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► **To cite this version:**

Andrew Elvey Price. A Cayley graph for  $F_2 \times F_2$  which is not minimally almost convex. *International Journal of Algebra and Computation*, 2022, 32 (1), pp.115-126. 10.1142/S0218196722500059 . hal-03886498

**HAL Id: hal-03886498**

**<https://hal.science/hal-03886498>**

Submitted on 6 Dec 2022

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# A Cayley graph for $F_2 \times F_2$ which is not minimally almost convex.

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## Abstract

mkmk We give an example of a Cayley graph  $\Gamma$  for the group  $F_2 \times F_2$  which is not minimally almost convex (MAC). On the other hand, the standard Cayley graph for  $F_2 \times F_2$  does satisfy the falsification by fellow traveler property (FFTP), which is strictly stronger. As a result, any Cayley graph property  $K$  lying between FFTP and MAC (i.e.,  $\text{FFTP} \Rightarrow K \Rightarrow \text{MAC}$ ) is dependent on the generating set. This includes the well known properties FFTP and almost convexity, which were already known to depend on the generating set as well as Poénaru's condition  $P(2)$  and the basepoint loop shortening property for which dependence on the generating set was previously unknown. We also show that the Cayley graph  $\Gamma$  does not have the loop shortening property, so this property also depends on the generating set.

## 1 Introduction

Due to the unsolvability of even the word problem on general Cayley graphs or finitely generated groups, numerous classes of Cayley graphs have been considered in the literature which simple sounding problems like this can be easily solved. Typically these classes are defined by geometric properties of the Cayley graphs, such as all triangles being thin in the case of hyperbolic-groups. Whenever such a new class is defined a natural question arises: Is the Cayley graph property independent of the generating set? In this article we use the group  $F_2 \times F_2$  to show that the answer is no for the falsification by fellow traveller property (FFTP), the loop shortening property and every non-trivial almost-convexity condition. In order to prove this, we consider two different presentations for the group  $F_2 \times F_2$ , whose Cayley graphs exhibit quite different geometry. First we consider the standard presentation

$$(G, S_1) = \langle x, y, c, d \mid xc = cx, yc = cy, xd = dx, yd = dy \rangle.$$

Note that this is a right angled Artin group (with standard presentation), so by [5, Theorem 3.1], the pair  $(G, S_1)$  satisfies the falsification by fellow traveller property. The other presentation that we will consider is

$$(G, S_2) = \langle a, b, c, d \mid ac = ca, bc = cb, acd = dac, bcd = dbc \rangle,$$

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which can be seen to define the same group due to the Tietze transformations  $y = bc$  and  $x = ac$ . Our main theorem is that the pair  $(G, S_2)$  is not minimally almost convex. As a corollary, the following properties all depend on the generating set:

- The falsification by fellow traveler property
- The basepoint loop shortening property
- Every non-trivial almost convexity condition. This includes almost convexity, Poénaru's condition  $P(2)$ , minimal almost convexity (MAC) and the slightly stronger condition M'AC.

In the final section we show that  $(G, S_2)$  does not satisfy the loop shortening property, so this property also depends on the generating set.

There are many known examples of groups which satisfy FFTP with respect to one generating set but not others. For example, Neumann and Shapiro gave an example in [7] when they introduced the property, and Elder gave another example in [3]. To the author's knowledge, the only previous example for almost convexity was given by Theil [9]. Elder and Hermiller used solvable Baumslag-Solitar groups to show that minimal almost convexity is not a quasi-isometry invariant, nor is M'AC [4], so it is not surprising that these both depend on the generating set. It was previously unknown, however, whether Poénaru's condition  $P(2)$  or either of the loop shortening properties were dependent on the generating set.

