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# On the Polytope of the Vertex Separator Problem

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**Abstract:** Given  $G = (V, E)$  a connected undirected graph and a positive integer  $\beta(|V|)$ , the vertex separator problem is to find a partition of  $V$  into three nonempty subsets  $A, B, C$  such that (i) there is no edge between the nodes of  $A$  and those of  $B$ , (ii)  $\max\{|A|, |B|\} \leq \beta(|V|)$  and (iii)  $|C|$  is minimum. In this paper, we consider the problem from a polyhedral point of view. We first propose a new integer programming formulation for the problem. Then we provide several valid inequalities for the polytope which generalize those introduced by Balas and De Souza [1], and give conditions under which these inequalities define facets.

**Key words.** graph partitioning, vertex separator, polyhedral approach.

## 1 Introduction

Let  $G = (V, E)$  be a connected undirected graph and  $\beta(n)$  a positive integer, where  $n = |V|$ . The *vertex separator problem* (VSP for short) is, given  $G$  and  $\beta(n)$ , to find a partition of  $V$  into three nonempty sets  $A, B, C$  such that

1. there is no edge between the nodes of  $A$  and those of  $B$ ;
2.  $1 \leq |A| \leq \beta(n)$  and  $1 \leq |B| \leq \beta(n)$ ;
3.  $|C|$  is minimum (or  $|A| + |B|$  is maximum).

The subset  $C$  is called a *separator*. For convenience, we will alternatively call a separator either a partition  $\{A, B, C\}$  of  $V$  satisfying conditions 1) and 2), or the pair  $(A, B)$ . Note that in this latter definition, the node set  $C$  is given by  $C = V \setminus (A \cup B)$ .

The VSP is NP-hard [5] and appears in a wide range of applications, including telecommunications network protection, graph theory algorithmic and differential equation systems. When  $\beta(n) = n - k$ , for some positive constant

$k$ , the problem reduces to computing a particular matching in a bipartite graph, which can be done in polynomial time (see [1]). When  $\beta(n) = 1$ , the VSP is trivial and can be solved in polynomial time when  $\beta(n) \geq n - 1$ . Despite the fact that the VSP has been the subject of extensive research, the first polyhedral approach was done only in 2005 by Balas and De Souza [1, 6]. In 2007, Cavalcante et De Souza [2] proposed an algorithm that combines Lagrangian relaxation with cutting plane techniques. In 2011, Didi Biha and Meurs [4] introduced new classes of valid inequalities. Cornaz and al. [3] studied a variant of the VSP in which we are look for a minimum size node set  $C$  whose removal partitions the node set into  $k$  subsets  $V_1, \dots, V_k$ , for some  $k \geq 2$ , and such that  $\delta(V_i, V_j) = \emptyset$ , for every  $i, j \in \{1, \dots, k\}$ . In particular, they introduced two integer programming formulations and devised a Branch-and-Cut and a Branch-and-Price algorithms for this problem.

In this paper, we consider the VSP from a polyhedral point of view. We present two integer programming formulations for the problem. The first formulation, called basic formumation, is known from the literature. The second formulation, called distance-based formulation, is, to the best of our knowledge, new. Then, we investigate the polytope associated with the problem and introduce several class of valid inequalities. We also discuss the conditions under which they define facets. Finally, we devise a Branch-and-Cut algorithm using the inequalities we have introduced and present some computational results.

The paper is organized as follows. In Section 2, we present the basic and distance-based integer programming formulations for the VSP. In Section 3, we investigate the polytope associated with the basic formulation, present several classes of valid inequalities and conditions for these inequalities to define facets. Then, in Section ??, we present our Branch-and-Cut algorithm for the VSP. The computational results are given in Section ??. Finally, we give some concluding remarks in Section 4. The remainder of this section is devoted to the main definitions and notations we use in the paper.

The graphs we consider are finite, undirected and connected. We denote a graph by  $G = (V, E)$ , where  $V$  is the node set and  $E$  the edge set. Let  $n = |V|$  and  $m = |E|$ . If  $e$  is an edge with end-nodes  $u$  and  $v$ , then we write  $e = uv$ . If  $S$  and  $T$  are disjoint subsets of  $V$ , then  $\delta(S, T)$  denotes the set of edges of  $G$  which have one node in  $S$  and the other one in  $T$ , that is  $\delta(S, T) = \{uv \in E \text{ for all } u \in S, v \in T\}$ . For a node set  $S \subseteq V$  with  $\emptyset \neq S \neq V$ ,  $\delta(S) = \delta(S, V \setminus S)$  is *the cut* induced by  $S$ . When  $S = \{u\}$ , we alternatively denote by  $\delta(u)$  the cut induced by  $S = \{u\}$ . For a node set  $S \subseteq V$  with  $|S| \geq 2$ , we denote by  $E(S)$  the set of edges having both end-nodes in  $S$ , that is  $E(S) = \{uv \in E : u, v \in S\}$ , and by  $G(S) = (S, E(S))$  the subgraph of  $G$  induced by  $S$ . Also, for an edge set  $F \subseteq E$ , we denote by

$V(F)$  the set of nodes incident to the edges of  $F$ , and by  $G(F) = (V(F), F)$  the subgraph of  $G$  induced by the edge set  $F$ .

For a node set  $S \subseteq V$  such that  $\emptyset \neq S \neq V$ ,  $N(S)$  is the set of *neighbors of  $S$* , that is the set of nodes of  $V \setminus S$  that are incident to a node of  $S$ . Namely,  $N(S) = \{u \in V \setminus S \text{ such that } uv \in E, \text{ for some } v \in S\}$ . When  $S = \{u\}$ , we alternatively denote  $N(S) = N(\{u\})$  by  $N(u)$ . For every node  $u \in V$ , we call *degree of  $u$* , denoted by  $d(u)$ , the number of neighbors of  $u$  in  $G$ , that is  $d(u) = |N(u)| = |\delta(u)|$ .

A graph  $G = (V, E)$  is said to be *complete* if there exists an edge between every pair of nodes of  $G$ , and is said *empty* if  $E = \emptyset$ . A node set  $S \subseteq V$  is a *stable set* of  $G$  if  $G(S)$  is empty. A node set  $Q \subseteq V$  is a *clique* of  $G$  if  $G(Q)$  is complete. A graph is *connected* if there exists a path between all the pairs of nodes of  $G$ . A set  $S$  is said to be connected if the subgraph  $G(S)$  is connected. When  $G$  is connected, a node  $u \in V$  is an *articulation point* (or cut vertex) if and only if its removal from  $G$  disconnects the graph.

Let  $u, v \in V$  be two nonadjacent nodes of  $G$ . We denote by  $\alpha_{uv}$  the maximum number of node-disjoint paths between  $u$  and  $v$ . We also let  $\alpha^* = \min\{\alpha_{uv} : u, v \in V, uv \notin E\}$ . The graph  $G$  is called  $\alpha$ -connected if there are  $\alpha$  node-disjoint paths between all pairs of nodes in the graph. By convention, if  $G$  is complete,  $\alpha^* = n$ . If  $\alpha \geq 1$ ,  $G$  is connected. Let  $S \subseteq V$  be a subset of  $V$  and  $\alpha^S = \min\{\alpha_{uv}^S : u, v \in S, uv \notin E\}$ , where  $\alpha_{uv}^S$  is the maximum number of node-disjoint paths between  $u$  and  $v$  in  $G(S)$ .  $G(S)$  is  $\alpha$ -connected if there are  $\alpha$  node-disjoint paths between every pair of nodes in the graph  $G(S)$ . By convention, if  $S$  is a clique,  $\alpha^S = |S|$ .

Note that there is a separator  $(A, B)$  in  $G$  only if  $G$  is not complete. Thereafter, we will write  $\beta$  instead of  $\beta(n)$ , and call a  $\beta$ -separator of  $G$ , a separator  $(A, B)$  of  $G$  such that  $\max\{|A|, |B|\} \leq \beta$ .

## 2 Definitions and Preliminary Results

In this section, we present two integer programming formulations for the VSP. The first one, called basic formulation, has been introduced by Balas and De Souza [1]. The second one, called distance-based formulation, to the best of our knowledge, is new and is presented for the first time in this paper.

### 2.1 Basic formulation

The formulation below is given by Balas and De Souza [1] and has been improved by Didi Biha and Meurs [4]. Let  $(A, B)$  be a  $\beta$ -separator of  $G$ . and  $x, y \in \mathbb{R}^V$  two 0-1 vectors such that  $x_v = 1$  (resp.  $y_v = 1$ ) if  $v \in A$  (resp.

