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# VARIATIONAL METHODS FOR ACOUSTIC RADIATION IN A DUCT WITH A SHEAR FLOW AND AN ABSORBING BOUNDARY

J-F. MERCIER\*

**Abstract.** The well-posedness of the acoustic radiation in a 2D duct in presence of both a shear flow and an absorbing wall described by the Myers boundary condition is studied thanks to variational methods. Without flow the problem is found well-posed for any impedance value. The presence of a flow complicates the results. With a uniform flow the problem is proven to be always of the Fredholm type but is found well-posed only when considering a dissipative radiation problem. With a general shear flow, the Fredholm property is recovered for a weak enough shear and the dissipative radiation problem requires to introduce extra conditions to be well-posed: enough dissipation, a large enough frequency and non-intuitive conditions on the impedance value.

**Key words.** Acoustics, time-harmonic radiation, shear flow, impedance boundary condition, variational methods, Fredholm alternative

**AMS subject classifications.** 35J20,35J50,35Q35,76N10

**1. Introduction.** Wave propagation in a steady flow over an acoustically lined wall has been widely studied due to its applications to noise damping. To reduce fan and engine noise, an important feature is the inclusion of sections of acoustic lining at the intake or exhaust of an aeroengine. These acoustic linings are usually made of a honeycomb of small resonators designed to dampen tonal noise. Alternatively, bulk linings (e.g. foam or wool-type materials) can be used, with the characteristic to dampen broadband noise. The first case is called locally reacting liners because it is characterized by a local impedance defined to relate the acoustic pressure to the normal acoustic displacement at the liner surface. On the contrary the second kind of lining is modelled by a non local impedance through Fourier transforms along the interface air/material.

In this paper we consider the first case characterized by a local impedance and we focus more particularly on the influence of a flow. The boundary conditions often used in numerical simulations incorporate both the impedance of the lining and the effect of the slipping mean flow. Ingard [1] proposed a boundary condition accounting for the continuity of the acoustic normal displacement at the liner surface, between the fluid and the acoustic liner and considering the effect of a mean flow parallel to the surface. Myers [2] extended this result to any arbitrary mean flow along a curved wall. This Ingard-Myers impedance boundary condition has been extensively used and has been verified to correctly represent the limit of a vanishingly-thin inviscid boundary layer over the acoustic lining [3, 4] although the boundary layer needs to be very thin in some cases to give a good approximation [5].

The Ingard-Myers boundary condition is known to have drawbacks. In time-domain simulations, there have been several reports on issues of stability when applying the Myers boundary condition to an acoustically-lined duct with flow [6, 7, 8, 9, 10, 11]. The boundary condition leads to numerical simulations becoming unstable at the grid scale, even in the uniform flow case. However due to a lack of alternatives, the Myers boundary condition is still used, with any instability artifacts considered spurious and filtered out using an artificial selective filter [6, 12, 13, 14, 15].

In this paper we focus on the Myers condition in the time-harmonic regime. It has

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45 been less studied theoretically because numerical stability issues are in practice less  
 46 problematic when the frequency is fixed, and the well-posedness of the time-harmonic  
 47 radiation in presence of a Myers condition is still an open question. The aim is to bring  
 48 a comprehensive mathematical study of the time-harmonic propagation in a waveguide  
 49 with an absorbing boundary and to study the influence of various complexities of  
 50 the flow: no flow, uniform flow and shear flow. Our approach is different from the  
 51 ones developed in the references already cited because they mostly focus on the time  
 52 domain and on the behavior of the guided modes. Indeed, these studies restrict to  
 53 a problem without any source which enables them to use a spatial Fourier transform  
 54 of the linearized Euler equations. It leads to the Pridmore-Brown equation [16],  
 55 whose solutions are the guided modes. When one mode is found unstable, in the  
 56 sense exponentially growing in time, the problem is deduced to be ill-posed. On the  
 57 contrary in the frequency-domain, other tools must be used. Indeed even if a mode is  
 58 found unstable in the sense exponentially spatially growing, the ill-posedness cannot  
 59 be deduced since the modes do not form a complete basis [17], even for a uniform  
 60 flow. To avoid this procedural problem, in this paper we don't study the guided modes  
 61 individually. To recover a controlled mathematical framework we prefer to consider  
 62 a radiation problem due to the presence of a source and to use variational arguments  
 63 and coercivity properties to study the well-posedness.

64 The paper is organized as follows. Section 2 concerns a fluid at rest. The acoustic  
 65 radiation problem and the impedance boundary condition are presented. Thanks  
 66 to the absorbing boundary, the solution is found of finite energy and thus can be  
 67 directly sought in an unbounded waveguide. The problem is shown to be well-posed:  
 68 the proof is easy at low frequencies and more involved at larger frequencies. To  
 69 conclude in this latter case, we draw inspiration from works on the scattering from  
 70 unbounded rough surfaces [18]. The extension to a fluid in motion is done in section  
 71 3. First is treated in subsection 3.2 the case of a uniform flow. Thanks to the  
 72 introduction of PMLs to select the outgoing solution, the problem is proven to always  
 73 be of Fredholm type but to prove well-posedness, a dissipative radiation problem  
 74 must be considered. The problem is proven to be well-posed, including for weak  
 75 dissipations. Eventually subsection 3.3 deals with the case of a varying flow. Then  
 76 the problem is found to remain of Fredholm type for a weak shear and is proven to be  
 77 well-posed under some extra constraints: the dissipation must be large enough and  
 78 the physical parameters must follow some laws, a small impedance modulus or a high  
 79 frequency. These constraints are illustrated numerically.

## 80 2. Case of a fluid at rest.

81 **2.1. Geometry and equation.** We consider a 2D infinite duct  $(X, Y) \in \mathbb{R} \times$   
 82  $(0, h)$  of height  $h$  filled with a compressible fluid. The real acoustic pressure  $\tilde{P}(X, Y, t)$   
 83 satisfies the wave equation

$$84 \quad \Delta \tilde{P} - \frac{1}{c_0^2} \frac{\partial^2 \tilde{P}}{\partial t^2} = \tilde{F}(X, Y, t),$$

85 where  $\tilde{F}$  is a real source term and  $c_0$  is the sound speed. To use non-dimensional  
 86 equations, we introduce  $x = X/h$ ,  $y = Y/h$  and the new unknowns  $P(x, y, t) =$   
 87  $\tilde{P}(X, Y, t)$  and  $F(x, y, t) = \tilde{F}(X, Y, t)$ . The guide is defined by  $\Omega = \{(x, y) \in \mathbb{R} \times$   
 88  $(0, 1)\}$ . We note  $\Gamma_0 = \{(x, y) \in \mathbb{R}^2, y = 0\}$  the lower boundary that we suppose  
 89 rigid. The upper boundary  $\Gamma = \{(x, y) \in \mathbb{R}^2, y = 1\}$  is supposed to absorb the  
 90 sound and is characterized by a complex impedance  $Z$ . For a time harmonic source

91 term  $F(x, y, t) = f(x, y)e^{-i\omega t}$  with a frequency  $\omega > 0$ , we look for the pressure  
 92  $P(x, y, t) = p(x, y)e^{-i\omega t}$  satisfying the following Helmholtz problem: for  $f \in L^2(\Omega)$   
 93 and  $k = \omega/c_0 > 0$ , find  $p \in H^1(\Omega)$  such that

$$94 \quad (1) \quad \begin{cases} (\Delta + k^2)p = f & \text{in } \Omega, \\ \partial p / \partial y = ikYp & \text{on } \Gamma \text{ and } \partial p / \partial y = 0 \text{ on } \Gamma_0, \end{cases}$$

95 where  $Y = 1/Z$  is the admittance, that we consider constant with  $\Re(Y) > 0$  to  
 96 produce sound absorption.

97 **REMARK 1.** *The sign of the real part of the admittance is easy to understand when*  
 98 *considering the time domain and restricting to a real admittance: in the transient*  
 99 *regime, the pressure  $P(x, y, t)$  satisfies the time version of (1) (considered without*  
 100 *source term but with some initial conditions not precised here):*

$$101 \quad (2) \quad \begin{cases} c_0^2 \Delta P = \partial^2 P / \partial t^2 & \text{in } \Omega, \\ c_0 \partial P / \partial y = -Y \partial P / \partial t & \text{on } \Gamma \text{ and } \partial P / \partial y = 0 \text{ on } \Gamma_0. \end{cases}$$

102 *Multiplying (2) by  $\partial P / \partial t$  is easily deduced the energy balance*

$$103 \quad (3) \quad \frac{dE}{dt} = -\frac{Y}{c_0} \int_{\Gamma} \left( \frac{\partial P}{\partial t} \right)^2 dx \quad \text{with } E = \frac{1}{2} \int_{\Omega} \left[ \left( \frac{1}{c_0} \frac{\partial P}{\partial t} \right)^2 + |\nabla P|^2 \right] dx dy.$$

104 *It is clear that the energy decreases only if  $Y > 0$ .*

105 **REMARK 2.** *In [19] is considered a Generalized Impedance Boundary Condition*  
 106 *(GIBC) of the form*

$$107 \quad \frac{\partial p}{\partial y} + \frac{\partial p}{\partial x} \left( \mu \frac{\partial p}{\partial x} \right) + \lambda p = 0.$$

108 *It corresponds to our case with  $\mu = 0$  and  $\lambda = -ikY$ , thus with  $\Im(\lambda) < 0$  for our*  
 109 *acoustics applications. On the contrary [19] treats the opposite case  $\Im(\lambda) > 0$  (with*  
 110 *the same  $e^{-i\omega t}$  convention), well-adapted to electromagnetism applications [20]. Note*  
 111 *that our case is less favorable to prove the coercivity of the radiation problem.*

112 For the rest of the paper we consider the time-harmonic regime. Due to the  
 113 absorption and to the energy balance (3), the pressure decays away from the source  
 114 and looking for a solution of finite energy, thus being in  $H^1(\Omega)$ , we will prove that the  
 115 radiation problem is well-posed, first for low frequencies and then for all frequencies.  
 116 To do so, we will use the Lax-Milgram theorem. We first notice that the problem (1)  
 117 is equivalent to the following variational formulation: find  $p \in H^1(\Omega)$  such that

$$118 \quad (4) \quad a(k; p, q) = -(f, q)_{L^2(\Omega)} \quad \forall q \in H^1(\Omega),$$

119 where we have introduced the sesquilinear form for all  $p, q \in H^1(\Omega)$

$$120 \quad (5) \quad a(k; p, q) = \int_{\Omega} (\nabla p \cdot \nabla \bar{q} - k^2 p \bar{q}) dx dy - ikY \int_{\Gamma} p \bar{q} dx.$$

121 **2.2. Well-posedness at low frequencies.** To prove the coercivity of  $a(k; p, q)$ ,  
 122 we start with a Poincaré-like inequality.

123 **LEMMA 1.** *For all  $p \in H^1(\Omega)$  and all  $\lambda > 0$ ,*

$$124 \quad (6) \quad \int_{\Omega} |p|^2 dx dy \leq (1 + \lambda) \int_{\Gamma} |p|^2 dy + \frac{1}{2} \left( 1 + \frac{1}{\lambda} \right) \int_{\Omega} \left| \frac{\partial p}{\partial y} \right|^2 dx dy.$$

125 **Proof.** For all  $p \in C^\infty(\bar{\Omega}) \cap H^1(\Omega)$  and for all  $y \in [0, 1]$  we have

$$126 \quad |p(x, y)| \leq |p(x, 1)| + \int_y^1 \left| \frac{\partial p}{\partial y}(x, t) \right| dt.$$

127 We use Young's inequality: for all  $\lambda > 0$ ,

$$128 \quad |p(x, y)|^2 \leq (1 + \lambda) |p(x, 1)|^2 + \left(1 + \frac{1}{\lambda}\right) \left(\int_y^1 \left| \frac{\partial p}{\partial y}(x, t) \right| dt\right)^2.$$

129 Then thanks to Cauchy-Schwarz inequality, the final result is obtained by integrating  
130 on  $\Omega$  and is extended to  $p \in H^1(\Omega)$  by density.  $\square$

131 Now we prove the well-posedness at low frequencies:

132 **LEMMA 2.** *If*

$$133 \quad (7) \quad k < \frac{\sqrt{1 + 2|Y|^2} - 1}{|Y|},$$

134 *then problem (4) is well-posed.*

135 **Proof.** We use the Lax-Milgram theorem.  $a(k; \cdot, \cdot)$  is continuous on  $H^1(\Omega) \times$   
136  $H^1(\Omega)$  because

$$137 \quad |a(k; p, q)| \leq \int_{\Omega} (|\nabla p| |\nabla \bar{q}| + k^2 |p| |\bar{q}|) dx dy + k|Y| \int_{\Gamma} |p| |\bar{q}| dx.$$

138 We conclude thanks to the continuity of the trace application:  $\exists C_{tr} > 0$  such that  
139  $\forall p \in H^1(\Omega)$ ,  $\|p\|_{L^2(\Gamma)} \leq C_{tr} \|p\|_{H^1(\Omega)}$ .

