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Transonic limit of traveling waves of the Euler-Korteweg system

Marc-Antoine Vassenet *

December 5, 2022

Abstract

We prove the convergence in the transonic limit of two-dimensional traveling waves of the E-K system, up to rescaling, toward a ground state of the Kadomtsev-Petviashvili Equation. Similarly, in dimension one we prove the convergence in the transonic limit of solitons toward the soliton of the Korteweg de Vries equation.

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1 Introduction

The Euler Korteweg system, in dimension d , reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + u \cdot \nabla u + \nabla g(\rho) = \nabla \left(K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right), \end{cases} \quad t \in \mathbb{R}, x \in \mathbb{R}^d, \quad (\text{E-K})$$

where $\rho > 0$ is the density of the fluid, $u \in \mathbb{R}^d$ is its velocity field, the right hand side of the second line is the capillary tensor. The functions K, g are defined on \mathbb{R}_*^+ and smooth, the function K is positive. When the velocity is irrotational i.e $u = \nabla \phi$ for some ϕ that cancels at infinity, the momentum equation rewrites

$$\partial_t \phi + \frac{|\nabla \phi|^2}{2} + g(\rho) = K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2.$$

There is a formally conserved energy

$$E(\rho, \phi) = \int_{\mathbb{R}^d} \frac{K(\rho) |\nabla \rho|^2 + \rho |\nabla \phi|^2}{2} + G(\rho) dx, \quad (1.1)$$

where G is a primitive of g with a later specified integration constant. Moreover we have a momentum

$$P(\rho, \phi) = \int_{\mathbb{R}^d} (\rho - \rho_0) \partial_1 \phi dx. \quad (1.2)$$

which makes sense when $\rho - \rho_0 \in L^2, \nabla \phi \in L^2$. The energy makes sense for (ρ, u) localized near the constant state $(\rho_0, 0)$, which will be our framework. We call traveling wave a solution of (E-K) of the form

$$(\rho(x \cdot \mathbf{n} - ct, x_\perp), u(x \cdot \mathbf{n} - ct, x_\perp)), \quad x_\perp = (x_1, x_2, \dots, x_d) - x \cdot \mathbf{n},$$

where c is the speed of propagation and \mathbf{n} the direction of the speed. The direction of the speed does not matter, thus we let $\mathbf{n} = \mathbf{e}_1$. A traveling wave solves

$$\begin{cases} -c \partial_1 \rho + \operatorname{div}(\rho \nabla \phi) = 0, \\ -c \partial_1 \phi + g(\rho) - K(\rho) \Delta(\rho) + \frac{|\nabla \phi|^2}{2} - K'(\rho) \frac{|\nabla \rho|^2}{2} = 0. \end{cases} \quad (1.3)$$

In [1], Audiard proves the existence of traveling waves in dimension two, localized near the constant state $(\rho_0, 0)$ with $g(\rho_0) = 0, g'(\rho_0) > 0$. Their speed is close but less than the speed of sound that we define now. When neglecting the capillary tensor and linearizing this system near $(\rho_0, 0)$ i.e. $\rho = \rho_0 + r$, we obtain the Euler equation

$$\begin{cases} \partial_t r + \rho_0 \operatorname{div}(u) = 0, \\ \partial_t u + g'(\rho_0) \nabla r = 0. \end{cases} \quad (1.4)$$

The speed of sound is

$$c_s = \sqrt{\rho_0 g'(\rho_0)}.$$

For simplification, we use the following rescaling

$$\begin{aligned} (\rho, \phi) &= \left(\rho_0 \rho_r \left(\sqrt{\frac{g'(\rho_0)}{\rho_0}} x \right), \phi_r \left(\sqrt{\frac{g'(\rho_0)}{\rho_0}} x \right) \right), K_r(\rho_r) = \frac{K(\rho_0 \rho_r)}{\rho_0}, \\ g_r(\rho_r) &= \frac{g(\rho_0 \rho_r)}{g'(\rho_0) \rho_0}, c_r = \frac{c}{\sqrt{\rho_0 g'(\rho_0)}}. \end{aligned}$$

Then (1.3) becomes

$$\begin{cases} -c_r \partial_1 \rho_r + \operatorname{div}(\rho_r \nabla \phi_r) = 0, \\ -c_r \partial_1 \phi_r + g_r(\rho_r) - K_r(\rho_r) \Delta(\rho_r) + \frac{|\nabla \phi_r|^2}{2} - K_r'(\rho_r) \frac{|\nabla \rho_r|^2}{2} = 0. \end{cases} \quad (1.5)$$

In this system the constant state is 1, $g_r(1) = 0$, $g_r'(1) = 1$ and the speed of sound is $\sqrt{1g_r'(1)} = 1$. We will forget the subscript r and focus on the rescaled system, i.e. we will assume through the rest of the paper that

$$g(1) = 0, g'(1) = 1 \text{ and } c_s = 1. \quad (1.6)$$

Solutions of (1.5) with speed near the speed of sound are known to exist, the precise existence statement of [1] is the following:

Theorem 1. (*[1] Theorem 1.1, proposition 3.3 and proposition 2.3*) *In dimension two, under the assumption $\Gamma := 3 + g''(1) \neq 0$, there exists $p_0 > 0$ such that for any $0 \leq p \leq p_0$ we have $(\rho_p, \phi_p) \in \cap_{j \geq 0} (1 + H^j) \times \dot{H}^{j+1}$, solution of (1.5) for some $c_p > 0$, with $P(\rho_p, \phi_p) = p$.*

Moreover there exists $\alpha, \beta, C > 0$ such that, for any $0 \leq p \leq p_0$

$$p - \beta p^3 \leq E(\rho_p, \phi_p) \leq p - \alpha p^3, \quad (1.7)$$

$$1 - \beta p^2 \leq c_p \leq 1 - \alpha p^2, \quad (1.8)$$

$$\|\rho_p - 1\|_\infty \leq C \sqrt{E(\rho_p, \phi_p)} \quad \text{for } p \ll 1. \quad (1.9)$$

$$\|\rho_p - 1\|_\infty \geq C p^2 \quad \text{for } p \ll 1. \quad (1.10)$$

Remark 1.1. *As has been proven for the Schrödinger equation (see [14]), it is possible that such solitons exist in higher dimensions.*

To construct the traveling waves of theorem 1 the author in [1] solves a minimization problem. On the space $\mathcal{H} = \left\{ (\rho, \phi) \in (1 + H^1) \times \dot{H}^1 \right\}$ the momentum is well-defined, however the energy (1.1) does not make sense. For example the term $\rho |\nabla \phi|^2$ is not necessarily integrable. The solution is to work with a modified energy \tilde{E} which has nice coercive properties and such that

$\tilde{E} = E$ if $|\rho - 1| \ll 1$. Then the author finds (ρ, ϕ) solution of the minimization problem

$$\inf \left\{ \tilde{E}(\rho, \phi), (\rho, \phi) \in \mathcal{H} : P(\rho, \phi) = p \right\}, \quad (1.11)$$

for small p , such that the minimizer is smooth and satisfies $|\rho - 1| \ll 1$. Our aim is to describe the asymptotic behaviour, as $p \rightarrow 0$, of the traveling waves (ρ_p, ϕ_p) . It is instructive to compare our problem to the extensive litterature studying the nonlinear Schrödinger equation. Indeed in the case $K = \kappa/\rho$, with κ a positive constant, up to a rescaling there exists a formal correspondance with the nonlinear Schrödinger equation

$$i\partial_t \Psi + \Delta \Psi = g(|\Psi|^2)\Psi, \quad (\text{NLS})$$

using the *Madelung transform* $(\rho, \nabla \phi) \mapsto \Psi := \sqrt{\rho}e^{i\phi}$ (see [8] for more details). In the case $g(\rho) = \rho - 1$ (NLS) is called the Gross-Pitaevskii equation

$$i\partial_t \Psi + \Delta \Psi = \Psi(|\Psi|^2 - 1) \text{ on } \mathbb{R}^d \times \mathbb{R}. \quad (\text{GP})$$

The counterpart of (1.1) is

$$E(\Psi) = \int_{\mathbb{R}^d} |\nabla \Psi|^2 + \frac{1}{2} \int_{\mathbb{R}^d} (1 - |\Psi|^2)^2, \quad (1.12)$$

and that of (1.2) is

$$P(\Psi) = \frac{1}{2} \int_{\mathbb{R}^d} \Re \mathfrak{e}(i\nabla \Psi(\overline{\Psi - 1})). \quad (1.13)$$

We also call traveling wave a solution of (GP) of the form

$$\Psi(x, t) = v(x_1 - ct, x_\perp), x_\perp = (x_2, \dots, x_d),$$

where c is the speed of propagation. The traveling waves play an important role in the long time dynamics of (GP) (see e.g [10, 7, 23, 16, 14]). The profile v solves the equation

$$-ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \quad (\text{TWc})$$

Using the Madelung transform, the associated speed of sound for (GP), around the constant solution $\psi = 1$, is $\sqrt{2}$ (a rescaling changes the quantity $\sqrt{1g'(1)}$ into 1). The transonic limit was first studied by physicists (see [24, 25]). In the one-dimensional case, equation (TWc) is integrable with elementary computations. Solutions to (TWc) are related to the soliton of the Korteweg-de Vries equation

$$\partial_t \psi + \psi \partial_1 \psi + \partial_1^3 \psi = 0. \quad (\text{KdV})$$

Indeed in the transonic limit $c \rightarrow \sqrt{2}$, the traveling waves converge, up to rescaling, to the (KdV) soliton (see [2, 9]). A similar result exists in the two-dimensional case: for any $p > 0$, there exists a non-constant finite energy solution v_p to (TWc) with $P(v_p) = \frac{1}{2} \int_{\mathbb{R}^d} \Re \mathfrak{e}(i\nabla \Psi(\overline{\Psi - 1})) = p$ (see [4] theorem

1 and the survey [2] for properties of traveling waves). The convergence, in the transonic limit, of those minimizing traveling waves for the two-dimensional Gross-Pitaevskii equation towards a ground state of the Kadomtsev-Petviashvili equation was obtained by Bethuel-Gravejat-Saut (see [3]). The Kadomtsev-Petviashvili equation

$$\partial_t \psi + \psi \partial_1 \psi + \partial_1^3 \psi - \partial_1^{-1} (\partial_2^2 \psi) = 0, \quad (\text{KPI})$$

is a higher dimensional generalization of the Korteweg de Vries equation with energy

$$E_{KP}(\psi) = \frac{1}{2} \int_{\mathbb{R}^2} (\partial_1 \psi)^2 + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_1^{-1} (\partial_2 \psi))^2 - \frac{1}{6} \int_{\mathbb{R}^2} (\psi)^3. \quad (1.14)$$

It is a well-known asymptotic model for the propagation of weakly transverse dispersive waves [26]. Solitary waves are localized solutions to (KPI) of the form $\psi(x, t) = \omega(x_1 - \sigma t, x_2)$, where ω belongs to the energy space for (KPI), i.e. the space $Y(\mathbb{R}^2)$ (see [19]) defined as the closure of $\partial_1 C_c^\infty(\mathbb{R}^2)$ for the norm

$$\|\partial_1 f\|_{Y(\mathbb{R}^2)} = (\|\nabla f\|_{L^2(\mathbb{R}^2)}^2 + \|\partial_1^2 f\|_{L^2(\mathbb{R}^2)}^2)^{\frac{1}{2}}.$$

The equation of a solitary wave ω of speed $\sigma = 1$ is given by

$$\partial_1 \omega - \omega \partial_1 \omega - \partial_1^3 \omega + \partial_1^{-1} (\partial_2^2 \omega) = 0. \quad (\text{SW})$$

Given any $\sigma > 0$, the scale change $\tilde{\omega}(x, y) = \sigma \omega(x\sqrt{\sigma}, y\sigma)$ transforms any solution of equation (SW) into a solitary wave $\tilde{\omega}$ of speed σ . The strategy in [3] is to rewrite (GP) as an hydrodynamical system using Madelung transform, then rewrite the new equations as a Kadomtsev-Petviashvili equation with some remainder. This transonic limit convergence result has been generalized in dimension two and three by Chiron and Mariş in [13] for a large class of nonlinearities.

In the same spirit, it is proved in [12] that solutions of (E-K) with well-prepared initial data converge, in a long wave asymptotic regime, to a solution of the Kadomtsev-Petviashvili (Korteweg de Vries in the one-dimensional case) equation (See also [6, 5, 17, 11]).

The main focus of this paper is the convergence in the transonic limit of the Euler Korteweg two dimensional traveling waves to a ground state of (KPI). We also obtain with a more elementary argument, in dimension one, the convergence of the Euler-Korteweg soliton toward the soliton to the Korteweg-de-Vries equation.

Heuristic Let us do the following formal computation. We let $\rho_p(x_1, x_2) = 1 + \varepsilon_p^2 \eta_p(z_1, z_2)$, $\phi_p = \varepsilon_p \theta_p(z_1, z_2)$, with $z_1 = \varepsilon_p x_1$, $z_2 = \varepsilon_p^2 x_2$ and $c_p = \sqrt{1 - \varepsilon_p^2}$. Then the first line of (1.5) rewrites

$$-c_p \partial_1 \eta_p + \partial_1^2 \theta_p + \varepsilon_p^2 (\partial_2^2 \theta_p + \eta_p \partial_1^2 \theta_p + \partial_1 \eta_p \partial_1 \theta_p) = O(\varepsilon_p^4),$$

furthermore, by Taylor expansion we have $g = \varepsilon_p^2 \eta_p + g''(1) \varepsilon_p^4 \frac{\eta_p^2}{2} + O(\varepsilon_p^4)$, then the second lines of (1.5) rewrites

$$-c_p \partial_1 \theta_p + \eta_p + \varepsilon_p^2 \left(\frac{g''(1) \eta_p^2 + (\partial_1 \theta_p)^2}{2} - K(1) \partial_1^2 \eta_p \right) = O(\varepsilon_p^4).$$

At first order we have $\partial_1 \theta_p = \eta_p + O(\varepsilon_p^2)$, so that these functions should have the same limit. Then, multiplying the first equation by c_p and applying the operator ∂_1 to the second equation, we obtain

$$\partial_1 \eta_p + \partial_2^2 \partial_1^{-1} \eta_p + (3 + g''(1)) \eta_p \partial_1 \eta_p - K(1) \partial_1^3 \eta_p = O(\varepsilon_p^2).$$

Finally, we let

$$\eta_p = -\frac{1}{3 + g''(1)} N_p \left(\frac{x_1}{\sqrt{K(1)}}, \frac{x_2}{\sqrt{K(1)}} \right).$$

Then N_p is an (approximate) solution to (SW).

Main result Let (ρ_p, ϕ_p) be the solution given by theorem 1, we consider

$$\eta_p(x_1, x_2) = \rho_p(x_1, x_2) - 1, \quad (1.15)$$

and the rescaled functions

$$\begin{cases} \theta_p(x_1, x_2) &= -\frac{1}{\gamma \varepsilon_p \sqrt{K(1)}} \phi_p \left(\frac{\sqrt{K(1)} x_1}{\varepsilon_p}, \frac{\sqrt{K(1)} x_2}{\varepsilon_p^2} \right), \\ N_p(x_1, x_2) &= -\frac{1}{\gamma \varepsilon_p^2} \eta_p \left(\frac{\sqrt{K(1)} x_1}{\varepsilon_p}, \frac{\sqrt{K(1)} x_2}{\varepsilon_p^2} \right). \end{cases} \quad (1.16)$$

where

$$\varepsilon_p = \sqrt{1 - c_p^2} \quad \text{and} \quad \gamma = \frac{1}{g''(1) + 3}. \quad (1.17)$$

Our main theorem is

Theorem 2. *Under the assumption $\Gamma := 3 + g''(1) \neq 0$, let $(p_n)_{n \in \mathbb{N}}$ such that $p_n \rightarrow 0$. Then, there exists a ground state N_0 of (KPI), such that, up to a subsequence, we have*

$$N_{p_n} \rightarrow N_0 \text{ in } W^{k,q}(\mathbb{R}^2), \text{ when } n \rightarrow \infty. \quad (1.18)$$

and

$$\partial_1 \theta_{p_n} \rightarrow N_0 \text{ in } W^{k,q}(\mathbb{R}^2), \text{ when } n \rightarrow \infty. \quad (1.19)$$

for any $k \in \mathbb{N}$ and any $1 < q \leq \infty$.