$v \in B$ ) and 0 otherwise, for every  $v \in V$ . The VSP can be equivalent to the following integer linear program, called  $F^{\text{basic}}$

$$\begin{aligned} & \text{Max } \sum_{v \in V} (x_v + y_v) \\ & \text{s.t.} \\ & \left. \begin{aligned} x_u + y_v &\leq 1 \\ x_v + y_u &\leq 1 \end{aligned} \right\}, \quad \text{for all edge } uv \in E, & (1) \\ & x_v + y_v \leq 1, \quad \text{for all } v \in V, & (2) \\ & 1 \leq \sum_{v \in V} x_v \leq \beta, & (3) \\ & 1 \leq \sum_{v \in V} y_v \leq \beta, & (4) \\ & x_v \geq 0, y_v \geq 0, \quad \text{for all } v \in V, & (5) \\ & x_v \in \{0, 1\}, \quad \text{for all } v \in V. & (6) \end{aligned}$$

From a solution of  $F^{\text{basic}}$ , we obtain a separator  $(A, B)$  defined by  $A = \{v \in V \text{ such that } x_v = 1\}$  and  $B = \{v \in V \text{ such that } y_v = 1\}$ . Constraints (1) ensure that there is no edge between  $A$  and  $B$ , while Constraints (2) ensure that  $A \cap B = \emptyset$ . Constraint (3) and (4) guarantee that both  $A$  and  $B$  are not empty and contains at most  $\beta$  nodes. Notice that in this formulation, only the variables  $x_v$  are restricted to be integer.

Didi Biha and Meurs [4] improved this formulation by observing that  $\alpha^V = \alpha^*$  gives a lower bound of the cardinality of any separator  $C$  of  $G$ . Using this, they proposed to reinforce formulation  $F^{\text{basic}}$  by adding the inequality (7) below.

$$\sum_{v \in V} (x_v + y_v) \leq n - \alpha^*. \quad (7)$$

Inequality (7) indicates that at most  $n - \alpha^*$  nodes are in  $A \cup B$ .

In order to avoid issues related to symmetries in the VSP, one can impose that  $|A| \leq |B|$  in a separator. Thus, the following inequality is valid for the VSP

$$\sum_{v \in V} x_v \leq \sum_{v \in V} y_v. \quad (8)$$

By combining inequalities (7) and (8) using Chvatal-Gomory procedure, one can easily show that Inequality (9) below is valid for the VSP.

$$\sum_{v \in V} x_v \leq \lfloor \frac{n - \alpha^*}{2} \rfloor. \quad (9)$$

In our objective, we can add a different cost  $c_v$  associated to each node  $v$ , see [6].

## 2.2 Distance-based formulation

This formulation is based on distance variables between all pairs of nodes: if the distance is null, then the two nodes are in the same set ( $A$  or  $B$ ). A dummy node  $a$  (respectively  $b$ ) is added to  $G$  and is placed in the set  $A$  (respectively  $B$ ). We define the variables  $d_{av} = 1$  if the node  $v$  is not in  $A$ , 0 otherwise; and  $d_{bv} = 1$  if the node  $v$  is not in  $B$ , 0 otherwise.  $d_{uv} = 1$  if the nodes  $u$  and  $v$  are not in the same set ( $A$  or  $B$ ), 0 otherwise. We obtain the following formulation:

$$[F^{\text{dist}}] \quad \max \sum_{v \in V} (2 - d_{av} - d_{bv}),$$

$$\text{s.t. } d_{au} + d_{bv} \geq 1, \quad d_{av} + d_{bu} \geq 1, \quad \text{for all edge } uv \in E, \quad (10)$$

$$d_{av} + d_{bv} \geq 1, \quad \text{for all } v \in V, \quad (11)$$

$$1 \leq \sum_{v \in V} (1 - d_{av}) \leq \min \left\{ \beta, \left\lfloor \frac{n - \alpha^*}{2} \right\rfloor \right\}, \quad (12)$$

$$1 \leq \sum_{v \in V} (1 - d_{bv}) \leq \beta, \quad (13)$$

$$\sum_{v \in V} (2 - d_{av} - d_{bv}) \leq n - \alpha^*, \quad (14)$$

$$\sum_{v \in V} d_{av} \geq \sum_{v \in V} d_{bv}, \quad (15)$$

$$d_{uv} \geq |d_{au} + d_{bu} - d_{av} - d_{bv}|, \quad \text{for all edge } uv \in E, \quad (16)$$

$$d_{uv} + d_{au} + d_{bu} + d_{av} + d_{bv} \leq 4, \quad \text{for all } u, v \in V, \quad (17)$$

$$d_{uw} \leq d_{uv} + d_{vw}, \quad \text{for all } u, v, w \in V, \quad (18)$$

$$d_{av}, d_{bv}, d_{uv} \in \{0, 1\}, \quad \text{for all } u < v \in V \quad (19)$$

$$d_{ab} = 1. \quad (20)$$

Constraints (10-15) have the same signification that the constraints (1-9). Constraints (16) imply that if  $u$  and  $v$  are in the same set, then there is impossible that one node is in  $A$  and the other is in  $B$ . Constraints (17) impose for each pair of nodes that either at least one of the two nodes is in  $A$  or  $B$ , nor they are in the same set. Constraints (18) come from the triangular inequality: if  $u$  and  $w$  are not together, then  $v$  can not be with  $u$  and  $w$ . Notice that for each pair of nodes  $u, v$ ,  $d_{av}$  and  $d_{bv}$  can be relaxed. To impose coherence between the distances for nonadjacent nodes, we can add the inequalities:

$$d_{uv} + d_{av} + d_{bv} \geq 1, \quad \forall uv \notin E.$$

These inequalities are redundant for adjacent nodes.

### 2.3 Comparisons

Let  $(\hat{d}_{av}, \hat{d}_{bv}, \hat{d}_{uv})$  be a feasible solution of  $F^{\text{dist}}$ . By setting  $\hat{x}_v = 1 - \hat{d}_{av}$  and  $\hat{y}_v = 1 - \hat{d}_{bv}$  for each node  $v$  of  $V$ , we obtain a feasible solution of  $F^{\text{basic}}$ . Hence, all feasible solutions of  $F^{\text{dist}}$  are feasible for  $F^{\text{basic}}$ .

Conversely, let  $\hat{x}_v$  and  $\hat{y}_v$  be a feasible solution of  $F^{\text{basic}}$ . By setting for each node  $v \in V$ ,  $\hat{d}_{av} = 1 - \hat{x}_v$ ,  $\hat{d}_{bv} = 1 - \hat{y}_v$  and for each pair of nodes  $u, v \in V$ ,  $\hat{d}_{uv} = |\hat{x}_u + \hat{y}_u - \hat{x}_v - \hat{y}_v|$ , we obtain a feasible solution of  $F^{\text{dist}}$ . At first, note that  $\hat{d}_{uv} \leq 1$  et  $\hat{d}_{uv} \geq 0$  because  $0 \leq |\hat{x}_u + \hat{y}_u - \hat{x}_v - \hat{y}_v| \leq 1$ . Indeed, the constraints (10-15) are obviously satisfied. Then, the constraints (16) is satisfied by definition of  $\hat{d}_{uv}$ . The constraints (17) are satisfied because:

$$\begin{aligned} 4 - \hat{d}_{uv} &= 4 - |\hat{x}_u + \hat{y}_u - \hat{x}_v - \hat{y}_v| \\ &\geq 1 - |\hat{x}_u| + 1 - |\hat{y}_u| + 1 - |\hat{x}_v| + 1 - |\hat{y}_v| = \hat{d}_{au} + \hat{d}_{bu} + \hat{d}_{av} + \hat{d}_{bv} \end{aligned}$$

Endly, the constraints (18) are satisfied because:

$$\begin{aligned} \hat{d}_{uw} &= |\hat{x}_u + \hat{y}_u - \hat{x}_w - \hat{y}_w| = |\hat{x}_u + \hat{y}_u - \hat{x}_v - \hat{y}_v + \hat{x}_v + \hat{y}_v - \hat{x}_w - \hat{y}_w| \\ &\leq |\hat{x}_u + \hat{y}_u - \hat{x}_v - \hat{y}_v| + |\hat{x}_v + \hat{y}_v - \hat{x}_w - \hat{y}_w| = \hat{d}_{uv} + \hat{d}_{vw} \end{aligned}$$

Hence, all feasible solutions of  $F^{\text{basic}}$  are feasible for  $F^{\text{dist}}$  and finally the both formulations are equivalent. In the next section, we only consider inequalities for  $F^{\text{basic}}$ .

## 3 Valid Inequalities and Facets

In this section, we consider the basic formulation for the VSP. We denote by  $P(G, \beta)$  the polytope associated with the problem that  $P(G, \beta)$  is the convex hull of the incidence vectors of  $\beta$ -separators of  $G$ . By Section 2.1, we have that

$$\begin{aligned} P(G, \beta) = \text{conv}\{(x, y) \in \mathbb{R}^{V \times V} \text{ such that } (x, y) \\ \text{satisfies (1) - (5) and } x \in \mathbb{Z}^V\}. \end{aligned}$$

In the remainder of this section, we present several classes of valid inequalities for  $P(G, \beta)$  and investigate the conditions under which they define facets.

### 3.1 The dimension of $P(G, \beta)$

Here we discuss the dimension of  $P(G, \beta)$  and present, in particular, conditions under which the polytope is full dimensional. We first make the following observations which can be seen from [1].

- $P(G, \beta) = \emptyset$  if  $G$  is complete.

- When  $\beta = 1$ , Inequalities (3) and (4) become equations  $x(V) = y(V) = 1$  which implies that  $P(G, \beta)$  is not full dimensional. Moreover, the VSP in this case reduces to computing  $\alpha^*$ , which can be done in polynomial time.
- When  $\beta \geq n - 1$ , the VSP is polynomially solvable, and as we will see below,  $P(G, \beta)$  may be full dimensional in this case.

