

Selecting generalized continuum theories for nonlinear periodic solids based on the instabilities of the underlying microstructure

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1. Context

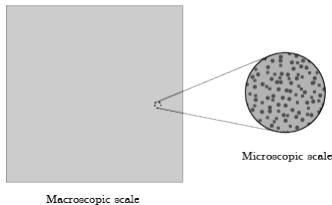
2. Research Question

3. Our Study

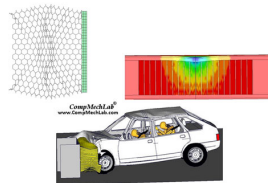
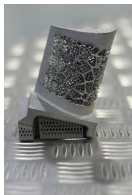
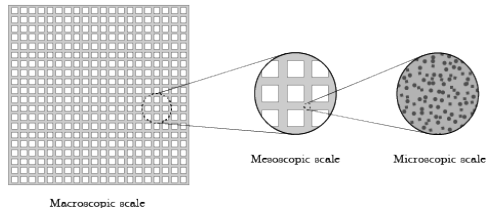
Architected Materials

Definition and applications

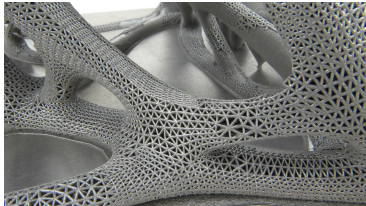
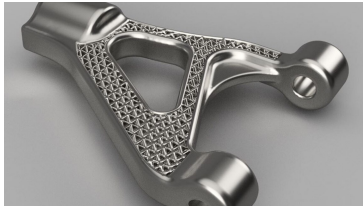
Scale separation in conventional materials



Scale separation in architected materials



Architected Materials

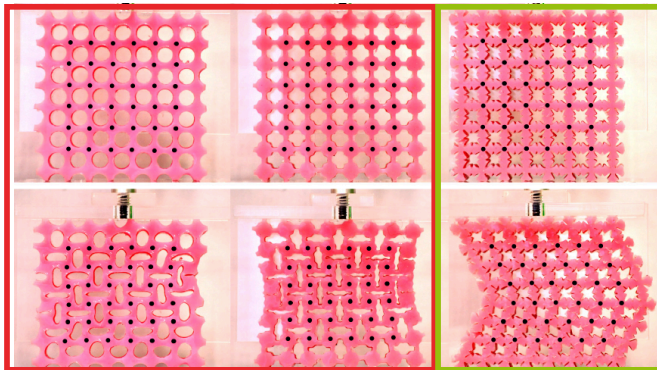


Prohibitive computational cost for full modeling
⇒ Need for equivalent continuum media

Architected Materials & Instabilities

Buckling when compressed

Effect of cell geometry : commensurate vs structural

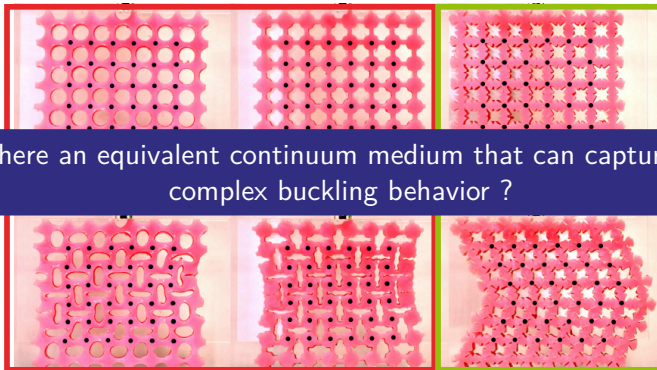


[Overvelde J.T.B. & Bertoldi K. 2014]