Remark 1.2. *The proof in the manuscript follows the same lines of [3], but is more involved at a technical level because of the extension to arbitrary nonlinearities g and K . It provides an extension of recent results on the nonlinear Schrödinger equations with non-zero condition at infinity towards the wider, but*

still physically relevant, class of the Euler-Korteweg systems. See [15] for the existence of smooth branch of travelling waves for the Euler-Korteweg equation converging to the first lump in the transonic limit. It is not known that the solitons of Theorem 1 are the same as those in [15], thus the result of theorem 2 is not contained in [15] (and conversely). To obtain the convergence of the full sequence, it is sufficient that the limit N_0 is unique but it is a difficult and open problem (see [28]).

Remark 1.3. A condition similar to $\Gamma \neq 0$ is highlighted in [14] for the Schrödinger equation. As has been proven for the Schrödinger equation (see [13]), it is possible that a similar result exist in dimension three. Let us recall that the existence of solitons in dimension 3 is an open problem for Euler-Korteweg.

We will also prove a similar result in the one-dimensional case. That is, the (E-K) solitons converge, up to rescaling, to the (KdV) soliton, in the transonic limit (see the appendix A for a precise statement). Moreover as the computation are simpler in dimension 1 we are able to compute the transonic limit for $\Gamma = 0$ and a new nondegeneracy condition (see proposition 1.6). In this case, the limit is not a solution of (KdV) but of (gKdV).

Proposition 1.4. *Under the conditions*

$$g(1) = 0, \quad g'(1) = 1, \quad \Gamma \neq 0.$$

there exists (ρ, u) global solution of

$$\begin{cases} -c\rho' + (\rho u)' = 0, & (1) \\ cu' + \left(\frac{u^2}{2}\right)' + g' = \left(K(\rho)\rho'' + \frac{1}{2}K'(\rho)\rho'^2\right)' & (2), \quad x \in \mathbb{R}, \end{cases} \quad (\text{E-K})$$

with $\|\rho - 1\|_{L^\infty} \xrightarrow{\varepsilon \rightarrow 0} 0$. Moreover if we let

$$\rho - 1 = -\varepsilon^2 \gamma r_\varepsilon \left(\frac{\varepsilon x}{\sqrt{K(1)}} \right), \quad (1.20)$$

then for any $k \in \mathbb{N}$

$$\|r_\varepsilon^{(k)} - N^{(k)}\|_{L^\infty(\mathbb{R})} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where

$$N(x) = \frac{3}{ch^2\left(\frac{x}{2}\right)}, \quad (1.21)$$

is the classical soliton to the Korteweg-de-Vries equation

$$-\psi' + \psi\psi' + \psi''' = 0. \quad (\text{KdV})$$

Remark 1.5. The argument proposed in [9] for the Schrödinger equation should extend to our framework, nevertheless we propose an alternative proof in section A.

It is also possible to describe the case $\Gamma = 0$:

Proposition 1.6. *Under the conditions*

$$g(1) = 0, g'(1) = 1, \Gamma = 0, \Gamma' := g'''(1) - 12 < 0,$$

There exists (ρ^\pm, u^\pm) global solution of (E-K) with $\|\rho^\pm - 1\|_{L^\infty} \xrightarrow{\varepsilon \rightarrow 0} 0$. Moreover if we let

$$\rho^\pm - 1 = \varepsilon \gamma' r_\varepsilon^\pm \left(\frac{\varepsilon x}{\sqrt{K(1)}} \right),$$

with $\gamma' = \frac{1}{\sqrt{12 - g'''(1)}}$ then for any $k \in \mathbb{N}$

$$\|r_\varepsilon^{\pm(k)} - (w^{\pm(k)})\|_{L^\infty(\mathbb{R})} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

where $w^\pm(x) = \pm \frac{\sqrt{12}}{ch(x)}$ are the two opposite soliton of the focusing modified Korteweg de Vries equation

$$\psi' - \frac{1}{2}\psi^2\psi' = \psi'''. \quad (\text{mKdV})$$

Organization of the article In section 2 we introduce the notations and recall the properties of solitary waves solutions to (KPI). In section 3 we prove that N_p and $\partial_1 \theta_p$ converge and have the same limit. In section 4, we prove that Sobolev bounds for N_p give bound for θ_p . In section 5, using the Taylor expansion in (1.5) with respect to ε we obtain the (SW) equation with some remainder. Using Fourier transform we obtain Sobolev bounds for N_p and $\partial_1 \theta_p$, in section 6 and 7. Finally, we end the proof of theorem 2 in section 8.

2 Notations, functional spaces and properties of solution to (KPI)

Functional spaces Let $p \in [1, +\infty]$, $k \in \mathbb{N}$, $0 < \alpha < 1$ and Ω be an smooth open subset of \mathbb{R}^d . We denote by $W^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,2}(\Omega)$ the usual Sobolev spaces. For $s \geq 0$, we define

$$H^s(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) \left| \|u\|_{H^s(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty \right. \right\}.$$

We define

$$\dot{H}^s(\mathbb{R}^d) = \left\{ u \left| \hat{u} \in L^1_{loc}(\mathbb{R}^d), \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < \infty \right. \right\}.$$

$C^{0,\alpha}(\Omega)$ is the space of bounded α -Hölder continuous function on Ω . We define

$$C_0^k(\mathbb{R}^d) = \{ u \in C^k(\mathbb{R}^d) : \lim_{|x| \rightarrow \infty} |\partial^\alpha u(x)| = 0, \forall \alpha \in \mathbb{N}^d, |\alpha| \leq k \},$$

with the norm

$$\|u\|_{C_0^k(\mathbb{R}^d)} = \sum_{|\alpha| \leq k} \sup_{\mathbb{R}^d} |\partial^\alpha u|.$$

Sobolev embedding We recall the Sobolev embeddings,

$$\forall kq_1 < d, W^{k,q_1}(\mathbb{R}^d) \hookrightarrow L^{q_2}(\mathbb{R}^d), q_2 = \frac{dq_1}{d - kq_1}, \quad (2.1)$$

$$\forall d < q < \infty, W^{1,q}(\Omega) \hookrightarrow C^{0,1-\frac{d}{q}}(\Omega). \quad (2.2)$$

$$\forall (k - k')q > d, W^{k,q}(\mathbb{R}^d) \hookrightarrow C_0^{k'}(\mathbb{R}^d). \quad (2.3)$$

Moreover if Ω is bounded the embedding

$$W^{1,q}(\Omega) \hookrightarrow C^{0,1-\frac{d}{\alpha}}(\Omega), d < \alpha < q, \quad (2.4)$$

is compact. In particular

$$H^{\frac{d}{2}}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d), 2 \leq q < \infty, \quad (2.5)$$

$$\forall 0 \leq s < \frac{d}{2}, H^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d), 2 \leq q \leq \frac{2d}{d - 2s}, \quad (2.6)$$

$$\forall s > \frac{d}{2} + k, H^s(\mathbb{R}^d) \hookrightarrow C_0^k(\mathbb{R}^d). \quad (2.7)$$

Basic convolution results let $(f, g) \in H^s(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$, we have

$$\|f * g\|_{H^s(\mathbb{R}^d)} \leq \|f\|_{H^s(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}. \quad (2.8)$$

We recall a result on Fourier multipliers due to Lizorkin.

Theorem 3. ([29]). *Let \widehat{K} be a bounded function in $C^2(\mathbb{R}^2 \setminus \{0\})$ and assume that*

$$\xi_1^{k_1} \xi_2^{k_2} \partial_1^{k_1} \partial_2^{k_2} \widehat{K}(\xi) \in L^\infty(\mathbb{R}^2),$$

for any integer $0 \leq k_1, k_2 \leq 1$ such that $k_1 + k_2 \leq 2$. Then, \widehat{K} is a multiplier from $L^q(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ for any $1 < q < \infty$, i.e. there exists a constant $C(q)$, depending only on q , such that

$$\|K * f\|_{L^q(\mathbb{R}^2)} \leq C(q)M(\widehat{K})\|f\|_{L^q(\mathbb{R}^2)}, \quad \forall f \in L^q(\mathbb{R}^2).$$

where we denote

$$M(\widehat{K}) = \sup\{|\xi_1|^{k_1} |\xi_2|^{k_2} |\partial_1^{k_1} \partial_2^{k_2} \widehat{K}(\xi)|, \xi \in \mathbb{R}^2, 0 \leq k_1 \leq 1, 0 \leq k_2 \leq 1, 0 \leq k_1 + k_2 \leq 2\}.$$

Existence and properties of solitary wave solutions to (KPI) We recall some results on the Kadomtsev-Petviashvili Equation. A ground state is a solitary wave that minimizes the action

$$S(\omega) = E_{KP}(\omega) + \frac{\sigma}{2} \int_{\mathbb{R}^2} \omega^2,$$

among all non-constant solitary waves of speed σ . The constant S_{KP} denotes the action $S(\omega)$ of the ground states ω of speed $\sigma = 1$. We will denote by \mathcal{G}_σ the set of the ground state of speed σ . The ground states solutions are characterized as minimizers of energy constrained by constant L^2 -norm. Let $\mu > 0$, then the minimization problem

$$\mathcal{E}^{KP}(\mu) = \inf \left\{ E_{KP}(\omega), \omega \in Y(\mathbb{R}^2), \int_{\mathbb{R}^2} |\omega|^2 = \mu \right\}, \quad (P_{KP}(\mu))$$

has at least one solution. Moreover there exists σ such that the set of minima $\mathcal{E}^{KP}(\mu)$ is exactly equal to \mathcal{G}_σ (see De Bouard and Saut [18]). For $\sigma = 1$ we have $\mu = \mu^* = 3S_{KP}$. Since it was proved by making use of the concentration-compactness principle of P.L. Lions (see [27]), we have the compactness of minimizing sequences.

Theorem 4. ([18]) *Let $\mu > 0$, and let $(\omega_n)_{n \in \mathbb{N}}$ be a minimizing sequence to $(P_{KP}(\mu))$ in $Y(\mathbb{R}^2)$. Then, there exists some points $(a_n)_{n \in \mathbb{N}}$ and a function $N \in Y(\mathbb{R}^2)$ such that up to some subsequence,*

$$\omega_n(\cdot - a_n) \rightarrow N \text{ in } Y(\mathbb{R}^2), \text{ as } n \rightarrow \infty.$$

N is solution to the minimization problem $(P_{KP}(\mu))$ and thus is a ground state for (KPI).

Using a scaling argument it is possible to compute \mathcal{E}^{KP} .

Lemma 2.1. ([3]) *Let $N \in Y(\mathbb{R}^2)$. Given any $\sigma > 0$, the function $N_\sigma(x_1, x_2) = \sigma N(\sqrt{\sigma}x_1, \sigma x_2)$ is a minimizer for $\mathcal{E}^{KP}(\sqrt{\sigma}\mu^*)$ if and only if N is a minimizer for $\mathcal{E}^{KP}(\mu^*)$. In particular we have*

$$\mathcal{E}^{KP}(\mu) = -\frac{\mu^3}{54S_{KP}^2}, \quad \forall \mu > 0.$$

Moreover, N_σ and N are ground states for (KPI), with respective speed σ , and 1. We have the relation $\sigma = \frac{\mu^2}{(\mu^*)^2}$.

Finally, in our proof, we will have sequences $(\omega_n)_{n \in \mathbb{N}}$ which are not minimizing sequences for $(P_{KP}(\mu))$ but satisfy

$$E_{KP}(\omega_n) \rightarrow \mathcal{E}^{KP}(\mu), \text{ and } \int_{\mathbb{R}^2} \omega_n^2 \rightarrow \mu, \text{ as } n \rightarrow \infty, \quad (2.9)$$

for some positive number μ .

Proposition 2.2. ([3]) *Let $\mu > 0$, and $(\omega_n)_{n \in \mathbb{N}}$ denote a sequence of functions in $Y(\mathbb{R}^2)$ satisfying (2.9) for μ . Then, there exists some sequence $(a_n)_{n \in \mathbb{N}}$ and a ground state solution N , with speed $\sigma = \frac{\mu^2}{(\mu^*)^2}$ such that, up to some subsequence,*

$$\omega_n(\cdot - a_n) \rightarrow N \text{ in } Y(\mathbb{R}^2), \text{ as } n \rightarrow \infty. \quad (2.10)$$

Reformulation In [20], the authors use a new formulation for the solitary wave equation. Applying operator ∂_1 to (SW) we have

$$\partial_1^4 \omega - \Delta \omega + \frac{1}{2} \partial_1^2 (\omega^2) = 0.$$

Let

$$\widehat{K}_0(\xi) = \frac{\xi_1^2}{|\xi|^2 + \xi_1^4}, \quad \xi \in \mathbb{R}^2. \quad (2.11)$$

and $\omega \in Y(\mathbb{R}^2)$. Then, ω is a solution to (SW) if and only if

$$\omega = \frac{1}{2} K_0 * \omega^2. \quad (2.12)$$

3 Weak convergence in L^2

The aim of this section is to prove that N_p and $\partial_1 \theta_p$ have the same limit, and compute the convergence speed of $N_p - \partial_1 \theta_p$ towards 0.

3.1 Rescaling and energy

Let (ρ_p, ϕ_p) the solutions given by theorem 1. As in [3], we consider rescaled functions and use anisotropic space variables. let

$$\varepsilon_p = \sqrt{1 - c_p^2} \quad \text{and} \quad \gamma = \frac{1}{g''(1) + 3}, \quad (3.1)$$

$$\theta_p(x_1, x_2) = -\frac{1}{\gamma \varepsilon_p \sqrt{K(1)}} \phi_p \left(\frac{\sqrt{K(1)} x_1}{\varepsilon_p}, \frac{\sqrt{K(1)} x_2}{\varepsilon_p^2} \right), \quad (3.2)$$

and

$$N_p(x_1, x_2) = -\frac{1}{\gamma \varepsilon_p^2} \eta_p \left(\frac{\sqrt{K(1)} x_1}{\varepsilon_p}, \frac{\sqrt{K(1)} x_2}{\varepsilon_p^2} \right). \quad (3.3)$$

We can assume, up to a translation, using (1.10)

$$\exists C > 0, \quad N_p(0) > C, \quad \forall p_0 > p > 0. \quad (3.4)$$

Proposition 3.1. *Let $(p_n)_{n \in \mathbb{N}}$ a sequence such that $p_n \rightarrow 0$. Then, there exists $N_0 \in L^2(\mathbb{R}^2)$ such that, up to a subsequence,*

$$N_{p_n} \rightharpoonup N_0 \text{ in } L^2(\mathbb{R}^2) \quad (3.5)$$

$$\partial_1 \theta_{p_n} \rightharpoonup N_0 \text{ in } L^2(\mathbb{R}^2) \quad (3.6)$$

Moreover there exists some positive constant C , not depending on p , such that

$$\int_{\mathbb{R}^2} (N_p - \partial_1 \theta_p)^2 dx \leq C \varepsilon_p^{\frac{1}{2}}. \quad (3.7)$$

Remark 3.2. In section 7, we will prove that N_0 is a ground state of (KPI).