## 2 Almost convexity conditions

The properties considered in this article require us to interpret the Cayley graph  $\Gamma$  as a metric space, which we denote  $\tilde{\Gamma}$ . Edges are then considered to be segments of length 1 between the vertices that they join and the points in this metric space are the vertices of  $\Gamma$  as well the interior points of the edges. Assuming  $\Gamma$  is a connected graph, any two points  $u, v$  in  $\tilde{\Gamma}$  are joined by at least one continuous path, so the distance  $d(u, v)$  between  $u$  and  $v$  is defined to be the shortest such path. This is sometimes called the *path metric*.

Since the elements of the underlying group  $G$  are precisely the vertices of  $\Gamma$ , these are also points in the metric space  $\tilde{\Gamma}$ . In particular the identity 1 of  $G$  is a point in  $\tilde{\Gamma}$ .

In the following, we define the *sphere*  $Sph(r)$  and the *ball*  $\overline{B(r)}$  of radius  $r$  around 1 in the following standard way:

$$Sph(r) = \{z \in \tilde{\Gamma} : d(z, 1) = r\} \quad \text{and} \quad \overline{B(r)} = \{z \in \tilde{\Gamma} : d(z, 1) \leq r\}.$$

We are now ready to define *almost convexity*.

**Definition 2.1.** Let  $G$  be a group and let  $S$  be a finite generating set for  $G$ . Let  $r_0 \in \mathbb{N}$  be a positive integer and let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a function. The pair  $(G, S)$  is said to satisfy the almost convexity condition  $AC_{f, r_0}$  if the following holds: If  $r > r_0$  and  $u, v \in Sph(r)$  satisfy  $d(u, v) \leq 2$ , then there is some path  $p$  between  $u$  and  $v$  such that  $p$  is contained in  $\overline{B(r)}$  and  $p$  has length at most  $f(r)$ .

The pair  $(G, S)$  is said to satisfy

- *Almost Convexity (AC)* if it satisfies  $AC_{f,r_0}$  for some constant function  $f$  and some  $r_0 > 0$ ,
- *Poénaru's condition P(2)* if it satisfies  $AC_{f,r_0}$  for some sub-linear function  $f$  and some  $r_0 > 0$ ,
- *Minimal Almost Convexity (MAC)* if it satisfies  $AC_{f,r_0}$  for some  $r_0 > 0$ , where  $f(x) = 2x - 1$ ,
- *M'AC* if it satisfies  $AC_{f,r_0}$  for some  $r_0 > 0$ , where  $f(x) = 2x - 2$ .

Using this definition, almost convexity (AC) imposes the strongest possible restriction on  $f$ , namely that  $f$  is a constant function.

Note that the points  $u$  and  $v$  in the definition above are joined via the origin by a path of length  $2r$ , so every finitely generated group trivially satisfies any almost convexity condition where  $f(r) \geq 2r$ . Hence, the requirement that  $f(r) = 2r - 1$  for MAC is the weakest possible non-trivial almost convexity condition. Due to the successive strengthening of these conditions, we have the implication chain

$$AC \Rightarrow P(2) \Rightarrow M'AC \Rightarrow MAC.$$

The last two conditions M'AC and MAC were introduced by I. Kapovich in [6], where they are called  $K(2)$  and  $K'(2)$ , respectively. These conditions are discussed in greater detail in [4], where Elder and Hermiller show that M'AC does not imply  $P(2)$ . It is still an open question, however, whether the properties M'AC and MAC are equivalent. The property AC was introduced by Cannon in [1], and remains the most widely studied almost convexity condition. [references??](#)

### 3 The pair $(G, S_2)$ does not satisfy MAC

The main theorem in this section is that the group  $(G, S_2)$  with presentation

$$\langle a, b, c, t \mid ac = ca, bc = cb, act = tac, bct = tbc \rangle$$

does not satisfy MAC. As a consequence it does not satisfy AC,  $P(2)$  or M'AC.

We start by interpreting  $(G, S_2)$  as a HNN extension, as this will allow us to understand the paths in the Cayley graph using Britton's Lemma, which we state below. Consider the group  $P = F_2 \times C_\infty = \langle a, b, c \mid ac = ca, bc = cb \rangle$  and let  $H$  be the subgroup generated by  $ac$  and  $bc$ . Then  $G$  is the HNN extension  $P*_\phi$  where  $\phi : H \rightarrow H$  is the identity.

