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# The Emerton–Gee stack of rank one $(\varphi, \Gamma)$ -modules

DAT PHAM

## Abstract

We give a classification of rank one  $(\varphi, \Gamma)$ -modules with coefficients in a  $p$ -adically complete  $\mathbf{Z}_p$ -algebra. As a consequence, we obtain a new proof of [EG20, Prop. 7.2.17], which gives an explicit description of the Emerton–Gee stack of  $(\varphi, \Gamma)$ -modules in the rank one case.

## 1 Introduction

For simplicity, we fix our local field to be  $\mathbf{Q}_p$  in this introduction. Let  $\mathbf{A}'_{\mathbf{Q}_p}$  be the  $p$ -adic completion of the Laurent series ring  $\mathbf{Z}_p((T))$ , endowed with the usual commuting semilinear actions of  $\varphi$  and  $\Gamma = \mathbf{Z}_p^\times$ . An étale  $(\varphi, \Gamma)$ -module is, by definition, a finite  $\mathbf{A}'_{\mathbf{Q}_p}$ -module endowed with commuting semilinear actions of  $\varphi$  and  $\Gamma$ , with the property that the linearized action of  $\varphi$  is an isomorphism. The most important feature of étale  $(\varphi, \Gamma)$ -modules is that they are naturally equivalent to continuous representations of  $G_{\mathbf{Q}_p}$  on finite  $\mathbf{Z}_p$ -modules (cf. [Fon90]).

In [EG20], Emerton and Gee define and study moduli stacks parametrizing families of étale  $(\varphi, \Gamma)$ -modules. More specifically, they consider the stack  $\mathcal{X}_d$  over  $\mathrm{Spf} \mathbf{Z}_p$  whose groupoid of  $A$ -valued points, for any  $p$ -adically complete  $\mathbf{Z}_p$ -algebra  $A$ , is given by the groupoid of rank  $d$  projective étale  $(\varphi, \Gamma)$ -modules over  $\mathbf{A}'_{\mathbf{Q}_p, A} := \mathbf{A}'_{\mathbf{Q}_p} \widehat{\otimes}_{\mathbf{Z}_p} A$ . The geometry of the stack  $\mathcal{X}_d$  has been studied extensively in [EG20]. In particular, the authors show that  $\mathcal{X}_d$  is a Noetherian formal algebraic stack, and moreover, its underlying reduced substack is an algebraic stack of finite type over  $\mathbf{F}_p$ , whose irreducible components admit a natural labelling by Serre weights.

The goal of this note is to prove the following classification of families of rank one étale  $(\varphi, \Gamma)$ -modules.

**Theorem 1.1** (Theorem (3.1)). *Let  $A$  be a  $p$ -adically complete  $\mathbf{Z}_p$ -algebra. Let  $M$  be a rank one étale  $(\varphi, \Gamma)$ -module with  $A$ -coefficients. Then  $M$  has the form  $\mathbf{A}'_{\mathbf{Q}_p, A}(\delta) \otimes_A L$  for some character  $\delta : \mathbf{Q}_p^\times \rightarrow A^\times$  and some invertible  $A$ -module  $L$ . Here,  $\mathbf{A}'_{\mathbf{Q}_p, A}(\delta)$  denotes the free  $(\varphi, \Gamma)$ -module of rank 1 with a basis  $v$  for which  $\varphi(v) = \delta(p)v$  and  $\gamma(v) = \delta(\gamma)v$  for  $\gamma \in \mathbf{Z}_p^\times$ .*

As a consequence, we deduce the following explicit description of the stack  $\mathcal{X}_1$ .

**Corollary 1.2** (Corollary (3.2)). *There is an isomorphism of stacks*

$$\left[ \left( \mathbf{Z}_p[[\mathbf{Z}_p^\times]] \times \widehat{\mathbf{G}}_m \right) / \widehat{\mathbf{G}}_m \right] \xrightarrow{\sim} \mathcal{X}_1,$$

where  $\widehat{\mathbf{G}}_m$  denotes the  $p$ -adic completion of  $\widehat{\mathbf{G}}_{m, \mathbf{Z}_p}$ , and in the formation of the quotient stack, the action of  $\widehat{\mathbf{G}}_m$  is taken to be trivial.

We emphasize that the above description is already given in [EG20, Prop. 7.2.17]. However, our argument here is different; in particular it avoids the use of uniform bounds on the ramification of families of characters valued in finite Artinian algebras (see [EG20, §7.3]).

**Notation.** We mostly follow the notation in [EG20]. In particular, we fix a finite extension  $K/\mathbf{Q}_p$  with residue field  $k$  and inertia degree  $f$ . Fix also an algebraic closure  $\overline{K}$  of  $K$ , with absolute Galois group  $G_K$ , Weil group  $W_K$ , and inertia group  $I_K$ . As usual,  $W_K^{\mathrm{ab}}$  denotes the abelianization of  $W_K$ , while  $I_K^{\mathrm{ab}}$  denotes the image of  $I_K$  in  $W_K^{\mathrm{ab}}$ . We denote by  $\mathbf{C}^{\flat}$  the tilt of the completion  $\mathbf{C} := \widehat{\overline{K}}$ , by  $K_{\mathrm{cyc}}$  the

cyclotomic  $\mathbf{Z}_p$ -extension of  $K$  and by  $k_\infty$  its residue field. We also fix a finite extension  $E/\mathbf{Q}_p$  with ring of integers  $\mathcal{O}$ , which will serve as the base of our coefficients. As usual,  $\varpi$  (resp.  $\mathbf{F}$ ) denotes a uniformizer (resp. the residue field) of  $\mathcal{O}$ . We will assume throughout that  $\mathbf{F}$  contains  $k$ .