Proof of (3.5) and (3.6). Since $G''(1) = g'(1) = 1$, $G'(1) = g(1) = G(1) = 0$, there exists $\delta > 0$ such that $G(\rho) \geq \frac{(\rho-1)^2}{3}$ for any $\rho \in]1 + \delta, 1 - \delta[$. Then for p small enough, using (1.9) and the definition (1.1), we have

$$\int_{\mathbb{R}^2} (\eta_p(x_1, x_2))^2 dx \leq ME(\rho_p, \phi_p). \quad (3.8)$$

Thus, we deduce from (1.7) and (1.17) that

$$\begin{aligned} \int_{\mathbb{R}^2} (N_p)^2 &= \frac{1}{\gamma^2 \varepsilon_p^4} \int_{\mathbb{R}^2} \left(\eta_p \left(\frac{\sqrt{K(1)}x_1}{\varepsilon_p}, \frac{\sqrt{K(1)}x_2}{\varepsilon_p^2} \right) \right)^2 \\ &\leq \frac{1}{K(1)\gamma^2 \varepsilon_p} \int_{\mathbb{R}^2} (\eta_p(x_1, x_2))^2 \\ &\leq \frac{1}{\varepsilon_p} CE(\rho_p, \phi_p) \\ &\leq C. \end{aligned} \quad (3.9)$$

Then, using Banach-Alaoglu theorem, there exists a function $N_0 \in L^2(\mathbb{R}^2)$ such that, up to some subsequence

$$N_{p_n} \rightharpoonup N_0 \text{ in } L^2(\mathbb{R}^2). \quad (3.10)$$

The convergence of $\partial_1 \theta_{p_n}$ is a consequence of (3.7). In order to complete the proof of proposition 3.1, it only remains to prove (3.7). This requires to use rescaled energy and Pohozaev estimates, so that (3.7) is postponed to section 3.3. \square

Lemma 3.3. *The energy can be expressed in terms of the new functions as*

$$E(\rho_p, \phi_p) = \frac{K(1)\gamma^2 \varepsilon_p}{2} \left(E_0(N_p, \theta_p) + \varepsilon_p^2 (E_2(N_p, \theta_p)) + \varepsilon_p^4 (E_4(N_p, \theta_p)) \right),$$

with

$$E_0(N_p, \theta_p) = \int_{\mathbb{R}^2} N_p^2 + (\partial_1 \theta_p)^2,$$

$$E_2(N_p, \theta_p) = 2 \left(\int_{\mathbb{R}^2} \frac{(\partial_1 N_p(x))^2}{2} + \frac{(\partial_2 \theta_p(x))^2}{2} - \frac{\gamma}{6} (3N_p (\partial_1 \theta_p)^2 + g''(1)N_p^3) dx \right),$$

$$\begin{aligned} E_4(N_p, \theta_p) &= \int_{\mathbb{R}^2} K \left(1 - \gamma \varepsilon_p^2 N_p(x) \right) \frac{1}{\sqrt{K(1)}^2} (\partial_2 N_p(x))^2 - \gamma N_p(x) (\partial_2 \theta_p(x))^2 \\ &\quad + N_p(x) j_p(x) \left((\partial_1 N_p(x))^2 + (\varepsilon_p)^2 (\partial_2 N_p(x))^2 \right) \\ &\quad + \gamma^4 N_p^4(x) l_p(x) dx, \end{aligned}$$

with j_p and l_p some functions smooth and bounded in L^∞ uniformly in p . For the momentum we have

$$p = \varepsilon_p \gamma^2 K(1) \int_{\mathbb{R}^2} N_p(x) \partial_1 \theta_p(x) dx.$$

Proof. Since $N_p \simeq \partial_1 \theta_p$, then passing to the limit in p we have $E_0(N_p, \theta_p) \simeq 2 \int_{\mathbb{R}^2} N_0^2$ and $E_2(N_p, \theta_p) \simeq 2E_{KP}(N_0)$. Later, to prove that the weak limit N_0 is a solution to (SW), we will use $(P_{KP}(\mu))$. This is a direct but tedious computation. The functions j_p and l_p are given by

$$\begin{aligned} G(1+x) &= G(1) + G'(1)x + G''(1)\frac{x^2}{2} + x^3 l(x) \\ &= \frac{x^2}{2} + x^3 l(x). \end{aligned}$$

So

$$G(\rho_p(x)) = \gamma^2 \frac{1}{2} \varepsilon_p^4 N_p^2(x') - \gamma^3 \varepsilon_p^6 \frac{g''(1) N_p^3(x')}{6} + \gamma^4 \varepsilon_p^8 N_p^4(x') l(-\varepsilon^2 \gamma N_p(x')),$$

where l is the third order remainder of Taylor expansion and

$$x' = \left(\frac{\varepsilon_p x_1}{\sqrt{K(1)}}, \frac{\varepsilon_p^2 x_2}{\sqrt{K(1)}} \right).$$

In view of (1.9), the function $l(-\varepsilon_p^2 \gamma N_p(x')) = l(\rho_p(x) - 1)$ is bounded independently of p . To simplify we write

$$G(\rho_p(x)) = \gamma^2 \frac{1}{2} \varepsilon_p^4 N_p^2(x') - \gamma^3 \varepsilon_p^6 \frac{g''(1) N_p^3(x')}{6} + \gamma^4 \varepsilon_p^8 N_p^4(x') l_p(x').$$

Similarly, we write

$$K\left(1 - \gamma \varepsilon_p^2 N_p(x')\right) = K(1) - \varepsilon_p^2 N_p(x') j_p(x').$$

□

3.2 Pohozaev's identities

We estimate now derivative terms in the energy using Pohozaev's identities.

Lemma 3.4.

$$\left| \int_{\mathbb{R}^2} (\partial_2 \phi_p)^2 + (\partial_2 \rho_p)^2 + (\partial_1 \rho_p)^2 dx \right| \leq C \varepsilon_p^3, \quad (3.11)$$

and

$$\left| \int_{\mathbb{R}^2} \eta_p (\partial_1 \phi_p)^2 dx \right| \leq C \varepsilon_p^{\frac{3}{2}}, \quad \left| \int_{\mathbb{R}^2} \eta_p (\partial_2 \phi_p)^2 dx \right| \leq C \varepsilon_p^{\frac{7}{2}}. \quad (3.12)$$

Proof. Thanks to (1.7), we have

$$|E(\rho_p, \phi_p) - P(\rho_p, \phi_p)| < CP(\rho_p, \phi_p)^3.$$

We recall Pohozaev's identities obtained in [1] proposition 5.1

$$E(\rho_p, \phi_p) = \int_{\mathbb{R}^2} \rho_p (\partial_2 \phi_p)^2 + K(\rho_p) (\partial_2 \rho_p)^2 dx + c_p P(\rho_p, \phi_p), \quad (3.13)$$

$$E(\rho_p, \phi_p) = \int_{\mathbb{R}^2} \rho_p (\partial_1 \phi_p)^2 + K(\rho_p) (\partial_1 \rho_p)^2 dx. \quad (3.14)$$

Moreover we have

$$c_p P(\rho_p, \phi_p) = \int_{\mathbb{R}^2} \rho_p (\nabla \phi_p)^2 dx. \quad (3.15)$$

It follows from (1.7) and (1.17) that

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \rho_p (\partial_2 \phi_p)^2 + K(\partial_2 \rho_p)^2 dx \right| &= |E(\rho_p, \phi_p) - cP(\rho_p, \phi_p)| \\ &= |E(\rho_p, \phi_p) - P(\rho_p, \phi_p) + (1 - \sqrt{1 - \varepsilon_p^2})P(\rho_p, \phi_p)| \\ &\leq |E(\rho_p, \phi_p) - P(\rho_p, \phi_p)| + \varepsilon_p^2 \left| \frac{1 - \sqrt{1 - \varepsilon_p^2}}{\varepsilon_p^2} P(\rho_p, \phi_p) \right| \\ &\leq CP(\rho_p, \phi_p)^3, \end{aligned}$$

and therefore using $\|\rho_p - 1\|_\infty \rightarrow 0$ when $p \rightarrow 0$ we get

$$\left| \int_{\mathbb{R}^2} (\partial_2 \phi_p)^2 + (\partial_2 \rho_p)^2 dx \right| \leq CP(\rho_p, \phi_p)^3. \quad (3.16)$$

In view of (3.14) and (3.15) we obtain

$$\left| \int_{\mathbb{R}^2} K(\rho_p) (\partial_1 \rho_p)^2 dx \right| = \left| \int_{\mathbb{R}^2} \rho_p (\partial_2 \phi_p)^2 dx + E(\rho_p, \phi_p) - cP(\rho_p, \phi_p) \right|,$$

and then

$$\left| \int_{\mathbb{R}^2} (\partial_1 \rho_p)^2 dx \right| \leq CP(\rho_p, \phi_p)^3. \quad (3.17)$$

Finally combining (3.17) and (3.16) we have

$$\left| \int_{\mathbb{R}^2} (\partial_2 \phi_p)^2 + (\partial_2 \rho_p)^2 + (\partial_1 \rho_p)^2 dx \right| \leq CP(\rho_p, \phi_p)^3. \quad (3.18)$$

Moreover by (3.14)

$$E(\rho_p, \phi_p) = \int_{\mathbb{R}^2} \rho_p (\partial_1 \phi_p)^2 + K(\rho_p) (\partial_1 \rho_p)^2 dx, \quad (3.19)$$

thus

$$\int_{\mathbb{R}^2} (\partial_1 \phi_p)^2 dx \leq CE(\rho_p, \phi_p).$$

Combining (3.19) with (1.9), we obtain

$$\left| \int_{\mathbb{R}^2} \eta_p (\partial_1 \phi_p)^2 dx \right| \leq C\varepsilon_p^{\frac{3}{2}}. \quad (3.20)$$

Similarly

$$\left| \int_{\mathbb{R}^2} \eta_p (\partial_2 \phi_p)^2 dx \right| \leq C\varepsilon_p^{\frac{7}{2}}, \quad (3.21)$$

this ends the proof of lemma 3.4. \square

3.3 Energy estimates

We are now in position to conclude the proof of proposition 3.1.

End of proof of proposition 3.1. First we have

$$\begin{aligned} \int_{\mathbb{R}^2} (\partial_1 N_p)^2 &= \frac{1}{\gamma^2 \varepsilon_p^3} \int_{\mathbb{R}^2} (\partial_1 \rho_p)^2, & \int_{\mathbb{R}^2} (\partial_2 \theta_p)^2 &= \frac{1}{K(1) \gamma^2 \varepsilon_p^3} \int_{\mathbb{R}^2} (\partial_2 \phi_p)^2, \\ \int_{\mathbb{R}^2} (\partial_2 N_p)^2 &= \frac{1}{\gamma^2 \varepsilon_p^5} \int_{\mathbb{R}^2} (\partial_2 \rho_p)^2, & \int_{\mathbb{R}^2} N_p (\partial_1 \theta_p)^2 &= -\frac{1}{K(1) \gamma^3 \varepsilon_p^3} \int_{\mathbb{R}^2} \eta_p (\partial_1 \phi_p)^2, \\ \int_{\mathbb{R}^2} N_p^3 &= -\frac{1}{\gamma^3 K(1) \varepsilon_p^3} \int_{\mathbb{R}^2} \eta_p^3, & \int_{\mathbb{R}^2} N_p (\partial_2 \theta_p)^2 &= -\frac{1}{\gamma^3 \varepsilon_p^5 K(1)} \int_{\mathbb{R}^2} \eta_p (\partial_2 \phi_p)^2. \end{aligned}$$

Using (3.18) and (1.17), we have

$$\int_{\mathbb{R}^2} (\partial_2 \theta_p)^2 + (\partial_1 N_p)^2 \leq C. \quad (3.22)$$

Then with (1.9), (3.8) and (3.20) we obtain

$$E_2(N_p, \theta_p) \leq \frac{C}{\varepsilon_p^{\frac{3}{2}}}. \quad (3.23)$$

Similarly, using (3.21), (3.18), (1.9) and by definition of E_4 , we have

$$E_4(N_p, \theta_p) \leq \frac{C}{\varepsilon_p^3}. \quad (3.24)$$

We deduce that

$$\begin{aligned} \left| E(\rho_p, \phi_p) - \frac{\gamma^2 \varepsilon_p K(1)}{2} E_0(\rho_p, \phi_p) \right| &= \left| E(\rho_p, \phi_p) - \frac{\gamma^2 \varepsilon_p K(1)}{2} \int_{\mathbb{R}^2} (N_p)^2 + (\partial_1 \theta_p)^2 \right| \\ &\leq C\varepsilon_p^{\frac{3}{2}}. \end{aligned} \quad (3.25)$$

Thus, by (1.7) we have

$$\begin{aligned} \int_{\mathbb{R}^2} (N_p - \partial_1 \theta_p)^2 &= E_0(N_p, \theta_p) - 2 \int_{\mathbb{R}^2} N_p \partial_1 \theta_p \\ &\leq \frac{2E(\rho_p, \phi_p)}{\gamma^2 K(1) \varepsilon_p} - \frac{2}{\varepsilon_p \gamma^2 K(1)} p + C \varepsilon_p^{\frac{1}{2}} \\ &\leq C_1 \varepsilon_p^{\frac{1}{2}}. \end{aligned}$$

This completes the proof of estimate (3.7) and Proposition 3.1. \square

4 Elliptic estimates on θ_p

In the present section, we aim to prove that N_p bounds θ_p in Sobolev spaces. More precisely, we prove the following proposition:

Proposition 4.1. *Let $1 < q < \infty$ there exists a constant $C(q)$ depending on q , but not on p , such that*

$$\|\partial_1 \theta_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial_2 \theta_p\|_{L^q(\mathbb{R}^2)} \leq C(q) \|N_p\|_{L^q(\mathbb{R}^2)}, \quad (4.1)$$

for p small enough. Moreover for any $\alpha \in \mathbb{N}^2$ let

$$\Gamma_p(q, \alpha) = \|\partial^\alpha \partial_1 \theta_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial^\alpha \partial_2 \theta_p\|_{L^q(\mathbb{R}^2)},$$

then there exists $C(q, \alpha)$ such that

$$\Gamma_p(q, \alpha) \leq C(q, \alpha) \left(\|\partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \sum_{0 \leq \beta < \alpha} \|\partial^\beta N_p\|_{L^\infty(\mathbb{R}^2)} \Gamma_p(q, \alpha - \beta) \right). \quad (4.2)$$

First of all in view of theorem 1 and lemma 4.2 in [1] we have:

Lemma 4.2. *For any $k \in \mathbb{N}$, $q \in [2, \infty[$ we have*

$$(\eta_p, \nabla \phi_p) \in W^{k,q}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2). \quad (4.3)$$

Moreover for any $\alpha \in \mathbb{N}^2$, there exists $C(\alpha) > 0$ not depending on p such that

$$\|(\partial^\alpha \eta_p, \partial^\alpha \nabla \phi_p)\|_{L^\infty(\mathbb{R}^2)} \leq C(\alpha). \quad (4.4)$$

Later, using Lizorkin theorem, we will have N_p , $\partial_1 \theta_p$ and $\partial_2 \theta_p$ in $W^{k,q}(\mathbb{R}^2)$, for any $1 < q < 2$ and $k \in \mathbb{N}$. Thus the quantity $\Gamma_p(q, \alpha)$ is finite for any $1 < q < \infty$, $\alpha \in \mathbb{N}^2$. This is the reason why in proposition 4.1, we let $1 < q < \infty$.