In the following, we assume that the graph  $G$  is not complete and is connected, and that  $2 \leq \beta \leq n - 2$ .

Now give some definitions, which can be found in [1], and then discuss the conditions for  $P(G, \beta)$  to be full dimensional.

**Definition 1.** A vertex  $u \in V$  is called **regular** if there exists a separator  $S \subset V \setminus \{u\}$  such that  $S \cup \{u\}$  is also a separator. A vertex that is not regular is called **irregular**.

**Definition 2.** A vertex  $u$  is **universal** if it is adjacent to every  $v \in V \setminus \{u\}$ . Such a node is obviously irregular.

**Definition 3.** If  $G$  has two nonadjacent vertices  $u$  and  $v$  such that  $N(u) = N(v) = V \setminus \{u, v\}$ , then  $u$  and  $v$  are irregular and they form a **polar** pair of irregular vertices.

In [1], Balas and De Souza proved that  $P(G, \beta)$  is full dimensional when all the nodes of  $G$  are regular. They also presented some cases in which the graph  $G$  contains irregular vertices.

One can also observe that when  $G$  contains a universal vertex or a polar pair of irregular vertices, the polytope  $P(G, \beta)$  is not full dimensional. Indeed, if  $u$  is universal, for some  $u \in V$ , then  $x_u = y_u = 0$ , for every solution  $(x, y) \in P(G, \beta)$ . In the case of a polar pair of irregular vertices  $(u, v)$ , we have that  $x_u + y_u = x_v + y_v = 1$ , for every solution  $(x, y) \in P(G, \beta)$ . Clearly,  $P(G, \beta)$  is not full dimensional in both cases.

**Definition 4.** A graph having a universal vertex or a polar pair of irregular vertices is called **degenerate**.

In [1], Balas and De Souza stated that in a nondegenerate graph,  $P(G, \beta)$  is full dimensional if and only if  $\bigcap_{k \in I} N(k)$  is not a clique, where  $I$  is the set of irregular vertices of  $G$ . **In fact, this result is not well stated. First, one can see that  $\bigcap_{k \in I} N(k)$  is never a clique, for all nondegenerate graphs. In fact, if  $\bigcap_{k \in I} N(k)$  is a clique, then each vertex of  $\bigcap_{k \in I} N(k)$  is universal, which is not possible for a nondegenerate**

**graph. Thus, stating in the theorem that  $\bigcap_{k \in I} N(k)$  is not a clique is not necessary. Also,** in the proof of their result, Balas and De Souza do not consider the case where  $\bigcap_{k \in I} N(k) = \emptyset$ . In fact, one can find a counter-example of their result, that is a graph in which  $\bigcap_{k \in I} N(k) = \emptyset$  but the polytope  $P(G, \beta)$  is not full dimensional. Figure 1 illustrates such a counter-example.

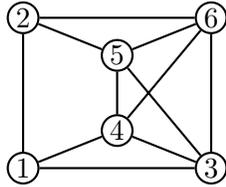


Figure 1: Counter-example:  $P(G, \beta)$  is not full dimensional

We can easily observe that the graph  $G$  is nondegenerate and that vertices 1, 2 are irregular whereas vertices 3, 4, 5, 6 are regular. Hence  $I = \{1, 2\}$  and  $\bigcap_{k \in I} N(k)$  is empty. To prove that  $P(G, \beta)$  is not full dimensional for this graph, it is sufficient to observe that  $x(I) + y(I) = 1$ , for every incidence vector  $(x, y)$  of a  $\beta$ -separator  $(A, B)$ . Indeed, if  $x(I) + y(I) = 0$  or 2, it is not possible to find  $\beta$ -separator in the remained graph. Consequently, in this nondegenerate graph,  $P(G, \beta)$  is not full dimensional whereas  $\bigcap_{k \in I} N(k)$  is not a clique.

Another counter-example can be obtained by considering a graph  $G = (V, E)$  where  $V = \{u_0\} \cup U$ ,  $G(U)$  is complete and  $u$  is incident to all the nodes of  $U$ . It is not hard to see that  $\bigcap_{k \in I} N(k)$  is not a clique and  $P(G, \beta)$  is not full dimensional.

**In Theorem 3.1, we give the correct statement of the result of Balas and De Souza [1]. We do not give an explicit proof of that theorem as it relies on that given by Balas and De Souza [1].**

**Theorem 3.1.** *Assume that the graph  $G = (V, E)$  is nondegenerate and let  $I \subseteq V$  be the set of irregular vertices of  $G$ . Then,  $P(G, \beta)$  is full dimensional if and only if*

1.  $I = \emptyset$  or
2.  $I \neq \emptyset$  and  $\bigcap_{k \in I} N(k) \neq \emptyset$ .

**One can notice that the conditions of Theorem 3.1 are met when  $d(u) \leq n - 3$ , for every  $u \in V$ . We prove it in Lemma 3.2.**

**Lemma 3.2.** *If  $d(u) \leq n - 3$ , for all  $u \in V$ , then all the nodes of  $G$  are regular.*

*Proof.* Suppose that the condition holds in  $G$  and consider a node  $u \in V$ . Since  $d(u) \leq n - 3$ , there exists a node  $v \in V \setminus \{u\}$  which is not incident to  $u$ . Also, as by assumption,  $d(v) \leq n - 3$ , there exists a node  $w \in V \setminus \{u, v\}$  which is not incident to  $v$ . Now, it is not hard to see that the pair of node sets  $(\{u, w\}, \{v\})$  and  $(\{w\}, \{v\})$  are  $\beta$ -separators of  $G$ . Thus,  $u$  is regular.  $\square$

**Notice that the condition of Lemma 3.2 also implies that the set of irregular vertices is empty. Thus, a consequence of Lemma 3.2 is that  $P(G, \beta)$  is full dimensional when  $d(u) \leq n - 3$ , for every  $u \in V$ .**

**Corollary 3.3.** *The polytope  $P(G, \beta)$  is full dimensional when  $d(u) \leq n - 3$ , for all  $u \in V$ .*

It should be noticed that if the graph  $G$  contains a node  $u \in V$  such that  $d(u) \geq n - 2$ , the VSP in  $G$  reduces to solving the VSP in the graph  $G'$  obtained by removing  $u$  from  $G$ . Indeed, consider a minimum size  $\beta$ -separator  $C$  of  $G$ , and suppose first that  $d(u) = n - 1$ . In this case, as  $u$  is incident to all the nodes of  $V \setminus \{u\}$ , we have that  $u \in C$ . It is not hard to see that  $C' = C \setminus \{u\}$  is a  $\beta$ -separator of  $G'$  whose size is minimum. Now, in the case where  $d(u) = n - 2$ , there is a node  $v \in V \setminus \{u\}$  which is not incident to  $u$ . If  $u \notin C$ , then it is not hard to see that  $|C| = 1 + \min\{\beta, n - d(v)\}$ . If  $u \in C$ , as in the previous case,  $C = C' \cup \{u\}$  where  $C'$  is a minimum size  $\beta$ -separator for  $G'$ .

Consequently, solving the VSP in  $G$  is equivalent to solving the VSP in  $G'$  where  $G'$  is obtained by iteratively removing from  $G$  the nodes  $u$  for which  $d(u) \geq n - 2$ . Obviously,  $G'$  can be built in polynomial time, and all the nodes of  $G'$  are of degree at most  $n - 3$ . From here and until the remainder of the paper, unless the contrary is mentioned, we assume that  $d(u) \leq n - 3$ , for every  $u \in V$ , and hence, all the nodes of  $G$  are regular and  $P(G, \beta)$  is full dimensional.

### 3.2 Facets of $P(G, \beta)$

This section examines valid inequalities for  $P(G, \beta)$  under assumption ?? and the conditions under which they are facet defining. For convenience, we will say that a  $\beta$ -separator  $(A, B)$  belongs to a face  $F$  of  $P(G, \beta)$  if its incidence vectors  $(x, y)$  belongs to  $F$ . A valid inequality  $a^T x + b^T y \leq \gamma$  of  $P(G, \beta)$  is symmetric if the coefficients  $a$  and  $b$  are equal, with  $a, b \in \mathbb{R}^n$  and  $\gamma \in \mathbb{R}$ .

**Proposition 3.4.** *Let  $F$  be a non empty face induced by a symmetric inequality:  $F = \{(x, y) \in P(G, \beta) : d^T(x + y) = \delta\}$  where  $d \in \mathbb{R}^n$  and  $\delta \in \mathbb{R}$ . Assume that  $F$  is in the facet  $\tilde{F} = \{(x, y) \in P(G, \beta) : a^T x + b^T y = \gamma\}$  with  $a, b \in \mathbb{R}^n$  and  $\gamma \in \mathbb{R}$ . Let  $R(a, b)$  be a unary relation by the element  $(a, b)$  of  $\tilde{F}$ . Thus, the relation  $R(b, a)$  is also satisfied.*

*Proof.* If  $(A, B)$  is a  $\beta$ -separator of  $F$ , then  $(B, A)$  is also a  $\beta$ -separator of  $F$ . Hence, by symmetry,  $R(a, b) \Rightarrow R(b, a)$ .  $\square$

The facet conditions are mainly based on the relationship between separators and dominators. We first give some definitions.

