Effective models for generalized Newtonian fluids through a thin porous media following the Carreau law

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Abstract

We consider the flow of generalized Newtonian fluid through a thin porous media Ω_ε of thickness ε which is perforated by periodically distribute solid cylinders of size ε. Generalizing (Anguiano *et al.*, Q. J. Mech. Math., 75(1), 2022, 1-27) we do a new and complete study on the fluid described by the 3D incompressible Stokes system with non-linear viscosity, following the Carreau law of flow index $1 < r < +\infty$, which is scaled by a factor ε^γ, where $\gamma \in \mathbb{R}$. Depending of γ and the flow index r, using homogenization techniques, we derive different effective linear and non-linear lower-dimensional Darcy's laws.

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1 Introduction

The incompressible generalized Newtonian fluids is a type of non-Newtonian fluids which are characterized by the viscosity depending on the principal invariants of the symmetric stretching tensor $\mathbb{D}[u]$. If u is the velocity, p the pressure and Du the gradient velocity tensor, $\mathbb{D}[u] = (Du + D^t u)/2$ denotes the symmetric stretching tensor and σ the stress tensor given by $\sigma = -pI + 2\eta_r \mathbb{D}[u]$. The viscosity η_r is constant for a Newtonian fluid but dependent on the shear rate, i.e. $\eta_r = \eta_r(\mathbb{D}[u])$, for viscous generalized Newtonian fluids. The deviatoric stress tensor τ, i.e. the part of the total stress tensor that is zero at equilibrium, is then a nonlinear function of the shear rate $\mathbb{D}[u]$, i.e. $\tau = \eta_r(\mathbb{D}[u])\mathbb{D}[u]$ (see Barnes *et al.* [19], Bird *et al.* [20] and Mikelić [32] for more details).

A commonly used viscosity formula η_r used for describing generalized Newtonian fluids is the *power law* or Ostwald-de Waele model (Ostwald, 1925; de Waele, 1923), which is defined by

$$\eta_r(\mathbb{D}[u]) = \mu|\mathbb{D}[u]|^{r-2}, \quad 1 < r < +\infty, \quad \mu > 0, \tag{1}$$

where $\mu > 0$ is the consistency of the fluid and r is the flow index. The matrix norm $|\cdot|$ is defined by $|\xi|^2 = Tr(\xi\xi^t)$ with $\xi \in \mathbb{R}^3$. We recall that the generalized Newtonian fluids are classified in two main categories (see Saramito [34, Chapter 2] for more details):

- *Pseudoplastic* or *shear thinning* fluids, where the viscosity decreases with the shear rate, which correspond to the case of flow index $1 < r < 2$.
- *Dilatant* or *shear thickening* fluids, where the viscosity increases with the shear rate, which correspond to the case of flow index $r > 2$.

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We also recall that the case $r = 2$ corresponds to the case of Newtonian fluids.

Power law model describes correctly the behavior of polymers at high shear rates, offering the advantage of allowing analytical calculations in simple geometries. However, it has the disadvantage of not describing a Newtonian plateau and even predicts an infinite viscosity as the shear rate goes to zero and $1 < r < 2$ (see Agassant et al. [2], p. 49), whereas for real fluids it tends to some constant value η_0 called the zero-shear-rate viscosity. For that reason, other viscosity models are used, which describe better the real behavior of the fluids but are more difficult to analyze mathematically.

An important model among these ones is the well-known *Carreau law*, which will be considered in this paper and is defined by

$$\eta_r(\mathbb{D}[u]) = (\eta_0 - \eta_\infty)(1 + \lambda|\mathbb{D}[u]|^2)^{\frac{r}{2}-1} + \eta_\infty, \quad 1 < r < +\infty, \quad \eta_0 > \eta_\infty > 0, \quad \lambda > 0, \tag{2}$$

where η_∞ is the high-shear-rate limit of the viscosity, the parameter λ is a time constant and $r - 1$ is describes the slope in the power law region.

Related to the generalized Newtonian fluids, the interest in their behavior through thin porous media has been increasing recently, mainly because of its importance in many industrial processes (see Prat and Agaësse [33] for more details). For instance, we refer to recent studies by Anguiano and Suárez-Grau [11, 14, 16], Frabricius *et al.* [28], Bunoiu and Timofte [24], Jouybari and T. S. Lundström [29], and Yeghiazarian *et al.* [38], where the behavior of Newtonian or power law fluids through thin porous media has been considered. However, the literature on Carreau fluid flows in this type of domains is far less complete, although these problems have now become of great practical relevance to Chemical Industry and Rheology, for instance in injection moulding of melted polymers, flow of oils, muds, etc.

In this paper, we consider a thin porous media $\Omega_\varepsilon = \omega_\varepsilon \times (0, \varepsilon) \subset \mathbb{R}^3$ of small height ε which is perforated by periodically distributed solid cylinders of diameter of size ε. Here, the bottom of the domain without perforations $\omega \subset \mathbb{R}^2$ is made of two parts, the fluid part ω_ε and the solid part $\omega \setminus \omega_\varepsilon$. Assuming that the flow is sufficiently slow to neglect inertial effects, we consider the following stationary Stokes system with a non-linear viscosity following the *Carreau law* (2) that we scale by a factor ε^γ, where $\gamma \in \mathbb{R}$, in the thin porous media:

$$\begin{cases} -\varepsilon^\gamma \operatorname{div}\left(\eta_r(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]\right) + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3}$$

Vey recently, this problem has been considered in Anguiano *et al.* [8] in the particular case of pseudoplastic fluids ($1 < r < 2$) and $\gamma = 1$. By using homogenization techniques, when ε tends to zero, it was proved that the mean global filtration velocity as a function of the pressure gradient is given by a non-linear 2D Darcy law of Carreau type

$$\begin{cases} V'(x') = \mathcal{U}\left(f'(x') - \nabla_{x'} p(x')\right), \quad V_3(x') = 0 \text{ in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \ \text{ in } \omega, \quad V'(x') \cdot n = 0 \ \text{ on } \partial\omega, \end{cases} \tag{4}$$

where n is the outward normal to $\partial\omega$, $V' = (V_1, V_2)$, $x' = (x_1, x_2)$, and the *permeability* function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined through the solutions of 3D local non-Newtonian Stokes problems with non-linear viscosity following the *Carreau law* (2) and posed in a reference cell.

In this paper, we obtain a new and complete study on the asymptotic behavior of Carreau fluids, given by (3), depending on the type of fluid and the value of γ. We generalize the results obtained in [8] by considering system (3) not only for pseudoplastic fluids, but also for dilatant fluids and Newtonian fluids, and moreover, we consider the case $\gamma \in \mathbb{R}$. Starting from problem (3), by using homogenization techniques when ε tends to zero, we derive different effective problems depending on the type of fluid and the value of γ. To do this, we use an adaptation of the unfolding method to define the unfolded velocity and pressure $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ which take into

account the microstructure of the domain. For instance, in the pseudoplastic case, it holds that $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ satisfies the following variational formulation (see (75) for more details)

$$
\begin{aligned}
&(\eta_0-\eta_\infty)\int_{\omega\times Y}(1+\lambda\varepsilon^{2(1-\gamma)}|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1}\mathbb{D}_y[\varphi]:\mathbb{D}_y[\varphi-\varepsilon^{\gamma-2}\hat{u}_\varepsilon]\,dx'dy\\
&+\eta_\infty\int_{\omega\times Y}\mathbb{D}_y[\varphi]:\mathbb{D}_y[\varphi-\varepsilon^{\gamma-2}\hat{u}_\varepsilon]dx'dy\\
&-\int_{\omega\times Y}\hat{P}_\varepsilon\,\mathrm{div}_{x'}(\varphi'-\varepsilon^{\gamma-2}\hat{u}'_\varepsilon)\,dx'dy\geq\int_{\omega\times Y}f'\cdot(\varphi'-\varepsilon^{\gamma-2}\hat{u}'_\varepsilon)\,dx'dy+O_\varepsilon,
\end{aligned}
$$

and that $(\varepsilon^{\gamma-2}\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ converges, as ε tends to zero, in appropriate Sobolev spaces to a pair of functions called $(\hat{u}, \tilde{P})$. Then, we observe that if $\gamma<1$, then $2(1-\gamma)>0$ and so, $\lambda\varepsilon^{2(1-\gamma)}|\mathbb{D}_y[\varphi]|^2$ tends to zero, and if $\gamma>1$, then $2(1-\gamma)<0$ and so, $(1+\lambda\varepsilon^{2(1-\gamma)}|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1}$ tends to zero. As consequence, the two first terms of the previous formulation converge to the linear term

$$
\eta\int_{\omega\times Y}\mathbb{D}_y[\varphi]:\mathbb{D}_y[\varphi-\hat{u}]dx'dy,
$$

with $\eta=\eta_0$ if $\gamma<1$ and $\eta=\eta_\infty$ if $\gamma>1$. In the case $\gamma=1$, the critical case that couples the nonlinear term and the linear one, they converge to

$$
(\eta_0-\eta_\infty)\int_{\omega\times Y}(1+\lambda|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1}\mathbb{D}_y[\varphi]:\mathbb{D}_y[\varphi-\hat{u}]\,dx'dy+\eta_\infty\int_{\omega\times Y}\mathbb{D}_y[\varphi]:\mathbb{D}_y[\varphi-\hat{u}]dx'dy.
$$

Thus, we obtain three different asymptotic behaviors depending on whether the value of γ is smaller, equal or greater than 1. For dilatant fluids, there exist three different convergences of the unfolding velocity depending on the value of γ and, as consequence, three different homogenized models are derived. In summary, we have the following asymptotic behaviors of Carreau fluids depending on the type of fluid and the value of γ:

(i) For pseudoplastic fluids, we obtain the following asymptotic behaviors depending on the value of γ:

- If $\gamma\neq1$, then the effective problem is a linear 2D Darcy law

$$
\begin{cases}
V'(x')=\dfrac{1}{\eta}\mathcal{A}\left(f'(x')-\nabla_{x'}p(x')\right), \quad V_3(x')=0 \quad \text{in } \omega,\\
\mathrm{div}_{x'}V'(x')=0 \ \text{ in } \omega, \quad V'(x')\cdot n=0 \ \text{ on } \partial\omega,
\end{cases}
$$

  where the *permeability* tensor $\mathcal{A}\in\mathbb{R}^{2\times2}$ is obtained by solving 3D local Newtonian Stokes local problems posed in a reference cell which contains the information of the geometry of the obstacles. The viscosity η is equal to η_0 if $\gamma<1$ and equal to η_∞ if $\gamma>1$.
- If $\gamma=1$, then the effective problem is the non-linear 2D Darcy law of *Carreau* type (4), which is obtained in [8].

(ii) For dilatant fluids, we obtain three different asymptotic behaviors depending on the value of γ:

- If $\gamma<1$, then the filtration velocity is zero.
- If $\gamma>1$, then the effective problem is a non-linear 2D Darcy law of power law type

$$
\begin{cases}
V'(x')=\dfrac{\lambda^{\frac{2-r}{2}}}{\eta_0-\eta_\infty}\mathcal{U}\left(f'(x')-\nabla_{x'}p(x')\right), \quad V_3(x')=0 \text{ in } \omega,\\
\mathrm{div}_{x'}V'(x')=0 \ \text{ in } \omega, \quad V'(x')\cdot n=0 \ \text{ on } \partial\omega,
\end{cases}
$$

  where the *permeability* function $\mathcal{U}:\mathbb{R}^2\to\mathbb{R}^2$ is defined through the solutions of 3D local non-Newtonian Stokes problems with non-linear viscosity following the *power law* (1) and posed in a reference cell.

– If $\gamma = 1$, then the effective problem is a non-linear 2D Darcy law of *Carreau* type as in the case of pseudoplastic fluids.

(iii) For Newtonian fluids, we obtain that, for $\gamma \in \mathbb{R}$, the effective problem is a linear 2D Darcy law

$$\begin{cases} V'(x') = \dfrac{1}{\eta_0}\mathcal{A}\left(f'(x') - \nabla_{x'}p(x')\right), \quad V_3(x') = 0 \quad \text{in } \omega, \\ \operatorname{div}_{x'}V'(x') = 0 \ \text{ in } \omega, \quad V'(x')\cdot n = 0 \ \text{ on } \partial\omega, \end{cases}$$

where the *permeability* tensor $\mathcal{A} \in \mathbb{R}^{2\times 2}$ is obtained by solving 3D local Newtonian Stokes local problems posed in a reference cell which contains the information of the geometry of the obstacles.

In Table 1, we summarize every asymptotic behavior of the Carreau fluid governing by (3) depending on the type of fluid and the value of γ:

	$1 < r < 2$	$r = 2$	$r > 2$
$\gamma < 1$	Linear 2D Darcy's law (viscosity η_0)	Linear 2D Darcy's law (viscosity η_0)	Zero filtration velocity
$\gamma = 1$	Non-linear 2D Darcy's law (Carreau type)		Non-linear 2D Darcy's law (Carreau type)
$\gamma > 1$	Linear 2D Darcy's law (viscosity η_∞)		Non-linear 2D Darcy's law (power law type)

Table 1: Asymptotic behaviors of Carreau fluids depending on the values of r and γ.

We finish the introduction with a list of references of other recent studies concerning thin porous media. Some stationary models for different fluids are obtained in Anguiano and Suárez-Grau [12, 13, 15] and Zhengan and Hongxing [39], and some non-stationary models are developed in Anguiano [3, 4, 5, 6, 7]. The case of a Bingham flow is considered in Anguiano and Bunoiu [9, 10] and the case of micropolar fluids in Suárez-Grau [36, 37]. The two-phase flow problem in thin porous media domains of Brinkman type has been considered in Armiti-Juber [17] and an approach for effective heat transport in thin porous media has been derived by Scholz and Bringedal [35].

The structure of the paper is as follows. In Section 2 we introduce the domain, make the statement of the problem and give the main results (Theorems 2.1, 2.3 and 2.5). The proofs of the main results are provided in Section 3. We finish the paper with a list of references.

2 Setting of the problem and main result

Geometrical setting. The periodic porous medium is defined by a domain ω and an associated microstructure, or periodic cell $Y' = (-1/2, 1/2)^2$, which is made of two complementary parts: the fluid part Y'_f, and the solid part T' ($Y'_f \bigcup T' = Y'$ and $Y'_f \bigcap T' = \emptyset$). More precisely, we assume that ω is a smooth, bounded, connected set in $\mathbb{R}^2$ with smooth enough boundary $\partial\omega$, that n is the outward normal to $\partial\omega$, and that T' is an open connected subset of Y' with a smooth boundary $\partial T'$, such that $\bar{T}'$ is strictly included in Y'.