Architected Materials & Instabilities

Buckling when compressed

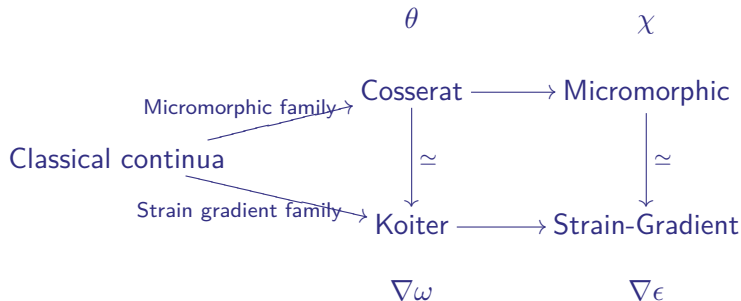
Effect of cell geometry : commensurate vs structural



[Overvelde J.T.B. & Bertoldi K. 2014]

Literature review

Family of equivalent media

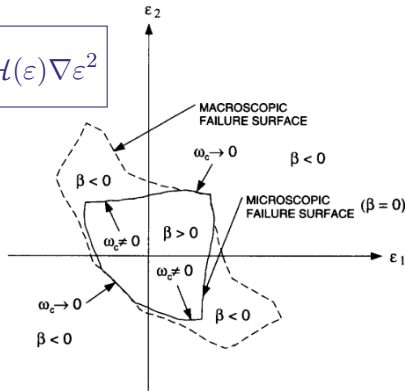


Basic extensions of a classical continuum.
Adapted from [Auffray et al. 2015].

Literature review

Higher-order gradient continuum model

$$W^s(\varepsilon) = \mathcal{E}(\varepsilon) + \frac{\ell^2}{2} \mathcal{H}(\varepsilon) \nabla \varepsilon^2$$



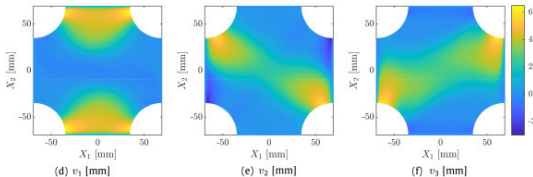
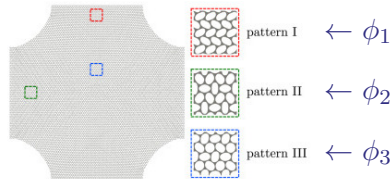
if $\mathcal{H}(\varepsilon) < 0$ at macroscopic buckling, commensurate instability happens before

[Triantafyllidis & Bardenhagen 1994-1996]

Literature review

Micromorphic continuum model

$$\mathcal{W}^m(\varepsilon, \phi_i)$$



[Rokos et al. 2014]



Research Question

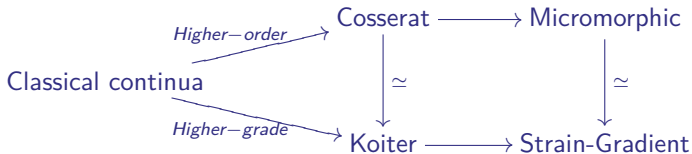


Research Question

[Auffray et al. 2015]

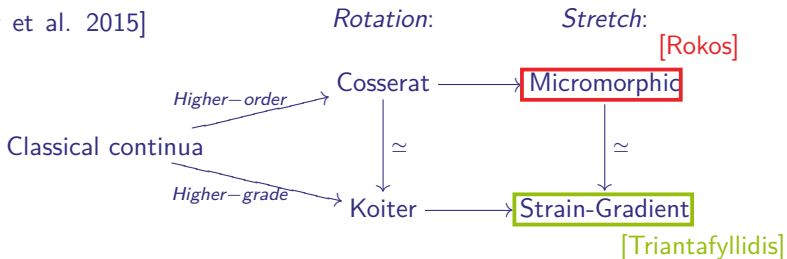
Rotation:

Stretch:



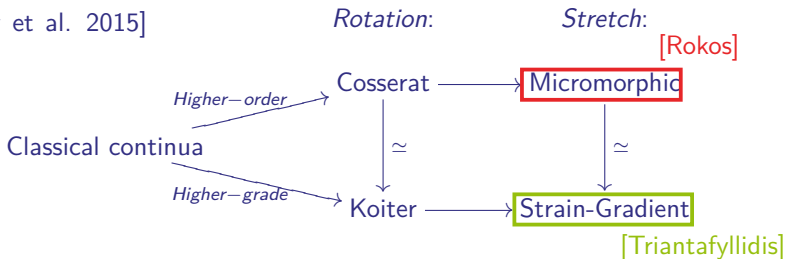
Research Question

[Auffray et al. 2015]



Research Question

[Auffray et al. 2015]



Which generalized continuum model is better suited to capture both commensurate and macroscopic instabilities in periodic media?

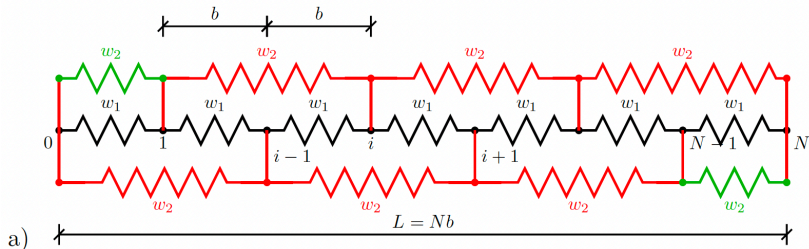
Our Study



Very simple example



Mesoscopic study



non-linear stored energy density in a spring $p = 1$ or $p = 2$

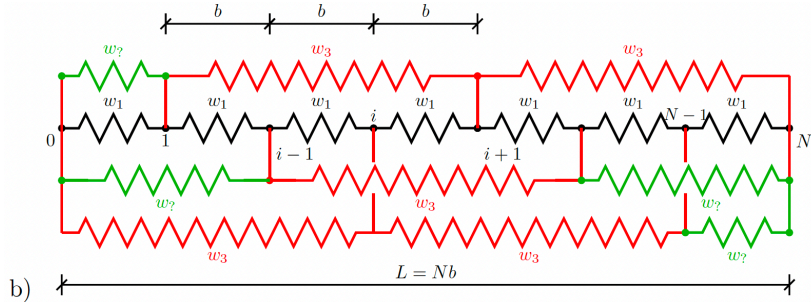
$$w_p(\varepsilon) = \frac{1}{2} K_p \varepsilon^2 - \frac{1}{4} M_p \varepsilon^4 + \frac{1}{6} N_p \varepsilon^6$$

inspired from [Triantafyllidis & Bardenhagen 1994]

[Truskinovsky & Vainchtein 2014]

Mesoscopic study

alternate example with springs on 3 nodes



Mesoscopic study

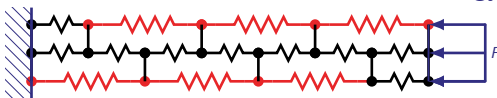
non-linear stored energy density in a spring $p = 1$ or $p = 2$

$$w_p(\varepsilon) = \frac{1}{2} K_p \varepsilon^2 - \frac{1}{4} M_p \varepsilon^4 + \frac{1}{6} N_p \varepsilon^6$$

Stored energy density of the full system

$$\mathcal{W} = \frac{1}{2b} \sum_{n=0}^{n=N} \sum_{p=1}^q p [w_p(\varepsilon_p^{n+}) + w_p(\varepsilon_p^{n-})]$$

Total energy



$$\mathcal{E} = \mathcal{W} - \frac{1}{Nb} F \cdot u_N$$

F end forces associated with end displacement u_N

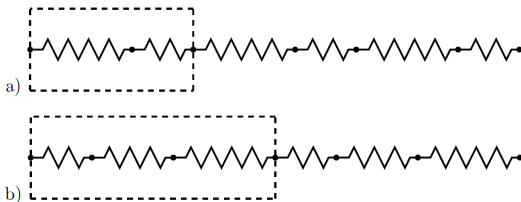
Mesoscopic study

Equilibrium \Rightarrow principal path $\varepsilon^0 =$ uniform strain of all springs