140 Now we prove that  $a(k; \cdot, \cdot)$  is coercive on  $H^1(\Omega)$ . Let us note  $iY = |Y| e^{i\zeta}$   
141 where  $\zeta = \arg(Y) + \pi/2 \in ]0, \pi[$ , since  $\Re e Y > 0$ . We introduce the decomposition  
142  $a(k; p, p) = \alpha(p) - e^{i\zeta} \beta(p)$ , where we have introduced the forms:

$$143 \quad \alpha(p) = \int_{\Omega} (|\nabla p|^2 - k^2 |p|^2) dx dy \quad \text{and} \quad \beta(p) = k|Y| \int_{\Gamma} |p|^2 dx.$$

144 The lower bound:  $|\alpha - e^{i\zeta} \beta| = |e^{-i\zeta/2}(\alpha - e^{i\zeta} \beta)| \geq |\Im m(e^{-i\zeta/2} \alpha - e^{i\zeta/2} \beta)| \geq$   
145  $-\Im m(e^{-i\zeta/2} \alpha - e^{i\zeta/2} \beta) = \sin(\zeta/2) (\alpha + \beta)$ , leads to the estimation:

$$146 \quad |a(k; p, p)| \geq \sin\left(\frac{\zeta}{2}\right) \left[ \int_{\Omega} (|\nabla p|^2 - k^2 |p|^2) dx dy + k|Y| \int_{\Gamma} |p|^2 dx \right].$$

147 Note that  $\sin(\zeta/2) \neq 0$  because  $\Re e(Y) > 0$ . Using (6) leads to

$$148 \quad |a(k; p, p)| \geq \sin\left(\frac{\zeta}{2}\right) \left\{ \left[1 - \frac{k^2}{2} \left(1 + \frac{1}{\lambda}\right)\right] \int_{\Omega} |\nabla p|^2 dx dy + k[|Y| - k(1 + \lambda)] \int_{\Gamma} |p|^2 dx \right\}. \blacksquare$$

149 To recover the  $H^1$ -norm on the right-hand side, we use again (6) leading to

$$150 \quad \|p\|_{H^1(\Omega)}^2 \leq (1 + \lambda) \int_{\Gamma} |p|^2 dx + \frac{1}{2} \left(3 + \frac{1}{\lambda}\right) \int_{\Omega} |\nabla p|^2 dx dy.$$

151 This leads to the lower bound

$$152 \quad |a(k; p, p)| \geq \sin\left(\frac{\zeta}{2}\right) \min \left[ \frac{1 - \frac{k^2}{2} \left(1 + \frac{1}{\lambda}\right)}{\frac{1}{2} \left(3 + \frac{1}{\lambda}\right)}, \frac{k[|Y| - k(1 + \lambda)]}{(1 + \lambda)} \right] \|p\|_{H^1(\Omega)}^2.$$

153 Therefore  $a$  is coercive if  $\lambda$  is chosen such that  $[(2/k^2) - 1]^{-1} < \lambda < (|Y|/k) - 1$ . This  
154 is possible if  $k$  is such that  $|Y|k^2 + 2k - 2|Y| < 0$ , and also if  $k^2 < 2$ , ensuring that  
155  $\lambda > 0$ . Since  $k > 0$ , both conditions are fulfilled if (7) applies.  $\square$

156 **2.3. Well-posedness at all frequencies.** The previous result can not be easily  
 157 extended to large frequencies. However it remains true and we have the following

158 **THEOREM 3.** *For all  $k > 0$ , the problem (4) is well-posed.*

159 To prove this theorem, we follow the procedure in [18], indicating that we just  
 160 have to find an a-priori estimate for a solution of (4). It corresponds to prove that  
 161 exists  $C > 0$  such that for all  $p \in H^1(\Omega)$  and  $f \in L^2(\Omega)$  satisfying (4), it holds that

$$162 \quad (8) \quad \|p\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

163 Then as in [18], we invoke Lemmas 4.4 and 4.5 in [21], which show that the a-priori  
 164 estimate implies an inf-sup condition for the sesquilinear form  $a(k; p, q)$  and also the  
 165 transposed inf-sup condition. Then [[22], Theorem 2.15] yields existence, uniqueness,  
 166 and boundedness of a solution of (4).

167 We now focus on establishing the a priori bound (8), which is a rather technical  
 168 task. First we prove the following lemmas:

169 **LEMMA 4.** *If  $p \in H^1(\Omega)$  is solution of the variational problem (4) then*

$$170 \quad (9) \quad \|\nabla p\|_{L^2(\Omega)}^2 - k^2 \|p\|_{L^2(\Omega)}^2 \leq (1 + |\Im m(Y)|/\Re e(Y)) \|f\|_{L^2(\Omega)} \|p\|_{L^2(\Omega)},$$

171 *and*

$$172 \quad (10) \quad k \Re e(Y) \|p\|_{L^2(\Gamma)}^2 \leq \|f\|_{L^2(\Omega)} \|p\|_{L^2(\Omega)}.$$

173 **Proof.** From (4) and (5) written for  $q = p$  is found

$$174 \quad a(k; p, p) = \int_{\Omega} (|\nabla p|^2 - k^2 |p|^2) \, dx dy - k [i \Re e(Y) - \Im m(Y)] \int_{\Gamma} |p|^2 \, dx = -(f, p)_{L^2(\Omega)}.$$

175 The real and imaginary parts read

$$176 \quad \begin{cases} k \Re e(Y) \int_{\Gamma} |p|^2 \, dx = \Im m(f, p)_{L^2(\Omega)}, \\ \int_{\Omega} (|\nabla p|^2 - k^2 |p|^2) \, dx dy + k \Im m(Y) \int_{\Gamma} |p|^2 \, dx = -\Re e(f, p)_{L^2(\Omega)}, \end{cases}$$

177 which gives the two inequalities. □

178 To go further, we need first to prove that  $p \in H^2(\Omega)$ .

179 **LEMMA 5.** *If  $p \in H^1(\Omega)$  is solution of the variational problem (4) then  $p \in$   
 180  $H^2(\Omega)$ .*

181 **Proof.**  $p \in H^1(\Omega)$  satisfies (1) which can be written

$$182 \quad \begin{cases} -\Delta p + p = g & \text{in } \Omega \text{ with } g \equiv -f + (k^2 + 1)p, \\ \partial p / \partial y = ikYp & \text{on } \Gamma \text{ and } \partial p / \partial y = 0 \text{ on } \Gamma_0. \end{cases}$$

183 The key point is to use the Fourier transform  $\hat{p}(\xi, y)$  of  $p(x, y)$  and a convenient  
 184 definition of the  $H^2$ -norm. In [23] is indicated that  $H^2(\Omega)$  can be equipped with the  
 185 following norm

$$186 \quad (11) \quad \|p\|_{H^2(\Omega)}^2 = \int_{\mathbb{R}} \left( \|\hat{p}(\xi, \cdot)\|_{H^2(0,1)}^2 + \xi^4 \|\hat{p}(\xi, \cdot)\|_{L^2(0,1)}^2 \right) d\xi,$$

187 equivalent to the usual norm in  $H^2(\Omega)$ . Taking the Fourier transform along the  $x$   
 188 axis,  $\hat{p}(\xi, y)$ , noted for simplicity  $\varphi(y)$  since  $\xi$  is just a parameter, is found to satisfy

$$189 \quad \begin{cases} -d^2\varphi/dy^2 + (1 + \xi^2)\varphi = h & \text{in } (0, 1), \\ d\varphi/dy = ikY\varphi & \text{at } y = 1 \text{ and } d\varphi/dy = 0 \text{ at } y = 0, \end{cases}$$

190 where  $h(y) = \hat{g}(\xi, y)$ . Deriving the corresponding variational form and choosing the  
 191 test field  $\psi = \varphi$ , we deduce

$$192 \quad \|d\varphi/dy\|_{L^2(0,1)}^2 + (1 + \xi^2)\|\varphi\|_{L^2(0,1)}^2 - ikY|\varphi(1)|^2 = (h, \varphi)_{L^2(0,1)}.$$

193 Proceeding as in the proof of lemma 2, we get the coercivity of the left-hand side  
 194 leading to

$$195 \quad \sin(\zeta/2) \left( \|d\varphi/dy\|_{L^2(0,1)}^2 + (1 + \xi^2)\|\varphi\|_{L^2(0,1)}^2 + k|Y|\varphi(1)|^2 \right) \leq |(h, \varphi)_{L^2(0,1)}|.$$

196 Therefore by the Lax-Milgram theorem we have that for all  $\xi \in \mathbb{R}$ ,  $\hat{p}(\xi, \cdot) = \varphi(\cdot) \in$   
 197  $H^1(0, 1)$  exists and is unique and we derive also two upper bounds:

$$198 \quad \sin(\zeta/2) (1 + \xi^2)\|\varphi\|_{L^2(0,1)} \leq \|h\|_{L^2(0,1)},$$

199 that we note

$$200 \quad (12) \quad (1 + \xi^2)\|\varphi\|_{L^2(0,1)} \leq C\|h\|_{L^2(0,1)},$$

201 where  $C$  designs a generic constant (same convention in the following). The other  
 202 upper bound is

$$203 \quad \sin(\zeta/2) \left( \|d\varphi/dy\|_{L^2(0,1)}^2 + \|\varphi\|_{L^2(0,1)}^2 \right) \leq \|h\|_{L^2(0,1)}\|\varphi\|_{L^2(0,1)},$$

204 from which we deduce a control of the  $H^1$ -norm of  $\varphi$ :  $\|\varphi\|_{H^1(0,1)} \leq C\|h\|_{L^2(0,1)}$ . To  
 205 go further and to control the  $L^2$ -norm of  $d^2\varphi/dy^2$ , we come back to the equation  
 206 satisfied by  $\varphi$  from which is deduced

$$207 \quad \|d^2\varphi/dy^2\|_{L^2(0,1)} \leq (1 + \xi^2)\|\varphi\|_{L^2(0,1)} + \|h\|_{L^2(0,1)} \leq C\|h\|_{L^2(0,1)},$$

208 where (12) has been used. Therefore we deduce the control of the  $H^2$ -norm

$$209 \quad \|\varphi\|_{H^2(0,1)}^2 = \|d^2\varphi/dy^2\|_{L^2(0,1)}^2 + \|\varphi\|_{H^1(0,1)}^2 \leq C\|h\|_{L^2(0,1)}^2.$$

210 Finally to prove that  $p \in H^2(\Omega)$ , we just need to show that

$$211 \quad \|\hat{p}(\xi, \cdot)\|_{H^2(0,1)}^2 + \xi^4\|\hat{p}(\xi, \cdot)\|_{L^2(0,1)}^2 \leq C\|\hat{g}(\xi, \cdot)\|_{L^2(0,1)}^2,$$

212 from which (11) leads, with Plancherel relation, to

$$213 \quad \|p\|_{H^2(\Omega)}^2 \leq C\|g\|_{L^2(\Omega)}^2.$$

214 This is achieved by noting that

$$215 \quad \|\varphi\|_{H^2(0,1)} + \xi^2\|\varphi\|_{L^2(0,1)} \leq \|\varphi\|_{H^2(0,1)} + (1 + \xi^2)\|\varphi\|_{L^2(0,1)} \leq C\|h\|_{L^2(0,1)},$$

216 from which is deduced

$$217 \quad \|\varphi\|_{H^2(0,1)}^2 + \xi^4\|\varphi\|_{L^2(0,1)}^2 \leq (\|\varphi\|_{H^2(0,1)} + \xi^2\|\varphi\|_{L^2(0,1)})^2 \leq C\|h\|_{L^2(0,1)}^2,$$

218 with  $\varphi(y) = \hat{p}(\xi, y)$ ,  $h(y) = \hat{g}(\xi, y)$ . □

219 Now we prove the following technical lemma:

220 LEMMA 6. *If  $p \in H^1(\Omega)$  is solution of the variational problem (4) then*

$$221 \quad (13) \quad 2 \left\| \frac{\partial p}{\partial y} \right\|_{L^2(\Omega)}^2 \leq \left( 1 + \frac{|\Im m(Y)|}{\Re e(Y)} + \frac{k}{\Re e(Y)}(1 + |Y|^2) + 2 \right) \|f\|_{L^2(\Omega)} \|p\|_{H^1(\Omega)}.$$

222 **Proof.** We follow closely the approach in [18], using in particular some Green's  
223 like identity [[24], Theorem 4.4, Theorem 3.34]. For any  $p \in H^2(\Omega)$ , using two times  
224 the Green formula, one gets:

$$(14)$$

$$225 \quad 2\Re e(y\partial p/\partial y, \Delta p)_{L^2(\Omega)} = -2 \|\partial p/\partial y\|_{L^2(\Omega)}^2 + \|\nabla p\|_{L^2(\Omega)}^2 + 2 \|\partial p/\partial y\|_{L^2(\Gamma)}^2 - \|\nabla p\|_{L^2(\Gamma)}^2.$$

226 On the other side,  $p \in H^1(\Omega)$  solution of the variational problem (4) belongs to  $H^2(\Omega)$   
227 thanks to lemma 5 and satisfies (1)

$$228 \quad \begin{cases} (\Delta + k^2)p = f & \text{in } \Omega, \\ \partial p/\partial y = ikYp & \text{on } \Gamma \text{ and } \partial p/\partial y = 0 \text{ on } \Gamma_0. \end{cases}$$

229 Multiplying  $\bar{f}$  by  $y\partial p/\partial y$  is obtained

$$230 \quad 2\Re e(y\partial p/\partial y, \Delta p)_{L^2(\Omega)} = k^2 \|p\|_{L^2(\Omega)}^2 - k^2 \|p\|_{L^2(\Gamma)}^2 + 2\Re e(y\partial p/\partial y, f)_{L^2(\Omega)},$$

231 while (14) becomes for the solution of problem (4):

$$232 \quad 2\Re e(y\partial p/\partial y, \Delta p)_{L^2(\Omega)} = -2 \|\partial p/\partial y\|_{L^2(\Omega)}^2 + \|\nabla p\|_{L^2(\Omega)}^2 + k^2 |Y|^2 \|p\|_{L^2(\Gamma)}^2 - \|\partial p/\partial x\|_{L^2(\Gamma)}^2. \blacksquare$$