The microscale of a porous medium is a small positive number ε. The domain ω is covered by a regular mesh of square of size ε: for $k' \in \mathbb{Z}^2$, each cell $Y'_{k',\varepsilon} = \varepsilon k' + \varepsilon Y'$ is divided in a fluid part $Y'_{f_{k',\varepsilon}}$ and a solid part $T'_{k',\varepsilon}$, i.e. is similar to the unit cell Y' rescaled to size ε. We define $Y = Y' \times (0,1) \subset \mathbb{R}^3$, which is divided in a fluid part $Y_f = Y'_f \times (0,1)$ and a solid part $T = T' \times (0,1)$, and consequently $Y_{k',\varepsilon} = Y'_{k',\varepsilon} \times (0,1) \subset \mathbb{R}^3$, which is also divided in a fluid part $Y_{f_{k',\varepsilon}}$ and a solid part $T_{k',\varepsilon}$ (see Figures 1 and 2).

We denote by $\tau(\overline{T}'_{k',\varepsilon})$ the set of all translated images of $\overline{T}'_{k',\varepsilon}$. The set $\tau(\overline{T}'_{k',\varepsilon})$ represents the obstacles in $\mathbb{R}^2$.

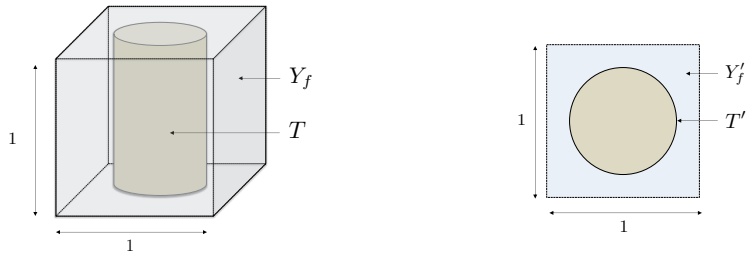

Figure 1: View of the 3D reference cells Y (left) and the 2D reference cell Y' (right).

Figure 2: View of the 3D reference cells $Y_{k',\varepsilon}$ (left) and the 2D reference cell $Y'_{k',\varepsilon}$ (right).

The fluid part of the bottom $\omega_\varepsilon \subset \mathbb{R}^2$ of a porous medium is defined by $\omega_\varepsilon = \omega \backslash \bigcup_{k' \in \mathcal{K}_\varepsilon} \overline{T}'_{k',\varepsilon}$, where $\mathcal{K}_\varepsilon = \{k' \in \mathbb{Z}^2 : Y'_{k',\varepsilon} \cap \omega \neq \emptyset\}$. The whole fluid part $\Omega_\varepsilon \subset \mathbb{R}^3$ in the thin porous medium is defined by (see Figure 3)

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \in \omega_\varepsilon \times \mathbb{R} : 0 < x_3 < \varepsilon\}. \tag{5}$$

We assume that the obstacles $\tau(\overline{T}'_{k',\varepsilon})$ do not intersect the boundary $\partial\omega$ and we denote by S_ε the set of the solid cylinders contained in Ω_ε, i.e. $S_\varepsilon = \bigcup_{k' \in \mathcal{K}_\varepsilon} T'_{k',\varepsilon} \times (0,\varepsilon)$.

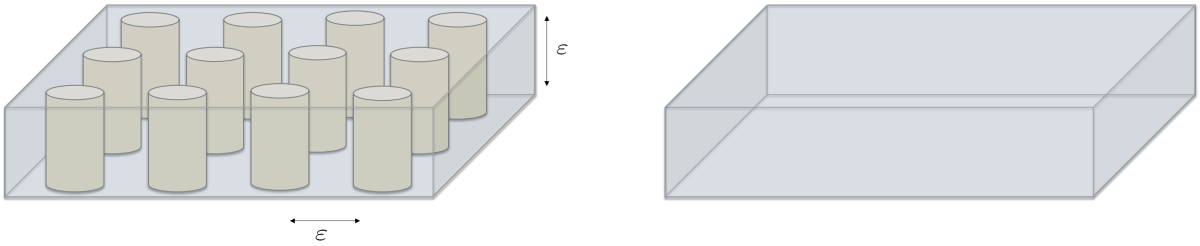

Figure 3: View of the thin porous media Ω_ε (left) and domain without perforations Q_ε (right).

We define

$$\widetilde{\Omega}_\varepsilon = \omega_\varepsilon \times (0,1), \quad \Omega = \omega \times (0,1), \quad Q_\varepsilon = \omega \times (0,\varepsilon). \tag{6}$$

We observe that $\widetilde{\Omega}_\varepsilon = \Omega \backslash \bigcup_{k' \in \mathcal{K}_\varepsilon} \bar{T}_{k',\varepsilon}$, and we define $T_\varepsilon = \bigcup_{k' \in \mathcal{K}_\varepsilon} T_{k',\varepsilon}$ as the set of the solid cylinders contained in $\widetilde{\Omega}_\varepsilon$.

To finish, we introduce some notation that will be useful along the paper. The points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' = (x_1, x_2) \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$. Let $C^\infty_\#(Y)$ be the space of infinitely differentiable functions in $\mathbb{R}^3$ that are Y'-periodic. By $L^q_\#(Y)$ (resp. $W^{1,q}_\#(Y)$), $1 < q < +\infty$, we denote its completion in the norm $L^q(Y)$ (resp. $W^{1,q}(Y)$) and by $L^q_{0,\#}(Y)$ the space of functions in $L^q_\#(Y)$ with zero mean value.

Statement of the problem. Let us consider the following stationary Stokes system with the non-linear viscosity following the *Carreau law* (2) in Ω_ε, with a Dirichlet boundary condition on the exterior boundary ∂Q_ε

and the cylinders ∂S_ε,

$$\begin{cases} -\varepsilon^\gamma \mathrm{div}\,(\eta_r(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]) + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \mathrm{div}\, u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial Q_\varepsilon \cup \partial S_\varepsilon, \end{cases} \tag{7}$$

where the second member f is of the form

$$f(x) = (f'(x'), 0) \quad \text{with } f' \in L^\infty(\omega)^2. \tag{8}$$

We remark that the assumptions of neglecting the vertical component of the exterior force and the independence of the vertical variable are usual when dealing with fluids in through thin domains (see [21] for more details).

Under previous assumptions, the classical theory (see for instance [21, 22, 30]), gives the existence of a unique weak solution $(u_\varepsilon, p_\varepsilon) \in H_0^1(\Omega_\varepsilon)^3 \times L_0^2(\Omega_\varepsilon)$, for $1 < r \leq 2$, and $(u_\varepsilon, p_\varepsilon) \in W_0^{1,r}(\Omega_\varepsilon)^3 \times L_0^{r'}(\Omega_\varepsilon)$ with $1/r + 1/r' = 1$, for $r > 2$, where L_0^2 (respectively $L_0^{r'}$) is the space of functions of L^2 (respectively $L^{r'}$) with zero mean value.

Our goal is to study the asymptotic behavior of u_ε and p_ε when ε tends to zero. For this purpose, we use the dilatation in the variable x_3 as follows

$$y_3 = \frac{x_3}{\varepsilon}, \tag{9}$$

in order to have the functions defined in the open set with fixed height $\widetilde{\Omega}_\varepsilon$. Namely, we define $\tilde{u}_\varepsilon$ and $\tilde{p}_\varepsilon$ by

$$\tilde{u}_\varepsilon(x', y_3) = u_\varepsilon(x', \varepsilon y_3), \quad \tilde{p}_\varepsilon(x', y_3) = p_\varepsilon(x', \varepsilon y_3), \quad a.e.\ (x', y_3) \in \widetilde{\Omega}_\varepsilon.$$

Let us introduce some notation which will be useful in the following. For a vectorial function $v = (v', v_3)$ and a scalar function w, we will denote $\mathbb{D}_{x'}[v] = \frac{1}{2}(D_{x'}v + D_{x'}^t v)$ and $\partial_{y_3}[v] = \frac{1}{2}(\partial_{y_3} v + \partial_{y_3}^t v)$, where we denote $\partial_{y_3} = (0, 0, \frac{\partial}{\partial y_3})^t$. Moreover, associated to the change of variables (9), we introduce the operators: $\mathbb{D}_\varepsilon$, D_ε, div_ε and ∇_ε, by

$$\mathbb{D}_\varepsilon[v] = \frac{1}{2}\left(D_\varepsilon v + D_\varepsilon^t v\right),$$

$$(D_\varepsilon v)_{i,j} = \partial_{x_j} v_i \text{ for } \ i = 1,2,3,\ j = 1,2, \quad (D_\varepsilon v)_{i,3} = \varepsilon^{-1}\partial_{y_3} v_i \text{ for } \ i = 1,2,3,$$

$$\mathrm{div}_\varepsilon v = \mathrm{div}_{x'} v' + \varepsilon^{-1}\partial_{y_3} v_3, \quad \nabla_\varepsilon w = (\nabla_{x'} w, \varepsilon^{-1}\partial_{y_3} w)^t.$$

Using the transformation (9), system (7) can be rewritten as

$$\begin{cases} -\varepsilon^\gamma \mathrm{div}_\varepsilon\,(\eta_r(\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon])\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]) + \nabla_\varepsilon \tilde{p}_\varepsilon = f \text{ in } \widetilde{\Omega}_\varepsilon, \\ \mathrm{div}_\varepsilon \tilde{u}_\varepsilon = 0 \text{ in } \widetilde{\Omega}_\varepsilon, \\ \tilde{u}_\varepsilon = 0 \text{ on } \partial\Omega \cup \partial T_\varepsilon. \end{cases} \tag{10}$$

Our goal then is to describe the asymptotic behavior of this new sequence $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$. The sequences of solutions $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ is not defined in a fixed domain independent of ε but rather in a varying set $\widetilde{\Omega}_\varepsilon$. In order to pass the limit if ε tends to zero, convergences in fixed Sobolev spaces (defined in Ω) are used which requires first that $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ be extended to the whole domain Ω. Then, an extension $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in W_0^{1,q}(\Omega)^3 \times L_0^{q'}(\Omega)$ is defined on Ω and coincides with $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ on $\widetilde{\Omega}_\varepsilon$ (we will use the same notation, $\tilde{u}_\varepsilon$, for the velocity in $\widetilde{\Omega}_\varepsilon$ and its continuation in Ω).

Our main results referred to the asymptotic behavior of the solution of (10) is given by the following theorems.

Theorem 2.1 (Pseudoplastic fluids)**.** *Consider $1 < r < 2$ and $\gamma \in \mathbb{R}$. Then, there exist $\tilde{u} \in H_0^1(0,1; L^2(\omega)^3)$ with $\tilde{u}_3 = 0$ and $\tilde{P} \in L_0^2(\omega)$, such that the extension $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ of the solution of (10) satisfies the following convergences*

$$\varepsilon^{\gamma-2}\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \quad \textit{weakly in } H^1(0,1; L^2(\omega)^3), \quad \tilde{P}_\varepsilon \to \tilde{P} \quad \textit{strongly in } L^2(\Omega).$$

Moreover, defining $V(x') = \int_0^1 \tilde{u}(x', y_3)\, dy_3$, it holds that $(V, \tilde{P}) \in L^2(\omega)^3 \times (L_0^2(\omega) \cap H^1(\omega))$ is the unique solution of a lower-dimensional effective Darcy's law depending on the value of γ. More precisely:

– *If $\gamma \neq 1$, then $(V, \tilde{P})$ is the unique solution of the linear 2D Darcy's law*

$$\begin{cases} V'(x') = \dfrac{1}{\eta}\mathcal{A}\left(f'(x') - \nabla_{x'}\tilde{P}(x')\right), \quad V_3(x') = 0 \quad \text{in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \quad \text{in } \omega, \quad V'(x')\cdot n = 0 \quad \text{on } \partial\omega, \end{cases} \tag{11}$$

where

$$\eta = \begin{cases} \eta_0 & \text{if } \gamma < 1, \\ \eta_\infty & \text{if } \gamma > 1, \end{cases}$$

and the permeability tensor $\mathcal{A} \in \mathbb{R}^{2\times 2}$ is defined by its entries

$$\mathcal{A}_{ij} = \int_{Y_f} (w_j^i)'(y)\, dy, \quad i,j = 1,2, \tag{12}$$

with $(w^i, \pi^i) \in H^1_{0,\#}(Y_f)^3 \times L^2_{0,\#}(Y_f)$, $i = 1,2$, the unique solution of the local Stokes system

$$\begin{cases} -\Delta_y w^i + \nabla_y \pi^i = e_i & \text{in } Y_f, \\ \operatorname{div}_y w^i = 0 & \text{in } Y_f, \\ w^i = 0 & \text{on } \partial T, \\ y \to w^i, \pi^i & Y-\text{periodic}, \end{cases} \tag{13}$$

and $\{e_i\}_{i=1,2,3}$ being the canonical basis of $\mathbb{R}^3$.

– *If $\gamma = 1$, then $(V, \tilde{P})$ is the unique solution of the non-linear 2D Darcy's law of Carreau type*

$$\begin{cases} V'(x') = \mathcal{U}\left(f'(x') - \nabla_{x'}\tilde{P}(x')\right), \quad V_3(x') = 0 \quad \text{in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \quad \text{in } \omega, \quad V'(x')\cdot n = 0 \quad \text{on } \partial\omega, \end{cases} \tag{14}$$

where the permeability function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathcal{U}(\xi') = \int_{Y_f} w'_{\xi'}(y)\, dy, \quad \forall\, \xi' \in \mathbb{R}^2, \tag{15}$$

with $(w_{\xi'}, \pi_{\xi'}) \in H^1_{0,\#}(Y_f)^3 \times L^2_{0,\#}(Y_f)$, for every $\xi' \in \mathbb{R}^2$, the unique solution of the local Stokes system

$$\begin{cases} -\operatorname{div}_y(\eta_r(\mathbb{D}_y[w_{\xi'}])\mathbb{D}_y[w_{\xi'}]) + \nabla_y \pi_{\xi'} = \xi' & \text{in } Y_f, \\ \operatorname{div}_y w_{\xi'} = 0 & \text{in } Y_f, \\ w_{\xi'} = 0 & \text{on } \partial T, \end{cases} \tag{16}$$

and the nonlinear viscosity η_r is given by the Carreau law (2).