Proof of proposition 4.1. We have

$$-c_p \partial_1 \rho_p = -\operatorname{div}(\rho_p \nabla \phi_p), \quad (4.5)$$

then

$$\Delta\phi_p = c_p\partial_1\eta_p - \operatorname{div}(\eta_p\nabla\phi_p),$$

and for any $\alpha \in \mathbb{N}^2$

$$\Delta(\partial^\alpha\phi_p) = c_p\partial_1\partial^\alpha\eta_p - \operatorname{div}(\partial^\alpha(\eta_p\nabla\phi_p)).$$

Thus, using elliptic estimates (see [21]), there exists some constant $C(q)$ such that

$$\|\nabla(\partial^\alpha\phi_p)\|_{L^q(\mathbb{R}^2)} \leq C(q)(\|\partial^\alpha\eta_p\|_{L^q(\mathbb{R}^2)} + \|\partial^\alpha(\eta_p\nabla\phi_p)\|_{L^q(\mathbb{R}^2)}). \quad (4.6)$$

Using (1.9) we have $C(q)\|\eta_p\nabla\phi_p\|_{L^q(\mathbb{R}^2)} \leq \frac{1}{2}\|\nabla\phi_p\|_{L^q(\mathbb{R}^2)}$ so that from (4.6) we have

$$\|\nabla\phi_p\|_{L^q(\mathbb{R}^2)} \leq C(q)\|\eta_p\|_{L^q(\mathbb{R}^2)}. \quad (4.7)$$

Next using Leibniz formula we obtain

$$\begin{aligned} \|\partial^\alpha(\eta_p\nabla\phi_p)\|_{L^q(\mathbb{R}^2)} &\leq C(q, \alpha) \left(\|\partial^\alpha\eta_p\|_{L^q(\mathbb{R}^2)} \|\nabla\phi_p\|_{L^\infty(\mathbb{R}^2)} \right. \\ &\quad \left. + \sum_{0 \leq \beta < \alpha} \|\partial^\beta\eta_p\|_{L^\infty(\mathbb{R}^2)} \|\partial^{\alpha-\beta}(\nabla\phi_p)\|_{L^q(\mathbb{R}^2)} \right). \end{aligned} \quad (4.8)$$

Then, using (4.4) and (4.6), we have

$$\begin{aligned} \|\partial^\alpha(\nabla\phi_p)\|_{L^q(\mathbb{R}^2)} &\leq C(q, \alpha) \left(\|\partial^\alpha\eta_p\|_{L^q(\mathbb{R}^2)} \right. \\ &\quad \left. + \sum_{0 \leq \beta < \alpha} \|\partial^\beta\eta_p\|_{L^\infty(\mathbb{R}^2)} \|\partial^{\alpha-\beta}(\nabla\phi_p)\|_{L^q(\mathbb{R}^2)} \right). \end{aligned} \quad (4.9)$$

We observe that (as a direct consequence of the rescaling (3.2), (3.3)) there exists some positive constants $C_1(q, \alpha)$, $C_2(q, \alpha)$ and $C_3(q, \alpha)$, such that

$$\begin{aligned} \|\partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} &= \frac{C_1(q, \alpha)}{\varepsilon_p^{2+\alpha_1+2\alpha_2-\frac{3}{q}}} \|\partial^\alpha\eta_p\|_{L^q(\mathbb{R}^2)}, \\ \|\partial^\alpha\partial_1\theta_p\|_{L^q(\mathbb{R}^2)} &= \frac{C_2(q, \alpha)}{\varepsilon_p^{2+\alpha_1+2\alpha_2-\frac{3}{q}}} \|\partial^\alpha\partial_1\phi_p\|_{L^q(\mathbb{R}^2)}, \\ \|\partial^\alpha\partial_2\theta_p\|_{L^q(\mathbb{R}^2)} &= \frac{C_3(q, \alpha)}{\varepsilon_p^{3+\alpha_1+2\alpha_2-\frac{3}{q}}} \|\partial^\alpha\partial_2\phi_p\|_{L^q(\mathbb{R}^2)}. \end{aligned}$$

Combining with (4.7) and (4.9), we obtain (4.1) and (4.2). \square

Remark 4.3. *Assuming we have bounds for N_p in $W^{k,q}$ for any $k \in \mathbb{N}$, $1 < q \leq \infty$ then, by induction using proposition 4.1 and Sobolev embedding, we can bound θ_p in $W^{k,q}$ for any $k \in \mathbb{N}$, $1 < q \leq \infty$.*

5 Convolution equation

5.1 Reformulation

The aim of this section is to rewrite equation (E-K) as a convolution equation for N_p , similar to (2.12). The first two lemmas are direct computations.

Lemma 5.1. *We have*

$$\partial_1 N_p - \partial_1^2 \theta_p = \varepsilon_p^2 (\mathbf{L}_1(N_p, \theta_p) + R(N_p, \theta_p)), \quad (5.1)$$

with

$$\begin{aligned} \mathbf{L}_1(N_p, \theta_p) &= \frac{1}{\varepsilon_p^2} (1 - c_p) \partial_1 N_p + \partial_2^2 \theta_p, \\ R(N_p, \theta_p) &= -\gamma \partial_1 (N_p \partial_1 \theta_p) - \gamma \varepsilon_p^2 \partial_2 (N_p \partial_2 \theta_p). \end{aligned}$$

Lemma 5.2. *The function N_p and θ_p satisfy*

$$\begin{aligned} N_p(x) - c_p \partial_1(\theta_p(x)) &= + \varepsilon_p^2 \left[\frac{\gamma g''(1)}{2} N_p^2 - \varepsilon_p^2 \gamma^2 N_p^3 l_1 \right. \\ &\quad + K \left(1 - \gamma \varepsilon_p^2 N_p(x) \right) \left(\frac{\partial_1^2 N_p}{K(1)} + \varepsilon_p^2 \frac{\partial_2^2 N_p}{K(1)} \right) \\ &\quad + \gamma \frac{(\partial_1 \theta_p)^2}{2} + \gamma \frac{\varepsilon_p^2 (\partial_2 \theta_p)^2}{2} \\ &\quad \left. - \gamma \frac{K' \left(1 - \gamma \varepsilon_p^2 N_p(x) \right)}{2K(1)} (\varepsilon_p^2 (\partial_1 N_p)^2 + \varepsilon_p^4 (\partial_2 N_p)^2) \right], \end{aligned} \quad (5.2)$$

where l_1 is defined by

$$g(\rho_p(x)) = -\gamma \varepsilon_p^2 N_p(x') + \varepsilon_p^4 \frac{g''(1)}{2} \gamma^2 N_p^2(x') - \gamma^3 \varepsilon_p^6 N_p^3(x') l_1(-\varepsilon_p^2 \gamma N_p(x')).$$

Remark 5.3. *We recall that*

$$x' = \left(\frac{\varepsilon_p x_1}{\sqrt{K(1)}}, \frac{\varepsilon_p^2 x_2}{\sqrt{K(1)}} \right).$$

In view of (1.9), the function $l_1(-\varepsilon_p^2 \gamma N_p(x')) = l_1(\rho_p(x) - 1)$ is smooth and bounded in L^∞ independently of p . To simplify we write

$$g(\rho_p(x)) = -\gamma \varepsilon_p^2 N_p(x') + \varepsilon_p^4 \frac{g''(1)}{2} \gamma^2 N_p^2(x') - \gamma^3 \varepsilon_p^6 N_p^3(x') l_1(x').$$

Now we combine these two lemmas to obtain a Kadomtsev-Petviashvili equation with some remainder.

Proposition 5.4.

$$\begin{aligned} \partial_1^4 N_p - \Delta N_p - \mathbf{L}(N_p, \theta_p) &= -\partial_1^2 f_p \\ &\quad - \varepsilon_p^2 \sum_{i+j=2} \partial_1^i \partial_2^j R_{\varepsilon_p}^{i,j}, \end{aligned} \quad (5.3)$$

with

$$\begin{aligned} \mathbf{L}(N_p, \theta_p) &= -(2\varepsilon_p^2 \partial_1^2 \partial_2^2 N_p + \varepsilon_p^4 \partial_2^4 N_p), \quad (5.4) \\ \frac{1}{\gamma} R_{\varepsilon_p}^{2,0}(N_p, \theta_p) &= \left(\frac{c_p - 1}{\varepsilon_p^2} N_p \partial_1 \theta_p - \gamma N_p^3 l_1 + \frac{1}{2} (\partial_2 \theta_p)^2 - \frac{K'(\rho_p)}{K(1)} (\partial_1 N_p)^2 \right) \\ &\quad - \varepsilon_p^2 \left(\frac{K'(\rho_p)}{K(1)} (\partial_2 N_p)^2 \right) - \frac{h_1}{\gamma \varepsilon_p^4} h_2, \end{aligned} \quad (5.5)$$

$$\begin{aligned} \frac{1}{\gamma} R_{\varepsilon_p}^{0,2}(N_p, \theta_p) &= \left(\frac{g''(1)}{2} N_p^2 + \frac{(\partial_1 \theta_p)^2}{2} \right) + \varepsilon_p^2 \left[-\gamma N_p^3 l_1 + \frac{(\partial_2 \theta_p)^2}{2} - \frac{K'(\rho_p) (\partial_1 N_p)^2}{2K(1)} \right] \\ &\quad - \varepsilon_p^4 \frac{K'(\rho_p) (\partial_2 N_p)^2}{2K(1)} - \frac{1}{\gamma \varepsilon_p^2} h_1 h_2, \end{aligned} \quad (5.6)$$

$$\frac{1}{\gamma} R_{\varepsilon_p}^{1,1}(N_p, \theta_p) = c_p N_p \partial_2 \theta_p, \quad (5.7)$$

$$f_p = \gamma \left(N_p \partial_1 \theta_p + \frac{g''(1)}{2} N_p^2 + \frac{1}{2} (\partial_1 \theta_p)^2 \right). \quad (5.8)$$

Where h_1 and h_2 are some functions defined later.

Remark 5.5. We observe that

$$\gamma \left(1 + \frac{g''(1)}{2} + \frac{1}{2} \right) = \gamma \left(\frac{3 + g''(1)}{2} \right) = \frac{1}{2}. \quad (5.9)$$

Then passing, formally in (5.3), to the limit in p we have

$$\partial_1^4 N_0 - \Delta N_0 + \frac{1}{2} \partial_1^2 (N_0^2) = 0, \quad (5.10)$$

i.e. N_0 is a solution of (KPI).

Proof. Let

$$\frac{1}{K(1)} h_1(x) = \frac{1}{K(1 - \gamma \varepsilon_p^2 N_p(x))} - \frac{1}{K(1)} = h_3(-\varepsilon_p^2 \gamma N_p(x)) \varepsilon_p^2 N_p(x), \quad (5.11)$$

and

$$\begin{aligned} h_2 &= N_p - c_p \partial_1 \theta_p - \varepsilon_p^2 \left[\gamma \frac{g''(1)}{2} N_p^2 - \gamma^2 \varepsilon_p^2 N_p^3 l_1 + \frac{\gamma (\partial_1 \theta_p)^2}{2} + \gamma \frac{\varepsilon_p^2 (\partial_2 \theta_p)^2}{2} \right. \\ &\quad \left. - \gamma \frac{K' \left(1 - \varepsilon_p^2 N_p \right)}{2K(1)} \left(\varepsilon_p^2 (\partial_1 N_p)^2 + \varepsilon_p^4 (\partial_2 N_p)^2 \right) \right]. \end{aligned} \quad (5.12)$$

First of all, multiplying (5.2) by $K(1)\left(\frac{1}{K(1)} + \frac{1}{K(1-\gamma\varepsilon_p^2 N_p)} - \frac{1}{K(1)}\right)$ we obtain

$$\begin{aligned}
N_p - c_p \partial_1 \theta_p = & +\varepsilon_p^2 \left[\gamma \frac{g''(1)}{2} N_p^2 - \gamma^2 \varepsilon_p^2 N_p^3 l + \left(\partial_1^2 N_p + \varepsilon_p^2 \partial_2^2 N_p \right) \right. \\
& + \gamma \frac{(\partial_1 \theta_p)^2}{2} + \gamma \frac{\varepsilon_p^2 (\partial_2 \theta_p)^2}{2} \\
& \left. - \gamma \frac{K'(1-\gamma\varepsilon_p^2 N_p)}{2K(1)} (\varepsilon_p^2 (\partial_1 N_p)^2 + \varepsilon_p^4 (\partial_2 N_p)^2) \right] \\
& - h_1 h_2.
\end{aligned} \tag{5.13}$$

We have computing $-(\partial_1^2 + \varepsilon_p^2 \partial_2^2)(5.13) + c \partial_1(5.1)$

$$\begin{aligned}
\partial_1^4 N_p - \Delta N_p = & -\partial_1^2 \left[\gamma \left(N_p \partial_1 \theta_p + \frac{g''(1)}{2} N_p^2 + \frac{1}{2} (\partial_1 \theta_p)^2 \right) \right] \\
& + \mathbf{L}(N_p, \theta_p) - \varepsilon_p^2 \sum_{i+j=2} \partial_1^i \partial_2^j R_{\varepsilon_p}^{i,j},
\end{aligned} \tag{5.14}$$

this completes the proof of Claim (5.3). \square

We now recast (5.3) as a convolution equation.

Proposition 5.6. *Let*

$$\widehat{K_{\varepsilon_p}^{i,j}}(\xi) = \frac{\xi_1^i \xi_2^j}{|\xi|^2 + \xi_1^4 + 2\varepsilon_p^2 \xi_1^2 \xi_2^2 + \varepsilon_p^4 \xi_2^4}, \tag{5.15}$$

we have

$$N_p = K_{\varepsilon_p}^{2,0} * f_p + \sum_{i+j=2} \varepsilon_p^2 K_{\varepsilon_p}^{i,j} * R_{\varepsilon_p}^{i,j}. \tag{5.16}$$

Proof. It is a direct computation. \square

Next we define

$$Q(\xi) = |\xi|^2 + \xi_1^4 + 2\varepsilon_p^2 \xi_1^2 \xi_2^2 + \varepsilon_p^4 \xi_2^4. \tag{5.17}$$

We need to establish that the remainder term R_{ε_p} are small enough in some Sobolev spaces.

Proposition 5.7. *There exists some positive constant C , not depending on p , such that*

$$\int_{\mathbb{R}^2} |R_{\varepsilon_p}^{1,1}| \leq C, \quad \int_{\mathbb{R}^2} |f_p| \leq C, \tag{5.18}$$

$$\int_{\mathbb{R}^2} |R_{\varepsilon_p}^{0,2}| \leq C, \quad \int_{\mathbb{R}^2} |R_{\varepsilon_p}^{2,0}| \leq \frac{C}{\varepsilon_p^2}. \tag{5.19}$$

Proof. Proposition 5.7 is a direct consequence of (1.9), identities in subsection 3.3, estimates in subsection 3.2, (3.9) and Hölder's inequality. For example we have

$$\int_{\mathbb{R}^2} (\partial_2 \theta_p)^2 = \frac{1}{K(1)\gamma^2 \varepsilon_p^3} \int_{\mathbb{R}^2} (\partial_2 \phi_p)^2 \leq C,$$

thus

$$\int_{\mathbb{R}^2} |R_{\varepsilon_p}^{1,1}| \leq C \|N_p\|_{L^2(\mathbb{R}^2)} \|\partial_2 \theta_p\|_{L^2(\mathbb{R}^2)} \leq C.$$

Similarly we obtain the estimate on f_p and (5.19). \square

5.2 Kernel estimates

In order to use (5.16) we need to control $K_{\varepsilon_p}^{i,j}$. In this section we use computations of [3] section 5.

Proposition 5.8. (*[3] lemma 5.1*) *Let $0 \leq s < 1$, there exists a constant $C(s)$ depending possibly on s , but not on p , such that*

$$\|K_{\varepsilon_p}^{2,0}\|_{\dot{H}^s(\mathbb{R}^2)} \leq C(s)(1 + \varepsilon_p^{\frac{1}{2}-2s}), \quad \|K_{\varepsilon_p}^{1,1}\|_{\dot{H}^s(\mathbb{R}^2)} \leq C(s)(1 + \varepsilon_p^{-\frac{1}{2}-2s}), \quad (5.20)$$

$$\|K_{\varepsilon_p}^{0,2}\|_{\dot{H}^s(\mathbb{R}^2)} \leq C(s)(1 + \varepsilon_p^{-\frac{3}{2}-2s}). \quad (5.21)$$

Thus we have

$$\|K_{\varepsilon_p}^{2,0}\|_{H^s(\mathbb{R}^2)} + \varepsilon_p \|K_{\varepsilon_p}^{1,1}\|_{H^s(\mathbb{R}^2)} + \varepsilon_p^2 \|K_{\varepsilon_p}^{0,2}\|_{H^s(\mathbb{R}^2)} \leq C(s), \quad (5.22)$$

for any $0 \leq s \leq \frac{1}{4}$.

