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Efoevi Angelo Koudou, Jacek Wesolowski

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INDEPENDENCES OF KUMMER LAWS

ANGELO EFOÉVI KOUDOU AND JACEK WESOŁOWSKI

ABSTRACT. We prove that if X, Y are positive, independent, non-Dirac random variables and if $\alpha, \beta \geq 0$, $\alpha \neq \beta$, then the random variables U and V defined by

$$U = Y \frac{1+\beta(X+Y)}{1+\alpha X+\beta Y} \quad \text{and} \quad V = X \frac{1+\alpha(X+Y)}{1+\alpha X+\beta Y}$$

are independent if and only if X and Y follow Kummer distributions with suitable parameters. In other words, the Kummer distributions are the only invariant measures for lattice recursion models introduced by Croydon and Sasada in [3]. The result extends earlier characterizations of Kummer and gamma laws by independence of

$$U = \frac{Y}{1+X} \quad \text{and} \quad V = X \left(1 + \frac{Y}{1+X}\right),$$

which is the case of $(\alpha, \beta) = (1, 0)$.

1. INTRODUCTION

Consider, for $b, c > 0$, the gamma distribution $\mathcal{G}(b, c)$ with density proportional to

$$y^{b-1} e^{-cy} I_{(0, \infty)}(y),$$

for $p \in \mathbb{R}, a > 0, b > 0$, the generalized inverse Gaussian (GIG) distribution $\text{GIG}(p, a, b)$ with density proportional to

$$x^{p-1} e^{-(-ax-b/x)} I_{(0, \infty)}(x)$$

and, for $a, c > 0$ and $b \in \mathbb{R}$, the Kummer distribution $\mathcal{K}(a, b, c)$ with density proportional to

$$\frac{x^{a-1} e^{-cx}}{(1+x)^b} I_{(0, \infty)}(x).$$

Following [3], we say that a quadruplet of probability measures $(\mu, \nu, \tilde{\mu}, \tilde{\nu})$ on $\mathcal{U}, \mathcal{V}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}}$, respectively, satisfy the *detailed balance equation* for a map $F : \mathcal{U} \times \mathcal{V} \rightarrow \tilde{\mathcal{U}} \times \tilde{\mathcal{V}}$ if

$$F(\mu \otimes \nu) = \tilde{\mu} \otimes \tilde{\nu},$$

where $F(\mu \otimes \nu)$ means $(\mu \otimes \nu) \circ F^{-1}$.

The Matsumoto-Yor property is the following: for $p, a, b > 0$, given two independent, positive random variables X and Y such that $X \sim \text{GIG}(-p, a, b)$ and $Y \sim \mathcal{G}(p, a)$, the random variables $\frac{1}{X+Y}$ and $\frac{1}{X} - \frac{1}{X+Y}$ are independent (and follow the $\text{GIG}(-p, b, a)$ and $\mathcal{G}(p, b)$, respectively). Using the terminology of [3], the Matsumoto-Yor property says that the quadruplet of probability measures $\mu = \text{GIG}(-p, a, b)$, $\nu = \mathcal{G}(p, a)$, $\tilde{\mu} = \text{GIG}(-p, b, a)$, $\tilde{\nu} = \mathcal{G}(p, b)$ satisfy the *detailed balance equation* for the map

$$F : (0, \infty)^2 \rightarrow (0, \infty)^2$$

$$(x, y) \xrightarrow{F} \left(\frac{1}{x+y}, \frac{1}{x} - \frac{1}{x+y} \right).$$

This property was discovered by [10] in the case $a = b$, while studying some functionals of exponential Brownian motion. In [7] the authors noticed that it is true also if $a \neq b$ and proved that the independence property to be in fact a characterization: for two non-Dirac, positive and independent random variables

X and Y , the random variables $\frac{1}{X+Y}$ and $\frac{1}{X} - \frac{1}{X+Y}$ are independent if and only if $X \sim \text{GIG}(-p, a, b)$ while $Y \sim \mathcal{G}(p, a)$ for some $p, a, b > 0$.

In [6] the authors studied the question of finding decreasing and bijective functions $f : (0, \infty) \rightarrow (0, \infty)$ such that there exists a quadruplet of probability measures $(\mu, \nu, \tilde{\mu}, \tilde{\nu})$ on $(0, \infty)$ satisfying the detailed balance equation for the map

$$T_f : (0, \infty)^2 \rightarrow (0, \infty)^2 \\ (x, y) \mapsto (f(x+y), f(x) - f(x+y)).$$

This led, at the cost of some regularity assumptions, to other independence properties of the Matsumoto-Yor type (of course, one retrieves the original Matsumoto-Yor case for $f(x) = 1/x$), among which a property involving the Kummer distribution. More precisely, it was proved in [6] that if X and Y are independent Kummer and gamma with suitably related parameters then

$$U = X + Y \quad \text{and} \quad V = \frac{1+(X+Y)^{-1}}{1+X^{-1}}$$

are independent Kummer and beta random variables. This was the starting point of a number of works on Matsumoto-Yor type characterizations of the Kummer distribution. Firstly, starting from the latter property and looking for an involutive version of it, i.e. trying to find an involutive map $F : (X, Y) \mapsto (U, V)$ for which the Kummer distribution is involved in a detailed balance equation, the following interesting property was discovered in [4]: let X and Y be independent, X have the distribution $\mathcal{K}(a, b, c)$ and Y have the gamma distribution $\mathcal{G}(b, c)$, then

$$(1) \quad U = \frac{Y}{1+X} \quad \text{and} \quad V = X \frac{1+X+Y}{1+X}$$

are independent, $U \sim \mathcal{K}(b, a, c)$ and $V \sim \mathcal{G}(a, c)$.

In [13] this independence property was proved to give a characterization result with no assumption of existence of densities. Related characterizations were considered in [14] and [12]. In [5] an extension to the matrix-variate case was established, while in [11] a free probability version of the property and characterization was given. The latter needed a definition of a new distribution, a free analogue of the Kummer distribution.