233 Combining these two equalities together gives:

$$234 \quad 2 \|\partial p/\partial y\|_{L^2(\Omega)}^2 = \|\nabla p\|_{L^2(\Omega)}^2 - k^2 \|p\|_{L^2(\Omega)}^2 + T - 2\Re e(y\partial p/\partial y, f)_{L^2(\Omega)},$$

235 where

$$236 \quad T = k^2 |Y|^2 \|p\|_{L^2(\Gamma)}^2 - \|\partial p/\partial x\|_{L^2(\Gamma)}^2 + k^2 \|p\|_{L^2(\Gamma)}^2.$$

237 Using (10) we get an upper bound for  $T$

$$238 \quad T \leq [k/\Re e(Y)](1 + |Y|^2) \|f\|_{L^2(\Omega)} \|p\|_{L^2(\Omega)},$$

239 and combined with (9), it gives (13). □

240 Now we can prove the required a priori estimation (8) from which theorem 3 is  
241 directly deduced:

242 LEMMA 7. *If  $p \in H^1(\Omega)$  is solution of the variational problem (4) then (8) holds  
243 with*

$$244 \quad C = (1 + k^2) \left[ \frac{2}{k\Re e(Y)} + \frac{1}{2} \left( 1 + \frac{|\Im m(Y)|}{\Re e(Y)} + \frac{k}{\Re e(Y)}(1 + |Y|^2) + 2 \right) \right] + 1 + \frac{|\Im m(Y)|}{\Re e(Y)}.$$

245 **Proof.** We start from (6) with  $\lambda = 1$ . Using (10) combined with (13) leads to

$$246 \quad \|p\|_{L^2(\Omega)}^2 \leq \left[ \frac{2}{k\Re e(Y)} + \frac{1}{2} \left( 1 + \frac{|\Im m(Y)|}{\Re e(Y)} + \frac{k}{\Re e(Y)}(1 + |Y|^2) + 2 \right) \right] \|f\|_{L^2(\Omega)} \|p\|_{H^1(\Omega)}. \blacksquare$$

247 Then with the control of  $\|\nabla p\|_{L^2(\Omega)}^2$  with (9), we get

$$248 \quad \|p\|_{H^1(\Omega)}^2 \leq (1 + k^2) \|p\|_{L^2(\Omega)}^2 + (1 + |\Im m(Y)|/\Re e(Y)) \|f\|_{L^2(\Omega)} \|p\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \|p\|_{H^1(\Omega)}. \blacksquare$$

249 □



250 **3. The case of a fluid in motion.** Now we extend the previous study by  
 251 adding a motion of the fluid. Restricting to a bounded domain, we will prove that  
 252 the radiation problem is always of Fredholm type, for no shear or at most for a weak  
 253 shear (see paragraphs 3.2.5 and 3.3.4) but to prove the well-posedness in an unbounded  
 254 guide, a main difference with the no-flow case is that we need to introduce some extra  
 255 dissipation (limiting absorption principle). We consider in the duct  $\Omega$  an horizontal  
 256 subsonic shear flow of velocity  $U(y)\mathbf{e}_x$  with  $|U| < c_0$ , which in non-dimensional form  
 257 becomes  $M(y)\mathbf{e}_x$  with  $M(y) = U(y)/c_0$  the Mach number. The acoustics equations  
 258 are more complicated than in the no-flow case and are detailed now.

259 **3.1. Impedance boundary condition in presence of a flow.** In the flow  
 260 case, the impedance boundary condition reads

$$261 \quad \partial u_y / \partial t = Y c_0 p \text{ at } y = 1,$$

262 where  $\mathbf{u}$  is the acoustic displacement linked to the velocity by  $\mathbf{v} = D_t \mathbf{u}$  with the  
 263 convective derivative

$$264 \quad (15) \quad D_t = (1/c_0) \partial / \partial t + M \partial / \partial x.$$

265 Without flow the condition expressed versus the velocity and the pressure is simply  
 266  $v_y = c_0 Y p$  but in presence of a uniform flow it becomes  $\partial v_y / \partial t = Y c_0 D_t p$  at  $y = 1$ .  
 267 In the time-harmonic regime it reads

$$268 \quad v_y = (iY/k) D_k p,$$

269 where  $D_k$  is the convective operator

$$270 \quad (16) \quad D_k = M(y) \partial / \partial x - ik.$$

271 Note that in the no flow case,  $D_k$  reduces to  $-ik$  and since the Linearized Euler  
 272 Equations give  $\nabla p = ik\mathbf{v}$ , we recover the no-flow condition in Eq. (1).

273 For a fluid in motion, the difficulty of the study is weaker when the Mach number  
 274 is constant. Therefore we present first the case of a uniform flow and then we consider  
 275 the most difficult case of a varying flow.

### 276 3.2. Uniform flow case.

277 **3.2.1. Equations of the problem.** We consider a uniform flow  $M = \text{cst} \neq 0$ .  
 278 Then the Linearized Euler Equations read

$$279 \quad (17) \quad \begin{cases} D_k \mathbf{v} + \nabla p = 0, \\ D_k p + \text{div } \mathbf{v} = f, \end{cases}$$

280 with  $D_k$  defined in (16). The first relation of (17) implies that  $\text{curl}(D_k \mathbf{v} + \nabla p) = 0$   
 281 where has been used the scalar curl operator defined by  $\text{curl } \mathbf{v} = \partial_x v_y - \partial_y v_x$ . Thus  
 282 it implies that exists a velocity potential  $\varphi$  such that  $\mathbf{v} = \nabla \varphi$  with  $p = -D_k \varphi$ .  
 283 Indeed the solution of  $D_k(\text{curl } \mathbf{v}) = 0$  is  $\text{curl } \mathbf{v} = A(y) \exp(ikx/M)$  and the only  
 284 causal solution ( $\text{curl } \mathbf{v} = 0$  when  $x \rightarrow -\infty$ ) is  $\text{curl } \mathbf{v} = 0$ . Expressing the impedance  
 285 boundary condition versus the velocity potential leads to the new equations replacing  
 286 (1):

$$287 \quad (18) \quad \begin{cases} \Delta \varphi - D_k^2 \varphi = f & \text{in } \Omega, \\ ik \partial \varphi / \partial y = Y D_k^2 \varphi & \text{on } \Gamma \text{ and } \partial \varphi / \partial y = 0 \text{ on } \Gamma_0. \end{cases}$$

288 For a fluid at rest  $M = 0$ , since  $p = ik\varphi$  we recover (1). Note that even when  $M \neq 0$ ,  
 289 the equations (18) can be expressed versus the pressure (in fact, since the admittance  
 290  $Y$  is constant,  $p$  satisfies the same equations). But it will no longer be the case when  
 291  $M$  is not constant. Since the velocity potential will be a natural unknown in the shear  
 292 case, we prefer to formulate the problem with the velocity potential in the uniform  
 293 flow case.

294 We keep on considering  $\Re(Y) > 0$  although contrary to the case of a fluid at rest,  
 295 we don't know how to establish an energy balance for the problem (18) as explained  
 296 in the two following remarks.

297 **REMARK 3.** *The transient version of (18), without any source term and restricted*  
 298 *to a real admittance is:*

$$299 \quad \begin{cases} \Delta\Phi - \left(\frac{1}{c_0}\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)^2 \Phi = 0 & \text{in } \Omega, \\ \frac{1}{c_0}\frac{\partial^2\Phi}{\partial y\partial t} = -Y\left(\frac{1}{c_0}\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)^2 \Phi & \text{on } \Gamma \text{ and } \frac{\partial\Phi}{\partial y} = 0 & \text{on } \Gamma_0, \end{cases}$$

300 with  $\Phi(x, y, t) = \varphi(x, y)e^{-i\omega t}$ . Multiplying the volume equation by  $\partial\Phi/\partial t$ , we did not  
 301 succeed in deriving an energy balance. This is due to the term  $\partial^2\Phi/\partial y\partial t$  which does  
 302 not appear naturally when applying the Green formula. In fact we suspect that it is  
 303 not possible to establish an energy balance because in the following we will be able to  
 304 prove the well-posedness of the time-harmonic problem only when introducing some  
 305 extra dissipation.

306 **REMARK 4.** *Eliminating all the unknowns to work with the velocity only, it is*  
 307 *possible to derive an equality close to an energy balance. We start from (17) without*  
 308 *any source term and expressed in the time domain:*

$$309 \quad \begin{cases} D_t\mathbf{v} + \nabla p = 0 & \text{and } D_t p + \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \partial v_y/\partial t = Yc_0 D_t p & \text{on } \Gamma \text{ and } v_y = 0 & \text{on } \Gamma_0, \end{cases}$$

310 with  $D_t$  defined in (15). Eliminating the pressure leads to

$$311 \quad \begin{cases} D_t^2\mathbf{v} - \nabla \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \partial v_y/\partial t = -Yc_0 \operatorname{div} \mathbf{v} & \text{on } \Gamma \text{ and } v_y = 0 & \text{on } \Gamma_0. \end{cases}$$

312 Multiplying by  $\partial\mathbf{v}/\partial t$  is easily deduced the equation

$$313 \quad \frac{dE}{dt} = -Yc_0 \int_{\Gamma} (\operatorname{div} \mathbf{v})^2 dx \text{ with } E = \frac{1}{2} \int_{\Omega} \left[ \left(\frac{1}{c_0}\frac{\partial\mathbf{v}}{\partial t}\right)^2 - \left(M\frac{\partial\mathbf{v}}{\partial x}\right)^2 + (\operatorname{div} \mathbf{v})^2 \right] dx dy. \blacksquare$$

314 Thus  $E$  decreases only if  $Y > 0$  and as in the no flow case, we recover that an  
 315 admittance with a positive real part corresponds to an absorbing boundary condition.  
 316 Unfortunately the sign of  $E$  is not known as soon as  $M \neq 0$ , which prevents from  
 317 assuring that it is an energy.

318 The consequence of these remarks is that we are not allowed to look for a solution  
 319 of problem (18) in  $H^1(\Omega)$ . In fact we think that such solution doesn't exist, only a  
 320 solution in  $H_{\text{loc}}^1(\Omega)$  should exist. To characterized this solution, we introduce some  
 321 extra dissipation, as detailed now.

322 **3.2.2. The dissipative problem.** We consider a dissipative radiation problem  
 323 by extending the frequency  $k$  to the upper complex plane. Indeed, to define uniquely  
 324 the good physical solution of a radiation problem, a usual approach (see [25] for  
 325 instance) is to use the limiting absorption principle [26]: the frequency  $k$  is extended  
 326 to the complex plane by  $k + i\varepsilon$  with  $\varepsilon > 0$ , which defines the so-called dissipative  
 327 problem. Then the physical solution is defined as the limit, as  $\varepsilon$  goes to 0, of the  
 328 unique  $H^1$ -solution of the dissipative problem. In [27] is given an interpretation of  
 329 the limiting absorption principle: it is shown that to extend the frequency to the  
 330 complex plane in the Helmholtz equation corresponds to add a slight dissipation in  
 331 the medium in the wave equation (for the problem in time). To apply the limiting  
 332 absorption principle, we use the following transformation to extend the frequency to  
 333 the complex plane with  $\Im m(k) > 0$

$$334 \quad (19) \quad k \rightarrow k_\theta = ke^{i\theta}, \quad \text{with } 0 < \theta < \pi/2,$$

335 more convenient than the usual transformation  $k \rightarrow k + i\varepsilon$ . In the following we will  
 336 determine how  $\theta$  must be chosen to get well-posedness.

337 The variational formulation of (18) for  $k = k_\theta$  is to find  $\varphi \in V$  such that  $\forall \psi \in V$

$$338 \quad (20) \quad a_M(k_\theta; \varphi, \psi) = -(f, \psi)_{L^2(\Omega)},$$

339 where the sesquilinear form reads

$$340 \quad (21) \quad \begin{aligned} a_M(k_\theta; \varphi, \psi) &= \int_{\Omega} \left[ \nabla \varphi \cdot \nabla \bar{\psi} - \left( M \frac{\partial \varphi}{\partial x} - ik_\theta \varphi \right) \left( M \frac{\partial \bar{\psi}}{\partial x} + ik_\theta \bar{\psi} \right) \right] dx dy, \\ &+ \frac{Y}{ik_\theta} \int_{\Gamma} \left( M \frac{\partial \varphi}{\partial x} - ik_\theta \varphi \right) \left( M \frac{\partial \bar{\psi}}{\partial x} + ik_\theta \bar{\psi} \right) dx. \end{aligned}$$

341 The boundary term on  $\Gamma$  implies that  $H^1(\Omega)$  is no longer the good framework and we  
 342 must choose the Hilbert space

$$343 \quad (22) \quad V = \{ \varphi \in H^1(\Omega), \partial \varphi / \partial x \in L^2(\Gamma) \},$$

344 equipped with the norm  $\|\varphi\|_V^2 = \|\nabla \varphi\|_{\Omega}^2 + \|\varphi\|_{\Omega}^2 + \|\partial \varphi / \partial x\|_{\Gamma}^2$ .