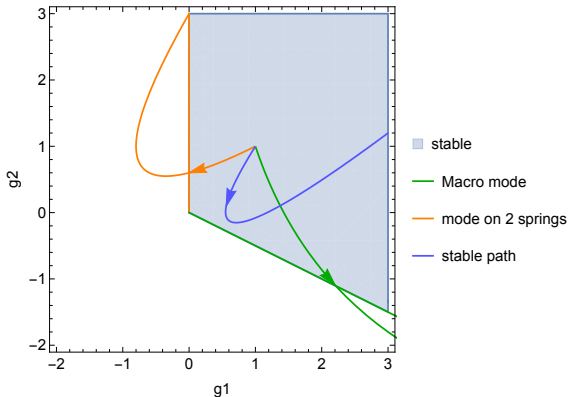
Bloch-wave perturbation predicts 2 possible unstable regimes

- macroscopic : $g_1(\varepsilon^0) + \rho g_2(\varepsilon^0) = 0$
- commensurate ($\lambda = \rho \ell$) : $g_1(\varepsilon^0) = 0$

$$\text{with } g_p(\varepsilon) = K_p - 3M_p\varepsilon^2 + 5N_p\varepsilon^4$$

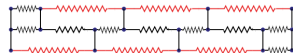


Mesoscopic study

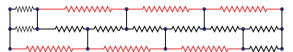


Stability of paths depends on the values of parameters K_p , M_p and N_p

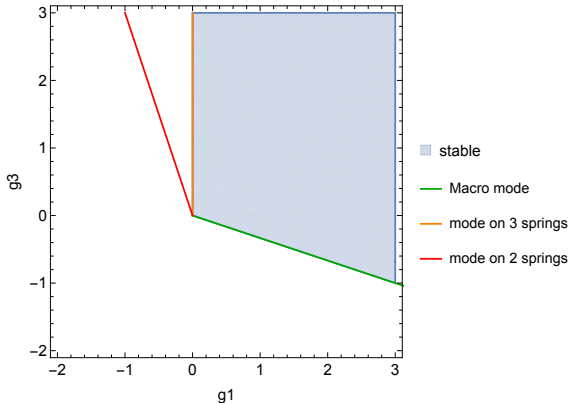
commensurate mode :



Macro mode :

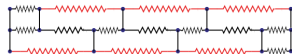


Mesoscopic study

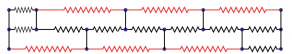


Stability of paths depends on the values of parameters K_p , M_p and N_p

commensurate mode :

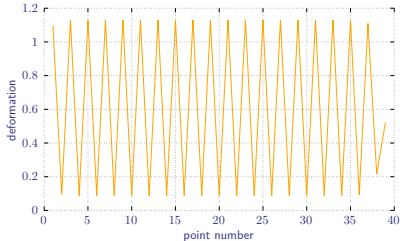
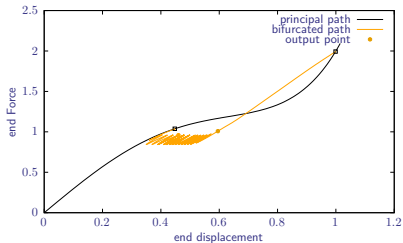


Macro mode :

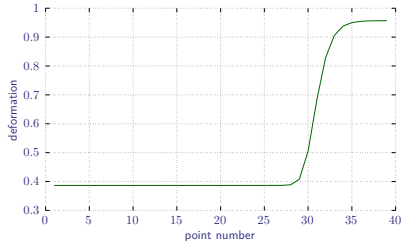
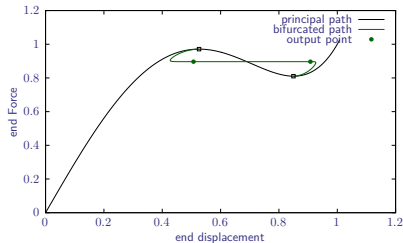


