



**HAL**  
open science

# Effective models for generalized Newtonian fluids through a thin porous media following the Carreau law

María Anguiano, Matthieu Bonnivard, Francisco J. Suárez-Grau

► **To cite this version:**

María Anguiano, Matthieu Bonnivard, Francisco J. Suárez-Grau. Effective models for generalized Newtonian fluids through a thin porous media following the Carreau law. 2023. hal-03885537v2

**HAL Id: hal-03885537**

**<https://hal.science/hal-03885537v2>**

Preprint submitted on 21 Mar 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Effective models for generalized Newtonian fluids through a thin porous medium following the Carreau law

María ANGUIANO<sup>1</sup>, Matthieu BONNIVARD<sup>2</sup> and Francisco J. SUÁREZ-GRAU<sup>3</sup>

## Abstract

We consider the flow of a generalized Newtonian fluid through a thin porous medium of thickness  $\varepsilon$ , perforated by periodically distributed solid cylinders of size  $\varepsilon$ . We assume that the fluid is described by the 3D incompressible Stokes system, with a non-linear viscosity following the Carreau law of flow index  $1 < r < +\infty$ , and scaled by a factor  $\varepsilon^\gamma$ , where  $\gamma \in \mathbb{R}$ . Generalizing (Anguiano *et al.*, Q. J. Mech. Math., 75(1), 2022, 1-27), where the particular case  $r < 2$  and  $\gamma = 1$  was addressed, we perform a new and complete study on the asymptotic behaviour of the fluid as  $\varepsilon$  goes to zero. Depending on  $\gamma$  and the flow index  $r$ , using homogenization techniques, we derive and rigorously justify different effective linear and non-linear lower-dimensional Darcy's laws. Finally, using a finite element method, we study numerically the influence of the rheological parameters of the fluid and of the shape of the solid obstacles on the behaviour of the effective systems.

**AMS classification numbers:** 76-10, 76A05, 76M50, 76A20, 76S05, 35B27, 35Q35.

**Keywords:** Homogenization, non-Newtonian fluid, Carreau law, thin porous media.

## 1 Introduction

An incompressible generalized Newtonian fluid is a type of non-Newtonian fluid which is characterized by a viscosity depending on the principal invariants of the symmetric stretching tensor  $\mathbb{D}[u]$ . If  $u$  is the velocity,  $p$  the pressure and  $Du$  the gradient velocity tensor,  $\mathbb{D}[u] = (Du + D^t u)/2$  denotes the symmetric stretching tensor and  $\sigma$  the stress tensor given by  $\sigma = -pI + 2\eta_r \mathbb{D}[u]$ . The viscosity  $\eta_r$  is constant for a Newtonian fluid but dependent on the shear rate, i.e.  $\eta_r = \eta_r(\mathbb{D}[u])$ , for generalized Newtonian fluids. The deviatoric stress tensor  $\tau$ , i.e. the part of the total stress tensor that is zero at equilibrium, is then a nonlinear function of the shear rate  $\mathbb{D}[u]$ , i.e.  $\tau = \eta_r(\mathbb{D}[u])\mathbb{D}[u]$  (see Barnes *et al.* [20], Bird *et al.* [21] and Mikelić [36] for more details).

A commonly used viscosity formula  $\eta_r$  for describing generalized Newtonian fluids is the *power law* or Ostwald-de Waele model (Ostwald, 1925; de Waele, 1923), which is defined by

$$\eta_r(\mathbb{D}[u]) = \mu |\mathbb{D}[u]|^{r-2}, \quad 1 < r < +\infty, \quad \mu > 0, \quad (1.1)$$

---

<sup>1</sup>Departamento de Análisis Matemático. Facultad de Matemáticas. Universidad de Sevilla, 41012 Sevilla (Spain) [anguiano@us.es](mailto:anguiano@us.es)

<sup>2</sup>Université Paris Cité and Sorbonne Université, CNRS, Laboratoire J-L. Lions / LJLL, F-75006 Paris, France, [bonnivard@ljl.univ-paris-diderot.fr](mailto:bonnivard@ljl.univ-paris-diderot.fr)

<sup>3</sup>Departamento de Ecuaciones Diferenciales y Análisis Numérico. Facultad de Matemáticas. Universidad de Sevilla, 41012 Sevilla (Spain) [grau@us.es](mailto:grau@us.es)

where  $\mu > 0$  is the consistency of the fluid and  $r$  is the flow index. The matrix norm  $|\cdot|$  is defined by  $|\xi|^2 = \text{Tr}(\xi\xi^t)$  with  $\xi \in \mathbb{R}^3$ . We recall that the generalized Newtonian fluids are classified in two main categories (see Saramito [39, Chapter 2] for more details):

- *pseudoplastic* or *shear thinning* fluids, where the viscosity decreases with the shear rate, which corresponds to the case of a flow index  $1 < r < 2$ ;
- *dilatant* or *shear thickening* fluids, where the viscosity increases with the shear rate, which correspond to the case  $r > 2$ .

We also recall that the case  $r = 2$  corresponds to a Newtonian fluid.

The power law model describes correctly the behaviour of polymers at high shear rates, offering the advantage of allowing analytical calculations in simple geometries. However, it has the disadvantage of not describing a Newtonian plateau and even predicts an infinite viscosity as the shear rate goes to zero and  $1 < r < 2$  (see Agassant *et al.* [2, p. 49]), whereas for real fluids it tends to some constant value  $\eta_0$  called the zero-shear-rate viscosity. For these reasons, other viscosity models are used, which better describe the real behaviour of pseudoplastic or dilatant fluids, but are more difficult to analyze mathematically.

**The Carreau law** An important model among these ones is the well-known *Carreau law*, which will be considered in this paper and is defined by

$$\eta_r(\mathbb{D}[u]) = (\eta_0 - \eta_\infty)(1 + \lambda|\mathbb{D}[u]|^2)^{\frac{r}{2}-1} + \eta_\infty, \quad 1 < r < +\infty, \quad \eta_0 > \eta_\infty > 0, \quad \lambda > 0. \quad (1.2)$$

In this relation,  $r$  is the flow index of the fluid,  $\eta_0$  is the low-shear-rate limit of the viscosity, and for  $1 < r < 2$ ,  $\eta_\infty$  is the high-shear-rate limit of the viscosity. Parameter  $\lambda$  is a time constant and  $r - 1$  describes the slope in the power law region. For  $r = 2$ , as in the power-law model, one recovers a Newtonian fluid model, with viscosity  $\eta_0$ .

The interest in the behaviour of generalized Newtonian fluids through thin porous media has increased recently, mainly because of their use in many industrial processes (see Prat and Agaësse [38] for more details). For instance, we refer to recent studies by Anguiano and Suárez-Grau [12, 15, 17], Frabricius *et al.* [30], Bunoiu and Timofte [25, 26], Jouybari and T. S. Lundström [33], and Yeghiazarian *et al.* [45], where the behaviour of Newtonian or power law fluids through a thin porous medium is considered. However, the literature on Carreau fluid flows in this type of domains is far less complete, although these problems have now become of great practical relevance to Chemical Industry and Rheology, for instance in injection moulding of melted polymers, flow of oils, muds, etc. (see for example Pereire and Lecampion [37] and Wrobel *et al.* [43, 44]).

In this paper, we consider a thin porous medium  $\Omega_\varepsilon = \omega_\varepsilon \times (0, \varepsilon) \subset \mathbb{R}^3$  of small height  $\varepsilon$  which is perforated by periodically distributed solid cylinders of diameter of size  $\varepsilon$ . Here, the bottom of the domain without perforations  $\omega \subset \mathbb{R}^2$  is made of two parts, the fluid part  $\omega_\varepsilon$  and the solid part  $\omega \setminus \omega_\varepsilon$ . Assuming that the flow is sufficiently slow to neglect inertial effects, we consider the following stationary Stokes system with a non-linear viscosity following the *Carreau law* (1.2) that we scale by a

factor  $\varepsilon^\gamma$ , where  $\gamma \in \mathbb{R}$ , in the thin porous medium:

$$\begin{cases} -\varepsilon^\gamma \operatorname{div}(\eta_r(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]) + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \quad (1.3)$$

This problem has been very recently considered in Anguiano *et al.* [9] in the particular case of a pseudoplastic fluid ( $1 < r < 2$ ) with  $\gamma = 1$ . Using homogenization techniques, it was proved that when  $\varepsilon$  tends to zero, the mean global filtration velocity is given, as a function of the pressure gradient, by a non-linear 2D Darcy law of Carreau type:

$$\begin{cases} V'(x') = \mathcal{U}(f'(x') - \nabla_{x'} p(x')), \quad V_3(x') = 0 & \text{in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 & \text{in } \omega, \quad V'(x') \cdot n = 0 & \text{on } \partial\omega. \end{cases} \quad (1.4)$$

In the above system,  $n$  is the outward normal to  $\partial\omega$ ,  $V' = (V_1, V_2)$ ,  $x' = (x_1, x_2)$ , and the *permeability* function  $\mathcal{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined through the solutions of 3D local non-Newtonian Stokes problems with non-linear viscosity following the *Carreau law* (1.2) and posed in a reference cell.

In this paper, we perform a new and complete study on the asymptotic behaviour of Carreau fluids, modeled by the equations of motion (1.3) in the thin porous medium  $\Omega_\varepsilon$ , depending on the type of fluid and the value of  $\gamma$ . We generalize the results obtained in [9] by considering system (1.3) not only for pseudoplastic fluids, but also for dilatant fluids and Newtonian fluids, and moreover, for any exponent  $\gamma \in \mathbb{R}$ . Starting from problem (1.3) and using homogenization techniques, we derive different effective problems depending on the type of fluid and the value of  $\gamma$ , describing the asymptotic behaviour of the model as  $\varepsilon$  tends to zero. The approach that we use relies strongly on an adaptation of the periodic unfolding method introduced by Cioranescu *et al.* [27, 29].

In order to give a taste of the kind of arguments that will allow us to distinguish between the different regimes related to  $r$  and  $\gamma$ , let us give the heuristics of the obtention of the effective system in the pseudoplastic case  $1 < r < 2$ . The so-called *unfolded velocity and pressure*  $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$  (defined in Section 3.2) satisfy the following variational formulation: for any admissible test function  $\varphi$ ,

$$\begin{aligned} & (\eta_0 - \eta_\infty) \int_{\omega \times Y} (1 + \lambda \varepsilon^{2(1-\gamma)} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\gamma-2} \hat{u}_\varepsilon] dx' dy \\ & + \eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\gamma-2} \hat{u}_\varepsilon] dx' dy \\ & - \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'}(\varphi' - \varepsilon^{\gamma-2} \hat{u}'_\varepsilon) dx' dy \geq \int_{\omega \times Y} f' \cdot (\varphi' - \varepsilon^{\gamma-2} \hat{u}'_\varepsilon) dx' dy + O_\varepsilon. \end{aligned}$$

Also,  $(\varepsilon^{\gamma-2} \hat{u}_\varepsilon, \hat{P}_\varepsilon)$  converges in appropriate Sobolev spaces to a pair of functions called  $(\hat{u}, \tilde{P})$ . We refer to Sections 3.3 and 3.4 (in particular Equation (3.54)) for more details.

Then, we observe that if  $\gamma < 1$ ,  $2(1-\gamma) > 0$  so  $\lambda \varepsilon^{2(1-\gamma)} |\mathbb{D}_y[\varphi]|^2$  tends to zero, whereas for  $\gamma > 1$ ,  $2(1-\gamma) < 0$  so  $(1 + \lambda \varepsilon^{2(1-\gamma)} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1}$  tends to zero. As a consequence, the sum of the two first terms in the previous formulation converges to the linear term

$$\eta \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] dx' dy,$$

with  $\eta = \eta_0$  if  $\gamma < 1$  and  $\eta = \eta_\infty$  if  $\gamma > 1$ . In the case  $\gamma = 1$ , the critical case that couples the nonlinear term and the linear one, their sum converges to

$$(\eta_0 - \eta_\infty) \int_{\omega \times Y} (1 + \lambda |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] dx' dy + \eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] dx' dy.$$

Thus, we obtain three different asymptotic behaviours depending on whether the value of  $\gamma$  is smaller, equal or greater than 1.

For dilatant fluids, there exist three different convergences of the unfolding velocity depending on the value of  $\gamma$  and, as consequence, three different homogenized models are derived.

**Summary of the different asymptotic regimes** In summary, we have the following asymptotic behaviours of Carreau fluids depending on the type of fluid and the value of  $\gamma$ :

(i) **Pseudoplastic fluids** ( $1 < r < 2$ )

- If  $\gamma \neq 1$ , then the effective problem is a linear 2D Darcy law

$$\begin{cases} V'(x') = \frac{1}{\eta} \mathcal{A} (f'(x') - \nabla_{x'} p(x')), & V_3(x') = 0 \quad \text{in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \quad \text{in } \omega, & V'(x') \cdot n = 0 \quad \text{on } \partial\omega, \end{cases}$$

where the *permeability* tensor  $\mathcal{A} \in \mathbb{R}^{2 \times 2}$  is obtained by solving 3D local Newtonian Stokes problems posed in a reference cell which contains the information on the obstacles' geometry. The viscosity  $\eta$  is equal to  $\eta_0$  if  $\gamma < 1$  and to  $\eta_\infty$  if  $\gamma > 1$ .

- If  $\gamma = 1$ , then the effective problem is the non-linear 2D Darcy law of *Carreau* type (1.4), which is obtained in [9].

(ii) **Dilatant fluids** ( $r > 2$ )

- If  $\gamma < 1$ , then the filtration velocity is zero.
- If  $\gamma > 1$ , then the effective problem is a non-linear 2D Darcy law of *power law* type

$$\begin{cases} V'(x') = \frac{1}{\lambda^{\frac{2-r'}{2}} (\eta_0 - \eta_\infty)^{r'-1}} \mathcal{U} \left( f'(x') - \nabla_{x'} \tilde{P}(x') \right), & V_3(x') = 0 \quad \text{in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \quad \text{in } \omega, & V'(x') \cdot n = 0 \quad \text{on } \partial\omega, \end{cases}$$

where the *permeability* function  $\mathcal{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined through the solutions of 3D local non-Newtonian Stokes problems with non-linear viscosity following the *power law* (1.1) and posed in a reference cell.

- If  $\gamma = 1$ , then the effective problem is a non-linear 2D Darcy law of *Carreau* type as in the case of pseudoplastic fluids.

(iii) **Newtonian fluids** ( $r = 2$ ): for  $\gamma \in \mathbb{R}$ , the effective problem is a linear 2D Darcy law

$$\begin{cases} V'(x') = \frac{1}{\eta_0} \mathcal{A} (f'(x') - \nabla_{x'} p(x')), & V_3(x') = 0 \quad \text{in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \quad \text{in } \omega, & V'(x') \cdot n = 0 \quad \text{on } \partial\omega, \end{cases}$$

where the *permeability* tensor  $\mathcal{A} \in \mathbb{R}^{2 \times 2}$  is obtained by solving 3D local Newtonian Stokes local problems posed in a reference cell which contains the information on obstacles' geometry.

In Table 1, we summarize every asymptotic behaviour of the Carreau fluid governed by (1.3) depending on the type of fluid and the value of  $\gamma$ :

	$1 < r < 2$	$r = 2$	$r > 2$
$\gamma < 1$	Linear 2D Darcy's law (viscosity $\eta_0$ )	Linear 2D Darcy's law (viscosity $\eta_0$ )	Zero filtration velocity
$\gamma = 1$	Non-linear 2D Darcy's law (Carreau type)		Non-linear 2D Darcy's law (Carreau type)
$\gamma > 1$	Linear 2D Darcy's law (viscosity $\eta_\infty$ )		Non-linear 2D Darcy's law (power law type)

Table 1: Asymptotic behaviours of Carreau fluids depending on the values of  $r$  and  $\gamma$ .

We conclude the introduction with a list of references of other recent studies concerning thin porous media. Some stationary models for different fluids are obtained in Anguiano and Suárez-Grau [13, 14, 16] and Zhengan and Hongxing [46], and some non-stationary models are developed in Anguiano [3, 4, 5, 6, 7, 8]. The case of a Bingham flow is considered in Anguiano and Bunoiu [10, 11] and the case of micropolar fluids in Suárez-Grau [41, 42]. The two-phase flow problem in thin porous media domains of Brinkman type has been considered in Armiti-Juber [18] and an approach for effective heat transport in thin porous media has been derived by Scholz and Bringedal [40].

The structure of the paper is as follows. In Section 2 we introduce the domain, make the statement of the problem and give the main results (Theorems 2.1, 2.3 and 2.5). The proofs of the main results are provided in Section 3. Finally, we perform in Section 4 a numerical study of the different effective systems described in Theorems 2.1, 2.3 and 2.5, based on the computation of permeability tensors  $\mathcal{A}$  and permeability functions  $\mathcal{U}$  using a finite element method. We finish the paper with a list of references.

## 2 Setting of the problem and main result

**Geometrical setting.** The periodic porous medium is defined by a domain  $\omega$  and an associated microstructure, or periodic cell  $Y' = (-1/2, 1/2)^2$ , which is made of two complementary parts: the fluid part  $Y'_f$ , and the solid part  $T'$  ( $Y'_f \cup T' = Y'$  and  $Y'_f \cap T' = \emptyset$ ). More precisely, we assume that  $\omega$  is a smooth, bounded, connected set in  $\mathbb{R}^2$  with smooth enough boundary  $\partial\omega$ , that  $n$  is the outward normal to  $\partial\omega$ , and that  $T'$  is an open connected subset of  $Y'$  with a smooth boundary  $\partial T'$ , such that  $\bar{T}'$  is strictly included in  $Y'$ .

The microscale of a porous medium is a small positive number  $\varepsilon$ . The domain  $\omega$  is covered by a regular mesh of square of size  $\varepsilon$ : for  $k' \in \mathbb{Z}^2$ , each cell  $Y'_{k',\varepsilon} = \varepsilon k' + \varepsilon Y'$  is divided in a fluid part  $Y'_{f_{k',\varepsilon}}$

and a solid part  $T'_{k',\varepsilon}$ , i.e. is similar to the unit cell  $Y'$  rescaled to size  $\varepsilon$ . We define  $Y = Y' \times (0, 1) \subset \mathbb{R}^3$ , which is divided in a fluid part  $Y_f = Y'_f \times (0, 1)$  and a solid part  $T = T' \times (0, 1)$ , and consequently  $Y_{k',\varepsilon} = Y'_{k',\varepsilon} \times (0, 1) \subset \mathbb{R}^3$ , which is also divided in a fluid part  $Y_{f,k',\varepsilon}$  and a solid part  $T_{k',\varepsilon}$  (see Figures 1 and 2).

We denote by  $\tau(\bar{T}'_{k',\varepsilon})$  the set of all translated images of  $\bar{T}'_{k',\varepsilon}$ . The set  $\tau(\bar{T}'_{k',\varepsilon})$  represents the obstacles in  $\mathbb{R}^2$ .

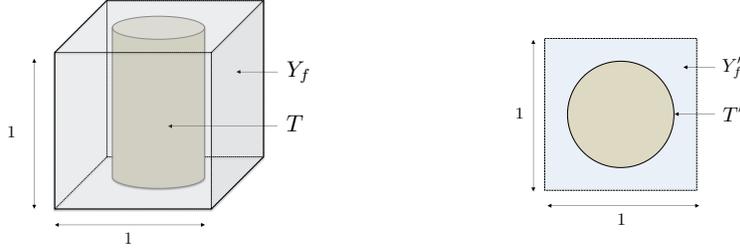


Figure 1: View of the 3D reference cells  $Y$  (left) and the 2D reference cell  $Y'$  (right).

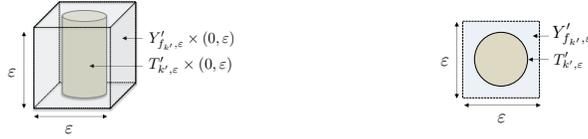


Figure 2: View of the 3D reference cells  $Y_{k',\varepsilon}$  (left) and the 2D reference cell  $Y'_{k',\varepsilon}$  (right).

The fluid part of the bottom  $\omega_\varepsilon \subset \mathbb{R}^2$  of a porous medium is defined by  $\omega_\varepsilon = \omega \setminus \bigcup_{k' \in \mathcal{K}_\varepsilon} \bar{T}'_{k',\varepsilon}$ , where  $\mathcal{K}_\varepsilon = \{k' \in \mathbb{Z}^2 : Y'_{k',\varepsilon} \cap \omega \neq \emptyset\}$ . The whole fluid part  $\Omega_\varepsilon \subset \mathbb{R}^3$  in the thin porous medium is defined by (see Figure 3)

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \in \omega_\varepsilon \times \mathbb{R} : 0 < x_3 < \varepsilon\}. \quad (2.1)$$

We assume that the obstacles  $\tau(\bar{T}'_{k',\varepsilon})$  do not intersect the boundary  $\partial\omega$  and we denote by  $S_\varepsilon$  the set of the solid cylinders contained in  $\Omega_\varepsilon$ , i.e.  $S_\varepsilon = \bigcup_{k' \in \mathcal{K}_\varepsilon} T'_{k',\varepsilon} \times (0, \varepsilon)$ .