Proposition 5.9. (*[3] lemma 5.2*) *let $1 < q < \infty$ and $0 \leq i, j \leq 4$ integers such that $2 \leq i + j \leq 4$ we denote by*

$$\kappa_{i,j} = \max \{i + 2j - 4, 0\}.$$

Then there exists a constant $C(q)$, depending possibly on q , but not on p such that

$$\|K_{\varepsilon_p}^{i,j} * f\|_{L^q(\mathbb{R}^2)} \leq \frac{C(q)}{\varepsilon_p^{\kappa_{i,j}}} \|f\|_{L^q(\mathbb{R}^2)},$$

for any $f \in L^q(\mathbb{R}^2)$ and $\varepsilon_p > 0$.

Remark 5.10. *As a consequence N_p and all its derivatives, belong to $L^q(\mathbb{R}^2)$ for any $1 < q < 2$. Indeed, we have*

$$N_p = K_{\varepsilon_p}^{2,0} * f_p + \sum_{i+j=2} \varepsilon_p^2 K_{\varepsilon_p}^{i,j} * R_{\varepsilon_p}^{i,j}. \quad (5.23)$$

Thus, using lemma 4.2 we obtain

$$\partial^\alpha N_p = K_{\varepsilon_p}^{2,0} * \partial^\alpha f_p + \sum_{i+j=2} \varepsilon_p^2 K_{\varepsilon_p}^{i,j} * \partial^\alpha R_{\varepsilon_p}^{i,j}.$$

Then, combining the definition of $R_{\varepsilon_p}^{i,j}$, lemma 4.2 with the proposition above, we have $\partial^\alpha N_p \in L^q$ for any $1 < q < 2$. Therefore using (4.1), $\partial_1 \theta_p$ and $\partial_2 \theta_p$ belong to $L^q(\mathbb{R}^2)$ for any $1 < q < 2$. Finally (5.2) and (5.1) give, respectively, $\partial_1 \theta_p \in W^{k,q}$ and $\partial_2 \theta_p \in W^{k,q}$ for any $k \in \mathbb{N}$.

6 Bounds in Sobolev spaces

This section is devoted to the proof by induction of the following proposition:

Proposition 6.1. *There exists $p_0 > 0$ such that for any $0 \leq p \leq p_0$, $\alpha \in \mathbb{N}^2$ and $1 < q < \infty$, there exists $C(q, \alpha)$ such that*

$$\begin{aligned} & \|\partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} + \|\partial_1 \partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} + \|\partial_2 \partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} \\ & + \|\partial_1^2 \partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial_1 \partial_2 \partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial_2^2 \partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} \leq C(q, \alpha). \end{aligned} \quad (6.1)$$

We have the following consequence.

Theorem 5. *There exists $p_0 > 0$ such that for any $0 \leq p \leq p_0$, $k \in \mathbb{N}$ and $1 < q \leq \infty$ there exists $C(k, q)$ such that*

$$\|N_p\|_{W^{k,q}(\mathbb{R}^2)} + \|\partial_1 \theta_p\|_{W^{k,q}(\mathbb{R}^2)} + \varepsilon_p \|\partial_2 \theta_p\|_{W^{k,q}(\mathbb{R}^2)} \leq C(k, q). \quad (6.2)$$

Proof of theorem 5 assuming proposition 6.1. By proposition 6.1, we have for any $k \in \mathbb{N}$, $1 < q < \infty$, there exists a constant $C(k, q)$ not depending on p such that

$$\|N_p\|_{W^{k,q}(\mathbb{R}^2)} \leq C(k, q). \quad (6.3)$$

Thus using Sobolev embedding (2.7), we obtain

$$\forall k \in \mathbb{N}, \|N_p\|_{C_0^k(\mathbb{R}^2)} \leq C(k, q). \quad (6.4)$$

By proposition 4.1 we have

$$\Gamma_p(q, \alpha) \leq C(q, \alpha) \left(1 + \varepsilon_p^2 \sum_{0 \leq \beta < \alpha} \Gamma_p(q, \alpha - \beta) \right). \quad (6.5)$$

Combining (4.1) and (6.3), the term $\Gamma_p(q, (0, 0))$ is bounded independently of p . Then by induction and (6.5) the quantity $\Gamma_p(q, \alpha)$ is bounded independently of p , for any $1 < q < \infty$, $\alpha \in \mathbb{N}^2$. Finally, using Sobolev embedding (2.7) this result is also true for $q = \infty$. \square

Preliminary We begin the proof of proposition 6.1. First of all, we have:

Lemma 6.2. *For any $1 < q < \infty$, there exists a constant $C(q)$, independent of p , such that*

$$\|N_p(\partial_1\theta_p)^2\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2\|N_p(\partial_2\theta_p)^2\|_{L^q(\mathbb{R}^2)} \leq C(q)\|N_p\|_{L^{3q}(\mathbb{R}^2)}^3, \quad (6.6)$$

and

$$\begin{aligned} & \|(\partial_1\theta_p)^2\|_{L^q(\mathbb{R}^2)} + \varepsilon_p\|N_p\partial_2\theta_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2\|(\partial_2\theta_p)^2\|_{L^q(\mathbb{R}^2)} + \|N_p\partial_1\theta_p\|_{L^q(\mathbb{R}^2)} \\ & \leq C(q)\|N_p\|_{L^{2q}(\mathbb{R}^2)}^2. \end{aligned} \quad (6.7)$$

Proof. Hölder inequality and (4.1) leads to

$$\begin{aligned} & \|N_p(\partial_1\theta_p)^2\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2\|N_p(\partial_2\theta_p)^2\|_{L^q(\mathbb{R}^2)} \\ & \leq C(q)\|N_p\|_{L^{3q}(\mathbb{R}^2)}(\|\partial_1\theta_p\|_{L^{3q}(\mathbb{R}^2)}^2 + \varepsilon_p^2\|\partial_2\theta_p\|_{L^{3q}(\mathbb{R}^2)}^2) \\ & \leq C(q)\|N_p\|_{L^{3q}(\mathbb{R}^2)}^3, \end{aligned}$$

a similar computations gives us (6.7). \square

Lemma 6.3. *For any $1 < q < \infty$, there exists a constant $C(q)$, such that*

$$\begin{aligned} \left\| \frac{h_1 h_2}{\varepsilon_p^2} \right\|_{L^q(\mathbb{R}^2)} & \leq C(q) \left(\|N_p\|_{L^{2q}(\mathbb{R}^2)}^2 + \varepsilon_p^2 \|N_p\|_{L^{3q}(\mathbb{R}^2)}^3 \right. \\ & \quad \left. + \varepsilon_p^4 \|N_p\|_{L^\infty(\mathbb{R}^2)} \|\partial_1 N_p\|_{L^{2q}(\mathbb{R}^2)}^2 \right. \\ & \quad \left. + \varepsilon_p^6 \|N_p\|_{L^\infty(\mathbb{R}^2)} \|\partial_2 N_p\|_{L^{2q}(\mathbb{R}^2)}^2 \right), \end{aligned} \quad (6.8)$$

$$\|f_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|R_{\varepsilon_p}^{1,1}\|_{L^q(\mathbb{R}^2)} \leq C(q)\|N_p\|_{L^{2q}(\mathbb{R}^2)}^2, \quad (6.9)$$

$$\begin{aligned} \|R_{\varepsilon_p}^{2,0}\|_{L^q(\mathbb{R}^2)} & \leq \frac{C(q)}{\varepsilon_p^2} \left(\|N_p\|_{L^{2q}(\mathbb{R}^2)}^2 + \varepsilon_p^2 \|N_p\|_{L^{3q}(\mathbb{R}^2)}^3 \right. \\ & \quad \left. + \varepsilon_p^2 \|\partial_1 N_p\|_{L^{2q}(\mathbb{R}^2)}^2 \right. \\ & \quad \left. + \varepsilon_p^4 \|\partial_2 N_p\|_{L^{2q}(\mathbb{R}^2)}^2 + \left\| \frac{h_1(x)h_2(x)}{\varepsilon_p^2} \right\|_{L^q(\mathbb{R}^2)} \right), \end{aligned} \quad (6.10)$$

$$\begin{aligned} \|R_{\varepsilon_p}^{0,2}\|_{L^q(\mathbb{R}^2)} & \leq C(q) \left(\|N_p\|_{L^{2q}(\mathbb{R}^2)}^2 + \varepsilon_p^2 \|N_p\|_{L^{3q}(\mathbb{R}^2)}^3 \right. \\ & \quad \left. + \varepsilon_p^2 \|\partial_1 N_p\|_{L^{2q}(\mathbb{R}^2)}^2 \right. \\ & \quad \left. + \varepsilon_p^4 \|\partial_2 N_p\|_{L^{2q}(\mathbb{R}^2)}^2 + \left\| \frac{h_1(x)h_2(x)}{\varepsilon_p^2} \right\|_{L^q(\mathbb{R}^2)} \right). \end{aligned} \quad (6.11)$$

Where h_1, h_2 are defined in proposition 5.4.

Proof. The proof is a direct computation combined with lemma 6.2, estimate (1.9) and observing that

$$\begin{aligned} \left\| \frac{(h_1 h_2)}{\varepsilon_p^2} \right\|_{L^q(\mathbb{R}^2)} &\leq C(q) \left(\|N_p^2\|_{L^q(\mathbb{R}^2)} + \|N_p \partial_1 \theta_p\|_{L^q(\mathbb{R}^2)} \right. \\ &\quad + \varepsilon_p^2 \|N_p^3\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|N_p (\partial_1 \theta_p)^2\|_{L^q(\mathbb{R}^2)} \\ &\quad + \varepsilon_p^4 \|N_p^4\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^4 \|(\partial_2 \theta_p)^2 N_p\|_{L^q(\mathbb{R}^2)} \\ &\quad + \varepsilon_p^4 \|(\partial_1 N_p)^2 N_p\|_{L^q(\mathbb{R}^2)} \\ &\quad \left. + \varepsilon_p^6 \|(\partial_2 N_p)^2 N_p\|_{L^q(\mathbb{R}^2)} \right). \end{aligned}$$

□

Lemma 6.4. *For any $1 < q < \infty$, there exists a constant $C(q)$, such that*

$$\begin{aligned} &\|N_p\|_{L^q(\mathbb{R}^2)} + \|\partial_1 N_p\|_{L^q(\mathbb{R}^2)} + \|\partial_2 N_p\|_{L^q(\mathbb{R}^2)} \\ &+ \|\partial_1^2 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial_1 \partial_2 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial_2^2 N_p\|_{L^q(\mathbb{R}^2)} \\ &\leq C(q) \left(\|N_p\|_{L^{2q}(\mathbb{R}^2)}^2 + \varepsilon_p^2 \|N_p\|_{L^{3q}(\mathbb{R}^2)}^3 + \varepsilon_p^2 \|\partial_1 N_p\|_{L^{2q}(\mathbb{R}^2)}^2 \right. \\ &\quad + \varepsilon_p^4 \|\partial_2 N_p\|_{L^{2q}(\mathbb{R}^2)}^2 + \varepsilon_p^4 \|\partial_1 N_p\|_{L^{2q}(\mathbb{R}^2)}^2 \|N_p\|_{L^\infty(\mathbb{R}^2)} \\ &\quad \left. + \varepsilon_p^6 \|\partial_2 N_p\|_{L^{2q}(\mathbb{R}^2)}^2 \|N_p\|_{L^\infty(\mathbb{R}^2)} \right). \end{aligned} \quad (6.12)$$

Proof. First we observe that

$$\partial^\alpha K_{\varepsilon_p}^{j,k} = \varepsilon_p^{\alpha_1 + \alpha_2} K_{\varepsilon_p}^{j+\alpha_1, k+\alpha_2}, \quad (6.13)$$

for any $|\alpha| \leq 2$. Applying the operator ∂^α to (5.16) we obtain

$$\|\partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} \leq \|\partial^\alpha K_{\varepsilon_p}^{2,0} * f_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \sum_{i+j=2} \|\partial^\alpha K_{\varepsilon_p}^{i,j} * R_{\varepsilon_p}^{i,j}\|_{L^q(\mathbb{R}^2)}, \quad (6.14)$$

then, with proposition 5.9, we deduce

$$\begin{aligned} \|N_p\|_{L^q(\mathbb{R}^2)} &\leq \|K_{\varepsilon_p}^{2,0} * f_p\|_{L^q(\mathbb{R}^2)} + \sum_{i+j=2} \varepsilon_p^2 \|K_{\varepsilon_p}^{i,j} * R_{\varepsilon_p}^{i,j}\|_{L^q(\mathbb{R}^2)} \\ &\leq C(q) \left(\|f_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \sum_{i+j=2} \|R_{\varepsilon_p}^{i,j}\|_{L^q(\mathbb{R}^2)} \right), \end{aligned}$$

$$\begin{aligned} \|\partial_1 N_p\|_{L^q(\mathbb{R}^2)} &\leq \|\partial_1 K_{\varepsilon_p}^{2,0} * f_p\|_{L^q(\mathbb{R}^2)} + \sum_{i+j=2} \varepsilon_p^2 \|\partial_1 K_{\varepsilon_p}^{i,j} * R_{\varepsilon_p}^{i,j}\|_{L^q(\mathbb{R}^2)} \\ &\leq C(q) \left(\|f_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 (\|R_{\varepsilon_p}^{2,0}\|_{L^q(\mathbb{R}^2)} + \|R_{\varepsilon_p}^{1,1}\|_{L^q(\mathbb{R}^2)}) \right. \\ &\quad \left. + \varepsilon_p \|R_{\varepsilon_p}^{0,2}\|_{L^q(\mathbb{R}^2)} \right). \end{aligned}$$

Similar estimates hold for $\partial_2 N_p$, $\partial_1^2 N_p$, $\partial_1 \partial_2 N_p$ and $\partial_2^2 N_p$. Finally using lemma 6.3, we obtain lemma 6.4. \square

Lemma 6.5. *Let $2 \leq q \leq \infty$, there exists a positive constant $C(q)$, depending possibly on q , but not on p , such that*

$$\|\partial_1 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial_2 N_p\|_{L^q(\mathbb{R}^2)} \leq C(q) \varepsilon_p^{\frac{6}{q}-3}, \quad (6.15)$$

and

$$\|N_p\|_{L^q(\mathbb{R}^2)} \leq C(q), \quad \forall 2 \leq q \leq \frac{8}{3}. \quad (6.16)$$

Moreover, for $\frac{8}{3} < q < 8$, there exists $C(q)$ such that the following bound holds.

$$\varepsilon_p^{\frac{2}{3}} \|N_p\|_{L^q(\mathbb{R}^2)} \leq C(q). \quad (6.17)$$

Proof. Using (4.4), we have

$$\|\partial_1 N_p\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{\varepsilon_p^3}, \quad \text{and} \quad \|\partial_2 N_p\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{\varepsilon_p^4}. \quad (6.18)$$

Combining with (3.18) and (3.22) we obtain (6.15) using standard interpolations inequalities. Applying (2.8) on (5.16) for any $0 \leq s \leq \frac{1}{4}$ we obtain

$$\begin{aligned} \|N_p\|_{H^s(\mathbb{R}^2)} &\leq \|K_{\varepsilon_p}^{2,0}\|_{H^s(\mathbb{R}^2)} (\|f_p\|_{L^1(\mathbb{R}^2)} + \varepsilon_p^2 \|R_{\varepsilon_p}^{2,0}\|_{L^1(\mathbb{R}^2)}) \\ &\quad + \varepsilon_p^2 \|R_{\varepsilon_p}^{1,1}\|_{L^1(\mathbb{R}^2)} \|K_{\varepsilon_p}^{1,1}\|_{H^s(\mathbb{R}^2)} \\ &\quad + \varepsilon_p^2 \|R_{\varepsilon_p}^{0,2}\|_{L^1(\mathbb{R}^2)} \|K_{\varepsilon_p}^{0,2}\|_{H^s(\mathbb{R}^2)}. \end{aligned} \quad (6.19)$$

By proposition 5.7, lemma 6.3 and claim (5.22), we deduce

$$\|N_p\|_{H^s(\mathbb{R}^2)} \leq C(s). \quad (6.20)$$

Thus, by Sobolev embedding (2.6) we have

$$\|N_p\|_{L^q(\mathbb{R}^2)} \leq C(q), \quad \forall 2 \leq q \leq \frac{8}{3}. \quad (6.21)$$

Let $\nu > 0$ combining (6.16) and (6.15) there exists a constant $C(\nu)$ and $q > 2$ such that

$$\|N_p\|_{W^{1,q}(\mathbb{R}^2)} \leq C(\nu) \left(1 + \varepsilon_p^{-1-\nu}\right).$$

Thus, by Sobolev embedding (2.2), we obtain

$$\|N_p\|_{L^\infty(\mathbb{R}^2)} \leq C(\nu) \left(1 + \varepsilon_p^{-1-\nu}\right).$$

Combining with (6.16) we have (6.17) by interpolation between Lebesgue spaces. \square

We are now able to prove the first step of our induction.