Mesoscopic study

commensurate mode :



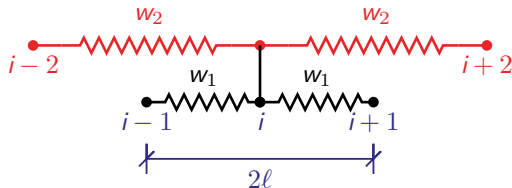
Macro mode :



Equivalent media

Micromorphic

from mesoscopic study : commensurate wavelength mode $\lambda = 2\ell$



Cauchy-Born kinematics :

$$u(\vec{x}) = \mathbf{E} \cdot \vec{x} + \mathbf{s}(\vec{x})$$

point position

macroscopic strain

internal shift
 \approx bif. mode

Equivalent media

Micromorphic

Micromorphic energy : $u(X)$ macroscopic displ. & $s(X)$ internal shift

$$\mathcal{W}^m(u, s) = \int_0^L w_1(u, x) + 2w_2(u, x) + G_1(u, x) \left(\frac{s}{\ell}\right)^2 + k_1(u, x) \left(\frac{s}{\ell}\right)^4 + \frac{N_1}{6} \left(\frac{s}{\ell}\right)^6 dX$$

Equilibrium equations

$$\frac{\partial \mathcal{W}^m}{\partial u} \quad \text{and} \quad \frac{\partial \mathcal{W}^m}{\partial s}$$

Macro

internal

Stability condition

$$\left[\begin{array}{cc} \frac{\partial^2 \mathcal{W}^s}{\partial u^2} & \frac{\partial^2 \mathcal{W}^s}{\partial u \partial s} \\ \frac{\partial^2 \mathcal{W}^s}{\partial u \partial s} & \frac{\partial^2 \mathcal{W}^s}{\partial s^2} \end{array} \right] \Big|_{s=0}$$

def positive

Equivalent media

Micromorphic

Stability condition

$$\det \begin{bmatrix} \frac{\partial^2 W^s}{\partial u^2} & \frac{\partial^2 W^s}{\partial u \partial s} \\ \frac{\partial^2 W^s}{\partial u \partial s} & \frac{\partial^2 W^s}{\partial s^2} \end{bmatrix} \Bigg|_{s=0} = 0 \quad \Leftrightarrow \quad g_1(u, x) (g_1(u, x) + 2g_2(u, x)) = 0$$

Exact same 2 conditions as for the mesoscopic case:

- $g_1(u, x) + 2g_2(u, x) = 0$
- $g_1(u, x) = 0$

Micromorphic medium able to capture both instabilities

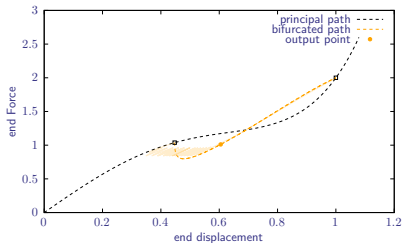
≈ vibration where micromorphic media can capture optical branches

Equivalent media

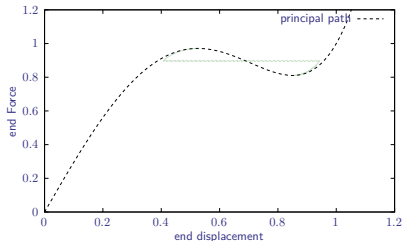
Micromorphic

Numerical solutions : discrete vs micromorphic

commensurate mode :



Macro mode :