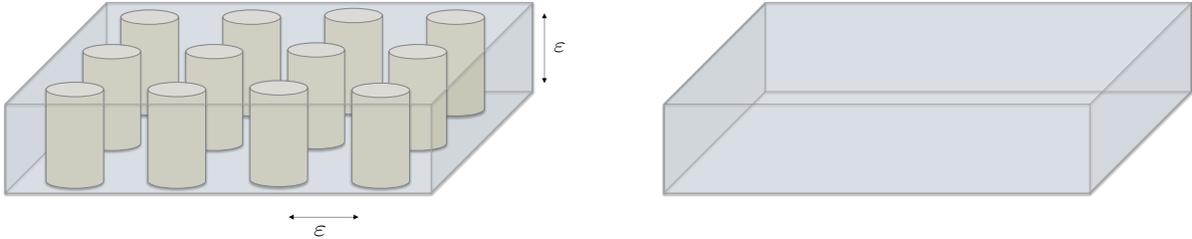


Figure 3: View of the thin porous media  $\Omega_\varepsilon$  (left) and domain without perforations  $Q_\varepsilon$  (right).

We define

$$\tilde{\Omega}_\varepsilon = \omega_\varepsilon \times (0, 1), \quad \Omega = \omega \times (0, 1), \quad Q_\varepsilon = \omega \times (0, \varepsilon). \quad (2.2)$$

We observe that  $\tilde{\Omega}_\varepsilon = \Omega \setminus \bigcup_{k' \in \mathcal{K}_\varepsilon} \bar{T}'_{k',\varepsilon}$ , and we define  $T_\varepsilon = \bigcup_{k' \in \mathcal{K}_\varepsilon} T'_{k',\varepsilon}$  as the set of the solid cylinders contained in  $\tilde{\Omega}_\varepsilon$ .

To finish, we introduce some notation that will be useful throughout the paper. The points  $x \in \mathbb{R}^3$  will be decomposed as  $x = (x', x_3)$  with  $x' = (x_1, x_2) \in \mathbb{R}^2$ ,  $x_3 \in \mathbb{R}$ . We also use the notation  $x'$  to denote a generic vector of  $\mathbb{R}^2$ . Let  $C_{\#}^{\infty}(Y)$  be the space of infinitely differentiable functions in  $\mathbb{R}^3$  that are  $Y'$ -periodic. By  $L_{\#}^q(Y)$  (resp.  $W_{\#}^{1,q}(Y)$ ),  $1 < q < +\infty$ , we denote its completion in the norm  $L^q(Y)$  (resp.  $W^{1,q}(Y)$ ) and by  $L_{0,\#}^q(Y)$  the space of functions in  $L_{\#}^q(Y)$  with zero mean value. Finally, we denote by  $W_{0,\#}^{1,q}(Y_f)$  the subspace of  $W^{1,q}(Y)$  composed of functions vanishing in  $T$ , with zero trace on  $Y' \times \{0, 1\}$ . For  $q = 2$ , we set  $H^1(Y) = W^{1,2}(Y)$  and  $H_{0,\#}^1(Y_f) = W_{0,\#}^{1,2}(Y_f)$ .

**Statement of the problem.** Let us consider the following stationary Stokes system with the non-linear viscosity following the *Carreau law* (1.2) in  $\Omega_{\varepsilon}$ , with a Dirichlet boundary condition on the exterior boundary  $\partial Q_{\varepsilon}$  and the cylinders  $\partial S_{\varepsilon}$ ,

$$\begin{cases} -\varepsilon^{\gamma} \operatorname{div}(\eta_r(\mathbb{D}[u_{\varepsilon}])\mathbb{D}[u_{\varepsilon}]) + \nabla p_{\varepsilon} = f & \text{in } \Omega_{\varepsilon}, \\ \operatorname{div} u_{\varepsilon} = 0 & \text{in } \Omega_{\varepsilon}, \\ u_{\varepsilon} = 0 & \text{on } \partial Q_{\varepsilon} \cup \partial S_{\varepsilon}, \end{cases} \quad (2.3)$$

where the second member  $f$  is of the form

$$f(x) = (f'(x'), 0) \quad \text{with } f' \in L^{\infty}(\omega)^2. \quad (2.4)$$

We remark that the assumptions of neglecting the vertical component of the exterior force and the independence of the vertical variable are usual when dealing with fluids in through thin domains (see [22] for more details).

Under previous assumptions, the classical theory (see for instance [22, 23, 34]), gives the existence of a unique weak solution  $(u_{\varepsilon}, p_{\varepsilon}) \in H_0^1(\Omega_{\varepsilon})^3 \times L_0^2(\Omega_{\varepsilon})$ , for  $1 < r \leq 2$ , and  $(u_{\varepsilon}, p_{\varepsilon}) \in W_0^{1,r}(\Omega_{\varepsilon})^3 \times L_0^{r'}(\Omega_{\varepsilon})$  with  $1/r + 1/r' = 1$ , for  $r > 2$ , where  $L_0^2$  (respectively  $L_0^{r'}$ ) is the space of functions of  $L^2$  (respectively  $L^{r'}$ ) with zero mean value.

Our goal is to study the asymptotic behaviour of  $u_{\varepsilon}$  and  $p_{\varepsilon}$  when  $\varepsilon$  tends to zero. For this purpose, we use the dilatation in the variable  $x_3$  as follows

$$y_3 = \frac{x_3}{\varepsilon}, \quad (2.5)$$

in order to have the functions defined in the open set with fixed height  $\tilde{\Omega}_{\varepsilon}$ . Namely, we define  $\tilde{u}_{\varepsilon}$  and  $\tilde{p}_{\varepsilon}$  by

$$\tilde{u}_{\varepsilon}(x', y_3) = u_{\varepsilon}(x', \varepsilon y_3), \quad \tilde{p}_{\varepsilon}(x', y_3) = p_{\varepsilon}(x', \varepsilon y_3), \quad \text{a.e. } (x', y_3) \in \tilde{\Omega}_{\varepsilon}.$$

Let us introduce some notation which will be useful in the following. For a vectorial function  $v = (v', v_3)$  and a scalar function  $w$ , we will denote  $\mathbb{D}_{x'}[v] = \frac{1}{2}(D_{x'}v + D_{x'}^t v)$  and  $\partial_{y_3}[v] = \frac{1}{2}(\partial_{y_3}v + \partial_{y_3}^t v)$ , where we denote  $\partial_{y_3} = (0, 0, \frac{\partial}{\partial y_3})^t$ . Moreover, associated to the change of variables (2.5), we introduce the operators:  $\mathbb{D}_{\varepsilon}$ ,  $D_{\varepsilon}$ ,  $\operatorname{div}_{\varepsilon}$  and  $\nabla_{\varepsilon}$ , by

$$\mathbb{D}_{\varepsilon}[v] = \frac{1}{2}(D_{\varepsilon}v + D_{\varepsilon}^t v),$$

$$(D_{\varepsilon}v)_{i,j} = \partial_{x_j} v_i \text{ for } i = 1, 2, 3, j = 1, 2, \quad (D_{\varepsilon}v)_{i,3} = \varepsilon^{-1} \partial_{y_3} v_i \text{ for } i = 1, 2, 3,$$

$$\operatorname{div}_\varepsilon v = \operatorname{div}_{x'} v' + \varepsilon^{-1} \partial_{y_3} v_3, \quad \nabla_\varepsilon w = (\nabla_{x'} w, \varepsilon^{-1} \partial_{y_3} w)^t.$$

Using the transformation (2.5), system (2.3) can be rewritten as

$$\left\{ \begin{array}{l} -\varepsilon^\gamma \operatorname{div}_\varepsilon (\eta_r(\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon])\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]) + \nabla_\varepsilon \tilde{p}_\varepsilon = f \text{ in } \tilde{\Omega}_\varepsilon, \\ \operatorname{div}_\varepsilon \tilde{u}_\varepsilon = 0 \text{ in } \tilde{\Omega}_\varepsilon, \\ \tilde{u}_\varepsilon = 0 \text{ on } \partial\Omega \cup \partial T_\varepsilon. \end{array} \right. \quad (2.6)$$

Our goal then is to describe the asymptotic behaviour of this new sequence  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ . The sequences of solutions  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$  is not defined in a fixed domain independent of  $\varepsilon$  but rather in a varying set  $\tilde{\Omega}_\varepsilon$ . In order to pass the limit if  $\varepsilon$  tends to zero, convergences in fixed Sobolev spaces (defined in  $\Omega$ ) are used which requires first that  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$  be extended to the whole domain  $\Omega$ . Then, an extension  $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in W_0^{1,q}(\Omega)^3 \times L_0^q(\Omega)$  is defined on  $\Omega$  and coincides with  $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$  on  $\tilde{\Omega}_\varepsilon$  (we will use the same notation,  $\tilde{u}_\varepsilon$ , for the velocity in  $\tilde{\Omega}_\varepsilon$  and its continuation in  $\Omega$ ).

Our main results referred to the asymptotic behaviour of the solution of (2.6) is given by the following theorems.

**Theorem 2.1** (Pseudoplastic fluids). *Consider  $1 < r < 2$  and  $\gamma \in \mathbb{R}$ . Then, there exist  $\tilde{u} \in H_0^1(0, 1; L^2(\omega)^3)$  with  $\tilde{u}_3 = 0$  and  $\tilde{P} \in L_0^2(\omega)$ , such that the extension  $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$  of the solution of (2.6) satisfies the following convergences:*

$$\varepsilon^{\gamma-2} \tilde{u}_\varepsilon \rightharpoonup \tilde{u} \text{ weakly in } H^1(0, 1; L^2(\omega)^3), \quad \tilde{P}_\varepsilon \rightarrow \tilde{P} \text{ strongly in } L^2(\Omega).$$

Moreover, defining  $V(x') = \int_0^1 \tilde{u}(x', y_3) dy_3$ , the pair  $(V, \tilde{P}) \in L^2(\omega)^3 \times (L_0^2(\omega) \cap H^1(\omega))$  is the unique solution of a lower-dimensional effective Darcy's law depending on the value of  $\gamma$ . More precisely:

- If  $\gamma \neq 1$ , then  $(V, \tilde{P})$  is the unique solution of the linear 2D Darcy's law

$$\left\{ \begin{array}{l} V'(x') = \frac{1}{\eta} \mathcal{A} \left( f'(x') - \nabla_{x'} \tilde{P}(x') \right), \quad V_3(x') = 0 \text{ in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \text{ in } \omega, \quad V'(x') \cdot n = 0 \text{ on } \partial\omega, \end{array} \right. \quad (2.7)$$

where

$$\eta = \begin{cases} \eta_0 & \text{if } \gamma < 1, \\ \eta_\infty & \text{if } \gamma > 1. \end{cases}$$

In system (2.7), the permeability tensor  $\mathcal{A} \in \mathbb{R}^{2 \times 2}$  is defined by its entries

$$\mathcal{A}_{ij} = \int_{Y_f} w_j^i(y) dy, \quad i, j = 1, 2, \quad (2.8)$$

where for  $i = 1, 2$ , the pair  $(w^i, \pi^i) \in H_{0,\#}^1(Y_f)^3 \times L_{0,\#}^2(Y_f)$  is the unique solution of the local Stokes system

$$\left\{ \begin{array}{l} -\Delta_y w^i + \nabla_y \pi^i = e_i \text{ in } Y_f, \\ \operatorname{div}_y w^i = 0 \text{ in } Y_f, \\ w^i = 0 \text{ on } \partial T \cup (Y_f' \times \{0, 1\}), \\ y \rightarrow w^i, \pi^i \text{ } Y \text{ - periodic,} \end{array} \right. \quad (2.9)$$

and  $\{e_i\}_{i=1,2,3}$  being the canonical basis of  $\mathbb{R}^3$ .

– If  $\gamma = 1$ , then  $(V, \tilde{P})$  is the unique solution of the non-linear 2D Darcy's law of Carreau type

$$\begin{cases} V'(x') = \mathcal{U} \left( f'(x') - \nabla_{x'} \tilde{P}(x') \right), & V_3(x') = 0 \quad \text{in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \quad \text{in } \omega, & V'(x') \cdot n = 0 \quad \text{on } \partial\omega. \end{cases} \quad (2.10)$$

The permeability function  $\mathcal{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  appearing in system (2.10) is defined by

$$\mathcal{U}(\xi') = \int_{Y_f} w_{\xi'}(y) dy, \quad \forall \xi' \in \mathbb{R}^2, \quad (2.11)$$

where, for any  $\xi' \in \mathbb{R}^2$ , the pair  $(w_{\xi'}, \pi_{\xi'}) \in H_{0,\#}^1(Y_f)^3 \times L_{0,\#}^2(Y_f)$  is the unique solution of the local Stokes system

$$\begin{cases} -\operatorname{div}_y(\eta_r(\mathbb{D}_y[w_{\xi'}])\mathbb{D}_y[w_{\xi'}]) + \nabla_y \pi_{\xi'} = \xi' & \text{in } Y_f, \\ \operatorname{div}_y w_{\xi'} = 0 & \text{in } Y_f, \\ w_{\xi'} = 0 & \text{on } \partial T \cup (Y_f' \times \{0, 1\}), \end{cases} \quad (2.12)$$

and the nonlinear viscosity  $\eta_r$  is given by the Carreau law (1.2).

**Remark 2.2.** According to [1, Theorem 1.1], the permeability tensor  $\mathcal{A}$  is symmetric and definite positive.

**Theorem 2.3** (Dilatant fluids). Consider  $r > 2$  and  $\gamma \in \mathbb{R}$ . We divide the theorem depending on the value of  $\gamma$ :

(i) If  $\gamma < 1$ , then there exist  $\tilde{u} \in W_0^{1,r}(0, 1; L^r(\omega)^3)$  and  $\tilde{P} \in L_0^{r'}(\omega)$ , such that the extension  $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$  of the solution of (2.6) satisfies the following convergences:

$$\varepsilon^{-1+\frac{2}{r}(\gamma-1)} \tilde{u}_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } W^{1,r}(0, 1; L^r(\omega)^3), \quad \tilde{P}_\varepsilon \rightarrow \tilde{P} \quad \text{strongly in } L^{r'}(\Omega).$$

Moreover, defining  $V(x') = \int_0^1 \tilde{u}(x', y_3) dy_3$ , there holds

$$V(x') = 0 \quad \text{in } \omega. \quad (2.13)$$

(ii) If  $\gamma > 1$ , then there exist  $\tilde{u} \in W_0^{1,r}(0, 1; L^r(\omega)^3)$  with  $\tilde{u}_3 = 0$  and  $\tilde{P} \in L_0^{r'}(\omega)$ , such that the extension  $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$  of the solution of (2.6) satisfies the following convergences:

$$\varepsilon^{\frac{\gamma-r}{r-1}} \tilde{u}_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } W^{1,r}(0, 1; L^r(\omega)^3), \quad \tilde{P}_\varepsilon \rightarrow \tilde{P} \quad \text{strongly in } L^{r'}(\Omega).$$

Moreover, defining  $V(x') = \int_0^1 \tilde{u}(x', y_3) dy_3$ , the pair  $(V, \tilde{P}) \in L^r(\omega)^3 \times (L_0^{r'}(\omega) \cap W^{1,r'}(\omega))$  is the unique solution of the lower-dimensional effective non-linear Darcy's law

$$\begin{cases} V'(x') = \frac{1}{\lambda^{\frac{2-r'}{2}} (\eta_0 - \eta_\infty)^{r'-1}} \mathcal{U} \left( f'(x') - \nabla_{x'} \tilde{P}(x') \right), & V_3(x') = 0 \quad \text{in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \quad \text{in } \omega, & V'(x') \cdot n = 0 \quad \text{on } \partial\omega. \end{cases} \quad (2.14)$$

The permeability function  $\mathcal{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  appearing in system (2.14) is defined by (2.11) where, for any  $\xi' \in \mathbb{R}^2$ ,  $w_{\xi'}$  is the unique solution of the local Stokes system (2.12) with nonlinear viscosity of type power law given by  $\eta_r(\mathbb{D}_y[w_{\xi'}]) = |\mathbb{D}_y[w_{\xi'}]|^{r-2}$ .

(iii) If  $\gamma = 1$ , then there exist  $\tilde{u} \in W_0^{1,r}(0,1;L^r(\omega)^3)$  with  $\tilde{u}_3 = 0$  and  $\tilde{P} \in L_0^r(\omega)$ , such that the extension  $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$  of the solution of (2.6) satisfies the following convergences:

$$\varepsilon^{-1}\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } W^{1,r}(0,1;L^r(\omega)^3), \quad \tilde{P}_\varepsilon \rightarrow \tilde{P} \quad \text{strongly in } L^r(\Omega).$$

Moreover, defining  $V(x') = \int_0^1 \tilde{u}(x', y_3) dy_3$ , the pair  $(V, \tilde{P}) \in L^r(\omega)^3 \times (L_0^r(\omega) \cap W^{1,r'}(\omega))$  is the unique solution of the lower-dimensional effective non-linear 2D Darcy's law of Carreau type (2.10), where for every  $\xi' \in \mathbb{R}^2$ ,  $\mathcal{U}(\xi')$  is defined by (2.11) where  $(w_{\xi'}, \pi_{\xi'}) \in W_{0,\#}^{1,r}(Y_f)^3 \times L_{0,\#}^r(Y_f)$  is the unique solution of the local Stokes system (2.12) with nonlinear viscosity given by the Carreau law (1.2).

**Remark 2.4.** According to [24, Lemma 2], the permeability function  $\mathcal{U}$  is coercive and strictly monotone.

**Theorem 2.5** (Newtonian fluids). Consider  $r = 2$  and  $\gamma \in \mathbb{R}$ . Then, there exist  $\tilde{u} \in H_0^1(0,1;L^2(\omega)^3)$  with  $\tilde{u}_3 = 0$  and  $\tilde{P} \in L_0^2(\omega)$ , such that the extension  $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$  of the solution of (2.6) satisfies the following convergences:

$$\varepsilon^{\gamma-2}\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } H^1(0,1;L^2(\omega)^3), \quad \tilde{P}_\varepsilon \rightarrow \tilde{P} \quad \text{strongly in } L^2(\Omega).$$

Moreover, defining  $V(x') = \int_0^1 \tilde{u}(x', y_3) dy_3$ , the pair  $(V, \tilde{P}) \in L^2(\omega)^3 \times (L_0^2(\omega) \cap H^1(\omega))$  is the unique solution of the linear 2D Darcy's law

$$\begin{cases} V'(x') = \frac{1}{\eta_0} \mathcal{A} \left( f'(x') - \nabla_{x'} \tilde{P}(x') \right), & V_3(x') = 0 \quad \text{in } \omega, \\ \operatorname{div}_{x'} V'(x') = 0 \quad \text{in } \omega, & V'(x') \cdot n = 0 \quad \text{on } \partial\omega. \end{cases} \quad (2.15)$$

The permeability tensor  $\mathcal{A} \in \mathbb{R}^{2 \times 2}$  appearing in system (2.15) is defined by its entries

$$\mathcal{A}_{ij} = \int_{Y_f} w_j^i(y) dy, \quad i, j = 1, 2, \quad (2.16)$$

where, setting  $\{e_k\}_{k=1,2,3}$  the canonical basis of  $\mathbb{R}^3$ , for  $i = 1, 2$ , the pair  $(w^i, \pi^i) \in H_{0,\#}^1(Y_f)^3 \times L_{0,\#}^2(Y_f)$ , is the unique solution of the local Stokes system

$$\begin{cases} -\Delta_y w^i + \nabla_y \pi^i = e_i & \text{in } Y_f, \\ \operatorname{div}_y w^i = 0 & \text{in } Y_f, \\ w^i = 0 & \text{on } \partial T \cup (Y_f' \times \{0, 1\}), \\ y \rightarrow w^i, \pi^i & Y - \text{periodic}. \end{cases} \quad (2.17)$$

**Remark 2.6.** According to [1, Theorem 1.1], the permeability tensor  $\mathcal{A}$  is symmetric and definite positive.

### 3 Proof of the main results

In this section we provide the proof of the main results (Theorems 2.1, 2.3 and 2.5). To this aim, we first establish some *a priori* estimates of the solution of (2.6) and we define its extension. Second, we introduce the version of the unfolding method depending on  $\varepsilon$ . Next, we establish a compactness result, which is the main key for passing to the limit in the system, and conclude the proofs of the Theorems.

### 3.1 *A priori estimates.*

In this subsection, we establish sharp *a priori* estimates of the dilated solution in  $\tilde{\Omega}_\varepsilon$ . To do this, we first need the Poincaré and Korn inequalities in  $\tilde{\Omega}_\varepsilon$ , which can be found in [12].