We refer the reader to [EG20, §2.2] for the definition of the coefficient rings  $\mathbf{A}_{K,A}$  of our  $(\varphi, \Gamma)$ -modules. Finally, as the field  $K$  is fixed throughout, we will often drop  $K$  from the notation in what follows.

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## 2 $(\varphi, \Gamma)$ -modules associated to characters of the Weil group

In this section, we explain how to associate a free étale  $(\varphi, \Gamma)$ -module of rank 1 to any character of  $W_K$ .

First recall the following result of Dee, which is a generalization of Fontaine's equivalence between Galois representations on finite  $\mathbf{Z}_p$ -modules and étale  $(\varphi, \Gamma)$ -modules to the context with coefficients. For simplicity, we only state the result for Artinian coefficients.

**Theorem 2.1** ([Dee01]). *Let  $A$  be a finite Artinian local  $\mathcal{O}$ -algebra, and let  $W(\mathbf{C}^b)_A := W(\mathbf{C}^b) \otimes_{\mathbf{Z}_p} A$ . Then the functor*

$$V \mapsto \mathbf{D}_A(V) := (W(\mathbf{C}^b)_A \otimes_A V)^{G_{K_{\text{cyc}}}}$$

*defines an equivalence between the category of finite free  $A$ -modules with a continuous action of  $G_K$ , and the category of finite projective étale  $(\varphi, \Gamma)$ -modules with  $A$ -coefficients. A quasi-inverse functor is given by  $M \mapsto T_A(M) := (W(\mathbf{C}^b)_A \otimes_{\mathbf{A}_{K,A}} M)^{\varphi=1}$ .*

We want to extend the above construction of rank one étale  $(\varphi, \Gamma)$ -modules from Galois character to the case where  $A$  is an arbitrary  $\varpi$ -adically complete  $\mathcal{O}$ -algebra. We begin with the case of unramified characters.

**Lemma 2.2.** *Let  $A$  be an  $\mathcal{O}$ -algebra, and let  $a \in A^\times$ . Then, up to isomorphism, there is a unique free étale  $\varphi$ -module  $D_{K,a}$  of rank one over  $W(k) \otimes_{\mathbf{Z}_p} A$  with the property that  $\varphi^f = 1 \otimes a$  on  $D_{K,a}$ .*

*Proof.* We need to show that the norm map  $(W(k) \otimes_{\mathbf{Z}_p} A)^\times \rightarrow A^\times, x \mapsto N(x) := x\varphi(x) \dots \varphi^{f-1}(x)$  is surjective with kernel given by the set  $\{\varphi(y)/y \mid y \in (W(k) \otimes_{\mathbf{Z}_p} A)^\times\}$ . Since  $\mathbf{F}$  is assumed to contain  $k$ ,  $\mathcal{O}$  (and hence  $A$ ) is naturally a  $W(k)$ -algebra. In particular, we have an  $A$ -algebra isomorphism  $W(k) \otimes_{\mathbf{Z}_p} A \rightarrow \prod A, x \otimes 1 \mapsto (x, \varphi(x), \dots, \varphi^{f-1}(x))$ . The lemma then follows easily using this isomorphism.  $\square$

**Definition 2.3.** Let  $A$  be a  $\varpi$ -adically complete  $\mathcal{O}$ -algebra, and let  $a \in A^\times$ . Define  $\mathbf{A}_{K,A}(\text{ur}_a) := D_{K,a} \otimes_{W(k) \otimes_{\mathbf{Z}_p} A} \mathbf{A}_{K,A}$ . This is a rank one étale  $(\varphi, \Gamma)$ -module with  $A$ -coefficients, where we let  $\varphi$  act diagonally, and  $\Gamma$  act on the second factor.

**Lemma 2.4.** *Let  $A$  be a finite Artinian local  $\mathcal{O}$ -algebra, and let  $a \in A^\times$ . Then, under Dee's equivalence (2.1),  $\mathbf{A}_{K,A}(\text{ur}_a)$  corresponds to the unramified character  $\text{ur}_a$  of  $G_K$  sending geometric Frobenii to  $a$ .*

*Proof.* By definition, the rank one  $A$ -representation of  $G_K$  corresponding to  $\mathbf{A}_{K,A}(\text{ur}_a)$  is given by

$$\begin{aligned} V &:= (W(\mathbf{C}^b)_A \otimes_{\mathbf{A}_{K,A}} \mathbf{A}_{K,A}(\text{ur}_a))^{\varphi=1} \\ &\cong \{hv \mid h \in W(\mathbf{C}^b) \otimes_{\mathbf{Z}_p} A \text{ such that } \varphi(hv) = hv\}, \end{aligned}$$

where  $v$  is a basis of  $D_{K,a}$ . Assume  $V$  has a basis  $hv$  with  $h \in W(\overline{\mathbf{F}}_p) \otimes_{\mathbf{Z}_p} A$ . We verify that  $G_K$  acts on this basis via the unramified character taking geometric Frobenii to  $a$ . First, for  $\sigma \in I_K$ , we have  $\sigma(hv) = \sigma(h)v = hv$  (note that  $\sigma(h) = h$  as  $h \in W(\overline{\mathbf{F}}_p) \otimes_{\mathbf{Z}_p} A$ ). Now, let  $\sigma = \varphi_q^{-1}$  be an arithmetic

Frobenius. From the relation  $\varphi(hv) = hv$  and the fact that  $\varphi^f(v) = av$ , we obtain  $\sigma(h)av = \varphi^f(hv) = hv$  whence  $\sigma(hv) = \sigma(h)v = a^{-1}(hv)$ , as desired.