### 3.2.1 Definitions

A node set  $D \subseteq V$  is a *dominating set*, also called *dominator*, of  $G$  if  $D \cup N(D) = V$ . We say that  $D$  is a *connected dominator* if  $D$  is a dominator and  $G[D]$  is connected.

A node set  $D \subseteq V$  is called  $\nu$ -**dominator** of  $G$  if all the nodes of  $V \setminus D$  are incident to at least  $\nu$  nodes of  $D$ . A dominator  $D$  is a  $\nu$ -dominator with  $\nu = 1$ . We denote by  $\nu_D$  the dominance of the set  $D$  for the graph  $G$ .

Let  $S \subseteq V$  be a subset of  $V$  such that  $\emptyset \neq S \neq V$ . A node set  $D \subseteq S$  is called a  $\nu^S$ -**dominant** of  $G[S]$  if all nodes of  $S \setminus D$  are incident to at least  $\nu^S$  nodes of  $D$  (see figure 2).

Note that in the case where  $V \setminus S = \emptyset$ ,  $D$  is a  $\nu$ -dominating set for  $G$ . We denote by  $\nu_{D,S}$  the dominance of the set  $D$  for the graph  $G[S]$ . We say that the vertices of the set  $D$  are  $\nu$ -dominated by  $D$  if every vertex is adjacent to  $\nu$  vertices of  $D$ .

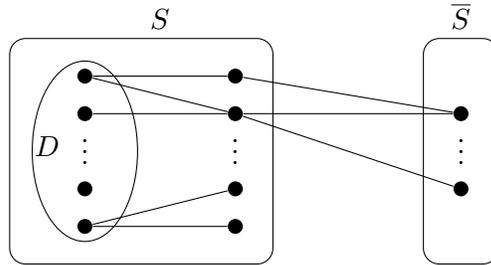


Figure 2: Bi-partition of the graph  $G$  into two sets  $S$  and  $\bar{S}$  with  $D \subset S$  a dominator of  $G[S]$

Let  $D \subseteq V$  be a dominator of  $G$ . We denote by  $P(u)$  the set of vertices

of  $V \setminus D$  only dominated by the vertex  $u \in D$  such that

$$P(u) = \{v \in V \setminus D : N_D(v) = \{u\}\}, \quad \forall u \in D. \quad (21)$$

$P(u)$  is the set of **pendent** vertices of  $u$ .

Let  $D \subseteq V$  be a dominator of  $G$ . The vertex  $u \in D$  is called **self-dominating** vertex if the set  $D \setminus \{u\}$  is a dominator of  $G[V \setminus \{u\}]$  but is not a dominator of  $G$ .

In [1], Balas and De Souza characterize the self-dominating vertices of a minimal dominator.

**Lemma 3.5.** [1]

*Let  $D \subseteq V$  be a minimal dominator of  $G$ . A vertex  $u \in D$  is self-dominator if and only if  $P(u) = \emptyset$ .*

Lemma 3.5 implies that, if  $D \subseteq V$  is a minimal dominator, then each vertex  $u \in D$  is either self-dominator (i.e.  $P(u) = \emptyset$ ), or adjacent to at least one pendent vertex (i.e.  $P(u) \neq \emptyset$ ).

### 3.2.2 The basic inequalities

Now we study the conditions under which the inequalities (1)-(5) define facets. We first give

**Theorem 3.6.** *For each vertex  $u \in V$ , the inequalities  $x_u \geq 0$  and  $y_u \geq 0$  are facets defining.*

*Proof.* We establish the proof for inequality  $x_u \geq 0$ . For  $y_u \geq 0$ , the proof follows the same lines by symmetry.

Let  $u \in V$  and  $F_u$  be the face induced by  $x_u \geq 0$  that is

$$F_u = \{(x, y) \in P(G, \beta) : x_u = 0\}.$$

First, we show that  $F_u \neq \emptyset$  which implies that  $F_u$  is proper face of  $P(G, \beta)$ . To see this, note that as, by assumption, all the nodes of  $G$  are regular, there exists a  $\beta$ -separator  $(A, B)$  of  $G$  such that  $u$  is in  $V \setminus (A \cup B)$ . This implies that the incidence vector of  $(A, B)$  satisfies  $x_u \geq 0$  with equality. Thus,  $F_u \neq \emptyset$  and is a proper face of  $P(G, \beta)$ .

Now we show that  $F_u$  is a facet of  $P(G, \beta)$ . Let  $a^T x + b^T y \leq \gamma$  be a facet-defining inequality for  $P(G, \beta)$  and suppose that

$$F_u \subseteq \tilde{F} = \{(x, y) \in P(G, \beta) : a^T x + b^T y = \gamma\}.$$

To prove that  $F_u$  is a facet, it suffices to show that  $b_v = 0$ , for all  $v \in V$ ,  $a_v = 0$ , for all  $v \in V \setminus \{u\}$ ,  $a_u > 0$ . Let  $v_0 \in V$ . As  $d(v) \leq n - 3$ , for all  $v \in V$ , there exist two distinct nodes  $w, w' \in V \setminus \{v_0\}$  such that  $v_0w \notin E$  and  $ww' \notin E$ . Clearly, the pair of subsets  $(\{w\}, \{v_0, w'\})$  and  $(\{w\}, \{w'\})$  are  $\beta$ -separators of  $G$  and their incidence vectors are in  $F_u \subseteq \tilde{F}$ . Thus, we have that  $b_{v_0} = 0$ , for all  $v_0 \in V$ .

Now let  $v \in V \setminus \{u\}$  and suppose first that  $uv \notin E$ . As  $d(u) \leq n - 3$ , there exists a node  $w \in V \setminus \{u, v\}$  such that  $uw \notin E$ . As before, the pairs  $(\{v, w\}, \{u\})$  and  $(\{w\}, \{u\})$  are  $\beta$ -separators of  $G$  whose incidence vectors are in  $F_u$ , which implies that  $a_v = 0$ , for all  $v \in V \setminus \{u\}$ . Now if  $uv \in E$ , we distinguish two cases.

**Case 1.**

- Let  $v \in V$ . As all vertices are regular, there exist  $w, w' \in V \setminus \{v\}$ ,  $w \neq w'$ , such that  $vw \notin E$  and  $ww' \notin E$ . So, the  $\beta$ -separators  $(\{w\}, \{v, w'\})$  et  $(\{w\}, \{w'\})$  belong to the face  $F_u$ , thus  $b_v = 0$ .
- Let  $v \in V \setminus \{u\}$  such that  $uv \notin E$ . As  $d(u) < |V| - 2$ , there exists  $w \in V \setminus \{u, v\}$  such that  $uw \notin E$ . So, the  $\beta$ -separators  $(\{v, w\}, \{u\})$  and  $(\{w\}, \{u\})$  belong to the face  $F_u$ , thus  $a_v = 0$ .
- Let  $v \in V \setminus \{u\}$  such that  $uv \in E$ . As  $d(v) < |V| - 2$ , there exist  $w_1, w_2 \in V \setminus \{u, v\}$  such that  $vw_1 \notin E$  and  $vw_2 \notin E$ .
  - The  $\beta$ -separator  $(\{v\}, \{w_1\})$  belongs to the face  $F_u$ , thus  $a_v = \gamma$  ( $b_{w_1} = 0$ ); and by symmetry,  $a_{w_1} = \gamma$ .
  - The  $\beta$ -separators  $(\{v\}, \{w_1, w_2\})$  and  $(\{v\}, \{w_2\})$  belong to the face  $F_u$ , thus  $a_{w_1} = 0$ .

Consequently,  $a_v = \gamma = 0$ .

Thus,  $F_u$  is a facet. □

**Theorem 3.7.** *For each vertex  $u \in V$ , the inequality  $x_u + y_u \leq 1$  induces a facet different from the trivial inequalities if and only if for all  $v \in N(u)$ , the node set  $\{u, v\}$  is not a dominator of  $G$ .*

*Proof.* Let  $u \in V$  and  $F_u$  be the face induced by  $x_u + y_u \leq 1$ , that is

$$F_u = \{(x, y) \in P(G, \beta) : x_u + y_u = 1\}.$$

Note that the solutions of  $F_u$  corresponds to those  $\beta$ -separators  $(A, B)$  of  $G$  such that  $u \in A \cup B$ . Since all the nodes of  $G$  are assumed to be regular, there exists a  $\beta$ -separator  $(A, B)$  of  $G$  with  $u \in A \cup B$ . Thus,  $F_u \neq \emptyset$  and is a proper face of  $P(G, \beta)$ .

*Necessity.* Assume that there exists a node  $v \in N(u)$  such that  $\{u, v\}$  is a dominator of  $G$ . Let  $(x, y) \in F_u$  and denote by  $(A, B)$  the corresponding  $\beta$ -separator. Recall that  $u \in A \cup B$ . As  $\{u, v\}$  is a dominator of  $G$  and  $u \in A \cup B$ , the node  $v$  is neither in  $A$  nor in  $B$ , which implies that  $x_v + y_v = 0$ , that is  $x_v = y_v = 0$ , for every solution  $(x, y) \in F_u$ . Thus,  $F_u$  cannot induce a facet different from those induced by the trivial inequalities.

