

# Selecting generalized continuum theories for nonlinear periodic solids based on the instabilities of the underlying microstructure

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## 1. Context

## 2. Research Question

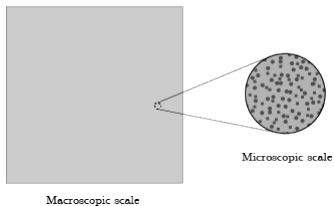
## 3. Our Study

# Architected Materials

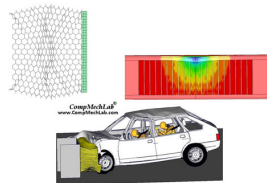
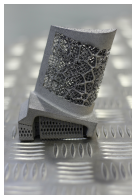
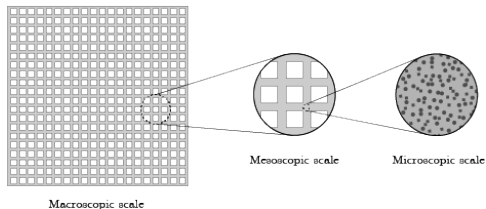
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## Definition and applications

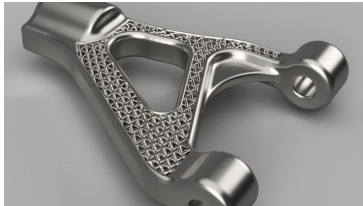
Scale separation in conventional materials



Scale separation in architected materials



# Architected Materials

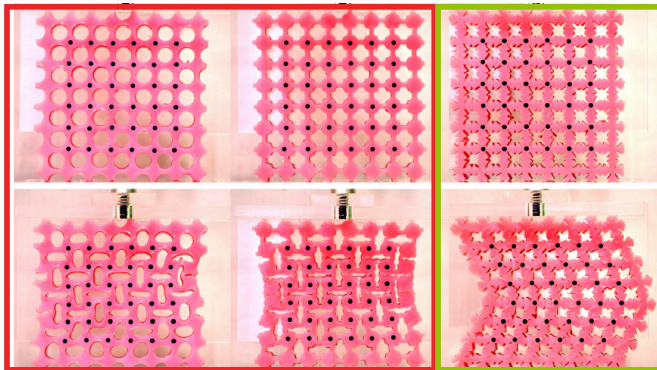


Prohibitive computational cost for full modeling  
⇒ Need for equivalent continuum media

# Architected Materials & Instabilities

## Buckling when compressed

Effect of cell geometry : commensurate vs structural

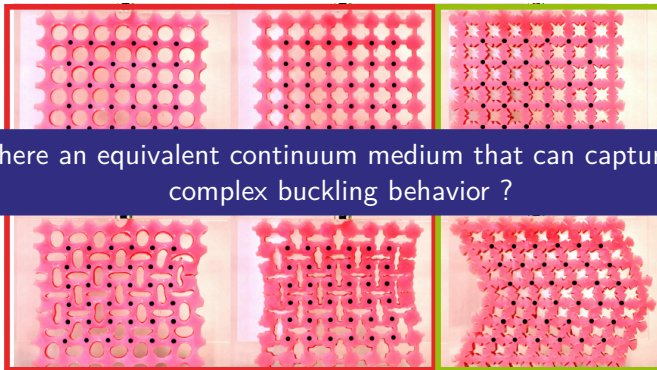


[Overvelde J.T.B. & Bertoldi K. 2014]

# Architected Materials & Instabilities

## Buckling when compressed

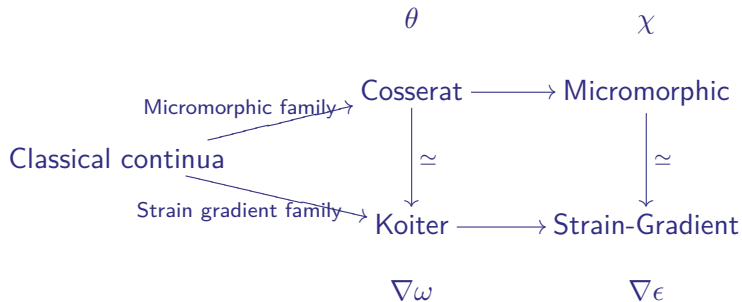
Effect of cell geometry : commensurate vs structural



[Overvelde J.T.B. & Bertoldi K. 2014]

# Literature review

## Family of equivalent media

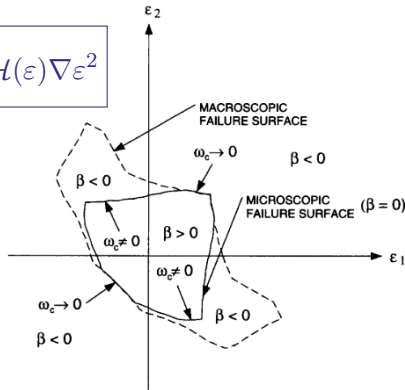


Basic extensions of a classical continuum.  
Adapted from [Auffray et al. 2015].