Let us come back to the definition of the detailed balance equation and recall its context as described in [3], which considers models assuming the following dynamics: for (n, t) in \mathbb{Z}^2 , n is the spatial coordinate and t the temporal one. For fixed $t \in \mathbb{Z}$, $(x_n^t)_{n \in \mathbb{Z}} \in (0, \infty)^{\mathbb{Z}}$ is the configuration of the system at time t , and $(y_n^t)_{n \in \mathbb{Z}} \in (0, \infty)^{\mathbb{Z}}$ a collection of auxiliary variables through which the dynamics from t to $t+1$ are defined. Namely, (x_n^t, y_n^t) depends on (x_n^{t-1}, y_{n-1}^t) only, through the formula

$$(x_n^t, y_n^t) = G(x_n^{t-1}, y_{n-1}^t),$$

where for a bijection $F : \mathcal{X} \times \mathcal{Y} \rightarrow \tilde{\mathcal{X}} \times \tilde{\mathcal{Y}}$ either $G = F$, when $n+t$ is even or $G = F^{-1}$ when $n+t$ is odd. The case when F is involutive is referred to as type I model, while the general case is referred to as type II model. Let $x = (x_n)_{n \in \mathbb{Z}}$ be such that the above recursion with the initial condition $x_n^0 = x_n$, $n \in \mathbb{Z}$, has a unique solution $(x_n^t(x), y_n^t(x))_{n, t \in \mathbb{Z}}$. Let \mathcal{X}^* denote set of all such x 's. According to Theorem 1.1 in [3] in type I model for a sequence of iid random variables $X = (X_n)_{n \in \mathbb{Z}}$ with $X_1 \sim \mu$ we have that $X \stackrel{d}{=} (x_n^1(X))_{n \in \mathbb{Z}}$ iff there exists a probability measure ν such that the pair (μ, ν) satisfies the detailed balance condition with respect to F . In this sense $\mu \otimes \nu$ is the invariant measure for this lattice recursion. In case of the type II model similar alternating invariance holds for pairs $\mu \otimes \nu$ and $\tilde{\mu} \otimes \tilde{\nu}$. In [3] the authors identified four such type I and/or type II models:

- (1) ultra-discrete KdV (Korteweg-de Vries) type I model for

$$F(x, y) := F_{udK}^{(J, K)} = (y - (x + y - J)_+ + (x + y - K)_+, x - (x + y - K)_+ + (x + y - J)_+)$$

with μ and ν the shifted truncated exponential or shifted scaled truncated geometric laws;

(2) discrete KdV type I model for

$$F(x, y) := F_{dK}^{(\alpha, \beta)}(x, y) = \left(\frac{y(1+\beta xy)}{1+\alpha xy}, \frac{x(1+\alpha xy)}{1+\beta xy} \right)$$

with μ the GIG law and ν the GIG (gamma) law which, when $\alpha\beta = 0$, has a direct reference to the Matsumoto-Yor property and related characterization of the GIG and gamma laws;

(3) ultra-discrete Toda type II model for

$$F(x, y) := F_{udT^*}(x \wedge y, x - y)$$

with μ, ν $\tilde{\mu}$ the shifted exponential, $\tilde{\nu}$ asymmetric Laplace or μ, ν $\tilde{\mu}$ shifted scaled geometric, $\tilde{\nu}$ scaled discrete Laplace laws;

(4) discrete Toda type II model for

$$F(x, y) := F_{dT^*}(x, y) = \left(x + y, \frac{x}{x+y} \right)$$

with μ, ν $\tilde{\mu}$ the gamma, $\tilde{\nu}$ beta laws having a direct reference to Lukacs (1955) characterization of the gamma distribution.

In the context of the discrete KdV model in [3] the authors observed that if the laws of X and Y , as well as U and V , are independent GIG with suitable parameters, then (X, Y) and (U, V) satisfy the detailed balance equation for the map $F_{dK}^{(\alpha, \beta)}$, and they conjectured that the GIG distributions are the only possible ones which let this $F_{dK}^{(\alpha, \beta)}$ -detailed balance equation be satisfied. Recently, in [8] this conjecture was proved without the assumptions of existence and regularity of densities made by [2] in their proof of the same conjecture. More precisely, [8] established the following extension of the Matsumoto-Yor property: if A and B are non-degenerate, positive and independent random variables, and if α and β are two positive and distinct numbers, then the random variables

$$S = \frac{1}{B} \frac{\beta A + B}{\alpha A + B}, \quad T = \frac{1}{A} \frac{\beta A + B}{\alpha A + B}$$

are independent if and only if A and B have GIG distributions with suitable parameters.

In this paper we reveal one more candidate for invariant measure for the lattice recursion model. We derive the detailed balance equation for the Kummer distributions. Our main result (Theorem 3.3) gives a characterization of the Kummer laws, which is a result of a similar nature as the one in [8] for the GIG laws, i.e. it says that the Kummer distributions are the only possible ones which let the detailed balance equation be satisfied for the map

$$(2) \quad F(x, y) = \left(y \frac{1+\beta(x+y)}{1+\alpha x+\beta y}, x \frac{1+\alpha(x+y)}{1+\alpha x+\beta y} \right).$$

The proof uses a suitably designed "Laplace-type" transform and leads to a special second order linear differential equation for an unknown function of such form. In this sense the general methodology (a Laplace type transform and a second order ordinary linear differential equation) resembles one of the proof from [8]. However, at the technical level, the challenges to overcome were of quite a different nature. Interpreting this result in the context of the lattice system of recursions described above, it says that the Kummer distributions are the only relevant invariant measures for the type I model governed by F defined in (2).

The paper is organized as follows: in Section 2 we introduce a scaled version of the Kummer distribution, we express and prove the considered independence property in terms of that scaled Kummer distribution (Theorem 2.1). In Section 3 we define and analyze the *Kummer transform*, an extended Laplace transform that will help us to prove the characterization theorem formulated at the end of this section. Section 4 contains the proof of the characterization splitted in several steps (subsections) of the proof, among which the crucial observation is that the unknown *Kummer transform* satisfies the Kummer differential equation (see [1]).

2. THE INDEPENDENCE PROPERTY

For the purpose of this paper it will be convenient to introduce a scaled version of the Kummer distribution.

Definition 2.1. Let $\mathcal{K}_\alpha(a, b; c)$ for $\alpha \geq 0$, $a, c > 0$ and $b \in \mathbb{R}$ be the probability distribution defined by the density

$$f(x) \propto \frac{x^{\alpha-1} e^{-cx}}{(1+\alpha x)^b} I_{(0, \infty)}(x)$$

Remark 2.1. Note that $\mathcal{K}_0(a, b; c) = \mathcal{G}(a; c)$. Also $\mathcal{K}_\alpha(a, 0; c) = \mathcal{G}(a; c)$ - see e.g. Remark 3.1 below. Moreover, for $\alpha > 0$ and $X \sim \mathcal{K}_\alpha(a, b; c)$ we have

$$\alpha X \sim \mathcal{K}(a, b; c/\alpha).$$

Theorem 2.1. Assume that

$$(X, Y) \sim \mathcal{K}_\alpha(a, b; c) \otimes \mathcal{K}_\beta(b, a; c)$$

for $a, b, c > 0$ and $\alpha, \beta \geq 0$, $\alpha \neq \beta$. Let

$$(3) \quad U = Y \frac{1+\beta(X+Y)}{1+\alpha X+\beta Y} \quad \text{and} \quad V = X \frac{1+\alpha(X+Y)}{1+\alpha X+\beta Y}.$$

Then

$$(U, V) \sim \mathcal{K}_\alpha(b, a; c) \otimes \mathcal{K}_\beta(a, b; c).$$

Remark 2.2. Note that in view of Remark 2.1 the above result gives a straightforward extension of the property observed in [4] and recalled in the beginning, see (1). It suffices to take $(\alpha, \beta) = (1, 0)$.