**Lemma 3.1** (Remark 4.3-(i) in [12]). *We have the following two estimates in thin domains:*

(i) *For every  $\tilde{v} \in W_0^{1,q}(\tilde{\Omega}_\varepsilon)^3$ ,  $1 \leq q < +\infty$ , there exists a positive constant  $C$ , independent of  $\varepsilon$ , such that*

$$\|\tilde{v}\|_{L^q(\tilde{\Omega}_\varepsilon)^3} \leq C\varepsilon \|D_\varepsilon \tilde{v}\|_{L^q(\tilde{\Omega}_\varepsilon)^{3 \times 3}}, \quad (\text{Poincaré's inequality}). \quad (3.1)$$

(ii) *For every  $\tilde{v} \in W_0^{1,q}(\tilde{\Omega}_\varepsilon)^3$ ,  $1 < q < +\infty$ , there exists a positive constant  $C$ , independent of  $\varepsilon$ , such that*

$$\|D_\varepsilon \tilde{v}\|_{L^q(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C \|\mathbb{D}_\varepsilon[\tilde{v}]\|_{L^q(\tilde{\Omega}_\varepsilon)^{3 \times 3}}, \quad (\text{Korn's inequality}). \quad (3.2)$$

Using the previous estimates, we give *a priori* estimates for velocity  $\tilde{u}_\varepsilon$  in  $\tilde{\Omega}_\varepsilon$ .

**Lemma 3.2.** *Let  $\tilde{u}_\varepsilon$  the velocity solution of (2.6). Then, we have the following estimates:*

(i) *(Pseudoplastic fluid and Newtonian fluid) Consider  $1 < r \leq 2$ . There exists a positive constant  $C$ , independent of  $\varepsilon$ , such that, for every value of  $\gamma$ , we have*

$$\|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{2-\gamma}, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C\varepsilon^{1-\gamma}, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C\varepsilon^{1-\gamma}. \quad (3.3)$$

(ii) *(Dilatant fluid) Consider  $r > 2$ . There exists a positive constant  $C$ , independent of  $\varepsilon$ , such that estimates (3.3) hold and also, depending on the value of  $\gamma$ , we have*

– *If  $\gamma < 1$ , it holds*

$$\|\tilde{u}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)+1}, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)}, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)}, \quad (3.4)$$

– *If  $\gamma > 1$ , it holds*

$$\|\tilde{u}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^3} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}+1}, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}}, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}}. \quad (3.5)$$

– *If  $\gamma = 1$ , it holds*

$$\|\tilde{u}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^3} \leq C\varepsilon, \quad \|D_\varepsilon \tilde{u}_\varepsilon\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C, \quad \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C. \quad (3.6)$$

*Proof.* Multiplying (2.6) by  $\tilde{u}_\varepsilon$  and integrating over  $\tilde{\Omega}_\varepsilon$ , we get

$$\varepsilon^\gamma(\eta_0 - \eta_\infty) \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{\gamma}{2}-1} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx' dy_3 + \varepsilon^\gamma \eta_\infty \int_{\tilde{\Omega}_\varepsilon} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx' dy_3 = \int_{\tilde{\Omega}_\varepsilon} f' \cdot \tilde{u}'_\varepsilon dx' dy_3. \quad (3.7)$$

We divide the proof in two steps. First, we derive estimates (3.3) for every  $r > 1$  and then, for  $r > 2$  we establish estimates (3.4)-(3.6) depending on the value of  $\gamma$ .

*Step 1.* We consider  $r > 1$ . Taking into account that  $\eta_0 > \eta_\infty$ , and  $\lambda > 0$ , we have

$$\varepsilon^\gamma (\eta_0 - \eta_\infty) \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx' dy_3 \geq 0,$$

and then, from Cauchy-Schwarz's inequality and the assumption on  $f'$  given in (2.4), we get from (3.7) that

$$\varepsilon^\gamma \eta_\infty \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}}^2 \leq C \|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3}.$$

Applying Poincaré's inequality (3.1) and Korn's inequality (3.2) in the right-hand side, we have

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}}^2 \leq C \varepsilon^{1-\gamma} \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}},$$

which gives (3.3)<sub>3</sub>. Finally, applying again (3.1) and (3.2), we obtain (3.3)<sub>1</sub> and (3.3)<sub>2</sub>.

*Step 2.* For  $r > 2$ , since

$$\varepsilon^\gamma \eta_\infty \int_{\tilde{\Omega}_\varepsilon} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx' dy_3 \geq 0,$$

from (3.7) and Cauchy-Schwarz's inequality, we get

$$\varepsilon^\gamma (\eta_0 - \eta_\infty) \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx' dy_3 \leq C \|\tilde{u}_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)^3}. \quad (3.8)$$

Since it holds

$$\varepsilon^\gamma \lambda^{\frac{r-2}{2}} (\eta_0 - \eta_\infty) \int_{\tilde{\Omega}_\varepsilon} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^r dx' dy_3 \leq \varepsilon^\gamma (\eta_0 - \eta_\infty) \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2 dx' dy_3,$$

then, applying this and also Poincaré's inequality (3.1) and Korn's inequality (3.2) in the right-hand side in (3.8), we get

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}}^r \leq C \varepsilon^{1-\gamma} \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}}. \quad (3.9)$$

On the one hand, applying estimate (3.3)<sub>3</sub>, we get

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C \varepsilon^{-\frac{2}{r}(\gamma-1)}.$$

On the other hand, from the continuity of embedding  $L^r(\tilde{\Omega}_\varepsilon) \hookrightarrow L^2(\tilde{\Omega}_\varepsilon)$  in (3.9), we also have

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}}^r \leq C \varepsilon^{1-\gamma} \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}},$$

which gives

$$\|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C \varepsilon^{-\frac{\gamma-1}{r-1}}.$$

As a result, we have derived two different estimates of  $\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]$  in  $L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}$ , that we may now compare in order to obtain the more accurate one, depending on the value of  $\gamma$ . Since  $-\frac{2}{r}(\gamma-1) > -\frac{\gamma-1}{r-1}$  if  $\gamma < 1$  and  $-\frac{2}{r}(\gamma-1) < -\frac{\gamma-1}{r-1}$  if  $\gamma > 1$ , we deduce estimates (3.4)<sub>3</sub> and (3.5)<sub>3</sub>. In the case  $\gamma = 1$ , both estimates give (3.6)<sub>3</sub>. Finally, from Poincaré's inequality (3.1) and Korn's inequality (3.2) we derive the remaining estimates (3.4), (3.5) and (3.6).  $\square$

**Remark 3.3.** We extend the velocity  $\tilde{u}_\varepsilon$  by zero in  $\Omega \setminus \tilde{\Omega}_\varepsilon$  (this is compatible with the homogeneous boundary condition on  $\partial\Omega \cup \partial T_\varepsilon$ ), and denote the extension by same symbol. Obviously, estimates given in Lemma 3.2 remain valid and the extension  $\tilde{u}_\varepsilon$  is divergence free too.

In order to extend the pressure  $\tilde{p}_\varepsilon$  to the whole domain  $\Omega$  and obtain *a priori* estimates, we recall a result in which is concerned with the extension of the pressure  $p_\varepsilon$  to the whole domain  $Q_\varepsilon$ , which define a restriction operator  $R_q^\varepsilon$  from  $W_0^{1,q}(Q_\varepsilon)^3$  into  $W_0^{1,q}(\Omega_\varepsilon)^3$ ,  $1 < q < +\infty$ , introduced in [12].

**Lemma 3.4** (Lemma 4.5-(i) in [12]). *There exists a (restriction) operator  $R_q^\varepsilon$  acting from  $W_0^{1,q}(Q_\varepsilon)^3$  into  $W_0^{1,q}(\Omega_\varepsilon)^3$ ,  $1 < q < +\infty$ , such that*

1.  $R_q^\varepsilon v = v$ , if  $v \in W_0^{1,q}(\Omega_\varepsilon)^3$  (elements of  $W_0^{1,q}(\Omega_\varepsilon)$  are extended by 0 to  $Q_\varepsilon$ ).
2.  $\operatorname{div} R_q^\varepsilon v = 0$  in  $\Omega_\varepsilon$ , if  $\operatorname{div} v = 0$  on  $Q_\varepsilon$ .
3. For every  $v \in W_0^{1,q}(Q_\varepsilon)^3$ , there exists a positive constant  $C$ , independent of  $v$  and  $\varepsilon$ , such that

$$\|R_q^\varepsilon v\|_{L^q(\Omega_\varepsilon)^3} + \varepsilon \|DR_q^\varepsilon v\|_{L^q(\Omega_\varepsilon)^{3 \times 3}} \leq C (\|v\|_{L^q(Q_\varepsilon)^3} + \varepsilon \|Dv\|_{L^q(Q_\varepsilon)^{3 \times 3}}). \quad (3.10)$$

In the next result, by using the restriction operator defined in Lemma 3.4, we extend the gradient of the pressure by duality in  $W^{-1,q'}(Q_\varepsilon)^3$  and then, by means of the dilatation, we extend  $\tilde{p}_\varepsilon$  to  $\Omega$ . Finally, we derive estimates of the extension of the pressure.

**Lemma 3.5.** *Let  $\tilde{p}_\varepsilon$  the pressure solution of (2.6). Then:*

- (i) (Pseudoplastic fluid and Newtonian fluid) Consider  $1 < r \leq 2$ . There exist an extension  $\tilde{P}_\varepsilon \in L_0^2(\Omega)$  of pressure  $\tilde{p}_\varepsilon$  and a positive constant  $C$  independent of  $\varepsilon$ , such that

$$\|\tilde{P}_\varepsilon\|_{L^2(\Omega)} \leq C, \quad \|\nabla_\varepsilon \tilde{P}_\varepsilon\|_{H^{-1}(\Omega)^3} \leq C. \quad (3.11)$$

- (ii) (Dilatant fluid) Consider  $r > 2$ . There exist an extension  $\tilde{P}_\varepsilon \in L_0^{r'}(\Omega)$  of pressure  $\tilde{p}_\varepsilon$  and a positive constant  $C$  independent of  $\varepsilon$ , such that

$$\|\tilde{P}_\varepsilon\|_{L^{r'}(\Omega)} \leq C, \quad \|\nabla_\varepsilon \tilde{P}_\varepsilon\|_{W^{-1,r'}(\Omega)^3} \leq C, \quad (3.12)$$

where  $r'$  is the conjugate exponent of  $r$ .

*Proof.* We divide the proof in three steps. First, we extend the pressure in all cases (pseudoplastic, Newtonian and dilatant). Then, we obtain the estimates for pseudoplastic and Newtonian fluids, before deriving the estimate for dilatant fluids.

*Step 1. Extension of the pressure.* Using the restriction operator  $R_q^\varepsilon$  given in Lemma 3.4, with  $q = \max\{2, r\}$ , we define  $F_\varepsilon$  in  $W^{-1,q'}(Q_\varepsilon)^3$ , where  $q'$  is the conjugate exponent of  $q$ , by the formula:

$$\langle F_\varepsilon, v \rangle_{W^{-1,q'}(Q_\varepsilon)^3, W_0^{1,q}(Q_\varepsilon)^3} = \langle \nabla p_\varepsilon, R_q^\varepsilon v \rangle_{W^{-1,q'}(\Omega_\varepsilon)^3, W_0^{1,q}(\Omega_\varepsilon)^3}, \quad \text{for any } v \in W_0^{1,q}(Q_\varepsilon)^3. \quad (3.13)$$

Using the variational formulation of problem (2.3), the right hand side of (3.13) can be rephrased as follows:

$$\begin{aligned} \langle F_\varepsilon, v \rangle_{W^{-1,q'}(Q_\varepsilon)^3, W_0^{1,q}(Q_\varepsilon)^3} &= -\varepsilon^\gamma (\eta_0 - \eta_\infty) \int_{\Omega_\varepsilon} (1 + \lambda |\mathbb{D}[u_\varepsilon]|^2)^{\frac{r}{2}-1} \mathbb{D}[u_\varepsilon] : DR_q^\varepsilon v \, dx \\ &\quad - \varepsilon^\gamma \eta_\infty \int_{\Omega_\varepsilon} \mathbb{D}[u_\varepsilon] : DR_q^\varepsilon v \, dx + \int_{\Omega_\varepsilon} f' \cdot (R_q^\varepsilon v)' \, dx. \end{aligned} \quad (3.14)$$

Using Lemma 3.2 for fixed  $\varepsilon$ , we see that it is a bounded functional on  $W_0^{1,q}(Q_\varepsilon)$  (see *Steps 2 and 3* below), and in fact  $F_\varepsilon \in W^{-1,q'}(Q_\varepsilon)^3$ . Moreover,  $\operatorname{div} v = 0$  implies  $\langle F_\varepsilon, v \rangle = 0$ , and De Rham theorem gives the existence of  $P_\varepsilon$  in  $L_0^{q'}(Q_\varepsilon)$  such that  $F_\varepsilon = \nabla P_\varepsilon$ .

Next, we get for every  $\tilde{v} \in W_0^{1,q}(\Omega)^3$  where  $\tilde{v}(x', y_3) = v(x', \varepsilon y_3)$ , using the change of variables (2.5), that

$$\langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v} \rangle_{W^{-1,q'}(\Omega)^3, W_0^{1,q}(\Omega)^3} = - \int_{\Omega} \tilde{P}_\varepsilon \operatorname{div}_\varepsilon \tilde{v} \, dx' dy_3 = -\varepsilon^{-1} \int_{Q_\varepsilon} P_\varepsilon \operatorname{div} v \, dx = \varepsilon^{-1} \langle \nabla P_\varepsilon, v \rangle_{W^{-1,q'}(Q_\varepsilon)^3, W_0^{1,q}(Q_\varepsilon)^3}.$$

Using the identification (3.14) of  $F_\varepsilon$ , we have

$$\begin{aligned} \langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v} \rangle_{W^{-1,q'}(\Omega)^3, W_0^{1,q}(\Omega)^3} &= \varepsilon^{-1} \left( -\varepsilon^\gamma (\eta_0 - \eta_\infty) \int_{\Omega_\varepsilon} (1 + \lambda |\mathbb{D}[u_\varepsilon]|^2)^{\frac{r}{2}-1} \mathbb{D}[u_\varepsilon] : DR_q^\varepsilon v \, dx \right. \\ &\quad \left. - \varepsilon^\gamma \eta_\infty \int_{\Omega_\varepsilon} \mathbb{D}[u_\varepsilon] : DR_q^\varepsilon v \, dx + \int_{\Omega_\varepsilon} f' \cdot (R_q^\varepsilon v)' \, dx \right), \end{aligned}$$

and applying the change of variables (2.5), we obtain

$$\begin{aligned} \langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v} \rangle_{W^{-1,q'}(\Omega)^3, W_0^{1,q}(\Omega)^3} &= -\varepsilon^\gamma (\eta_0 - \eta_\infty) \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_q^\varepsilon \tilde{v} \, dx' dy_3 \\ &\quad - \varepsilon^\gamma \eta_\infty \int_{\tilde{\Omega}_\varepsilon} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_q^\varepsilon \tilde{v} \, dx' dy_3 + \int_{\tilde{\Omega}_\varepsilon} f'(x') \cdot (\tilde{R}_q^\varepsilon \tilde{v})' \, dx' dy_3, \end{aligned} \quad (3.15)$$

where  $\tilde{R}_q^\varepsilon \tilde{v} = R_q^\varepsilon v$  for any  $\tilde{v} \in W_0^{1,q}(\Omega)^3$ .

*Step 2. Estimates of the extended pressure for pseudoplastic fluids and Newtonian fluids.* Applying the dilatation in (3.10) for  $q = 2$ , we have that  $\tilde{R}_2^\varepsilon \tilde{v}$  satisfies the following estimate

$$\|\tilde{R}_2^\varepsilon \tilde{v}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} + \varepsilon \|D_\varepsilon \tilde{R}_2^\varepsilon \tilde{v}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C (\|\tilde{v}\|_{L^2(\Omega)^3} + \varepsilon \|D_\varepsilon \tilde{v}\|_{L^2(\Omega)^{3 \times 3}}), \quad (3.16)$$

and since  $\varepsilon \ll 1$ , we have

$$\|\tilde{R}_2^\varepsilon \tilde{v}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C \|\tilde{v}\|_{H_0^1(\Omega)^3}, \quad \|D_\varepsilon \tilde{R}_2^\varepsilon \tilde{v}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq \frac{C}{\varepsilon} \|\tilde{v}\|_{H_0^1(\Omega)^3}. \quad (3.17)$$

Taking into account that  $1 < r \leq 2$ , we have that

$$(1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} \leq 1,$$

and then, from Cauchy-Schwarz's inequality, we deduce

$$\begin{aligned} & \left| \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_2^\varepsilon \tilde{v} \, dx' dy_3 \right| \leq \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]| : |D_\varepsilon \tilde{R}_2^\varepsilon \tilde{v}| \, dx' dy_3 \\ & \leq \int_{\tilde{\Omega}_\varepsilon} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]| : |D_\varepsilon \tilde{R}_2^\varepsilon \tilde{v}| \, dx' dy_3 \leq \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \|D_\varepsilon \tilde{R}_2^\varepsilon \tilde{v}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}}. \end{aligned}$$

Using the last estimate in (3.3) and the last estimate of the dilated restricted operator given in (3.17), we obtain

$$\left| \varepsilon^\gamma (\eta_0 - \eta_\infty) \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_2^\varepsilon \tilde{v} \, dx' dy_3 \right| \leq C \|\tilde{v}\|_{H_0^1(\Omega)^3}. \quad (3.18)$$

Moreover, from Cauchy-Schwarz's inequality, the last estimate in (3.3), the assumption of  $f'$  given in (2.4) and estimates of the dilated restricted operator given in (3.17), we obtain

$$\left| \varepsilon^\gamma \eta_\infty \int_{\tilde{\Omega}_\varepsilon} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_2^\varepsilon \tilde{v} \, dx' dy_3 \right| \leq C \varepsilon^\gamma \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \|D_\varepsilon \tilde{R}_2^\varepsilon \tilde{v}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C \|\tilde{v}\|_{H_0^1(\Omega)^3}, \quad (3.19)$$

$$\left| \int_{\tilde{\Omega}_\varepsilon} f' \cdot (\tilde{R}_2^\varepsilon \tilde{v})' \, dx' dy_3 \right| \leq C \|\tilde{R}_2^\varepsilon \tilde{v}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C \|\tilde{v}\|_{H_0^1(\Omega)^3}. \quad (3.20)$$

Then, taking into account (3.18)-(3.20) in (3.15), we get the second estimate in (3.11) and finally, using the Nečas inequality, there exists a representative  $\tilde{P}_\varepsilon \in L_0^2(\Omega)$  such that

$$\|\tilde{P}_\varepsilon\|_{L^2(\Omega)} \leq C \|\nabla \tilde{P}_\varepsilon\|_{H^{-1}(\Omega)^3} \leq C \|\nabla_\varepsilon \tilde{P}_\varepsilon\|_{H^{-1}(\Omega)^3},$$

which implies the first estimate in (3.11).