The first step Let $1 < q < \infty$, there exists a constant $C(q)$, such that

$$\begin{aligned} & \|N_p\|_{L^q(\mathbb{R}^2)} + \|\partial_1 N_p\|_{L^q(\mathbb{R}^2)} + \|\partial_2 N_p\|_{L^q(\mathbb{R}^2)} + \\ & \|\partial_1^2 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial_1 \partial_2 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial_2^2 N_p\|_{L^q(\mathbb{R}^2)} \leq C(q). \end{aligned} \quad (6.22)$$

Proof. First of all for any $1 < q \leq \frac{4}{3}$ we have, using lemma 6.4, inequalities (6.17), (6.15) and (6.16),

$$\begin{aligned} & \|N_p\|_{L^q(\mathbb{R}^2)} + \|\partial_1 N_p\|_{L^q(\mathbb{R}^2)} + \|\partial_2 N_p\|_{L^q(\mathbb{R}^2)} + \\ & \|\partial_1^2 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial_1 \partial_2 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial_2^2 N_p\|_{L^q(\mathbb{R}^2)} \leq C(q). \end{aligned} \quad (6.23)$$

Then, by Sobolev embedding (2.1) and standard interpolations inequalities,

$$\|N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial_1 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial_2 N_p\|_{L^q(\mathbb{R}^2)} \leq C(q), \quad (6.24)$$

for any $1 < q \leq 4$. Then, for any $1 < q \leq 2$ by lemma 6.4, we obtain

$$\begin{aligned} & \|N_p\|_{L^q(\mathbb{R}^2)} + \|\partial_1 N_p\|_{L^q(\mathbb{R}^2)} + \|\partial_2 N_p\|_{L^q(\mathbb{R}^2)} + \\ & \|\partial_1^2 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial_1 \partial_2 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial_2^2 N_p\|_{L^q(\mathbb{R}^2)} \leq C(q). \end{aligned} \quad (6.25)$$

We deduce, by Sobolev embedding (2.1), that

$$\|N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial_1 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial_2 N_p\|_{L^q(\mathbb{R}^2)} \leq C(q), \quad (6.26)$$

$\forall 1 < q < \infty$. Finally using lemma 6.4 we obtain claim (6.22). \square

Inductive step We fix $k \in \mathbb{N}$. In this part we assume that (6.1) is true for any $1 < q < \infty$ and any $\alpha \in \mathbb{N}^2$ such that $|\alpha| \leq k$. We will prove that (6.1) holds for any $|\alpha| \leq k + 1$.

Lemma 6.6. *There exists $p_0 > 0$ such that for any $0 \leq p \leq p_0$, $1 < q < \infty$ and $\alpha \in \mathbb{N}^2$ such that $|\alpha| \leq k + 1$, then there exists a constant $C(q, \alpha)$ such that*

$$\|\partial^\alpha \partial_1 \theta_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial^\alpha \partial_2 \theta_p\|_{L^q(\mathbb{R}^2)} \leq C(q, \alpha). \quad (6.27)$$

Proof. Let $\alpha \in \mathbb{N}^2$, $|\alpha| \leq k + 1$. Using Sobolev embedding (2.3) and our hypothesis, we have

$$\|N_p\|_{C_0^k(\mathbb{R}^2)} \leq C(k), \quad (6.28)$$

where $C(k)$ is a constant not depending on p . Then, by (4.2), we obtain

$$\begin{aligned} & \|\partial^\alpha \partial_1 \theta_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial^\alpha \partial_2 \theta_p\|_{L^q(\mathbb{R}^2)} \leq C(q, \alpha) \left(\|\partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} \right. \\ & \left. + \varepsilon_p^2 \sum_{0 \leq \beta < \alpha} (\|\partial^{\alpha-\beta} \partial_1 \theta_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial^{\alpha-\beta} \partial_2 \theta_p\|_{L^q(\mathbb{R}^2)}) \right). \end{aligned} \quad (6.29)$$

If we denote

$$S_k^q = \sum_{|\alpha| \leq k+1} \left(\|\partial^\alpha \partial_1 \theta_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial^\alpha \partial_2 \theta_p\|_{L^q(\mathbb{R}^2)} \right),$$

summing the previous inequalities, we obtain

$$S_k^q \leq C(q, \alpha) \left(\varepsilon_p^2 S_k^q + \sum_{|\alpha| \leq k+1} \|\partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} \right).$$

We conclude by using (6.1). \square

Lemma 6.7. *There exists a positive constant $C(q, \alpha)$ depending possibly on q and α , but not on p , such that*

$$\begin{aligned} & \|\partial^\alpha f_p\|_{L^q(\mathbb{R}^2)} + \|\partial^\alpha R_{\varepsilon_p}^{0,2}\|_{L^q(\mathbb{R}^2)} \\ & + \varepsilon_p \|\partial^\alpha R_{\varepsilon_p}^{1,1}\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial^\alpha R_{\varepsilon_p}^{2,0}\|_{L^q(\mathbb{R}^2)} \leq C(q, \alpha), \end{aligned} \quad (6.30)$$

for any $1 \leq q < \infty$, $|\alpha| \leq k+1$.

Proof. We will detail the computation for $h_1 h_2$ defined in proposition 5.4. First of all we have using (5.11), (5.12)

$$\begin{aligned} \left\| \frac{\partial^\alpha(h_1 h_2)}{\varepsilon_p^2} \right\|_{L^q(\mathbb{R}^2)} & \leq C \left(\|\partial^\alpha(\delta N_p^2)\|_{L^q(\mathbb{R}^2)} + \|\partial^\alpha(\delta N_p \partial_1 \theta_p)\|_{L^q(\mathbb{R}^2)} \right. \\ & \quad + \varepsilon_p^2 \|\partial^\alpha(\delta N_p^3)\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial^\alpha(N_p(\partial_1 \theta_p)^2 \delta)\|_{L^q(\mathbb{R}^2)} \\ & \quad + \varepsilon_p^4 \|\partial^\alpha(N_p^4 l \delta)\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^4 \|\partial^\alpha((\partial_2 \theta_p)^2 N_p \delta)\|_{L^q(\mathbb{R}^2)} \\ & \quad + \varepsilon_p^4 \|\partial^\alpha((\partial_1 N_p)^2 N_p \delta K')\|_{L^q(\mathbb{R}^2)} \\ & \quad \left. + \varepsilon_p^6 \|\partial^\alpha((\partial_2 N_p)^2 N_p \delta K')\|_{L^q(\mathbb{R}^2)} \right), \end{aligned}$$

where $\delta = \delta(-\varepsilon_p^2 \gamma N_p)$, $K' = K'(-\varepsilon_p^2 \gamma N_p)$, $l = l(-\varepsilon_p^2 \gamma N_p)$. Using our hypothesis and the chain rule we have for any $\beta \in \mathbb{N}^2$, $|\beta| \leq k+1$, $1 < q < \infty$

$$\|\partial^\beta(K'(-\varepsilon_p^2 \gamma N_p))\|_{L^q(\mathbb{R}^2)} + \|\partial^\beta(\delta(-\varepsilon_p^2 \gamma N_p))\|_{L^q(\mathbb{R}^2)} + \|\partial^\beta(l(-\varepsilon_p^2 \gamma N_p))\|_{L^q(\mathbb{R}^2)} \leq C.$$

Thus using Leibniz formula, our hypothesis and (6.27) we have

$$\left\| \frac{\partial^\alpha(h_1 h_2)}{\varepsilon^2} \right\|_{L^q(\mathbb{R}^2)} \leq C.$$

Applying the operator ∂^α on (5.5), (5.6), (5.7) and using our hypothesis and (6.27) we obtain claim (6.30). \square

Finally let $|\alpha| \leq k+1$ using (5.16) we have

$$\partial^\alpha N_p = K_{\varepsilon_p}^{2,0} * \partial^\alpha f_p + \sum_{i+j=2} \varepsilon_p^2 K_{\varepsilon_p}^{i,j} * \partial^\alpha R_{\varepsilon_p}^{i,j}. \quad (6.31)$$

Then using proposition 5.9, there exists $C(q)$ such that

$$\|\partial^\alpha N_p\|_{L^q(\mathbb{R}^2)} \leq C(q) \left(\|\partial^\alpha f_p\|_{L^q(\mathbb{R}^2)} + \sum_{i+j=2} \varepsilon_p^2 \|\partial^\alpha R_{\varepsilon_p}^{i,j}\|_{L^q(\mathbb{R}^2)} \right), \quad (6.32)$$

$$\begin{aligned} \|\partial^\alpha \partial_1 N_p\|_{L^q(\mathbb{R}^2)} &\leq C(q) \left(\|\partial^\alpha f_p\|_{L^q(\mathbb{R}^2)} \right. \\ &\quad \left. + \varepsilon_p^2 (\|\partial^\alpha R_{\varepsilon_p}^{2,0}\|_{L^q(\mathbb{R}^2)} + \|\partial^\alpha R_{\varepsilon_p}^{1,1}\|_{L^q(\mathbb{R}^2)}) \right. \\ &\quad \left. + \varepsilon_p \|\partial^\alpha R_{\varepsilon_p}^{0,2}\|_{L^q(\mathbb{R}^2)} \right), \end{aligned} \quad (6.33)$$

and

$$\begin{aligned} &\|\partial^\alpha \partial_2 N_p\|_{L^q(\mathbb{R}^2)} + \|\partial^\alpha \partial_1^2 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial^\alpha \partial_1 \partial_2 N_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial^\alpha \partial_2^2 N_p\|_{L^q(\mathbb{R}^2)} \\ &\leq C(q) \left(\|\partial^\alpha f_p\|_{L^q(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial^\alpha R_{\varepsilon_p}^{2,0}\|_{L^q(\mathbb{R}^2)} + \varepsilon_p \|\partial^\alpha R_{\varepsilon_p}^{1,1}\|_{L^q(\mathbb{R}^2)} + \|\partial^\alpha R_{\varepsilon_p}^{0,2}\|_{L^q(\mathbb{R}^2)} \right). \end{aligned} \quad (6.34)$$

We conclude the induction by using (6.30).

Conclusion We have proved proposition 6.1, theorem 5 follows from our discussion at the beginning of the section.

7 Strong convergence

7.1 Strong local convergence

Proposition 7.1. *Let $(p_n)_{n \in \mathbb{N}}$ such that $p_n \rightarrow 0$. Then there exists N_0 a non constant solution of (SW) such that, up to a subsequence,*

$$\forall 1 < q < \infty, \forall k \in \mathbb{N}, N_{p_n} \rightharpoonup N_0 \text{ in } W^{k,q}(\mathbb{R}^2), \text{ when } n \rightarrow \infty. \quad (7.1)$$

Thus for any $0 \leq \gamma < 1$ and any compact subset \mathcal{K} , we obtain

$$N_{p_n} \rightarrow N_0 \text{ in } C^{0,\gamma}(\mathcal{K}), \text{ when } n \rightarrow \infty. \quad (7.2)$$

Proof. Combining (6.2) with Banach-Alaoglu theorem, there exists a subsequence $(p_n)_{n \in \mathbb{N}}$ and a function N_0 such that $N_{p_n} \rightharpoonup N_0$ in $W^{k,q}(\mathbb{R}^2)$ for any $1 < q < \infty, k \in \mathbb{N}$. Then (7.2) is a consequence of (7.1) and (2.4). Thus N_0 is non constant using (3.4). We will now prove that N_0 is a solution of (SW). We recall that

$$f_p = \gamma \left(N_p \partial_1 \theta_p + \frac{g''(1)}{2} N_p^2 + \frac{1}{2} (\partial_1 \theta_p)^2 \right), \quad (7.3)$$

with

$$\gamma \left(1 + \frac{g''(1)}{2} + \frac{1}{2} \right) = \gamma \left(\frac{3 + g''(1)}{2} \right) = \frac{1}{2}, \quad (7.4)$$

and

$$\begin{aligned} \partial_1^4 N_p - \Delta N_p &= -\partial_1^2 f_p + \mathbf{L}(N_p, \theta_p) \\ &\quad - \varepsilon_p^2 \sum_{i+j=2} \partial_1^i \partial_2^j R_{\varepsilon_p}^{i,j}. \end{aligned} \quad (7.5)$$

Using (5.4) and theorem 5 we have

$$\|\mathbf{L}(N_p, \theta_p)\|_{L^2(\mathbb{R}^2)} \xrightarrow[n \rightarrow \infty]{} 0.$$

Using (5.6) and theorem 5 we have

$$\varepsilon_p^2 \|\partial_1^2 R_{\varepsilon_p}^{2,0}\|_{L^1(\mathbb{R}^2)} \xrightarrow[n \rightarrow \infty]{} 0,$$

then using (6.30) we have

$$\varepsilon_p^2 \|\partial_2^2 R_{\varepsilon_p}^{0,2}\|_{L^1(\mathbb{R}^2)} + \varepsilon_p^2 \|\partial_1 \partial_2 R_{\varepsilon_p}^{1,1}\|_{L^1(\mathbb{R}^2)} \xrightarrow[n \rightarrow \infty]{} 0.$$

On the other hand, using (7.2), we have

$$\|N_{p_n}^2 - N_0^2\|_{L^\infty(K)} \xrightarrow[n \rightarrow \infty]{} 0,$$

for any compact K . Thus

$$N_{p_n}^2 \xrightarrow[n \rightarrow \infty]{} N_0^2 \text{ in } D'(\mathbb{R}^2).$$

Combining (3.7) and

$$\|(\partial_1 \theta_p)^2 - N_p^2\|_{L^1(\mathbb{R}^2)} \leq \|\partial_1 \theta_p - N_p\|_{L^2(\mathbb{R}^2)} \|\partial_1 \theta_p + N_p\|_{L^2(\mathbb{R}^2)},$$

we obtain

$$\|N \partial_1 \theta_p - N_p^2\|_{L^1(\mathbb{R}^2)} \xrightarrow[p \rightarrow 0]{} 0, \quad \|(\partial_1 \theta_p)^2 - N_p^2\|_{L^1(\mathbb{R}^2)} \xrightarrow[p \rightarrow 0]{} 0.$$

Thus, we deduce

$$\left\| \frac{1}{2} N_p^2 - f_p \right\|_{L^1(\mathbb{R}^2)} \leq (|\gamma| \|N_p \partial_1 \theta_p - N_p^2\|_{L^1(\mathbb{R}^2)} + \frac{|\gamma|}{2} \|(\partial_1 \theta_p)^2 - N_p^2\|_{L^1(\mathbb{R}^2)}) \xrightarrow[p \rightarrow 0]{} 0.$$

Finally passing to the limit in (7.5) we have

$$\partial_1^4 N_0 - \Delta N_0 + \frac{1}{2} \partial_1^2 (N_0^2) = 0. \quad (7.6)$$

□

We will now prove the strong global convergence.