Proof of Theorem 2.1. Denote

$$\psi(x, y) = \left(y \frac{1+\beta(x+y)}{1+\alpha x+\beta y}, x \frac{1+\alpha(x+y)}{1+\alpha x+\beta y} \right) =: (u, v), \quad x, y > 0.$$

Note that $\psi : (0, \infty)^2 \rightarrow (0, \infty)^2$ is an involution. Moreover, the following identities hold true:

$$(4) \quad x + y = u + v,$$

$$(5) \quad \frac{x}{1+\beta y} = \frac{v}{1+\alpha u},$$

$$(6) \quad \frac{y}{1+\alpha x} = \frac{u}{1+\beta v}.$$

Now we compute the Jacobian $J_{\psi^{-1}}(u, v)$ of $\psi^{-1} = \psi$. Note that (4) implies

$$(7) \quad \frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} = 1 \quad \text{and} \quad \frac{\partial x}{\partial v} + \frac{\partial y}{\partial v} = 1.$$

Therefore,

$$(8) \quad J_{\psi^{-1}}(u, v) = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ 1 - \frac{\partial x}{\partial u} & 1 - \frac{\partial x}{\partial v} \end{array} \right| = \frac{\partial x}{\partial u} - \frac{\partial x}{\partial v}$$

Rewrite (5) as $x(1 + \alpha u) = v(1 + \beta v)$ and differentiate it with respect to u to get

$$\frac{\partial x}{\partial u}(1 + \alpha u) + \alpha x = \beta v \frac{\partial y}{\partial u}$$

Combining the latter equality with (7) we get

$$(9) \quad \frac{\partial x}{\partial u} = \frac{\beta v - \alpha x}{1 + \alpha u + \beta v}.$$

Similarly, rewrite (6) as $y(1 + \beta v) = u(1 + \alpha x)$ and differentiate with respect to v to get

$$\frac{\partial y}{\partial v}(1 + \beta v) + \beta y = \alpha u \frac{\partial x}{\partial v}.$$

Combining the latter equality with (7) we get

$$(10) \quad \frac{\partial x}{\partial v} = \frac{1 + \beta(v + y)}{1 + \alpha u + \beta v}.$$

Now we insert (9) and (10) into (8) getting

$$(11) \quad |J_{\psi^{-1}}(u, v)| = \frac{1+\alpha x+\beta y}{1+\alpha u+\beta v}.$$

Multiplying (5) and (6) sidewise after cancellation we get

$$\frac{1+\alpha x+\beta y}{xy} = \frac{1+\alpha u+\beta v}{uv}.$$

Referring to (11) we get

$$(12) \quad |J_{\psi^{-1}}(u, v)| = \frac{xy}{uv}.$$

Now we are ready to find the joint density of (U, V) . We have

$$\begin{aligned} f_{(U,V)}(u, v) &= |J_{\psi^{-1}}(u, v)| f_X(x(u, v)) f_Y(y(u, v)) \propto \frac{xy}{uv} \frac{x^{\alpha-1}}{(1+\alpha x)^b} e^{-cx} \frac{y^{\beta-1}}{(1+\beta y)^a} e^{-cy} I_{(0,\infty)^2}(u, v) \\ &= \frac{1}{uv} \left(\frac{x}{1+\beta y}\right)^a \left(\frac{y}{1+\alpha x}\right)^b e^{-c(x+y)} I_{(0,\infty)^2}(u, v). \end{aligned}$$

In view of (4), (5) and (6) we thus get

$$f_{(U,V)}(u, v) \propto \frac{1}{uv} \left(\frac{v}{1+\alpha u}\right)^a \left(\frac{u}{1+\beta v}\right)^b e^{-c(u+v)} I_{(0,\infty)^2}(u, v)$$

and the result follows. \square

For $\alpha, \beta \neq 0$ denote $\tilde{X} = \alpha X$, $\tilde{Y} = \beta Y$, $\tilde{U} = \alpha U$, $\tilde{V} = \beta V$. Then, for $\gamma = \alpha/\beta \neq 1$, in view of Remark 2.1, Theorem 2.1 yields the following independence property for pure (i.e. $\alpha = \beta = 1$) Kummer variables.

Corollary 2.2. *Let $(\tilde{X}, \tilde{Y}) \sim \mathcal{K}(a, b; c) \otimes \mathcal{K}(b, a; \gamma c)$ for $1 \neq \gamma > 0$. Then*

$$(\tilde{U}, \tilde{V}) := \left(\tilde{Y} \frac{\tilde{X} + \gamma(1 + \tilde{Y})}{1 + \tilde{X} + \tilde{Y}}, \tilde{X} \frac{\tilde{Y} + \gamma^{-1}(1 + \tilde{X})}{1 + \tilde{X} + \tilde{Y}} \right) \sim \mathcal{K}(b, a; c) \otimes \mathcal{K}(a, b; \gamma c).$$

3. THE KUMMER TRANSFORM AND THE CHARACTERIZATION

For a positive random variable W and $\gamma \geq 0$ consider an extended Laplace transform $L_W^{(\gamma)}$ of the form

$$L_W^{(\gamma)}(s, t, z) = \mathbb{E} \frac{W^s}{(1+\gamma W)^t} e^{-zW}.$$

We will call it the Kummer transform. Note that the Kummer transform is well defined at least for $s, z > 0$ and $t \in \mathbb{R}$. Moreover, for any fixed $s > 0$, $t \in \mathbb{R}$, the Kummer transform as a function of $z > 0$, is just the Laplace transform of the measure $\frac{w^s}{(1+\gamma w)^t} \mathbb{P}_W(dw)$, so it uniquely determines the distribution of W . Note also that

$$(13) \quad L_W^{(\gamma)}(s, t, z) + \gamma L_W^{(\gamma)}(s+1, t, z) = L_W^{(\gamma)}(s, t-1, z)$$

and for any $k = 1, 2, \dots$

$$(14) \quad \frac{\partial^k L_W^{(\gamma)}(s, t, z)}{\partial z^k} = -L_W^{(\gamma)}(s+k, t, z).$$

Proposition 3.1. *Let $X \sim \mathcal{K}_\alpha(a, b, c)$, $a, c > 0$, $b \in \mathbb{R}$. Then*

$$(15) \quad L_X^{(\alpha)}(s, t, z) = \frac{\Gamma(a+s) U\left(a+s, a+s-b-t+1, \frac{c+z}{\alpha}\right)}{\alpha^s \Gamma(a) U\left(a, a-b+1, \frac{c}{\alpha}\right)},$$

$s > 0$, $t \in \mathbb{R}$, $z > -c$, where U is the Kummer function (see 13.2.5 in [1]) defined by

$$(16) \quad U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty \frac{x^{a-1}}{(1+x)^{a-b+1}} e^{-zx} dx, \quad a, z > 0, b \in \mathbb{R}.$$

Proof. It is a simple consequence of that fact that due to the definition of the Kummer function U in (16) the normalizing constant of the Kummer distribution $\mathcal{K}_\alpha(a, b, c)$ has the form

$$\frac{\alpha^a}{\Gamma(a)U(a, a-b+1, \frac{c}{\alpha})}.$$

□

Remark 3.1. Note that when $b = a + 1$ in view of (16) we have

$$U(a, a + 1, z) = z^{-a},$$

whence (15) gives

$$L_X^{(\alpha)}(0, 0, z) = \frac{c^a}{(c+z)^a},$$

which implies that X is a Gamma random variable, $\mathcal{G}(a, c)$.

