  in  $(0, 1)$  is a compact interval (possibly reduced to one point), and we let  $\xi_0$  denote its rightmost point. Note that  $\xi_0$  is fixed by  $\phi(\tilde{g})$  and that  $\tilde{g}.\xi > \xi$  for any  $\xi > \xi_0$ . Consider the interval  $I_0 = (x_0, h_0.x_0)$ , and for  $n > 0$  set  $I_{-n} = g^{-n}.I_0 = (x_0, h_{-n}.x_0)$ . Observe that if  $f_1, f_2 \in L_{-n}$  are distinct, then the intervals  $f_1.I_{-n-1}$  and  $f_2.I_{-n-1}$  are disjoint (this follows from the fact that the interval  $I_{-n-1}$  is contained in a connected component of  $\text{Supp}^\psi(h_{-n-1})$ , and that the subgroup  $L_{-n}$  preserves the set of such connected components and acts freely on it). Set  $J_0$  as the minimal open interval such that  $\tau(J_0) = I_0$ , and define  $J_{-n} = \tilde{g}^{-n}.J_0$ . Note that  $J_{-n}$  has  $\xi_0$  as leftmost point and  $\tau.J_{-n} = I_{-n}$  for  $n \geq 0$ .

Choose  $0 < \lambda < |D\varphi(\tilde{g})(\xi_0)|^{-1}$ . Then there exists a constant  $C > 0$  such that

$$(7.1) \quad |J_{-n}| \geq C\lambda^n.$$

Fix  $N > \frac{1}{\lambda}$  and let  $\varepsilon > 0$  be such that

$$(7.2) \quad (1 - \varepsilon)^{2N+2}N\lambda > 1.$$

Let  $\eta \in (0, 1)$  be such that for any  $\xi \in (\eta, 1]$  and  $s \in \{\tilde{h}_0, \tilde{g}, \tilde{g}^{-1}\}$ , one has  $\varphi(s)'(\xi) > 1 - \varepsilon$ . Since the subgroup  $L$  acts without fixed points in the Plante's action, we can find an element  $a \in L$  such that  $a.x_0 > \tau(\eta)$ ; choosing a preimage  $\tilde{a} \in \rho^{-1}(a)$  we have  $\tilde{a}.\xi_0 > \eta$ . Given  $N \in \mathbb{N}$  and an  $n$ -tuple of integers  $\underline{i} = (i_0, \dots, i_{n-1}) \in \{1, \dots, N\}^n$ , set  $\tilde{f}_{\underline{i}} = \tilde{h}_{-n+1}^{i_0} \dots \tilde{h}_0^{i_{n-1}}$  and  $f_{\underline{i}} = \pi(\tilde{f}_{\underline{i}})$ . Note that the intervals of the form  $\tilde{f}_{\underline{i}}\tilde{a}.J_{-n}$  are pairwise disjoint when  $\underline{i}$  varies. Indeed, applying the semi-conjugacy  $\tau$  maps each such interval to the corresponding  $f_{\underline{i}}a.I_{-n} = af_{\underline{i}}.I_{-n}$ , and these are pairwise disjoint since the elements  $f_{\underline{i}}$  all belong to the subgroup  $L_{-n-1}$ . We shall estimate from below the size of  $\tilde{f}_{\underline{i}}\tilde{a}.J_{-n}$ . For this, note that  $\tilde{f}_{\underline{i}}$  may be rewritten as

$$\begin{aligned} \tilde{f}_{\underline{i}} &= \left(\tilde{g}^{-n+1}\tilde{h}_0^{i_{n-1}}\tilde{g}^{n-1}\right) \dots \left(\tilde{g}^{-2}\tilde{h}_0^{i_2}\tilde{g}^2\right) \left(\tilde{g}^{-1}\tilde{h}_0^{i_1}\tilde{g}\right) \tilde{h}_0^{i_0} \\ &= \tilde{g}^{-n+1}\tilde{h}_0^{i_{n-1}}\tilde{g}^{i_{n-1}-n+2}\tilde{g} \dots \tilde{g}^{i_1}\tilde{h}_0^{i_1}\tilde{g}^{i_1-1}\tilde{h}_0^{i_0}. \end{aligned}$$