**Lemma 3.1.** (*Britton's Lemma*) Assume  $w \in G$  is given by the product

$$w = p_0 t^{\epsilon_1} p_1 t^{\epsilon_2} \cdots t^{\epsilon_n} p_n,$$

where  $n \geq 1$ , each  $p_j \in P$  and each  $\epsilon_j = \pm 1$ . Assume further that there is no  $1 \leq j \leq n$  satisfying both  $p_j \in H$  and  $\epsilon_j = -\epsilon_{j+1}$ . Then  $w \neq 1$ .

Note that if some  $j$  satisfied both  $p_j \in H$  and  $\epsilon_j = -\epsilon_{j+1}$ , then the word defining  $w$  could be reduced, as we would have  $t^{\epsilon_j} p_j t^{\epsilon_{j+1}} = p_j$ . Hence Britton's Lemma gives a simple way to determine whether a word represents the identity in  $G$ .

In order to use Britton's Lemma to understand the metric  $d$  in  $\tilde{\Gamma}$ , we first need to understand the subgroup  $H$  of  $P$ , which we do in the following Lemma:

**Lemma 3.2.**  $H$  is the subgroup of  $G$  consisting of all elements  $wc^k$ , where  $w$  is a word in  $\{a, b, a^{-1}, b^{-1}\}^*$  and  $k$  is the sum of the exponents in  $w$ .

*Proof.* Let  $w$  be a word in  $\{a, b, a^{-1}, b^{-1}\}^*$  and let  $k$  be the sum of the exponents in  $w$ . We will show that  $wc^k \in H$ . Let  $w = s_1^{p_1} s_2^{p_2} \dots s_n^{p_n}$ , where each  $s_i \in \{a, b\}$  and each  $p_i \in \{-1, 1\}$ . Then

$$wc^k = wc^{p_1 + \dots + p_n} = (s_1 c)^{p_1} (s_2 c)^{p_2} \dots (s_n c)^{p_n} \in H.$$

Now let  $h \in H$ . We will show that  $h = wc^k$  for some word  $w \in \{a, b, a^{-1}, b^{-1}\}^*$ , where  $k$  is the sum of the exponents in  $w$ . Since  $h \in H = \langle ac, bc \rangle$ , we can write

$$h = (s_1 c)^{p_1} (s_2 c)^{p_2} \dots (s_n c)^{p_n},$$

where each  $s_i \in \{a, b\}$  and each  $p_i \in \{-1, 1\}$ . Then we have

$$h = s_1^{p_1} s_2^{p_2} \dots s_n^{p_n} c^{p_1 + p_2 + \dots + p_n} = wc^k,$$

where  $w = s_1^{p_1} s_2^{p_2} \dots s_n^{p_n}$  and  $k$  is the sum of the exponents in  $w$ .  $\square$

Before proving the main theorem, we will prove some Lemmas regarding distances in the metric space  $\tilde{\Gamma}$ :

**Lemma 3.3.** Let  $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{-1, 1\}$ , let  $w$  be a word of length  $n$  on the alphabet  $\{a^{\epsilon_1}, b^{\epsilon_2}, c^{\epsilon_3}, t^{\epsilon_4}\}$ , and let  $\bar{w} \in G$  be the element represented by  $w$ . Then  $d(1, \bar{w}) = n$ .

*Proof.* Since the word  $w$  itself determines a path of length  $n$  between 1 and  $\bar{w}$ , it is clear that  $d(1, \bar{w}) \leq n$ .