*Step 3. Estimates of the extended pressure for dilatant fluids.* Applying the dilatation in (3.10) for  $q = r$ , we have that  $\tilde{R}_r^\varepsilon \tilde{v}$  satisfies the following estimate

$$\|\tilde{R}_r^\varepsilon \tilde{v}\|_{L^r(\tilde{\Omega}_\varepsilon)^3} + \varepsilon \|D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v}\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C (\|\tilde{v}\|_{L^r(\Omega)^3} + \varepsilon \|D_\varepsilon \tilde{v}\|_{L^r(\Omega)^{3 \times 3}}), \quad (3.21)$$

and since  $\varepsilon \ll 1$ , we have

$$\|\tilde{R}_r^\varepsilon \tilde{v}\|_{L^r(\tilde{\Omega}_\varepsilon)^3} \leq C \|\tilde{v}\|_{W_0^{1,r}(\Omega)^3}, \quad \|D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v}\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq \frac{C}{\varepsilon} \|\tilde{v}\|_{W_0^{1,r}(\Omega)^3}. \quad (3.22)$$

Taking into account that  $r > 2$ , we have the continuous embedding  $L^r(\tilde{\Omega}_\varepsilon) \hookrightarrow L^2(\tilde{\Omega}_\varepsilon)$ , and from Hölder's inequality, we can deduce

$$\begin{aligned} & \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v} \, dx' dy_3 \\ & \leq \int_{\tilde{\Omega}_\varepsilon} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v} \, dx' dy_3 + \lambda^{\frac{r}{2}-1} \int_{\tilde{\Omega}_\varepsilon} |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^{r-2} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v} \, dx' dy_3 \\ & \leq \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \|D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v}\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} + C \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}}^{r-1} \|D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v}\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \\ & \leq \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \|D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v}\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} + C \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}}^{r-1} \|D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v}\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}}. \end{aligned}$$

Observe that if  $\gamma < 1$ , taking into account that  $-\frac{2}{r}(\gamma - 1) > -\frac{\gamma-1}{(r-1)}$ , using the last estimates in (3.3) and (3.4), and the last estimate of the dilated restricted operator given in (3.22), we obtain

$$\begin{aligned}
 & \left| \varepsilon^\gamma (\eta_0 - \eta_\infty) \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v} \, dx' dy_3 \right| \\
 & \leq \varepsilon^\gamma (\eta_0 - \eta_\infty) \left( \varepsilon^{1-\gamma} \varepsilon^{-1} + \varepsilon^{-\frac{2}{r}(\gamma-1)(r-1)} \varepsilon^{-1} \right) \|\tilde{v}\|_{W_0^{1,r}(\tilde{\Omega}_\varepsilon)^3} \\
 & \leq \varepsilon^\gamma (\eta_0 - \eta_\infty) \left( \varepsilon^{1-\gamma} \varepsilon^{-1} + \varepsilon^{-\frac{\gamma-1}{r-1}(r-1)} \varepsilon^{-1} \right) \|\tilde{v}\|_{W_0^{1,r}(\tilde{\Omega}_\varepsilon)^3} \\
 & \leq C \|\tilde{v}\|_{W_0^{1,r}(\tilde{\Omega}_\varepsilon)^3}.
 \end{aligned}$$

If  $\gamma \geq 1$ , using the last estimates in (3.3), (3.5) for  $\gamma > 1$  and (3.6) for  $\gamma = 1$ , and the last estimate of the dilated restricted operator given in (3.22), we obtain

$$\begin{aligned}
 & \left| \varepsilon^\gamma (\eta_0 - \eta_\infty) \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v} \, dx' dy_3 \right| \\
 & \leq \varepsilon^\gamma (\eta_0 - \eta_\infty) \left( \varepsilon^{1-\gamma} \varepsilon^{-1} + \varepsilon^{-\frac{\gamma-1}{r-1}(r-1)} \varepsilon^{-1} \right) \|\tilde{v}\|_{W_0^{1,r}(\tilde{\Omega}_\varepsilon)^3} \\
 & \leq C \|\tilde{v}\|_{W_0^{1,r}(\tilde{\Omega}_\varepsilon)^3}.
 \end{aligned}$$

In summary, we obtain for all  $\gamma$ :

$$\left| \varepsilon^\gamma (\eta_0 - \eta_\infty) \int_{\tilde{\Omega}_\varepsilon} (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{r}{2}-1} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v} \, dx' dy_3 \right| \leq C \|\tilde{v}\|_{W_0^{1,r}(\Omega)^3}. \quad (3.23)$$

Moreover, from Cauchy-Schwarz's inequality, the last estimate in (3.3), the continuous embedding  $L^r(\tilde{\Omega}_\varepsilon) \hookrightarrow L^2(\tilde{\Omega}_\varepsilon)$ , the assumption of  $f'$  given in (2.4) and estimates of the dilated restricted operator given in (3.22), we obtain

$$\left| \varepsilon^\gamma \eta_\infty \int_{\tilde{\Omega}_\varepsilon} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v} \, dx' dy_3 \right| \leq C \varepsilon^\gamma \|\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]\|_{L^2(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \|D_\varepsilon \tilde{R}_r^\varepsilon \tilde{v}\|_{L^r(\tilde{\Omega}_\varepsilon)^{3 \times 3}} \leq C \|\tilde{v}\|_{W_0^{1,r}(\Omega)^3}, \quad (3.24)$$

$$\left| \int_{\tilde{\Omega}_\varepsilon} f' \cdot (\tilde{R}_r^\varepsilon \tilde{v})' \, dx' dy_3 \right| \leq C \|\tilde{R}_r^\varepsilon \tilde{v}\|_{L^2(\tilde{\Omega}_\varepsilon)^3} \leq C \|\tilde{R}_r^\varepsilon \tilde{v}\|_{L^r(\tilde{\Omega}_\varepsilon)^3} \leq C \|\tilde{v}\|_{W_0^{1,r}(\Omega)^3}. \quad (3.25)$$

Then, taking into account (3.23)-(3.25) in (3.15), we get

$$\left| \langle \nabla_\varepsilon \tilde{P}_\varepsilon, \tilde{v} \rangle_{W^{-1,r'}(\Omega)^3, W_0^{1,r}(\Omega)^3} \right| \leq C \|\tilde{v}\|_{W_0^{1,r}(\Omega)^3}.$$

This implies the second estimate in (3.12) and then, using the Nečas inequality, there exists a representative  $\tilde{P}_\varepsilon \in L_0^{r'}(\Omega)$  such that

$$\|\tilde{P}_\varepsilon\|_{L^{r'}(\Omega)} \leq C \|\nabla \tilde{P}_\varepsilon\|_{W^{-1,r'}(\Omega)^3} \leq C \|\nabla_\varepsilon \tilde{P}_\varepsilon\|_{W^{-1,r'}(\Omega)^3},$$

which implies the first estimate in (3.12).  $\square$

### 3.2 Adaptation of the unfolding method.

The change of variables (2.5) does not provide the information we need about the behaviour of  $\tilde{u}_\varepsilon$  in the microstructure associated to  $\tilde{\Omega}_\varepsilon$ . To solve this difficulty, we use an adaptation introduced in [12] of the unfolding method from [27].

Let us recall that this adaptation of the unfolding method divides the domain  $\tilde{\Omega}_\varepsilon$  in cubes of lateral length  $\varepsilon$  and vertical length 1. Thus, given  $(\tilde{\varphi}_\varepsilon, \tilde{\psi}_\varepsilon) \in W_0^{1,q}(\Omega)^3 \times L_0^{q'}(\Omega)$ ,  $1 < q < +\infty$  and  $1/q + 1/q' = 1$ , we define  $(\hat{\varphi}_\varepsilon, \hat{\psi}_\varepsilon)$  by

$$\hat{\varphi}_\varepsilon(x', y) = \tilde{\varphi}_\varepsilon \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right) + \varepsilon y', y_3 \right), \quad \hat{\psi}_\varepsilon(x', y) = \tilde{\psi}_\varepsilon \left( \varepsilon \kappa \left( \frac{x'}{\varepsilon} \right) + \varepsilon y', y_3 \right), \quad \text{a.e. } (x', y) \in \omega \times Y, \quad (3.26)$$

assuming  $\tilde{\varphi}_\varepsilon$  and  $\tilde{\psi}_\varepsilon$  are extended by zero outside  $\omega$ , where the function  $\kappa : \mathbb{R}^2 \rightarrow \mathbb{Z}^2$  is defined by

$$\kappa(x') = k' \iff x' \in Y'_{k',1}, \quad \forall k' \in \mathbb{Z}^2.$$

**Remark 3.6.** *We make the following comments:*

- The function  $\kappa$  is well defined up to a set of zero measure in  $\mathbb{R}^2$  (the set  $\cup_{k' \in \mathbb{Z}^2} \partial Y'_{k',1}$ ). Moreover, for every  $\varepsilon > 0$ , we have

$$\kappa \left( \frac{x'}{\varepsilon} \right) = k' \iff x' \in Y'_{k',\varepsilon}.$$

- For  $k' \in \mathcal{K}_\varepsilon$ , the restrictions of  $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$  to  $Y'_{k',\varepsilon} \times Y$  does not depend on  $x'$ , whereas as a function of  $y$  it is obtained from  $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$  by using the change of variables  $y' = \frac{x' - \varepsilon k'}{\varepsilon}$ , which transforms  $Y'_{k',\varepsilon}$  into  $Y$ .

Following the proof of [12, Lemma 4.9], we have the following estimates relating  $(\hat{\varphi}_\varepsilon, \hat{\psi}_\varepsilon)$  and  $(\tilde{\varphi}_\varepsilon, \tilde{\psi}_\varepsilon)$ .

**Lemma 3.7.** *The sequence  $(\hat{\varphi}_\varepsilon, \hat{\psi}_\varepsilon)$  defined by (3.26) satisfies the following estimates:*

$$\begin{aligned} \|\hat{\varphi}_\varepsilon\|_{L^q(\omega \times Y)^3} &\leq \|\tilde{\varphi}_\varepsilon\|_{L^q(\Omega)^3}, \\ \|D_{y'} \hat{\varphi}_\varepsilon\|_{L^q(\omega \times Y)^{3 \times 2}} &\leq \varepsilon \|D_{x'} \tilde{\varphi}_\varepsilon\|_{L^q(\Omega)^{3 \times 2}}, \quad \|\partial_{y_3} \hat{\varphi}_\varepsilon\|_{L^q(\omega \times Y)^3} \leq \|\partial_{y_3} \tilde{\varphi}_\varepsilon\|_{L^q(\Omega)^3}, \\ \|\mathbb{D}_{y'} [\hat{\varphi}_\varepsilon]\|_{L^q(\omega \times Y)^{3 \times 2}} &\leq \varepsilon \|\mathbb{D}_{x'} [\tilde{\varphi}_\varepsilon]\|_{L^q(\Omega)^{3 \times 2}}, \quad \|\partial_{y_3} [\hat{u}_\varepsilon]\|_{L^q(\omega \times Y)^3} \leq \|\partial_{y_3} [\tilde{\varphi}_\varepsilon]\|_{L^q(\Omega)^3}, \\ \|\hat{\psi}_\varepsilon\|_{L^{q'}(\omega \times Y)} &\leq \|\tilde{\psi}_\varepsilon\|_{L^{q'}(\Omega)}. \end{aligned} \quad (3.27)$$

**Definition 3.8** (Unfolded velocity and pressure). *Let us define the unfolded velocity and pressure  $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$  from  $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$  depending on the type of fluid:*

- (Pseudoplastic fluids and Newtonian fluids) From  $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ , we define  $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$  by using (3.26) with  $\tilde{\varphi}_\varepsilon = \tilde{u}_\varepsilon$ ,  $\tilde{\psi}_\varepsilon = \tilde{P}_\varepsilon$  and  $q = 2$ .
- (Dilatant fluids) From  $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon) \in W_0^{1,r}(\Omega)^3 \times L_0^{r'}(\Omega)$ , we define  $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$  by using (3.26) with  $\tilde{\varphi}_\varepsilon = \tilde{u}_\varepsilon$ ,  $\tilde{\psi}_\varepsilon = \tilde{P}_\varepsilon$  and  $q = r$ .

Now, from estimates of the extended velocity (3.3)-(3.6) and pressure (3.11)-(3.12) together with Lemma 3.7, we have the following estimates for  $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ .

**Lemma 3.9.** *We have the following estimates for the unfolded functions  $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$  depending on the type of fluid:*

(i) *(Pseudoplastic fluids and Newtonian fluids) Consider  $1 < r \leq 2$ . There exists a constant  $C > 0$  independent of  $\varepsilon$ , such that, for every value of  $\gamma$ , we have*

$$\|\hat{u}_\varepsilon\|_{L^2(\omega \times Y)^3} \leq C\varepsilon^{2-\gamma}, \quad \|D_y \hat{u}_\varepsilon\|_{L^2(\omega \times Y)^{3 \times 3}} \leq C\varepsilon^{2-\gamma}, \quad \|\mathbb{D}_y[\hat{u}_\varepsilon]\|_{L^2(\omega \times Y)^{3 \times 3}} \leq C\varepsilon^{2-\gamma}, \quad (3.28)$$

$$\|\hat{P}_\varepsilon\|_{L^2(\omega \times Y)} \leq C. \quad (3.29)$$

(ii) *(Dilatant fluids) Consider  $r > 2$ . There exists a constant  $C > 0$  independent of  $\varepsilon$ , such that estimates (3.28) hold and also, depending on the value of  $\gamma$ , we have*

– *If  $\gamma < 1$ , it holds*

$$\|\hat{u}_\varepsilon\|_{L^r(\omega \times Y)^3} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)+1}, \quad \|D_y \hat{u}_\varepsilon\|_{L^r(\omega \times Y)^{3 \times 3}} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)+1}, \quad (3.30)$$

$$\|\mathbb{D}_y[\hat{u}_\varepsilon]\|_{L^r(\omega \times Y)^{3 \times 3}} \leq C\varepsilon^{-\frac{2}{r}(\gamma-1)+1}.$$

– *If  $\gamma > 1$ , it holds*

$$\|\hat{u}_\varepsilon\|_{L^r(\omega \times Y)^3} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}+1}, \quad \|D_y \hat{u}_\varepsilon\|_{L^r(\omega \times Y)^{3 \times 3}} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}+1}, \quad (3.31)$$

$$\|\mathbb{D}_y[\hat{u}_\varepsilon]\|_{L^r(\omega \times Y)^{3 \times 3}} \leq C\varepsilon^{-\frac{\gamma-1}{r-1}+1}.$$

– *If  $\gamma = 1$ , it holds*

$$\|\hat{u}_\varepsilon\|_{L^r(\omega \times Y)^3} \leq C\varepsilon, \quad \|D_y \hat{u}_\varepsilon\|_{L^r(\omega \times Y)^{3 \times 3}} \leq C\varepsilon, \quad (3.32)$$

$$\|\mathbb{D}_y[\hat{u}_\varepsilon]\|_{L^r(\omega \times Y)^{3 \times 3}} \leq C\varepsilon.$$

Moreover, we have the following estimate for the pressure

$$\|\hat{P}_\varepsilon\|_{L^{r'}(\omega \times Y)} \leq C, \quad (3.33)$$

where  $r'$  is the conjugate exponent of  $r$ .

### 3.3 Compactness results.

We analyze the asymptotic behaviour of sequences of the extension of  $(\tilde{u}_\varepsilon, \tilde{P}_\varepsilon)$  and  $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$ , when  $\varepsilon$  tends to zero.

**Lemma 3.10.** *We have the following convergences results for the velocity:*

(i) (*Pseudoplastic fluids and Newtonian fluids*) Consider  $1 < r \leq 2$ . There exist  $\tilde{u} \in H_0^1(0, 1; L^2(\omega)^3)$  where  $\tilde{u}_3 = 0$ ,  $\hat{u} \in L^2(\omega; H_{0,\#}^1(Y)^3)$ , with  $\hat{u} = 0$  on  $\omega \times T$  such that  $\int_Y \hat{u}(x', y) dy = \int_0^1 \tilde{u}(x', y_3) dy_3$  with  $\int_Y \hat{u}_3(x', y) dy = 0$ , such that, up to a subsequence,

$$\varepsilon^{\gamma-2} \tilde{u}_\varepsilon \rightharpoonup (\tilde{u}', 0) \text{ weakly in } H^1(0, 1; L^2(\omega)^3), \quad (3.34)$$

$$\varepsilon^{\gamma-2} \hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ weakly in } L^2(\omega; H^1(Y)^3). \quad (3.35)$$

(ii) (*Dilatant fluids*) Consider  $r > 2$ . There exist  $\tilde{u} \in W_0^{1,r}(0, 1; L^r(\omega)^3)$  where  $\tilde{u}_3 = 0$ ,  $\hat{u} \in L^r(\omega; W_{0,\#}^{1,r}(Y)^3)$ , with  $\hat{u} = 0$  on  $\omega \times T$  such that  $\int_Y \hat{u}(x', y) dy = \int_0^1 \tilde{u}(x', y_3) dy_3$  with  $\int_Y \hat{u}_3(x', y) dy = 0$ , such that, up to a subsequence,

– if  $\gamma < 1$ , it holds

$$\varepsilon^{-1+\frac{2}{r}(\gamma-1)} \tilde{u}_\varepsilon \rightharpoonup (\tilde{u}', 0) \text{ weakly in } W^{1,r}(0, 1; L^r(\omega)^3), \quad (3.36)$$

$$\varepsilon^{-1+\frac{2}{r}(\gamma-1)} \hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ weakly in } L^r(\omega; W^{1,r}(Y)^3), \quad (3.37)$$

– if  $\gamma > 1$ , it holds

$$\varepsilon^{\frac{\gamma-r}{r-1}} \tilde{u}_\varepsilon \rightharpoonup (\tilde{u}', 0) \text{ weakly in } W^{1,r}(0, 1; L^r(\omega)^3), \quad (3.38)$$

$$\varepsilon^{\frac{\gamma-r}{r-1}} \hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ weakly in } L^r(\omega; W^{1,r}(Y)^3). \quad (3.39)$$

– if  $\gamma = 1$ , it holds

$$\varepsilon^{-1} \tilde{u}_\varepsilon \rightharpoonup (\tilde{u}', 0) \text{ weakly in } W^{1,r}(0, 1; L^r(\omega)^3), \quad (3.40)$$

$$\varepsilon^{-1} \hat{u}_\varepsilon \rightharpoonup \hat{u} \text{ weakly in } L^r(\omega; W^{1,r}(Y)^3). \quad (3.41)$$

Moreover,  $\tilde{u}$  and  $\hat{u}$  satisfy the following divergence conditions in both cases

$$\operatorname{div}_{x'} \left( \int_0^1 \tilde{u}'(x', y_3) dy_3 \right) = 0 \text{ in } \omega, \quad \left( \int_0^1 \tilde{u}'(x', y_3) dy_3 \right) \cdot n = 0 \text{ in } \partial\omega, \quad (3.42)$$

$$\operatorname{div}_y \hat{u}(x', y) = 0 \text{ in } \omega \times Y_f, \quad \operatorname{div}_{x'} \left( \int_{Y_f} \hat{u}'(x', y) dy \right) = 0 \text{ in } \omega, \quad \left( \int_{Y_f} \hat{u}'(x', y) dy \right) \cdot n = 0 \text{ on } \partial\omega. \quad (3.43)$$

*Proof.* Arguing as in [12, Lemma 5.2.-(i)], we obtain convergence (3.34) (resp. (3.36), (3.38) and (3.40)) and divergence condition (3.42). Moreover, proceeding similarly as in [12, Lemma 5.4.-(i)] we deduce convergence (3.35) (resp. (3.37), (3.39) and (3.41)) and divergence conditions (3.43).  $\square$

**Lemma 3.11.** *We have the following convergences results for the pressure:*

(i) (*Pseudoplastic fluids and Newtonian fluids*) Consider  $1 < r \leq 2$ . There exists  $\tilde{P} \in L_0^2(\omega)$  such that

$$\tilde{P}_\varepsilon \rightarrow \tilde{P} \text{ strongly in } L^2(\Omega), \quad (3.44)$$

$$\hat{P}_\varepsilon \rightarrow \tilde{P} \text{ strongly in } L^2(\omega \times Y). \quad (3.45)$$

(ii) (*Dilatant fluids*) Consider  $r > 2$ . There exists  $\tilde{P} \in L_0^2(\omega)$  such that

$$\tilde{P}_\varepsilon \rightarrow \tilde{P} \text{ strongly in } L^{r'}(\Omega), \quad (3.46)$$

$$\hat{P}_\varepsilon \rightarrow \tilde{P} \text{ strongly in } L^{r'}(\omega \times Y), \quad (3.47)$$

where  $r'$  is the conjugate exponent of  $r$ .

*Proof.* We give some remarks concerning the case (i) (proceed similarly for the case (ii)). The first estimate in (3.11) implies, up to a subsequence, the existence of  $\tilde{P} \in L_0^2(\Omega)$  such that

$$\tilde{P}_\varepsilon \rightharpoonup \tilde{P} \text{ weakly in } L^2(\Omega). \quad (3.48)$$

Also, from the second estimate in (3.11), by noting that  $\partial_{y_3} \tilde{P}_\varepsilon / \varepsilon$  also converges weakly in  $H^{-1}(\Omega)$ , we obtain  $\partial_{y_3} \tilde{P} = 0$  and so  $\tilde{P}$  is independent of  $y_3$ . Moreover, if we argue as in [23, Lemma 4.4], we have that the convergence (3.48) of the pressure  $\tilde{P}_\varepsilon$  is in fact strong. Since  $\tilde{P}_\varepsilon$  has null mean value in  $\Omega$ , then  $\tilde{P}$  has null mean value in  $\omega$ , which concludes the proof of (3.44). Finally, the strong convergence of  $\hat{P}_\varepsilon$  given in (3.45) follows from [29, Proposition 1.9-(ii)] and the strong convergence of  $\tilde{P}_\varepsilon$  given in (3.44).  $\square$

### 3.4 Proof of Theorems 2.1, 2.3 and 2.5.

By using monotonicity arguments together with Minty's lemma, let us derive the variational inequality will be useful in the proofs.

We choose a test function  $v(x', y) \in \mathcal{D}(\omega; C_\#^\infty(Y)^3)$  with  $v(x', y) = 0$  in  $\omega \times T$ . Multiplying (2.6) by  $v(x', x'/\varepsilon, y_3)$ , integrating by parts, and taking into account the extension of  $\tilde{u}_\varepsilon$  and  $\tilde{P}_\varepsilon$ , we have

$$\begin{aligned} & \varepsilon^\gamma (\eta_0 - \eta_\infty) \int_\Omega (1 + \lambda |\mathbb{D}_\varepsilon[\tilde{u}_\varepsilon]|^2)^{\frac{\gamma}{2}-1} \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : (\mathbb{D}_{x'}[v] + \varepsilon^{-1} \mathbb{D}_y[v]) \, dx' dy_3 \\ & + \varepsilon^\gamma \eta_\infty \int_\Omega \mathbb{D}_\varepsilon[\tilde{u}_\varepsilon] : (\mathbb{D}_{x'}[v] + \varepsilon^{-1} \mathbb{D}_y[v]) \, dx' dy_3 \\ & - \int_\Omega \tilde{P}_\varepsilon (\operatorname{div}_{x'} v' + \varepsilon^{-1} \operatorname{div}_y v) \, dx' dy_3 = \int_\Omega f' \cdot v' \, dx' dy + O_\varepsilon, \end{aligned}$$

where  $O_\varepsilon$  is a generic real sequence depending on  $\varepsilon$  that can change from line to line.