Proposition 3.2. Let $b \in \mathbb{R}$, $a, c, \alpha > 0$. Assume that for some fixed $(s, t) \in (0, \infty) \times \mathbb{R}$ and all $z > 0$

$$(17) \quad L_X^{(\alpha)}(s, t, z) = k(s, t)U(a + s, a + s - b - t + 1, \frac{c+z}{\alpha}),$$

where $k(s, t)$ is a constant (depending also on α, a, b, c) Then $X \sim \mathcal{K}_\alpha(a, b, c)$.

Proof. Fix some $z_0 > 0$. Then

$$(18) \quad \frac{L_X^{(\alpha)}(s, t, z+z_0)}{L_X^{(\alpha)}(s, t, z_0)} = \frac{U(a+s, a+s-b-t+1, \frac{c+z_0+z}{\alpha})}{U(a+s, a+s-b-t+1, \frac{c+z_0}{\alpha})}, \quad z \geq 0,$$

implies that it is the Laplace transform of a random variable Y with distribution

$$(19) \quad \mathbb{P}_Y(dy) = \frac{y^s (1+\alpha y)^{-t} e^{-z_0 y} \mathbb{P}_X(dy)}{L_X^{(\alpha)}(s, t, z_0)}.$$

In view of Proposition 3.1, by (15), the random variable Y has the Kummer distribution $\mathcal{K}_\alpha(a + s, b + t, c + z_0)$, i.e.

$$\mathbb{P}_Y(dy) \propto \frac{y^{a+s-1}}{(1+\alpha y)^{b+t}} e^{-(c+z_0)y} dy.$$

The result follows by comparing the last formula with (19). □

Remark 3.2. Let X and Y be independent. Assume also that U and V as defined in (3) are also independent. In view of (4), (5) and (6) we then have

$$(20) \quad L_X^{(\alpha)}(s, t, z) L_Y^{(\beta)}(t, s, z) = L_U^{(\alpha)}(t, s, z) L_V^{(\beta)}(s, t, z), \quad (s, t, z) \in (0, \infty) \times \mathbb{R} \times (0, \infty).$$

Due to the result of Theorem 2.1 equation (20) has to hold true in case when $(X, Y) \sim \mathcal{K}_\alpha(a, b; c) \otimes \mathcal{K}_\beta(b, a; c)$ and $(U, V) \sim \mathcal{K}_\alpha(b, a; c) \otimes \mathcal{K}_\beta(a, b; c)$. Indeed, using (15) we see that (20) reduces to

$$(21) \quad \frac{U(a+s, a+s-b-t+1, \frac{c+z}{\alpha}) U(b+t, b+t-a-s+1, \frac{c+z}{\beta})}{\alpha^s \beta^t U(a, a-b+1, \frac{c}{\alpha}) U(b, b-a+1, \frac{c}{\beta})} = \frac{U(b+t, b+t-a-s+1, \frac{c+z}{\alpha}) U(a+s, a+s-b-t+1, \frac{c+z}{\beta})}{\alpha^t \beta^s U(b, b-a+1, \frac{c}{\alpha}) U(a, a-b+1, \frac{c}{\beta})}.$$

To see that the above equality holds true we rely on the following identity for the Kummer function U (see (13.1.29) in [1])

$$(22) \quad U(a, b, z) = z^{1-b} U(1 + a - b, 2 - b, z).$$

Note also, that in view of (20) and (15) of Proposition 3.1 we have

$$\frac{L_X^{(\alpha)}(s, t, z)}{L_U^{(\alpha)}(t, s, z)} = \frac{L_V^{(\beta)}(s, t, z)}{L_Y^{(\beta)}(t, s, z)} = c^{a-b} \frac{\Gamma(b)}{\Gamma(a)} \frac{\Gamma(a+s)}{\Gamma(b+t)} (c+z)^{b-a+t-s}.$$

Now we are ready to formulate the main result which is a characterization of Kummer laws by the detailed balance condition with respect to the function F given in (2).

Theorem 3.3. *Let $\alpha, \beta \geq 0$, $\alpha \neq \beta$. Let X, Y be positive, independent, non-Dirac random variables and define*

$$U = Y \frac{1+\beta(X+Y)}{1+\alpha X+\beta Y} \quad \text{and} \quad V = X \frac{1+\alpha(X+Y)}{1+\alpha X+\beta Y}.$$

If U and V are independent, then there exist $a, b, c > 0$ such that

$$(X, Y) \sim \mathcal{K}_\alpha(a, b; c) \otimes \mathcal{K}_\beta(b, a; c).$$

We then have

$$(U, V) \sim \mathcal{K}_\alpha(b, a; c) \otimes \mathcal{K}_\beta(a, b; c).$$

Remark 3.3. *Recall that Theorem 2.6 of [13] says that for independent, positive, non-Dirac random variables X and Y if U and V given by (1) are also independent that $X \sim \mathcal{K}(a, b, c)$ and $Y \sim \mathcal{G}(b, c)$. (Note the change of parametrization of the Kummer distribution: instead of $b - a$ in [13] we write here just b .) In view of the first part of Remark 2.1 this result covers the case $(\alpha, \beta) = (1, 0)$ in Theorem 3.3. Due to the second part of Remark 2.1 and symmetry with respect to α and β we conclude that the cases $\beta = 0$, $\alpha > 0$ and $\beta > 0$, $\alpha = 0$ also follow immediately from Theorem 2.6. of [13]. That is, we need only to prove Theorem 3.3 for $\alpha > 0$ and $\beta > 0$. The proof of this case is given in several steps in the next section.*

4. THE PROOF OF THE CHARACTERIZATION FOR $\alpha > 0$ AND $\beta > 0$

4.1. Independence through Kummer transforms. In view of Remark 3.2 the equality (20) is equivalent to independence of X, Y and of U, V . We work below with $(s, t, z) \in (0, \infty) \times \mathbb{R} \times (0, \infty)$. Differentiating (20) with respect to z and dividing side-wise by (20) we get

$$(23) \quad \frac{L_X^{(\alpha)}(s+1, t, z)}{L_X^{(\alpha)}(s, t, z)} + \frac{L_Y^{(\beta)}(t+1, s, z)}{L_Y^{(\beta)}(t, s, z)} = \frac{L_U^{(\alpha)}(t+1, s, z)}{L_U^{(\alpha)}(t, s, z)} + \frac{L_V^{(\beta)}(s+1, t, z)}{L_V^{(\beta)}(s, t, z)}.$$