To prove a lower bound on  $d(1, \bar{w})$ , we consider a homomorphism  $h : G \rightarrow \langle x \rangle$  from  $G$  to the infinite cyclic group  $\langle x \rangle$  defined by  $h(a) = x^{\epsilon_1}$ ,  $h(b) = x^{\epsilon_2}$ ,  $h(c) = x^{\epsilon_3}$  and  $h(t) = x^{\epsilon_4}$ . To check that this is a homomorphism it suffices to verify that it satisfies the relations of  $(G, S_2)$ , that is  $h(ac) = h(ca)$ ,  $h(bc) = h(cb)$ ,  $h(act) = h(tac)$  and  $h(bct) = h(tbc)$ . Indeed these all follow from the fact that  $\langle x \rangle$  is a commutative group. Now, by the definition of this homomorphism,  $h(\bar{w}) = x^n$ .

Now suppose  $w_1 = \alpha_1 \alpha_2 \dots \alpha_m$  is another word representing  $\bar{w}$ , where each  $\alpha_j \in \{a, a^{-1}, b, b^{-1}, c, c^{-1}, t, t^{-1}\}$ . Then by applying  $h$ , we find

$$h(\alpha_1)h(\alpha_2) \dots h(\alpha_m) = x^n.$$

Since each term  $h(\alpha_j) \in \{x, x^{-1}\}$ , we must have  $m \geq n$ . Since this is true for any word representing  $\bar{w}$ , it follows that  $d(1, \bar{w}) \geq n$   $\square$

**Theorem 3.4.** The pair  $(G, S_2)$  does not satisfy MAC.

*Proof.* Consider the elements  $a^n b^{-n}$  and  $ta^n b^{-n+1}$  in  $G$  (See Figure 1). Then

$$|a^n b^{-n}| = 2n = |ta^n b^{-n+1}|$$

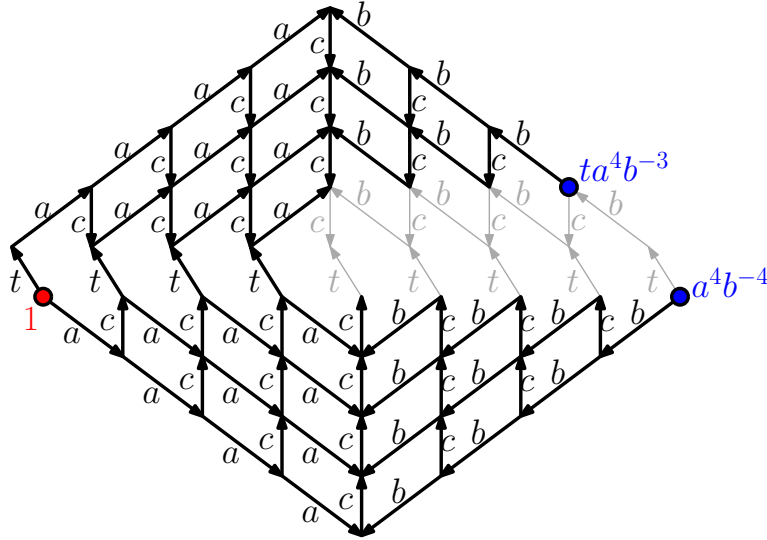


Figure 1: Part of the Cayley graph  $\Gamma(G, S_2)$ . The bold vertices and edges are those within the ball of radius 8.

and

$$d(a^n b^{-n}, ta^n b^{-n+1}) = |b^n a^{-n} ta^n b^{-n+1}| = |tb^n a^{-n} a^n b^{-n+1}| = |tb| = 2.$$

So it suffices to prove that if  $p$  is a path in  $B(2n)$  between  $a^n b^{-n}$  and  $ta^n b^{-n+1}$ , then  $p$  has length at least  $4n$ .

Let  $p$  be a path in  $B(2n)$  between  $a^n b^{-n}$  and  $ta^n b^{-n+1}$ .