Consider the sequence of intervals obtained by applying successively the terms in this expression to the interval  $\tilde{a}.J_{-n}$ . Observe that any such interval stays inside  $(\eta, 1]$ : indeed, each application of the generator  $\tilde{g}$  or  $\tilde{h}_0$  moves the interval to the right, and the final term  $\tilde{g}^{-n+1}$  moves it to the left but is compensated by the previous  $n-1$  instances of  $\tilde{g}$  with a positive power. By the mean value theorem and the choice of  $\eta$ , at each application of  $\tilde{g}, \tilde{g}^{-1}$  or  $\tilde{h}_0$  the size of the interval may decrease by at most a factor  $1 - \varepsilon$ . Hence

$$\left|\tilde{f}_{\underline{i}}\tilde{a}.J_{-n}\right| \geq (1 - \varepsilon)^{nN+2n-2} |\tilde{a}.J_{-n}| \geq CC_1(1 - \varepsilon)^{nN+2n-2}\lambda^n,$$

where we have used (7.1), and the constant  $C_1$  is  $\min_{\xi \in [0, 1]} \varphi(\tilde{a})'(\xi)$ . Summing over  $\underline{i}$  and using that the intervals are disjoint, we have

$$1 \geq \sum_{\underline{i} \in \{1, \dots, N\}^n} \left|\tilde{f}_{\underline{i}}\tilde{a}.J_{-n}\right| \geq CC_1(1 - \varepsilon)^{-2} \left((1 - \varepsilon)^{2N+2}N\lambda\right)^n,$$



which is impossible, since by (7.2) the right-hand side tends to  $+\infty$  as  $n \rightarrow \infty$ . This is the desired contradiction.  $\square$

We will also use the following result, whose proof readily follows from the discussion in [3, §4.2].

**Lemma 7.2.** *Let  $G$  be a group and  $\varphi: G \rightarrow \text{Diff}_0^1([0, 1])$  be an irreducible action. Suppose that the  $\varphi$ -action on  $(0, 1)$  is semi-conjugate to an action  $\psi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ , and that there exist elements  $a, b \in G$  such that  $\psi(a)$  is a homothety  $x \mapsto \lambda x$  with  $\lambda > 1$  and  $\psi(b)$  is a translation  $x \mapsto x + \alpha$  with  $\alpha \neq 0$ . Then  $\varphi$  is minimal on  $(0, 1)$ , and thus conjugate to  $\psi$ .*

*Proof of Theorem A.* Let  $\varphi: G \rightarrow \text{Diff}^1([0, 1])$  be an irreducible action. If by contradiction the restriction of  $\varphi$  to  $(0, 1)$  is not semi-conjugate to any affine action, by Theorem 5.2 it is semi-conjugate to a minimal  $\mathbb{R}$ -focal action  $\psi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ , whose kernel we denote by  $N$ . By Proposition 6.4 we have that  $G/N$  contains a subgroup  $\mathbb{Z} \wr \mathbb{Z}$  that acts via an action which is semi-conjugate to a Plante's action. If we choose two preimages  $\tilde{h}_0, \tilde{g} \in G$  of the corresponding generators of  $\mathbb{Z} \wr \mathbb{Z}$ , the restriction of  $\varphi$  to the group  $H = \langle \tilde{h}_0, \tilde{g} \rangle$  satisfies all assumptions of Proposition 7.1, giving a contradiction. The last sentence of the theorem follows from Lemma 7.2.  $\square$

## 8. THE METANILPOTENT CASE

In this section we consider the class of virtually metanilpotent groups. Recall that  $G$  is virtually metanilpotent if it admits a finite index subgroup  $G_0 \leq G$  which is metanilpotent, meaning that there is a normal subgroup  $N \trianglelefteq G_0$  such that  $N$  and  $G_0/N$  are both nilpotent.

*Remark 8.1.* The class of virtually metanilpotent groups contains in particular metabelian groups and all virtually solvable linear groups, i.e. virtually solvable subgroups of  $\text{GL}(n, \mathbb{K})$  where  $\mathbb{K}$  is a field. Indeed by a classical theorem of Mal'cev [17], every solvable subgroup of  $\text{GL}(n, \mathbb{K})$  has a finite index subgroup which is triangularizable over the algebraic closure of  $\mathbb{K}$ , and thus is nilpotent-by-abelian.