*Sufficiency.* Now assume that  $\{u, v\}$  is not a dominator of  $G$ , for every  $v \in N(u)$ , and suppose that  $F_u \subseteq \tilde{F}$  where  $\tilde{F}$  is a facet of  $P(G, \beta)$  induced by an inequality  $ax + by \leq \gamma$ . We show that  $F_u = \tilde{F}$ . To do this, we will show that  $a_v = b_v = 0$ , for all  $v \in V \setminus \{u\}$ , and that  $a_u = b_u = \gamma$ . Thus, let  $v \in V \setminus \{u\}$  and suppose first that  $uv \notin E$ . As  $d(u) \leq n - 3$ , there exists a node  $w \in V \setminus \{u, v\}$  such that  $uw \notin E$ . Thus, the pairs  $(\{v, w\}, \{u\})$  and  $(\{w\}, \{u\})$  are  $\beta$ -separators of  $G$  whose incidence vectors are in  $F_u \subseteq \tilde{F}$ . This implies that  $a_v + a_w + b_u = a_w + b_u = \gamma$ , and hence,  $a_v = 0$ . By symmetry, we also have that  $b_v = 0$ .

If  $uv \notin E$ , since  $\{u, v\}$  is not a dominator of  $G$ , there exists a node  $w \in V \setminus \{u, v\}$  such that  $uw \notin E$  and  $vw \notin E$ . Thus, the pairs  $(\{u, v\}, \{w\})$  and  $(\{u\}, \{w\})$  are  $\beta$ -separators of  $G$  whose incidence vectors are in  $F_u \subseteq \tilde{F}$ . As before, this implies that  $a_v = 0$ , and by symmetry,  $b_v = 0$ .

Thus,  $a_v = b_v = 0$ , for all  $v \in V \setminus \{u\}$ .

Now, we show that  $a_u = b_u = \gamma$ . For this, let  $(A, B)$  be a  $\beta$ -separator of  $G$  whose incidence vector  $(x^A, y^B)$  is in  $F_u \subseteq \tilde{F}$ . W.l.o.g., we assume that  $u \in A$ . Since, as shown before,  $a_v = b_v = 0$ , for all  $v \in V \setminus \{u\}$ ,  $(x^A, y^B) \in F_u$  implies that  $a_u = \gamma$ . Also, by symmetry, we get  $b_u = \gamma$ .

Consequently,  $F_u = \tilde{F}$  and hence  $F_u$  is a facet of  $P(G, \beta)$ .  $\square$

**Theorem 3.8.** *For each edge  $uv \in E$ , the inequalities  $x_u + y_v \leq 1$  and  $x_v + y_u \leq 1$  define facets different from those induced by the trivial inequalities if and only if*

- 1)  $\{u, v\}$  is not a dominator of  $G$ ,
- 2) for all  $w \in N(u)$ ,  $\{v, w\}$  is not a dominator of  $G$ ,
- 3) for all  $w \in N(v)$ ,  $\{u, w\}$  is not a dominator of  $G$ .

*Proof.* Let  $uv \in E$  and  $F_{uv}$  be the face induced by  $x_u + y_v \leq 1$ , that is

$$F_{uv} = \{(x, y) \in P(G, \beta) : x_u + y_v = 1\}.$$

As  $u$  is regular and  $uv \in E$ , there exists a  $\beta$ -separator  $(A, B)$  of  $G$  such that  $u \in A$  and  $v \notin B$ . Thus,  $F_{uv} \neq \emptyset$  and is a proper face of

$P(G, \beta)$ .

*Necessity.*

1) Assume first that  $\{u, v\}$  is a dominator of  $G$  and let  $(A, B)$  be a  $\beta$ -separator of  $G$  whose incidence vector  $(x^A, x^B)$  is in  $F_{uv}$ . As  $\{u, v\}$  is a dominator of  $G$ , we have that  $|\{u, v\} \cap (A \cup B)| = 1$ . W.l.o.g., we will assume that  $u \in A$  and, hence,  $v \notin (A \cup B)$ . This implies that  $(x^A, x^B)$  satisfies both equations  $x_u + y_v = 1$  and  $y_u = x_v = 0$ . Consequently,  $F_{uv}$  cannot define a facet different from those induced by the trivial inequalities.

2) Assume that there exists a node  $w \in N(u)$  such that  $\{v, w\}$  is a dominator of  $G$ . Let  $(A, B)$  be a  $\beta$ -separator whose incidence vector is in  $F_{uv}$ , that is  $|\{u, v\} \cap (A \cup B)| = 1$ . We distinguish two cases. If  $u \in A$ , then, as  $uw \in E$ , we obtain that  $w \in V \setminus (A \cup B)$ . Now if  $v \in B$ , as  $\{v, w\}$  is a dominator of  $G$ , we still obtain that  $w \notin (A \cup B)$ . Thus, the incidence vector of  $(A, B)$  also satisfies  $y_w = 0$ . Thus,  $F_{uv}$  does not define a facet different from those induced by the trivial inequalities.

3) Similar to 2).

*Sufficiency.* Assume that conditions 1), 2) and 3) hold. We will show that  $F_{uv}$  is a facet of  $P(G, \beta)$ . Assume that  $F_{uv} \subseteq \tilde{F}$  where  $\tilde{F}$  defines a facet of  $P(G, \beta)$  induced by an inequality  $ax + by \leq \gamma$ , that is

$$F_{uv} \subseteq \tilde{F} = \{(x, y) \in P(G, \beta) : a^T x + b^T y = \gamma\}.$$

We are going to show that  $F_{uv} = \tilde{F}$ . First notice that by condition 1),  $\{u, v\}$  is not a dominator of  $G$ . Thus, there exists a node  $w_0 \in V \setminus \{u, v\}$  such that  $uw_0 \notin E$  and  $vw_0 \notin E$ . Thus, the pairs  $(\{u, v\}, \{w_0\})$  and  $(\{u\}, \{w_0\})$  are  $\beta$ -separators whose incidence vectors are in  $F_{uv} \subseteq \tilde{F}$ . This implies that  $a_u + a_v + b_{w_0} = a_u + a_{w_0} = \gamma$ , and hence,  $a_v = 0$ . Similarly, by considering the pairs  $(\{w\}, \{u, v\})$  and  $(\{w_0\}, \{v\})$ , we show that  $b_u = 0$ .

Now let  $w \in V \setminus \{u, v\}$ . We distinguish two cases. If  $w \notin N(v)$ , as  $d(v) \leq |V| - 3$ , there exists a node  $w' \notin V \setminus \{u, v, w\}$  such that  $vw' \notin E$ . Clearly, the pairs  $(\{w, w'\}, \{v\})$  and  $(\{w'\}, \{v\})$  are  $\beta$ -separators of  $G$  whose incidence vectors are in  $F_{uv} \subseteq \tilde{F}$ . Hence,  $a_w = 0$ . Now if  $w \in N(v)$ , as by condition 3),  $\{u, w\}$  is not a dominator of  $G$ , there exists a node  $w' \notin V \setminus \{u, v, w\}$  such that  $uw' \notin E$  and  $ww' \notin E$ . Then, clearly, the pairs  $(\{u, w\}, \{w'\})$  and  $(\{u\}, \{w'\})$  are  $\beta$ -separators of  $G$  whose incidence vectors are in  $F_{uv} \subseteq \tilde{F}$ , which implies that  $a_w = 0$ . Thus,  $a_w = 0$ , for all  $w \in V \setminus \{u, v\}$ . Using similar arguments, we also

show that  $b_w = 0$ , for all  $w \in V \setminus \{u, v\}$ .

Finally, by considering the separators  $(\{u\}, \{w_0\})$  and  $(\{w_0\}, \{v\})$  together with the above result, we obtain  $a_u = b_v = \gamma$ . Consequently,  $F_{uv} = \tilde{F}$  and  $F_{uv}$  defines a facet of  $P(G, \beta)$ .  $\square$

**Theorem 3.9.** *The inequalities  $\sum_{u \in V} x_u \geq 1$  and  $\sum_{u \in V} y_u \geq 1$  define facets of  $P(G, \beta)$ .*

*Proof.* Let  $F$  be the face induced by  $\sum_{u \in V} x_u \geq 1$ , that is

$$F = \{(x, y) \in P(G, \beta) : \sum_{u \in V} x_u = 1\}.$$

Note that the solutions of  $F$  corresponds to those separators  $(A, B)$  such that  $|A| = 1$ . As  $d(u) \leq n - 3$ , for all  $u \in V$ , one can easily exhibit a separator  $(\{u\}, B)$ , for some  $u \in V$ , which implies that  $F \neq \emptyset$ .

Assume that  $F \subseteq \tilde{F}$  where

$$F \subseteq \tilde{F} = \{(x, y) \in P(G, \beta) : a^T x + b^T y = \gamma\}$$

is a facet of  $P(G, \beta)$  induced by an inequality  $ax + by \geq \gamma$ . We are going to show that  $F = \tilde{F}$ . Let  $u \in V$ . As  $d(u) \leq n - 3$ , there exists two distinct nodes  $v, w \in V \setminus \{u\}$ , such that  $uv \notin E$  and  $vw \notin E$ . Clearly, the pairs  $(\{v\}, \{u, w\})$  and  $(\{v\}, \{w\})$  are  $\beta$ -separators whose incidence vectors are in  $F \subseteq \tilde{F}$ . This implies that  $a_v + b_u + b_w = a_v + b_w = \gamma$ , and hence  $b_u = 0$ , for all  $u \in V$ .