Remark 2.2. *According to [1, Theorem 1.1], the permeability tensor $\mathcal{A}$ is symmetric and definite positive.*

Theorem 2.3 (Dilatant fluids)**.** *Consider $r > 2$ and $\gamma \in \mathbb{R}$. We divide the theorem depending on the value of γ:*

(i) If $\gamma < 1$, then it holds there exist $\tilde{u} \in W^{1,r}_0(0,1; L^r(\omega)^3)$ and $\tilde{P} \in L^{r'}_0(\omega)$, such that the extension $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ of the solution of (10) satisfies the following convergences

$$\varepsilon^{-1+\frac{2}{r}(\gamma-1)}\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } W^{1,r}(0,1; L^r(\omega)^3), \quad \tilde{P}_\varepsilon \to \tilde{P} \quad \text{strongly in } L^{r'}(\Omega).$$

Moreover, defining $V(x') = \int_0^1 \tilde{u}(x', y_3)\, dy_3$, it holds that

$$V(x') = 0 \quad \text{in} \quad \omega. \tag{17}$$

(ii) If $\gamma > 1$, then there exist $\tilde{u} \in W_0^{1,r}(0,1;L^r(\omega)^3)$ with $\tilde{u}_3 = 0$ and $\tilde{P} \in L_0^{r'}(\omega)$, such that the extension $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ of the solution of (10) satisfies the following convergences

$$\varepsilon^{\frac{\gamma-r}{r-1}}\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } W^{1,r}(0,1;L^r(\omega)^3), \quad \tilde{P}_\varepsilon \to \tilde{P} \quad \text{strongly in } L^{r'}(\Omega).$$

Moreover, defining $V(x') = \int_0^1 \tilde{u}(x', y_3)\, dy_3$, it holds that $(V, \tilde{P}) \in L^r(\omega)^3 \times (L_0^{r'}(\omega) \cap W^{1,r'}(\omega))$ is the unique solution of the lower-dimensional effective non-linear Darcy's law

$$\begin{cases} V'(x') = \dfrac{\lambda^{\frac{2-r}{2}}}{\eta_0 - \eta_\infty}\mathcal{U}\left(f'(x') - \nabla_{x'}\tilde{P}(x')\right), \quad V_3(x') = 0 \quad \text{in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \quad \text{in } \omega, \quad V'(x')\cdot n = 0 \quad \text{on } \partial\omega, \end{cases} \tag{18}$$

where the permeability function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by (15) with $w_{\xi'}$, for every $\xi' \in \mathbb{R}^2$, the unique solution of the local Stokes system (16) with nonlinear viscosity of type power law given by $\eta_r(\mathbb{D}_y[w_{\xi'}]) = |\mathbb{D}_y[w_{\xi'}]|^{r-2}$.

(iii) If $\gamma = 1$, then there exist $\tilde{u} \in W_0^{1,r}(0,1;L^r(\omega)^3)$ with $\tilde{u}_3 = 0$ and $\tilde{P} \in L_0^{r'}(\omega)$, such that the extension $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ of the solution of (10) satisfies the following convergences

$$\varepsilon^{-1}\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } W^{1,r}(0,1;L^r(\omega)^3), \quad \tilde{P}_\varepsilon \to \tilde{P} \quad \text{strongly in } L^{r'}(\Omega).$$

Moreover, defining $V(x') = \int_0^1 \tilde{u}(x', y_3)\, dy_3$, it holds that $(V, \tilde{P}) \in L^r(\omega)^3 \times (L_0^{r'}(\omega) \cap W^{1,r'}(\omega))$ is the unique solution of the lower-dimensional effective non-linear 2D Darcy's law of Carreau type (14), where the permeability function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by (15) with $(w_{\xi'}, \pi_{\xi'}) \in W^{1,r}_{0,\#}(Y_f)^3 \times L^{r'}_{0,\#}(Y_f)$, for every $\xi' \in \mathbb{R}^2$, the unique solution of the local Stokes system (16) with nonlinear viscosity given by the Carreau law (2).

Remark 2.4. *According to [23, Lemma 2], the permeability function $\mathcal{U}$ is coercive and strictly monotone.*

Theorem 2.5 (Newtonian fluids)**.** *Consider $r = 2$ and $\gamma \in \mathbb{R}$. Then, there exist $\tilde{u} \in H_0^1(0,1;L^2(\omega)^3)$ with $\tilde{u}_3 = 0$ and $\tilde{P} \in L_0^2(\omega)$, such that the extension $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ of the solution of (10) satisfies the following convergences*

$$\varepsilon^{\gamma-2}\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } H^1(0,1;L^2(\omega)^3), \quad \tilde{P}_\varepsilon \to \tilde{P} \quad \text{strongly in } L^2(\Omega).$$

Moreover, defining $V(x') = \int_0^1 \tilde{u}(x', y_3)\, dy_3$, it holds that $(V, \tilde{P}) \in L^2(\omega)^3 \times (L_0^2(\omega) \cap H^1(\omega))$ is the unique solution of the linear 2D Darcy's law

$$\begin{cases} V'(x') = \dfrac{1}{\eta_0}\mathcal{A}\left(f'(x') - \nabla_{x'}\tilde{P}(x')\right), \quad V_3(x') = 0 \quad \text{in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \quad \text{in } \omega, \quad V'(x')\cdot n = 0 \quad \text{on } \partial\omega, \end{cases} \tag{19}$$

where the permeability tensor $\mathcal{A} \in \mathbb{R}^{2\times 2}$ is defined by its entries

$$\mathcal{A}_{ij} = \int_{Y_f} (w_j^i)'(y)\, dy, \quad i, j = 1, 2, \tag{20}$$

with $(w^i, \pi^i) \in H^1_{0,\#}(Y_f)^3 \times L^2_{0,\#}(Y_f)$, $i = 1, 2$, the unique solution of the local Stokes system

$$\begin{cases} -\Delta_y w^i + \nabla_y \pi^i = e_i & \text{in } Y_f, \\ \operatorname{div}_y w^i = 0 & \text{in } Y_f, \\ w^i = 0 & \text{on } \partial T, \\ y \to w^i, \pi^i & Y\text{-periodic}, \end{cases} \tag{21}$$

and $\{e_i\}_{i=1,2,3}$ being the canonical basis of $\mathbb{R}^3$.

Remark 2.6. *According to [1, Theorem 1.1], the permeability tensor $\mathcal{A}$ is symmetric and definite positive.*

3 Proof of the main results

In this section we provide the proof of the main results (Theorems 2.1, 2.3 and 2.5). To to this, first we establish some *a priori* estimates of the solution of (10) and we define the extension of the solution. Second, we introduce the version of the unfolding method depending on ε. Next, a compactness result, which is the main key when we will pass to the limit later, is addressed and finally, the proofs of Theorems 2.1, 2.3 and 2.5 are given.

3.1 *A priori* estimates.

In this subsection, we establish sharp *a priori* estimates of the dilated solution in $\widetilde{\Omega}_\varepsilon$. To do this, we first need the Poincaré and Korn inequalities in $\widetilde{\Omega}_\varepsilon$, which can be found in [11].

Lemma 3.1 (Remark 4.3-(i) in [11])**.** *We have the following two estimates in thin domains:*

(i) For every $\tilde{v} \in W_0^{1,q}(\widetilde{\Omega}_\varepsilon)^3$, $1 \leq q < +\infty$, there exists a positive constant C, independent of ε, such that

$$\|\tilde{v}\|_{L^q(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon \|D_\varepsilon \tilde{v}\|_{L^q(\widetilde{\Omega}_\varepsilon)^{3\times 3}}, \quad (\textit{Poincaré's inequality}). \tag{22}$$

(ii) For every $\tilde{v} \in W_0^{1,q}(\widetilde{\Omega}_\varepsilon)^3$, $1 < q < +\infty$, there exists a positive constant C, independent of ε, such that

$$\|D_\varepsilon \tilde{v}\|_{L^q(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\|\mathbb{D}_\varepsilon[\tilde{v}]\|_{L^q(\widetilde{\Omega}_\varepsilon)^{3\times 3}}, \quad (\textit{Korn's inequality}). \tag{23}$$

Using the previous estimates, we give *a priori* estimates for velocity $\tilde{u}_\varepsilon$ in $\widetilde{\Omega}_\varepsilon$.

Lemma 3.2. *Let $\tilde{u}_\varepsilon$ the velocity solution of (10). Then, we have the following estimates:*

(i) (Pseudoplastic fluid and Newtonian fluid) Consider $1 < r \leq 2$. There exists a positive constant C, independent of ε, such that, for every value of γ, we have

$$\|\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{2-\gamma}, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{1-\gamma}, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{1-\gamma}. \tag{24}$$

(ii) (Dilatant fluid) Consider $r > 2$. There exists a positive constant C, independent of ε, such that estimates (24) hold and also, depending on the value of γ, we have

- *If $\gamma < 1$, it holds*

$$\|\tilde{u}_\varepsilon\|_{L^r(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)+1}, \ \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)}, \ \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)}, \tag{25}$$

- *If $\gamma > 1$, it holds*

$$\|\tilde{u}_\varepsilon\|_{L^r(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}+1}, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}}, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}}. \tag{26}$$

- *If $\gamma = 1$, it holds*

$$\|\tilde{u}_\varepsilon\|_{L^r(\widetilde{\Omega}_\varepsilon)^3} \leq C\varepsilon, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C. \tag{27}$$

Proof. Multiplying (10) by $\tilde{u}_\varepsilon$ and integrating over $\widetilde{\Omega}_\varepsilon$, we get

$$\varepsilon^\gamma(\eta_0 - \eta_\infty) \int_{\widetilde{\Omega}_\varepsilon} \left(1 + \lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2\right)^{\frac{r}{2}-1} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx'dy_3 + \varepsilon^\gamma \eta_\infty \int_{\widetilde{\Omega}_\varepsilon} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx'dy_3 = \int_{\widetilde{\Omega}_\varepsilon} f' \cdot \tilde{u}'_\varepsilon \, dx'dy_3. \tag{28}$$

We divide the proof in two steps. First, we derive estimates (24) for every $r > 1$ and then, for $r > 2$ we establish estimates (25)-(27) depending on the value of γ.

Step 1. We consider $r > 1$. Taking into account that $\eta_0 > \eta_\infty$, and $\lambda > 0$, we have

$$\varepsilon^\gamma(\eta_0 - \eta_\infty) \int_{\widetilde{\Omega}_\varepsilon} \left(1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2\right)^{\frac{r}{2}-1} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx' dy_3 \geq 0,$$

and then, from Cauchy-Schwarz's inequality and the assumption on f' given in (8), we get from (28) that

$$\varepsilon^\gamma \eta_\infty \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C \|\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}.$$

Applying Poincaré's inequality (22) and Korn's inequality (23) in the right-hand side, we have

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^2_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{1-\gamma} \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}},$$

which gives $(24)_3$. Finally, applying again (22) and (23), we obtain $(24)_1$ and $(24)_2$.

Step 2. For $r > 2$, since

$$\varepsilon^\gamma \eta_\infty \int_{\widetilde{\Omega}_\varepsilon} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx' dy_3 \geq 0,$$

from (28) and Cauchy-Schwarz's inequality, we get

$$\varepsilon^\gamma(\eta_0 - \eta_\infty) \int_{\widetilde{\Omega}_\varepsilon} \left(1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2\right)^{\frac{r}{2}-1} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx' dy_3 \leq C \|\tilde{u}_\varepsilon\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}. \tag{29}$$

Since it holds

$$\varepsilon^\gamma \lambda^{\frac{r-2}{2}} (\eta_0 - \eta_\infty) \int_{\widetilde{\Omega}_\varepsilon} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^r dx' dy_3 \leq \varepsilon^\gamma(\eta_0 - \eta_\infty) \int_{\widetilde{\Omega}_\varepsilon} \left(1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2\right)^{\frac{r}{2}-1} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx' dy_3,$$

then, applying this and also Poincaré's inequality (22) and Korn's inequality (23) in the right-hand side in (29), we get

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^r_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{1-\gamma} \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}. \tag{30}$$

On the one hand, applying estimate $(24)_3$, we get

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)}.$$

On the other hand, from the continuity of embedding $L^r(\widetilde{\Omega}_\varepsilon) \hookrightarrow L^2(\widetilde{\Omega}_\varepsilon)$ in (30), we also have

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^r_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{1-\gamma} \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}},$$

which gives

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}}.$$

Then, we have two estimates of $\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]$ in $L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}$ and we have to compare them in order to consider the more accurate estimate depending on the value of γ. Since $-\frac{2}{r}(\gamma-1) > -\frac{\gamma-1}{r-1}$ if $\gamma < 1$ and $-\frac{2}{r}(\gamma-1) < -\frac{\gamma-1}{r-1}$ if $\gamma > 1$, then we have estimates $(25)_3$ and $(26)_3$. In the case $\gamma = 1$, both estimates give $(27)_3$. Finally, from Poincaré's inequality (22) and Korn's inequality (23) we derive the rest of estimates in (25), (26) and (27). □

Remark 3.3. *We extend the velocity $\tilde{u}_\varepsilon$ by zero in $\Omega \setminus \widetilde{\Omega}_\varepsilon$ (this is compatible with the homogeneous boundary condition on $\partial\Omega \cup \partial T_\varepsilon$), and denote the extension by same symbol. Obviously, estimates given in Lemma 3.2 remain valid and the extension $\tilde{u}_\varepsilon$ is divergence free too.*

In order to extend the pressure $\tilde{p}_\varepsilon$ to the whole domain Ω and obtain *a priori* estimates, we recall a result in which is concerned with the extension of the pressure p_ε to the whole domain Q_ε, which define a restriction operator R^ε_q from $W^{1,q}_0(Q_\varepsilon)^3$ into $W^{1,q}_0(\Omega_\varepsilon)^3$, $1 < q < +\infty$, introduced in [11].

Lemma 3.4 (Lemma 4.5-(i) in [11]). *There exists a (restriction) operator R_q^ε acting from $W_0^{1,q}(Q_\varepsilon)^3$ into $W_0^{1,q}(\Omega_\varepsilon)^3$, $1 < q < +\infty$, such that*

1. *$R_q^\varepsilon v = v$, if $v \in W_0^{1,q}(\Omega_\varepsilon)^3$ (elements of $W_0^{1,q}(\Omega_\varepsilon)$ are extended by 0 to Q_ε).*
2. *$\mathrm{div} R_q^\varepsilon v = 0$ in Ω_ε, if $\mathrm{div}\, v = 0$ on Q_ε.*
3. *For every $v \in W_0^{1,q}(Q_\varepsilon)^3$, there exists a positive constant C, independent of v and ε, such that*

$$\|R_q^\varepsilon v\|_{L^q(\Omega_\varepsilon)^3} + \varepsilon \|DR_q^\varepsilon v\|_{L^q(\Omega_\varepsilon)^{3\times 3}} \leq C \left(\|v\|_{L^q(Q_\varepsilon)^3} + \varepsilon \|Dv\|_{L^q(Q_\varepsilon)^{3\times 3}} \right) . \tag{31}$$

In the next result, by using the restriction operator defined in Lemma 3.4, we extend the gradient of the pressure by duality in $W^{-1,q'}(Q_\varepsilon)^3$ and then, by means of the dilatation, we extend $\tilde{p}_\varepsilon$ to Ω. Finally, we derive estimates of the extension of the pressure.