For this class of groups, the results in Section 5 become much stronger. Theorem 5.1 specifies as follows.

**Corollary 8.2.** *Let  $G$  be a virtually metanilpotent group and  $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft)$  a  $d$ -minimal action on a directed tree. Then  $\Phi$  can be horograded by an action by translations  $j: G \rightarrow (\mathbb{R}, +)$ . Equivalently,  $\Phi$  preserves a compatible  $\mathbb{R}$ -tree metric.*

*Proof.* Let  $G_0$  be a normal metanilpotent subgroup of  $G$ . By Theorem 5.1,  $\Phi$  can be horograded by a minimal action  $j: G \rightarrow \text{Homeo}_0(\mathbb{R})$  which factors through  $G/\text{Fit}(G)$ . The fact that  $G$  be virtually metanilpotent implies that  $G/\text{Fit}(G)$  is virtually nilpotent (using Lemma 2.4). By a classical result of Plante [23] (see also Navas [22, Theorem 2.2.39]), we deduce that  $j: G \rightarrow \text{Homeo}_0(\mathbb{R})$  is conjugate to an action by translations. The equivalent formulation in terms of invariant metrics is a consequence of Proposition 3.19.  $\square$

*Remark 8.3.* The reader may notice that the proof above works more generally for groups with virtual Fitting length  $\text{vf}(G) = 2$ . However one can check that for finitely generated groups, this condition is equivalent to being virtually metanilpotent.

As a consequence, Theorem 5.2 translates here as follows.

**Corollary 8.4.** . *Let  $G$  be a finitely generated, virtually metanilpotent group. Then every minimal action  $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$  is either conjugate to an affine action, or it is  $\mathbb{R}$ -focal and can be horograded by an action by translations  $j: G \rightarrow (\mathbb{R}, +)$ .*

It is interesting to note that the previous results imply that actions on the line of a virtually metanilpotent group always arise from an isometric action on a planar hyperbolic metric space, namely the hyperbolic plane, in the case of affine actions, or an  $\mathbb{R}$ -tree, in the case of  $\mathbb{R}$ -focal actions.

**8.1. Perturbations of affine actions.** Here we prove Theorem C. It will follow from the results above and from some general results on perturbations of affine actions of a finitely generated group.

**Proposition 8.5.** *Let  $G$  be a finitely generated group. Then for any irreducible action  $\varphi_0 \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$  which is semi-conjugate to a non-abelian affine action, there exists an open neighborhood in  $\text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$  which contains no action which is semi-conjugate to an action by translations.*

*Proof.* Let  $S$  be a finite symmetric generating set of  $G$  containing the identity. As  $\varphi_0$  is semi-conjugate to a non-abelian affine action  $\bar{\varphi}_0: G \rightarrow \text{Aff}(\mathbb{R})$ , we can choose  $s \in S$  such that  $\bar{\varphi}_0(s)$  is an expanding homothety. Thus  $\varphi_0(s)$  is an expanding pseudohomothety, and let  $K$  be its compact set of fixed points. Choose  $\xi_- < \min K, \xi_+ > \max K$ . Since  $\varphi_0(s^n)(\xi_+)$  tends to  $+\infty$ , we can choose  $n > 0$  such that  $\varphi_0(s^n)(\xi_+) > \max_{t \in S} \varphi_0(t)(\xi_+)$ . Let  $\mathcal{U}$  be the set of actions  $\psi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$  such that  $\psi(s^n)(\xi_+) > \max_{t \in S} \psi(t)(\xi_+)$  and  $\psi(s)(\xi_-) < \xi_-$ . Then  $\mathcal{U}$  is an open neighborhood of  $\varphi_0$  and we claim that it satisfies the desired conclusion. Assume by contradiction that  $\psi \in \mathcal{U}$  is semi-conjugate to an action given by a non-trivial homomorphism  $\bar{\psi}: G \rightarrow (\mathbb{R}, +)$  by translations. Since  $\psi(s)(\xi_-) < \xi_-$  and  $\psi(s)(\xi_+) > \xi_+$ , we must have  $\bar{\psi}(s) = 0$ . On the other hand since  $S$  is symmetric, there exists  $t \in S$  such that  $\bar{\psi}(t) > 0$ . For such a  $t$  we must have  $\psi(t)(\xi_+) < \psi(s^n)(\xi_+)$  for every  $n \in \mathbb{Z}$ , a contradiction with the definition of  $\mathcal{U}$ .  $\square$