It remains to find a basis as stated. If  $A$  is a field, say  $A = \mathbf{F}_q$  for some finite extension  $\mathbf{F}_q/\mathbf{F}$ , this can be done by using the ring isomorphism  $\mathbf{C}^b \otimes_{\mathbf{F}_p} \mathbf{F}_q \xrightarrow{\sim} \prod \mathbf{C}^b$ . Indeed,  $h$  is a vector in  $\prod \mathbf{C}^b$  whose coordinates satisfy a finite set of polynomial equations with coefficients in  $\overline{\mathbf{F}}_p$ , so it necessarily lies in  $\prod \overline{\mathbf{F}}_p$ . In general, by factoring  $A \rightarrow A/\mathfrak{m}_A$  as a chain of square-zero thickenings, we may assume that, for some ideal  $I$  with  $I^2 = 0$ , there is a basis  $\overline{h}v$  of  $V/IV$  with  $\overline{h} \in W(\overline{\mathbf{F}}_p) \otimes_{\mathbf{Z}_p} (A/I)$ . Let  $h \in W(\overline{\mathbf{F}}_p) \otimes_{\mathbf{Z}_p} A$  be a lift of  $\overline{h}$ . Then  $\varphi(hv) = g(hv)$  for some  $g \in 1 + W(\overline{\mathbf{F}}_p) \otimes I$ , say  $g = 1 + g_1 \otimes m_1 + \dots + g_n \otimes m_n$  with  $g_i \in W(\overline{\mathbf{F}}_p)$  and  $m_i \in I$ . For each  $i$ , choose  $h_i \in W(\overline{\mathbf{F}}_p)$  such that  $\varphi(h_i) - h_i = -g_i$ . Let  $f := 1 + h_1 \otimes m_1 + \dots + h_n \otimes m_n \in 1 + W(\overline{\mathbf{F}}_p) \otimes I$ . Then  $\varphi(f) = f/g$ , and hence  $\varphi(hfv) = hfv$ . Finally, as  $(hf)v$  lifts a basis of  $V/IV$ , it is a basis of  $V$  by Nakayama's lemma.  $\square$

Recall that  $\widehat{\mathbf{G}}_m$  denotes the  $\varpi$ -adic completion of  $\mathbf{G}_{m,\mathcal{O}}$ . We denote the resulting map  $\widehat{\mathbf{G}}_m \rightarrow \mathcal{X}_1, a \mapsto \mathbf{A}_{K,A}(\text{ur}_a)$  by  $\text{ur}_x$  (where we can think of  $x$  as the coordinate on  $\widehat{\mathbf{G}}_m = \text{Spf } \mathcal{O}[x, x^{-1}]$ ), and refer to it simply as the universal unramified character<sup>1</sup>.

We now consider the case of a general character of the Weil group  $W_K$ . It is convenient to introduce some notation.

**Definition 2.5.** Let  $X^{\text{an}}$  be the functor on  $\varpi$ -adically complete  $\mathcal{O}$ -algebras taking  $A$  to the set of (continuous) characters  $\delta : W_K \rightarrow A^\times$ .

By choosing a geometric Frobenius  $\sigma \in G_K$ , i.e. an isomorphism  $W_K^{\text{ab}} \cong I_K^{\text{ab}} \times \mathbf{Z}$ , we see that  $X^{\text{an}}$  is represented by the Noetherian affine formal scheme  $\text{Spf } \mathcal{O}[[I_K^{\text{ab}}]] \times \widehat{\mathbf{G}}_m$  over  $\text{Spf } \mathcal{O}$ . We endow  $X^{\text{an}}$  with the trivial action of  $\widehat{\mathbf{G}}_m$ .

**Lemma 2.6.** *There is a morphism  $X^{\text{an}} \rightarrow \mathcal{X}_1, \delta \mapsto \mathbf{A}_{K,A}(\delta)$  with the property that for any finite Artinian  $\mathcal{O}$ -algebra  $A$ , the  $(\varphi, \Gamma)$ -module  $\mathbf{A}_{K,A}(\delta)$  corresponds, under Dee's equivalence (2.1), to the character  $\delta$ .*

*Proof.* Fix a geometric Frobenius  $\sigma \in G_K$ , and hence an isomorphism  $X^{\text{an}} \cong \text{Spf } \mathcal{O}[[I_K^{\text{ab}}]] \times \widehat{\mathbf{G}}_m$ . For each finite Artinian quotient  $A$  of  $\mathcal{O}[[I_K^{\text{ab}}]]$ , we extend the map  $I_K^{\text{ab}} \rightarrow A^\times$  to a character  $G_K \rightarrow A^\times$  by taking  $\sigma$  to 1. Under Dee's equivalence (2.1), this gives rise to a rank one étale  $(\varphi, \Gamma)$ -module with  $A$ -coefficients, i.e. an object of  $\mathcal{X}_1(A)$ . As  $\mathcal{O}[[I_K^{\text{ab}}]]$  is the inverse limit of all such quotients  $A$ , we obtain a map  $\text{Spf } \mathcal{O}[[I_K^{\text{ab}}]] \rightarrow \mathcal{X}_1$ . We can now define  $X^{\text{an}} \rightarrow \mathcal{X}_1$  as the composite

$$X^{\text{an}} \cong \text{Spf } \mathcal{O}[[I_K^{\text{ab}}]] \times \widehat{\mathbf{G}}_m \rightarrow \mathcal{X}_1 \times \widehat{\mathbf{G}}_m \rightarrow \mathcal{X}_1,$$

where the last map is given by taking tensor product with the universal unramified character  $\text{ur}_x$ . That the map is compatible with Dee's equivalence in case of Artinian coefficients follows immediately from construction and Lemma (2.4).  $\square$