Lemma 3.5. *Let $\tilde{p}_\varepsilon$ the pressure solution of (10). Then:*

(i) (Pseudoplastic fluid and Newtonian fluid) Consider $1 < r \leq 2$. There exist an extension $\tilde{P}_\varepsilon \in L_0^2(\Omega)$ of pressure $\tilde{p}_\varepsilon$ and a positive constant C independent of ε, such that

$$\|\tilde{P}_\varepsilon\|_{L^2(\Omega)} \leq C\,, \quad \|\nabla_\varepsilon \tilde{P}_\varepsilon\|_{H^{-1}(\Omega)^3} \leq C. \tag{32}$$

(ii) (Dilatant fluid) Consider $r > 2$. There exist an extension $\tilde{P}_\varepsilon \in L_0^{r'}(\Omega)$ of pressure $\tilde{p}_\varepsilon$ and a positive constant C independent of ε, such that

$$\|\tilde{P}_\varepsilon\|_{L^{r'}(\Omega)} \leq C\,, \quad \|\nabla_\varepsilon \tilde{P}_\varepsilon\|_{W^{-1,r'}(\Omega)^3} \leq C, \tag{33}$$

where r' is the conjugate exponent of r.

Proof. We divide the proof in three steps. First, we extend the pressure in all the cases, next we obtain the estimates in the case of pseudoplastic fluids and Newtonian fluis, and finally, we get the estimates in the case of dilatant fluids.

Step 1. Extension of the pressure. Using the restriction operator with $q = \max\{2, r\}$ given in Lemma 3.4, i.e. R_q^ε, we introduce F_ε in $W^{-1,q'}(Q_\varepsilon)^3$, where q' is the conjugate exponent of q, in the following way

$$\langle F_\varepsilon, v\rangle_{W^{-1,q'}(Q_\varepsilon)^3, W_0^{1,q}(Q_\varepsilon)^3} = \langle \nabla p_\varepsilon, R_q^\varepsilon v\rangle_{W^{-1,q'}(\Omega_\varepsilon)^3, W_0^{1,q}(\Omega_\varepsilon)^3}\,, \quad \text{for any } v \in W_0^{1,q}(Q_\varepsilon)^3\,, \tag{34}$$

and calculate the right hand side of (34) by using the variational formulation of problem (7), which gives

$$\begin{aligned} \langle F_\varepsilon, v\rangle_{W^{-1,q'}(Q_\varepsilon)^3, W_0^{1,q}(Q_\varepsilon)^3} = \quad & -\varepsilon^\gamma(\eta_0 - \eta_\infty) \int_{\Omega_\varepsilon} (1 + \lambda|\mathbb{D}[u_\varepsilon]|^2)^{\frac{r}{2}-1} \mathbb{D}[u_\varepsilon] : DR_q^\varepsilon v\, dx \\ & -\varepsilon^\gamma \eta_\infty \int_{\Omega_\varepsilon} \mathbb{D}[u_\varepsilon] : DR_q^\varepsilon v\, dx + \int_{\Omega_\varepsilon} f' \cdot (R_q^\varepsilon v)'\, dx\,. \end{aligned} \tag{35}$$

Using Lemma 3.2 for fixed ε, we see that it is a bounded functional on $W_0^{1,q}(Q_\varepsilon)$ (see *Steps* 2 and 3 below), and in fact $F_\varepsilon \in W^{-1,q'}(Q_\varepsilon)^3$. Moreover, $\mathrm{div}\, v = 0$ implies $\langle F_\varepsilon, v\rangle = 0$, and the DeRham theorem gives the existence of P_ε in $L_0^{q'}(Q_\varepsilon)$ with $F_\varepsilon = \nabla P_\varepsilon$.

Next, we get for every $\tilde{v} \in W_0^{1,q}(\Omega)^3$ where $\tilde{v}(x', y_3) = v(x', \varepsilon y_3)$, using the change of variables (9), that

$$\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\rangle_{W^{-1,q'}(\Omega)^3, W_0^{1,q}(\Omega)^3} = -\int_\Omega \tilde{P}_\varepsilon\, \mathrm{div}_\varepsilon\, \tilde{v}\, dx' dy_3 = -\varepsilon^{-1} \int_{Q_\varepsilon} P_\varepsilon\, \mathrm{div}\, v\, dx = \varepsilon^{-1} \langle \nabla P_\varepsilon, v\rangle_{W^{-1,q'}(Q_\varepsilon)^3, W_0^{1,q}(Q_\varepsilon)^3}\,.$$

Using the identification (35) of F_ε, we have

$$\begin{aligned}\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\rangle_{W^{-1,q'}(\Omega)^3, W_0^{1,q}(\Omega)^3} = \quad & \varepsilon^{-1}\left(-\varepsilon^\gamma(\eta_0-\eta_\infty)\int_{\Omega_\varepsilon}(1+\lambda|\mathbb{D}[u_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}[u_\varepsilon]:DR_q^\varepsilon v\,dx\right.\\ & \left.-\varepsilon^\gamma\eta_\infty\int_{\Omega_\varepsilon}\mathbb{D}[u_\varepsilon]:DR_q^\varepsilon v\,dx+\int_{\Omega_\varepsilon}f'\cdot(R_q^\varepsilon v)'\,dx\right),\end{aligned}$$

and applying the change of variables (9), we obtain

$$\begin{aligned}\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\rangle_{W^{-1,q'}(\Omega)^3, W_0^{1,q}(\Omega)^3} = \quad & -\varepsilon^\gamma(\eta_0-\eta_\infty)\int_{\widetilde{\Omega}_\varepsilon}(1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]:D_\varepsilon\tilde{R}_q^\varepsilon\tilde{v}\,dx'dy_3\\ & -\varepsilon^\gamma\eta_\infty\int_{\widetilde{\Omega}_\varepsilon}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]:D_\varepsilon\tilde{R}_q^\varepsilon\tilde{v}\,dx'dy_3+\int_{\widetilde{\Omega}_\varepsilon}f'(x')\cdot(\tilde{R}_q^\varepsilon\tilde{v})'\,dx'dy_3,\end{aligned}\tag{36}$$

where $\tilde{R}_q^\varepsilon\tilde{v}=R_q^\varepsilon v$ for any $\tilde{v}\in W_0^{1,q}(\Omega)^3$.

Step 2. Estimates of the extended pressure for pseudoplastic fluids and Newtonian fluids. Applying the dilatation in (31) for $q=2$, we have that $\tilde{R}_2^\varepsilon\tilde{v}$ satisfies the following estimate

$$\|\tilde{R}_2^\varepsilon\tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}+\varepsilon\|D_\varepsilon\tilde{R}_2^\varepsilon\tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}\leq C\left(\|\tilde{v}\|_{L^2(\Omega)^3}+\varepsilon\|D_\varepsilon\tilde{v}\|_{L^2(\Omega)^{3\times3}}\right),\tag{37}$$

and since $\varepsilon\ll1$, we have

$$\|\tilde{R}_2^\varepsilon\tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\leq C\|\tilde{v}\|_{H_0^1(\Omega)^3},\quad \|D_\varepsilon\tilde{R}_2^\varepsilon\tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}\leq\frac{C}{\varepsilon}\|\tilde{v}\|_{H_0^1(\Omega)^3}.\tag{38}$$

Taking into account that $1<r\leq2$, we have that

$$(1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\leq1,$$

and then, from Cauchy-Schwarz's inequality, we deduce

$$\begin{aligned}&\left|\int_{\widetilde{\Omega}_\varepsilon}(1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]:D_\varepsilon\tilde{R}_2^\varepsilon\tilde{v}\,dx'dy_3\right|\leq\int_{\widetilde{\Omega}_\varepsilon}(1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|:|D_\varepsilon\tilde{R}_2^\varepsilon\tilde{v}|\,dx'dy_3\\ &\leq\int_{\widetilde{\Omega}_\varepsilon}|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|:|D_\varepsilon\tilde{R}_2^\varepsilon\tilde{v}|\,dx'dy_3\leq\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}\|D_\varepsilon\tilde{R}_2^\varepsilon\tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}.\end{aligned}$$

Using the last estimate in (24) and the last estimate of the dilated restricted operator given in (38), we obtain

$$\left|\varepsilon^\gamma(\eta_0-\eta_\infty)\int_{\widetilde{\Omega}_\varepsilon}(1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]:D_\varepsilon\tilde{R}_2^\varepsilon\tilde{v}\,dx'dy_3\right|\leq C\|\tilde{v}\|_{H_0^1(\Omega)^3}.\tag{39}$$

Moreover, from Cauchy-Schwarz's inequality, the last estimate in (24), the assumption of f' given in (8) and estimates of the dilated restricted operator given in (38), we obtain

$$\left|\varepsilon^\gamma\eta_\infty\int_{\widetilde{\Omega}_\varepsilon}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]:D_\varepsilon\tilde{R}_2^\varepsilon\tilde{v}\,dx'dy_3\right|\leq C\varepsilon^\gamma\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}\|D_\varepsilon\tilde{R}_2^\varepsilon\tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times3}}\leq C\|\tilde{v}\|_{H_0^1(\Omega)^3},\tag{40}$$

$$\left|\int_{\widetilde{\Omega}_\varepsilon}f'\cdot(\tilde{R}_2^\varepsilon\tilde{v})'\,dx'dy_3\right|\leq C\|\tilde{R}_2^\varepsilon\tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^3}\leq C\|\tilde{v}\|_{H_0^1(\Omega)^3}.\tag{41}$$

Then, taking into account (39)-(41) in (36), we get the second estimate in (32) and finally, using the Nečas inequality, there exists a representative $\tilde{P}_\varepsilon \in L^2_0(\Omega)$ such that

$$\|\tilde{P}_\varepsilon\|_{L^2(\Omega)} \leq C\|\nabla \tilde{P}_\varepsilon\|_{H^{-1}(\Omega)^3} \leq C\|\nabla_\varepsilon \tilde{P}_\varepsilon\|_{H^{-1}(\Omega)^3},$$

which implies the first estimate in (32).

Step 3. Estimates of the extended pressure for dilatant fluids. Applying the dilatation in (31) for $q = r$, we have that $\tilde{R}^\varepsilon_r \tilde{v}$ satisfies the following estimate

$$\|\tilde{R}^\varepsilon_r \tilde{v}\|_{L^r(\widetilde{\Omega}_\varepsilon)^3} + \varepsilon\|D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\left(\|\tilde{v}\|_{L^r(\Omega)^3} + \varepsilon\|D_\varepsilon \tilde{v}\|_{L^r(\Omega)^{3\times 3}}\right), \tag{42}$$

and since $\varepsilon \ll 1$, we have

$$\|\tilde{R}^\varepsilon_r \tilde{v}\|_{L^r(\widetilde{\Omega}_\varepsilon)^3} \leq C\|\tilde{v}\|_{W^{1,r}_0(\Omega)^3}, \quad \|D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq \frac{C}{\varepsilon}\|\tilde{v}\|_{W^{1,r}_0(\Omega)^3}. \tag{43}$$

Taking into account that $r > 2$, we have the continuous embedding $L^r(\widetilde{\Omega}_\varepsilon) \hookrightarrow L^2(\widetilde{\Omega}_\varepsilon)$, and from Hölder's inequality, we can deduce

$$\begin{aligned}
&\int_{\widetilde{\Omega}_\varepsilon} (1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\, dx'dy_3 \\
&\leq \int_{\widetilde{\Omega}_\varepsilon} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\, dx'dy_3 + \lambda^{\frac{r}{2}-1}\int_{\widetilde{\Omega}_\varepsilon} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^{r-2}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\, dx'dy_3 \\
&\leq \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}\|D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} + C\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^{r-1}_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}}\|D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \\
&\leq \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}}\|D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} + C\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|^{r-1}_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}}\|D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}}.
\end{aligned}$$

Observe that if $\gamma < 1$, taking into account that $-\frac{2}{r}(\gamma-1) > -\frac{\gamma-1}{(r-1)}$, using the last estimates in (24) and (25), and the last estimate of the dilated restricted operator given in (43), we obtain

$$\begin{aligned}
&\left|\varepsilon^\gamma(\eta_0-\eta_\infty)\int_{\widetilde{\Omega}_\varepsilon} (1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\, dx'dy_3\right| \\
&\leq \varepsilon^\gamma(\eta_0-\eta_\infty)\left(\varepsilon^{1-\gamma}\varepsilon^{-1} + \varepsilon^{-\frac{2}{r}(\gamma-1)(r-1)}\varepsilon^{-1}\right)\|\tilde{v}\|_{W^{1,r}_0(\widetilde{\Omega}_\varepsilon)^3} \\
&\leq \varepsilon^\gamma(\eta_0-\eta_\infty)\left(\varepsilon^{1-\gamma}\varepsilon^{-1} + \varepsilon^{-\frac{\gamma-1}{r-1}(r-1)}\varepsilon^{-1}\right)\|\tilde{v}\|_{W^{1,r}_0(\widetilde{\Omega}_\varepsilon)^3} \\
&\leq C\|\tilde{v}\|_{W^{1,r}_0(\widetilde{\Omega}_\varepsilon)^3}.
\end{aligned}$$

If $\gamma \geq 1$, using the last estimates in (24), (26) for $\gamma > 1$ and (27) for $\gamma = 1$, and the last estimate of the dilated restricted operator given in (43), we obtain

$$\begin{aligned}
&\left|\varepsilon^\gamma(\eta_0-\eta_\infty)\int_{\widetilde{\Omega}_\varepsilon} (1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\, dx'dy_3\right| \\
&\leq \varepsilon^\gamma(\eta_0-\eta_\infty)\left(\varepsilon^{1-\gamma}\varepsilon^{-1} + \varepsilon^{-\frac{\gamma-1}{r-1}(r-1)}\varepsilon^{-1}\right)\|\tilde{v}\|_{W^{1,r}_0(\widetilde{\Omega}_\varepsilon)^3} \\
&\leq C\|\tilde{v}\|_{W^{1,r}_0(\widetilde{\Omega}_\varepsilon)^3}.
\end{aligned}$$

In summary, we obtain for all γ:

$$\left|\varepsilon^\gamma(\eta_0-\eta_\infty)\int_{\widetilde{\Omega}_\varepsilon} (1+\lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}^\varepsilon_r \tilde{v}\, dx'dy_3\right| \quad \leq C\|\tilde{v}\|_{W^{1,r}_0(\Omega)^3}. \tag{44}$$

Moreover, from Cauchy-Schwarz's inequality, the last estimate in (24), the continuous embedding $L^r(\widetilde{\Omega}_\varepsilon) \hookrightarrow L^2(\widetilde{\Omega}_\varepsilon)$, the assumption of f' given in (8) and estimates of the dilated restricted operator given in (43), we obtain

$$\left|\varepsilon^\gamma \eta_\infty \int_{\widetilde{\Omega}_\varepsilon} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v}\, dx' dy_3\right| \leq C\varepsilon^\gamma \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \|D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v}\|_{L^r(\widetilde{\Omega}_\varepsilon)^{3\times 3}} \leq C\|\tilde{v}\|_{W_0^{1,r}(\Omega)^3}, \tag{45}$$

$$\left|\int_{\widetilde{\Omega}_\varepsilon} f' \cdot (\tilde{R}_r^\varepsilon \tilde{v})'\, dx' dy_3\right| \leq C\|\tilde{R}_r^\varepsilon \tilde{v}\|_{L^2(\widetilde{\Omega}_\varepsilon)^3} \leq C\|\tilde{R}_r^\varepsilon \tilde{v}\|_{L^r(\widetilde{\Omega}_\varepsilon)^3} \leq C\|\tilde{v}\|_{W_0^{1,r}(\Omega)^3}. \tag{46}$$

Then, taking into account (44)-(46) in (36), we get

$$\left|\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v}\rangle_{W^{-1,r'}(\Omega)^3, W_0^{1,r}(\Omega)^3}\right| \leq C\|\tilde{v}\|_{W_0^{1,r}(\Omega)^3}.$$

This implies the second estimate in (33) and then, using the Nečas inequality, there exists a representative $\tilde{P}_\varepsilon \in L_0^{r'}(\Omega)$ such that

$$\|\tilde{P}_\varepsilon\|_{L^{r'}(\Omega)} \leq C\|\nabla \tilde{P}_\varepsilon\|_{W^{-1,r'}(\Omega)^3} \leq C\|\nabla_\varepsilon \tilde{P}_\varepsilon\|_{W^{-1,r'}(\Omega)^3},$$

which implies the first estimate in (33).