**Lemma 8.6.** *Let  $G$  be a finitely generated group, and let  $S \subset G$  be a finite symmetric generating subset. Let  $\varphi: G \rightarrow \text{Homeo}_0(\mathbb{R})$  be a minimal  $\mathbb{R}$ -focal action horograded by an action by translations  $j: G \rightarrow \text{Homeo}_0(\mathbb{R})$ . Then, for every element  $g \in \ker j$  and  $p \in \mathbb{R}$ , there exists  $h \in S \cup \{sgs^{-1} : s \in S\}$  such that*

$$h^{-1}.p < g^n.p < h.p$$

for every  $n \in \mathbb{Z}$ .

*Proof.* Take an element  $g \in \ker j$  and note that by Proposition 3.17,  $\varphi(g)$  is totally bounded. Fix a point  $p \in \mathbb{R}$ . If  $g.p = p$  the lemma follows from the fact that  $\varphi$  has no global fixed points. Suppose now that  $g.p \neq p$  and let  $J \subset \text{Supp}^\varphi(g)$  be the connected component containing  $p$ . By Proposition 3.19, we can find an action  $\Phi: G \rightarrow \text{Aut}(\mathbb{T}, \triangleleft, \prec)$  of  $G$  on a planar directed tree representing  $\varphi$  and which is isometric for a compatible  $\mathbb{R}$ -tree metric. Denote by  $\sigma: (\partial^*\mathbb{T}, \prec) \rightarrow (\mathbb{R}, <)$  the map given in Lemma 3.14 and set  $U_z := \{w \in \partial^*\mathbb{T} : w \triangleleft z\}$ . Set also  $\tilde{J} = \sigma^{-1}(J)$  and define  $v := \min\{w \in \mathbb{T} : \tilde{J} \subseteq U_w\}$ . In order to show that  $v$  is well defined, notice that the interval  $\tilde{J}$  is bounded above and below in  $\partial^*\mathbb{T}$ , thus we can choose  $\xi_0, \xi_1 \in \partial^*\mathbb{T}$  such that  $\xi_0 \prec \xi \prec \xi_1$  for every  $\xi \in \tilde{J}$ . Then the point  $w = \xi_0 \wedge \xi_1$  satisfies  $\tilde{J} \subseteq U_w$ , as it follows from the compatibility condition (CC) for the order  $\prec$ . Finally, the existence of a  $\triangleleft$ -smallest point follows easily from the definition of directed tree. Also notice that, by the  $\Phi(g)$ -invariance of  $\tilde{J}$  and the construction of  $v$ , it holds that  $g.v = v$ .

**Claim 1.** *Suppose that  $w \triangleleft v$  and  $g.w = w$ . Then, the intersection  $U_w \cap \tilde{J}$  is empty.*

*Proof of claim.* Suppose by contradiction that there exists  $\xi_0 \in U_w \cap \tilde{J}$ . Note that, after the choice of  $v$ , the subset  $U_w$  cannot contain  $\tilde{J}$ . Therefore, we can consider  $\xi_1 \in \tilde{J} \setminus U_w$ . Assume without loss of generality that  $\xi_0 \prec \xi_1$ . By  $\prec$ -convexity of  $U_w$ , we have that  $U_w \prec \xi_1$ . Since  $\sigma$  is non-decreasing (Lemma 3.14), it follows that  $\sigma(U_w) \leq \sigma(\xi_1)$ . Therefore, we conclude that  $\sup \sigma(U_w) \in J$ . On the other hand, since  $g.w = w$  we have that  $g.\sigma(U_w) = \sigma(U_w)$  and therefore  $\sup \sigma(U_w)$  is a fixed point of  $\varphi(g)$  inside  $J$ , which is a contradiction.  $\square$

**Claim 2.** *Suppose that for some  $h \in G$  one of the following holds:*

- (1) *either  $v$  and  $h.v$  are  $\prec$ -related, or*
- (2)  *$h.v \triangleleft v$  and  $g.(h.v) = h.v$ .*