Using identity (13) we obtain

$$\beta \frac{L_X^{(\alpha)}(s, t-1, z)}{L_X^{(\alpha)}(s, t, z)} + \alpha \frac{L_Y^{(\beta)}(t, s-1, z)}{L_Y^{(\beta)}(t, s, z)} = \beta \frac{L_U^{(\alpha)}(t, s-1, z)}{L_U^{(\alpha)}(t, s, z)} + \alpha \frac{L_V^{(\beta)}(s, t-1, z)}{L_V^{(\beta)}(s, t, z)}.$$

Changing in the above formula s to $s+1$ and t to $t+1$ we arrive at

$$(24) \quad \beta \frac{L_X^{(\alpha)}(s+1, t, z)}{L_X^{(\alpha)}(s+1, t+1, z)} + \alpha \frac{L_Y^{(\beta)}(t+1, s, z)}{L_Y^{(\beta)}(t+1, s+1, z)} = \beta \frac{L_U^{(\alpha)}(t+1, s, z)}{L_U^{(\alpha)}(t+1, s+1, z)} + \alpha \frac{L_V^{(\beta)}(s+1, t, z)}{L_V^{(\beta)}(s+1, t+1, z)}.$$

Subtracting side-wise (23) (multiplied by $\alpha\beta$) from (24), in view of (13), we get

$$(25) \quad \beta M_X(s, t, z) + \alpha M_Y(t, s, z) = \beta M_U(t, s, z) + \alpha M_V(s, t, z),$$

where

$$M_W(s, t, z) = \frac{L_W^{(\gamma)}(s+1, t, z)L_W^{(\gamma)}(s, t+1, z)}{L_W^{(\gamma)}(s, t, z)L_W^{(\gamma)}(s+1, t+1, z)}.$$

Note also that (20) implies

$$(26) \quad M_X(s, t, z)M_Y(t, s, z) = M_U(t, s, z)M_V(s, t, z).$$

Combining (25) with (26) we get

$$(27) \quad (\beta M_X(s, t, z) - \alpha M_V(s, t, z))(M_X(s, t, z) - M_U(t, s, z)) = 0$$

$$(28) \quad (\beta M_U(t, s, z) - \alpha M_Y(t, s, z))(M_V(s, t, z) - M_Y(t, s, z)) = 0.$$

Since M_X, M_Y, M_U, M_V all extend uniquely to meromorphic functions in a common domain in \mathbb{C}^3 it follows from (27) that either $\beta M_X \equiv \alpha M_V$ or $M_X \equiv M_U$ and from (28) that either $\beta M_U \equiv \alpha M_Y$ or $M_V \equiv M_Y$.

In Section 4.2 we will prove that $\beta M_X \equiv \alpha M_V$ is impossible. It will follow by symmetry that also $\beta M_U \equiv \alpha M_Y$ is impossible. Then, in Section 4.3 we will consider the case $M_X \equiv M_U$. The case $M_V \equiv M_Y$ will follow by the analogous approach.

4.2. The case $\beta M_X \equiv \alpha M_V$ is impossible. Assume

$$(29) \quad \beta M_X(s, t, z) = \alpha M_V(s, t, z), \quad s, t \geq 0, \quad z > 0.$$

Define

$$(30) \quad A(s, t, z) := \frac{L_X(s+1, t)L_V(s, t)}{L_X(s, t)L_V(s+1, t)}$$

and

$$(31) \quad C(s, t, z) := \frac{L_X(s, t)L_V(s, t+1)}{L_X(s, t+1)L_V(s, t)},$$

where we suppressed the argument z and superscripts (α) and (β) in L_X and L_V , respectively. Note that (29) implies

$$A(s, t+1, z) = \frac{\beta}{\alpha} A(s, t, z) \quad \text{and} \quad C(s+1, t, z) = \frac{\alpha}{\beta} C(s, t, z), \quad s, t \in \mathbb{N}.$$

Consequently,

$$A(s, t, z) = \left(\frac{\beta}{\alpha}\right)^t a(s, z), \quad \text{and} \quad C(s, t, z) = \left(\frac{\alpha}{\beta}\right)^s c(t, z),$$

where $a(s, z) = A(s, 0, z)$ and $c(t, z) = C(0, t, z)$.

Note that (29) implies also

$$\alpha A(s, t, z) \frac{L_X(s, t)}{L_X(s+1, t+1)} = \beta C(s, t, z) \frac{L_V(s, t)}{L_V(s+1, t+1)}.$$

Consequently,

$$h(s, t, z) := \frac{a(s, z)}{c(t, z)} = \left(\frac{\alpha}{\beta}\right)^{t-s-1} \frac{L_X(s, t)L_V(s+1, t+1)}{L_X(s+1, t+1)L_V(s, t)}.$$

Then

$$\left(\frac{\alpha}{\beta}\right)^{s-t+1} \frac{\partial h(s, t, z)}{\partial z} = \frac{\text{Num}}{[L_X(s+1, t+1)L_V(s, t)]^2},$$

where the numerator

$$\begin{aligned} \text{Num} &= - [L_X(s+1, t)L_V(s+1, t+1) + L_X(s, t)L_V(s+2, t+1)] L_X(s+1, t+1)L_V(s, t) \\ &\quad + L_X(s, t)L_V(s+1, t+1) [L_X(s+2, t+1)L_V(s, t) + L_X(s+1, t+1)L_V(s+1, t)] \\ &= L_X(s, t)L_V(s, t) [L_X(s+2, t+1)L_V(s+1, t+1) - L_X(s+1, t+1)L_V(s+2, t+1)] \\ &\quad + L_X(s, t)L_X(s+1, t+1)L_V(s+1, t)L_V(s+1, t+1) \\ &\quad - L_X(s+1, t)L_X(s+1, t+1)L_V(s, t)L_V(s+1, t+1) = I_1 + I_2 - I_3. \end{aligned}$$

Note that the last two summands in the above expression can be rewritten with the help of (13) as follows:

$$I_2 = \frac{1}{\beta} L_X(s, t)L_X(s+1, t+1)L_V(s+1, t) [L_V(s, t) - L_V(s, t+1)]$$

and

$$I_3 = \frac{1}{\alpha} L_X(s+1, t) [L_X(s, t) - L_X(s, t+1)] L_V(s, t)L_V(s+1, t+1).$$

Consequently,

$$\begin{aligned} I_1 + I_2 - I_3 &= L_X(s, t)L_V(s, t) \left\{ \frac{1}{\alpha} L_V(s+1, t+1) [\alpha L_X(s+2, t+1) - L_X(s+1, t)] \right. \\ &\quad \left. - \frac{1}{\beta} L_X(s+1, t+1) [\beta L_V(s+2, t+1) - L_V(s+1, t)] \right\} \\ &\quad + \frac{1}{\alpha} L_X(s+1, t)L_X(s, t+1)L_V(s, t)L_V(s+1, t+1) \\ &\quad - \frac{1}{\beta} L_X(s, t)L_X(s+1, t+1)L_V(s+1, t)L_V(s, t+1). \end{aligned}$$

Note that (29) implies that the two last terms cancel. Therefore referring again to (13) in the first two expressions above we get

$$\text{Num} = L_X(s, t)L_V(s, t)L_X(s+1, t+1)L_V(s+1, t+1) \frac{\beta-\alpha}{\alpha\beta}$$

whence

$$\frac{\partial h(s, t, z)}{\partial z} = h(s, t, z) \frac{\beta-\alpha}{\alpha\beta},$$

i.e.

$$h(s, t, z) = C(s, t) \exp\left(\frac{\beta-\alpha}{\alpha\beta} z\right),$$

where $C(s, t)$ does not depend on z . Consequently,

$$\frac{a(s, z)}{c(t, z)} = \frac{a(s)}{c(t)} e^{z \frac{\beta-\alpha}{\alpha\beta}},$$

where $a(s) = a(s, 0)$ and $c(t) = c(0, t)$.