# Literature review

## Higher-order gradient continuum model

$$W^s(\varepsilon) = \mathcal{E}(\varepsilon) + \frac{\ell^2}{2} \mathcal{H}(\varepsilon) \nabla \varepsilon^2$$



if  $\mathcal{H}(\varepsilon) < 0$  at macroscopic buckling, commensurate instability happens before

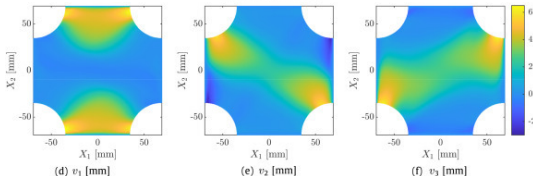
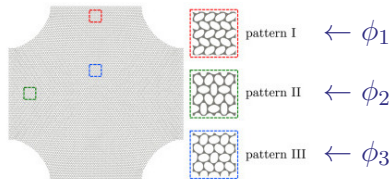
[Triantafyllidis & Bardenhagen 1994-1996]



# Literature review

## Micromorphic continuum model

$$\mathcal{W}^m(\varepsilon, \phi_i)$$



[Rokos et al. 2014]



# Research Question



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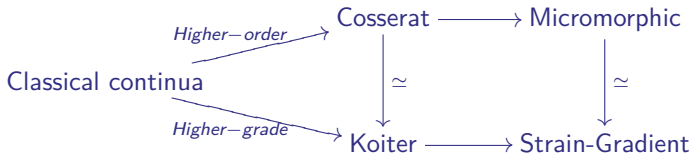
# Research Question

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[Auffray et al. 2015]

*Rotation:*

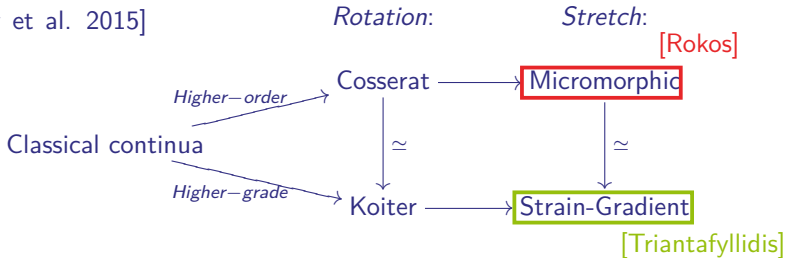
*Stretch:*



# Research Question

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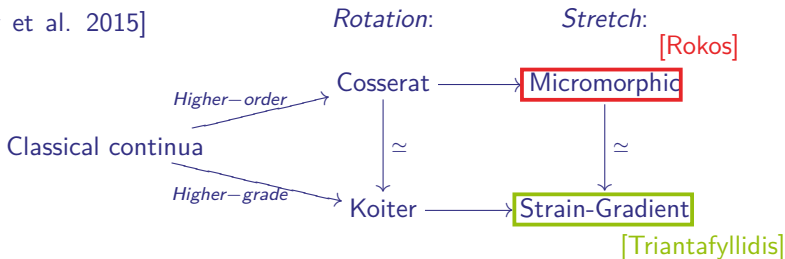
[Auffray et al. 2015]



# Research Question

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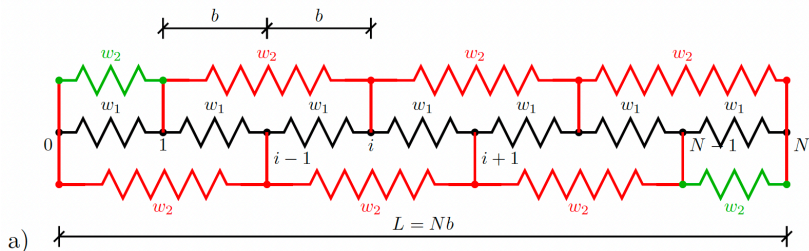
[Auffray et al. 2015]



Which generalized continuum model is better suited to capture both commensurate and macroscopic instabilities in periodic media?