*Then, up to changing  $h$  with its inverse, it holds that  $h^{-1}.p < g^n.p < h.p$  for every  $n \in \mathbb{Z}$ .*

*Proof of claim.* Suppose first that (1) holds. Up to changing  $h$  with its inverse, we can assume that  $h^{-1}.v \prec v \prec h.v$ . Therefore, the compatibility condition (CC) implies that  $U_{h^{-1}.v} \prec U_v \prec U_{h.v}$ , which in turn implies that  $h^{-1}.\tilde{J} \prec \tilde{J} \prec h.\tilde{J}$ . Thus, since  $\sigma$  is non-decreasing we obtain that  $h^{-1}.J < J < h.J$  which concludes the proof, since  $J$  is the component of the support of  $\varphi(g)$  containing  $p$ . Assume next that (2) holds. By Claim 1, we have  $U_{h.v} \cap \tilde{J} = \emptyset$  is empty and therefore, since  $\tilde{J} \subset U_v$ , we get that  $h.\tilde{J} \cap \tilde{J} = \emptyset$ . Since  $\tilde{J}$  is  $\prec$ -convex, up to changing  $h$  with its inverse, we can assume that  $h^{-1}.\tilde{J} \prec \tilde{J} \prec h.\tilde{J}$ . Therefore, applying  $\sigma$  we get that  $h^{-1}.J < J < h.J$ .  $\square$

Let  $s \in S$  be a generator such that  $j(s)$  is a positive translation. If  $s.v$  and  $v$  are  $\prec$ -related, then we can conclude the proof using Claim 2, as condition (1) is satisfied. In the remaining case, since  $j(s)$  is a positive translation, we must have  $v \triangleleft s.v$ . Assume first that  $sgs^{-1}.v \neq v$ . In this case, since  $\Phi$  preserves an  $\mathbb{R}$ -tree metric and  $s.v \in \text{Fix}^\Phi(sgs^{-1})$ , it must hold that  $sgs^{-1}.v$  and  $v$  are  $\prec$ -related. Thus, we conclude again using Claim 2, as condition (1) is satisfied. For the remaining case, suppose that  $sgs^{-1}.v = v$ . Rewrite the equality as  $gs^{-1}.v = s^{-1}.v$  and notice that  $s^{-1}.v \triangleleft v$ . In this case, we conclude using Claim 2, setting  $h = s^{-1}$  in condition (2).  $\square$

**Proposition 8.7.** *Let  $G$  be a finitely generated group. Let  $\mathcal{X} \subset \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$  be the set of minimal actions of  $G$  which are  $\mathbb{R}$ -focal and can be horograded by an action by translations. Let  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  be an action which is semi-conjugate to a non-abelian affine action. Then  $\varphi$  has a neighborhood  $\mathcal{U}$  such that no element of  $\mathcal{U}$  is semi-conjugate to an action in  $\mathcal{X}$ .*

*Proof.* Let  $\varphi_0 : G \rightarrow \mathbb{R}$  be an action which is semi-conjugate to a non-abelian affine action  $\bar{\varphi}_0$ , fix a point  $p \in \mathbb{R}$ . and a finite symmetric generating subset  $S \subset G$ . Since  $G$  is non-abelian, there exists  $g \in [G, G]$  such that  $\bar{\varphi}_0(g)$  acts as a translation, and thus  $\varphi_0(g)$  acts without fixed points. We can assume without loss of generality that  $g.p > p$ . Thus, for any finite subset  $F \subset G$ , there exists  $K = K_F \in \mathbb{N}$  such that for any  $f \in F$  we have

$$\varphi_0(g)^{-K}(p) < \varphi_0(f)(p) < \varphi_0(g^K)(p).$$

This is an open condition in the space of actions  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  with respect to the compact-open topology, so we can find a neighborhood  $\mathcal{U}_F$  of  $\varphi_0$  in the space of irreducible actions of  $G$  such that the condition above is satisfied. Assume  $\varphi : G \rightarrow \text{Homeo}_0(\mathbb{R})$  is an action which is semi-conjugate to an action  $\bar{\varphi} \in \mathcal{X}$ . Using Lemma 8.6, we find an element  $h \in F := S \cup \{sgs^{-1} : s \in S\}$  such that  $\bar{\varphi}(h)(\bar{p}) < \bar{\varphi}(g)^n(\bar{p})$  for any  $n \in \mathbb{Z}$ . Therefore the