345 **3.2.3. Well-posedness conditions.** As in the no flow case, to prove the well-  
 346 posedness of the problem (20) we will use the Lax-Milgram theorem. In this aim it is  
 347 sufficient to show that  $\exists C > 0$  such that  $\forall \varphi \in V$ ,

$$348 \quad |a_M(k_\theta; \varphi, \varphi)| \geq C \|\varphi\|_V^2.$$

349 To simplify the notations, we introduce the admittance argument  $-\pi/2 < \gamma < \pi/2$   
 350 (let us recall that  $\Re \epsilon(Y) > 0$ ) such that

$$351 \quad (23) \quad Y = |Y|e^{i\gamma}.$$

352

353 **THEOREM 8.** *For all Mach number  $0 < M < 1$  and for all admittance  $Y = |Y|e^{i\gamma}$*   
 354 *defined in (23), a critical angle  $0 < \theta_c < \theta_{max} \equiv (2\gamma + \pi)/4$  exists such that the problem*  
 355 *(20) is well posed for all dissipations associated to an angle  $0 < \theta < \theta_c$  where  $\theta$  is*  
 356 *defined in (19) and  $\theta_c$  is defined in (31).*

357 **REMARK 5.** *Since the lower bound for  $\theta$  is zero, the dissipation is allowed to*  
 358 *take very small values. It will not be the case when  $M' \neq 0$  for which we will get*  
 359  *$\theta > \theta_{\min} > 0$ .*

360 To prove latter this theorem, we introduce

$$361 \quad A_M(\varphi) = a \|\partial\varphi/\partial x\|_\Omega^2 + b \|\partial\varphi/\partial y\|_\Omega^2 + c \|\varphi\|_\Omega^2 + d \|\partial\varphi/\partial x\|_\Gamma^2 + 2e \Im m(\varphi, \partial\varphi/\partial x)_\Gamma + f \|\varphi\|_\Gamma^2, \blacksquare$$

362 with the parameters defined by

$$363 \quad (24) \quad \begin{aligned} a &= (1 - M^2) \sin(\theta)/k, & b &= \sin(\theta)/k, & c &= k \sin(\theta), \\ d &= |Y|M^2 \cos(\gamma - 2\theta)/k^2, & e &= |Y|M \cos(\gamma - \theta)/k, & f &= |Y| \cos(\gamma). \end{aligned}$$

364 The quantity  $A_M(\varphi)$  is useful since we have the

365 **LEMMA 9.** *If  $\exists C > 0$  such that  $\forall \varphi \in V$  defined in (22),*

$$366 \quad A_M(\varphi) \geq C \left( \|\nabla\varphi\|_\Omega^2 + \|\varphi\|_\Omega^2 + \|\partial\varphi/\partial x\|_\Gamma^2 \right),$$

367 *then  $a_M(k_\theta; \cdot, \cdot)$  is coercive and consequently problem (20) is well posed.*

368 **Proof.** As we will show hereafter,  $A_M(\varphi) = \Im m \left( \overline{a_M(k_\theta; \varphi, \varphi)} / k_\theta \right)$ . Thus the  
 369 coercivity comes from

$$370 \quad \left| \frac{a_M(k_\theta; \varphi, \varphi)}{k_\theta} \right| \geq \Im m \left[ \overline{\left( \frac{a_M(k_\theta; \varphi, \varphi)}{k_\theta} \right)} \right] \quad \text{and} \quad |a_M(k_\theta; \varphi, \varphi)| \geq |k_\theta| \left| \frac{a_M(k_\theta; \varphi, \varphi)}{k_\theta} \right|.$$

371 Now we evaluate  $\Im m \left( \overline{a_M(k_\theta; \varphi, \varphi)} / k_\theta \right)$ . Noting  $\|\cdot\|_\Omega = \|\cdot\|_{L^2(\Omega)}$  and  $\|\cdot\|_\Gamma = \|\cdot\|_{L^2(\Gamma)}$ ,  
 372 from (21) we get

$$373 \quad \left( \frac{\overline{a_M(k_\theta; \varphi, \varphi)}}{k_\theta} \right) = \frac{(1 - M^2)}{\overline{k_\theta}} \left\| \frac{\partial\varphi}{\partial x} \right\|_\Omega^2 + \frac{1}{\overline{k_\theta}} \left\| \frac{\partial\varphi}{\partial y} \right\|_\Omega^2 - \overline{k_\theta} \|\varphi\|_\Omega^2 - 2M \Im m \left( \varphi, \frac{\partial\varphi}{\partial x} \right)_\Omega, \\ 374 \\ 375 \quad + i \frac{\overline{Y}M^2}{\overline{k_\theta}^2} \left\| \frac{\partial\varphi}{\partial x} \right\|_\Gamma^2 + i \frac{\overline{Y}M}{\overline{k_\theta}} 2 \Im m \left( \varphi, \frac{\partial\varphi}{\partial x} \right)_\Gamma + i \overline{Y} \|\varphi\|_\Gamma^2.$$

376 Taking the imaginary part, we get  $A_M(\varphi)$ . □

377 Now we look for conditions on  $\theta$  under which lemma 9 applies. If  $M = 0$ , we  
 378 get from (24) the simplifications  $d = 0 = e$ ,  $a = \sin(\theta)/k = b$  and thus  $A_M(\varphi) =$   
 379  $a \|\nabla\varphi\|_\Omega^2 + c \|\varphi\|_\Omega^2 + f \|\varphi\|_\Gamma^2$  has all its coefficients positive. Therefore  $A_M(\varphi) \geq$   
 380  $a \|\nabla\varphi\|_\Omega^2 + c \|\varphi\|_\Omega^2$  and the dissipative radiation problem is well-posed in  $H^1(\Omega)$  for  
 381 all  $\theta > 0$  ( $V \neq H^1(\Omega)$  has to be introduced only when  $M > 0$ ). Note that we have  
 382 proven in the previous section that it is also true without dissipation ( $\theta = 0$ ).

383 Now we focus on the case  $M > 0$  and to go further we need to establish some  
 384 lower bounds. First we show the

385 **LEMMA 10.** *For all  $\mu > 0$ ,*

$$386 \quad (25) \quad \|\varphi\|_\Gamma^2 \leq (1 + \mu) \|\varphi\|_\Omega^2 + (1/\mu) \|\partial\varphi/\partial y\|_\Omega^2.$$

387 **Proof.** For all  $\varphi \in C^\infty(\overline{\Omega}) \cap H^1(\Omega)$ ,  $\forall y \in [0, 1]$ ,

$$388 \quad |\varphi(x, 1)|^2 - |\varphi(x, y)|^2 = \int_y^1 \frac{\partial}{\partial t} (|\varphi(x, t)|^2) dt,$$

389 which is developed in

$$390 \quad |\varphi(x, 1)|^2 = |\varphi(x, y)|^2 + 2\Re e \left( \int_y^1 \bar{\varphi}(x, t) \frac{\partial \varphi}{\partial t}(x, t) dt \right).$$

391 We use Young's inequality: for all  $\mu > 0$  and for  $y = 0$  in the integral,

$$392 \quad |\varphi(x, 1)|^2 \leq |\varphi(x, y)|^2 + \mu \int_0^1 |\varphi(x, y)|^2 dy + (1/\mu) \int_0^1 |\partial \varphi / \partial y(x, y)|^2 dy,$$

393 The result is obtained by integrating for  $y$  between 0 and 1 and then for  $x$  on  $\mathbb{R}$  and  
394 is finally extended to  $\varphi \in H^1(\Omega)$  by density.  $\square$

395 Then we show the

396 LEMMA 11. For all  $\lambda > 0$  and  $\mu > 0$ ,

$$397 \quad A_M(\varphi) \geq a \|\partial \varphi / \partial x\|_{\Omega}^2 + C_1 \|\partial \varphi / \partial y\|_{\Omega}^2 + C_2 \|\varphi\|_{\Omega}^2 + C_3 \|\partial \varphi / \partial x\|_{\Gamma}^2,$$

398 with  $g = e\lambda - f$ ,  $C_1 = b - (g/\mu)$ ,  $C_2 = c - g(1 + \mu)$ ,  $C_3 = d - (e/\lambda)$  and other constants  
399 defined in (24).

400 **Proof.** We use the Young inequality:

$$401 \quad \forall \lambda > 0, |2 \Im m(\varphi, \partial \varphi / \partial x)_{\Gamma}| \leq \lambda \|\varphi\|_{\Gamma}^2 + (1/\lambda) \|\partial \varphi / \partial x\|_{\Gamma}^2,$$

402 to deduce:

$$403 \quad A_M(\varphi) \geq a \|\partial \varphi / \partial x\|_{\Omega}^2 + b \|\partial \varphi / \partial y\|_{\Omega}^2 + c \|\varphi\|_{\Omega}^2 + (d - e/\lambda) \|\partial \varphi / \partial x\|_{\Gamma}^2 - g \|\varphi\|_{\Gamma}^2.$$

404 Then using (25) leads to the relation in lemma 11.  $\square$

405 Now we can prove theorem 8 by finding conditions on  $\theta$  under which lemma 9  
406 applies, thanks to lemma 11.

407 **Proof of theorem 8.** We want all the coefficients  $a$ ,  $C_1$ ,  $C_2$  and  $C_3$  in lemma  
408 11 to be strictly positive. First we consider the case  $C_3 = 0$  thus

$$409 \quad (26) \quad \lambda = \lambda_0(\theta) \equiv \frac{e}{d} = \frac{k \cos(\gamma - \theta)}{M \cos(\gamma - 2\theta)}.$$

410 From lemma 11,  $\lambda$  must be strictly positive which implies  $2\theta < \gamma + \pi/2$  and which  
411 defines the maximum angle

$$412 \quad (27) \quad \theta_{\max} = (2\gamma + \pi)/4.$$

413 This upper bound for  $\theta$  becomes a strong constraint only when  $\gamma \rightarrow -\pi/2$ . For  
414  $\lambda = \lambda_0$ ,  $g = g_0 \equiv e\lambda_0 - f$  is found to be equal to

$$415 \quad (28) \quad g_0(\theta) = |Y| \sin^2(\theta) / \cos(\gamma - 2\theta),$$

416 and thus  $g_0 > 0$ . To get  $C_1 > 0$  and  $C_2 > 0$  we must satisfy  $(g_0/b) < \mu < (c/g_0) - 1$   
417 with  $\mu > 0$ . Thus  $\theta \in ]0, \theta_{\max}[$  must be chosen such that  $(g_0/b) < (c/g_0) - 1$  and  
418 this is obtained for  $\theta$  below a critical value. Indeed  $(g_0/b)(g_0/c) + (g_0/c) < 1$  can be  
419 written as  $\tilde{P}_0(u) < 0$  with the polynomial

$$420 \quad (29) \quad \tilde{P}_0(u) = u^2 + (u/k) - 1,$$

421 where

$$422 \quad (30) \quad u(\theta) = g_0(\theta)/\sin(\theta) = |Y| \sin(\theta)/\cos(\gamma - 2\theta).$$

423 Since  $\tilde{P}_0(0) < 0$ , this is achieved for  $u < u_c$  where  $\tilde{P}_0(u_c) = 0$  with the explicit critical  
 424 value  $2u_c = -(1/k) + \sqrt{(1/k)^2 + 4}$ . Finally,  $\theta \in (0, \theta_{\max}) \rightarrow u(\theta)$  is found to be an  
 425 increasing function of range  $(0, \infty)$ .  $u < u_c$  corresponds to  $\theta < \theta_c$ , where the critical  
 426 angle  $\theta_c$  is defined as the unique solution  $\theta \in (0, \theta_{\max})$  of  $u(\theta_c) = u_c$  which reads more  
 427 explicitly

$$428 \quad (31) \quad 2 \frac{|Y| \sin(\theta_c)}{\cos(\gamma - 2\theta_c)} = 2u_c \equiv -\frac{1}{k} + \sqrt{\frac{1}{k^2} + 4}.$$

429 Note that  $\theta_c$  is surprisingly independent of  $M$ . Eventually to satisfy lemma 9 we need  
 430 conditions under which  $C_3$  is strictly positive. This is achieved by slightly perturbing  
 431 the condition under which  $C_3 = 0$ . We take  $\lambda_\varepsilon = (e/d) + \varepsilon$  for any  $\varepsilon > 0$ . Then  $C_3$   
 432 becomes  $C_3^\varepsilon$  such that  $C_3^\varepsilon = \varepsilon d / [(e/d) + \varepsilon] > 0$ .  $g$  becomes  $g_\varepsilon \equiv e\lambda_\varepsilon - f = g_0 + e\varepsilon > 0$ .  
 433 The conditions  $C_1^\varepsilon \equiv b - (g_\varepsilon/\mu) > 0$  and  $C_2^\varepsilon \equiv c - g_\varepsilon(1 + \mu) > 0$  lead to  $(g_\varepsilon/b) < \mu <$   
 434  $(c/g_\varepsilon) - 1$ , which implies the condition  $\tilde{P}_\varepsilon(u) < 0$  with

$$435 \quad \tilde{P}_\varepsilon(u) = \tilde{P}_0(u) + \frac{2e\varepsilon g_0}{\sin(\theta)^2} + \left( \frac{e\varepsilon}{\sin(\theta)} \right)^2 + \frac{e\varepsilon}{k \sin(\theta)}.$$

436 Since  $\tilde{P}_\varepsilon(u) \geq \tilde{P}_0(u)$ ,  $\tilde{P}_\varepsilon(u_c(\varepsilon)) = 0$  for  $u_c(\varepsilon) \leq u_c(0) = u_c$ . Therefore the condition  
 437  $\tilde{P}_\varepsilon(u) < 0$  is satisfied for  $\theta < \theta_c(\varepsilon) \leq \theta_c(0)$ , with  $\theta_c(0)$  noted previously  $\theta_c$ .  $\theta_c(\varepsilon)$  can  
 438 be as close as we want to  $\theta_c$  by taking  $\varepsilon$  small enough which ensures that the problem  
 439 is well posed for any  $\theta < \theta_c$ , as stated in theorem 8.