Then let  $h$  and  $ht$  be adjacent vertices in  $p$ , such that  $h$  is in the same sheet of the HNN extension as 1 and  $ht$  is in the same sheet as  $t$ . Then  $h \in H$ . Moreover, since  $p$  is contained in  $B(2n)$ , the length  $|h| + 1 = |ht| \leq 2n$ , so  $|h| \leq 2n - 1$ . Let  $h = wc^k$  where  $w$  is a word in  $\{a, b, a^{-1}, b^{-1}\}^*$  and  $k$  is the sum of the exponents in  $w$ . Then

$$d(a^n b^{-n}, h) = |b^n a^{-n} h| = |b^n a^{-n} wc^k| = |b^n a^{-n} w| + |k|.$$

Let  $x \in \mathbb{Z}_{\geq 0}$  be maximal such that the word  $w$  splits as  $w = a^x w_1$ . In other words,  $x$  is the number of  $a$ 's at the start of  $w$ . Then the sum of the exponents in  $w_1$  is  $k - x$ , so  $|w_1| \geq |k - x|$ . Therefore,

$$2n - 1 \geq |h| = |w| + |k| = x + |w_1| + |k| \geq x + |k - x| + |k| \geq 2x,$$

so  $x < n$ . Therefore,  $b^n a^{x-n} w_1$  is freely reduced, so

$$|b^n a^{-n} a^x w_1| = |b^n a^{x-n} w_1| = 2n - x + |w_1|.$$

Therefore,

$$d(a^n b^{-n}, h) = |b^n a^{-n} w| + |k| = 2n - x + |w_1| + |k| \geq 2n - x + x = 2n.$$

Hence, the path  $p$  satisfies

$$|p| \geq d(a^n b^{-n}, h) + 1 + d(ta^n b^{-n}, ht) \geq d(a^n b^{-n}, h) + d(ta^n b^{-n}, ht) = 2d(a^n b^{-n}, h) \geq 4n,$$

as required. So  $(G, S_2)$  does not satisfy MAC.  $\square$

## 4 loop shortening properties

In [2] Elder introduced the loop shortening and basepoint loop shortening properties as a natural generalisation of the falsification by fellow traveller property. Where FFTP gives a simple way to check if a word is a geodesic, each of the loop shortening properties gives a somewhat simple way to check if a word represents the identity in the group.

**Definition 4.1.** Let  $G$  be a group with finite generating set  $S$ .  $(G, S)$  has the (synchronous) loop shortening property (LSP) if there is a constant  $k$  such that for any loop  $v_0, v_1, \dots, v_n$  in  $\Gamma(G, S)$  with  $n \geq 1$ , there is a shorter loop  $u_0, u_1, \dots, u_m$  such that  $d(u_j, v_j) < k$  for each  $j \leq m$ , and  $d(u_m, v_j) < k$  for  $m \leq j \leq n$ . In other words, the paths (synchronously)  $k$ -fellow travel.

**Definition 4.2.** Let  $G$  be a group with finite generating set  $S$ .  $(G, S)$  has the (synchronous) basepoint loop shortening property (BLSP) if there is a constant  $k$  such that for any loop  $v_0, v_1, \dots, v_n$  in  $\Gamma(G, S)$  with  $n \geq 1$ , there is a shorter loop  $(v_0 = u_0), u_1, \dots, u_m$  such that  $d(u_j, v_j) < k$  for each  $j \leq m$ , and  $d(u_m, v_j) < k$  for  $m \leq j \leq n$ . In other words, the paths (synchronously)  $k$ -fellow travel.

Note that the only difference between these two properties is that for the basepoint loop shortening property, the initial loop is around a basepoint which the shorter loop has to pass through, whereas for the loop shortening property no such restriction is imposed. Hence, it is clear that

$$\text{BLSP} \Rightarrow \text{LSP}.$$

Elder also showed that the basepoint loop shortening property is strictly stronger than almost convexity and strictly weaker than the falsification by fellow traveller property (FFTP), so we have the long implication chain

$$\text{FFTP} \Rightarrow \text{BLSP} \Rightarrow \text{AC} \Rightarrow \text{P}(2) \Rightarrow \text{M}'\text{AC} \Rightarrow \text{MAC}.$$

Elder asked two questions about the two loop shortening properties. The first is whether they are equivalent, and this remains an open problem. The second is whether either or both of these properties depend on the generating set. We have already shown that the group  $G = F_2 \times F_2$  satisfies FFTP with respect to one generating set, but fails MAC with another, which implies that BLSP depends on the generating set. Our final theorem settles the other half of this question, namely that the loop shortening property also depends on the generating set.