□

3.2 Adaptation of the unfolding method.

The change of variables (9) does not provide the information we need about the behavior of $\tilde{u}_\varepsilon$ in the microstructure associated to $\widetilde{\Omega}_\varepsilon$. To solve this difficulty, we use an adaptation introduced in [11] of the unfolding method from [25].

Let us recall that this adaptation of the unfolding method divides the domain $\widetilde{\Omega}_\varepsilon$ in cubes of lateral length ε and vertical length 1. Thus, given $(\tilde{\varphi}_\varepsilon, \tilde{\psi}_\varepsilon) \in W_0^{1,q}(\Omega)^3 \times L_0^{q'}(\Omega)$, $1 < q < +\infty$ and $1/q + 1/q' = 1$, we define $(\hat{\varphi}_\varepsilon, \hat{\psi}_\varepsilon)$ by

$$\hat{\varphi}_\varepsilon(x', y) = \tilde{\varphi}_\varepsilon\left(\varepsilon\kappa\left(\frac{x'}{\varepsilon}\right) + \varepsilon y', y_3\right), \quad \hat{\psi}_\varepsilon(x', y) = \tilde{\psi}_\varepsilon\left(\varepsilon\kappa\left(\frac{x'}{\varepsilon}\right) + \varepsilon y', y_3\right), \quad \text{a.e. } (x', y) \in \omega \times Y, \tag{47}$$

assuming $\tilde{\varphi}_\varepsilon$ and $\tilde{\psi}_\varepsilon$ are extended by zero outside ω, where the function $\kappa : \mathbb{R}^2 \to \mathbb{Z}^2$ is defined by

$$\kappa(x') = k' \Longleftrightarrow x' \in Y'_{k',1}, \quad \forall\, k' \in \mathbb{Z}^2.$$

Remark 3.6. *We make the following comments:*

- *The function κ is well defined up to a set of zero measure in $\mathbb{R}^2$ (the set $\cup_{k'\in\mathbb{Z}^2} \partial Y'_{k',1}$). Moreover, for every $\varepsilon > 0$, we have*
$$\kappa\left(\frac{x'}{\varepsilon}\right) = k' \Longleftrightarrow x' \in Y'_{k',\varepsilon}.$$
- *For $k' \in \mathcal{K}_\varepsilon$, the restrictions of $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ to $Y'_{k',\varepsilon} \times Y$ does not depend on x', whereas as a function of y it is obtained from $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ by using the change of variables $y' = \dfrac{x' - \varepsilon k'}{\varepsilon}$, which transforms $Y_{k',\varepsilon}$ into Y.*

Following the proof of [11, Lemma 4.9], we have the following estimates relating $(\hat{\varphi}_\varepsilon, \hat{\psi}_\varepsilon)$ and $(\tilde{\varphi}_\varepsilon, \tilde{\psi}_\varepsilon)$.

Lemma 3.7. *The sequence $(\hat{\varphi}_\varepsilon, \hat{\psi}_\varepsilon)$ defined by (47) satisfies the following estimates:*

$$\begin{gathered}\|\hat{\varphi}_\varepsilon\|_{L^q(\omega\times Y)^3} \leq \|\tilde{\varphi}_\varepsilon\|_{L^q(\Omega)^3},\\ \|D_{y'}\hat{\varphi}_\varepsilon\|_{L^q(\omega\times Y)^{3\times 2}} \leq \varepsilon\|D_{x'}\tilde{\varphi}_\varepsilon\|_{L^q(\Omega)^{3\times 2}}, \quad \|\partial_{y_3}\hat{\varphi}_\varepsilon\|_{L^q(\omega\times Y)^3} \leq \|\partial_{y_3}\tilde{\varphi}_\varepsilon\|_{L^q(\Omega)^3},\\ \|\mathbb{D}_{y'}[\hat{\varphi}_\varepsilon]\|_{L^q(\omega\times Y)^{3\times 2}} \leq \varepsilon\|\mathbb{D}_{x'}[\tilde{\varphi}_\varepsilon]\|_{L^q(\Omega)^{3\times 2}}, \quad \|\partial_{y_3}[\hat{u}_\varepsilon]\|_{L^q(\omega\times Y)^3} \leq \|\partial_{y_3}[\tilde{\varphi}_\varepsilon]\|_{L^q(\Omega)^3},\\ \|\hat{\psi}_\varepsilon\|_{L^{q'}(\omega\times Y)} \leq \|\tilde{\psi}_\varepsilon\|_{L^{q'}(\Omega)}.\end{gathered} \tag{48}$$

Definition 3.8 (Unfolded velocity and pressure)**.** *Let us define the unfolded velocity and pressure $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ from $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ depending on the type of fluid:*

- *(Pseudoplastic fluids and Newtonian fluids) From $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in H^1_0(\Omega)^3 \times L^2_0(\Omega)$, we define $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ by using (47) with $\tilde{\varphi}_\varepsilon = \tilde{u}_\varepsilon$, $\tilde{\psi}_\varepsilon = \tilde{P}_\varepsilon$ and $q = 2$.*

- *(Dilatant fluids) From $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in W^{1,r}_0(\Omega)^3 \times L^{r'}_0(\Omega)$, we define $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ by using (47) with $\tilde{\varphi}_\varepsilon = \tilde{u}_\varepsilon$, $\tilde{\psi}_\varepsilon = \tilde{P}_\varepsilon$ and $q = r$.*

Now, from estimates of the extended velocity (24)-(27) and pressure (32)-(33) together with Lemma 3.7, we have the following estimates for $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$.

Lemma 3.9. *We have the following estimates for the unfolded functions $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ depending on the type of fluid:*

(i) (Pseudoplastic fluids and Newtonian fluids) Consider $1 < r \leq 2$. There exists a constant $C > 0$ independent of ε, such that, for every value of γ, we have

$$\|\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^3} \leq C\varepsilon^{2-\gamma}, \quad \|D_y\hat{u}_\varepsilon\|_{L^2(\omega\times Y)^{3\times 3}} \leq C\varepsilon^{2-\gamma}, \quad \|\mathbb{D}_y[\hat{u}_\varepsilon]\|_{L^2(\omega\times Y)^{3\times 3}} \leq C\varepsilon^{2-\gamma}, \tag{49}$$

$$\|\hat{P}_\varepsilon\|_{L^2(\omega\times Y)} \leq C. \tag{50}$$

(ii) (Dilatant fluids) Consider $r > 2$. There exists a constant $C > 0$ independent of ε, such that estimates (49) hold and also, depending on the value of γ, we have

- *If $\gamma < 1$, it holds*

$$\begin{gathered}\|\hat{u}_\varepsilon\|_{L^r(\omega\times Y)^3} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)+1}, \quad \|D_y\hat{u}_\varepsilon\|_{L^r(\omega\times Y)^{3\times 3}} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)+1},\\ \|\mathbb{D}_y[\hat{u}_\varepsilon]\|_{L^r(\omega\times Y)^{3\times 3}} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)+1}.\end{gathered} \tag{51}$$

- *If $\gamma > 1$, it holds*

$$\begin{gathered}\|\hat{u}_\varepsilon\|_{L^r(\omega\times Y)^3} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}+1}, \quad \|D_y\hat{u}_\varepsilon\|_{L^r(\omega\times Y)^{3\times 3}} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}+1},\\ \|\mathbb{D}_y[\hat{u}_\varepsilon]\|_{L^r(\omega\times Y)^{3\times 3}} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}+1}.\end{gathered} \tag{52}$$

- *If $\gamma = 1$, it holds*

$$\begin{gathered}\|\hat{u}_\varepsilon\|_{L^r(\omega\times Y)^3} \leq C\varepsilon, \quad \|D_y\hat{u}_\varepsilon\|_{L^r(\omega\times Y)^{3\times 3}} \leq C\varepsilon,\\ \|\mathbb{D}_y[\hat{u}_\varepsilon]\|_{L^r(\omega\times Y)^{3\times 3}} \leq C\varepsilon.\end{gathered} \tag{53}$$

Moreover, we have the following estimate for the pressure

$$\|\hat{P}_\varepsilon\|_{L^{r'}(\omega\times Y)} \leq C, \tag{54}$$

where r' is the conjugate exponent of r.

3.3 Compactness results.

We analyze the asymptotic behavior of sequences of the extension of $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$ and $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$, when ε tends to zero.

Lemma 3.10. *We have the following convergences results for the velocity:*

(i) (Pseudoplastic fluids and Newtonian fluids) Consider $1 < r \leq 2$. There exist $\tilde{u} \in H_0^1(0,1;L^2(\omega)^3)$ where $\tilde{u}_3 = 0$, $\hat{u} \in L^2(\omega; H^1_{0,\#}(Y)^3)$, with $\hat{u} = 0$ on $\omega \times T$ such that $\int_Y \hat{u}(x',y)\,dy = \int_0^1 \tilde{u}(x',y_3)\,dy_3$ with $\int_Y \hat{u}_3(x',y)\,dy = 0$, such that, up to a subsequence,

$$\varepsilon^{\gamma-2}\tilde{u}_\varepsilon \rightharpoonup (\tilde{u}',0) \text{ weakly in } H^1(0,1;L^2(\omega)^3), \tag{55}$$

$$\varepsilon^{\gamma-2}\hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ weakly in } L^2(\omega; H^1(Y)^3). \tag{56}$$

(ii) (Dilatant fluids) Consider $r > 2$. There exist $\tilde{u} \in W_0^{1,r}(0,1;L^r(\omega)^3)$ where $\tilde{u}_3 = 0$, $\hat{u} \in L^r(\omega; W^{1,r}_{0,\#}(Y)^3)$, with $\hat{u} = 0$ on $\omega \times T$ such that $\int_Y \hat{u}(x',y)\,dy = \int_0^1 \tilde{u}(x',y_3)\,dy_3$ with $\int_Y \hat{u}_3(x',y)\,dy = 0$, such that, up to a subsequence,

- *if $\gamma < 1$, it holds*

$$\varepsilon^{-1+\frac{2}{r}(\gamma-1)}\tilde{u}_\varepsilon \rightharpoonup (\tilde{u}',0) \text{ weakly in } W^{1,r}(0,1;L^r(\omega)^3), \tag{57}$$

$$\varepsilon^{-1+\frac{2}{r}(\gamma-1)}\hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ weakly in } L^r(\omega; W^{1,r}(Y)^3), \tag{58}$$

- *if $\gamma > 1$, it holds*

$$\varepsilon^{\frac{\gamma-r}{r-1}}\tilde{u}_\varepsilon \rightharpoonup (\tilde{u}',0) \text{ weakly in } W^{1,r}(0,1;L^r(\omega)^3), \tag{59}$$

$$\varepsilon^{\frac{\gamma-r}{r-1}}\hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ weakly in } L^r(\omega; W^{1,r}(Y)^3). \tag{60}$$

- *if $\gamma = 1$, it holds*

$$\varepsilon^{-1}\tilde{u}_\varepsilon \rightharpoonup (\tilde{u}',0) \text{ weakly in } W^{1,r}(0,1;L^r(\omega)^3), \tag{61}$$

$$\varepsilon^{-1}\hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ weakly in } L^r(\omega; W^{1,r}(Y)^3). \tag{62}$$

Moreover, $\tilde{u}$ and $\hat{u}$ satisfy the following divergence conditions in both cases

$$\mathrm{div}_{x'}\left(\int_0^1 \tilde{u}'(x',y_3)\,dy_3\right) = 0 \quad \text{in } \omega, \qquad \left(\int_0^1 \tilde{u}'(x',y_3)\,dy_3\right)\cdot n = 0 \quad \text{in } \partial\omega, \tag{63}$$

$$\mathrm{div}_y\,\hat{u}(x',y) = 0 \quad \text{in } \omega \times Y_f, \quad \mathrm{div}_{x'}\left(\int_{Y_f} \hat{u}'(x',y)\,dy\right) = 0 \quad \text{in } \omega, \quad \left(\int_{Y_f} \hat{u}'(x',y)\,dy\right)\cdot n = 0 \quad \text{on } \partial\omega. \tag{64}$$

Proof. Arguing as in [11, Lemma 5.2.-(i)], we obtain convergence (55) (resp. (57), (59) and (61)) and divergence condition (63). Moreover, proceeding similarly as in [11, Lemma 5.4.-(i)] we deduce convergence (56) (resp. (58), (60) and (62)) and divergence conditions (64). □

Lemma 3.11. *We have the following convergences results for the pressure:*

(i) (Pseudoplastic fluids and Newtonian fluids) Consider $1 < r \leq 2$. There exists $\tilde{P} \in L^2_0(\omega)$ such that

$$\tilde{P}_\varepsilon \to \tilde{P} \text{ strongly in } L^2(\Omega), \tag{65}$$

$$\hat{P}_\varepsilon \to \tilde{P} \text{ strongly in } L^2(\omega \times Y). \tag{66}$$

(ii) (Dilatant fluids) Consider $r > 2$. There exists $\tilde{P} \in L^2_0(\omega)$ such that

$$\tilde{P}_\varepsilon \to \tilde{P} \text{ strongly in } L^{r'}(\Omega), \tag{67}$$

$$\hat{P}_\varepsilon \to \tilde{P} \text{ strongly in } L^{r'}(\omega \times Y), \tag{68}$$

where r' is the conjugate exponent of r.