By the change of variables given in Remark 3.6, we obtain

$$\begin{aligned} & \varepsilon^{\gamma-1} (\eta_0 - \eta_\infty) \int_{\omega \times Y} (1 + \lambda |\varepsilon^{-1} \mathbb{D}_y[\hat{u}_\varepsilon]|^2)^{\frac{\gamma}{2}-1} (\varepsilon^{-1} \mathbb{D}_y[\hat{u}_\varepsilon]) : \mathbb{D}_y[v] \, dx' dy \\ & + \varepsilon^{\gamma-1} \eta_\infty \int_{\omega \times Y} \varepsilon^{-1} \mathbb{D}_y[\hat{u}_\varepsilon] : \mathbb{D}_y[v] \, dx' dy \\ & - \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'} v' \, dx' dy - \varepsilon^{-1} \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_y v \, dx' dy = \int_{\omega \times Y} f' \cdot v' \, dx' dy + O_\varepsilon, \end{aligned} \quad (3.49)$$

with  $|O_\varepsilon| \leq C\varepsilon$  for every  $\gamma \in \mathbb{R}$ .

Now, let us define the functional  $J_r$  by

$$J_r(v) = \frac{\eta_0 - \eta_\infty}{r\lambda} \int_{\omega \times Y} (1 + \lambda |\mathbb{D}_y[v]|^2)^{\frac{r}{2}} dx' dy + \frac{\eta_\infty}{2} \int_{\omega \times Y} |\mathbb{D}_y[v]|^2 dx' dy.$$

Observe that  $J_r$  is convex and Gateaux differentiable on  $L^q(\omega; W_{\#}^{1,q}(Y)^3)$  with  $q = \max\{2, r\}$ , (see [19, Proposition 2.1 and Section 3] for more details) and  $A_r = J_r'$  is given by

$$(A_r(w), v) = (\eta_0 - \eta_\infty) \int_{\omega \times Y} (1 + \lambda |\mathbb{D}_y[w]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[w] : \mathbb{D}_y[v] dx' dy + \eta_\infty \int_{\omega \times Y} \mathbb{D}_y[w] : \mathbb{D}_y[v] dx' dy.$$

Applying [34, Proposition 1.1., p.158], in particular, we have that  $A_r$  is monotone, i.e.

$$(A_r(w) - A_r(v), w - v) \geq 0, \quad \forall w, v \in L^q(\omega; W_{\#}^{1,q}(Y)^3). \quad (3.50)$$

On the other hand, for all  $\varphi \in \mathcal{D}(\omega; C_{\#}^{\infty}(Y)^3)$  with  $\varphi = 0$  in  $\omega \times T$ , satisfying the divergence conditions  $\operatorname{div}_{x'} \int_Y \varphi' dy = 0$  in  $\omega$ ,  $\int_{Y_f} \varphi(x', y) dy \cdot n = 0$  on  $\partial\omega$ , and  $\operatorname{div}_y \varphi = 0$  in  $\omega \times Y$ , we choose  $v_\varepsilon$  defined by

$$v_\varepsilon = \varphi - \varepsilon^{-1} \hat{u}_\varepsilon,$$

as a test function in (3.49).

Taking into account that  $\operatorname{div}_\varepsilon \tilde{u}_\varepsilon = 0$ , we get that  $\varepsilon^{-1} \operatorname{div}_y \hat{u}_\varepsilon = 0$ , and then we obtain

$$\varepsilon^{\gamma-1} (A_r(\varepsilon^{-1} \hat{u}_\varepsilon), v_\varepsilon) - \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'} v'_\varepsilon dx' dy = \int_{\omega \times Y} f' \cdot v'_\varepsilon dx' dy + O_\varepsilon,$$

which is equivalent to

$$\varepsilon^{\gamma-1} (A_r(\varphi) - A_r(\varepsilon^{-1} \hat{u}_\varepsilon), v_\varepsilon) - \varepsilon^{\gamma-1} (A_r(\varphi), v_\varepsilon) + \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'} v'_\varepsilon dx' dy = - \int_{\omega \times Y} f' \cdot v'_\varepsilon dx' dy + O_\varepsilon.$$

Due to (3.50), we can deduce

$$\varepsilon^{\gamma-1} (A_r(\varphi), v_\varepsilon) - \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'} v'_\varepsilon dx' dy \geq \int_{\omega \times Y} f' \cdot v'_\varepsilon dx' dy + O_\varepsilon,$$

i.e.

$$\begin{aligned} & \varepsilon^{\gamma-1} (\eta_0 - \eta_\infty) \int_{\omega \times Y} (1 + \lambda |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[v_\varepsilon] dx' dy + \varepsilon^{\gamma-1} \eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[v_\varepsilon] dx' dy \\ & - \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'} v'_\varepsilon dx' dy \geq \int_{\omega \times Y} f' \cdot v'_\varepsilon dx' dy + O_\varepsilon. \end{aligned} \quad (3.51)$$

Here, in the case  $1 < r \leq 2$ , it holds  $|O_\varepsilon| \leq C\varepsilon^\alpha$  with  $\alpha = 1$  if  $\gamma \leq 1$  and  $\alpha = 2 - \gamma$  if  $\gamma > 1$ . In the case  $r > 2$ , it holds  $|O_\varepsilon| \leq C\varepsilon^\alpha$  with  $\alpha = \gamma + \frac{2}{r}(1 - \gamma)$  if  $\gamma \leq 1$  and  $\alpha = 1 - \frac{\gamma-1}{r-1}$  if  $\gamma > 1$ .

*Proof of Theorem 2.1.* We recall that  $1 < r < 2$ . The case  $\gamma = 1$  is developed in [9], so we omit it. As consequence, in the rest of the proof we consider that  $\gamma \neq 1$ . The proof will be divided in two steps. In the first step, we obtain the homogenized behaviour given by a coupled system, with a constant macroviscosity, and in the second step we decouple it to obtain the macroscopic law.

*Step 1.* From Lemmas 3.10 and 3.11, we prove that the sequence  $(\varepsilon^{\gamma-2}\hat{u}_\varepsilon, \tilde{P}_\varepsilon)$  converges to  $(\hat{u}, \tilde{P}) \in L^2(\omega; H_{\#}^1(Y_f)^3) \times (L_0^2(\omega) \cap H^1(\omega))$ , which are the unique solutions of the following two-pressures Newtonian Stokes problem with the linear viscosity  $\eta$  equal to  $\eta_0$  if  $\gamma < 1$  and  $\eta_\infty$  if  $\gamma > 1$ :

$$\left\{ \begin{array}{l} -\eta \operatorname{div}_y \mathbb{D}_y[\hat{u}] + \nabla_y \hat{\pi} = f' - \nabla_{x'} \tilde{P} \quad \text{in } \omega \times Y_f, \\ \operatorname{div}_y \hat{u} = 0 \quad \text{in } \omega \times Y_f, \\ \operatorname{div}_{x'} \left( \int_{Y_f} \hat{u}' dy \right) = 0 \quad \text{in } \omega, \\ \left( \int_{Y_f} \hat{u}' dy \right) \cdot n = 0 \quad \text{on } \partial\omega, \\ \hat{u} = 0 \quad \text{in } \omega \times T, \\ \hat{\pi} \in L^2(\omega; L_{0,\#}^2(Y_f)). \end{array} \right. \quad (3.52)$$

Divergence conditions (3.52)<sub>2,3,4</sub> and condition (3.52)<sub>5</sub> follow from Lemma 3.10. To prove that  $(\hat{u}, \tilde{P})$  satisfies the momentum equation given in (3.52), we follow the lines of the proof to obtain (3.51) but choosing now  $v_\varepsilon$  and  $\varphi$  such that  $v_\varepsilon = \varepsilon^{1-\gamma}\varphi - \varepsilon^{-1}\hat{u}_\varepsilon$  with  $\varphi \in \mathcal{D}(\omega; C_{\#}^\infty(Y)^3)$  with  $\varphi = 0$  in  $\omega \times T$ , satisfying the divergence conditions  $\operatorname{div}_{x'} \int_Y \varphi' dy = 0$  in  $\omega$ ,  $\int_Y \varphi' dy \cdot n = 0$  on  $\partial\omega$ , and  $\operatorname{div}_y \varphi = 0$  in  $\omega \times Y$ . Then, we get

$$\begin{aligned} & \varepsilon^{1-\gamma}(\eta_0 - \eta_\infty) \int_{\omega \times Y} (1 + \lambda \varepsilon^{2(1-\gamma)} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\gamma-2}\hat{u}_\varepsilon] dx' dy \\ & + \varepsilon^{1-\gamma} \eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\gamma-2}\hat{u}_\varepsilon] dx' dy \\ & - \varepsilon^{1-\gamma} \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'}(\varphi' - \varepsilon^{\gamma-2}\hat{u}'_\varepsilon) dx' dy \geq \varepsilon^{1-\gamma} \int_{\omega \times Y} f' \cdot (\varphi' - \varepsilon^{\gamma-2}\hat{u}'_\varepsilon) dx' dy + O_\varepsilon, \end{aligned} \quad (3.53)$$

where  $|O_\varepsilon| \leq C\varepsilon^{2-\gamma}$ . Dividing by  $\varepsilon^{1-\gamma}$ , we have

$$\begin{aligned} & (\eta_0 - \eta_\infty) \int_{\omega \times Y} (1 + \lambda \varepsilon^{2(1-\gamma)} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\gamma-2}\hat{u}_\varepsilon] dx' dy \\ & + \eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\gamma-2}\hat{u}_\varepsilon] dx' dy \\ & - \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'}(\varphi' - \varepsilon^{\gamma-2}\hat{u}'_\varepsilon) dx' dy \geq \int_{\omega \times Y} f' \cdot (\varphi' - \varepsilon^{\gamma-2}\hat{u}'_\varepsilon) dx' dy + O_\varepsilon, \end{aligned} \quad (3.54)$$

where  $|O_\varepsilon| \leq C\varepsilon$ , which tends to zero when  $\varepsilon \rightarrow 0$ . Now, we can pass to the limit depending on the value of  $\gamma$ :

- If  $\gamma < 1$ , then  $2(1 - \gamma) > 0$  and so,  $\lambda \varepsilon^{2(1-\gamma)} |\mathbb{D}_y[\varphi]|^2$  tends to zero. From convergence (3.35), passing to the limit when  $\varepsilon$  tends to zero in (3.54), we have that the first and second terms converge to

$$\eta_0 \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] \, dx' dy.$$

From convergences (3.35) and (3.45), the third term converges to

$$\int_{\omega \times Y} \tilde{P} \operatorname{div}_{x'}(\varphi' - \hat{u}') \, dx' dy.$$

Since  $\tilde{P}$  does not depend on  $y$ , by using the divergences conditions  $\operatorname{div}_{x'} \int_Y \varphi' \, dy = 0$  and (3.43)<sub>2</sub>, we have

$$\int_{\omega \times Y} \tilde{P} \operatorname{div}_{x'}(\varphi' - \hat{u}') \, dx' dy = \int_{\omega} \tilde{P} \operatorname{div}_{x'} \left( \int_Y (\varphi' - \hat{u}') \, dy \right) \, dx' = 0.$$

Thus, we deduce that the variational inequality (3.54) converges to the following one

$$\eta_0 \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] \, dx' dy \geq \int_{\omega \times Y} f' \cdot (\varphi' - \hat{u}') \, dx' dy.$$

Since  $\varphi$  is arbitrary, by Minty's lemma, see [34, Chapter 3, Lemma 1.2], we deduce

$$\eta_0 \int_{\omega \times Y} \mathbb{D}_y[\hat{u}] : \mathbb{D}_y[\varphi] \, dx' dy = \int_{\omega \times Y} f' \cdot \varphi' \, dx' dy.$$

- If  $\gamma > 1$ , then  $2(1 - \gamma) < 0$  and so,  $(1 + \lambda \varepsilon^{2(1-\gamma)} |\mathbb{D}_y[\varphi]|^2)^{\frac{\gamma}{2}-1}$  tends to zero. From convergence (3.35), passing to the limit when  $\varepsilon$  tends to zero in (3.54), we have that the first and second terms converge to

$$\eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] \, dx' dy.$$

From convergences (3.35) and (3.45), the third term converges to

$$\int_{\omega \times Y} \tilde{P} \operatorname{div}_{x'}(\varphi' - \hat{u}') \, dx' dy.$$

Since  $\tilde{P}$  does not depend on  $y$ , by using the divergences conditions  $\operatorname{div}_{x'} \int_Y \varphi' \, dy = 0$  and (3.43)<sub>2</sub>, we have

$$\int_{\omega \times Y} \tilde{P} \operatorname{div}_{x'}(\varphi' - \hat{u}') \, dx' dy = \int_{\omega} \tilde{P} \operatorname{div}_{x'} \left( \int_Y (\varphi' - \hat{u}') \, dy \right) \, dx' = 0.$$

Thus, we deduce that the variational inequality (3.54) converges to the following one

$$\eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] \, dx' dy \geq \int_{\omega \times Y} f' \cdot (\varphi' - \hat{u}') \, dx' dy,$$

and by the same argument as above, we get

$$\eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\hat{u}] : \mathbb{D}_y[\varphi] \, dx' dy = \int_{\omega \times Y} f' \cdot \varphi' \, dx' dy.$$

In summary, considering  $\eta$  equal to  $\eta_0$  if  $\gamma < 1$  or  $\eta_\infty$  if  $\gamma > 1$ , we have obtained that by density, the equality

$$\eta \int_{\omega \times Y} \mathbb{D}_y[\hat{u}] : \mathbb{D}_y[v] dx' dy = \int_{\omega \times Y} f' \cdot v' dx' dy, \quad (3.55)$$

holds for every  $v$  in the Hilbert space  $\mathcal{V}$  defined by

$$\mathcal{V} = \left\{ \begin{array}{l} v(x', y) \in L^2(\omega; H_{\#}^1(Y)^3) \text{ such that} \\ \operatorname{div}_{x'} \left( \int_{Y_f} v(x', y) dy \right) = 0 \text{ in } \omega, \quad \left( \int_{Y_f} v(x', y) dy \right) \cdot n = 0 \text{ on } \partial\omega \\ \operatorname{div}_y v(x', y) = 0 \text{ in } \omega \times Y_f, \quad v(x', y) = 0 \text{ in } \omega \times T \end{array} \right\}. \quad (3.56)$$

Reasoning as in [1, Lemma 1.5], the orthogonal of  $\mathcal{V}$ , a subset of  $L^2(\omega; H_{\#}^{-1}(Y)^3)$ , is made of gradients of the form  $\nabla_{x'} \tilde{\pi}(x') + \nabla_y \hat{\pi}(x', y)$ , with  $\tilde{\pi}(x') \in H^1(\omega)/\mathbb{R}$  and  $\hat{\pi}(x', y) \in L^2(\omega; L_{\#}^2(Y_f)/\mathbb{R})$ . Thus, integrating by parts, the variational formulation (3.55) is equivalent to the two-pressures Newtonian Stokes problem (3.52). It remains to prove that  $\tilde{\pi}$  coincides with pressure  $\tilde{P}$ . This can be easily done passing to the limit similarly as above by considering the test function  $\varphi$ , which is divergence-free only in  $y$ , and by identifying limits. It holds then that  $\tilde{P} \in L_0^2(\omega) \cap H^1(\omega)$ . From [1], problem (3.52) admits a unique solution  $(\hat{u}, \hat{\pi}, \tilde{P}) \in L^2(\omega; H_{\#}^1(Y_f)^3) \times L^2(\omega; L_{0,\#}^2(Y_f)) \times (L_0^2(\omega) \cap H^1(\omega))$ , which implies that the entire sequence  $(\hat{u}_\varepsilon, \tilde{P}_\varepsilon)$  converges to  $(\hat{u}, \tilde{P})$ .

*Step 2.* To prove (2.7), it remains to eliminate the microscopic variable  $y$  in the effective problem (3.52). The procedure is standard and can be viewed for instance in [15], but for reader's convenience, we give some details of the proof. From the first equation of (3.52), the velocity  $\hat{u}$  is computed in terms of the macroscopic forces and the local velocities

$$\hat{u}(x', y) = \frac{1}{\eta} \sum_{i=1}^2 \left( f_i(x') - \partial_{x_i} \tilde{P}(x') \right) w^i(y), \quad \hat{\pi}(x', y) = \sum_{i=1}^2 \left( f_i(x') - \partial_{x_i} \tilde{P}(x') \right) \pi^i(y).$$

Integrating this expression on  $Y$  and taking into account that  $V(x') = \int_0^1 \tilde{u}(x', y_3) dy_3 = \int_{Y_f} \hat{u}(x', y) dy$  and  $\int_{Y_f} \hat{u}_3 dy = 0$ , we get the Darcy relationship (2.7)<sub>1</sub> because the matrix  $\mathcal{A} \in \mathbb{R}^{2 \times 2}$  satisfies (2.8). Combining the expression of  $V$  with the divergence-free condition on  $V$  given by (3.52)<sub>3</sub> yields the lower-dimensional homogenized Darcy's law (2.7)<sub>2</sub>. □

*Proof of Theorem 2.3.* We recall that in this case  $r > 2$ . The proof will be divided in four steps. In the first step, we obtain the homogenized behaviour in the case  $\gamma < 1$  and as a result, we obtain that the filtration velocity  $V = 0$  in  $\omega$ . In the second step, we obtain the homogenized behaviour for  $\gamma > 1$  giving by a coupled system with a non-linear macroviscosity of power law type, which will be decoupled to obtain the macroscopic law in the third step. Finally, in the fourth step, we consider the case  $\gamma = 1$ .

*Step 1.* We consider the case  $\gamma < 1$ . We will prove that the filtration velocity  $V(x') = \int_0^1 \tilde{u} dy_3 = 0$  in  $\omega$ . To do this, we first prove that  $\hat{u}$  does not depend on the variable  $y$  and next, taking into account the boundary conditions on  $\hat{u}$  and the relation of  $\hat{u}$  and  $\tilde{u}$ , we prove that  $V = 0$ .

- We start by proving that  $\hat{u}$  does not depend on  $y$ . Reasoning similarly to the case  $1 < r < 2$  and considering  $v_\varepsilon = \varepsilon^{-\frac{2}{r}(\gamma-1)}\varphi - \varepsilon^{-1}\hat{u}_\varepsilon$  with  $\varphi \in \mathcal{D}(\omega; C^\infty_\#(Y)^3)$  with  $\varphi = 0$  in  $\omega \times T$ , satisfying only the divergence condition  $\operatorname{div}_y \varphi = 0$  in  $\omega \times Y$ , we get

$$\begin{aligned}
 & \varepsilon^{(-1+\frac{4}{r})(1-\gamma)}(\eta_0 - \eta_\infty) \int_{\omega \times Y} (1 + \lambda \varepsilon^{\frac{4}{r}(1-\gamma)} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}_\varepsilon] dx' dy \\
 & + \varepsilon^{(-1+\frac{4}{r})(1-\gamma)} \eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}_\varepsilon] dx' dy \\
 & - \varepsilon^{\frac{2}{r}(1-\gamma)} \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'}(\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}'_\varepsilon) dx' dy \\
 & \geq \varepsilon^{\frac{2}{r}(1-\gamma)} \int_{\omega \times Y} f' \cdot (\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}'_\varepsilon) dx' dy + O_\varepsilon,
 \end{aligned} \tag{3.57}$$

where  $|O_\varepsilon| \leq C\varepsilon^{\gamma+\frac{4}{r}(1-\gamma)}$ . Dividing by  $\varepsilon^{(-1+\frac{4}{r})(1-\gamma)}$ , we have

$$\begin{aligned}
 & (\eta_0 - \eta_\infty) \int_{\omega \times Y} (1 + \lambda \varepsilon^{\frac{4}{r}(1-\gamma)} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}_\varepsilon] dx' dy \\
 & + \eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}_\varepsilon] dx' dy \\
 & - \varepsilon^{(1-\frac{2}{r})(1-\gamma)} \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'}(\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}'_\varepsilon) dx' dy \\
 & \geq \varepsilon^{(1-\frac{2}{r})(1-\gamma)} \int_{\omega \times Y} f' \cdot (\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}'_\varepsilon) dx' dy + O_\varepsilon,
 \end{aligned} \tag{3.58}$$

with  $|O_\varepsilon| \leq C\varepsilon^{1+(1-\gamma)(1-\frac{2}{r})}$ , which tends to zero when  $\varepsilon \rightarrow 0$ , because  $1 + (1 - \frac{2}{r})(1 - \gamma) > 0$ .

Since  $\gamma < 1$ , then  $\frac{4}{r}(1-\gamma) > 0$  and so,  $\lambda \varepsilon^{\frac{4}{r}(1-\gamma)} |\mathbb{D}_y[\varphi]|^2$  tends to zero. From convergence (3.37), passing to the limit when  $\varepsilon$  tends to zero in (3.58), we have that the first and second terms converge to

$$\eta_0 \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] dx' dy.$$

Since  $r > 2$ , then  $(1 - \frac{2}{r})(1 - \gamma) > 0$ . This together with convergences (3.37) and (3.47), implies that

$$\varepsilon^{(1-\frac{2}{r})(1-\gamma)} \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'}(\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}'_\varepsilon) dx' dy \quad \text{and} \quad \varepsilon^{(1-\frac{2}{r})(1-\gamma)} \int_{\omega \times Y} f' \cdot (\varphi' - \varepsilon^{\frac{2}{r}(\gamma-1)-1} \hat{u}'_\varepsilon) dx' dy$$

tend to zero.