440 □

441 **3.2.4. Numerical illustration.** It is not possible to get  $\theta_c$  more explicitly than  
 442 the solution of the fixed point equation (31) but we can characterize it numerically.  
 443 Some dependences of  $\theta_c$  versus several physical parameters are explicit: since  $u$  is an  
 444 increasing function of  $\theta$ , from (31) is deduced that  $\theta_c(k, Y)$  increases when  $k$  increases  
 445 or  $|Y|$  decreases. Moreover  $\theta_c \rightarrow 0$  when  $k \rightarrow 0$  or when  $|Y| \rightarrow \infty$ . However the  
 446 variations of  $\theta_c(\gamma)$  are not easy to guess and it is why we plot them now numerically.

447 For  $k = 2$ ,  $\theta_c(\gamma)$  solution of (31) is plotted in Fig. 1 for two modulus of the  
 448 admittance:  $|Y| = 1.4$  and  $|Y| = 1.6$ . The maximum of  $\theta_c$  is located at  $\gamma_{\max} =$   
 449  $2 \arcsin(u_c/|Y|)$  with  $u_c$  defined in (31). For  $|Y| = 1.4$  we get  $\gamma_{\max} = 1.18$  and  
 450  $\theta_c(\gamma_{\max}) = 0.59$  whereas  $|Y| = 1.6$  leads to  $\gamma_{\max} = 1.02$  and thus  $\theta_c(\gamma_{\max}) = 0.51$ .  
 451 Moreover the values of  $\theta_c(\gamma)$  at  $\gamma = \pm\pi/2$  are expected. When  $\gamma \rightarrow -\pi/2$ , since  $0 <$   
 452  $\theta_c < \theta_{\max}$  with  $\theta_{\max}(\gamma) \rightarrow 0$  when  $\gamma \rightarrow -\pi/2$ , naturally  $\theta_c(\gamma) \rightarrow 0$ . For  $\gamma \rightarrow \pi/2$ , we  
 453 get directly from (31) that  $\theta_c(\pi/2) = \arccos(|Y|/2u_c)$  if  $|Y|/2u_c \leq 1$  and  $\theta_c(\pi/2) = 0$   
 454 for  $|Y|/2u_c \geq 1$ . For  $|Y| = 1.4$  we get  $|Y|/2u_c = 0.90$  and  $\theta_c(\pi/2) = 0.46$  whereas  
 455  $|Y| = 1.6$  leads to  $|Y|/2u_c = 1.03$  and thus  $\theta_c(\pi/2) = 0$ .

456 **3.2.5. Case without dissipation.** Without dissipation ( $\theta = 0$  and  $k_\theta = k$ ),  
 457 we are not able to prove that the problem (20) is well-posed (and we suspect it is  
 458 not true), as we did in the no-flow case, but we can at least prove that the prob-  
 459 lem is of Fredholm-type. To do so, we restrict the problem to a bounded domain  
 460 and we close it with appropriate radiation conditions. The outgoing solution is se-  
 461 lected thanks to the introduction of PMLs: the problem is set in a bounded domain  
 462  $\Omega_\alpha = \bar{\Omega}_d \cup \Omega_\pm^L$  composed of the central domain  $\Omega_d = \{(x, y); |x| < d, 0 < y < 1\}$

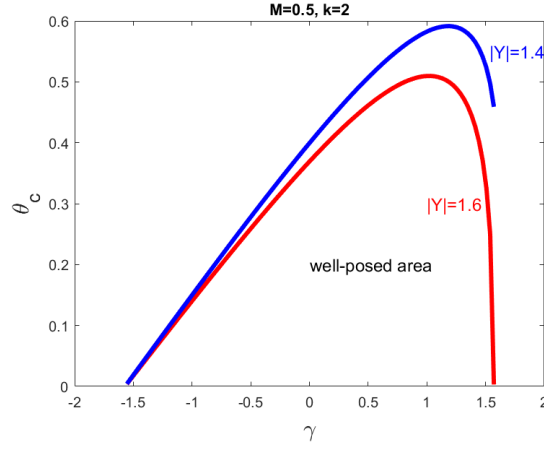


FIG. 1. For  $k = 2$ ,  $\theta_c$  in (31) versus  $\gamma = \arg Y$  for  $|Y| = 1.4$  in blue and  $|Y| = 1.6$  in red

463 containing the source (for  $d$  large enough) and of surrounding absorbing layers  $\Omega_{\pm}^L$   
 464 of length  $L$ :  $\Omega_{\pm}^L = \{(x, y); d < \pm x < d + L, 0 < y < 1\}$ . The introduction of PMLs  
 465 amounts to the transformation of the differential operator  $\partial/\partial x \rightarrow \alpha(x) \partial/\partial x$  in  
 466 the governing equations of the problem. The complex function  $\alpha$  is assumed to be  
 467 unity in  $\Omega_d$  and constant and equal to the complex scalar  $\alpha^*$ , satisfying the following  
 468 hypotheses  $\text{Re}(\alpha^*) > 0$ ,  $\text{Im}(\alpha^*) < 0$  to produce absorption (see [28] for a more thor-  
 469 ough description and justification). For a source  $f \in L^2(\Omega)$ , the radiation problem in  
 470 presence of PMLs reads

$$(32) \quad \begin{cases} \alpha \frac{\partial}{\partial x} \left( \alpha \frac{\partial \varphi}{\partial x} \right) + \frac{\partial^2 \varphi}{\partial y^2} - D_{\alpha}^2 \varphi = f & \text{in } \Omega_{\alpha}, \\ \frac{\partial \varphi}{\partial y} = (Y/ik) D_{\alpha}^2 \varphi & \text{on } \Gamma_{\alpha} = \{(x, 1); |x| < d + L\}, \\ \frac{\partial \varphi}{\partial y} = 0 & \text{on } \Gamma_{\alpha}^0 = \{(x, 0); |x| < d + L\}, \\ \varphi = 0 & \text{on } \Sigma_{\pm}, \end{cases}$$

472 where  $D_{\alpha} = M\alpha \partial/\partial x - ik$  and where the purpose of the Dirichlet condition on  
 473  $\Sigma_{\pm} = \{(x, y); \pm x = d + L, 0 < y < 1\}$  is to select the outgoing solution. This problem  
 474 has the equivalent variational form:

$$(33) \quad \begin{cases} \text{Find } \varphi \in U = \{\varphi \in H_{\Sigma, 0}^1(\Omega_{\alpha}) \text{ with } \partial \varphi / \partial x \in L^2(\Gamma_{\alpha})\} \\ \text{such that } a_{\alpha}(\varphi, \psi) = -(f/\alpha, \psi)_{L^2(\Omega_{\alpha})} \text{ for all } \psi \in U, \end{cases}$$

476 where  $H_{\Sigma, 0}^1(\Omega_{\alpha}) = \{\varphi \in H^1(\Omega_{\alpha}), \varphi|_{\Sigma_{\pm}} = 0\}$  and where the sesquilinear form  $a_{\alpha}(\varphi, \psi)$   
 477 is defined as:

$$478 \quad \int_{\Omega_{\alpha}} \frac{1}{\alpha} (\nabla_{\alpha} \varphi \cdot \nabla_{\alpha} \bar{\psi} - D_{\alpha} \varphi \bar{D}_{\alpha} \bar{\psi}) \, dx dy + \frac{Y}{ik} \int_{\Gamma_{\alpha}} \frac{1}{\alpha} (D_{\alpha} \varphi \bar{D}_{\alpha} \bar{\psi}) \, dx,$$

479 where  $\bar{D}_{\alpha} = M\alpha \partial/\partial x + ik$  and  $\nabla_{\alpha} = (\alpha \partial/\partial x, \partial/\partial y)$ .

480 **LEMMA 12.** *Problem (33) is of Fredholm type.*

481 **Proof.** We show that  $a_{\alpha}(\varphi, \psi)$  is the sum of a compact part and a coercive  
 482 part. The proof of compactness is classic (remember that  $\varphi|_{\Gamma_{\alpha}} \in H^1(\Gamma_{\alpha})$ ) and the

483 coerciveness is obtained by proving that  $\exists C > 0$  such that  $\forall \varphi \in U$  defined in (33):

484 
$$|b_\alpha(\varphi)| \geq C \left( \int_{\Omega_\alpha} |\nabla \varphi|^2 dx dy + \int_{\Gamma_\alpha} |\partial \varphi / \partial x|^2 dx \right),$$

485 with the non-compact part of  $a_\alpha(\varphi, \psi)$  defined as

486 (34) 
$$b_\alpha(\varphi) = \int_{\Omega_\alpha} \left[ (1 - M^2) \alpha \left| \frac{\partial \varphi}{\partial x} \right|^2 + \frac{1}{\alpha} \left| \frac{\partial \varphi}{\partial y} \right|^2 \right] dx dy + \frac{Y}{ik} \int_{\Gamma_\alpha} M^2 \alpha \left| \frac{\partial \varphi}{\partial x} \right|^2 dx.$$

487 It is true for any value of the admittance as soon as the numerical parameter  $\alpha^*$  is  
 488 chosen such that  $-2 \arg(\alpha^*) < \pi/2 + \arg Y$  (remember that  $\arg(\alpha^*) < 0$ ).  $\square$

489 Note that this condition on the PML parameter  $\alpha^*$  becomes hard to fulfill in the  
 490 limit  $\gamma = \arg Y \rightarrow -\pi/2$ . It is consistent with the results for the dissipative problem  
 491 set in an unbounded domain of the previous paragraphs: then it was hard to find a  
 492 good dissipation to get a well-posed problem in the same limit.

493 To go further, we are not able to prove uniqueness of problem (33), which would  
 494 imply well-posedness from Fredholm alternative. A classic approach to prove unique-  
 495 ness in a waveguide is to perform a Fourier transform along  $x$  and then to use the  
 496 completeness of the transverse modes of the guide. Here the transverse modes are  
 497 easy to determine but the associated theoretical framework is not well suited to prove  
 498 completeness. The difficulty is that the transverse modes satisfy a quadratic and not  
 499 self-adjoint eigenvalue problem. Completeness is proven only in the no flow case: then  
 500 the problem reduces to a linear eigenvalue problem, still not self-adjoint but at least  
 501 symmetric. Then we recover the same transverse modes than when studying water  
 502 waves propagation and, for a fixed  $k$ , excepting for a countable sequence of values of  
 503  $Y$ , the transverse modes have been proven [29, 30] to form a basis of  $H^{1/2}(0, 1)$ .

504 Although we don't know how to prove it, we postulate that problem (33) is well-  
 505 posed outside a countable sequence of frequencies tending to infinity. This is typical  
 506 in acoustic radiation problems [31] and it would explain why the problem (20) is  
 507 well-posed only for a dissipation  $\theta > 0$ , preventing to consider the limit  $\theta \rightarrow 0$ : it  
 508 is because the limit would not exist on a set of frequencies (even though this set is  
 509 small, of zero measure).

510 **3.3. Shear flow case.** Now we study the general case of a varying flow. The  
 511 effect of a mean shear flow on the acoustic perturbations has already been studied  
 512 [32, 33] but with other tools. It has been done in the absence of source, thanks to a  
 513 Fourier transform of the linearised Euler equations. Then compared to the uniform  
 514 flow case, the novelty is that among the numerical solutions of the Pridmore-Brown  
 515 [16] equation, unstable hydrodynamic modes (spatially exponentially growing) can  
 516 appear [34]. Thanks to our variational approach, we can consider a radiation problem  
 517 and thus study a realistic solution combining all the Pridmore-Brown modes together.  
 518 The main novelty compared to the uniform flow case is that it will not be always  
 519 possible to find a dissipation value for which the acoustic problem is well-posed.  
 520 This is expected since enough dissipation must be introduced to attenuate a possible  
 521 unstable mode.

522 **3.3.1. Equations of the problem.** We consider a shear flow  $M(y)e_x$  of regu-  
 523 larity  $M \in C^1([0, 1])$ . We suppose also that  $0 < M(y) < 1$ , the case of a vanishing  
 524 flow leading to specific difficulties hard to handle ( $M \rightarrow 0$  is a singular limit, see (37)).  
 525 When  $M \neq \text{cst}$ , (18) is not valid and we choose to use the Goldstein equations [35]



526 because they are convenient since they are a direct extension of (18). The velocity  
 527 has no longer a potential but reads  $\mathbf{v} = \nabla\varphi + \boldsymbol{\xi}$  associated to the pressure  $p = -D_{k_\theta}\varphi$   
 528 with  $D_{k_\theta} = M(y)\partial/\partial x - ik_\theta$ . The acoustic unknown  $\varphi \in V$  defined in (22) and  
 529 the hydrodynamic unknown  $\boldsymbol{\xi} \in W = \{\boldsymbol{\xi} \in (L^2(\Omega))^2, \partial\boldsymbol{\xi}/\partial x \in (L^2(\Omega))^2\}$  (such that  
 530  $D_{k_\theta}\boldsymbol{\xi} \in (L^2(\Omega))^2$ ) satisfy the Goldstein equations: they are made of the following  
 531 acoustic propagation equation for  $\varphi$

$$532 \quad (35) \quad \begin{cases} \operatorname{div}(\nabla\varphi + \boldsymbol{\xi}) - D_{k_\theta}^2\varphi = f & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial y} + \xi_y = \frac{Y}{ik_\theta}D_{k_\theta}^2\varphi & \text{on } \Gamma \text{ and } \frac{\partial\varphi}{\partial y} + \xi_y = 0 & \text{on } \Gamma_0, \end{cases}$$

533 (if  $\boldsymbol{\xi} = \mathbf{0}$  we recover (18)) coupled to the transport equations for  $\boldsymbol{\xi}$

$$534 \quad (36) \quad \begin{cases} D_{k_\theta}\xi_x = -M'(y)(\partial\varphi/\partial y + \xi_y) & \text{in } \Omega, \\ D_{k_\theta}\xi_y = M'(y)\partial\varphi/\partial x & \text{in } \Omega. \end{cases}$$

535 The transport equations are solved explicitly and we prove the

536 LEMMA 13. *The  $(L^2(\Omega))^2$  solution  $\boldsymbol{\xi}(\varphi)$  of (36) is*

$$537 \quad (37) \quad \begin{cases} \xi_y(x, y) = \frac{M'}{M} \int_{-x}^x e^{i\frac{k_\theta}{M}(x-s)} \frac{\partial\varphi}{\partial x}(s, y) ds, \\ \xi_x(x, y) = -\frac{M'}{M} \int_{-x}^x e^{i\frac{k_\theta}{M}(x-s)} \left( \frac{\partial\varphi}{\partial y} + \xi_y \right) (s, y) ds, \end{cases}$$

538 and satisfies for any  $\tau > 0$  and any  $\varphi \in H^1(\Omega)$ :

$$539 \quad (38) \quad \left| \int_{\Omega} (\boldsymbol{\xi} \cdot \nabla\bar{\varphi}) dx dy \right| \leq \left( \frac{s_1}{c} \right)^2 \left\| \frac{\partial\varphi}{\partial x} \right\|_{\Omega}^2 + \left( \frac{s_1}{c} \right) \left( \tau \left\| \frac{\partial\varphi}{\partial x} \right\|_{\Omega}^2 + \frac{1}{\tau} \left\| \frac{\partial\varphi}{\partial y} \right\|_{\Omega}^2 \right),$$

540 where  $s_1 = \sup_{y \in [0,1]} |M'(y)|$  is the maximum flow shear and  $c = -\Im m(k_\theta) = k \sin(\theta)$   
 541 defined in (24).