Proof. We give some remarks concerning the case *(i)* (proceed similarly for the case *(ii)*). The first estimate in (32) implies, up to a subsequence, the existence of $\tilde{P} \in L^2_0(\Omega)$ such that

$$\tilde{P}_\varepsilon \rightharpoonup \tilde{P} \text{ weakly in } L^2(\Omega). \tag{69}$$

Also, from the second estimate in (32), by noting that $\partial_{y_3}\tilde{P}_\varepsilon/\varepsilon$ also converges weakly in $H^{-1}(\Omega)$, we obtain $\partial_{y_3}\tilde{P} = 0$ and so $\tilde{P}$ is independent of y_3. Moreover, if we argue as in [22, Lemma 4.4], we have that the convergence (69) of the pressure $\tilde{P}_\varepsilon$ is in fact strong. Since $\tilde{P}_\varepsilon$ has null mean value in Ω, then $\tilde{P}$ has null mean value in ω, which concludes the proof of (65). Finally, the strong convergence of $\hat{P}_\varepsilon$ given in (66) follows from [27, Proposition 1.9-(ii)] and the strong convergence of $\tilde{P}_\varepsilon$ given in (65). □

3.4 Proof of Theorems 2.1, 2.3 and 2.5.

By using monotonicity arguments together with Minty's lemma, let us derive the variational inequality will be useful in the proofs.

We choose a test function $v(x', y) \in \mathcal{D}(\omega; C^\infty_\#(Y)^3)$ with $v(x', y) = 0$ in $\omega \times T$. Multiplying (10) by $v(x', x'/\varepsilon, y_3)$, integrating by parts, and taking into account the extension of $\tilde{u}_\varepsilon$ and $\tilde{P}_\varepsilon$, we have

$$\begin{aligned}
&\varepsilon^\gamma(\eta_0 - \eta_\infty)\int_\Omega (1 + \lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : \left(\mathbb{D}_{x'}[v] + \varepsilon^{-1}\mathbb{D}_y[v]\right)\, dx'dy_3 \\
&+\varepsilon^\gamma\eta_\infty\int_\Omega \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : \left(\mathbb{D}_{x'}[v] + \varepsilon^{-1}\mathbb{D}_y[v]\right)\, dx'dy_3 \\
&-\int_\Omega \tilde{P}_\varepsilon\left(\mathrm{div}_{x'}v' + \varepsilon^{-1}\mathrm{div}_y v\right)\, dx'dy_3 = \int_\Omega f'\cdot v'\, dx'dy + O_\varepsilon,
\end{aligned}$$

where O_ε is a generic real sequence depending on ε that can change from line to line.

By the change of variables given in Remark 3.6, we obtain

$$\begin{aligned}
&\varepsilon^{\gamma-1}(\eta_0 - \eta_\infty)\int_{\omega\times Y} (1 + \lambda|\varepsilon^{-1}\mathbb{D}_y[\hat{u}_\varepsilon]|^2)^{\frac{r}{2}-1}\left(\varepsilon^{-1}\mathbb{D}_y[\hat{u}_\varepsilon]\right) : \mathbb{D}_y[v]\, dx'dy \\
&+\varepsilon^{\gamma-1}\eta_\infty\int_{\omega\times Y} \varepsilon^{-1}\mathbb{D}_y[\hat{u}_\varepsilon] : \mathbb{D}_y[v]\, dx'dy \\
&-\int_{\omega\times Y} \hat{P}_\varepsilon\,\mathrm{div}_{x'}v'\, dx'dy - \varepsilon^{-1}\int_{\omega\times Y} \hat{P}_\varepsilon\,\mathrm{div}_y v\, dx'dy = \int_{\omega\times Y} f'\cdot v'\, dx'dy + O_\varepsilon,
\end{aligned} \tag{70}$$

with $|O_\varepsilon| \le C\varepsilon$ for every $\gamma \in \mathbb{R}$.

Now, let us define the functional J_r by

$$J_r(v) = \frac{\eta_0 - \eta_\infty}{r\lambda}\int_{\omega\times Y}(1 + \lambda|\mathbb{D}_y[v]|^2)^{\frac{r}{2}}\, dx'dy + \frac{\eta_\infty}{2}\int_{\omega\times Y}|\mathbb{D}_y[v]|^2 dx'dy.$$

Observe that J_r is convex and Gateaux differentiable on $L^q(\omega; W^{1,q}_{\#}(Y)^3)$ with $q = \max\{2, r\}$, (see [18, Proposition 2.1 and Section 3] for more details) and $A_r = J_r'$ is given by

$$(A_r(w), v) = (\eta_0 - \eta_\infty) \int_{\omega\times Y} (1 + \lambda|\mathbb{D}_y[w]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[w] : \mathbb{D}_y[v] dx'dy + \eta_\infty \int_{\omega\times Y} \mathbb{D}_y[w] : \mathbb{D}_y[v] dx'dy.$$

Applying [30, Proposition 1.1., p.158], in particular, we have that A_r is monotone, i.e.

$$(A_r(w) - A_r(v), w - v) \geq 0, \quad \forall w, v \in L^q(\omega; W^{1,q}_{\#}(Y)^3). \tag{71}$$

On the other hand, for all $\varphi \in \mathcal{D}(\omega; C^\infty_{\#}(Y)^3)$ with $\varphi = 0$ in $\omega \times T$, satisfying the divergence conditions $\mathrm{div}_{x'} \int_Y \varphi' \, dy = 0$ in ω, $\int_{Y_f} \varphi(x', y) \, dy \cdot n = 0$ on $\partial\omega$, and $\mathrm{div}_y \varphi = 0$ in $\omega \times Y$, we choose v_ε defined by

$$v_\varepsilon = \varphi - \varepsilon^{-1} \hat{u}_\varepsilon,$$

as a test function in (70).

Taking into account that $\mathrm{div}_\varepsilon \tilde{u}_\varepsilon = 0$, we get that $\varepsilon^{-1} \mathrm{div}_y \hat{u}_\varepsilon = 0$, and then we obtain

$$\varepsilon^{\gamma-1}(A_r(\varepsilon^{-1}\hat{u}_\varepsilon), v_\varepsilon) - \int_{\omega\times Y} \hat{P}_\varepsilon \, \mathrm{div}_{x'} v'_\varepsilon \, dx'dy = \int_{\omega\times Y} f' \cdot v'_\varepsilon \, dx'dy + O_\varepsilon,$$

which is equivalent to

$$\varepsilon^{\gamma-1}(A_r(\varphi) - A_r(\varepsilon^{-1}\hat{u}_\varepsilon), v_\varepsilon) - \varepsilon^{\gamma-1}(A_r(\varphi), v_\varepsilon) + \int_{\omega\times Y} \hat{P}_\varepsilon \, \mathrm{div}_{x'} v'_\varepsilon \, dx'dy = -\int_{\omega\times Y} f' \cdot v'_\varepsilon \, dx'dy + O_\varepsilon.$$

Due to (71), we can deduce

$$\varepsilon^{\gamma-1}(A_r(\varphi), v_\varepsilon) - \int_{\omega\times Y} \hat{P}_\varepsilon \, \mathrm{div}_{x'} v'_\varepsilon \, dx'dy \geq \int_{\omega\times Y} f' \cdot v'_\varepsilon \, dx'dy + O_\varepsilon,$$

i.e.

$$\begin{aligned} &\varepsilon^{\gamma-1}(\eta_0 - \eta_\infty) \int_{\omega\times Y} (1 + \lambda|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[v_\varepsilon] dx'dy + \varepsilon^{\gamma-1}\eta_\infty \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[v_\varepsilon] dx'dy \\ &- \int_{\omega\times Y} \hat{P}_\varepsilon \, \mathrm{div}_{x'} v'_\varepsilon \, dx'dy \geq \int_{\omega\times Y} f' \cdot v'_\varepsilon \, dx'dy + O_\varepsilon. \end{aligned} \tag{72}$$

Here, in the case $1 < r \leq 2$, it holds $|O_\varepsilon| \leq C\varepsilon^\alpha$ with $\alpha = 1$ if $\gamma \leq 1$ and $\alpha = 2 - \gamma$ if $\gamma > 1$. In the case $r > 2$, it holds $|O_\varepsilon| \leq C\varepsilon^\alpha$ with $\alpha = \gamma + \frac{2}{r}(1 - \gamma)$ if $\gamma \leq 1$ and $\alpha = 1 - \frac{\gamma-1}{r-1}$ if $\gamma > 1$.

Proof of Theorem 2.1. We recall that $1 < r < 2$. The case $\gamma = 1$ is developed in [8], so we omit it. As consequence, in the rest of the proof we consider that $\gamma \neq 1$. The proof will be divided in two steps. In the first step, we obtain the homogenized behavior given by a coupled system, with a constant macroviscosity, and in the second step we decouple it to obtain the macroscopic law.

Step 1. From Lemmas 3.10 and 3.11, we prove that the sequence $(\varepsilon^{\gamma-2}\hat{u}_\varepsilon, \tilde{P}_\varepsilon)$ converges to $(\hat{u}, \tilde{P}) \in L^2(\omega; H^1_{\#}(Y_f)^3) \times (L^2_0(\omega) \cap H^1(\omega))$, which are the unique solutions of the following two-pressures Newtonian

Stokes problem with the linear viscosity η equal to η_0 if $\gamma < 1$ and η_∞ if $\gamma > 1$:

$$\begin{cases} -\eta \operatorname{div}_y \mathbb{D}_y[\hat{u}] + \nabla_y \hat{\pi} = f' - \nabla_{x'} \tilde{P} & \text{in} \quad \omega \times Y_f, \\ \operatorname{div}_y \hat{u} = 0 & \text{in} \quad \omega \times Y_f, \\ \operatorname{div}_{x'} \left(\displaystyle\int_{Y_f} \hat{u}' \, dy \right) = 0 & \text{in } \omega, \\ \left(\displaystyle\int_{Y_f} \hat{u}' \, dy \right) \cdot n = 0 & \text{on } \partial\omega, \\ \hat{u} = 0 & \text{in } \omega \times T, \\ \hat{\pi} \in L^2(\omega; L^2_{0,\#}(Y_f)). \end{cases} \tag{73}$$

Divergence conditions $(73)_{2,3,4}$ and condition $(73)_5$ follow from Lemma 3.10. To prove that $(\hat{u}, \tilde{P})$ satisfies the momentum equation given in (73), we follows the lines of the proof to obtain (72) but choosing now v_ε and φ such that $v_\varepsilon = \varepsilon^{1-\gamma}\varphi - \varepsilon^{-1}\hat{u}_\varepsilon$ with $\varphi \in \mathcal{D}(\omega; C^\infty_\#(Y)^3)$ with $\varphi = 0$ in $\omega \times T$, satisfying the divergence conditions $\operatorname{div}_{x'} \int_Y \varphi' \, dy = 0$ in ω, $\int_Y \varphi' \, dy \cdot n = 0$ on $\partial\omega$, and $\operatorname{div}_y \varphi = 0$ in $\omega \times Y$. Then, we get

$$\begin{aligned} &\varepsilon^{1-\gamma}(\eta_0 - \eta_\infty) \int_{\omega\times Y} (1 + \lambda\varepsilon^{2(1-\gamma)} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\gamma-2}\hat{u}_\varepsilon] \, dx'dy \\ &+\varepsilon^{1-\gamma}\eta_\infty \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\gamma-2}\hat{u}_\varepsilon] dx'dy \\ &-\varepsilon^{1-\gamma} \int_{\omega\times Y} \hat{P}_\varepsilon \operatorname{div}_{x'}(\varphi' - \varepsilon^{\gamma-2}\hat{u}'_\varepsilon) \, dx'dy \geq \varepsilon^{1-\gamma} \int_{\omega\times Y} f' \cdot (\varphi' - \varepsilon^{\gamma-2}\hat{u}'_\varepsilon) \, dx'dy + O_\varepsilon, \end{aligned} \tag{74}$$

where $|O_\varepsilon| \leq C\varepsilon^{2-\gamma}$. Dividing by $\varepsilon^{1-\gamma}$, we have

$$\begin{aligned} &(\eta_0 - \eta_\infty) \int_{\omega\times Y} (1 + \lambda\varepsilon^{2(1-\gamma)} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\gamma-2}\hat{u}_\varepsilon] \, dx'dy \\ &+ \eta_\infty \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\gamma-2}\hat{u}_\varepsilon] dx'dy \\ &- \int_{\omega\times Y} \hat{P}_\varepsilon \operatorname{div}_{x'}(\varphi' - \varepsilon^{\gamma-2}\hat{u}'_\varepsilon) \, dx'dy \geq \int_{\omega\times Y} f' \cdot (\varphi' - \varepsilon^{\gamma-2}\hat{u}'_\varepsilon) \, dx'dy + O_\varepsilon, \end{aligned} \tag{75}$$

where $|O_\varepsilon| \leq C\varepsilon$, which tends to zero when $\varepsilon \to 0$. Now, we can pass to the limit depending on the value of γ:

- If $\gamma < 1$, then $2(1-\gamma) > 0$ and so, $\lambda\varepsilon^{2(1-\gamma)}|\mathbb{D}_y[\varphi]|^2$ tends to zero. From convergence (56), passing to the limit when ε tends to zero in (75), we have that the first and second terms converge to

$$\eta_0 \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] \, dx'dy.$$

  From convergences (56) and (66), the third term converges to

$$\int_{\omega\times Y} \tilde{P} \operatorname{div}_{x'}(\varphi' - \hat{u}') \, dx'dy.$$

  Since $\tilde{P}$ does not depend on y, by using the divergences conditions $\operatorname{div}_{x'} \int_Y \varphi' \, dy = 0$ and $(64)_2$, we have

$$\int_{\omega\times Y} \tilde{P} \operatorname{div}_{x'}(\varphi' - \hat{u}') \, dx'dy = \int_\omega \tilde{P} \operatorname{div}_{x'} \left(\int_Y (\varphi' - \hat{u}') dy \right) dx' = 0.$$

Thus, we deduce that the variational inequality (75) converges to the following one

$$\eta_0 \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}]\, dx'dy \geq \int_{\omega\times Y} f' \cdot (\varphi' - \hat{u}')\, dx'dy,$$

which by Minty's lemma, see [30, Chapter 3, Lemma 1.2], is equivalent to

$$-\eta_0 \operatorname{div}_y (\mathbb{D}_y[\hat{u}]) = f' \quad \text{in} \quad \omega \times Y.$$