Thus, we deduce that the variational inequality (3.58) converges to the following one

$$\eta_0 \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] dx' dy \geq 0,$$

which by Minty's lemma, see [34, Chapter 3, Lemma 1.2], implies that

$$\eta_0 \int_{\omega \times Y} \mathbb{D}_y[\hat{u}] : \mathbb{D}_y[v] dx' dy = 0, \tag{3.59}$$

holds, by density arguments, for every  $v$  in the Banach space  $\mathcal{V}$  defined by

$$\mathcal{V} = \left\{ \begin{array}{l} v(x', y) \in L^r(\omega; W_{\#}^{1,r}(Y)^3) \text{ such that} \\ \operatorname{div}_y v(x', y) = 0 \text{ in } \omega \times Y_f, \quad v(x', y) = 0 \text{ in } \omega \times T \end{array} \right\}.$$

Then, there exists  $\hat{\pi}(x', y) \in L^{r'}(\omega; L_{\#}^{r'}(Y_f)/\mathbb{R})$  such that, integrating by parts, the variational formulation (3.59) is equivalent to the Stokes system with respect to the variable  $y$

$$-\eta_0 \operatorname{div}_y (\mathbb{D}_y[\hat{u}]) + \nabla_y \hat{\pi} = 0 \quad \text{in } \omega \times Y.$$

We immediately find (see the boundary conditions with respect to  $y$ ) that

$$D_y \hat{u} = 0 \quad \text{and} \quad \nabla_y \hat{\pi} = 0,$$

i.e. that  $\hat{u}$  does not depend on  $y$ . So, we can not obtain microscopic effects in the limit. Next, we will derive the macroscopic asymptotic behaviour of  $\tilde{u}$ .

- We prove that the filtration velocity  $V = 0$  in  $\omega$ . To do this, taking into account that  $\hat{u}$  only depends on  $x'$  and from the boundary conditions  $\hat{u} = 0$  on  $y_3 = \{0, 1\}$ , we deduce that  $\hat{u} = 0$  in  $\omega \times Y$ . Finally, taking into account that  $\hat{u} = 0$ , that the filtration velocity is defined by  $V(x') = \int_0^1 \tilde{u}(x', y_3) dy_3$  and that  $\int_0^1 \tilde{u}(x', y_3) dy_3 = \int_Y \hat{u}(x', y) dy$  (see Lemma 3.10), we then deduce that the filtration velocity  $V(x') = 0$  in  $\omega$ , i.e. (2.13).

*Step 2.* We consider the case  $\gamma > 1$ . From Lemmas 3.10 and 3.11, we prove that the sequence  $(\hat{u}_\varepsilon, \tilde{P}_\varepsilon)$  converges to  $(\hat{u}, \tilde{P}) \in L^r(\omega; W_{\#}^{1,r}(Y_f)^3) \times (L_0^{r'}(\omega) \cap W^{1,r'}(\omega))$ , which are the unique solutions of the following two-pressures non-Newtonian Stokes problem with the non-linear power law viscosity

$$\left\{ \begin{array}{l} -(\eta_0 - \eta_\infty) \lambda^{\frac{r-2}{2}} \operatorname{div}_y (|\mathbb{D}_y[\hat{u}]|^{r-2} \mathbb{D}_y[\hat{u}]) + \nabla_y \hat{\pi} = f' - \nabla_{x'} \tilde{P} \quad \text{in } \omega \times Y_f, \\ \operatorname{div}_y \hat{u} = 0 \quad \text{in } \omega \times Y_f, \\ \operatorname{div}_{x'} \left( \int_{Y_f} \hat{u}' dy \right) = 0 \quad \text{in } \omega, \\ \left( \int_{Y_f} \hat{u}' dy \right) \cdot n = 0 \quad \text{on } \partial\omega, \\ \hat{u} = 0 \quad \text{in } \omega \times T, \\ \hat{\pi} \in L^{r'}(\omega; L_{0,\#}^{r'}(Y_f)). \end{array} \right. \quad (3.60)$$

Divergence conditions (3.60)<sub>2,3,4</sub> and condition (3.60)<sub>5</sub> follow from Lemma 3.10. To prove that  $(\hat{u}, \tilde{P})$  satisfies the momentum equation given in (3.60), we follow the lines of the proof of (3.51) but considering now  $v_\varepsilon$  and  $\varphi$  as  $v_\varepsilon = \varepsilon^{\frac{1-\gamma}{r-1}} \varphi - \varepsilon^{-1} \hat{u}_\varepsilon$  with  $\varphi \in \mathcal{D}(\omega; C_{\#}^\infty(Y)^3)$  with  $\varphi = 0$  in  $\omega \times T$ , satisfying the divergence conditions  $\operatorname{div}_{x'} \int_Y \varphi' dy = 0$  in  $\omega$ ,  $\int_Y \varphi' dy \cdot n = 0$  on  $\partial\omega$ , and  $\operatorname{div}_y \varphi = 0$  in  $\omega \times Y$ . Then,

we get

$$\begin{aligned}
 & \varepsilon^{\gamma-1+2\frac{1-\gamma}{r-1}}(\eta_0 - \eta_\infty) \int_{\omega \times Y} (1 + \lambda \varepsilon^{2\frac{1-\gamma}{r-1}} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{\gamma-r}{r-1}} \hat{u}_\varepsilon] dx' dy \\
 & + \varepsilon^{\gamma-1+2\frac{1-\gamma}{r-1}} \eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{\gamma-r}{r-1}} \hat{u}_\varepsilon] dx' dy \\
 & - \varepsilon^{\frac{1-\gamma}{r-1}} \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'}(\varphi' - \varepsilon^{\frac{\gamma-r}{r-1}} \hat{u}'_\varepsilon) dx' dy \geq \varepsilon^{\frac{1-\gamma}{r-1}} \int_{\omega \times Y} f' \cdot (\varphi' - \varepsilon^{\frac{\gamma-r}{r-1}} \hat{u}'_\varepsilon) dx' dy + O_\varepsilon,
 \end{aligned} \tag{3.61}$$

where  $|O_\varepsilon| \leq C\varepsilon^{1+\frac{1-\gamma}{r-1}}$ . Dividing by  $\varepsilon^{\frac{1-\gamma}{r-1}}$  we deduce that

$$\begin{aligned}
 & \varepsilon^{(\gamma-1)\frac{r-2}{r-1}}(\eta_0 - \eta_\infty) \int_{\omega \times Y} (1 + \lambda \varepsilon^{2\frac{1-\gamma}{r-1}} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{\gamma-r}{r-1}} \hat{u}_\varepsilon] dx' dy \\
 & + \varepsilon^{(\gamma-1)\frac{r-2}{r-1}} \eta_\infty \int_{\omega \times Y} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \varepsilon^{\frac{\gamma-r}{r-1}} \hat{u}_\varepsilon] dx' dy \\
 & - \int_{\omega \times Y} \hat{P}_\varepsilon \operatorname{div}_{x'}(\varphi' - \varepsilon^{\frac{\gamma-r}{r-1}} \hat{u}'_\varepsilon) dx' dy \geq \int_{\omega \times Y} f' \cdot (\varphi' - \varepsilon^{\frac{\gamma-r}{r-1}} \hat{u}'_\varepsilon) dx' dy + O_\varepsilon,
 \end{aligned} \tag{3.62}$$

with  $|O_\varepsilon| \leq C\varepsilon$ , which tends to zero when  $\varepsilon \rightarrow 0$ .

Since  $\gamma > 1$  and  $r > 2$ , we have that  $(\gamma - 1)\frac{r-2}{r-1} > 0$ , and from

$$\varepsilon^{(\gamma-1)\frac{r-2}{r-1}}(1 + \lambda \varepsilon^{2\frac{1-\gamma}{r-1}} |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1} = (\varepsilon^{2\frac{\gamma-1}{r-1}} + \lambda |\mathbb{D}_y[\varphi]|^2)^{\frac{r}{2}-1}$$

and convergences (3.39) and (3.47), passing to the limit in (3.62) when  $\varepsilon$  tends to zero, we deduce

$$\begin{aligned}
 & (\eta_0 - \eta_\infty) \lambda^{\frac{r-2}{2}} \int_{\omega \times Y} |\mathbb{D}_y[\varphi]|^{r-2} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] dx' dy \\
 & - \int_{\omega \times Y} \tilde{P} \operatorname{div}_{x'}(\varphi' - \hat{u}') dx' dy \geq \int_{\omega \times Y} f' \cdot (\varphi' - \hat{u}') dx' dy.
 \end{aligned} \tag{3.63}$$

Since  $\tilde{P}$  does not depend on  $y$ , by using the divergences conditions  $\operatorname{div}_{x'} \int_Y \varphi' dy = 0$  and (3.60)<sub>3</sub>, we have

$$\int_{\omega \times Y} \tilde{P} \operatorname{div}_{x'}(\varphi' - \hat{u}') dx' dy = \int_{\omega} \tilde{P} \operatorname{div}_{x'} \left( \int_Y (\varphi' - \hat{u}') dy \right) dx' = 0.$$

Thus, we deduce that the variational inequality (3.63) reads

$$(\eta_0 - \eta_\infty) \lambda^{\frac{r-2}{2}} \int_{\omega \times Y} |\mathbb{D}_y[\varphi]|^{r-2} \mathbb{D}_y[\varphi] : \mathbb{D}_y[\varphi - \hat{u}] dx' dy \geq \int_{\omega \times Y} f' \cdot (\varphi' - \hat{u}') dx' dy,$$

which by Minty's lemma, see [34, Chapter 3, Lemma 1.2], is equivalent to

$$(\eta_0 - \eta_\infty) \lambda^{\frac{r-2}{2}} \int_{\omega \times Y} |\mathbb{D}_y[\hat{u}]|^{r-2} \mathbb{D}_y[\hat{u}] : \mathbb{D}_y[v] dx' dy = \int_{\omega \times Y} f' \cdot v' dx' dy. \tag{3.64}$$

By density, the previous equality holds for every  $v$  in the Banach space  $\mathcal{V}$  defined by

$$\mathcal{V} = \left\{ \begin{array}{l} v(x', y) \in L^r(\omega; W_{\#}^{1,r}(Y)^3) \text{ such that} \\ \operatorname{div}_{x'} \left( \int_{Y_f} v(x', y) dy \right) = 0 \text{ in } \omega, \quad \left( \int_{Y_f} v(x', y) dy \right) \cdot n = 0 \text{ on } \partial\omega \\ \operatorname{div}_y v(x', y) = 0 \text{ in } \omega \times Y_f, \quad v(x', y) = 0 \text{ in } \omega \times T \end{array} \right\}.$$

Reasoning as in [1, Lemma 1.5], the orthogonal of  $\mathcal{V}$ , a subset of  $L^r(\omega; W_{\#}^{-1,r'}(Y)^3)$ , is made of gradients of the form  $\nabla_{x'} \tilde{\pi}(x') + \nabla_y \hat{\pi}(x', y)$ , with  $\tilde{\pi}(x') \in W^{1,r'}(\omega)/\mathbb{R}$  and  $\hat{\pi}(x', y) \in L^{r'}(\omega; L_{\#}^{r'}(Y_f)/\mathbb{R})$ . Thus, integrating by parts, the variational formulation (3.64) is equivalent to the two-pressures non-Newtonian Stokes problem (3.60). It remains to prove that  $\tilde{\pi}$  coincides with pressure  $\tilde{P}$ . This can be easily done passing to the limit similarly as above by considering the test function  $\varphi$ , which is divergence-free only in  $y$ , and by identifying limits. It holds then that  $\tilde{P} \in L_0^{r'}(\omega) \cap W^{1,r'}(\omega)$ . From [23, Theorem 2], problem (3.60) admits a unique solution  $(\hat{u}, \hat{\pi}, \tilde{P}) \in L^r(\omega; W_{\#}^{1,r}(Y_f)^3) \times L^{r'}(\omega; L_{0,\#}^{r'}(Y_f)) \times (L_0^{r'}(\omega) \cap W^{1,r'}(\omega))$ , hence the entire sequence  $(\hat{u}_\varepsilon, \hat{P}_\varepsilon)$  converges to  $(\hat{u}, \tilde{P})$ .

*Step 3.* In this step we give an approximation of the model (3.60), where the macroscopic scale is totally decoupled from the microscopic one. To do this, we seek a global filtration velocity of the form given in (2.14), i.e.

$$V'(x') = \frac{1}{\lambda^{\frac{2-r'}{2}}(\eta_0 - \eta_\infty)^{r'-1}} \mathcal{U}(f'(x') - \nabla_{x'} \tilde{P}(x')) \quad \text{in } \omega, \quad (3.65)$$

where  $\mathcal{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is a permeability function, not necessary linear, and  $V(x') = \int_0^1 \tilde{u}(x', y_3) dy_3 = \int_Y \hat{u}(x', y) dy$  with  $\operatorname{div}_{x'} V' = 0$  in  $\omega$  and  $V' \cdot n = 0$  on  $\partial\omega$ .

Using the idea from [24] to decouple the homogenized problems of *power law* type, for every  $\xi' \in \mathbb{R}^2$  we consider the function  $\mathcal{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  given by

$$\mathcal{U}(\xi') = \int_{Y_f} w_{\xi'}(y) dy,$$

where  $w_{\xi'}$  denotes the unique solution of the local Stokes problem given by (2.12), see [23, Theorem 2]. Thus,  $(\hat{u}, \hat{\pi})$  takes the form

$$\hat{u}(x', y) = \frac{1}{\lambda^{\frac{2-r'}{2}}(\eta_0 - \eta_\infty)^{r'-1}} w_{f'(x') - \nabla_{x'} \tilde{P}(x')}(y), \quad \hat{\pi}(x', y) = \pi_{f'(x') - \nabla_{x'} \tilde{P}(x')}(y) \quad \text{in } \omega \times Y.$$

Then, from the relation  $V(x') = \int_Y \hat{u}(x', y) dy$  with  $\int_Y \hat{u}_3(x', y) dy = 0$  given in Lemma 3.10, we deduce the filtration velocity (3.65), where  $V_3 = 0$ . Moreover, from second and third conditions given in (3.60) together with (3.65), we deduce

$$\operatorname{div}_{x'} V' = 0 \quad \text{in } \omega, \quad V' \cdot n = 0 \quad \text{on } \partial\omega.$$

Since  $V_3 = 0$ , to simplify the notation, we redefine the definition of  $\mathcal{U}$  by the expression given in (2.11) and then, we get  $\mathcal{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , which concludes the proof of (2.14). Finally, from [24, Theorem

1], the macroscopic problem (2.14) has a unique solution  $(V, \tilde{P}) \in L^r(\omega)^3 \times (L_0^{r'}(\omega) \cap W^{1,r}(\omega))$  and Theorem 2.3 is proved.

*Step 4.* We consider the case  $\gamma = 1$ . Reasoning as in the *Step 2* with  $\gamma = 1$  and using convergences (3.41) and (3.47), we deduce that the sequence  $(\hat{u}_\varepsilon, \tilde{P}_\varepsilon)$  converges to  $(\hat{u}, \tilde{P}) \in L^r(\omega; W_{0,\#}^{1,r}(Y_f)^3) \times (L_0^{r'}(\omega) \cap W^{1,r'}(\omega))$ , which are the unique solutions of the following two-pressures non-Newtonian Stokes problem with the non-linear Carreau viscosity (1.2)

$$\left\{ \begin{array}{l} -\operatorname{div}_y(\eta_r(\mathbb{D}_y[\hat{u}])\mathbb{D}_y[\hat{u}]) + \nabla_y \hat{\pi} = f' - \nabla_{x'} \tilde{P} \quad \text{in } \omega \times Y_f, \\ \operatorname{div}_y \hat{u} = 0 \quad \text{in } \omega \times Y_f, \\ \operatorname{div}_{x'} \left( \int_{Y_f} \hat{u}' dy \right) = 0 \quad \text{in } \omega, \\ \left( \int_{Y_f} \hat{u}' dy \right) \cdot n = 0 \quad \text{on } \partial\omega, \\ \hat{u} = 0 \quad \text{in } \omega \times T, \\ \hat{\pi} \in L^{r'}(\omega; L_{0,\#}^{r'}(Y_f)). \end{array} \right.$$

Proceeding as in *Step 3*, we deduce the non-linear 2D Darcy's law of Carreau type (2.10), where the permeability function  $\mathcal{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by (2.11) with  $(w_{\xi'}, \pi_{\xi'}) \in W_{0,\#}^{1,r}(Y_f)^3 \times L_{0,\#}^{r'}(Y_f)$ , for every  $\xi' \in \mathbb{R}^2$ , the unique solution of the local Stokes system (2.12) with nonlinear viscosity given by the Carreau law (1.2). □

*Proof of Theorem 2.5.* We recall that in this case  $r = 2$ , the Carreau law (1.2) reduces to  $\eta_0$ . Thus, by the linearity, the variational formulation (3.49) can be written as follows

$$\varepsilon^{\gamma-2} \eta_0 \int_{\omega \times Y} \mathbb{D}_y[\hat{u}_\varepsilon] : \mathbb{D}_y[\varphi] dx' dy - \int_{\omega \times Y} \tilde{P}_\varepsilon \operatorname{div}_{x'} \varphi' dx' dy = \int_{\omega \times Y} f' \cdot \varphi' dx' dy + O_\varepsilon, \quad (3.66)$$

for all  $\varphi \in \mathcal{D}(\omega; C_{\#}^\infty(Y)^3)$  with  $\varphi = 0$  in  $\omega \times T$ , satisfying the divergence conditions  $\operatorname{div}_{x'} \int_Y \varphi' dy = 0$  in  $\omega$ ,  $\int_Y \varphi' dy \cdot n = 0$  on  $\partial\omega$ , and  $\operatorname{div}_y \varphi = 0$  in  $\omega \times Y$ .

Passing to the limit in (3.66) by using convergences (3.35) and (3.45), we take into account that the second term converges to

$$\int_{\omega \times Y} \tilde{P} \operatorname{div}_{x'} \varphi' dx' dy.$$

Since  $\tilde{P}$  does not depend on  $y$ , by using the divergences conditions  $\operatorname{div}_{x'} \int_Y \varphi' dy = 0$ , we have

$$\int_{\omega \times Y} \tilde{P} \operatorname{div}_{x'} \varphi' dx' dy = \int_{\omega} \tilde{P} \operatorname{div}_{x'} \left( \int_Y \varphi' dy \right) dx' = 0.$$

Thus, we deduce that the variational inequality (3.66) converges to the following one

$$\eta_0 \int_{\omega \times Y} \mathbb{D}_y[\hat{u}] : \mathbb{D}_y[\varphi] dx' dy = \int_{\omega \times Y} f' \cdot \varphi' dx' dy,$$

which by density arguments it holds for every  $\varphi \in \mathcal{V}$  with  $\mathcal{V}$  given by (3.56). Proceeding similarly to the end of *Step 1* of the proof of Theorem 2.1, this variational formulation is equivalent to the system (3.52) with the linear viscosity  $\eta_0$ , which admits a unique solution  $(\hat{u}, \hat{\pi}, \tilde{P}) \in L^2(\omega; H^1_{\#}(Y_f)^3) \times L^2(\omega; L^2_{0,\#}(Y_f)) \times (L^2_0(\omega) \cap H^1(\omega))$ . Then, the limits do not depend on the subsequence.

Finally, reasoning as in *Step 2* of the proof of Theorem 2.1, we get the linear effective 2D Darcy's law (2.15), which concludes the proof of Theorem 2.5.  $\square$

## 4 Numerical simulations of the effective models

In this section, we perform a numerical study of the asymptotic behaviour of a flow of a Carreau fluid between two parallel plates, separated by a thin layer of porous medium, as described in Section 2.

We assume that the flow is driven by a constant pressure gradient. This amounts to imposing a constant external force  $f = (f', 0)$  with  $f' \in \mathbb{R}^2$ , which is a realistic assumption that is used in many applications such as enhanced oil recovery [21, Chapter 4]. For simplicity, we also assume that  $\omega$  is the unit square  $\omega = (-1, 1)^2$  and impose periodic boundary conditions on  $\partial Q_\varepsilon = \partial\omega \times (0, \varepsilon)$ . System (2.3) is thus rewritten

$$\left\{ \begin{array}{l} -\varepsilon^\gamma \operatorname{div}(\eta_r(\mathbb{D}[u_\varepsilon])\mathbb{D}[u_\varepsilon]) + \nabla p_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \partial S_\varepsilon, \\ u_\varepsilon(x_1, -1) = u_\varepsilon(x_1, 1), \quad x_1 \in (-1, 1), \\ u_\varepsilon(-1, x_2) = u_\varepsilon(1, x_2), \quad x_2 \in (-1, 1), \end{array} \right. \quad (4.1)$$

As observed in [9, Section 4], Lemma 3.10 needs to be slightly modified in that case: conditions (3.42) and (3.43) are respectively replaced by

$$\operatorname{div}_{x'} \left( \int_0^1 \tilde{u}'(x', y_3) dy_3 \right) = 0 \quad \text{in } \omega, \quad (4.2)$$

$$\operatorname{div}_y \hat{u}(x', y) = 0 \quad \text{in } \omega \times Y_f, \quad \operatorname{div}_{x'} \left( \int_{Y_f} \hat{u}'(x', y) dy \right) = 0 \quad \text{in } \omega. \quad (4.3)$$

As a result, by the periodicity hypothesis on the flow, the boundary condition  $V' \cdot n' = 0$  does not hold anymore on  $\partial\omega$ . Hence, in this particular configuration, we will discuss numerical simulations of Darcy's laws of the form as (2.7), (2.10), (2.14) and (2.15), but without the aforementioned boundary condition on  $V'$ .