Therefore, for $s = t = 0$ we obtain

$$(32) \quad e^{-\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)z} \int_{(0, \infty)^2} \frac{x}{1+\alpha x} e^{-z(x+v)} \mathbb{P}_X(dx) \mathbb{P}_V(dv) = \frac{\beta c(0)}{\alpha a(0)} \int_{(0, \infty)^2} \frac{v}{1+\beta v} e^{-z(x+v)} \mathbb{P}_X(dx) \mathbb{P}_V(dv).$$

Note that the support of $\tilde{X} + V$, where $\mathbb{P}_{\tilde{X}}(dx) \propto \frac{x}{1+\alpha x} \mathbb{P}_X(dx)$ for \tilde{X} and V independent is the same as the support \mathfrak{W} of $X' + V'$, where X' and V' are independent, $X' \stackrel{d}{=} X$ and $V' \stackrel{d}{=} V$. Similarly, the support of $X + \tilde{V}$, where $\mathbb{P}_{\tilde{V}}(dv) \propto \frac{v}{1+\beta v} \mathbb{P}_V(dv)$, is \mathfrak{W} . But (32) implies that $\mathfrak{W} + \frac{1}{\alpha} - \frac{1}{\beta} = \mathfrak{W}$ which is impossible since $\inf \mathfrak{W} \geq 0$ and $\alpha \neq \beta$.

4.3. The case of $M_X \equiv M_U$ and functions a, b and f . We consider the equation

$$(33) \quad M_X(s, t, z) = M_U(t, s, z), \quad s, t \in \{0, 1, \dots\}, \quad z > 0.$$

Denote

$$(34) \quad A(s, t, z) := \frac{L_X(s+1, t)L_U(t, s)}{L_X(s, t)L_U(t, s+1)},$$

and

$$(35) \quad B(t, s, z) := \frac{L_U(t+1, s)L_X(s, t)}{L_U(t, s)L_X(s, t+1)},$$

where we skipped the superscript (α) and the argument z in L_X and L_U .

Note that (33) implies that for all $s, t \in \mathbb{N} = \{0, 1, \dots\}$ we have

$$A(s, t, z) = A(s, t+1, z) \quad \text{and} \quad B(t, s, z) = B(t, s+1, z).$$

Consequently, for $(s, t) \in \mathbb{N}^2$ we have $A(s, t, z) = A(s, 0, z) =: A(s, z)$ and $B(t, s, z) = B(t, 0, z) =: B(t, z)$.

Now (33) can be written as

$$(36) \quad A(s, z) \frac{L_X(s, t)}{L_X(s+1, t+1)} = B(t, z) \frac{L_U(t, s)}{L_U(t+1, s+1)}.$$

Consider now the quotient

$$\frac{A(s, z)}{B(t, z)} = \frac{L_X(s+1, t+1)L_U(t, s)}{L_X(s, t)L_U(t+1, s+1)}.$$

Then the numerator Num of the derivative

$$\frac{\partial}{\partial z} \frac{A(s, z)}{B(t, z)}$$

assumes the form

$$\begin{aligned} \text{Num} &= [L_X(s+2, t+1)L_U(t, s) + L_X(s+1, t+1)L_U(t+1, s)]L_X(s, t)L_U(t+1, s+1) \\ &\quad - L_X(s+1, t+1)L_U(t, s)[L_X(s+1, t)L_U(t+1, s+1) + L_X(s, t)L_U(t+2, s+1)] \\ &= L_X(s, t)L_U(t, s)[L_X(s+2, t+1)L_U(t+1, s+1) - L_X(s+1, t+1)L_U(t+2, s+1)] \\ &\quad + L_X(s, t)L_X(s+1, t+1)L_U(t+1, s)\frac{1}{\alpha}[L_U(t, s) - L_U(t, s+1)] \\ &\quad - L_X(s+1, t)\frac{1}{\alpha}[L_X(s, t) - L_X(s, t+1)]L_U(t, s)L_U(t+1, s+1), \end{aligned}$$

where we twice used (13). Referring again to (33), after cancellation, we get

$$\begin{aligned} \frac{\alpha \text{Num}}{L_X(s, t)L_U(t, s)} &= L_U(t+1, s+1)[\alpha L_X(s+2, t+1) - L_X(s+1, t)] \\ &\quad - L_X(s+1, t+1)[\alpha L_U(t+2, s+1) - L_U(t+1, s)]. \end{aligned}$$

Note that (13) applied to the expressions in square brackets above gives $L_X(s+1, t+1)$ for the first square bracket and $L_U(t+1, s+1)$ for the second. Consequently, $\text{Num} = 0$ and thus $\frac{A(s, z)}{B(t, z)} = \frac{a(s)}{b(t)}$, where $a(s) := A(s, 0)$ and $b(t) := B(t, 0)$. Consequently, we have the representations:

$$(37) \quad A(s, z) = f(z)a(s) \quad \text{and} \quad B(t, z) = f(z)b(t), \quad z > 0, \quad s, t \in \mathbb{N},$$

where $f = \frac{A(0, z)}{a(0)} = \frac{B(0, z)}{b(0)}$.

Note that (36) can be rewritten as

$$(38) \quad a(s) \frac{L_X(s, t, z)}{L_X(s+1, t+1, z)} = b(t) \frac{L_U(t, s, z)}{L_U(t+1, s+1, z)}.$$

4.4. Computing a , b and f . Taking logarithms of (34) sidwise and differentiating with respect to z , in view of (14), we obtain

$$\frac{f'}{f} = -\frac{L_X(s+2, t)}{L_X(s+1, t)} + \frac{L_X(s+1, t)}{L_X(s, t)} - \frac{L_U(t+1, s)}{L_U(t, s)} + \frac{L_U(t+1, s+1)}{L_U(t, s+1)}.$$

Note that

$$\frac{\partial^2}{\partial z^2} \log L_X(s, t) = \frac{L_X(s+2, t)}{L_X(s, t)} - \left(\frac{L_X(s+1, t)}{L_X(s, t)} \right)^2.$$

Using the above formula, the identity (14) and recalling the definition of M_U we finally get

$$\frac{f'}{f} \frac{\partial}{\partial z} \log L_X(s, t) = \frac{\partial^2}{\partial z^2} \log L_X(s, t) + \frac{L_X(s+1, t)L_U(t+1, s)}{L_X(s, t)L_U(t, s)} (1 - M_U^{-1}(t, s, z)).$$