- If $\gamma > 1$, then $2(1-\gamma) < 0$ and so, $(1+\lambda\varepsilon^{2(1-\gamma)}|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1}$ tends to zero. From convergence (56), passing to the limit when ε tends to zero in (75), we have that the first and second terms converge to

$$\eta_\infty \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}]\, dx'dy.$$

From convergences (56) and (66), the third term converges to

$$\int_{\omega\times Y} \tilde{P} \operatorname{div}_{x'}(\varphi' - \hat{u}')\, dx'dy.$$

Since $\tilde{P}$ does not depend on y, by using the divergences conditions $\operatorname{div}_{x'} \int_Y \varphi'\, dy = 0$ and $(64)_2$, we have

$$\int_{\omega\times Y} \tilde{P} \operatorname{div}_{x'}(\varphi' - \hat{u}')\, dx'dy = \int_\omega \tilde{P} \operatorname{div}_{x'} \left(\int_Y (\varphi' - \hat{u}')dy \right) dx' = 0.$$

Thus, we deduce that the variational inequality (75) converges to the following one

$$\eta_\infty \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}]\, dx'dy \geq \int_{\omega\times Y} f' \cdot (\varphi' - \hat{u}')\, dx'dy,$$

which by Minty's lemma, see [30, Chapter 3, Lemma 1.2], is equivalent to

$$-\eta_\infty \operatorname{div}_y (\mathbb{D}_y[\hat{u}]) = f' \quad \text{in} \quad \omega \times Y.$$

In summary, considering η equal to η_0 if $\gamma < 1$ or η_∞ of $\gamma > 1$, we have obtained that by density

$$\eta \int_{\omega\times Y} \mathbb{D}_y[\hat{u}] : \mathbb{D}_y[v]\, dx'dy = \int_{\omega\times Y} f' \cdot v'\, dx'dy, \tag{76}$$

holds for every v in the Hilbert space $\mathcal{V}$ defined by

$$\mathcal{V} = \left\{ \begin{array}{l} v(x',y) \in L^2(\omega; H^1_\#(Y)^3) \text{ such that} \\ \operatorname{div}_{x'} \left(\displaystyle\int_{Y_f} v(x',y)\, dy \right) = 0 \text{ in } \omega, \quad \left(\displaystyle\int_{Y_f} v(x',y)\, dy \right) \cdot n = 0 \text{ on } \partial\omega \\ \operatorname{div}_y v(x',y) = 0 \text{ in } \omega \times Y_f, \quad v(x',y) = 0 \text{ in } \omega \times T \end{array} \right\}. \tag{77}$$

Reasoning as in [1, Lemma 1.5], the orthogonal of $\mathcal{V}$, a subset of $L^2(\omega; H^{-1}_\#(Y)^3)$, is made of gradients of the form $\nabla_{x'}\tilde{\pi}(x') + \nabla_y\hat{\pi}(x',y)$, with $\tilde{\pi}(x') \in H^1(\omega)/\mathbb{R}$ and $\hat{\pi}(x',y) \in L^2(\omega; L^2_\#(Y_f)/\mathbb{R})$. Thus, integrating by parts, the variational formulation (76) is equivalent to the two-pressures Newtonian Stokes problem (73). It remains to prove that $\tilde{\pi}$ coincides with pressure $\tilde{P}$. This can be easily done passing to the limit similarly as above by considering the test function φ, which is divergence-free only in y, and by identifying limits. It holds then that $\tilde{P} \in L^2_0(\omega) \cap H^1(\omega)$. From [1], problem (73) admits a unique solution $(\hat{u}, \hat{\pi}, \tilde{P}) \in L^2(\omega; H^1_\#(Y_f)^3) \times L^2(\omega; L^2_{0,\#}(Y_f)) \times (L^2_0(\omega) \cap H^1(\omega))$

and then, the entire sequence $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ converges to $(\hat{u}, \tilde{P})$.

Step 2. To prove (11), it remains to eliminate the microscopic variable y in the effective problem (73). The procedure is standard and can be viewed for instance in [14], but for reader's convenience, we give some details of the proof. From the first equation of (73), the velocity $\hat{u}$ is computed in terms of the macroscopic forces and the local velocities

$$\hat{u}(x', y) = \frac{1}{\eta} \sum_{i=1}^{2} \Big(f_i(x') - \partial_{x_i} \tilde{P}(x') \Big) w_i(y), \quad \hat{\pi}(x', y) = \sum_{i=1}^{2} \Big(f_i(x') - \partial_{x_i} \tilde{P}(x') \Big) \pi_i(y).$$

Integrating this expression on Y and taking into account that $V(x') = \int_0^1 \tilde{u}(x', y_3) dy_3 = \int_{Y_f} \hat{u}(x', y) dy$ and $\int_{Y_f} \hat{u}_3 \, dy = 0$, we get the Darcy relationship $(11)_1$ because the matrix $\mathcal{A} \in \mathbb{R}^{2\times 2}$ satisfies (12). Combining the expresion of V with the divergence-free condition on V given by $(73)_3$ yields the lower-dimensional homogenized Darcy's law $(11)_2$.

□

Proof of Theorem 2.3. We recall that in this case $r > 2$. The proof will be divided in four steps. In the first step, we obtain the homogenized behavior in the case $\gamma < 1$ and as a result, we obtain that the filtration velocity $V = 0$ in ω. In the second step, we obtain the homogenized behavior for $\gamma > 1$ giving by a coupled system with a non-linear macroviscosity of power law type, which will be decoupled to obtain the macroscopic law in the third step. Finally, in the fourth step, we consider the case $\gamma = 1$.

Step 1. We consider the case $\gamma < 1$. We will prove that the filtration velocity $V(x') = \int_0^1 \tilde{u} \, dy_3 = 0$ in ω. To do this, we first prove that $\hat{u}$ does not depend on the variable y and next, taking into account the boundary conditions on $\hat{u}$ and the relation of $\hat{u}$ and $\tilde{u}$, we prove that $V = 0$.

– We start by proving that $\hat{u}$ does not depend on y. Reasoning similarly to the case $1 < r < 2$ and considering $v_\varepsilon = \varepsilon^{-\frac{2}{r}(\gamma-1)} \varphi - \varepsilon^{-1} \hat{u}_\varepsilon$ with $\varphi \in \mathcal{D}(\omega; C^\infty_\#(Y)^3)$ with $\varphi = 0$ in $\omega \times T$, satisfying only the divergence condition $\mathrm{div}_y \varphi = 0$ in $\omega \times Y$, we get

$$\begin{aligned}
&\varepsilon^{(-1+\frac{4}{r})(1-\gamma)} (\eta_0 - \eta_\infty) \int_{\omega\times Y} (1 + \lambda \varepsilon^{\frac{4}{r}(1-\gamma)} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}_\varepsilon] \, dx' dy \\
&+ \varepsilon^{(-1+\frac{4}{r})(1-\gamma)} \eta_\infty \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}_\varepsilon] \, dx' dy \\
&- \varepsilon^{\frac{2}{r}(1-\gamma)} \int_{\omega\times Y} \hat{P}_\varepsilon \, \mathrm{div}_{x'} (\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}'_\varepsilon) \, dx' dy \\
&\geq \varepsilon^{\frac{2}{r}(1-\gamma)} \int_{\omega\times Y} f' \cdot (\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}'_\varepsilon) \, dx' dy + O_\varepsilon,
\end{aligned} \tag{78}$$

where $|O_\varepsilon| \leq C \varepsilon^{\gamma + \frac{4}{r}(1-\gamma)}$. Dividing by $\varepsilon^{(-1+\frac{4}{r})(1-\gamma)}$, we have

$$\begin{aligned}
&(\eta_0 - \eta_\infty) \int_{\omega\times Y} (1 + \lambda \varepsilon^{\frac{4}{r}(1-\gamma)} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}_\varepsilon] \, dx' dy \\
&+ \eta_\infty \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}_\varepsilon] dx' dy \\
&- \varepsilon^{(1-\frac{2}{r})(1-\gamma)} \int_{\omega\times Y} \hat{P}_\varepsilon \, \mathrm{div}_{x'} (\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}'_\varepsilon) \, dx' dy \\
&\geq \varepsilon^{(1-\frac{2}{r})(1-\gamma)} \int_{\omega\times Y} f' \cdot (\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}'_\varepsilon) \, dx' dy + O_\varepsilon,
\end{aligned} \tag{79}$$

with $|O_\varepsilon| \leq C\varepsilon^{1+(1-\gamma)(1-\frac{2}{r})}$, which tends to zero when $\varepsilon \to 0$, because $1+(1-\frac{2}{r})(1-\gamma) > 0$.

Since $\gamma < 1$, then $\frac{4}{r}(1-\gamma) > 0$ and so, $\lambda\varepsilon^{\frac{4}{r}(1-\gamma)}|\mathbb{D}_y[\varphi]|^2$ tends to zero. From convergence (58), passing to the limit when ε tends to zero in (79), we have that the first and second terms converge to

$$\eta_0 \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}]\, dx'dy.$$

Since $r > 2$, then $(1-\frac{2}{r})(1-\gamma) > 0$. This together with convergences (58) and (68), implies that

$$\varepsilon^{(1-\frac{2}{r})(1-\gamma)} \int_{\omega\times Y} \hat{P}_\varepsilon \,\mathrm{div}_{x'}(\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1}\hat{u}'_\varepsilon)\, dx'dy \quad \text{and} \quad \varepsilon^{(1-\frac{2}{r})(1-\gamma)} \int_{\omega\times Y} f' \cdot (\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1}\hat{u}'_\varepsilon)\, dx'dy$$

tend to zero.

Thus, we deduce that the variational inequality (79) converges to the following one

$$\eta_0 \int_{\omega\times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}]\, dx'dy \geq 0,$$

which by Minty's lemma, see [30, Chapter 3, Lemma 1.2], implies that

$$\eta_0 \int_{\omega\times Y} \mathbb{D}_y[\hat{u}] : \mathbb{D}_y[v]\, dx'dy = 0, \tag{80}$$

holds, by density arguments, for every v in the Banach space $\mathcal{V}$ defined by

$$\mathcal{V} = \left\{ \begin{array}{l} v(x',y) \in L^r(\omega; W^{1,r}_{\#}(Y)^3) \text{ such that} \\ \mathrm{div}_y v(x',y) = 0 \text{ in } \omega\times Y_f, \quad v(x',y) = 0 \text{ in } \omega\times T \end{array} \right\}.$$

Then, there exists $\hat{\pi}(x',y) \in L^{r'}(\omega; L^{r'}_{\#}(Y_f)/\mathbb{R})$ such that, integrating by parts, the variational formulation (80) is equivalent to the Stokes system with respect to the variable y

$$-\eta_0 \,\mathrm{div}_y\,(\mathbb{D}_y[\hat{u}]) + \nabla_y\hat{\pi} = 0 \quad \text{in} \quad \omega\times Y.$$

We immediately find (see the boundary conditions with respect to y) that

$$D_y\hat{u} = 0 \quad \text{and} \quad \nabla_y\hat{\pi} = 0,$$

i.e. that $\hat{u}$ does not depend on y. So, we can not obtain microscopic effects in the limit. Next, we will derive the macroscopic asymptotic behavior of $\tilde{u}$.

– We prove that the filtration velocity $V = 0$ in ω. To do this, taking into account that $\hat{u}$ only depends on x' and from the boundary conditions $\hat{u} = 0$ on $y_3 = \{0,1\}$, we dedue that $\hat{u} = 0$ in $\omega\times Y$. Finally, taking into account that $\hat{u} = 0$, that the filtration velocity is defined by $V(x') = \int_0^1 \tilde{u}(x',y_3)\, dy_3$ and that $\int_0^1 \tilde{u}(x',y_3)\, dy_3 = \int_Y \hat{u}(x',y)\, dy$ (see Lemma 3.10), we then deduce that the filtration velocity $V(x') = 0$ in ω, i.e. (17).

Step 2. We consider the case $\gamma > 1$. From Lemmas 3.10 and 3.11, we prove that the sequence $(\hat{u}_\varepsilon, \tilde{P}_\varepsilon)$ converges to $(\hat{u},\tilde{P}) \in L^r(\omega; W^{1,r}_{\#}(Y_f)^3) \times (L^{r'}_0(\omega) \cap W^{1,r'}(\omega))$, which are the unique solutions of the following

two-pressures non-Newtonian Stokes problem with the non-linear power law viscosity

$$\begin{cases} -(\eta_0-\eta_\infty)\lambda^{\frac{r-2}{2}}\,\mathrm{div}_y(|\mathbb{D}_y[\hat{u}]|^{r-2}\mathbb{D}_y[\hat{u}])+\nabla_y\hat{\pi}=f'-\nabla_{x'}\tilde{P} & \text{in} \quad \omega\times Y_f,\\ \mathrm{div}_y\hat{u}=0 & \text{in} \quad \omega\times Y_f,\\ \mathrm{div}_{x'}\left(\displaystyle\int_{Y_f}\hat{u}'\,dy\right)=0 & \text{in } \omega,\\ \left(\displaystyle\int_{Y_f}\hat{u}'\,dy\right)\cdot n=0 & \text{on } \partial\omega,\\ \hat{u}=0 & \text{in } \omega\times T,\\ \hat{\pi}\in L^{r'}(\omega;L^{r'}_{0,\#}(Y_f)). \end{cases} \tag{81}$$