analogue condition is satisfied for the action  $\varphi$ :  $\varphi(h)(p) < \varphi(g)^n(p)$  for any  $n \in \mathbb{Z}$ . We deduce that  $\varphi \notin \mathcal{U}_F$ . As the finite subset  $F$  does not depend on  $\varphi$ , this gives that  $\mathcal{U}_F$  contains no action which is semi-conjugate to an action in  $\mathcal{X}$ .  $\square$

*Proof of Theorem C.* Since  $G$  is finitely generated, every  $\varphi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$  is semi-conjugate to an action which is either minimal or cyclic. After Corollary 8.4, and by splitting further the affine actions into the non-abelian ones and actions by translations, we obtain that every  $\varphi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$  is semi-conjugate to either

- (1) a non-abelian affine action, or
- (2) an action by translations, or
- (3) a minimal  $\mathbb{R}$ -focal action which can be horograded by an action by translations.

By Propositions 8.5 and 8.7, every action satisfying (1) has a neighborhood consisting of actions which do not satisfy neither (2) nor (3). Thus the set of such actions is open.  $\square$

**8.2. A non-metaniipotent example.** Here we show that the conclusion of Theorem C fails for solvable groups which are not virtually metaniipotent. More precisely, we will construct a finitely generated 3-step solvable group  $G$  and a sequence  $(\varphi_n) \subset \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$  of minimal actions which are not semi-conjugate to affine actions, but converge to a limit  $\varphi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$  which is conjugate to a non-abelian affine action.

As a basis for our construction, let  $H \subseteq \text{Aff}(\mathbb{R})$ , be a finitely generated, non-abelian group of affine transformations acting irreducibly on  $\mathbb{R}$ . A precise choice of such an  $H$  will be irrelevant; for definiteness the reader may take  $H$  to be the group generated by the two transformations

$$x \mapsto ax, \quad x \mapsto x + 1 \quad (a > 0, a \neq 1).$$

We denote by  $j: H \rightarrow \text{Homeo}_0(\mathbb{R})$  the action given by the standard inclusion.

Let  $G$  be the wreath product  $G = \mathbb{Z} \wr H := \bigoplus_H \mathbb{Z} \rtimes H$ . We will simply write  $\mathbb{Z}^{(H)}$  for the direct sum  $\bigoplus_H \mathbb{Z}$ , and identify it with the group of finitely supported configurations  $f: H \rightarrow \mathbb{Z}$ . The semi-direct product is taken with respect to the left-regular action  $\lambda: H \rightarrow \text{Aut}(\mathbb{Z}^{(H)})$ , given by  $\lambda_h(f)(k) = f(h^{-1}k)$ . For later use, we also denote by  $\rho: H \rightarrow \text{Aut}(\mathbb{Z}^{(H)})$  the right-regular action, given by  $\rho_h(f)(k) = f(kh)$ . Note that these two actions commute.

We denote by  $\pi_H: G \rightarrow H$  the projection to the quotient and by  $(\delta_h)_{h \in H}$  the standard basis of  $\mathbb{Z}^{(H)}$ , where  $\delta_h \in \mathbb{Z}^{(H)}$  is the configuration taking the value 1 on  $h$  and 0 elsewhere. The identically zero configuration of  $\mathbb{Z}^{(H)}$  is denoted  $\bar{0}$ . Finally for  $f \in \mathbb{Z}^{(H)}$  we write  $\text{Supp}(f) = \{h \in H: f(h) \neq 0\}$ .