7.2 Strong global convergence

We recall that

$$E(\rho_p, \phi_p) = \frac{K(1)\gamma^2\varepsilon_p}{2} \left(E_0(N_p, \theta_p) + \varepsilon_p^2(E_2(N_p, \theta_p)) + \varepsilon_p^4(E_4(N_p, \theta_p)) \right),$$

with

$$E_0(N_p, \theta_p) = \int_{\mathbb{R}^2} N_p^2 + \partial_1 \theta_p^2,$$

$$E_2(N_p, \theta_p) = 2 \left(\int_{\mathbb{R}^2} \frac{(\partial_1 N_p(x))^2}{2} + \frac{|\partial_2 \theta_p(x)|^2}{2} - \frac{\gamma}{6} (3N_p(\partial_1 \theta_p)^2 + g''(1)N_p^3) dx \right),$$

$$\begin{aligned} E_4(N_p, \theta_p) &= \int_{\mathbb{R}^2} K \left(1 - \gamma \varepsilon_p^2 N_p(x) \right) \frac{1}{\sqrt{K(1)}} |\partial_2 N_p(x)|^2 - \gamma N_p(x) |\partial_2 \theta_p(x)|^2 \\ &\quad + N_p(x) j(x) \left((\partial_1 N_p(x))^2 + \varepsilon_p^2 (\partial_2 N_p(x))^2 \right) \\ &\quad + \gamma^4 N_p^4(x) l(x) dx, \end{aligned}$$

and

$$E_{KP}(\partial_1 \theta_p) = \frac{1}{2} \int_{\mathbb{R}^2} (\partial_1^2 \theta_p)^2 + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_2 \theta_p)^2 - \frac{1}{6} \int_{\mathbb{R}^2} (\partial_1 \theta_p)^3.$$

Proposition 7.2. *Let $(p_n)_{n \in \mathbb{N}}$ a sequence which converges to 0 and satisfies (7.1) and (7.2). Then, up to a subsequence, there exists a positive constant μ_0 such that*

$$E_{KP}(\partial_1 \theta_{p_n}) \rightarrow \mathcal{E}^{KP}(\mu_0), \quad \text{and} \quad \int_{\mathbb{R}^2} |\partial_1 \theta_{p_n}|^2 \xrightarrow{n \rightarrow \infty} \mu_0 > 0. \quad (7.7)$$

See section 2 for the definition of \mathcal{E}^{KP} .

First of all, we prove three lemmas.

Lemma 7.3. *There exists a constant C , not depending on p , such that*

$$E(\rho_p, \phi_p) - p \leq \frac{-1}{(K(1)\gamma^2)^2 54 S_{KP}^2} p^3 + Cp^4. \quad (7.8)$$

Proof. Let ω a ground state of (SW) and $p > 0$. According to [19, 20] ω is smooth, belongs to $L^q(\mathbb{R}^2)$ for any $q > 1$ its first order derivatives are in $L^2(\mathbb{R}^2)$. There exists v such that $\partial_1 v = \omega$ (see lemma 3.9 [4] and [22]) with v smooth. Moreover v is in $L^q(\mathbb{R}^2)$, $q > 2$ and its gradient belongs to $L^q(\mathbb{R}^2)$, $q > 1$. We let

$$\rho(x_1, x_2) = 1 - \varepsilon^2 \gamma \omega \left(\varepsilon \frac{x_1}{\sqrt{K(1)}}, \varepsilon^2 \frac{x_2}{\sqrt{K(1)}} \right), \quad (7.9)$$

$$\phi(x_1, x_2) = -\gamma\sqrt{K(1)}\varepsilon v \left(\varepsilon \frac{x_1}{\sqrt{K(1)}}, \varepsilon^2 \frac{x_2}{\sqrt{K(1)}} \right), \quad (7.10)$$

where the constant ε is chosen such that

$$p = K(1)\gamma^2\varepsilon \int_{\mathbb{R}^2} \omega^2.$$

We observe that $\rho \in 1 + H^1(\mathbb{R}^2)$ and $\phi \in \dot{H}^1(\mathbb{R}^2)$. Thus, by computation, we have

$$E(\rho, \phi) = K(1)\gamma^2\varepsilon \int_{\mathbb{R}^2} \omega^2 + K(1)\gamma^2\varepsilon^3 E_{KP}(\omega) + \frac{K(1)\gamma^2}{2}\varepsilon^5 E_4(\omega, v).$$

Then, using lemma 2.1, we obtain

$$\begin{aligned} E(\rho, \phi) - p &\leq K(1)\gamma^2\varepsilon^3 E_{KP}(\omega) + C\varepsilon^5 \\ &\leq K(1)\gamma^2 \left(\frac{p}{K(1)\gamma^2 \int \omega^2} \right)^3 E_{KP}(\omega) + Cp^4 \\ &\leq \frac{p^3}{(K(1)\gamma^2)^2} \frac{E_{KP}(\omega)}{(\int \omega^2)^3} + Cp^4 \\ &\leq \frac{p^3}{(K(1)\gamma^2)^2} \left(\frac{-1}{54S_{KP}^2} \right) + Cp^4. \end{aligned}$$

For $p \ll 1$, since $\omega \in L^\infty(\mathbb{R}^2)$ we have $|\rho - 1| \ll 1$ and $|\rho_p - 1| \ll 1$. Thus using (1.11) we have

$$\begin{aligned} E(\rho_p, \phi_p) - p &= E(\rho_p, \phi_p) - p \\ &\leq E(\rho, \phi) - p \\ &= E(\rho, \phi) - p \\ &\leq \frac{p^3}{(K(1)\gamma^2)^2} \left(\frac{-1}{54S_{KP}^2} \right) + Cp^4. \end{aligned}$$

□

Lemma 7.4. *We have*

$$E(N_p, \theta_p) - p = K(1)\gamma^2\varepsilon_p^3 E_{Kp}(\partial_1 \theta_p) + o(\varepsilon_p^3), \quad (7.11)$$

and

$$-\frac{1}{54S_{KP}^2} \left(\int_{\mathbb{R}^2} (\partial_1 \theta_p)^2 \right)^3 \leq E_{Kp}(\partial_1 \theta_p) \leq -\frac{1}{54S_{KP}^2} \left(\int_{\mathbb{R}^2} (\partial_1 \theta_p)^2 \right)^3 + o(1). \quad (7.12)$$

Proof. Combining (5.2) and (7.1) we obtain

$$\frac{1}{\varepsilon_p^2} \int_{\mathbb{R}^2} (N_p - \partial_1 \theta_p)^2 \xrightarrow{p \rightarrow 0} 0, \quad (7.13)$$

and

$$\|\partial_1 N_p - \partial_1^2 \theta_p\|_{L^2(\mathbb{R}^2)} \leq M\varepsilon_p. \quad (7.14)$$

Then, we have

$$\begin{aligned} E(N_p, \theta_p) - p &= \frac{K(1)\gamma^2\varepsilon_p}{2} \int_{\mathbb{R}^2} (N_p - \partial_1 \theta_p)^2 + K(1)\gamma^2\varepsilon_p^3 E_{KP}(\partial_1 \theta_p) \\ &+ \frac{K(1)\gamma^2\varepsilon_p^3}{2} \left(\int_{\mathbb{R}^2} ((\partial_1 N_p(x))^2 - (\partial_1^2 \theta_p)^2) dx + \right. \\ &+ \frac{\gamma g''(1)}{3} (-N_p^3 + (\partial_1 \theta_p)^3) + \gamma (-N_p(\partial_1 \theta_p)^2 + (\partial_1 \theta_p)^3) dx \left. \right) \\ &+ \frac{K(1)\gamma^2\varepsilon_p^5}{2} E_4(N_p, \theta_p). \end{aligned}$$

Thus

$$E(N_p, \theta_p) - p = K(1)\gamma^2\varepsilon_p^3 E_{KP}(\partial_1 \theta_p) + o(\varepsilon_p^3). \quad (7.15)$$

Using lemma 2.1

$$E_{KP}(\partial_1 \theta_p) \geq \varepsilon_{min}^{KP} \left(\int_{\mathbb{R}^2} (\partial_1 \theta_p)^2 \right) = -\frac{1}{54S_{KP}^2} \left(\int_{\mathbb{R}^2} (\partial_1 \theta_p)^2 \right)^3. \quad (7.16)$$

Then combining (7.11), lemma 7.3 and (7.13) we obtain

$$\begin{aligned} E_{KP}(\partial_1 \theta_p) &\leq -\frac{p^3}{(K(1)\gamma^2)^3 54S_{KP}^2 \varepsilon^3} + o(1) \\ &\leq \frac{-1}{54S_{KP}^2} \left(\int_{\mathbb{R}^2} N_p \partial_1 \theta_p \right)^3 + o(1) \\ &\leq \frac{-1}{54S_{KP}^2} \left(\int_{\mathbb{R}^2} (\partial_1 \theta_p)^2 \right)^3 + o(1). \end{aligned}$$

This ends the proof of lemma 7.4. \square

We are now able to prove proposition 7.2.

Proof of proposition 7.2 . Using (7.1) and (7.13) we have

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} (\partial_1 \theta_{p_n})^2 \geq \int_{\mathbb{R}^2} N_0^2, \quad (7.17)$$

thus, up to extraction,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} (\partial_1 \theta_{p_n})^2 \rightarrow \mu_0, \text{ when } n \rightarrow \infty, \quad (7.18)$$

where

$$\mu_0 \geq \int_{\mathbb{R}^2} N_0^2 > 0. \quad (7.19)$$

Combining lemma 7.4, (7.18) and lemma 2.1 we obtain

$$E_{KP}(\partial_1 \theta_{p_n}) \rightarrow \mathcal{E}_{min}^{KP}(\mu_0), \quad \text{and} \quad \int_{\mathbb{R}^2} |\partial_1 \theta_{p_n}|^2 \rightarrow \mu_0 > 0, \quad \text{when } n \rightarrow \infty. \quad (7.20)$$

□

We can now obtain global convergence thanks to proposition 2.2.

Proposition 7.5. *Let $(p_n)_{n \in \mathbb{N}}$ a sequence which converges to 0. Up to an extraction, there exists a ground state N_0 such that*

$$\partial_1 \theta_{p_n} \rightarrow N_0 \text{ in } Y(\mathbb{R}^2), \quad \text{and} \quad N_{p_n} \rightarrow N_0 \text{ in } L^2(\mathbb{R}^2), \quad \text{when } n \rightarrow \infty. \quad (7.21)$$

Proof. Using proposition 7.2 there exists a constant μ_0 , such that, up to a subsequence,

$$E_{KP}(\partial_1 \theta_{p_n}) \rightarrow \mathcal{E}_{min}^{KP}(\mu_0), \quad \text{and} \quad \int_{\mathbb{R}^2} |\partial_1 \theta_{p_n}|^2 \rightarrow \mu_0 > 0, \quad \text{when } n \rightarrow \infty. \quad (7.22)$$

Then using proposition 2.2, there exists $(a_n)_{n \in \mathbb{N}}$ and a ground state \tilde{N}_0 with speed $\sigma = \frac{\mu_0^2}{(\mu^*)^2}$, such that

$$\partial_1 \theta_{p_n}(\cdot - a_n) \xrightarrow[n \rightarrow \infty]{} \tilde{N}_0 \text{ in } Y(\mathbb{R}^2). \quad (7.23)$$

So by (7.13) we have

$$N_{p_n}(\cdot - a_n) \xrightarrow[n \rightarrow \infty]{} \tilde{N}_0 \text{ in } L^2(\mathbb{R}^2). \quad (7.24)$$

Using proposition 7.1 we have

$$\forall 1 < q < \infty, \forall k \in \mathbb{N}, N_{p_n}(\cdot - a_n) \rightarrow N_0 \text{ in } W^{k,q}(\mathbb{R}^2), \quad \text{when } n \rightarrow \infty,$$

with N_0 a solution of (SW). Thus by (7.24) $N_0 = \tilde{N}_0$, and \tilde{N}_0 is a ground state of speed 1. We will now prove the convergence of N_{p_n} and $\partial_1 \theta_{p_n}$. Using the continuity of the translation in $L^q(\mathbb{R}^2)$ for any $1 \leq q < \infty$, if $a_n \rightarrow a$ then

$$\partial_1 \theta_{p_n}(\cdot - a) \xrightarrow[n \rightarrow \infty]{} N_0 \text{ in } Y(\mathbb{R}^2), \quad \text{and} \quad N_{p_n}(\cdot - a) \xrightarrow[n \rightarrow \infty]{} N_0 \text{ in } L^2(\mathbb{R}^2).$$

Thus, it is sufficient to prove that $(a_n)_{n \in \mathbb{N}}$ is bounded. By contradiction assume, up to an extraction, that $(a_n)_{n \in \mathbb{N}}$ satisfies

$$a_n \xrightarrow[n \rightarrow \infty]{} \infty.$$

Then combining (7.2) and (1.10) there exists $C > 0$ not depending on n such that

$$\int_{B(0,1)} N_{p_n}^2 > 2C, \quad (7.25)$$

thus by (7.24) we have

$$\int_{B(0,1)} |N_0(x + a_n) - N_{p_n}(x)|^2 dx \xrightarrow{n \rightarrow \infty} 0. \quad (7.26)$$

Then for n large enough

$$\int_{B(0,1)} |N_0(x + a_n)|^2 dx \geq C, \quad (7.27)$$

which is absurd since for any $f \in L^2$ we have

$$\int_{B(0,1)} |f(x + a_n)|^2 dx \xrightarrow{n \rightarrow \infty} 0, \quad (7.28)$$

this concludes the proof of proposition 7.5. \square

Proof of theorem 2. Combining proposition 7.5 (L^2 convergence), theorem 5 (boundedness in $W^{k,q}$) and an interpolation argument we have

$$N_{p_n} \xrightarrow{n \rightarrow \infty} N_0 \text{ in } W^{k,q}(\mathbb{R}^2). \quad (7.29)$$

Since the embedding $Y(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$ is continuous the same argument gives

$$\partial_1 \theta_{p_n} \xrightarrow{n \rightarrow \infty} N_0 \text{ in } W^{k,q}(\mathbb{R}^2). \quad (7.30)$$

\square

A Soliton in the one dimensional case

We begin with some reminders about nonlinear Schrödinger equations in dimension 1. Traveling wave solutions to (TWc) are related to the soliton of the generic Korteweg-de Vries equation (see [2]). For traveling waves solution to (NLS) the transonic limit can lead to solitons of the modified (KdV) equation or even the generalized (KdV) (see [9]). We show in this section that traveling waves of (E-K) exhibit similar properties.