Since  $f'$  is constant, one gets that  $\tilde{P} \equiv 0$  and  $V'$  is also a constant vector in  $\mathbb{R}^2$ .

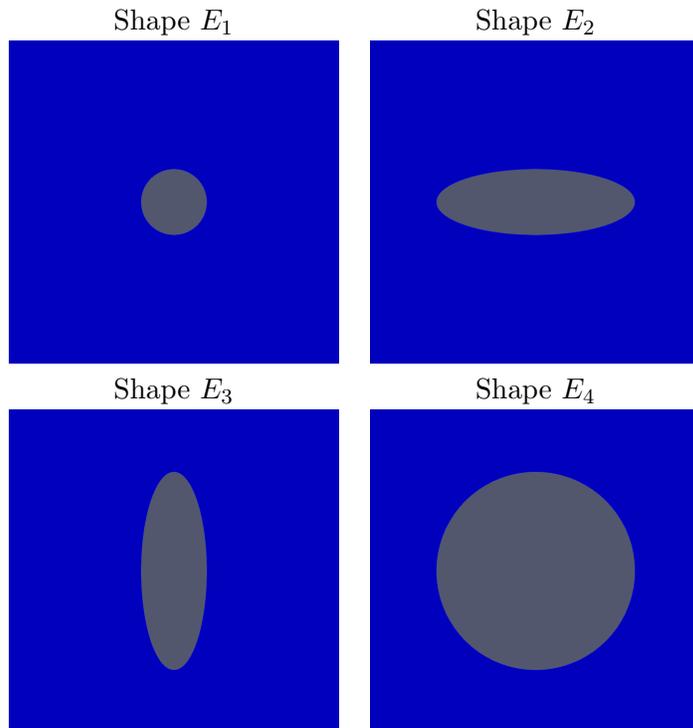


Figure 4: Representation of the 2D reference cells  $Y'$  used in the numerical simulations of the effective models. The inclusion  $T'$  (in grey) is surrounded by  $Y'_f$  (in blue). From left to right and top to bottom: disk or radius 0.1, ellipses of semi-major axis 0.3 and semi-minor axis 0.1, oriented respectively along the  $x$  and  $y$  directions, and disk of radius 0.3. These shapes are numbered  $E_1, E_2, E_3$  and  $E_4$ .

**Choice of rheological parameters.** Let us specify the range of parameters that we use in the numerical tests. In addition to the exponent  $\gamma \in \mathbb{R}$ , the model (4.1) depends on four rheological parameters:  $\eta_0, \eta_\infty, \lambda$  and  $r$ . Since in many applications (see for instance [21]),  $\eta_\infty$  is very small compared with  $\eta_0$ , we arbitrarily fix  $\eta_0 = 1$  and  $\eta_\infty = 10^{-3}$ . As regards  $\lambda$ , we take  $\lambda \in \{1, 10, 100\}$ . In the case where the effective model is nonlinear with respect to  $\lambda$ , the possibility of multiplying  $\lambda$  by a factor 10 from one simulation to another will give us access to a large panel of behaviours for the effective models. Finally, we consider a pseudoplastic case  $r = 1.7$ , the Newtonian case  $r = 2$  and two dilatant cases  $r = 2.3$  and  $r = 2.6$ .

**Shapes of inclusions  $T$ .** In order to illustrate the effect of a change of volume of the inclusion  $T$ , and the effect of anisotropy, we will consider four possible shapes for  $T'$ : two disks of respective radius 0.1 and 0.3, and the ellipse of semi-major axis 0.3 and semi-minor axis 0.1, parallel to the  $x$  or the  $y$  axis (see Fig. 4). These shapes will be numbered  $E_1, E_2, E_3$  and  $E_4$  in the rest of this section.

**Numerical resolution of the cell problems.** The solution of each cell problem of the form (2.9) or (2.12) is computed using a mixed formulation, that we solve by a finite element method, using FreeFem++ software [32]. In the nonlinear cases (2.12) where the viscosity  $\eta_r$  follows a Carreau law or a power law, we rely on a fixed point algorithm (see for instance [39, Section 2.8]). We consider

the Taylor-Hood approximation for the velocity-pressure pair, namely  $P_2$  elements for the velocity field and  $P_1$  elements for the pressure. This choice is well known to be compatible with the Babuška-Brezzi condition [31]. Each three-dimensional mesh of a cell  $Y_f$  is obtained by constrained Delaunay tetrahedralization, and contains approximately 8000 tetrahedra.

#### 4.1 Permeability tensor $\mathcal{A}$

In the cases where the effective system is described by a linear 2D Darcy law, the response of the fluid to a constant pressure gradient  $f'$  takes the form  $V' = \frac{1}{\eta} \mathcal{A} f'$ , where the constant viscosity  $\eta$  is either equal to  $\eta_0$  or  $\eta_\infty$ , depending on the values of  $r$  and  $\gamma$  (see Table 1). Hence, the asymptotic behaviour of the fluid is encoded in the permeability tensor  $\mathcal{A}$ .

In order to highlight certain properties of  $\mathcal{A}$ , we have summarized in Figure 5 the coefficients that we obtain numerically, for the different shape geometries  $E_1$  to  $E_4$ . We first notice that  $\mathcal{A}$  is perfectly symmetric, which comes from its very definition in the continuous setting. Indeed, testing against  $w^j$  in system (2.9) satisfied by  $w^i$ , or against  $w^i$  in the same system but satisfied by  $w^j$ , we obtain

$$\begin{aligned} \int_{Y_f} w_i^j(y) dy &= \int_{Y_f} \nabla_y w^i(y) : \nabla_y w^j(y) dy \\ &= \int_{Y_f} \nabla_y w^j(y) : \nabla_y w^i(y) dy \\ &= \int_{Y_f} w_j^i(y) dy, \end{aligned}$$

hence  $\mathcal{A}_{i,j} = \mathcal{A}_{j,i}$ .

One can also observe that, for isotropic inclusions  $E_1$  and  $E_4$ , up to numerical errors, the tensor  $\mathcal{A}$  is a diagonal matrix of the form  $aI$ . This means that, as expected, the filtration velocity  $V'$  is simply given by the product between  $f'$  and the positive constant  $a$  for such geometries. Also, the value of  $a$  appears to be a decreasing function of the area of the obstacle, which is quite intuitive as well. In the case of anisotropic geometries  $E_2$  and  $E_3$ ,  $\mathcal{A}$  is still diagonal, but its diagonal coefficients are not equal. Since  $E_3$  is obtained by applying a rotation of angle  $\pi/2$  to  $E_2$ , by symmetry, the associated matrix  $\mathcal{A}$  is the transposed of the one associated with shape  $E_2$ .

However, for more general geometries of inclusions, the permeability tensor  $\mathcal{A}$  is no longer symmetric. For instance, we have given in Fig. 6 the coefficients of  $\mathcal{A}$  computed when  $E_2$  is rotated by an angle  $\theta \in \{\pi/16, \pi/8, \pi/4\}$ . As expected, the diagonal coefficients  $\mathcal{A}_{1,2}$ ,  $\mathcal{A}_{2,1}$  are not equal to zero in such configurations.

#### 4.2 Carreau law ( $\gamma = 1$ )

In the case  $\gamma = 1$ , the behaviour of the effective model associated with a pseudoplastic fluid has been studied numerically in [9]. In this subsection, we complete these numerical results by considering dilatant fluids as well. In order to perform comparisons, we use the same parameters as in [9], namely  $\eta_0 = 1$ ,  $\eta_\infty = 10^{-3}$ , and  $\lambda \in \{1, 10, 100\}$ . We consider dilatant fluids with  $r \in \{2.3, 2.6\}$ , a Newtonian fluid ( $r = 2$ ) and a pseudoplastic fluid ( $r = 1.7$ ).

We explore numerically the influence of the amplitude of  $f'$  and of its orientation, on the computed value of the filtration velocity  $V'$ .

Inclusion	Permeability tensor $\mathcal{A}$
$E_1$	$\begin{pmatrix} 0.0697955 & -3.17061 \times 10^{-6} \\ -3.17061 \times 10^{-6} & 0.0697947 \end{pmatrix}$
$E_2$	$\begin{pmatrix} 0.054708 & -2.3436 \times 10^{-6} \\ -2.3436 \times 10^{-6} & 0.0210978 \end{pmatrix}$
$E_3$	$\begin{pmatrix} 0.0211078 & -5.69438 \times 10^{-7} \\ -5.69438 \times 10^{-7} & 0.0547038 \end{pmatrix}$
$E_4$	$\begin{pmatrix} 0.0153292 & -3.7408 \times 10^{-7} \\ -3.7408 \times 10^{-7} & 0.0153284 \end{pmatrix}$

Figure 5: Permeability tensor  $\mathcal{A}$  computed numerically in the case of an elliptic inclusion ( $E_1$  to  $E_4$ ).

Rotation angle $\theta$	Permeability tensor $\mathcal{A}$
$\pi/16$	$\begin{pmatrix} 0.0534164 & 0.00341334 \\ 0.00341334 & 0.0225729 \end{pmatrix}$
$\pi/8$	$\begin{pmatrix} 0.0498291 & 0.00653147 \\ 0.00653147 & 0.0266649 \end{pmatrix}$
$\pi/4$	$\begin{pmatrix} 0.0385604 & 0.00963438 \\ 0.00963438 & 0.0385636 \end{pmatrix}$

Figure 6: Permeability tensor  $\mathcal{A}$  computed numerically in the case where the  $E_2$  inclusion is rotated of an angle  $\theta$  with respect to the  $x_1$  axis.

#### 4.2.1 Influence of the amplitude of the pressure gradient $f'$

We impose a pressure gradient  $f'$  directed by  $e_1$ , i.e. of the form  $f' = (f_1, 0)$ , with  $f_1 \in [0, 1]$ . In that case, the computed filtration velocity  $V'$  is also directed by  $e_1$  and reads  $V' = (V_1, 0)$ . Fig. 7 represents  $V_1$  as a function of  $f_1$ , for the different obstacle shapes  $E_1$  top  $E_4$  and the choice of parameters  $r, \lambda$  specified above.

We observe that, for any choice of  $r$  and  $\lambda$ ,  $V_1$  is an increasing function of  $f_1$ . However, for  $r = 1.7$  (pseudoplastic case),  $V_1$  appears as a convex function of  $V_1$ , whereas for  $r > 2$  (dilatant case), it is a concave function of  $V_1$ . For  $r = 2$  (Newtonian case), the dependency on  $f_1$  is linear, as expected. Also, the separation between the curves gets more pronounced as  $\lambda$  increases, which comes from the fact that  $\lambda$  is the coefficient in front of the nonlinear term  $|D(u)|^{r-2}$  in the definition of the viscosity following the Carreau law (1.2). For any values of  $\lambda$ , and any geometry of obstacle shape  $E_1$  to  $E_4$ , for a given pressure gradient  $f'$ , the amplitude of the filtration velocity  $V'$  diminishes as the exponent  $r$  increases. This is coherent with the fact that, for high values of the shear rate, the viscosity of the dilatant fluid increases, whereas the behaviour of pseudoplastic fluids is the opposite.

#### 4.2.2 Influence of the orientation of $f'$

In order to test the impact of a rotation of  $f'$  on the behaviour of the effective system, we consider the anisotropic shape  $E_2$  and a family of pressure gradients  $f' = (\cos \theta, \sin \theta)$ , with the angle  $\theta \in [0, \pi/2]$ . The results that we obtain are represented in Fig. 8.

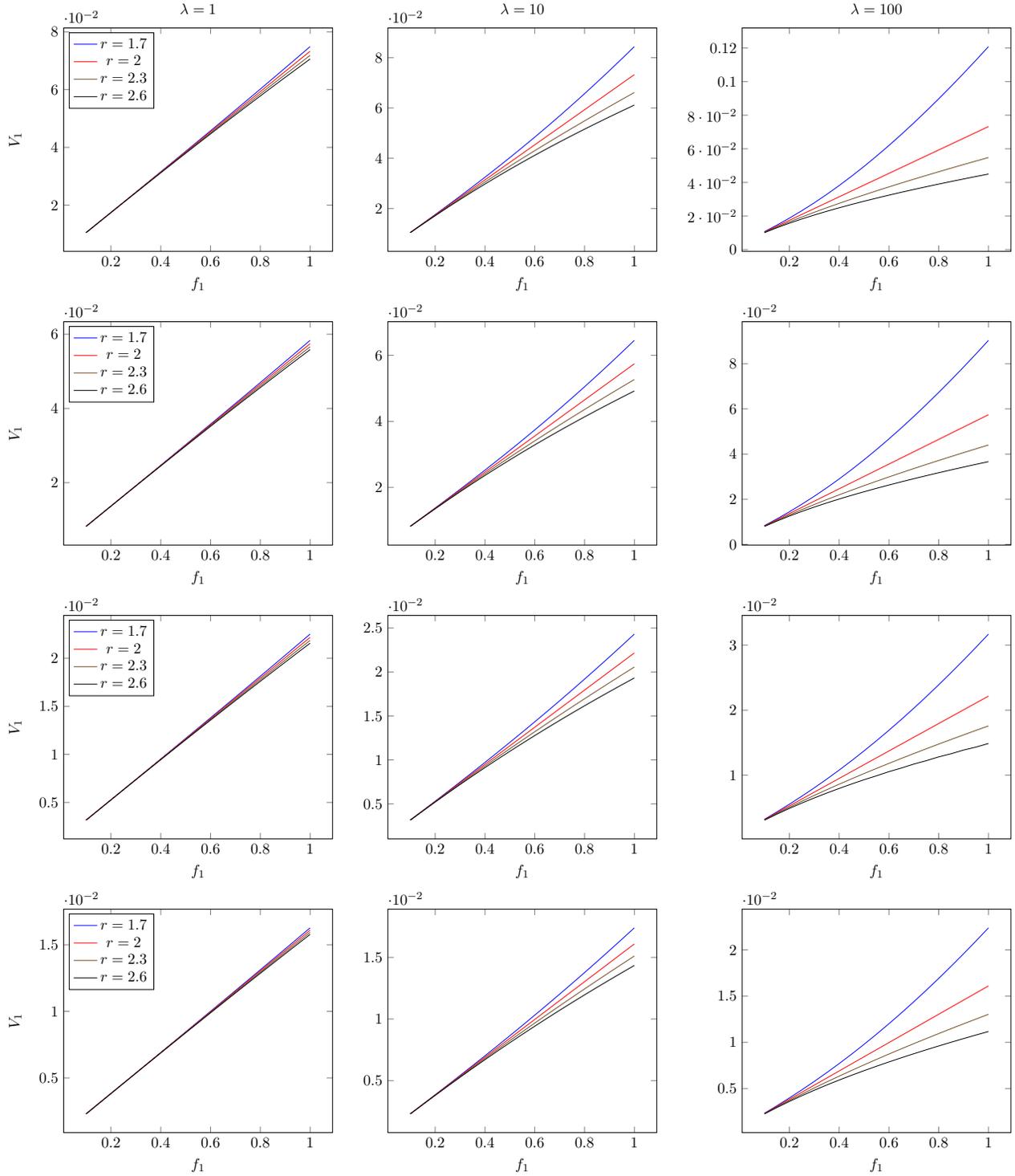


Figure 7: Case  $\gamma = 1$  (Carreau law). Component  $V_1$  of the mean filtration velocity  $V'$  plotted against  $f_1$ , with  $f' = (f_1, 0)$ , for  $r \in \{1.7, 2, 2.3, 2.6\}$ , in the case of elliptic inclusions  $E_1$  (first line),  $E_2$  (second line),  $E_3$  (third line) and  $E_4$  (fourth line). From left to right:  $\lambda = 1, \lambda = 10, \lambda = 100$ .

We notice that the orientation of  $V'$ , which is generally not parallel to the pressure gradient  $f'$  due to the anisotropy of the obstacle, does not appear to depend on the rheological parameters  $r, \lambda$ . In all the simulated configurations, the orientation of  $V'$  remains very close to what is observed in the Newtonian case  $r = 2$ . As regards the amplitude of  $V'$ , as observed in the previous paragraph, there is almost no observable effect for  $\lambda = 1$ . This can be explained by the fact that, for small values of  $\lambda$ , and an imposed pressure gradient of fixed size  $|f'| = 1$ , the viscosity described by the Carreau law is close to the constant value  $\eta_r = \eta_0$  corresponding to the Newtonian case  $r = 2$ . On the contrary, for  $\lambda = 100$ , the amplitude of  $V'$  is noticeably reduced as  $r$  increases: for instance, for  $r = 1.7$ , the maximal filtration velocity is about 0.085 while it reaches only about 0.035 for  $r = 2.6$ .

### 4.3 Power law ( $r > 2, \gamma > 1$ )

In order to allow for comparisons, we perform similar simulations for the power law regime ( $r > 2, \gamma > 1$ ) as we did for the Carreau regime ( $\gamma = 1$ ), taking  $\lambda \in \{1, 10, 100\}$  and  $r \in \{2.3, 2.6\}$ , to separately test the influence of the amplitude of  $f'$ , and of the angle formed by  $f'$  and the  $x_1$  axis. In each case, we provide the Newtonian behaviour corresponding to  $r = 2$ , as a reference model.

#### 4.3.1 Influence of the amplitude of $f'$

We have plotted in Fig. 9 the horizontal component of  $V'$  as a function of  $f_1$ , when  $f'$  takes the form  $f' = (f_1, 0)$ . Contrary to what we can observe in Fig. 7, for a given value of  $f_1$ , the order between the computed value of  $V_1$  for  $r$  in  $\{2, 2.3, 2.6\}$  depends on the choice of  $\lambda$ : the behaviour of the effective system is no longer monotonous with respect to  $r$ . In particular, for  $\lambda = 100$ , the different curves intersect for a specific value of  $f_1$ , which seems to be unique and depends on the shape of the obstacle. For instance, the  $f_1$  component of this intersection point is between 0.3 and 0.4 for  $E_1$  and close to 0.5 for  $E_4$ . Moreover, when  $f_1$  exceeds this value, we recover a behaviour that is very similar to what appeared in Fig. 7: for  $r \in \{2.3, 2.6\}$ ,  $V_1$  is an increasing concave function of  $f_1$ , and  $V_1$  is smaller for  $r = 2.6$  than for  $r = 2.3$ .

We may interpret these features as follows. When  $|f'|$  is small and  $\eta_r$  follows the power law  $\eta_r(\mathbb{D}_y[w_{\xi'}]) = |\mathbb{D}_y[w_{\xi'}]|^{r-2}$ , the deformation rate tensor  $D_y[w_{f'}]$  associated to the solution  $w_{f'}$  of system (2.12) (with  $\xi' = f'$ ) also has a small amplitude. Thus, as mentioned in the Introduction, the power law is not well-suited to capture the behaviour of a quasi-Newtonian fluid in such regime. On the opposite, for high values of parameter  $\lambda$  such as  $\lambda = 100$  in our simulations, and  $f_1$  large enough, the qualitative behaviour of the nonlinear 2D Darcy law associated with the Carreau law or with the power law become very similar, since the Carreau law behaves as a power law for large values of the deformation rate.

#### 4.3.2 Influence of the orientation of $f'$

Finally, we have represented in Fig. 10 the vector  $V'$  computed for different orientations of the imposed pressure gradient  $f'$ , similarly as in Fig. 8. We can observe for  $\lambda = 100$  and  $r \in \{2.3, 2.6\}$  a very similar behaviour of the effective systems (2.10) and (2.14), which seems to confirm the above interpretation. The differences between both systems appear for  $\lambda = 1$  and concern only the amplitude of  $V'$ , which increases as  $r$  increases in the case of the power law, while it was not affected by variations of  $r$  in the case of Carreau law and for this particular value of  $\lambda$ .