Chose a point  $\xi_0 \in \mathbb{R}$  which has a free orbit for the natural affine action  $j$  of  $H$ , and let  $<_H$  be the associated left-order on  $H$ , given by  $h_1 <_H h_2$  if  $j(h_1)(\xi_0) < j(h_2)(\xi_0)$ . We can use this order to construct an action of  $G$  on  $\mathbb{R}$  in a way which is analogous to the Plante's action (as in [5, Example 7.11]). Namely, consider the order  $\prec$  on  $\mathbb{Z}^{(H)}$  given by  $f_1 \prec f_2$  if  $f_1(k) < f_2(k)$ , where  $k = \max_{<_H} \{h: f_1(h) \neq f_2(h)\}$ . Let  $\alpha: G \rightarrow \text{Aut}(\mathbb{Z}^{(H)})$  be the "affine" action obtained by letting  $H$  act via the left-regular action and  $\mathbb{Z}^{(H)}$  act on itself by translations. Explicitly, for  $g = (r, h) \in \mathbb{Z}^{(H)} \rtimes H = G$  and  $f \in \mathbb{Z}^{(H)}$ , we have

$$\alpha(g)(f) = r + \lambda_h(f).$$

This action preserves the order  $\prec$  constructed above. By taking the dynamical realization of the action  $\alpha: G \rightarrow \text{Aut}(\mathbb{Z}^{(H)}, \prec)$ , we obtain an action  $\psi: G \rightarrow \text{Homeo}_0(\mathbb{R})$ . This action is minimal and  $\mathbb{R}$ -focal. A proof of this, and a more detailed discussion, can be found in [5, Example 7.11].

Let  $\iota: \mathbb{Z}^{(H)} \rightarrow \mathbb{R}$  be the associated equivariant order-preserving map from the dynamical realization  $\psi$ . We suppose that  $\iota(\bar{0}) = 0$ . Then 0 is a global fixed point for  $\psi(H)$ , and it is moved by all elements of  $\psi(\mathbb{Z}^{(H)})$ . Denote by  $I_h$  the connected component of the support of  $\psi(\delta_h)$  containing 0. Then each  $I_h$  is a bounded open interval and we have  $I_{h_1} \subset I_{h_2}$  if  $h_1 <_H h_2$ . Moreover since 0 is fixed by  $\psi(H)$  and  $h\delta_k h^{-1} = \delta_{hk}$ , we have  $\psi(h)(I_k) = I_{hk}$ . In particular if we denote by  $p_h$  the rightmost point of  $I_h$ , the map  $h \mapsto p_h$  is an order-preserving,  $H$ -equivariant embedding of  $(H, <_H)$  into  $\mathbb{R}$ . Finally we observe that the point  $p_h$  is fixed by  $\psi(\delta_k)$  for every  $k <_H h$ . Thus if  $f \in \mathbb{Z}^{(H)}$  is such that  $\max_{<_H} \text{Supp}(f) <_H h$ , then  $\psi(f)(p_h) = (p_h)$ .

Let now  $t \in H$  be any element such that  $j(t)$  is a positive translation. Let  $\tau: G \rightarrow G$  be the automorphism

$$\tau(f, h) = (\rho_t(f), h), \quad (f, h) \in \mathbb{Z}^{(H)} \rtimes H.$$

The fact that  $\tau$  is an automorphism follows from the fact that  $\rho_t$  commutes with the left-regular action of  $H$ . For every  $n \in \mathbb{Z}$  set  $\psi_n = \psi \circ \tau^n$ . Note that every  $\psi_n$  is minimal and  $\mathbb{R}$ -focal. We also point out to the reader that  $\tau$  is an exterior automorphisms of  $G$ , and the actions  $\psi_n$  are all pairwise non-conjugate. We will prove the following.

**Proposition 8.8.** *With notation as above, there exists a sequence of homeomorphisms  $(s_n) \subset \text{Homeo}_0(\mathbb{R})$  such that the sequence of conjugate actions  $\varphi_n(g) = s_n \psi_n(g) s_n^{-1}$  has a subsequence which converges to a limit  $\varphi \in \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$ , which is positively conjugate to the non-abelian affine action  $j \circ \pi_H$ .*

*Proof.* Before discussing the details of the proof we recall here a fundamental consequence of the work of Deroin, Kleptsyn, Navas and Parwani [9]. We follow the point of view in [5, §2.2]. For a fixed symmetric probability measure  $\mu$  supported on a finite generating set of  $G$ , one can consider the subset  $\text{Der}_\mu(G) \subset \text{Hom}_{\text{irr}}(G, \text{Homeo}_0(\mathbb{R}))$  of so-called “normalized  $\mu$ -harmonic actions” of  $G$ . Instead of discussing the way  $\text{Der}_\mu(G)$  is defined, let us simply list its fundamental properties.