542 **Proof.** The  $(L^2(\Omega))^2$  solution  $\boldsymbol{\xi}$  of (36) is obtained thanks to the causal Green  
 543 function  $G(x, y) = (H(x)/M) \exp(ik_\theta x/M(y))$  with  $H$  the Heaviside function. Then,  
 544 since  $\exp(ik_\theta x/M) \notin L^2(\Omega)$ , the only  $L^2$  solution is  $\xi_y(x, y) = G(\cdot, y) * (M'(y) \partial\varphi/\partial x(\cdot, y))$  ■  
 545 and using

$$546 \quad \|G(\cdot, y) * \partial\varphi/\partial x(\cdot, y)\|_{L^2(\mathbb{R})} \leq \|G(\cdot, y)\|_{L^1(\mathbb{R})} \|\partial\varphi/\partial x(\cdot, y)\|_{L^2(\mathbb{R})} \quad \text{with } \|G(\cdot, y)\|_{L^1(\mathbb{R})} = 1/c. \blacksquare$$

547 we get  $\int_{\mathbb{R}} |\xi_y(s, y)|^2 ds \leq |M'/c|^2 \int_{\mathbb{R}} |\partial\varphi/\partial x(s, y)|^2 ds$ . Using  $s_1 = \sup_{y \in [0,1]} |M'(y)|$   
 548 we finally get

$$549 \quad \begin{cases} \|\xi_y\|_{\Omega} \leq (s_1/c) \|\partial\varphi/\partial x\|_{\Omega}, \\ \|\xi_x\|_{\Omega} \leq (s_1/c) \|\partial\varphi/\partial y + \xi_y\|_{\Omega} \leq (s_1/c) \|\partial\varphi/\partial y\|_{\Omega} + (s_1/c)^2 \|\partial\varphi/\partial x\|_{\Omega}. \end{cases}$$

550 Eventually we deduce the upper bound

$$551 \quad \left| \int_{\Omega} (\boldsymbol{\xi} \cdot \nabla\bar{\varphi}) dx dy \right| \leq \|\xi_x\|_{\Omega} \|\partial\varphi/\partial x\|_{\Omega} + \|\xi_y\|_{\Omega} \|\partial\varphi/\partial y\|_{\Omega}, \\ 552 \quad \leq (s_1/c)^2 \|\partial\varphi/\partial x\|_{\Omega}^2 + 2(s_1/c) \|\partial\varphi/\partial x\|_{\Omega} \|\partial\varphi/\partial y\|_{\Omega},$$

553 and thus for any  $\tau > 0$ , thanks to the Young inequality:

$$554 \quad \left| \int_{\Omega} (\boldsymbol{\xi} \cdot \nabla\bar{\varphi}) dx dy \right| \leq (s_1/c)^2 \|\partial\varphi/\partial x\|_{\Omega}^2 + (s_1/c) \left( \tau \|\partial\varphi/\partial x\|_{\Omega}^2 + (1/\tau) \|\partial\varphi/\partial y\|_{\Omega}^2 \right).$$

555

□

556

Thanks to the resolution of the transport equation, the scattering problem for the unknown  $\varphi$  alone can be derived:

557

558

LEMMA 14. *The variational formulation of (35) is: find  $\varphi \in V$  defined in (22) such that  $\forall \psi \in V$*

559

560 (39)

$$\tilde{a}_M(k_\theta; \varphi, \psi) = -(f, \psi)_{L^2(\Omega)},$$

561 with

562

$$\tilde{a}_M(k_\theta; \varphi, \psi) = a_{M(y)}(k_\theta; \varphi, \psi) + \int_{\Omega} \boldsymbol{\xi}(\varphi) \cdot \nabla \bar{\psi} \, dx dy.$$

563

$a_{M(y)}(k_\theta; \cdot, \cdot)$  is  $a_M(k_\theta; \cdot, \cdot)$  defined in (21) evaluated for a varying Mach profile  $M = M(y)$  and  $\boldsymbol{\xi}(\varphi)$  is defined in (37).

564

565

**3.3.2. Well-posedness conditions.** Now we derive conditions under which

566

(39) is well-posed. The main novelty in the shear case is that  $\exists \theta_{\min} > 0$  such that the problem is well-posed only with enough dissipation: for  $\theta_{\min} < \theta < \theta_c$  (for a uniform flow,  $\theta_{\min} = 0$ ). The well-posedness conditions are given in the forthcoming theorem 16 which will be given later since several notations must be introduced before. To establish this theorem, we proceed as in the uniform case: we introduce

567

568

569

570

571 (40)

$$\tilde{A}_M(\varphi) = C'_4 \|\partial\varphi/\partial x\|_{\Omega}^2 + C'_1 \|\partial\varphi/\partial y\|_{\Omega}^2 + C'_2 \|\varphi\|_{\Omega}^2 + C'_3 \|\partial\varphi/\partial x\|_{\Gamma}^2,$$

572

and we will show in the proof of lemma 15 that it is a lower bound of  $\Im m \left( \overline{\tilde{a}_M(k_\theta; \varphi, \varphi)} / k_\theta \right)$ .

573

The constants are defined by

574 (41)

$$\begin{aligned} C'_4 &= a' - \frac{1}{k} \left( \frac{s_1}{c} \right)^2 - \left( \frac{s_1}{c} \right) \frac{\tau}{k}, & C'_1 &= b' - \frac{g'}{\mu} \text{ with } b' = b - \left( \frac{s_1}{c} \right) \frac{1}{k\tau}, \\ C'_2 &= c - g'(1 + \mu), & C'_3 &= d' - e'/\lambda, \end{aligned}$$

575

for all  $\lambda, \mu, \tau > 0$ , with  $b, c$  and  $f$  already defined in (24), with the new parameters

576

$$\begin{aligned} a' &= (1 - s_0^2) \sin(\theta)/k, & d' &= |Y| i_0^2 \cos(\gamma - 2\theta)/k^2, \\ e' &= |Y| s_0 \cos(\gamma - \theta)/k, & g' &= e' \lambda - f, \end{aligned}$$

with the upper and lower bounds of the flow velocity

$$s_0 = \sup_{y \in [0, h]} |M(y)| \quad \text{and} \quad i_0 = \inf_{y \in [0, h]} |M(y)|,$$

577

and with the shear  $s_1$  defined in lemma 13.  $\tilde{A}_M(\varphi)$  is introduced because we have the new lemma similar to lemma 9:

578

579

LEMMA 15. *If  $\exists C > 0$  such that  $\forall \varphi \in V$ ,*

580

$$\tilde{A}_M(\varphi) \geq C \left( \|\nabla \varphi\|_{\Omega}^2 + \|\varphi\|_{\Omega}^2 + \|\partial\varphi/\partial x\|_{\Gamma}^2 \right),$$

581

then  $\tilde{a}_M(k_\theta; \cdot, \cdot)$  is coercive and consequently problem (39) is well posed.

582

**Proof.** Starting from

583

$$\left( \frac{\overline{\tilde{a}_M(k_\theta; \varphi, \varphi)}}{k_\theta} \right) = \frac{1}{k_\theta} \left\| \sqrt{1 - M^2} \frac{\partial \varphi}{\partial x} \right\|_{\Omega}^2 + \frac{1}{k_\theta} \left\| \frac{\partial \varphi}{\partial y} \right\|_{\Omega}^2 - \bar{k}_\theta \|\varphi\|_{\Omega}^2 - 2 \Im m \left( M \varphi, \frac{\partial \varphi}{\partial x} \right)_{\Omega},$$

$$+i \frac{\bar{Y}}{k_\theta^2} \left\| M \frac{\partial \varphi}{\partial x} \right\|_\Gamma^2 + i \frac{\bar{Y}}{k_\theta} 2 \Im m \left( M \varphi, \frac{\partial \varphi}{\partial x} \right)_\Gamma + i \bar{Y} \|\varphi\|_\Gamma^2 + \frac{1}{k_\theta} \int_\Omega \overline{\xi(\varphi)} \cdot \nabla \varphi \, dx dy,$$

proceeding as for lemmas 9 and 11, we get for all  $\lambda, \mu > 0$  the lower bound

$$\Im m \left[ \left( \frac{\tilde{a}_M(k_\theta; \varphi, \varphi)}{k_\theta} \right) \right] \geq a' \left\| \frac{\partial \varphi}{\partial x} \right\|_\Omega^2 + \left( b - \frac{g'}{\mu} \right) \left\| \frac{\partial \varphi}{\partial y} \right\|_\Omega^2,$$

$$+ [c - g'(1 + \mu)] \|\varphi\|_\Omega^2 + \left( d' - \frac{e'}{\lambda} \right) \left\| \frac{\partial \varphi}{\partial x} \right\|_\Gamma^2 - \frac{1}{k} \left| \int_\Omega \overline{\xi(\varphi)} \cdot \nabla \varphi \, dx dy \right|.$$

Note that this is the same inequality than in lemma 11 with two differences: the constants are now written with a prime since they have been extended from a uniform flow to a varying flow ( $M$  is replaced by  $i_0$ ,  $s_0$  or  $s_1$ ) and the extra hydrodynamic unknown  $\xi$  is also involved.

Eventually, using (38) we can eliminate  $\xi$  and improve the lower bound: for all  $\tau > 0$ ,

$$\Im m \left[ \left( \frac{\tilde{a}_M(k_\theta; \varphi, \varphi)}{k_\theta} \right) \right] \geq \left[ a' - \frac{1}{k} \left( \frac{s_1}{c} \right)^2 - \left( \frac{s_1}{c} \right) \frac{\tau}{k} \right] \left\| \frac{\partial \varphi}{\partial x} \right\|_\Omega^2,$$

$$+ \left[ b - \frac{g'}{\mu} - \left( \frac{s_1}{c} \right) \frac{1}{k\tau} \right] \left\| \frac{\partial \varphi}{\partial y} \right\|_\Omega^2 + [c - g'(1 + \mu)] \|\varphi\|_\Omega^2 + \left( d' - \frac{e'}{\lambda} \right) \left\| \frac{\partial \varphi}{\partial x} \right\|_\Gamma^2,$$

with the right hand side noted  $\tilde{A}_M(\varphi)$  in (40).  $\square$

Now our aim is to write for a shear flow a theorem similar to theorem 8, providing the conditions on  $\theta$  under which lemma 15 applies. Thus we want all the coefficients  $C'_1$ ,  $C'_2$ ,  $C'_3$  and  $C'_4$  in (40) to be strictly positive. Compared to the no-flow case, we have the extra parameter  $\tau$  to adjust. First we choose  $C'_4 = 0$  defined in (41) and thus we choose

$$\tau = \tau_0(\theta) \equiv (1 - s_0^2) \frac{k \sin^2(\theta)}{s_1} - \frac{s_1}{k \sin(\theta)} = \frac{s_1}{k} \left( \frac{x^2}{\sigma^2} - \frac{1}{x} \right),$$

where we have noted  $x = \sin(\theta) \in ]0, 1[$  and where we have introduced the new parameter

$$(42) \quad \sigma = \frac{s_1}{k \sqrt{1 - s_0^2}},$$

which will be important in the following. We call it the instability parameter since it is linked to the possible existence of unstable solutions defined as solutions growing exponentially in time. More precisely we postulate that a velocity profile  $M(y)$  can be unstable only if  $\sigma$  is large enough. It is true if  $\sigma = 0$  since then the velocity is uniform and thus stable. It is also true for a shear flow with a maximum velocity  $s_0$  fixed: then in [36] is proven that a compressible velocity profile of fixed maximum velocity  $s_0$  can allow the development of instabilities only if the profile has an inflexion point (as for an incompressible flow) and if  $s_1$  is above a threshold.