It is easy to check that the construction  $\delta \mapsto \mathbf{A}_{K,A}(\delta)$  is independent on our choice of  $\sigma$  (see also Remark (3.10) below), and that  $\mathbf{A}_{K,A}(\delta_1 \delta_2) \cong \mathbf{A}_{K,A}(\delta_1) \otimes_{\mathbf{A}_{K,A}} \mathbf{A}_{K,A}(\delta_2)$  for all  $\delta_i$ . Of course, in case  $\delta = \text{ur}_a$  is an unramified character, this agrees with the notation introduced earlier. For a  $(\varphi, \Gamma)$ -module  $M$  with  $A$ -coefficients, we set  $M(\delta) := M \otimes_{\mathbf{A}_{K,A}} \mathbf{A}_{K,A}(\delta)$ .

### 3 The main result

Our main result in this text is the following.

**Theorem 3.1.** *Let  $A$  be a  $\varpi$ -adically complete  $\mathcal{O}$ -algebra. Let  $M$  be a rank one étale  $(\varphi, \Gamma)$ -module with  $A$ -coefficients. Then there exist a unique continuous character  $\delta : W_K \rightarrow A^\times$  and a unique (up to isomorphism) invertible  $A$ -module  $L$ , such that  $M \cong \mathbf{A}_{K,A}(\delta) \otimes_A L$ .*

<sup>1</sup>Note that the definition given in [EG20, §5.3] of the map  $\text{ur}_x$  is slightly incorrect in the sense that it does not agree with the construction of Dee for Artinian coefficients.

**Corollary 3.2** ([EG20, Prop. 7.2.17]). *The map  $X^{\text{an}} \rightarrow \mathcal{X}_1, \delta \mapsto \mathbf{A}_{K,A}(\delta)$  induces an isomorphism*

$$\left[ X^{\text{an}} / \widehat{\mathbf{G}}_m \right] \xrightarrow{\sim} \mathcal{X}_1.$$

*Proof of Corollary (3.2).* Since  $X^{\text{an}}$  is endowed with the trivial action of  $\widehat{\mathbf{G}}_m$ , the quotient stack  $[X^{\text{an}} / \widehat{\mathbf{G}}_m]$  is naturally identified with  $[\text{Spf } \mathcal{O} / \widehat{\mathbf{G}}_m] \times_{\text{Spf } \mathcal{O}} X^{\text{an}}$ . In other words, for any  $\varpi$ -adically complete  $\mathcal{O}$ -algebra  $A$ , its groupoid of  $A$ -valued points is equivalent to the groupoid of pairs  $(L, \delta)$  consisting of an invertible  $A$ -module  $L$ , and a character  $\delta \in X^{\text{an}}(A)$ . Define the map

$$\left[ X^{\text{an}} / \widehat{\mathbf{G}}_m \right] \rightarrow \mathcal{X}_1$$

by  $(L, \delta) \mapsto \mathbf{A}_{K,A}(\delta) \otimes_A L$ . The result now follows from Theorem (3.1), and the fact that the automorphism group of any rank one étale  $(\varphi, \Gamma)$ -module is given simply by the scalars:

$$\text{Aut}_{\mathbf{A}_{K,A}, \varphi, \Gamma}(M) = ((M \otimes M^\vee)^{\varphi, \Gamma=1})^\times = (\mathbf{A}_{K,A}^{\varphi, \Gamma=1})^\times = A^\times,$$

here we have used [EG20, Lem. 2.2.19 and Prop. 2.2.12] for the last equality.  $\square$

The rest of this section is devoted to proving Theorem (3.1). We begin with the uniqueness statement.

**Lemma 3.3.** *The uniqueness part of Theorem (3.1) holds.*

*Proof.* As  $(\mathbf{A}_{K,A})^{\varphi=1} = A$ , we necessarily have  $L \cong M(\delta^{-1})^{\varphi=1}$ . After passing to an *fppf* cover of  $A$  trivializing  $L$ , it remains to show that if  $\mathbf{A}_{K,A}(\delta) \cong \mathbf{A}_{K,A}(\delta')$  as  $(\varphi, \Gamma)$ -modules, then  $\delta = \delta'$ . This is equivalent to saying that the natural morphism

$$X^{\text{an}} \times_{\mathcal{X}_1} X^{\text{an}} \rightarrow X^{\text{an}} \times_{\text{Spf } \mathcal{O}} X^{\text{an}} \tag{3.3.1}$$

factors through the diagonal morphism  $X^{\text{an}} \hookrightarrow X^{\text{an}} \times_{\text{Spf } \mathcal{O}} X^{\text{an}}$ . After [EG20, Lem. 7.1.14], it suffices to check the factorization at the level of Artinian points. (Note that the map (3.3.1) is representable by algebraic spaces (even affine), and of finite presentation (being a base change of the diagonal  $\mathcal{X}_1 \rightarrow \mathcal{X}_1 \times_{\text{Spf } \mathcal{O}} \mathcal{X}_1$ , which has the same properties by [EG20, Prop. 3.2.17 (2)]), and hence limit preserving on objects by [EG20, Lem. A.2 (1)]. It is also easy to see that  $X^{\text{an}} \times_{\text{Spf } \mathcal{O}} X^{\text{an}}$  is limit preserving, so it is safe to apply [EG20, Lem. 7.1.14] here.) By design, the map  $\delta \mapsto \mathbf{A}_{K,A}(\delta)$  recovers the equivalence between rank one étale  $(\varphi, \Gamma)$ -modules and Galois characters for Artinian coefficients, so the claim follows immediately in this case.  $\square$