The existence of traveling waves in dimension one for the Euler-Korteweg equation follows from basic ode arguments that we sketch here. The Euler-Korteweg system, in dimension one, reads

$$\begin{cases} -c\rho' + (\rho u)' = 0, & (1) \\ -cu' + \left(\frac{u^2}{2}\right)' + g' = \left(K(\rho)\rho'' + \frac{1}{2}K'(\rho)\rho'^2\right)' & (2), \quad x \in \mathbb{R}. \end{cases} \quad (\text{E-K})$$

We assume

$$g(1) = 0, \quad g'(1) = 1, \quad \Gamma = g''(1) + 3 \neq 0, \quad \gamma = \frac{1}{g''(1) + 3}, \quad c \in]0, 1[. \quad (\text{A.1})$$

We will study solitons whose limits are

$$\rho_{\pm\infty} = 1; \rho'_{\pm\infty} = 0, \rho''_{\pm\infty} = 0, u_{\pm\infty} = 0; u'_{\pm\infty} = 0, u''_{\pm\infty} = 0. \quad (\text{A.2})$$

As in dimension two, we let

$$\varepsilon = \sqrt{1 - c^2}. \quad (\text{A.3})$$

By integrating (1) on $[x, +\infty[$ we obtain

$$-c\rho + \rho u = -c, \quad (\text{A.4})$$

so

$$u = \frac{c(\rho - 1)}{\rho}. \quad (\text{A.5})$$

Using (A.5) and (2), we have, by integration on $[x, +\infty[$,

$$-c^2 \frac{(\rho - 1)}{\rho} + \frac{c^2(\rho - 1)^2}{2\rho^2} + g(\rho) = K(\rho)\rho'' + \frac{1}{2}K'(\rho)\rho'^2. \quad (\text{A.6})$$

Multiplying by ρ' and integrating on $[x, +\infty[$, we have

$$\frac{1}{2}K(\rho)(\rho')^2 = \frac{-c^2}{2\rho}(\rho - 1)^2 + G(\rho). \quad (\text{A.7})$$

where G is a primitive of g such that $G(1) = 0$.

The case $\Gamma \neq 0$ We define the function

$$\frac{-c^2}{2\rho}(\rho - 1)^2 + G(\rho) = F_\varepsilon(\rho), \quad (\text{A.8})$$

and

$$\rho_{m,\varepsilon} = \begin{cases} \sup \{ \rho < 1, F_\varepsilon(\rho) = 0 \}, & \text{if } \gamma > 0. \\ \inf \{ \rho > 1, F_\varepsilon(\rho) = 0 \}, & \text{if } \gamma < 0. \end{cases} \quad (\text{A.9})$$

Then we have:

Lemma A.1. *For $\varepsilon \ll 1$, we have*

$$1 - (3\gamma\varepsilon^2 + |\gamma|\varepsilon^3) \leq \rho_{m,\varepsilon} \leq 1 - (3\gamma\varepsilon^2 - |\gamma|\varepsilon^3). \quad (\text{A.10})$$

Proof. The lemma is a direct consequence of the intermediate value theorem and the fact that

$$F_\varepsilon(1 - \gamma\alpha\varepsilon^2) = \varepsilon^6\alpha^2\gamma^2\left(1 - \frac{\alpha}{3} + \mathcal{O}(\varepsilon^2\alpha)\right).$$

□

Remark A.2. *For $\varepsilon \ll 1$,*

- If $\gamma > 0$, $0 < \rho_{m,\varepsilon} < 1$, $F_\varepsilon(\rho_{m,\varepsilon}) = 0$ and $F_\varepsilon(\rho) > 0$ for $\rho \in]\rho_{m,\varepsilon}, 1[$.
- If $\gamma < 0$, $1 < \rho_{m,\varepsilon} < 2$, $F_\varepsilon(\rho_{m,\varepsilon}) = 0$ and $F_\varepsilon(\rho) > 0$ for $\rho \in]1, \rho_{m,\varepsilon}[$.

To continue our reasoning, we need informations on the derivative.

$$\gamma F'_\varepsilon(\rho_{m,\varepsilon}) > 0, \quad (\text{if } 1 - c \ll 1).$$

Indeed, we will prove, if $\gamma > 0$, that the soliton decreases from 1 to $\rho_{m,\varepsilon}$ between $] - \infty, 0]$ then increases between $[0, \infty[$. If $\gamma < 0$, the soliton increases from 1 to $\rho_{m,\varepsilon}$ between $] - \infty, 0]$ then decreases between $[0, \infty[$. But if $F'_\varepsilon(\rho_{m,\varepsilon}) = 0$ then the Cauchy solution of (A.6) such that $\rho(0) = \rho_{m,\varepsilon}$, $\rho'(0) = 0$ is stationary. And in this case there is no soliton that converges to 1.

Lemma A.3. For $\varepsilon \ll 1$

$$\gamma F'_\varepsilon(\rho_{m,\varepsilon}) > 0. \quad (\text{A.11})$$

Proof. Consequence of lemma A.1 and elementary computation. \square

We can now conclude

Proposition A.4. Under the conditions

$$g(1) = 0, \quad g'(1) = 1, \quad \Gamma \neq 0.$$

For $\varepsilon \ll 1$, there exists ρ solution of (A.6) with

$$\rho(0) = \rho_{m,\varepsilon}, \quad \rho'(0) = 0,$$

and

$$u = \frac{c(\rho - 1)}{\rho}.$$

Then (ρ, u) is a global solution of

$$\begin{cases} -c\rho' + (\rho u)' = 0, & (1) \\ \left[cu' + \left(\frac{u^2}{2} \right)' + g' = \left(K(\rho)\rho'' + \frac{1}{2}K'(\rho)\rho'^2 \right)' \right] & (2), \quad x \in \mathbb{R}, \end{cases} \quad (\text{E-K})$$

such that

$$\rho_{\pm\infty} = 1; \quad \rho'_{\pm\infty} = 0, \quad \rho''_{\pm\infty} = 0, \quad u_{\pm\infty} = 0; \quad u'_{\pm\infty} = 0, \quad u''_{\pm\infty} = 0.$$

Moreover if $\gamma > 0$

- ρ is increasing on $[0, +\infty[$ and even.

If $\gamma < 0$

- ρ is decreasing on $[0, +\infty[$ and even.

Proof. Multiplying the equation (A.6) by ρ' and integrating on $[0, x]$ we obtain

$$\frac{1}{2}K(\rho)(\rho')^2 = \frac{-c^2}{2\rho}(\rho - 1)^2 + G(\rho) = F(\rho). \quad (\text{A.12})$$

We have, by (A.11), $\rho''(0) = F'(\rho_m)$ has the same sign as γ . So near 0, if $\gamma > 0$ ρ' is increasing. The rest of the proof is deduced using basic ode argument and phase portrait analysis (see figure 1, 2). \square

We let, as in the two-dimensional case

$$\rho - 1 = -\varepsilon^2 \gamma r_\varepsilon \left(\frac{\varepsilon x}{\sqrt{K(1)}} \right). \quad (\text{A.13})$$

Then we have

Proposition A.5. *Let*

$$N(x) = \frac{3}{ch^2(\frac{x}{2})}, \quad (\text{A.14})$$

the classical soliton to the Korteweg-de-Vries equation (KdV). Then

$$r_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} N, \text{ in } C^0(\mathbb{R}). \quad (\text{A.15})$$

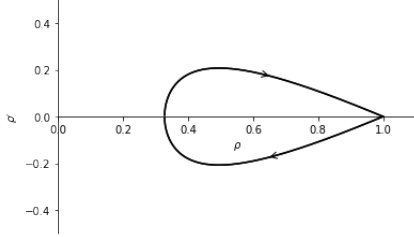


Figure 1: $K=1$, $\Gamma > 0$, $\varepsilon = 0.82$,
 $G(x) = \frac{(x-1)^2}{2}$.

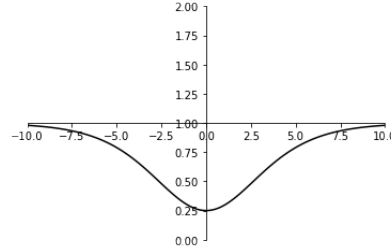


Figure 2: Graph of $\rho(x)$.

First of all we have

Lemma A.6. *For any $\varepsilon \ll 1$, r is even, increasing on $]-\infty, 0]$ then decreasing on $[0, +\infty[$. Moreover*

$$r_\varepsilon(0) \xrightarrow{\varepsilon \rightarrow 0} 3,$$

and there exists a constant C , not depending on ε , such that

$$\|r'_\varepsilon\|_\infty \leq C.$$

Proof. The variations of r are an immediate consequence of the proposition A.4. Using (A.13) and lemma A.1 we have for $\varepsilon \ll 1$

$$3 - \frac{\varepsilon}{|\gamma|} \leq r_\varepsilon(0) \leq 3 + \frac{\varepsilon}{|\gamma|}.$$

Using (A.12) we obtain

$$\frac{\gamma^2}{2K(1)} K(\rho) r'_\varepsilon{}^2 = \frac{r_\varepsilon^2 \gamma^2}{2} \left(1 - r_\varepsilon \gamma \frac{1}{3\gamma} \right) + O(\varepsilon^2 r_\varepsilon^3).$$

Since $K(1) \neq 0$ and $\rho \xrightarrow{\varepsilon \rightarrow 0} 1$ in $C^0(\mathbb{R})$ there exists M not depending on ε such that

$$\|r'_\varepsilon\|_\infty \leq M,$$

this ends the proof. \square

Proof of proposition A.5. We recall that

$$-c^2 \frac{(\rho-1)}{\rho} + \frac{c^2(\rho-1)^2}{2\rho^2} + g(\rho) = K(\rho)\rho'' + \frac{1}{2}K'(\rho)\rho'^2. \quad (\text{A.16})$$

After lengthy but simple computations, we find that $(r_\varepsilon, r'_\varepsilon)$ is a solution of

$$X' = f(t, X, \varepsilon), \quad X(0) = \begin{pmatrix} r_\varepsilon(0) \\ 0 \end{pmatrix},$$

with

$$f : \mathbb{R} \times U \times]-\delta, \delta[\rightarrow \mathbb{R}^2$$

$$f\left(t, \begin{pmatrix} r \\ r' \end{pmatrix}, \varepsilon\right) = \begin{pmatrix} r' \\ g(r, r', \varepsilon) \end{pmatrix},$$

where $U =]0, 4[\times]-M-1, M+1[$,

$$g(r, r', \varepsilon) = r - \frac{r^2}{2} + \varepsilon^2 R_1(\varepsilon^2 r) + \varepsilon^2 r^3 P_1(\varepsilon^2 r) + \varepsilon^2 R_2(\varepsilon^2 r) r'^2.$$

with $P_1, R_1, R_2 \in C^\infty(]-1, 1[)$. We recall that

$$N(x) = \frac{3}{ch^2(x/2)}.$$

Moreover N is a solution of

$$X' = f(t, X, 0), \quad X(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Then as r_ε is a solution of

$$X' = f(t, X, \varepsilon), \quad X(0) = \begin{pmatrix} r_\varepsilon(0) \\ 0 \end{pmatrix},$$

with $r_\varepsilon(0) \rightarrow 3$, we obtain, using the Cauchy-Lipschitz theorem with parameter, that for any compact set $[a, b]$

$$r_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} N, \text{ in } C^0([a, b]), \quad (\text{A.17})$$

and

$$r'_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} N', \text{ in } C^0([a, b]). \quad (\text{A.18})$$

We have using proposition A.4 and (A.13) that the functions N, r_ε are increasing on $] - \infty, 0]$, decreasing on $[0, \infty[$ and converge to 0 in $+\infty$ and $-\infty$. Thus we obtain

$$r_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} N, \text{ in } C^0(\mathbb{R}).$$

which concludes the proof. \square

The following proposition ends the proof of proposition 1.4

Proposition A.7. *We have for all $k \in \mathbb{N}$*

$$\|r_\varepsilon^{(k)} - N^{(k)}\|_{L^\infty(\mathbb{R})} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Proof. We already know the result for $k = 0$, we prove it for $k = 1$. We have by (A.18) uniform convergence on any compact. As r_ε converges uniformly to N and $N(x) \xrightarrow{|x| \rightarrow \infty} 0$ then $\sup_{\varepsilon' \leq \varepsilon} \sup_{M \leq |x|} r'_{\varepsilon'}(x) \xrightarrow{M \rightarrow \infty} 0$. Since we have

$$\frac{\gamma^2}{2K(1)} K(\rho) r'^2_\varepsilon = \frac{r_\varepsilon^2 \gamma^2}{2} \left(1 - r_\varepsilon \gamma \frac{1}{3\gamma} \right) + O(\varepsilon^2 r_\varepsilon^3). \quad (\text{A.19})$$

we obtain that $\sup_{\varepsilon' \leq \varepsilon} \sup_{M \leq |x|} r'_{\varepsilon'} \xrightarrow{|x| \rightarrow \infty} 0$. Moreover $N'(x) \xrightarrow{|x| \rightarrow \infty} 0$ so we obtain

$$\forall \eta > 0, \exists \varepsilon', \exists M, (|x| > M, \varepsilon < \varepsilon' \implies |r'_\varepsilon(x) - N'(x)| \leq |r'_\varepsilon(x)| + |N'(x)| \leq \eta),$$

combining with (A.15) we have

$$\|r_\varepsilon^{(1)} - N^{(1)}\|_{L^\infty(\mathbb{R})} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (\text{A.20})$$

The result for higher order follows by use of the ODE and a simple induction argument. \square

Remark A.8. *Using (A.5) and letting*

$$u(x) = -\varepsilon^2 \gamma v_\varepsilon \left(\frac{\varepsilon x}{\sqrt{K(1)}} \right), \quad (\text{A.21})$$

we obtain

$$v_\varepsilon(x) = \frac{\sqrt{1 - \varepsilon^2}}{1 - \varepsilon^2 \gamma r_\varepsilon(x)} r_\varepsilon(x),$$

so by the proposition A.7 we have that

$$\|r_\varepsilon - N\|_{L^\infty(\mathbb{R})} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

On the other hand by using Leibniz formula one has, for all $n \in \mathbb{N}$

$$v_\varepsilon(x) = \sum_{k=0}^{n-1} \left(\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon^2\gamma r_\varepsilon(x)} \right)^{(k)} r_\varepsilon^{(n-k)}(x) + \left(\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon^2\gamma r_\varepsilon(x)} \right) r_\varepsilon^{(n)}(x),$$

and therefore by proposition (A.7) we have for all $n \in \mathbb{N}$

$$\|v_\varepsilon^{(n)} - N^{(n)}\|_{L^\infty(\mathbb{R})} \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (\text{A.22})$$

a result similar to the one obtained in dimension 2.

The case $\Gamma = 0$ The proof of proposition 1.6 is similar to what was done earlier (see figure 3 for a phase portrait). let us give a heuristic argument. Substituting

$$\rho = 1 + \varepsilon r \left(\frac{\varepsilon x}{\sqrt{K(1)}} \right),$$

into (A.6) we find

$$\left(\frac{g'''(1) - 12}{6} \right) r^3 + r = r'' + O(\varepsilon).$$

Then differentiating this equation we have

$$\frac{1}{2} w^2 w' + w' = w''',$$

where

$$w = \frac{1}{\sqrt{g'''(1) - 12}}.$$

For a similar (and more general) result in the case of the non-linear Schrödinger equation, see [9].

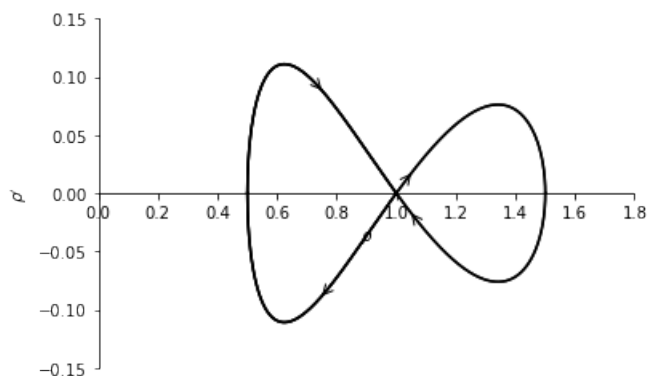


Figure 3: $K=1$, $\Gamma = 0$, $\varepsilon=0.5$, $G(x) = \frac{(x-1)^2}{2} - \frac{(x-1)^3}{2}$.

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