Starting with (35), in a similar way, we obtain the analogue of the above

$$\frac{f'}{f} \frac{\partial}{\partial z} \log L_U(t, s) = \frac{\partial^2}{\partial z^2} \log L_U(t, s) + \frac{L_U(t+1, s)L_X(s+1, t)}{L_U(t, s)L_X(s, t)} (1 - M_X^{-1}(s, t, z)).$$

Subtracting the last two equalities sidwise we obtain

$$\frac{f'}{f} = \frac{g'}{g}, \quad \text{where} \quad g = \frac{\partial}{\partial z} \log \frac{L_U(t, s)}{L_X(s, t)}.$$

Consequently,

$$(39) \quad K(s, t)f = \frac{\partial}{\partial z} \log \frac{L_U(t, s)}{L_X(s, t)} = \frac{L_X(s+1, t)}{L_X(s, t)} - \frac{L_U(t+1, s)}{L_U(t, s)}$$

for some function K which does not depend on z . Referring to (34) and (35) again we get

$$K(s, t) = a(s) \frac{L_U(t, s+1)}{L_U(t, s)} - b(t) \frac{L_X(s, t+1)}{L_X(s, t)}.$$

Now, (13) applied to $L_U(t, s+1)$ and $L_X(s, t+1)$ gives

$$K(s, t) = a(s) - b(t) + \alpha \left(a(s) \frac{L_U(t+1, s+1)}{L_U(t, s)} - b(t) \frac{L_X(s+1, t+1)}{L_X(s, t)} \right).$$

Referring to (38) we see that the expression in parenthesis above is zero, whence $K(s, t) = a(s) - b(t)$.

Now we write (39) for $s + 1$ and t , which gives

$$K(s + 1, t)f = \frac{\partial}{\partial z} \log \frac{L_U(t, s+1)}{L_X(s+1, t)}.$$

Subtracting sidewise this equality from (39) we get

$$(a(s) - a(s + 1))f = \frac{\partial}{\partial z} \log \frac{L_X(s+1, t)L_U(t, s)}{L_X(s, t)L_U(t, s+1)} = \frac{d \log f}{dz}.$$

Similarly, using (39) with s and $t + 1$ we get

$$(b(t) - b(t + 1))f = \frac{d \log f}{dz}.$$

Consequently $a(s) - a(s + 1) = b(t) - b(t + 1) =: \kappa \in \mathbb{R}$ and $a(s) = \kappa s + \tilde{a}$ and $b(t) = \kappa t + \tilde{b}$ for some constants $\tilde{a}, \tilde{b} \in \mathbb{R}$. Thus

$$-\kappa f = \frac{d \log f}{dz}.$$

whence $f(z) = (\kappa z + C)^{-1}$, where $C \in \mathbb{R}$ is a constant.

In case $\kappa \neq 0$ we have

$$A(s, z) = \frac{a+s}{c+z} \quad \text{and} \quad B(t, z) = \frac{b+t}{c+z},$$

where $a := \tilde{a}/\kappa$, $b := \tilde{b}/\kappa$ and $c = C/\kappa$. Since $A(s, z)$ and $B(t, z)$ are strictly positive for all $s \geq 0$, $t \geq 0$ and $z > 0$, we conclude that $a, b > 0$ and $c \geq 0$.

4.5. The Kummer ode and identification (up to parameters) of L_X , L_U , L_Y and L_V . Note that (39) can be rewritten as

$$\frac{K(s, t)}{z+c} = \frac{\partial}{\partial z} \log \frac{L_U(t, s)}{L_X(s, t)}, \quad z > 0.$$

Consequently,

$$(z + c)^{K(s, t)} = c(s, t) \frac{L_U(t, s)}{L_X(s, t)},$$

where $c(s, t)$ does not depend on z .

Rewrite the above as

$$(40) \quad (z + c)^{K(s, t)} L_X(s, t) = c(s, t) L_U(t, s)$$

and for $s + 1$ and t as

$$(41) \quad (z + c)^{K(s+1, t)} L_X(s + 1, t) = c(s + 1, t) L_U(t, s + 1).$$

Since $K(s + 1, t) - K(s, t) = 1$, dividing (41) by (40) and referring to (34) we get

$$(42) \quad \frac{c(s+1, t)}{c(s, t)} = \frac{L_X(s+1, t)L_U(t, s)}{L_X(s, t)L_U(t, s+1)}(z + s) = a(s).$$

Now we differentiate (40) with respect to z and get

$$(43) \quad K(s, t)(z + c)^{K(s, t)-1} L_X(s, t) - (z + c)^{K(s, t)} L_X(s + 1, t) = -c(s, t) L_U(t + 1, s).$$

For $L_X(s, t)$ and $L_X(s + 1, t)$ in (43) insert proper expression from (40) and (41), respectively. Together with (42) it yields

$$a(s) L_U(t, s + 1) - K(s, t) L_U(t, s) = (c + z) L_U(t + 1, s).$$

Regarding $L_U(t, s)$ and $L_U(t + 1, s)$ as the right hand side of identities (14) with suitable s and t we get

$$\alpha(c + z) L_U(t + 2, s + 1) + (c + z + \alpha K(s, t)) L_U(t + 1, s + 1) - b(t) L_U(t, s + 1) = 0.$$

The above equation transforms to the second order differential equation for the function $h := L_U(t, s + 1)$ as follows

$$\alpha(c + z) h''(z) + (\alpha b(t) - a(s)) - (c + z) h'(z) - b(t) h(z) = 0.$$

Consequently, for g defined by $g(z) = h(\alpha z - c)$ we get the Kummer equation

$$(44) \quad z g''(z) + (b(t) - a(s) - z) g'(z) - b(t) g(z) = 0.$$

It is well known that the general solution is of the form

$$g(z) = c_1 M(b(t), b(t) - a(s), z) + c_2 U(b(t), b(t) - a(s), z),$$

where (see 13.2.1 in [1]), for $b > a$,

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 \frac{t^{a-1}}{(1-t)^{a-b+1}} e^{zt} dt.$$

Recall that $M(a, b, z)$ is unbounded when $z \rightarrow \infty$ (see e.g. 13.1.4 in [1]) and $U(a, b, z) \rightarrow 0$ as $z \rightarrow \infty$. Since g , as a Laplace transform of a probability measure, is bounded, we necessarily have

$$g(z) = c_U(s, t)U(b(t), b(t) - a(s), z).$$

Returning to $L_U(t, s)$ (recall that g was defined through $L_U(t, s + 1)$) we get

$$L_U(t, s, z) = c_U(s, t)U(b + t, b + t - a - s + 1, \frac{c+z}{\alpha}),$$

with $a, b > 0$ and $c \geq 0$.