**Remark 3.4.** By Lemma (3.3) and the fact that the automorphism group of any rank one  $(\varphi, \Gamma)$  module is simply  $\widehat{\mathbf{G}}_m$ , we see that the map  $[X^{\text{an}} / \widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$  is at least a monomorphism. Showing that it is in fact essentially surjective (i.e. an isomorphism) is equivalent to showing the existence part of Theorem (3.1).

The next lemma allows us to reduce to the case where our test object  $A$  is a reduced  $\mathbf{F}$ -algebra.

**Lemma 3.5.** *If Theorem (3.1) holds for reduced finite type  $\mathbf{F}$ -algebras  $A$ , then it holds for any  $\varpi$ -adically complete  $\mathcal{O}$ -algebra  $A$ .*

*Proof.* Let  $A$  be an  $\mathcal{O}/\varpi^a$ -algebra for some  $a \geq 1$ , and let  $M$  be a rank one étale  $(\varphi, \Gamma)$ -module with  $A$ -coefficients. We want to show that  $M \cong \mathbf{A}_{K,A}(\delta) \otimes_A L$  for some  $\delta$  and  $L$ . As  $\mathcal{X}_1$  is limit preserving, we may assume  $A$  is Noetherian. We will induct on the nilpotency index  $e$  of the nilradical  $A^{\circ\circ}$ . The case  $e = 1$  is just our assumption. Assume now that  $e \geq 2$ . Let  $I := (A^{\circ\circ})^{e-1}$  and  $\bar{A} := A/I$ . By the induction hypothesis,  $M_{\bar{A}} := M \otimes_A \bar{A}$  has the form  $\mathbf{A}_{K,\bar{A}}(\bar{\delta}) \otimes_{\bar{A}} \tilde{L}$  for some character  $\bar{\delta}$  and some invertible  $\bar{A}$ -module  $\tilde{L}$ . Lifting  $\bar{\delta}$  to a character  $\delta \in X^{\text{an}}(A)$ ,  $\tilde{L}$  to an invertible  $A$ -module  $L$  (recall that finite projective modules always deform uniquely through nilpotent thickenings), and replacing  $M$  with  $M(\delta^{-1}) \otimes_A L^\vee$ , we may assume that  $M_{\bar{A}}$  is trivial. By [EG20, Prop. 5.1.33], the set of isomorphism classes of such  $M$  is

given by  $H^1(\mathcal{C}^\bullet(\mathbf{A}_{K,I}))$ , where  $\mathcal{C}^\bullet(\mathbf{A}_{K,I})$  is the Herr complex of  $\mathbf{A}_{K,I} := I\mathbf{A}_{K,A} \cong (W(k_\infty) \otimes_{\mathbf{Z}_p} I)((t))$ . Namely, given such  $M$ , there is an  $\mathbf{A}_{K,A}$ -basis  $v$  of  $M$  so that  $\varphi(v) = fv$  and  $\gamma(v) = gv$  for some  $f, g \in \ker((\mathbf{A}_{K,A})^\times \rightarrow (\mathbf{A}_{K,\bar{A}})^\times) = 1 + \mathbf{A}_{K,I}$ , where  $\gamma$  is a fixed topological generator of  $\Gamma_K \cong \mathbf{Z}_p$ . The corresponding cohomology class is then given by  $[(f-1, g-1)]$ .

Let  $(\mathcal{O}/\varpi^a)[I]$  be the usual square-zero thickening defined by  $I$ . Using the above description in terms of  $H^1(\mathcal{C}^\bullet(\mathbf{A}_{K,I}))$ , we see that  $M$  arises as the base change of some rank one  $(\varphi, \Gamma)$ -module with  $(\mathcal{O}/\varpi^a)[I]$ -coefficients via the natural map  $(\mathcal{O}/\varpi^a)[I] \rightarrow A$ . Thus we may reduce to the case  $A = (\mathcal{O}/\varpi^a)[I]$ . By writing  $I$  as the colimit of its finite sub- $\mathcal{O}$ -modules and using again the fact that  $\mathcal{X}_1$  is limit preserving, we may assume further that  $I$  is finite over  $\mathcal{O}$ . But in this case  $(\mathcal{O}/\varpi^a)[I]$  is a finite Artinian  $\mathcal{O}$ -algebra, so we are done by using (again) the fact that the construction  $\delta \mapsto \mathbf{A}_{K,A}(\delta)$  recovers the equivalence between Galois representations and  $(\varphi, \Gamma)$ -modules for Artinian coefficients.  $\square$

**Lemma 3.6.** *The map  $\text{ur}_x : \widehat{\mathbf{G}}_m \rightarrow \mathcal{X}_1$  induces a closed immersion  $[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$ .*