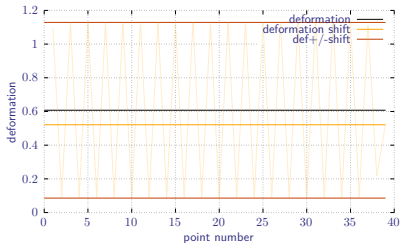
Equivalent media

Micromorphic

Numerical solutions : discrete vs micromorphic

commensurate mode :

Macro mode :



Equivalent media

Second gradient

[Triantafyllidis & Bardenhagen 1994]

Taylor-series expansion of displacement in strain expression

$$\epsilon_n^{p+} = \frac{u_{n+p} - u_n}{pl} = u_{,x} + \frac{1}{2}pl u_{,xx} + \frac{1}{6}(pl)^2 u_{,xxx} + \dots$$

$p = 1, 2$

$$\epsilon_n^{p-} = \frac{u_n - u_{n-p}}{pl} = u_{,x} - \frac{1}{2}pl u_{,xx} + \frac{1}{6}(pl)^2 u_{,xxx} - \dots$$

Second gradient energy density:

$$\mathcal{W}^s(u_{,x}) = w_1(u_{,x}) + 2w_2(u_{,x}) + (u_{,xx})^2 \frac{\ell^2}{2} h(u_{,x})$$

Equivalent media

Second gradient

[Triantafyllidis & Bardenhagen 1994]

Second gradient energy:

$$\mathcal{E}^s(u, x) = \int_0^L w_1(u, x) + 2w_2(u, x) + (u,_{xx})^2 \frac{\ell^2}{2} h(u, x) dx$$

Equilibrium equation: $\mathcal{E}_{,u}^s \delta u = 0$

$$w'_1(u, x) + 2w'_2(u, x) - \ell^2 \left(\frac{h'(u, x)}{2} (u,_{xx})^2 + h(u, x) u,_{xxx} \right) = F$$

Stability condition: $\mathcal{E}_{,uu}^s \delta u \delta u = 0$

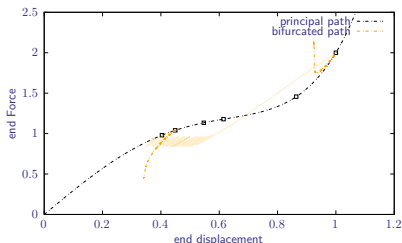
$$g_1(u, x) + 2g_2(u, x) + \left(\frac{\ell n \pi}{L} \right)^2 h(u, x) = 0 \quad \forall n \in \mathbb{Z}$$

Equivalent media

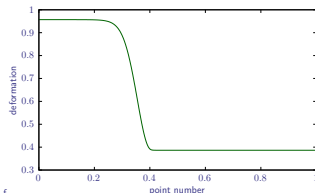
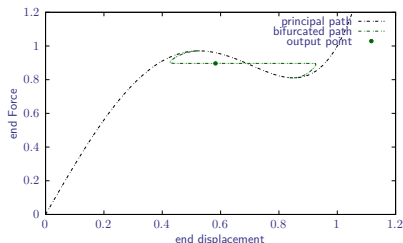
Second gradient

Numerical solutions : discrete vs second gradient (AUTO)

commensurate mode :



Macro mode :



Conclusions

Comparing generalized continua vs instabilities

Micromorphic:

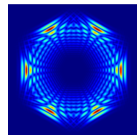
- Can capture patterning modes with criterion = mesoscopic
- Cannot capture localization
- Requires *a priori* knowledge of buckling mode wavelength

Second gradient:

- Can capture localization with criterion = mesoscopic
- Cannot capture commensurate mode
- Does not require any *a priori* knowledge & systematic construction

A mixte model might be useful to capture localization and commensurate mode !

C. Combescure, *Selecting generalized continuum theories for nonlinear periodic solids based on the instabilities of the underlying microstructure*, 2022, J. of El



Coss&Vita

MOMAP
ANR-19-CE10-0005



ANR-19-CE08-0005