We choose also  $C'_3 = 0$  defined in (41) thus

$$\lambda = \lambda'_0 \equiv \frac{e'}{d'} = k \frac{s_0 \cos(\gamma - \theta)}{i_0^2 \cos(\gamma - 2\theta)},$$

619 similar to (26), where  $M$  is replaced by  $i_0^2/s_0$ . To get  $\lambda'_0 > 0$ , we restrict to  $\theta < \theta_{\max}$   
 620 already defined in (27). The parameter  $g'_0$  defined by  $g'_0 = e'\lambda'_0 - f$  is found to be

$$621 \quad g'_0(\theta) = |Y| \left[ \zeta \frac{\sin^2(\theta)}{\cos(\gamma - 2\theta)} + (\zeta - 1) \cos(\gamma) \right],$$

622 with  $\zeta = (s_0/i_0)^2$  and is thus found positive. For  $\zeta = 1$  is recovered  $g_0$  defined in (28)  
 623 for a uniform flow.

624 Eventually we have to determine conditions under which  $C'_1 > 0$  and  $C'_2 > 0$ .  
 625  $C'_1 > 0$  requires at least  $b' = b - (s_1/ck\tau_0) > 0$ .  $\tau_0$  is an increasing function which  
 626 vanishes at  $x_\tau$  such that  $x_\tau^3 = \sigma^2$ . For  $x_\tau < x < x_{\max}$  with

$$627 \quad x_{\max} \equiv \sin(\theta_{\max}) = \sin[(2\gamma + \pi)/4],$$

628 we find

$$629 \quad b'(\theta) = \frac{1}{k} \left( x - \frac{1}{\frac{x^3}{\sigma^2} - 1} \right).$$

630  $b'$  is an increasing function vanishing at  $x_\sigma$  defined as the unique positive solution of

$$631 \quad (43) \quad \frac{x_\sigma^4}{x_\sigma + 1} = \sigma^2,$$

632 and  $b'$  is positive above the threshold  $x_\sigma > x_\tau$ . Then the remaining conditions to  
 633 fulfill for  $x > x_\sigma$  are  $C'_1 > 0$  and  $C'_2 > 0$  and these lead to a condition similar to the  
 634 one for a uniform flow

$$635 \quad (44) \quad \frac{g'_0}{b'} < \mu < \frac{c}{g'_0} - 1,$$

636 with a positive left-hand side for  $x_\sigma < x$  (we recall that  $\mu$  must be positive). The  
 637 existence of  $\mu$  satisfying the condition (44) is equivalent to  $P_\sigma < 0$  on  $(x_\sigma, x_{\max})$  with

$$638 \quad (45) \quad P_\sigma(x) = P_0(x)Q_\sigma(x) + \frac{\sigma^2}{x^4}v(x)^2,$$

639 where

$$640 \quad P_0(x) \equiv P_{\sigma=0}(x) = \tilde{P}_0(v(x)) = v(x)^2 + \frac{v(x)}{k} - 1,$$

641 with  $\tilde{P}_0$  recalled here but already defined in (29) and  $v$  defined by

$$642 \quad (46) \quad v(x) = \frac{g'_0}{x} = |Y| \left[ \zeta \frac{x}{\cos(\gamma - 2\theta(x))} + (\zeta - 1) \frac{\cos \gamma}{x} \right],$$

643 where  $\theta(x) = \arcsin(x)$  and with

$$644 \quad (47) \quad Q_\sigma(x) = 1 - \sigma^2 \frac{x + 1}{x^4}.$$

645 The sign of  $P_\sigma$  has to be determined on  $x \in (x_\sigma, x_{\max})$ . In the uniform flow case  
 646 ( $\sigma = 0$ ,  $\zeta = 1$ ),  $P_\sigma$  in (45) reduces to  $R_0$  defined by

$$647 \quad (48) \quad R_0(x) \equiv \tilde{P}_0(u(x)) = u(x)^2 + \frac{u(x)}{k} - 1,$$

648 where  $u$  has been defined in (30) versus  $\theta$ , that we can also write versus  $x$

$$649 \quad (49) \quad u(x) = |Y|x/\cos(\gamma - 2\theta(x)).$$

650 Then we found that  $R_0(x) < 0$  for  $0 < x_c^0 < x_{\max}$  with  $x_c^0$  the zero of  $R_0$  which, from  
651 (31), is also defined as the unique solution in  $(0, x_{\max})$  of

$$652 \quad (50) \quad 2u(x_c^0) = 2u_c \equiv -\frac{1}{k} + \sqrt{\frac{1}{k^2} + 4}.$$

653 For  $\sigma \neq 0$ , as for  $R_0(x)$  we still have  $\lim_{x \rightarrow x_{\max}} P_\sigma(x) = \infty$  since  $v(x) \rightarrow \infty$  but the  
654 main difference is that although for  $\sigma = 0$ ,  $R_0(0)$  was negative,  $P_\sigma(x_\sigma) = v(x_\sigma)^2 \sigma^2 / x_\sigma^4$   
655 is positive as soon as  $\sigma > 0$ . Therefore the existence of negative values of  $P_\sigma(x)$  is no  
656 longer guaranteed when  $\sigma \neq 0$ . Since  $Q_\sigma > 0$  for  $x > x_\sigma$  from (47), the existence of  $x$   
657 such that  $P_\sigma(x) < 0$  requires at least  $P_0(x) < 0$  from (45), thus  $v(x) < u_c$  defined in  
658 (31). Contrary to the behavior of  $u$ ,  $x \rightarrow v(x)$  is not an increasing function. Indeed  
659 from (46) is obtained that

$$660 \quad (51) \quad v(x) = \zeta u(x) + (\zeta - 1) \frac{|Y| \cos \gamma}{x},$$

661 with  $x \rightarrow u(x)$  an increasing function but  $x \rightarrow |Y| \cos(\gamma)/x$  is a decreasing function.  
662 Therefore the solutions of the inequality  $v(x) < u_c$  are not easy to characterize. The  
663 only easy result is that since  $u(x) < u_c$  for  $x < x_c^0$  defined in (50) and since  $v \geq u$   
664 from (51),  $v(x) < u_c$  implies that  $x < x_c^0$ .

665 Thanks to these notations, we can write the following theorem generalizing the-  
666 orem 8 to a varying flow:

667 **THEOREM 16.** *For all admittance  $Y = |Y|e^{i\gamma}$  defined in (23) and all instability  
668 parameter  $\sigma$  defined in (42), we define the set  $I_\sigma$  by*

$$669 \quad (52) \quad I_\sigma = \{x \in (x_\sigma, x_c^0), P_\sigma(x) < 0\},$$

670 with  $x_\sigma$  defined in (43),  $x_c^0$  defined in (50) and with the convention  $I_\sigma = \emptyset$  if  $x_c^0 \leq x_\sigma$ .  
671 If  $I_\sigma$  is not empty, then the problem (39) is well posed for all dissipation associated  
672 to the angle  $\theta$  defined in (19) such that  $\sin(\theta) \in I_\sigma$ .

673 **REMARK 6.** *Note that in the uniform flow case, the problem was well-posed as  
674 soon as  $\theta > 0$ . In the shear flow case, we need to introduce enough dissipation ( $\theta > \theta_{\min}$   
675 with  $\sin \theta_{\min} = x_\sigma$ ) to expect to get the well-posedness of (39).*

676 *Note also that the existence of  $\theta_{\min}$  is not a strong constraint since it is easy to  
677 get  $\theta_{\min}$  small:  $\theta_{\min} \rightarrow 0$  if  $\sigma \rightarrow 0$  from (43), thus for a small shear  $s_1$  and/or  $k$  large.  
678 As already mentioned,  $\sigma \rightarrow 0$  is expected to imply the existence of no instability and  
679 thus no need to introduce a strong dissipation.*

680 **Proof of theorem 16.**

681  $I_\sigma$  has been defined previously to impose the conditions  $C'_3 = 0 = C'_4$ . But as  
682 for the uniform case in the proof of theorem 8, we show now that it is easy to get  $C'_3$   
683 and  $C'_4$  strictly positive for a set as close as we want to  $I_\sigma$ . We take  $\lambda_\varepsilon = (e'/d) + \varepsilon$   
684 such that  $C'_3 > 0$  for all  $\varepsilon > 0$  from (41) and we take  $\tau_\eta = \tau - \eta$  such that  $C'_4 > 0$   
685 for any  $\eta > 0$  from (41). The constants  $C'_1$  and  $C'_2$  depend continuously on  $\varepsilon$  and  
686  $\eta$ . It is straightforward to check that the conditions  $C'_1(\varepsilon, \eta) > 0$  and  $C'_2(\varepsilon, \eta) > 0$   
687 lead to a slight perturbation of (44) and thus to a set  $I_\sigma^{\varepsilon, \eta} \subset I_\sigma$  with  $I_\sigma^{\varepsilon, \eta} \rightarrow I_\sigma$  when

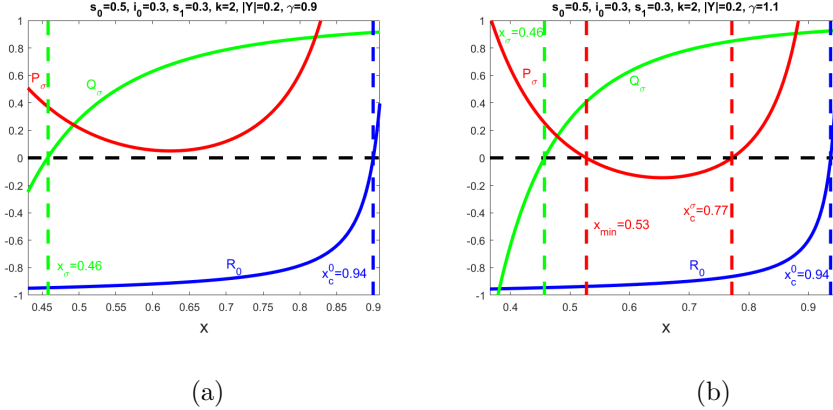


FIG. 2.  $P_\sigma(x)$  defined in (45) in red,  $Q_\sigma(x)$  defined in (47) in green and  $R_0(x)$  defined in (48) in blue for  $s_0 = 0.5$ ,  $i_0 = 0.3$ ,  $s_1 = 0.3$ ,  $k = 2$  and (a):  $Y = 0.2 e^{i0.9}$ , (b):  $Y = 0.2 e^{i1.1}$ .

688  $(\varepsilon, \eta) \rightarrow 0$ . Then the theorem is a consequence of lemma 15 since all the constants in  
 689 (40) are strictly positive.  $\square$

690 Let us analyze the theorem 16. We don't have general criteria for the existence  
 691 of a non-empty  $I_\sigma$  set ensuring the well-posedness of the problem (39) but we have  
 692 global tendencies given by the

693 LEMMA 17. *The set  $I_\sigma$  defined in (52) is empty if at least one of the following*  
 694 *condition is fulfilled:  $\sigma$  is large or  $k$  is small or  $|Y|$  is large.*

695 REMARK 7. *In other words, to get a non-empty set  $I_\sigma$ , necessary conditions are:*  
 696  *$\sigma$  small enough and  $k$  large enough and  $|Y|$  small enough. This will be confirmed by*  
 697 *the forthcoming numerical illustrations*

698 Note that the condition on the instability parameter  $\sigma$  was expected since the  
 699 velocity profile is expected to be stable for  $\sigma$  small enough.

700 **Proof.** The key point is that  $I_\sigma$  is empty if  $x_\sigma \geq x_c^0$ . From (43),  $x_\sigma$  is found  
 701 to be an increasing function of  $\sigma$  and tends to infinity when  $\sigma \rightarrow \infty$ . Moreover from  
 702 (49) and (50), we deduce that  $x_c^0(k, Y)$  decreases when  $k$  decreases or  $|Y|$  increases  
 703 and  $x_c^0(k, Y) \rightarrow 0$  when  $k \rightarrow 0$  or  $|Y| \rightarrow \infty$ . Therefore  $x_\sigma \geq x_c^0$  is necessarily satisfied  
 704 if  $\sigma$  too large or  $k$  too small or  $|Y|$  too large.  $\square$

705 **3.3.3. Numerical illustrations.** Now we illustrate numerically on some ex-  
 706 amples the theoretical derived bounds for the parameters given in lemma 17 for the  
 707 well-posedness of problem (39). In all the tested situations, when  $I_\sigma$  exists it has been  
 708 found as a one-piece interval, of the form  $I_\sigma = (x_{\min}, x_c^\sigma) \subset (x_\sigma, x_c^0)$ , with  $x_{\min}$  and  
 709  $x_c^\sigma$  the two zeros of  $P_\sigma$ . The upper zero  $x_c^\sigma < x_c^0$  is the generalization of  $x_c^0$  in the  
 710 sense:  $x_c^\sigma \rightarrow x_c^0$  when  $\sigma \rightarrow 0$ . We illustrate now numerically this empirical relation  
 711  $I_\sigma = (x_{\min}, x_c^\sigma)$ . In Fig. 2(a) and Fig. 2(b) we explain how we determine  $x_{\min}$  and  
 712  $x_c^\sigma$  and in this aim we represent the variations of  $P_\sigma(x)$  defined in (45) for the flow  
 713 parameters  $s_0 = 0.5$ ,  $i_0 = 0.3$ ,  $s_1 = 0.3$ , the frequency  $k = 2$  and the admittance  
 714  $|Y| = 0.2$  with two values of the admittance argument  $\gamma$ . For the argument  $\gamma = 0.9$   
 715 in Fig. 2(a), we look for zeros of  $P_\sigma$  on  $(x_\sigma, x_c^0)$  where the interval boundaries are  
 716 respectively the zeros of  $Q_\sigma$  defined in (47) and of  $R_0$  defined in (48).  $Q_\sigma(x)$  is plotted  
 717 in green and vanishes at  $x_\sigma = 0.46$  represented as a green vertical dashed line. We  
 718 plot also  $R_0(x)$  in blue which vanishes at  $x_c^0 = 0.94$  represented as a blue vertical

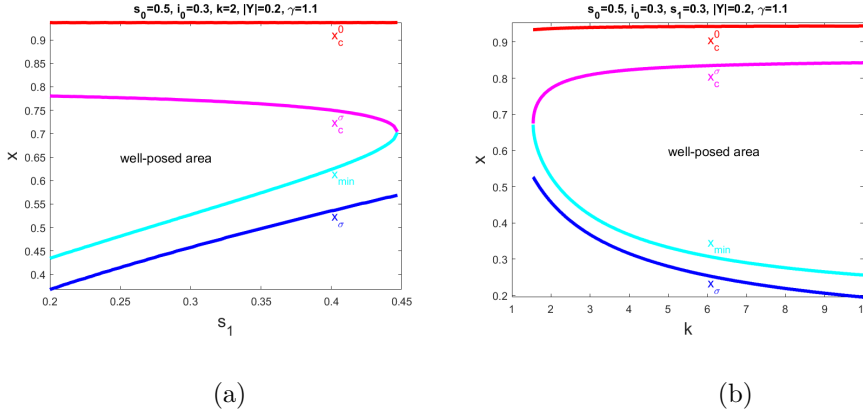


FIG. 3.  $x_\sigma \leq x_{\min} \leq x_c^\sigma \leq x_c^0$  for  $s_0 = 0.5$ ,  $i_0 = 0.3$ ,  $Y = 0.2 e^{i1.1}$ : (a) versus  $s_1$  for  $k = 2$ , (b) versus  $k$  for  $s_1 = 0.3$

719 dashed line.  $P_\sigma(x)$  is found to never vanish, leading to an empty  $I_\sigma$  set.