*Proof.* As in Remark (3.4), we see that  $\text{ur}_x$  induces a monomorphism  $[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$ . (More formally, this map is given by composing the monomorphism  $[X^{\text{an}}/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$  with the closed immersion  $[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow [X^{\text{an}}/\widehat{\mathbf{G}}_m]$  induced by the closed immersion  $\widehat{\mathbf{G}}_m \hookrightarrow X^{\text{an}}$  classifying unramified characters.) We want to show that this is in fact a closed immersion. First recall that by [EG20, Thm. 4.8.12],  $\mathcal{X}_1$  admits a closed  $\mathcal{O}$ -flat  $p$ -adic formal algebraic substack  $\mathcal{X}_1^{\text{ur}}$ , which is uniquely characterized by the property that, for any finite flat  $\mathcal{O}$ -algebra  $\Lambda$ ,  $\mathcal{X}_1^{\text{ur}}(\Lambda)$  is the subgroupoid of  $\mathcal{X}_1(\Lambda)$  consisting of characters  $G_K \rightarrow \Lambda^\times$  which are (after inverting  $p$ ) crystalline of Hodge–Tate weights 0, or equivalently, unramified characters  $G_K \rightarrow \Lambda^\times$ . (Note that we are free to enlarge the field of coefficients  $E$ ; in particular we may assume that it contains the Galois closure of  $K$  so that the running assumption of [EG20, Thm. 4.8.12] is satisfied.)

We claim that the map  $[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$  factors through the closed substack  $\mathcal{X}_1^{\text{ur}}$ , or equivalently, that the closed immersion

$$\mathcal{X}_1^{\text{ur}} \times_{\mathcal{X}_1} [\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow [\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m]$$

is an isomorphism. As the target is a  $p$ -adic formal algebraic stack of finite type and flat over  $\text{Spf } \mathcal{O}$  (since it admits a smooth cover by the  $p$ -adic formal algebraic space  $\widehat{\mathbf{G}}_m$ , which is of finite type and flat over  $\text{Spf } \mathcal{O}$ ), it follows from [LLLM20, Lem. 7.2.6 (3)] that it suffices to show that for any morphism  $\text{Spf } \Lambda \rightarrow [\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m]$  whose source is a finite flat  $\mathcal{O}$ -algebra (endowed with the  $p$ -adic topology), the composite  $\text{Spf } \Lambda \rightarrow [\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$  factors through  $\mathcal{X}_1^{\text{ur}}$ . This is clear as  $[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m](\Lambda)$  is also equivalent (via the monomorphism  $[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$ ) to the subgroupoid of  $\mathcal{X}_1(\Lambda)$  consisting of unramified characters  $G_K \rightarrow \Lambda^\times$ .

Thus, we have an induced monomorphism

$$[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1^{\text{ur}}.$$

We claim that this is in fact an isomorphism. As the source and target are both  $p$ -adic formal algebraic stacks which are of finite type and flat over  $\text{Spf } \mathcal{O}$ , and moreover  $\mathcal{X}_1^{\text{ur}}$  is analytically unramified by Lemma (3.8) below, it suffices, by [LLLM20, Lem. 7.2.6 (1)], to check that the above map induces an isomorphism on any finite flat  $\mathcal{O}$ -algebra  $\Lambda$ . This follows again from the fact that both sides admit the same moduli description (namely, as unramified characters) on these points.  $\square$

The following lemma is presumably standard, but for lack of a reference, we include a proof here.

**Lemma 3.7.** *Let  $\mathcal{X}$  be a  $p$ -adic formal algebraic stack locally of finite type over  $\text{Spf } \mathcal{O}$ . If  $\mathcal{X}$  admits reduced Noetherian versal rings at all finite type points, then  $\mathcal{X}$  is (residually Jacobson and) analytically unramified in the sense of [Eme, Rmk. 8.23].*

*Proof.* Let  $\mathcal{X}'$  be the associated reduced formal algebraic substack of  $\mathcal{X}$ , as defined in [Eme, Ex. 9.10]. We claim that  $\mathcal{X}'$  is analytically unramified. Choose a smooth (in particular, representable by algebraic spaces) surjection  $\coprod_i \text{Spf } B_i \rightarrow \mathcal{X}'$  where each  $B_i$  is a  $p$ -adically complete  $\mathcal{O}$ -algebra. By construction

of  $\mathcal{X}'$ , we know that the base change  $\mathcal{X}' \times_{\mathcal{X}} \mathrm{Spf} B_i$  is identified with  $\mathrm{Spf}(B_i)_{\mathrm{red}}$  (see [Eme, Ex. 9.10]), and furthermore that each  $\mathrm{Spf}(B_i)_{\mathrm{red}}$  is analytically unramified (e.g. by [Eme, Cor. 8.25], applied to the finite type adic map  $\mathrm{Spf}(B_i)_{\mathrm{red}} \rightarrow \mathrm{Spf} \mathcal{O}$ ). As  $\mathcal{X}'$  receives a smooth surjection from the disjoint union  $\coprod_i \mathrm{Spf}(B_i)_{\mathrm{red}}$  of analytically unramified affine formal algebraic spaces, it is analytically unramified by definition.