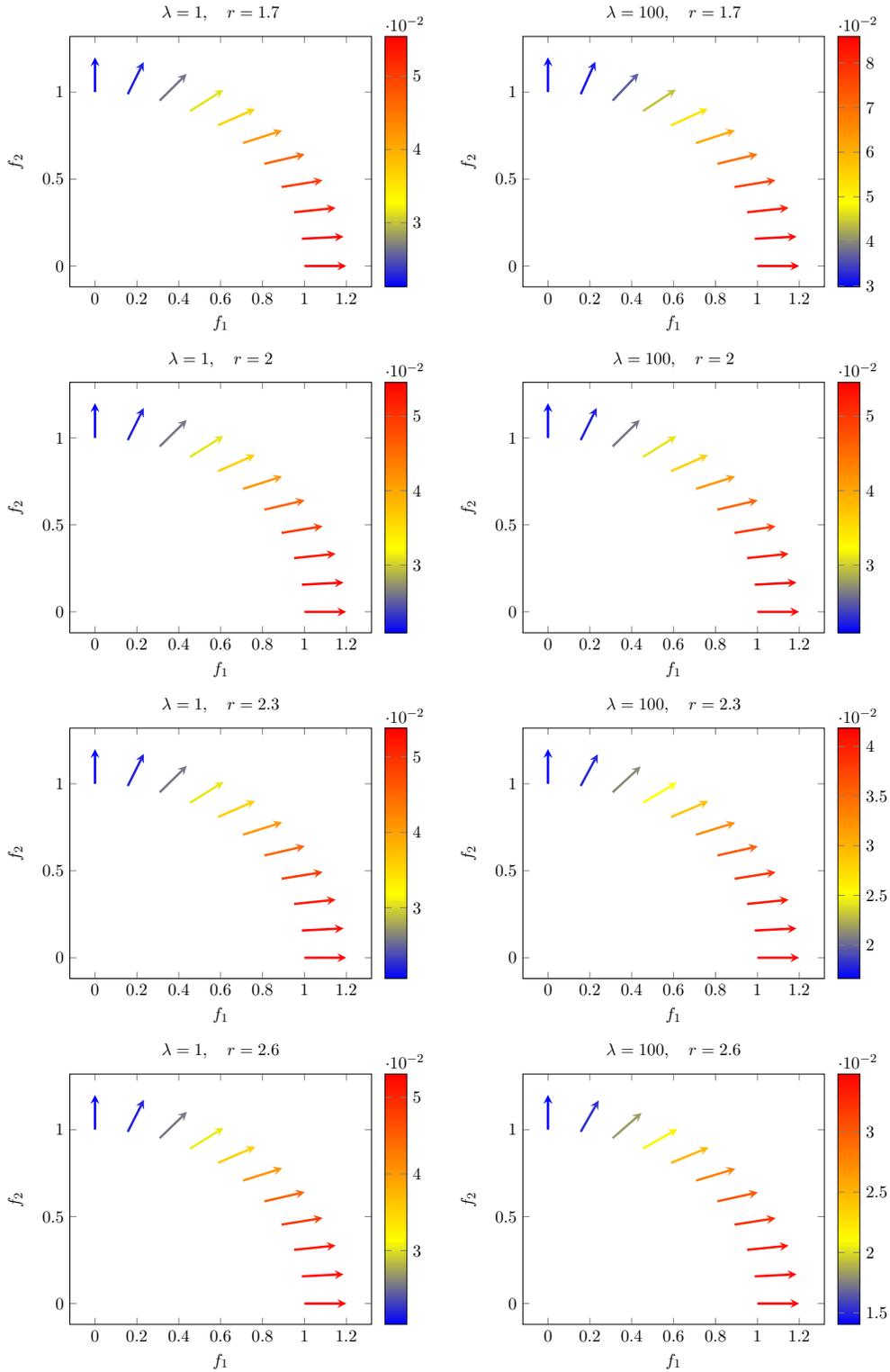


Figure 8: Case  $\gamma = 1$  (Carreau law). Representation of  $V'$  when  $f'$  is a unit vector of the form  $(f_1, f_2) = (\cos\theta, \sin\theta)$  with  $\theta \in [0, \pi/2]$ , in the case of an elliptic inclusion  $E_2$ . Each vector  $V'$  is represented by a vector of length 0.2, localized at point  $(f_1, f_2)$  and colored according to  $|V'|$ . The left column corresponds to  $\lambda = 1$  and the right one to  $\lambda = 100$ . Each line from top to bottom corresponds respectively to  $r = 1.7$ ,  $r = 2$ ,  $r = 2.3$  and  $r = 2.6$ .<sup>36</sup>

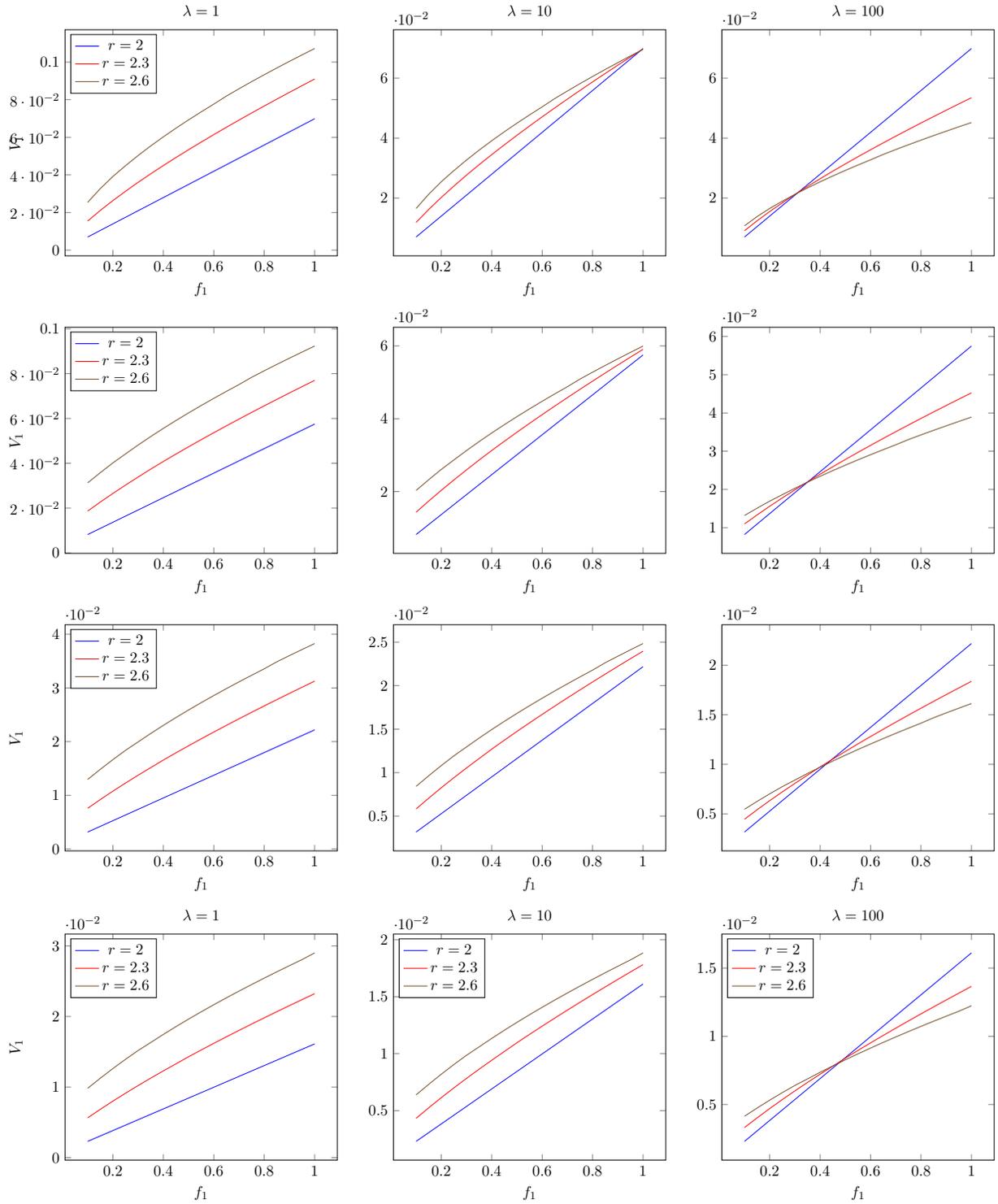


Figure 9: Case  $\gamma > 1$ ,  $r > 1$  (power law). Component  $V_1$  of the mean filtration velocity  $V'$  plotted against  $f_1$ , with  $f' = (f_1, 0)$ , for  $r \in \{2, 2.3, 2.6\}$  and elliptic inclusions  $E_1$  (first line),  $E_2$  (second line),  $E_3$  (third line) and  $E_4$  (fourth line). From left to right:  $\lambda = 1$ ,  $\lambda = 10$ ,  $\lambda = 100$ .

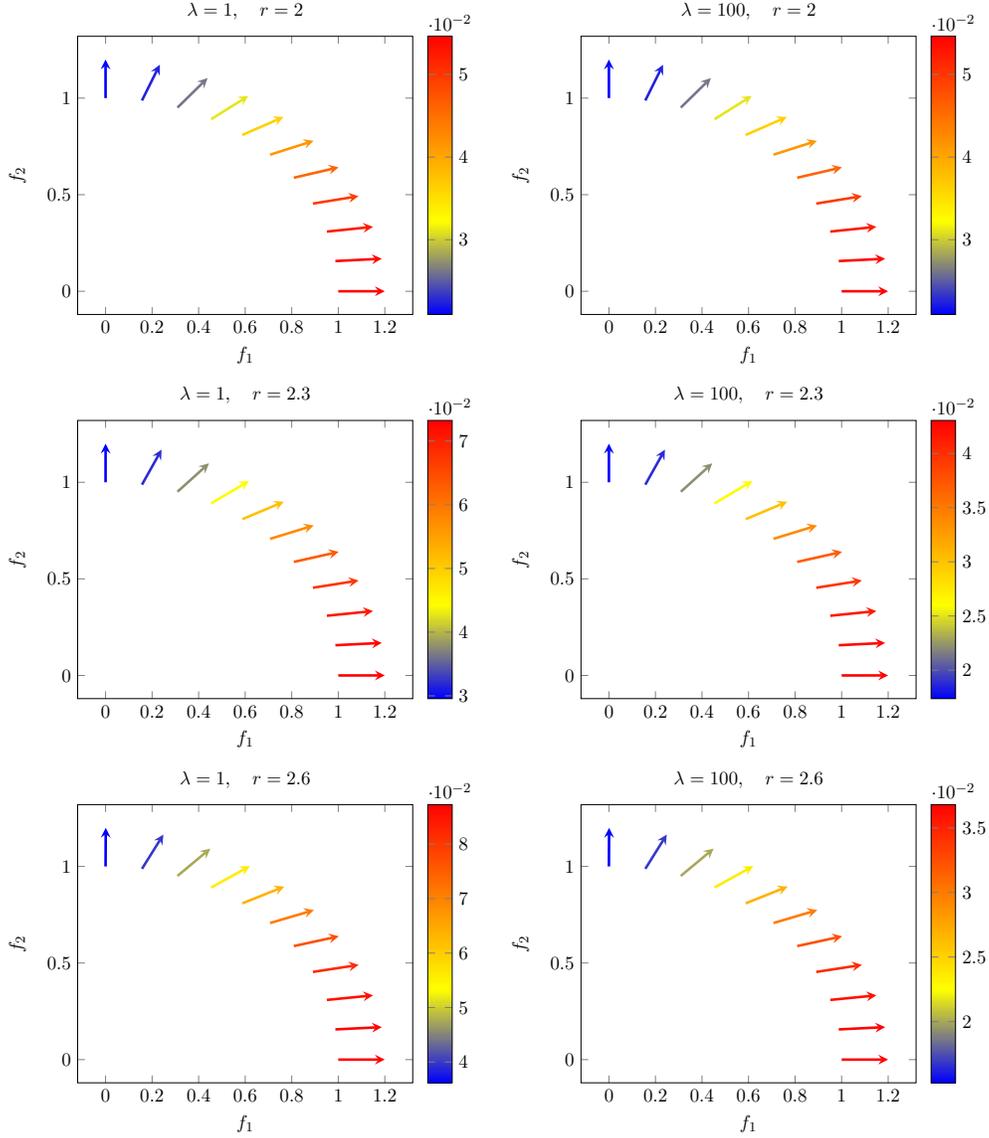


Figure 10: Case  $\gamma > 1$ ,  $r > 2$  (power law). Representation of  $V'$  when  $f' = (f_1, f_2)$  takes the form  $(f_1, f_2) = (\cos \theta, \sin \theta)$  with  $\theta \in [0, \pi/2]$ , in the case of the obstacle shape  $E_2$ . As in Fig. 8, each vector  $V'$  is represented by a vector of length 0.2, localized at point  $(f_1, f_2)$  and colored according to  $|V'|$ . The left column corresponds to  $\lambda = 1$ , the right one to  $\lambda = 100$ , and each line from top to bottom corresponds respectively to  $r = 2$ ,  $r = 2.3$  and  $r = 2.6$ .

**Acknowledgments:** The main idea of this paper was developed while María Anguiano and Francisco J. Suárez-Grau were visiting the Laboratoire Jacques-Louis Lions (in July 2022), at Sorbonne Université, Université Paris Cité and Centre National de la Recherche Scientifique in Paris (France), invited by Dr. Matthieu Bonnivard in order to develop a joint contribution on the thematic of the project “Análisis, homogeneización y simulación numérica de EDPs con origen en mecánica de fluidos”, financially supported by the Laboratoire Jacques-Louis Lions. María Anguiano and Francisco J. Suárez-Grau would like to thank people from this Laboratoire for their very kind hospitality and especially Dr. Matthieu Bonnivard.

## References

- [1] G. Allaire, One-phase Newtonian flow, in: *Homogenization and Porous Media, Interdisciplinary Applied Mathematics Series*, **6**, Springer-Verlag, New York, (1997), 77-94.
- [2] J.F. Agassant, P. Avenas, M. Vincent, B. Vergnes, P.J. Carreau, Polymer Processing, Principles and Modelling, Carl Hanser Verlag, Munich, 2017.
- [3] M. Anguiano, Darcy’s laws for non-stationary viscous fluid flow in a thin porous medium, *Math. Meth. Appl. Sci.*, **40**, No. 8 (2017), 2878–2895.
- [4] M. Anguiano, On the non-stationary non-Newtonian flow through a thin porous medium, *ZAMM-Z. Angew. Math. Mech.*, **97**, (2017), 895–915.
- [5] M. Anguiano, Derivation of a quasi-stationary coupled Darcy-Reynolds equation for incompressible viscous fluid flow through a thin porous medium with a fissure, *Math. Meth. Appl. Sci.*, **40**, (2017), 4738-4757.
- [6] M. Anguiano, Homogenization of a non-stationary non-Newtonian flow in a porous medium containing a thin fissure, *Eur. J. Appl. Math.*, **30** (2), (2019), 248–277.
- [7] M. Anguiano, Reaction-Diffusion Equation on Thin Porous Media, *Bull. Malays. Math. Sci. Soc.*, **44**, (2021), 3089-3110.
- [8] M. Anguiano, On p-Laplacian Reaction–Diffusion Problems with Dynamical Boundary Conditions in Perforated Media, *Mediterr. J. Math.*, (2023), 20:124.
- [9] M. Anguiano, M. Bonnivard and F. J. Suárez-Grau, Carreau law for non-Newtonian fluid flow through a thin porous media, *Q. J. Mech. Appl. Math.*, **75**(1), (2022), 1-27.
- [10] M. Anguiano and R. Bunoiu, On the flow of a viscoplastic fluid in a thin periodic domain, in *Integral Methods in Science and Engineering* (eds. C. Constanda and P. Harris), Birkhäuser, Cham, (2019), 15–24.
- [11] M. Anguiano and R. Bunoiu, Homogenization of Bingham flow in thin porous media, *Netw. Heterog. Media*, **15**, No. 1 (2020), 87–110.
- [12] M. Anguiano and F.J. Suárez-Grau, Homogenization of an incompressible non-newtonian flow through a thin porous medium, *Z. Angew. Math. Phys.*, (2017), 68:45.

- [13] M. Anguiano and F.J. Suárez-Grau, Derivation of a coupled Darcy-Reynolds equation for a fluid flow in a thin porous medium including a fissure *Z. Angew. Math. Phys.*, (2017), 68: 52.
- [14] M. Anguiano and F.J. Suárez-Grau, Analysis of the effects of a fissure for a non-Newtonian fluid flow in a porous medium, *Communications in Mathematical Sciences*, **16**, (2018), 273 – 292.
- [15] M. Anguiano and F.J. Suárez-Grau, The transition between the Navier-Stokes equations to the Darcy equation in a thin porous medium, *Mediterr. J. Math.*, (2018), 15:45.
- [16] M. Anguiano and F.J. Suárez-Grau, Newtonian fluid flow in a thin porous medium with non-homogeneous slip boundary conditions *Networks and Heterogeneous Media*, **14**, (2019) 289-316.
- [17] M. Anguiano and F.J. Suárez-Grau, Lower-dimensional nonlinear Brinkman’s law for non-Newtonian flows in a thin porous medium, *Mediterr. J. Math.*, (2021) 18: 175.
- [18] A. Armiti-Juber, On the limit of a two-phase flow problem in thin porous media domains of Brinkman type, *Math Meth Appl Sci.*, 45 (2022) 2563-2581.
- [19] J. Baranger and K. Najib, Analyse numerique des ecoulements quasi-Newtoniens dont la viscosite obeit a la loi puissance ou la loi de carreau, *Numer. Math.*, **58** (1990), 35-49.
- [20] H.A. Barnes, J.F. Hutton and K. Walters, *An introduction to rheology*, Amsterdam, Elsevier, 1989.
- [21] R.B. Bird, R.C. Armstrong and O. Hassager, *Dynamics of Polymeric Liquids, Volume 1: Fluid Mechanics*, Wiley, 1977.
- [22] F. Boughanim and R. Tapiéro, Derivation of the two-dimensional Carreau law for a quasi-newtonian fluid flow through a thin slab, *Appl. Anal.*, **57**, (1995), 243–269.
- [23] A. Bourgeat and A. Mikelić, Homogenization of a polymer flow through a porous medium, *Nonlinear Anal.*, **26**, (1996), 1221–1253.
- [24] A. Bourgeat, O. Gipouloux and E. Marušić-Paloka, Filtration law for polymer flow through porous media, *Multiscale Model. Sim.*, **1**, (2003), 432–457.
- [25] R. Bunoiu and C. Timofte, Upscaling of a double porosity problem with jumps in thin porous media, *Applicable Analysis*, **101** (2020), 1–18.
- [26] R. Bunoiu, K. Ramdani and C. Timofte, Upscaling of a double porosity problem with jumps in thin porous media, *Electronic Journal of Differential Equations*, **2021** (2021), 1–22.
- [27] D. Cioranescu, A. Damlamian and G. Griso, The periodic unfolding method in homogenization, *SIAM J. Math. Anal.*, **40**, No. 4 (2008), 1585–1620.
- [28] D. Cioranescu and P. Donato, *An introduction to homogeneization*, Oxford Lecture Series in Mathematics and its Applications, **17**, Oxford University Press, 1999.
- [29] D. Cioranescu, A. Damlamian and G. Griso, *The periodic unfolding method: theory and applications to partial differential problems*, Series in Contemporary Mathematics, **3**, Springer, Singapore, 2018.

- [30] J. Fabricius, J.G. I. Hellström, T.S. Lundström, E. Miroshnikova and P. Wall, Darcy’s law for flow in a periodic thin porous medium confined between two parallel plates, *Transp. Porous Med.*, **115**, (2016), 473–493.
- [31] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Springer Publishing Company, Incorporated (2011).
- [32] F. Hecht, New Development in FreeFem++, *J. Numer. Math.* **20** (2012) 251–265.
- [33] N. F. Jouybari and T. S. Lundström, Investigation of Post-Darcy Flow in Thin Porous Media, *Transport in Porous Media*, **138**, (2021), 157–184.
- [34] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.
- [35] A. Mikelić, Non-Newtonian Flow, in: *Homogenization and Porous Media, Interdisciplinary Applied Mathematics Series*, **6**, Springer-Verlag, New York, (1997), 45-68.
- [36] A. Mikelić, An introduction to the homogenization modeling of non-Newtonian and electrokinetic flows in porous media, in: *Non-Newtonian Fluid Mechanics and Complex Flows. Lecture Notes in Mathematics*, **2212**, Springer, 2018.
- [37] L. Pereira and B. Lecampion, A plane-strain hydraulic fracture driven by a shear-thinning Carreau fluid, *J. Numer. Anal. Meth. Geomech* (2021).
- [38] M. Prat and T. Agaësse, Thin Porous Media, in *Handbook of Porous Media* (ed. Kambiz Vafai), CRC PressEditors, (2015), 89-112.
- [39] P. Saramito, *Complex fluids: Modeling and Algorithms, Mathématiques et Applications*, Springer, 2016.
- [40] L. Scholz and C. Bringedal, A Three-Dimensional Homogenization Approach for Effective Heat Transport in Thin Porous Media, *Transport in Porous Media*, 141 (2022) 737-769.
- [41] F.J. Suárez-Grau, Mathematical modeling of micropolar fluid flows through a thin porous medium, *J. Eng. Math.*, 126: 7 (2021)
- [42] F.J. Suárez-Grau, Theoretical derivation of Darcy’s law for fluid flow in thin porous media, *Math. Nachr.*, 295 (2021) 607-623.
- [43] M. Wrobel, On the application of simplified rheological models of fluid in the hydraulic fracture problems, *International Journal of Engineering Science*, **150** (2020) 103275.
- [44] M. Wrobel, G. Mishuris and P. Papanastasiou, On the influence of fluid rheology on hydraulic fracture, *International Journal of Engineering Science*, **158** (2021) 103426.
- [45] L. Yeghiazarian, K. Pillai and R. Rosati, Thin Porous Media, *Transp. Porous Med.*, **115**, (2016), 407-410.
- [46] Y. Zhengan and Z. Hongxing, Homogenization of a stationary Navier-Stokes flow in porous medium with thin film, *Acta Math. Sci.*, **28**, (2008), 963–974.