Now we show that  $a_u = \gamma$ , for all  $u \in V$ . Let  $u \in V$ . As  $u$  is not a universal vertex, there exists  $v \in V \setminus \{u\}$  such that  $uv \notin E$ . Thus, the pair  $(\{u\}, \{v\})$  is a  $\beta$ -separator of  $G$  whose incidence vector is in  $F \subseteq \tilde{F}$ . Hence, we have  $a_u + b_v = \gamma$ . Since, by the above result,  $b_v = 0$ , we get  $a_u = \gamma$ .

Consequently,  $F = \tilde{F}$  and is a facet of  $P(G, \beta)$ .  $\square$

### 3.2.3 Inequalities associated with connected sets

Didi Biha and Meurs [4] have presented a class of valid inequalities for the VSP polytope based on the connectivity of node sets of  $V$ .

**Theorem 3.10.** [4] *Let  $S \subseteq V$  be a  $\alpha$ -connected non empty set. The inequality*

$$x(S) + y(S) \leq \max\{|S| - \alpha, \beta\}, \quad (22)$$

*is valid for  $P(G, \beta)$ .*

The interpretation of inequality (22) relates the number of nodes in  $A \cup B$ , for a  $\beta$ -separator  $\{A, B, C\}$ , with  $|S|$  and the node-connectivity  $\alpha$  of  $G(S)$ . It is particularly relevant when  $G(S)$  is connected and  $|S| > \beta$ . As  $|S| > \beta$ , we have that  $A \cap S \neq S \neq B \cap S$ , that is  $S$  is not neither a subset of  $A$  nor of  $B$ . If  $S \subseteq A \cup C$  (resp.  $S \subseteq B \cup C$ ), then  $|A \cap S| + |B \cap S| = |A \cap S| \leq \beta \leq \max\{\beta, |S| - \alpha\}$  (resp.  $|A \cap S| + |B \cap S| = |B \cap S| \leq \beta \leq \max\{\beta, |S| - \alpha\}$ ). Now if  $A \cap S \neq \emptyset \neq B \cap S$ , then let  $u$  and  $v$  be two nodes of  $A \cap S$  and  $B \cap S$  respectively with  $uv \notin E$ . As  $uv \notin E$ , each  $uv$ -path in  $G(S)$  has least one node in  $C$ , which implies that  $|C \cap S| \geq \alpha_{uv} \geq \alpha$ . Therefore,  $|A \cap S| + |B \cap S| \leq |S| - \alpha \leq \max\{|S| - \alpha, \beta\}$ .

One can notice that when the inequality (22) induced by a node set  $S$  and a connectivity  $\alpha$  does not induce a facet when  $\alpha \neq \alpha_S^*$ . Indeed, this latter inequality is redundant w.r.t. an inequality (22) induced by  $S$  and considering  $\alpha_S^*$ . Thus, thereafter, we consider that  $\alpha = \alpha_S^*$ . We can also remark that (22) may define a facet depending on the relation between  $|S|$ ,  $\alpha_S^*$  and  $\beta$ . In particular, we give the following necessary conditions for inequalities (22) to define facets.

**Theorem 3.11.** *An inequality (27) induced by a node set  $S \subseteq V$  induces a facet of  $P(G, \beta)$  only if  $|S| + 1 \leq \beta \leq 2\beta - \alpha_S^* - 1$  and*

1. *if  $\beta + 1 \leq |S| \leq \beta + \alpha_S^*$ , then for all  $S' \supsetneq S$ ,  $|S'| > \beta + \alpha_{S'}^*$ .*
2. *if  $\beta + \alpha_S^* + 1 \leq |S|$ , then for all  $S' \subsetneq S$ ,  $\alpha_S^* > \alpha_{S'}^*$ , or  $|S'| \leq \beta + \alpha_{S'}^*$ .*

*Proof.* First, we show that (22) defines a facet only if  $|S| + 1 \leq \beta \leq 2\beta - \alpha_S^* - 1$ . For this, consider the contrary, that is  $|S| \leq \beta$  or  $|S| \geq 2\beta + \alpha_S^*$ . It is not hard to see that, in this case, (22) is redundant w.r.t. (2) and (3)-(4).

Now, we show 1). Suppose that  $\beta + 1 \leq |S| \leq \beta + \alpha_S^*$ , and, on the contrary, there exists a node set  $S' \supsetneq S$  such that  $|S'| \leq \beta + \alpha_{S'}^*$ . Remark, in this case, the inequality (22) induced by  $S$  and  $S'$  are written as

$$x(S) + y(S) \leq \beta, \tag{23}$$

$$x(S') + y(S') \leq \beta \tag{24}$$

Since  $S' \supsetneq S$ , we have that  $x(S) + y(S) = x(S') + y(S') + x(S \setminus S') + y(S \setminus S')$ . Thus, (23) can be obtained by adding (24) with inequalities  $x(u) \geq 0$  and  $y(u) \geq 0$ , for all  $u \in S \setminus S'$ , which implies that (23) cannot define a facet.

Finally, we show 2) in the same way as above. Suppose that  $\beta + \alpha_S^* + 1 \leq |S|$  and assume that there exists a node set  $S' \subsetneq S$  such that  $\alpha_S^* \leq \alpha_{S'}^*$  and  $\beta + \alpha_{S'}^* + 1 \leq |S'|$ . In this case, both inequalities (22) induced by  $S$  and  $S'$  are written as

$$x(S) + y(S) \leq |S| - \alpha_S^*, \quad (25)$$

$$x(S') + y(S') \leq |S'| - \alpha_{S'}^* \quad (26)$$

As  $\alpha_S^* \leq \alpha_{S'}^*$ , we have

$$\begin{aligned} x(S) + y(S) &= x(S') + y(S') + x(S \setminus S') + y(S \setminus S') \\ &\leq |S'| - \alpha_{S'}^* + |S \setminus S'| = |S| - \alpha_{S'}^* \leq |S| - \alpha_S^*. \end{aligned}$$

Thus, (25) is redundant w.r.t. to (26), (2) and  $x(u) \geq 0$  and  $y(u) \geq 0$ , for all  $u \in S \setminus S'$ .  $\square$

In the remaining, we consider the inequalities (22) induced by a node set  $S \subseteq V$  such that  $|S| = \beta + \alpha_S^*$ . The next theorem gives conditions for inequalities (22) to define a facet when  $S \subseteq V$  is such that  $\alpha_S^* = 1$ .

**Theorem 3.12.** *Let  $S \subseteq V$  be a set of vertices such that  $|S| = \beta + \alpha_S^*$  and  $\alpha_S^* = 1$ ,  $G[S]$  is connected and  $S$  is not a dominator of the graph  $G$ . Hence the inequality*

$$x(S) + y(S) \leq \beta \quad (27)$$

*is facet defining if and only if for all  $u \in V \setminus S$ ,  $G[S \cup \{u\}]$  is not 2-connected.*

*Proof.* Let  $S$  be a set of vertices satisfying the conditions stated in the theorem and  $F$  be the face induced by the inequality (27) associated with  $S$ , that is

$$F = \{(x, y) \in P(G, \beta) : x(S) + y(S) = \beta\}.$$

We first have that  $F \neq \emptyset$ . Indeed, as  $S$  is not a dominator of  $G$ , there exists  $v \in V \setminus S$  non dominated by  $S$ . Hence, for all  $S' \subsetneq S$  such that  $|S'| = \beta$ , the pair of subsets  $(S', \{v\})$  induces a  $\beta$ -separator of  $G$  whose incidence vector is in  $F$ , which implies that  $F \neq \emptyset$ .

*Necessity.* Assume that there exists a vertex  $u \in V \setminus S$  such that the graph  $G[S']$ , where  $S' = S \cup \{u\}$ , is 2-connected. Note that since  $\alpha_S = 1$ ,  $\alpha_{S'} = 2$ . Consequently, the inequality (22) induced by  $S' = S \cup \{u\}$  is

$$x(S') + y(S') = x(S) + y(S) + x_u + y_u \leq \max\{|S| + 1 - 2, \beta\} = \beta. \quad (28)$$

Now let  $(\bar{x}, \bar{y}) \in F$ , that is  $\bar{x}(S) + \bar{y}(S) = \beta$ . As inequality (28) is valid for  $P(G, \beta)$ , it is also satisfied by  $(\bar{x}, \bar{y})$ . This implies that

$$\bar{x}(S') + \bar{y}(S') = \bar{x}(S) + \bar{y}(S) + \bar{x}_u + \bar{y}_u = \beta + \bar{x}_u + \bar{y}_u \leq \beta.$$

Thus,  $\bar{x}_u = \bar{y}_u = 0$ , for all  $(\bar{x}, \bar{y}) \in F$ . Thus,  $F$  cannot define a facet different from those induced by the trivial inequalities.