Changing the roles of L_X and L_U in the above argument starting with (40) we obtain

$$L_X(s, t, z) = c_X(s, t)U(a + s, a + s - b - t + 1, \frac{c+z}{\alpha}).$$

Assume that $c = 0$. Recalling (16) we see that: (1) if $a \neq b$ then either $U(a, a - b + 1, 0) = \infty$ or $U(b, b - a + 1, 0) = \infty$; (2) if $a = b$ then $U(a, 1, 0) = U(1, 1 - a, 0) = \infty$. Since $L_X(0, 0, 0) = L_U(0, 0, 0) = 1$ we obtain thus a contradiction. Therefore $c > 0$ and Proposition 3.2 implies that $X \sim \mathcal{K}_\alpha(a, b, c)$ and $U \sim \mathcal{K}_\alpha(b, a, c)$.

In case $\kappa = 0$ we have $f(z) = C \neq 0$ and $A(s, z) = a > 0$ and $B(t, z) = b > 0$ where $a = \tilde{a}C$ and $b = \tilde{b}/C$. We now show that this is impossible. Indeed, (39) then yields

$$a - b = \frac{L_X(s+1, t)}{L_X(s, t)} - \frac{L_U(t+1, s)}{L_U(t, s)}.$$

Combining this with (35) we get

$$(a - b)L_X(s, t) = L_X(s + 1, t) - bL_X(s, t + 1).$$

For $s = t = 0$ we thus get

$$\mathbb{E}((X - a + b)e^{-zX}) = b\mathbb{E}\frac{1}{1+\alpha X} e^{-zX}.$$

Consequently, $(x - a + b)P_X(dx) = \frac{b}{1+\alpha x} \mathbb{P}_X(dx)$. Equivalently,

$$\frac{(1+\alpha x)(x-a+b)}{b} \mathbb{P}_X(dx) = \mathbb{P}_X(dx).$$

Since $(1 + \alpha x)(x - a + b) = b$ is equivalent to $\alpha x^2 + (\alpha(b - a) + 1)x - a = 0$, so its roots have different signs. Since X is nonnegative this would imply that its support degenerates to a point, which contradicts our assumptions.

4.6. Identifying the parameters. We have proved that $X \sim \mathcal{K}_\alpha(a, b, c)$, $U \sim \mathcal{K}_\alpha(b, a, c)$, $Y \sim \mathcal{K}_\beta(\tilde{a}, \tilde{b}, \tilde{c})$ and $V \sim \mathcal{K}_\beta(\tilde{a}, \tilde{b}, \tilde{c})$ for some $a, b, c, \tilde{a}, \tilde{b}, \tilde{c} > 0$. Using (15) for each of the variables X, Y, U, V , equation (20) reads:

$$\begin{aligned} & \frac{\Gamma(a+s)U\left(a+s, a+s-b-t+1, \frac{c+z}{\alpha}\right)}{\alpha^s \Gamma(a)U\left(a, a-b+1, \frac{c}{\alpha}\right)} \times \frac{\Gamma(\tilde{a}+t)U\left(\tilde{a}+t, \tilde{a}+t-\tilde{b}-s+1, \frac{\tilde{c}+z}{\beta}\right)}{\beta^t \Gamma(\tilde{a})U\left(\tilde{a}, \tilde{a}-\tilde{b}+1, \frac{\tilde{c}}{\beta}\right)} \\ &= \frac{\Gamma(b+t)U\left(b+t, b+t-a-s+1, \frac{c+z}{\alpha}\right)}{\alpha^t \Gamma(b)U\left(b, b-a+1, \frac{c}{\alpha}\right)} \times \frac{\Gamma(\tilde{b}+s)U\left(\tilde{b}+s, \tilde{b}+s-\tilde{a}-t+1, \frac{\tilde{c}+z}{\beta}\right)}{\beta^s \Gamma(\tilde{b})U\left(\tilde{b}, \tilde{b}-\tilde{a}+1, \frac{\tilde{c}}{\beta}\right)} \end{aligned}$$

which, by applying identity (22) to the left-hand side, gives

$$\begin{aligned} & \frac{\Gamma(a+s) \left(\frac{c+z}{\alpha}\right)^{b+t-a-s} U\left(b+t, b+t-a-s+1, \frac{c+z}{\alpha}\right)}{\alpha^s \Gamma(a) \left(\frac{c}{\alpha}\right)^{b-a} U\left(b, b-a+1, \frac{c}{\alpha}\right)} \times \frac{\Gamma(\tilde{a}+t) \left(\frac{\tilde{c}+z}{\beta}\right)^{\tilde{b}+s-\tilde{a}-t} U\left(\tilde{b}+s, \tilde{b}+s-\tilde{a}-t+1, \frac{\tilde{c}+z}{\beta}\right)}{\beta^t \Gamma(\tilde{a}) \left(\frac{\tilde{c}}{\beta}\right)^{\tilde{b}-\tilde{a}} U\left(\tilde{b}, \tilde{b}-\tilde{a}+1, \frac{\tilde{c}}{\beta}\right)} \\ &= \frac{\Gamma(b+t) U\left(b+t, b+t-a-s+1, \frac{c+z}{\alpha}\right)}{\alpha^t \Gamma(b) U\left(b, b-a+1, \frac{c}{\alpha}\right)} \times \frac{\Gamma(\tilde{b}+s) U\left(\tilde{b}+s, \tilde{b}+s-\tilde{a}-t+1, \frac{\tilde{c}+z}{\beta}\right)}{\beta^s \Gamma(\tilde{b}) U\left(\tilde{b}, \tilde{b}-\tilde{a}+1, \frac{\tilde{c}}{\beta}\right)}. \end{aligned}$$

After cancellations we obtain

$$(45) \quad \frac{(c+z)^{b+t-a-s} (\tilde{c}+z)^{\tilde{b}+s-\tilde{a}-t}}{c^{b-a} \tilde{c}^{\tilde{b}-\tilde{a}}} \times \frac{\Gamma(a+s)}{\Gamma(a)} \times \frac{\Gamma(\tilde{a}+t)}{\Gamma(\tilde{a})} = \frac{\Gamma(b+t)}{\Gamma(b)} \times \frac{\Gamma(\tilde{b}+s)}{\Gamma(\tilde{b})}.$$

Taking the logarithm and differentiating in z gives

$$\frac{b+t-a-s}{c+z} + \frac{\tilde{b}+s-\tilde{a}-t}{\tilde{c}+z} = 0.$$

Since this holds for any $z > 0$, we conclude that $c = \tilde{c}$ and $b - a = \tilde{b} - \tilde{a}$. Returning to (45) we have

$$\frac{\Gamma(a+s)}{\Gamma(a)} \times \frac{\Gamma(\tilde{a}+t)}{\Gamma(\tilde{a})} = \frac{\Gamma(b+t)}{\Gamma(b)} \times \frac{\Gamma(\tilde{b}+s)}{\Gamma(\tilde{b})},$$

which, applied to $s = t = 1$, yields $a\tilde{a} = b\tilde{b}$. Since we also had $b - a = \tilde{b} - \tilde{a}$, we get $a = \tilde{a}$ and $b = \tilde{b}$.

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