Divergence conditions $(81)_{2,3,4}$ and condition $(81)_5$ follow from Lemma 3.10. To prove that $(\hat{u},\tilde{P})$ satisfies the momentum equation given in (81), we follow the lines of the proof of (72) but considering now v_ε and φ as $v_\varepsilon=\varepsilon^{\frac{1-\gamma}{r-1}}\varphi-\varepsilon^{-1}\hat{u}_\varepsilon$ with $\varphi\in\mathcal{D}(\omega;C^\infty_\#(Y)^3)$ with $\varphi=0$ in $\omega\times T$, satisfying the divergence conditions $\mathrm{div}_{x'}\int_Y\varphi'\,dy=0$ in ω, $\int_Y\varphi'\,dy\cdot n=0$ on $\partial\omega$, and $\mathrm{div}_y\varphi=0$ in $\omega\times Y$. Then, we get

$$\begin{aligned} &\varepsilon^{\gamma-1+2\frac{1-\gamma}{r-1}}(\eta_0-\eta_\infty)\int_{\omega\times Y}(1+\lambda\varepsilon^{2\frac{1-\gamma}{r-1}}|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1}\mathbb{D}_y[\varphi]:\mathbb{D}_y[\varphi-\varepsilon^{\frac{\gamma-r}{r-1}}\hat{u}_\varepsilon]\,dx'dy\\ &+\varepsilon^{\gamma-1+2\frac{1-\gamma}{r-1}}\eta_\infty\int_{\omega\times Y}\mathbb{D}_y[\varphi]:\mathbb{D}_y[\varphi-\varepsilon^{\frac{\gamma-r}{r-1}}\hat{u}_\varepsilon]\,dx'dy\\ &-\varepsilon^{\frac{1-\gamma}{r-1}}\int_{\omega\times Y}\hat{P}_\varepsilon\,\mathrm{div}_{x'}(\varphi'-\varepsilon^{\frac{\gamma-r}{r-1}}\hat{u}'_\varepsilon)\,dx'dy\geq\varepsilon^{\frac{1-\gamma}{r-1}}\int_{\omega\times Y}f'\cdot(\varphi'-\varepsilon^{\frac{\gamma-r}{r-1}}\hat{u}'_\varepsilon)\,dx'dy+O_\varepsilon, \end{aligned} \tag{82}$$

where $|O_\varepsilon|\leq C\varepsilon^{1+\frac{1-\gamma}{r-1}}$. Dividing by $\varepsilon^{\frac{1-\gamma}{r-1}}$ we deduce that

$$\begin{aligned} &\varepsilon^{(\gamma-1)\frac{r-2}{r-1}}(\eta_0-\eta_\infty)\int_{\omega\times Y}(1+\lambda\varepsilon^{2\frac{1-\gamma}{r-1}}|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1}\mathbb{D}_y[\varphi]:\mathbb{D}_y[\varphi-\varepsilon^{\frac{\gamma-r}{r-1}}\hat{u}_\varepsilon]\,dx'dy\\ &+\varepsilon^{(\gamma-1)\frac{r-2}{r-1}}\eta_\infty\int_{\omega\times Y}\mathbb{D}_y[\varphi]:\mathbb{D}_y[\varphi-\varepsilon^{\frac{\gamma-r}{r-1}}\hat{u}_\varepsilon]\,dx'dy\\ &-\int_{\omega\times Y}\hat{P}_\varepsilon\,\mathrm{div}_{x'}(\varphi'-\varepsilon^{\frac{\gamma-r}{r-1}}\hat{u}'_\varepsilon)\,dx'dy\geq\int_{\omega\times Y}f'\cdot(\varphi'-\varepsilon^{\frac{\gamma-r}{r-1}}\hat{u}'_\varepsilon)\,dx'dy+O_\varepsilon, \end{aligned} \tag{83}$$

with $|O_\varepsilon|\leq C\varepsilon$, which tends to zero when $\varepsilon\to 0$.

Since $\gamma>1$ and $r>2$, we have that $(\gamma-1)\frac{r-2}{r-1}>0$, and from

$$\varepsilon^{(\gamma-1)\frac{r-2}{r-1}}(1+\lambda\varepsilon^{2\frac{1-\gamma}{r-1}}|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1}=(\varepsilon^{2\frac{\gamma-1}{r-1}}+\lambda|\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1}$$

and convergences (60) and (68), passing to the limit in (83) when ε tends to zero, we deduce

$$\begin{aligned} &(\eta_0-\eta_\infty)\lambda^{\frac{r-2}{2}}\int_{\omega\times Y}|\mathbb{D}_y[\varphi]|^{r-2}\mathbb{D}_y[\varphi]:\mathbb{D}_y[\varphi-\hat{u}]\,dx'dy\\ &-\int_{\omega\times Y}\tilde{P}\,\mathrm{div}_{x'}(\varphi'-\hat{u}')\,dx'dy\geq\int_{\omega\times Y}f'\cdot(\varphi'-\hat{u}')\,dx'dy. \end{aligned} \tag{84}$$

Since $\tilde{P}$ does not depend on y, by using the divergences conditions $\mathrm{div}_{x'}\int_Y \varphi'\,dy = 0$ and $(81)_3$, we have

$$\int_{\omega\times Y} \tilde{P}\,\mathrm{div}_{x'}(\varphi'-\hat{u}')\,dx'dy = \int_\omega \tilde{P}\,\mathrm{div}_{x'}\left(\int_Y(\varphi'-\hat{u}')dy\right)\,dx' = 0.$$

Thus, we deduce that the variational inequality (84) reads

$$(\eta_0-\eta_\infty)\lambda^{\frac{r-2}{2}}\int_{\omega\times Y}|\mathbb{D}_y[\varphi]|^{r-2}\mathbb{D}_y[\varphi]:\mathbb{D}_y[\varphi-\hat{u}]\,dx'dy \geq \int_{\omega\times Y} f'\cdot(\varphi'-\hat{u}')\,dx'dy,$$

which by Minty's lemma, see [30, Chapter 3, Lemma 1.2], is equivalent to

$$-(\eta_0-\eta_\infty)\lambda^{\frac{r-2}{2}}\,\mathrm{div}_y\left(|\mathbb{D}_y[\hat{u}]|^{r-2}\mathbb{D}_y[\hat{u}]\right) = f' \quad \text{in} \quad \omega\times Y.$$

Then, by density we have that

$$(\eta_0-\eta_\infty)\lambda^{\frac{r-2}{2}}\int_{\omega\times Y}|\mathbb{D}_y[\hat{u}]|^{r-2}\mathbb{D}_y[\hat{u}]:\mathbb{D}_y[v]\,dx'dy = \int_{\omega\times Y} f'\cdot v'\,dx'dy, \tag{85}$$

holds for every v in the Banach space $\mathcal{V}$ defined by

$$\mathcal{V} = \left\{ \begin{array}{l} v(x',y)\in L^r(\omega;W^{1,r}_{\#}(Y)^3) \text{ such that} \\ \mathrm{div}_{x'}\left(\displaystyle\int_{Y_f} v(x',y)\,dy\right) = 0 \text{ in } \omega, \quad \left(\displaystyle\int_{Y_f} v(x',y)\,dy\right)\cdot n = 0 \text{ on } \partial\omega \\ \mathrm{div}_y v(x',y) = 0 \text{ in } \omega\times Y_f, \quad v(x',y) = 0 \text{ in } \omega\times T \end{array} \right\}.$$

Reasoning as in [1, Lemma 1.5], the orthogonal of $\mathcal{V}$, a subset of $L^r(\omega;W^{-1,r'}_{\#}(Y)^3)$, is made of gradients of the form $\nabla_{x'}\tilde{\pi}(x') + \nabla_y\hat{\pi}(x',y)$, with $\tilde{\pi}(x')\in W^{1,r'}(\omega)/\mathbb{R}$ and $\hat{\pi}(x',y)\in L^{r'}(\omega;L^{r'}_{\#}(Y_f)/\mathbb{R})$. Thus, integrating by parts, the variational formulation (85) is equivalent to the two-pressures non-Newtonian Stokes problem (81). It remains to prove that $\tilde{\pi}$ coincides with pressure $\tilde{P}$. This can be easily done passing to the limit similarly as above by considering the test function φ, which is divergence-free only in y, and by identifying limits. It holds then that $\tilde{P}\in L^{r'}_0(\omega)\cap W^{1,r'}(\omega)$. From [22, Theorem 2], problem (81) admits a unique solution $(\hat{u},\hat{\pi},\tilde{P}) \in L^r(\omega;W^{1,r}_{\#}(Y_f)^3)\times L^{r'}(\omega;L^{r'}_{0,\#}(Y_f))\times(L^{r'}_0(\omega)\cap W^{1,r'}(\omega))$ and then, the entire sequence $(\hat{u}_\varepsilon,\hat{P}_\varepsilon)$ converges to $(\hat{u},\tilde{P})$.

Step 3. In this step we give an approximation of the model (81), where the macroscopic scale is totally decoupled from the microscopic one. To do this, we seek a global filtration velocity of the form given in (18), i.e.

$$V(x') = \frac{\lambda^{\frac{2-r}{2}}}{\eta_0-\eta_\infty}\mathcal{U}(f'(x')-\nabla_{x'}\tilde{P}(x')) \quad \text{in } \omega, \tag{86}$$

where $\mathcal{U}:\mathbb{R}^2\to\mathbb{R}^3$ is a permeability function, not necessary linear, and $V(x') = \int_0^1 \tilde{u}(x',y_3)\,dy_3 = \int_Y \hat{u}(x',y)\,dy$ with $\mathrm{div}_{x'}V' = 0$ in ω and $V'\cdot n = 0$ on $\partial\omega$.

Using the idea from [23] to decouple the homogenized problems of *power law* type, for every $\xi'\in\mathbb{R}^2$ we consider the function $\mathcal{U}:\mathbb{R}^2\to\mathbb{R}^3$ given by

$$\mathcal{U}(\xi') = \int_{Y_f} w_{\xi'}(y)\,dy,$$

where $w_{\xi'}$ denotes the unique solution of the local Stokes problem given by (16), see [22, Theorem 2]. Thus, $(\hat{u}, \hat{\pi})$ takes the form

$$\hat{u}(x', y) = w_{f'(x')-\nabla_{x'}\tilde{P}(x')}(y), \quad \hat{\pi}(x', y) = \pi_{f'(x')-\nabla_{x'}\tilde{P}(x')}(y) \quad \text{in } \omega \times Y,$$

and then, from the relation $V(x') = \int_Y \hat{u}(x', y)\, dy$ with $\int_Y \hat{u}_3(x', y)\, dy = 0$ given in Lemma 3.10, we deduce the filtration velocity (86), where $V_3 = 0$. Moreover, from second and third conditions given in (81) together with (86), we deduce

$$\mathrm{div}_{x'} V' = 0 \quad \text{in } \omega, \quad V' \cdot n = 0 \quad \text{on } \partial\omega.$$

Since $V_3 = 0$, to simplify the notation, we redefine the definition of $\mathcal{U}$ by the expression given in (15) and then, we get $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$, which concludes the proof of (18). Finally, from [23, Theorem 1], the macroscopic problem (18) has a unique solution $(V, \tilde{P}) \in L^r(\omega)^3 \times (L_0^{r'}(\omega) \cap W^{1,r}(\omega))$ and Theorem 2.3 is proved.

Step 4. We consider the case $\gamma = 1$. Reasoning as in the *Step 2* with $\gamma = 1$ and using convergences (62) and (68), we deduce that the sequence $(\hat{u}_\varepsilon, \tilde{P}_\varepsilon)$ converges to $(\hat{u}, \tilde{P}) \in L^r(\omega; W^{1,r}_{0,\#}(Y_f)^3) \times (L_0^{r'}(\omega) \cap W^{1,r'}(\omega))$, which are the unique solutions of the following two-pressures non-Newtonian Stokes problem with the non-linear Carreau viscosity (2)

$$\begin{cases} -\mathrm{div}_y(\eta_r(\mathbb{D}_y[\hat{u}])\mathbb{D}_y[\hat{u}]) + \nabla_y \hat{\pi} = f' - \nabla_{x'}\tilde{P} & \text{in} \quad \omega \times Y_f, \\ \mathrm{div}_y \hat{u} = 0 & \text{in} \quad \omega \times Y_f, \\ \mathrm{div}_{x'} \left(\displaystyle\int_{Y_f} \hat{u}'\, dy \right) = 0 & \text{in } \omega, \\ \left(\displaystyle\int_{Y_f} \hat{u}'\, dy \right) \cdot n = 0 & \text{on } \partial\omega, \\ \hat{u} = 0 & \text{in } \omega \times T, \\ \hat{\pi} \in L^{r'}(\omega; L^{r'}_{0,\#}(Y_f)). \end{cases}$$

Proceeding as in *Step 3*, we deduce the non-linear 2D Darcy's law of Carreau type (14), where the permeability function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by (15) with $(w_{\xi'}, \pi_{\xi'}) \in W^{1,r}_{0,\#}(Y_f)^3 \times L^{r'}_{0,\#}(Y_f)$, for every $\xi' \in \mathbb{R}^2$, the unique solution of the local Stokes system (16) with nonlinear viscosity given by the Carreau law (2).

□

Proof of Theorem 2.5. We recall that in this case $r = 2$, the Carreau law (2) reduces to η_0. Thus, by the linearity, the variational formulation (70) can be written as follows

$$\varepsilon^{\gamma-2}\eta_0 \int_{\omega\times Y} \mathbb{D}_y[\hat{u}_\varepsilon] : \mathbb{D}_y[\varphi] dx' dy - \int_{\omega\times Y} \hat{P}_\varepsilon \,\mathrm{div}_{x'}\varphi'\, dx' dy = \int_{\omega\times Y} f' \cdot \varphi'\, dx' dy + O_\varepsilon, \tag{87}$$

for all $\varphi \in \mathcal{D}(\omega; C^\infty_\#(Y)^3)$ with $\varphi = 0$ in $\omega \times T$, satisfying the divergence conditions $\mathrm{div}_{x'} \int_Y \varphi'\, dy = 0$ in ω, $\int_Y \varphi'\, dy \cdot n = 0$ on $\partial\omega$, and $\mathrm{div}_y \varphi = 0$ in $\omega \times Y$.

Passing to the limit in (87) by using convergences (56) and (66), we take into account that the second term converges to

$$\int_{\omega\times Y} \tilde{P}\, \mathrm{div}_{x'}\varphi'\, dx' dy.$$

Since $\tilde{P}$ does not depend on y, by using the divergences conditions $\mathrm{div}_{x'} \int_Y \varphi' \, dy = 0$, we have

$$\int_{\omega \times Y} \tilde{P} \, \mathrm{div}_{x'} \varphi' \, dx' dy = \int_{\omega} \tilde{P} \, \mathrm{div}_{x'} \left(\int_Y \varphi' dy \right) dx' = 0.$$

Thus, we deduce that the variational inequality (87) converges to the following one

$$\eta_0 \int_{\omega \times Y} \mathbb{D}_y[\hat{u}] : \mathbb{D}_y[\varphi] dx' dy = \int_{\omega \times Y} f' \cdot \varphi' \, dx' dy,$$

which by density arguments it holds for every $\varphi \in \mathcal{V}$ with $\mathcal{V}$ given by (77). Proceeding similarly to the end of *Step 1* of the proof of Theorem 2.1, this variational formulation is equivalent to the system (73) with the linear viscosity η_0, which admits a unique solution $(\hat{u}, \hat{\pi}, \tilde{P}) \in L^2(\omega; H^1_{\#}(Y_f)^3) \times L^2(\omega; L^2_{0,\#}(Y_f)) \times (L^2_0(\omega) \cap H^1(\omega))$. Then, the limits do not depend on the subsequence.

Finally, reasoning as in *Step 2* of the proof of Theorem 2.1, we get the linear effective 2D Darcy's law (19), which concludes the proof of Theorem 2.5.

□

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