*Sufficiency.* Suppose that  $S$  satisfies the conditions of the theorem, that is  $|S| = \beta + 1$ ,  $\alpha_S = 1$ ,  $S$  is not a dominator of  $G$ , and for all  $u \in V \setminus S$ ,  $G[S \cup \{u\}]$  is not 2-connected. Also, assume that  $F \subseteq \tilde{F}$  where  $\tilde{F}$  is a facet of  $P(G, \beta)$  induced by an inequality  $ax + by \leq \gamma$ , that is

$$F \subseteq \tilde{F} = \{(x, y) \in P(G, \beta) : a^T x + b^T y = \gamma\}.$$

We are going to show that  $F = \tilde{F}$ . For this, we show first that  $a_u = b_u = 0$  for every  $u \in V \setminus S$ . Let  $u \in V \setminus S$ . Note that by assumption,  $G[S \cup \{u\}]$  is not 2-connected. Thus,  $G[S \cup \{u\}]$  has at least one articulation node, say  $v \in S \cup \{u\}$ . Note that, as  $\alpha_S = 1$ , we have that  $v \neq u$ . Let  $C_1, \dots, C_k$  be the connected components of  $G[(S \cup \{u\}) \setminus \{v\}]$ . Without loss of generality, we assume that  $u \in C_1$ . Also, notice that, as  $|S| = \beta + 1$ , we have that  $|C_1| \geq 1$  and  $|\bigcup_{i=2}^k C_i| \leq \beta$ . Thus, the pairs of subsets  $(C_1, \bigcup_{i=2}^k C_i)$  and  $(C_1 \setminus \{u\}, \bigcup_{i=2}^k C_i)$  induce  $\beta$ -separators of  $G$  whose incidence vectors are in  $F$ . Thus,  $a_u = 0$ . By symmetry, we also have that  $b_u = 0$ .

Now we show that  $a_u = b_u = \frac{\gamma}{\beta}$  for every  $u \in S$ . First, let  $u, v \in S$  with  $u \neq v$ . As  $S$  is not a dominator of  $G$ , there exists a vertex  $w \in V \setminus S$  non adjacent to any vertex of  $S$ . Thus,  $(\{w\}, S \setminus \{u\})$  and  $(\{w\}, S \setminus \{v\})$  are  $\beta$ -separators whose incidence vectors are in  $F$ , which implies that  $b_u = b_v$  for every pair of vertices  $u, v \in S$ . Moreover, as the  $\beta$ -separator  $(\{w\}, S \setminus \{u\})$  belongs to  $F$ , we have  $\sum_{i \in S \setminus \{u\}} b_i = (|S| - 1)b_u = \beta b_u = \gamma$ , for some  $u \in S$ , and thus  $b_u = \frac{\gamma}{\beta}$ , for every  $u \in S$ . By symmetry, we also have  $a_u = \frac{\gamma}{\beta}$ , for every  $u \in S$ . Finally, we have that  $a_u = b_u = \frac{\gamma}{\beta}$ , for all  $u \in S$ .

Consequently, we have  $F = \tilde{F}$ . □

### 3.2.4 Inequalities associated with connected dominators

Balas and De Souza [1] have given a family of valid inequalities based on connected dominators.

**Theorem 3.13.** *Let  $D$  be a non empty connected dominator of  $G$ . Then the following inequality is valid for  $P(G, \beta)$ :*

$$x(D) + y(D) \leq |D| - 1. \quad (29)$$

Balas and De Souza [1] have also provided conditions for inequalities (29) to define facets of  $P(G, \beta)$ . In the following, we propose a generalization of these inequalities to the case of connected dominators of an induced subgraph.

**Theorem 3.14.** *Let  $D \subseteq V$  be a non empty set such that  $G[D]$  is  $\alpha$ -connected, with  $\alpha \geq 1$ . Let  $S \subseteq V$  be a set of vertices such that  $D \subseteq S$  and  $D$  is a  $\nu$ -dominator of  $G[S]$  with  $0 \leq \nu \leq \alpha$ . Then the inequality*

$$x(D) + y(D) \leq |D| + \nu(x(V \setminus S) + y(V \setminus S) - 1) \quad (30)$$

is valid for  $P(G, \beta)$ .

*Proof.* Let  $D \subseteq V$  such that  $G[D]$  is  $\alpha$ -connected and  $(A, B)$  a  $\beta$ -separator where  $(x, y)$  is its incidence vector. Let  $S$  be a set of vertices which is  $\nu$ -dominated by  $D$ . If  $|(V \setminus S) \cap (A \cup B)| \geq 1$  (i.e.  $x(V \setminus S) + y(V \setminus S) \geq 1$ ), then the inequality (30) is satisfied by  $(x, y)$ . Assume now that  $|(V \setminus S) \cap (A \cup B)| = 0$  (i.e.  $x(V \setminus S) + y(V \setminus S) = 0$ ). Thus, we have  $A \cup B \subseteq S$ . We distinguish two cases.

**Case 1.**  $|D \cap A| \geq 1$  and  $|D \cap B| \geq 1$ . In this case, there exists  $u, v \in D$  such that  $u \in A$  and  $v \in B$ . However, as  $G[D]$  is  $\alpha$ -connected,  $|D \setminus (A \cup B)| \geq \alpha$ . Hence,  $|D \cap (A \cup B)| \leq |D| - \alpha \leq |D| - \nu$ .

**Case 2.**  $|D \cap A| = 0$  or  $|D \cap B| = 0$ . In this case, we have that  $A \subseteq S \setminus D$  or  $B \subseteq S \setminus D$ . Without loss of generality, assume that  $A \subseteq S \setminus D$ . Thus, there exists  $v \in S \setminus D$  such that  $v \in A$ . This implies that  $N_D(v) \subseteq D \setminus (A \cup B)$ . As  $D$  is a  $\nu$ -dominator of  $G[S]$ , we also have  $|N_D(v) \setminus (A \cup B)| \geq \nu$ . Hence,  $|D \cap (A \cup B)| \leq |D| - \nu$ .  $\square$

The inequality (30) indicates that for any  $\beta$ -separator  $(A, B)$ , if  $(V \setminus S) \cap (A \cup B) = \emptyset$ , then  $|(V \setminus (A \cup B)) \cap D| \geq \nu$ . Note that each vertex of  $D$  is adjacent to  $\alpha$  vertices of  $D$  as  $G[D]$  is  $\alpha$ -connected (the vertices of  $D$  are  $\alpha$ -dominated by  $D$ ).

We can observe that

- if  $\alpha = \nu = 1$  and  $S = V$ , then the inequality (30) induced by  $D$  and  $S$  corresponds the inequality (29) given by Balas and De Souza [1].
- if  $S = V$  and  $\alpha \neq 1$  or  $\nu \neq 1$ , then  $D$  is a  $\alpha$ -connected set and is a  $\nu$ -dominator of  $G$ . Thus, the inequality (30) associated with  $D$  and  $S$  is

$$x(D) + y(D) \leq |D| - \nu. \quad (31)$$

This inequality does not define a facet when  $D$  is not a  $\alpha$ -connected set and a *minimal*  $\nu$ -dominator of  $G$ . Moreover, inequality (31) dominates the inequality (29), given by Balas and De Souza [1]. Indeed, let  $D$  be

a  $\alpha$ -connected set and a  $\nu$ -dominator of  $G$ . Thus, for every  $u \in D$ , the set  $D \setminus \{u\}$  is at least a  $(\alpha - 1)$ -connected set and a  $(\nu - 1)$ -dominator of  $G$ . Thus, for every  $S \subset D$  such that  $|S| = \nu - 1$ , the set  $D \setminus S$  is a connected dominator of  $G$ , which yields the inequality

$$x(D \setminus S) + y(D \setminus S) \leq |D \setminus S| - 1 = |D| - \nu.$$

Therefore, inequality (29) is dominated by (31).

- It is possible to adjust the coefficients of the inequality (30) by considering the inequality (31). Let  $\alpha \geq 1$  be a positive integer,  $D$  be a  $\alpha$ -connected set and  $S$  be a set of vertices which is  $\nu$ -dominated by  $D$ . Now, let  $\nu^*$  be a positive integer such that  $D$  is  $\nu^*$ -dominator of  $G$ . Then the following inequality is valid for  $P(G, \beta)$

$$x(D) + y(D) \leq |D| - \nu^* + (\nu - \nu^*)(x(V \setminus S) + y(V \setminus S) - 1). \quad (32)$$

Note that this inequality is dominated by the inequality (22) if  $|D| - \nu \geq \beta$ .

### 3.2.5 Inequalities associated with dominators

Before starting this section, we notice that all the inequalities we present here have their symmetric versions. Thus, for each type of inequality, we will establish the proofs only for one version.

Balas and De Souza [1] have given a family of valid inequalities based on the dominators.

**Theorem 3.15.** [1]

*Let  $D$  be a non empty dominator of the graph  $G$ . Then, the following inequality is valid for the polyhedra  $P(G, \beta)$*

$$x(D) \leq |D| - 1. \quad (33)$$

Balas and De Souza [1] have defined three necessary and sufficient conditions of facets for the inequality (33). They have also proposed lifting procedures on these inequalities for the case where these conditions are not satisfied.

Here, we propose a generalization of these inequalities. For this, we define inequalities associated with subgraphs of  $G$  which induce dominators of  $G$ .

**Theorem 3.16.** *Let  $D \subseteq V$  be a non empty set of vertices of  $G$  and let  $\alpha = \min_{u \in D} \{d_D(u)\} + 1$ . Let  $S \subseteq V$  be a set of vertices such that  $D \subseteq S$  and  $D$  be a  $\alpha$ -dominator of  $G[S]$ . Then, the inequalities*

$$x(D) \leq |D| + \alpha(y(V \setminus S) - 1) \quad (34)$$

and

$$y(D) \leq |D| + \alpha(x(V \setminus S) - 1) \quad (35)$$

are valid for  $P(G, \beta)$ .