We now show that the closed immersion  $\mathcal{X}' \hookrightarrow \mathcal{X}$  is in fact an isomorphism (this will imply that  $\mathcal{X}$  is analytically unramified, as desired). As this can be checked at the level of Artinian points (e.g. by [EG20, Lem. 7.1.14]), it in turn suffices to work with versal rings. More precisely, let  $x \in \mathcal{X}(A)$  be a point valued in a finite Artinian local  $\mathcal{O}$ -algebra  $A$ . By assumption,  $\mathcal{X}$  admits a reduced Noetherian versal ring  $\mathrm{Spf} B$  at the finite type point induced by  $x$ . By versality, the map  $x : \mathrm{Spec} A \rightarrow \mathcal{X}$  factors through  $\mathrm{Spf} B \rightarrow \mathcal{X}$ , so it suffices to show that the latter map factors through  $\mathcal{X}'$ . Choose a smooth cover  $\coprod_i \mathrm{Spf} B_i \rightarrow \mathcal{X}$  as before. It suffices to show that each base change  $\mathrm{Spf} B \times_{\mathcal{X}} \mathrm{Spf} B_i \rightarrow \mathrm{Spf} B_i$  factors through  $\mathrm{Spf}(B_i)_{\mathrm{red}}$ , which in turn will follow once we show that given any smooth morphism  $\mathrm{Spf} C \rightarrow \mathrm{Spf} B \times_{\mathcal{X}} \mathrm{Spf} B_i$ , the composite  $\mathrm{Spf} C \rightarrow \mathrm{Spf} B \times_{\mathcal{X}} \mathrm{Spf} B_i \rightarrow \mathrm{Spf} B_i$  factors through  $\mathrm{Spf}(B_i)_{\mathrm{red}}$ . As any ring map from  $B_i$  to a reduced ring necessarily factors through  $(B_i)_{\mathrm{red}}$ , it suffices to show that  $C$  is reduced. As  $B$  is complete local Noetherian and reduced, it is analytically unramified by definition. The upshot is that we have a smooth morphism  $\mathrm{Spf} C \rightarrow \mathrm{Spf} B$  whose target is analytically unramified (and residually Jacobson, as  $(\mathrm{Spf} B)_{\mathrm{red}} = \mathrm{Spec} B/\mathfrak{m}_B$  is just a point). It then follows from [Eme, Lem. 8.20] that the source  $\mathrm{Spf} C$  is also analytically unramified, and in particular that  $C$  is reduced, as required.  $\square$

**Lemma 3.8.** *The stack  $\mathcal{X}_1^{\mathrm{ur}}$  is analytically unramified.*

*Proof.* This follows from Lemma (3.7), [EG20, Prop. 4.8.10] and [Kis08, Thm. 3.3.8].  $\square$

We can now finish our proof of Theorem (3.1).

*Proof of Theorem (3.1).* It remains to show that the monomorphism  $[X^{\mathrm{an}}/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$  is an isomorphism. In view of Lemma (3.5), it suffices to show that the induced map between underlying reduced substacks

$$\left[ X^{\mathrm{an}}/\widehat{\mathbf{G}}_m \right]_{\mathrm{red}} \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$$

is an isomorphism. As our stacks will all live over  $\mathrm{Spec} \mathbf{F}$  in the rest of this proof, we will drop the subscript  $\mathbf{F}$  for ease of notation. For each ‘‘Serre weight’’  $\delta : I_K \rightarrow \mathbf{F}^\times$  (recall that  $\mathbf{F}$  is assumed to contain  $k$ ), we abusively denote also by  $\delta$  the character of  $W_K$  extending  $\delta$  on  $I_K$  and taking our fixed choice of geometric Frobenius to 1. By twisting  $\delta : \mathrm{Spec} \mathbf{F} \rightarrow X^{\mathrm{an}}$  by unramified characters, we obtain a map  $\mathbf{G}_m \rightarrow X^{\mathrm{an}}$ . The induced map  $\coprod_{\delta} \mathbf{G}_m \rightarrow (X^{\mathrm{an}})_{\mathrm{red}}$  is then easily seen to be an isomorphism. In particular, we have an isomorphism  $\coprod_{\delta} [\mathbf{G}_m/\mathbf{G}_m] \xrightarrow{\sim} [X^{\mathrm{an}}/\widehat{\mathbf{G}}_m]_{\mathrm{red}}$ . Thus, it suffices to show that the map

$$\coprod_{\delta} [\mathbf{G}_m/\mathbf{G}_m] \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$$

is an isomorphism. Of course, by construction the component  $[\mathbf{G}_m/\mathbf{G}_m] \rightarrow (\mathcal{X}_1)_{\mathrm{red}}$  indexed by  $\delta$  is just obtained by twisting the residual gerbe  $[\mathrm{Spec} \mathbf{F}/\mathbf{G}_m] \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$  associated to  $\delta$  by unramified characters. In particular, for  $\delta = 1$ , we recover the map  $[\mathbf{G}_m/\mathbf{G}_m] \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$  induced by the universal unramified character  $\mathrm{ur}_x$ .