**Theorem 4.3.** *The group  $(G, S_2)$  with presentation*

$$\langle a, b, c, t \mid ac = ca, bc = cb, act = tac, bct = tbc \rangle$$

*does not satisfy the loop shortening property.*

*Proof.* Let  $k \in \mathbb{Z}_{>0}$ . We will show that there is a loop  $\ell$  in  $\Gamma(G, S_2)$  such that there is no shorter loop  $\ell'$  in  $\Gamma(G, S_2)$  which  $k$ -fellow travels with  $\ell$ . Hence this will show that  $\Gamma(G, S_2)$  does not satisfy the loop shortening property. Let  $\ell$  be the loop given by the word

$$w = a^{2k}b^{-4k}a^{2k}ta^{-2k}b^{4k}a^{-2k}t^{-1}.$$

We can easily check algebraically that  $\bar{w} = 1$ , so this is indeed a loop. Now let  $\ell'$  be a loop which  $k$ -fellow travels with  $\ell$ . Then we just need to show that the length of  $\ell'$  is at least the length of  $\ell$ , which is  $16k + 2$ . Since the four vertices

$$u_1 = a^k, \quad u_2 = a^{2k}b^{-4k}a^k, \quad u_3 = ta^{2k}b^{-4k}a^k \quad \text{and} \quad u_4 = ta^k,$$

appear in  $\ell$  in that order, there must be vertices  $v_1, v_2, v_3$  and  $v_4$  appearing in  $\ell'$  in that order which satisfy  $d(u_i, v_i) \leq k$  for each  $i \in \{1, 2, 3, 4\}$  (See Figure 2). Hence it suffices to prove that

$$d(v_1, v_2) + d(v_2, v_3) + d(v_3, v_4) + d(v_4, v_1) \geq 16k + 2.$$

For each  $i$ , let  $w_i$  be a word of minimal length from  $u_i$  to  $v_i$ . So  $|w_i| \leq k$  and  $\bar{w}_i = u_i^{-1}v_i$ . For each  $i \in \{1, 2, 3, 4\}$ , let  $x_i, y_i, z_i$  be the sums of the powers of  $a, b$  and  $c$ , respectively in the word  $w_i$ . Then we will show the following four inequalities, from which the desired result follows:

$$\begin{aligned} d(v_1, v_2) &\geq 6k - x_1 + x_2 + |y_1| + |y_2| + |z_1 - z_2|, \\ d(v_2, v_3) &\geq 2k + 1 - x_2 - x_3 - y_2 - y_3 + z_2 + z_3, \\ d(v_3, v_4) &\geq 6k - x_4 + x_3 + |y_4| + |y_3| + |z_4 - z_3|, \\ d(v_4, v_1) &\geq 2k + 1 + x_1 + x_4 + y_1 + y_4 - z_1 - z_4. \end{aligned}$$