720 Fig. 2(b) corresponds to the same parameters but for the argument  $\gamma = 1.1$ .  $P_\sigma$   
 721 is found to vanish two times and noting  $x_{\min} = 0.53$  and  $x_c^\sigma = 0.77$  the lower and  
 722 upper zeros of  $P_\sigma$ , represented as red vertical dashed lines, we find  $I_\sigma = (x_{\min}, x_c^\sigma)$  as  
 723 a one-piece set.

724 Now we extend the numerical illustrations and we consider the influence of the  
 725 flow parameters and of the acoustics parameters. We did not find general laws for  
 726 the existence of non-empty  $I_\sigma$  but we have checked numerically that the general  
 727 tendencies given by lemma 17 are relevant. Let us recall that necessary conditions for  
 728 the existence of a non-empty  $I_\sigma$  are:  $\sigma$  small enough and  $k$  large enough and  $|Y|$  small  
 729 enough (the influence of  $\gamma$  is not easy to characterize theoretically). We illustrate now  
 730 numerically these tendencies and in the following figures, we characterize the influence  
 731 of the parameters  $\sigma$ ,  $k$ ,  $|Y|$  and  $\gamma = \arg Y$ .

732 Fig. 3(a) studies the influence of the instability parameter  $\sigma$  for the parameters  
 733 of the flow  $s_0 = 0.5$ ,  $i_0 = 0.3$  and for  $k = 2$ ,  $Y = 0.2 e^{i1.1}$ .  $\sigma$  defined in (42) is changed  
 734 by varying the flow-shear  $s_1$ . We plot the four functions  $x_\sigma \leq x_{\min} \leq x_c^\sigma \leq x_c^0$   
 735 versus  $s_1$ .  $x_c^0$  is constant from (50) with  $x_c^0 = 0.90$ . There are two conclusions. First  
 736 and as already stated, we find that when  $I_\sigma$  exists, it is a one-piece set of the form  
 737  $I_\sigma = (x_{\min}, x_c^\sigma)$ . Then and as expected, it is found that  $I_\sigma$  exists only for  $\sigma$  small  
 738 enough,  $s_1 < 0.45$ , when the flow is more likely to be stable. The problem (39) has  
 739 been proven to be well-posed if  $x_{\min} < x < x_c^\sigma$ : this defines a “well-posed area” as  
 740 indicated on Fig. 3(a) such that if  $(s_1, x)$  is chosen in this area, then problem (39) is  
 741 well-posed. We recall that  $x = \sin \theta$  with  $\theta$  measuring the dissipation.

742 For the three coming illustrations, the parameters of the flow are fixed:  $s_0 = 0.5$ ,  
 743  $i_0 = 0.3$  and  $s_1 = 0.3$ . Fig. 3(b) studies the influence of the frequency  $k$  for  $Y =$   
 744  $0.2 e^{i1.1}$ . As expected,  $I_\sigma$  exists only for  $k$  large enough,  $k > 1.54$ . Fig. 4(a) studies  
 745 the influence of  $|Y|$  for  $k = 2$  and  $\gamma = 1.1$  and as expected,  $I_\sigma$  exists only for  $|Y|$  small  
 746 enough,  $|Y| < 0.23$ .

747 We finish with the influence of the argument  $\gamma$  of the admittance, for which we  
 748 don’t have general tendencies. It is illustrated in Fig. 4(b) for  $k = 2$  and  $|Y| = 0.2$ .  
 749 From (42),  $\sigma$  and thus  $x_\sigma$  are constant,  $x_\sigma = 0.46$ . The set  $I_\sigma = (x_{\min}, x_c^\sigma) \subset (x_\sigma, x_c^0)$   
 750 is found to exist only for  $\gamma > 0.95$ .

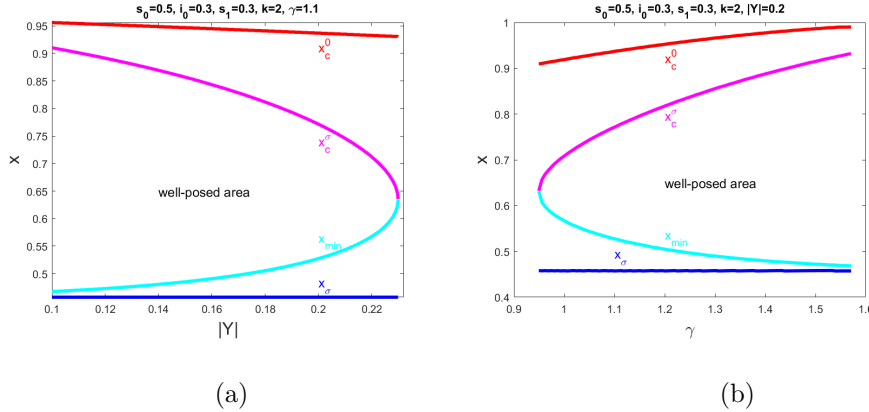


FIG. 4.  $x_\sigma \leq x_{\min} \leq x_c^\sigma \leq x_c^0$  for  $s_0 = 0.5$ ,  $i_0 = 0.3$ ,  $s_1 = 0.3$ ,  $k = 2$ , (a) versus  $|Y|$  for  $\gamma = 1.1$ , (b) versus  $\gamma$  for  $|Y| = 0.2$

751 **3.3.4. Case without dissipation.** As in the uniform flow case, without dissipation we don't know how to prove well-posedness of (39) but we can prove that the  
 752 problem is Fredholm. The problem with a shear flow (35) and (36), extended to the  
 753 presence of PMLs is  
 754

(53)

$$\left\{ \begin{array}{l} \alpha \frac{\partial}{\partial x} \left( \alpha \frac{\partial \varphi}{\partial x} + \xi_x \right) + \frac{\partial}{\partial y} \left( \frac{\partial \varphi}{\partial y} + \xi_y \right) - D_\alpha^2 \varphi = f \quad \text{in } \Omega_\alpha, \\ \left( M \alpha \frac{\partial}{\partial x} - ik \right) \xi_x = -M'(y) \left( \frac{\partial \varphi}{\partial y} + \xi_y \right) \quad \text{in } \Omega_\alpha, \\ \left( M \alpha \frac{\partial}{\partial x} - ik_\theta \right) \xi_y = M'(y) \alpha \frac{\partial \varphi}{\partial x} \quad \text{in } \Omega_\alpha, \\ ik \partial \varphi / \partial y = Y D_\alpha^2 \varphi \text{ on } \Gamma_\alpha \text{ and } \partial \varphi / \partial y = 0 \text{ on } \Gamma_\alpha^0, \\ \varphi = 0 \text{ on } \Sigma_\pm \text{ and } \xi = 0 \text{ on } \Sigma_-, \end{array} \right.$$

755

756 where  $\Gamma_\alpha$ ,  $\Gamma_\alpha^0$  and  $\Sigma_\pm$  are defined in (32). As in the absorbing case,  $\xi$  is explicitly  
 757 determined to get a problem depending only on  $\varphi$ .  $\xi \in (L^2(\Omega_\alpha))^2$  is given by an  
 758 expression similar to (37) but extended to the presence of PMLs. As for (37),  $\xi$  is  
 759 found proportional to  $M'$  and thus  $\|\xi\|_{L^2}$  is bounded by the flow shear  $s_1$ . This will  
 760 be important in the final estimate of the forthcoming proof.

761 The variational form of (53) is the same than in the uniform flow case (33) where  
 762 the sesquilinear form  $a_\alpha(\varphi, \psi)$  is replaced by:

$$\tilde{a}_\alpha(\varphi, \psi) = a_\alpha(\varphi, \psi) + \int_{\Omega_\alpha} \frac{1}{\alpha} \xi(\varphi) \cdot \nabla_\alpha \bar{\psi} \, dx dy.$$

763

764

765 **LEMMA 18.** *For a flow shear  $s_1$  small enough, problem (53) is of Fredholm type.*

766 **Proof.** We show that  $\tilde{a}_\alpha(\varphi, \psi)$  is the sum of a compact part and a coercive part.  
 767 As in the uniform flow case the proof of compactness is classical and coerciveness is  
 768 obtained by proving that  $\exists C > 0$  such that  $\forall \varphi \in U$  defined in (33):

$$(54) \quad \left| b_\alpha(\varphi) + \int_{\Omega_\alpha} \frac{1}{\alpha} \xi(\varphi) \cdot \nabla_\alpha \bar{\varphi} \, dx dy \right| \geq C \left( \int_{\Omega_\alpha} |\nabla \varphi|^2 \, dx dy + \int_{\Gamma_\alpha} \left| \frac{\partial \varphi}{\partial x} \right|^2 \, dx \right).$$

769



770 The term  $b_\alpha(\varphi)$  defined in (34) was already involved in the uniform flow case and  
 771 we already know from the proof of lemma 12 that under the condition  $-2 \arg(\alpha^*) <$   
 772  $\pi/2 + \arg Y$ ,  $\exists C^0 > 0$  such that  $\forall \varphi \in U$ :

$$773 \quad |b_\alpha(\varphi)| \geq C^0 \left( \int_{\Omega_\alpha} |\nabla \varphi|^2 dx dy + \int_{\Gamma_\alpha} \left| \frac{\partial \varphi}{\partial x} \right|^2 dx \right).$$

774 Therefore we just need to find an upper bound for  $\left| \int_{\Omega_\alpha} (1/\alpha) \boldsymbol{\xi}(\varphi) \cdot \nabla_\alpha \bar{\varphi} dx dy \right|$ . From  
 775 the explicit expression of  $\boldsymbol{\xi}(\varphi)$  is deduced a constant  $C_\alpha > 0$  (depending on the  
 776 geometry and on the flow parameters) such that

$$777 \quad \left| \int_{\Omega_\alpha} \left( \xi_x \frac{\partial \bar{\varphi}}{\partial x} + \xi_y \frac{1}{\alpha} \frac{\partial \bar{\varphi}}{\partial y} \right) dx dy \right| \leq C_\alpha \int_{\Omega_\alpha} |\nabla \varphi|^2 dx dy,$$

778 and finally is deduced  $C = C^0 - C_\alpha$  in (54). Coerciveness is obtained when  $C > 0$ .  
 779 As in (38),  $C_\alpha$  is proportional to the shear  $s_1$  and thus  $C_\alpha$  is small when  $s_1$  is small.  
 780 Therefore the problem is Fredholm for a flow shear  $s_1$  small enough: this condition of  
 781  $\sigma$  small was already involved when considering the problem without PMLs but with  
 782 dissipation.  $\square$

783 **4. Conclusion.** Thanks to variational methods, we have studied the well-posed-  
 784 ness of the time-harmonic radiation in a waveguide with a Myers absorbing boundary  
 785 condition on a boundary. The main tendencies are the followings. Without flow, the  
 786 radiation problem is always well-posed. In presence of a uniform flow, it is proven  
 787 to be always of Fredholm-type and well-posed as soon as just a little dissipation is  
 788 introduced. For a varying flow, the problem is Fredholm for a shear weak enough and  
 789 the well-posedness requires at least the introduction of enough dissipation, still with  
 790 moderate values of the flow shear.

791 To go further, let us mention that in the literature some progresses have been  
 792 made in the time domain to correct the illposedness induced by a uniform flow over  
 793 an impedance lining. Modifications to the Myers boundary condition have been sug-  
 794 gested, by incorporating a thin-but-nonzero thickness boundary layer over the lining,  
 795 leading to various so called modified Myers boundary conditions [34, 37, 38, 39]. These  
 796 boundary conditions remove the illposedness while still retaining the simplicity of a  
 797 uniform flow, with the thin boundary layer being incorporated within the boundary  
 798 condition. Moreover they match well [38] with solutions to the full linearised Euler  
 799 equations [40]. The extensions of the modified Myers boundary conditions to the  
 800 time-harmonic regime and their inclusion in our study would be interesting to in-  
 801 crease the domain of well-posedness of the considered radiation problem, but such  
 802 extensions are not straightforward since these conditions have complicated expression  
 803 preventing them from fitting naturally into a variational formulation, contrary to the  
 804 classical Myers condition.

805

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