*Proof.* We establish the proof only for (34). That for (35) is obtained by symmetry. Let  $D \subseteq V$  and  $(A, B)$  be a  $\beta$ -separator where  $(x, y)$  is its incidence vector. Let  $S$  be the set of vertices  $\alpha$ -dominated by  $D$ . If  $|(V \setminus S) \cap B| \geq 1$  (i.e.  $y(V \setminus S) \geq 1$ ), then the inequality is satisfied by the separator  $(A, B)$ . Assume that  $|(V \setminus S) \cap B| = 0$  (i.e.  $y(V \setminus S) = 0$ ), then  $B \subseteq S$ . We have two possible cases: either  $|(S \setminus D) \cap B| \geq 1$ , or  $|(S \setminus D) \cap B| = 0$ . The first case implies that there exists  $v \in S \setminus D$  such that  $v \in B$ . Thus  $(N(v) \cap D) \cap A = \emptyset$ . Then, by  $\alpha$ -dominance, we have  $|D \cap A| \leq |D| - \alpha$ . The second case implies that there exists  $v \in D$  such that  $v \in B$ . Thus  $(N_D(v) \cup \{v\}) \cap A = \emptyset$ . Then we have  $|D \cap A| \leq |D| - (d_D(v) + 1) \leq |D| - \alpha$  as  $\alpha = \min_{u \in D} \{d_D(u)\} + 1$ .  $\square$

Inequality (34) indicates that for any  $\beta$ -separator  $(A, B)$ , if none of the vertex of  $V \setminus S$  is in the set  $B$ , then there exists at least  $\alpha$  vertices of  $D$  which are not in the set  $A$ . Note that the vertices in  $D$  are  $\alpha$ -dominated by  $D$  as  $\alpha = \min_{u \in D} \{d_D(u)\} + 1$ .

We also make several observations.

- If  $\alpha = 1$  and  $V \setminus S = \emptyset$ , we obtain the inequality (33), given by Balas and De Souza [1].
- If  $V \setminus S = \emptyset$  and  $\alpha > 1$ , the corresponding inequality (34) is

$$x(D) \leq |D| - \alpha. \quad (36)$$

This latter inequality is clearly dominated by (3) when  $|D| - \alpha \geq \beta$ .

- If  $\alpha = 1$  and  $V \setminus S \neq \emptyset$  (i.e.  $D$  is a dominator of  $G[S]$ ), the inequality (34) becomes

$$x(D) \leq |D| - 1 + y(V \setminus S). \quad (37)$$

The following theorem gives necessary conditions for inequalities (34) to define facets.

**Theorem 3.17.** *The inequality (34) induced by two sets  $D$  and  $S$  defines a facet only if*

1.  $|D| < \beta$ .
2. The set  $D$  is a minimal dominator of  $G[S]$ .

*Proof.* 1) Suppose on the contrary that  $|D| \geq \beta + 1$ . One can then easily see that the inequality (34) is dominated by the constraint (3).

2) Suppose that  $D$  is not a minimal dominator, i.e. there exists  $u \in D$  such that  $D \setminus \{u\}$  is a dominator of  $G[S]$ . Thus, as before, one can see that inequality (37) with the dominator  $D \setminus \{u\}$  dominates the inequality (34) induced by  $D$ .  $\square$

The next theorem give conditions for inequalities (37) to be facet defining.

**Theorem 3.18.** *Let  $D \subseteq V$  be a non empty set of vertices such that  $|D| < \beta$ . Let  $S \subset V$  be a set of vertices such that  $D \subseteq S$ ,  $D$  be a minimal dominator of  $G[S]$  and  $|S|$  is maximal. Then, the inequality (37) induced by  $D$  and  $S$  is facet defining if and only if*

1. *Every vertex of  $S \setminus D$  is a pendent vertex of  $D$ , that is  $S \setminus D = \bigcup_{u \in D} P(u)$ ,*
2.  *$D$  contains no self-dominating vertices of  $G[S]$ ,*
3.  *$D$  is a stable set.*

Note that imposing the condition that  $|S|$  is maximal is not restrictive and implies that  $\delta(D, V \setminus S) = \emptyset$ . Indeed, for every  $v \in V \setminus S$  such that  $uv \in E$  and  $u \in D$ , just put  $v \in S$  and  $D$  remains a minimal dominator of  $G[S]$ .

*Proof.* Let  $F$  be the face induced by inequality (37), that is

$$F = \{(x, y) \in P(G, \beta) : x(D) = |D| - 1 + y(V \setminus S)\}.$$

1. *Necessity.* If any of the three conditions is not satisfied, then the result is the same as the proof of facet for the inequality (33) in [1] by setting  $u \notin A \cup B$  (i.e.  $x_u = y_u = 0$ ), for every  $u \in V \setminus S$ .
2. *Sufficiency.* To show that the conditions of the theorem implies that  $F$  is a facet we are going to explicit  $2|V|$  solutions of  $F$  which are affinely independent. Let the graph  $G[S]$ . Then  $D$  is a dominator of  $G[S]$ , a stable, without self-dominating vertex and such that  $S \setminus D = \bigcup_{u \in D} P(u)$ . Thus, the inequality  $x(D) \leq |D| - 1$  induces a facet in  $G[S]$ , noted  $F_S$ . It therefore exist  $2|S|$  solutions affinely independent on the face  $F_S$ , noted  $(\tilde{x}^k, \tilde{y}^k), \forall k = 1, \dots, 2|S|$  with  $\tilde{x}^k, \tilde{y}^k \in \{0, 1\}^{|S|}$ . From the previous solutions, we build  $2|S|$  solutions affinely independent on the face  $F$ . For every  $k = 1, \dots, 2|S|$ , let  $(x^k, y^k)$  be the following solutions, where  $x^k, y^k \in \{0, 1\}^{|V|}$ :

$$x_i^k = \begin{cases} \tilde{x}_i^k & \text{si } i \in S \\ 0 & \text{sinon} \end{cases} \quad \text{et} \quad y_i^k = \begin{cases} \tilde{y}_i^k & \text{si } i \in S \\ 0 & \text{sinon} \end{cases}, \quad \forall i \in V.$$

Then these solutions are trivially affinely independent. Moreover, for every vertex  $u \in V \setminus S$ , as there is no edge between  $D$  and  $V \setminus S$ , the  $\beta$ -separator  $(D, \{u\})$  is in  $F$ . Its incidence vector is affinely independent compared to the previous solutions. So we obtain a different vector for every vertex  $u \in V \setminus S$ . Therefore, we have built  $|V \setminus S|$  solutions affinely independent between them. Endly, let  $v \in D$  be a vertex. Then, for every vertex  $u \in V \setminus S$ , as  $D$  is a stable, the  $\beta$ -separator  $((D \setminus \{v\}) \cup \{u\}, \{v\})$  is in  $F$ . Its

incidence vector is affinely independent compared to the previous ones. So we obtain a different vector for every vertex  $u \in V \setminus S$ . Consequently, we have build  $|V \setminus S|$  solutions, which together with the previous solutions, form  $2|V|$  solutions of  $F$  which are affinely independent.  $\square$

Now we will give inequalities from the inequalities (37) when the condition 2 is not satisfied. Let  $D \subseteq V$  be a dominator of the graph  $G[S]$ , where  $D \subseteq S \subseteq V$ . Let  $D_a \subseteq D$  be the set of self-dominating vertices of  $D$ . Then, the set  $D \setminus D_a$  is a dominator of the graph  $G[S \setminus D_a]$ . The inequality (37) applied to the set  $D \setminus D_a$  therefore gives

$$x(D \setminus D_a) \leq |D \setminus D_a| - 1 + y((V \setminus S) \cup D_a) = |D| - 1 + y(V \setminus S) + (y(D_a) - |D_a|).$$

This inequality dominates the inequality (37) applied to  $D$  (because  $x(D_a) + y(D_a) \leq |D_a|$ ) and verifies the condition 2.

In particular, if  $D$  is a dominator of  $G$ , then the following inequality dominates (37) applied to  $D$  given by Balas and De Souza (when  $V \setminus S = \emptyset$ ):

$$x(D \setminus D_a) \leq |D| - |D_a| - 1 + y(D_a),$$

where  $D_a \subset D$  is the set of self-dominating vertices of  $D$ .

Finally, note that the lifting procedures of Balas and De Souza for the inequality (33) can be adapted to the inequality (37). Through these procedures, the conditions 3 and 1 are satisfied.

## 4 Concluding remarks

In this paper, we have investigated the polytope associated with the vertex separator problem. We have introduced several classes of inequalities and given conditions for these inequalities to define facets. The inequalities introduced here generalizes those given by Balas and De Souza [1]. It is important to notice that one objective of this work is to give a more complete description as possible of the polytope  $P(G, \beta)$ . Now, it should interesting to investigate the efficiency of the various inequalities we have given in terms of computations. Namely, we have to discuss the separation problem associated with each class of inequalities and include them in a Branch-and-Cut algorithm for the VSP. This will be the subject of a future work.

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