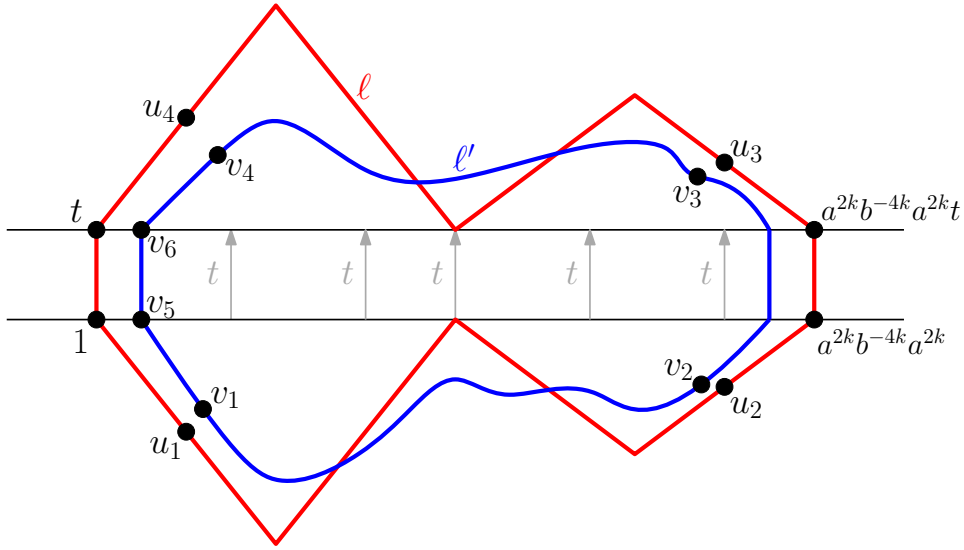


Figure 2: The loop  $\ell$  in red and a  $k$ -fellow travelling loop  $\ell'$  in blue.

Let  $w$  be a word of minimal length from  $v_1$  to  $v_2$ . So

$$\bar{w} = \bar{w}_1^{-1}u_1^{-1}u_2\bar{w}_2 = \overline{w_1^{-1}a^kb^{-4k}a^kw_2}.$$

Consider the quotient map  $f : G \rightarrow G$  defined by  $f(a) = a$ ,  $f(b) = b$ ,  $f(c) = c$  and  $f(t) = 1$ . Note that this is well defined since  $f(ac) = f(ca)$ ,  $f(bc) = f(cb)$ ,  $f(act) = f(tac)$  and  $f(bct) = f(tbc)$ . Since  $c$  commutes with  $a$  and  $b$ , we can write  $f(\bar{w}_1) = c^{z_1}\bar{r}_1$ , where



$r_1$  is a reduced word over the alphabet  $\{a, b, a^{-1}, b^{-1}\}$ . Then  $x_1$  and  $y_1$  are the sums of the powers of  $a$  and  $b$ , respectively, in  $r_1$  and

$$k \geq |w_1| \geq |r_1|.$$

Similarly we can write  $f(\overline{w_2}) = c^{z_2} \overline{r_2}$ . Now,

$$d(v_1, v_2) = |w| \geq d(1, f(\overline{w})) = d(1, f(\overline{w_1})^{-1} a^k b^{-4k} a^k f(\overline{w_2})) = d(1, c^{z_2 - z_1} \overline{r_1}^{-1} a^k b^{-4k} a^k \overline{r_2}).$$

Since  $r_1$  and  $r_2$  each have length at most  $k$ , they can only cancel with letters in  $a^k$  on either side of the last expression, and not the  $b^{-4k}$  term in the middle. In other words, if  $s_1$  is a reduced word for  $\overline{r_1}^{-1} a^k$  and  $s_2$  is a reduced word for  $a^k \overline{r_2}$ , then  $c^{z_2 - z_1} s_1 b^{-4k} s_2$  is a reduced word for  $c^{z_2 - z_1} \overline{r_1}^{-1} a^k b^{-4k} a^k \overline{r_2}$ . Moreover, the sums of the powers of  $a$  in  $s_1$  and  $s_2$  are  $k - x_1$  and  $k + x_2$  respectively, and the sums of the powers of  $b$  in  $s_1$  and  $s_2$  are  $-y_1$  and  $y_2$  respectively. Hence,  $|s_1| \geq k - x_1 + |y_1|$  and  $|s_2| \geq k + x_2 + |y_2|$ . Therefore,

$$\begin{aligned} d(v_1, v_2) &\geq d(1, c^{z_2 - z_1} \overline{r_1}^{-1} a^k b^{-4k} a^k \overline{r_2}) \\ &= |c^{z_2 - z_1} s_1 b^{-4k} s_2| \\ &= 4k + |z_2 - z_1| + |s_1| + |s_2| \\ &\geq 4k + |z_2 - z_1| + k - x_1 + |y_1| + k + x_2 + |y_2| \\ &= 6k - x_1 + x_2 + |y_1| + |y_2| + |z_1 - z_2|. \end{aligned}$$