By Lemma (3.6), this last map is a closed immersion. After twisting by  $\delta$ , we see that the same is true of the map  $[\mathbf{G}_m/\mathbf{G}_m] \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$  indexed by  $\delta$ . As any character  $\delta : G_K \rightarrow \overline{\mathbf{F}}_p^\times$  is an unramified twist of exactly one of the  $\delta$ , it is now straightforward (see Lemma (3.9) below) to deduce that the map  $\coprod_{\delta} [\mathbf{G}_m/\mathbf{G}_m] \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$  is indeed an isomorphism, as desired.  $\square$

**Lemma 3.9.** *Let  $\mathcal{Z}$  be a reduced algebraic stack locally of finite type over a field  $k$ . Let  $\mathcal{Z}_1, \dots, \mathcal{Z}_n$  be a family of closed algebraic substacks of  $\mathcal{Z}$  with the property that every  $\bar{k}$ -point of  $\mathcal{Z}$  factors through exactly one of the  $\mathcal{Z}_i$ . Then the natural map  $\coprod_i \mathcal{Z}_i \rightarrow \mathcal{Z}$  is an isomorphism.*

*Proof.* As usual, we denote by  $|\mathcal{Z}|$  the underlying topological space of  $\mathcal{Z}$ , and similarly for  $|\mathcal{Z}_i|$ . We first show that  $|\mathcal{Z}| = \coprod_i |\mathcal{Z}_i|$  set-theoretically. Let  $\mathcal{Z}'$  be the scheme-theoretic image of the map  $\coprod_i \mathcal{Z}_i \rightarrow \mathcal{Z}$ . Then  $\mathcal{Z}'$  is a closed algebraic substack of  $\mathcal{Z}$  with  $\mathcal{Z}'(\bar{k}) = \mathcal{Z}(\bar{k})$ . Since  $\mathcal{Z}$  is reduced, this forces  $\mathcal{Z}' = \mathcal{Z}$  (this can be checked after passing to a smooth cover of  $\mathcal{Z}$  by a reduced scheme, where the result is standard). In particular, we have  $|\mathcal{Z}| = |\mathcal{Z}'|$ . As  $|\mathcal{Z}'|$  is the closure of  $\bigcup_i |\mathcal{Z}_i|$  in  $|\mathcal{Z}|$  (cf. [Sta22, Tag 0CML]), and each  $|\mathcal{Z}_i|$  is a closed subset of  $|\mathcal{Z}|$ , we see that  $|\mathcal{Z}| = \bigcup_i |\mathcal{Z}_i|$ . Now for each  $i \neq j$ ,  $\mathcal{Z}_i \times_{\mathcal{Z}} \mathcal{Z}_j$  is an algebraic stack locally of finite type over  $k$  with  $(\mathcal{Z}_i \times_{\mathcal{Z}} \mathcal{Z}_j)(\bar{k}) = \emptyset$  by our assumption. This forces  $|\mathcal{Z}_i| \cap |\mathcal{Z}_j| = |\mathcal{Z}_i \times_{\mathcal{Z}} \mathcal{Z}_j| = \emptyset$ . Thus we have a disjoint decomposition  $|\mathcal{Z}| = \coprod_i |\mathcal{Z}_i|$ , and hence each  $|\mathcal{Z}_i|$  is also an open subset of  $|\mathcal{Z}|$ . Let  $\mathcal{U}_i$  be the unique *open* substack of  $\mathcal{Z}$  with underlying space  $|\mathcal{U}_i| = |\mathcal{Z}_i|$  (cf. [Sta22, Tag 06FJ]). In particular, we have a decomposition  $\mathcal{Z} = \coprod_i \mathcal{U}_i$  into open substacks. Now for each  $i$ , the map  $\mathcal{Z}_i \hookrightarrow \mathcal{Z}$  necessarily factors through a closed immersion  $\mathcal{Z}_i \hookrightarrow \mathcal{U}_i$ . Since  $\mathcal{U}_i$  is reduced (being an open substack of  $\mathcal{Z}$ ) and  $|\mathcal{Z}_i| \xrightarrow{\sim} |\mathcal{U}_i|$  by construction, this closed immersion is necessarily an isomorphism.  $\square$

**Remark 3.10.** Assume  $f : X^{\text{an}} \rightarrow \mathcal{X}_1$  is a morphism of stacks over  $\text{Spf } \mathcal{O}$  with the property that  $f(\delta) \cong \mathbf{A}_{K,A}(\delta)$  for all characters  $\delta$  valued in a finite Artinian  $\mathcal{O}$ -algebra. We claim that in fact  $f(\delta) \cong \mathbf{A}_{K,A}(\delta)$  everywhere, or equivalently, that the map  $g : X^{\text{an}} \rightarrow \mathcal{X}_1, \delta \mapsto f(\delta)\mathbf{A}_{K,A}(\delta)^{-1}$  satisfies  $g(\delta) \cong \mathbf{A}_{K,A}$  for any  $\delta$ . Indeed, by [EG20, Lem. 7.1.14], our assumption on  $f$  implies that  $g$  factors through the closed immersion  $[\text{Spf } \mathcal{O}/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$  (induced by the map  $\text{Spf } \mathcal{O} \hookrightarrow X^{\text{an}}$  classifying trivial characters). It therefore suffices to show that any map  $X^{\text{an}} \rightarrow [\text{Spf } \mathcal{O}/\widehat{\mathbf{G}}_m]$  is “trivial”, i.e. that any line bundle on the Noetherian affine formal scheme  $X^{\text{an}}$  is trivial. For this, it suffices to check the same result for the underlying reduced subscheme  $(X^{\text{an}})_{\text{red}}$ . But we have seen that  $(X^{\text{an}})_{\text{red}}$  is just a disjoint union of finitely many copies of  $\mathbf{G}_{m,\mathbf{F}} = \text{Spec } \mathbf{F}[x, x^{-1}]$ , and hence (as  $\mathbf{F}[x, x^{-1}]$  is a PID), we have the claimed result. Thus we see that there is a unique functorial way to extend the construction  $\delta \mapsto \mathbf{A}_{K,A}(\delta)$  appearing in Dee’s equivalence (2.1) from Artinian coefficients to all  $\varpi$ -adically complete  $\mathcal{O}$ -algebras  $A$ .

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