- (1)  $\text{Der}_\mu(G)$  is compact,
- (2)  $\text{Der}_\mu(G)$  contains a representative of any positive semi-conjugacy class of irreducible action of  $G$ , which moreover is either minimal or cyclic,
- (3)  $\text{Der}_\mu(G)$  is closed under conjugation by translations, and conversely any two representatives of the same positive semi-conjugacy class are conjugate by a translation.

Write  $p_0 := p_{\text{id}_H}$ . As  $\psi_n$  is minimal for any  $n \in \mathbb{N}$ , by the last two properties of the space  $\text{Der}_\mu(G)$  we can find a sequence  $(s_n)$  of homeomorphisms such that  $s_n(p_0) = p_0$ , and such that  $\varphi_n := s_n \psi_n \in \text{Der}_\mu(G)$ . As  $\text{Der}_\mu(G)$  is compact, upon extracting a subsequence we can suppose that  $\varphi_n$  converges to a limit  $\varphi \in \text{Der}_\mu(G)$ . In particular,  $\varphi$  is either minimal or cyclic. We claim that  $\varphi$  is positively conjugate to  $j \circ \pi_H$ . To see this, fix  $g = (f, h) \in G$ , with  $h = \pi_H(g)$  and compute

$$\begin{aligned} \varphi_n(g)(p_0) &= s_n \psi_n(g) s_n^{-1}(p_0) = s_n \psi_n(g)(p_0) \\ &= s_n \psi(\rho_{t^n}(f), h)(p_0) = s_n \psi(\rho_{t^n}(f))(p_h). \end{aligned}$$

Here we have used that the map  $h \mapsto p_h$  is  $H$ -equivariant, as observed above. Now note that  $\text{Supp}(\rho_{t^n}(f)) = \text{Supp}(f)t^{-n}$ . By the choice of  $t$ , the element  $k_n := \max_{<_H} \text{Supp}(f)t^{-n}$  tends to  $-\infty$  in  $(H, <_H)$ . In particular if  $n$  is large enough we have  $k_n < h$ . As observed earlier, this implies that  $\psi(\rho_{t^n}(f))(p_h) = p_h$ . Thus, for  $n$  large enough we have

$$\varphi_n(g)(p_0) = s_n(p_h) = s_n(p_{\pi_H(g)}).$$

Now, fix  $g_1, g_2 \in G$ . If  $n$  is large enough, the previous computation holds for both  $g_1$  and  $g_2$ . Therefore the following conditions are equivalent for sufficiently large  $n$ :

- (1)  $\varphi_n(g_1)(p_0) \leq \varphi_n(g_2)(p_0)$ ,
- (2)  $s_n(p_{\pi_H(g_1)}) \leq s_n(p_{\pi_H(g_2)})$ ,
- (3)  $\pi_H(g_1) \leq_H \pi_H(g_2)$ .

For the equivalence between the last two conditions, we use that  $s_n$  is a homeomorphism and that the map  $h \mapsto p_h$  is order preserving. Taking the limit, we get that if  $\pi_H(g_1) \leq_H \pi_H(g_2)$ , then  $\varphi(g_1)(p_0) \leq \varphi(g_2)(p_0)$ . We deduce that the map  $\sigma: (H, <_H) \rightarrow (\mathbb{R}, <)$  given by  $\sigma(h) = \varphi((\bar{0}, h))(p_0)$  is non-decreasing and  $G$ -equivariant, where  $G$  acts on  $(H, <_H)$  by translations via  $\pi_H$ , and on  $\mathbb{R}$  via  $\varphi$ . Note that the dynamical realization of the action on  $(H, <_H)$  is precisely  $j \circ \pi_H$ , and thus it is minimal. By Lemma 2.3 we deduce that  $\sigma$  is injective. In particular  $\varphi$  cannot be cyclic, since the ordered space  $(H, <_H)$  is not isomorphic to  $(\mathbb{Z}, <)$ . Thus  $\varphi$  is minimal, and Lemma 2.3 again implies that it is conjugate to  $j \circ \pi_H$ .  $\square$

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