Similarly,

$$d(v_3, v_4) \geq 6k - x_4 + x_3 + |y_4| + |y_3| + |z_4 - z_3|.$$

Now let  $r$  be a word of minimal length between  $v_2$  and  $v_3$ . Then  $w_2 r w_3^{-1}$  forms a path  $p$  from  $u_2$  to  $u_3$ . Since  $u_2$  is in the sheet of the HNN extension containing 1 and  $u_3$  is in the sheet containing  $t$ , there must be vertices  $v_5$  and  $v_6$  which are adjacent in  $p$  and such that  $v_5$  is in the sheet containing 1 and  $v_6$  is in the sheet containing  $t$ . So  $v_6 = v_5 t$  and  $v_5 \in H$ , the subgroup generated by  $ac$  and  $bc$ . Let  $C_1 = \langle e \rangle$  be the one generator cyclic group, and let  $h : G \rightarrow C_1$  be the group homomorphism defined by  $h(a) = h(b) = e$ ,  $h(c) = e^{-1}$  and  $h(t) = 1$ . Then  $H$  is in the kernel of  $h$ , so  $h(v_5) = h(v_6) = 1$ . Since  $h(u_2) = h(u_3) = e^{-k}$ , the distances  $d(u_2, v_5)$  and  $d(u_3, v_6)$  are both at least  $k$ . But the sections of  $p$  joining  $u_2$  and  $v_2$  and  $v_3$  to  $u_3$  both have length at most  $k$ . Hence  $v_5$  and  $v_6$  lie on the section of  $p$  between  $v_2$  and  $v_3$ . Hence, the distance

$$d(v_2, v_3) = d(v_2, v_5) + 1 + d(v_3, v_6).$$

Now  $h(v_2) = h(u_2)h(\overline{w_2}) = e^{-k} e^{x_2} e^{y_2} e^{-z_2} = e^{-k+x_2+y_2-z_2}$ . Therefore,

$$d(v_2, v_5) \geq k - x_2 - y_2 + z_2.$$

Similarly,

$$d(v_3, v_6) \geq k - x_3 - y_3 + z_3.$$

Putting these together gives

$$d(v_2, v_3) = d(v_2, v_5) + 1 + d(v_3, v_6) \geq 2k + 1 - x_2 - x_3 - y_2 - y_3 + z_2 + z_3.$$

In exactly the same way, we deduce the final inequality,

$$d(v_4, v_1) \geq 2k + 1 + x_1 + x_4 + y_1 + y_4 - z_1 - z_4.$$

So we now have lower bounds for all four of the distances  $d(v_1, v_2), d(v_2, v_3), d(v_3, v_4), d(v_4, v_1)$ . Therefore, the length of the loop  $\ell'$  is at least

$$\begin{aligned} & d(v_1, v_2) + d(v_2, v_3) + d(v_3, v_4) + d(v_4, v_1) \\ & \geq 6k - x_1 + x_2 + |y_1| + |y_2| + |z_1 - z_2| + 2k + 1 - x_2 - x_3 - y_2 - y_3 + z_2 + z_3 \\ & \quad + 6k - x_4 + x_3 + |y_4| + |y_3| + |z_4 - z_3| + 1 + x_1 + x_4 + y_1 + y_4 - z_1 - z_4 \\ & = 16k + 2 + |y_1| + y_1 + |y_2| - y_2 + |y_3| - y_3 + |y_4| + y_4 \\ & \quad + |z_1 - z_2| - (z_1 - z_2) + |z_4 - z_3| - (z_4 - z_3) \\ & \geq 16k + 2, \end{aligned}$$

which is the same as the length of  $\ell$ . Hence the pair  $(G, S_2)$  does not enjoy the loop shortening property.  $\square$

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