## Mesoscopic study



non-linear stored energy density in a spring  $p = 1$  or  $p = 2$

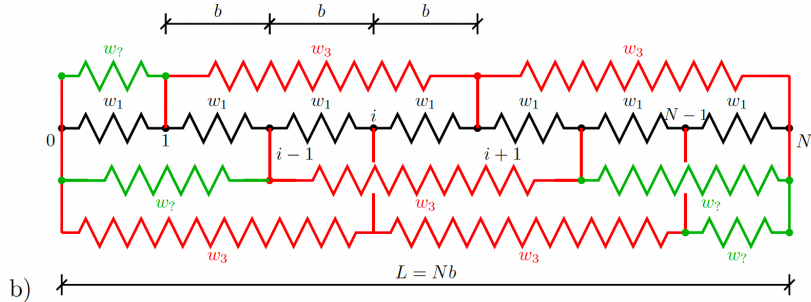
$$w_p(\varepsilon) = \frac{1}{2} K_p \varepsilon^2 - \frac{1}{4} M_p \varepsilon^4 + \frac{1}{6} N_p \varepsilon^6$$

*inspired from [Triantafyllidis & Bardenhagen 1994]*

*[Truskinovsky & Vainchtein 2014]*

# Mesoscopic study

alternate example with springs on 3 nodes





## Mesoscopic study

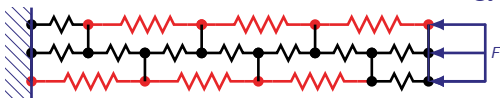
non-linear stored energy density in a spring  $p = 1$  or  $p = 2$

$$w_p(\varepsilon) = \frac{1}{2} K_p \varepsilon^2 - \frac{1}{4} M_p \varepsilon^4 + \frac{1}{6} N_p \varepsilon^6$$

Stored energy density of the full system

$$\mathcal{W} = \frac{1}{2b} \sum_{n=0}^{n=N} \sum_{p=1}^q p [w_p(\varepsilon_p^{n+}) + w_p(\varepsilon_p^{n-})]$$

Total energy



$$\mathcal{E} = \mathcal{W} - \frac{1}{Nb} F \cdot u_N$$

$F$  end forces associated with end displacement  $u_N$

## Mesoscopic study

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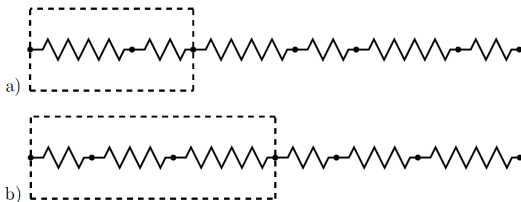
Equilibrium  $\Rightarrow$  principal path  $\varepsilon^0 =$  uniform strain of all springs

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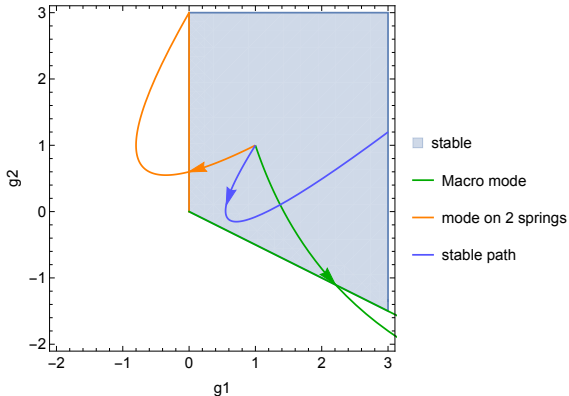
Bloch-wave perturbation predicts 2 possible unstable regimes

- macroscopic :  $g_1(\varepsilon^0) + \rho g_2(\varepsilon^0) = 0$
- commensurate ( $\lambda = \rho \ell$ ) :  $g_1(\varepsilon^0) = 0$

$$\text{with } g_p(\varepsilon) = K_p - 3M_p\varepsilon^2 + 5N_p\varepsilon^4$$

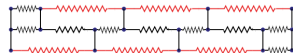


# Mesososcopic study

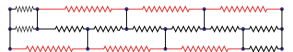


Stability of paths depends on the values of parameters  $K_p$ ,  $M_p$  and  $N_p$

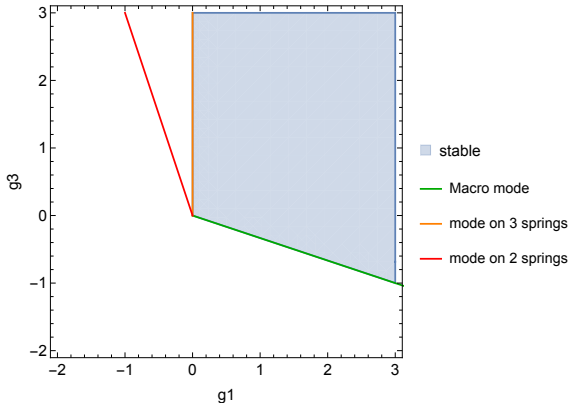
commensurate mode :



Macro mode :

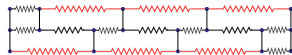


# Mesoscopic study

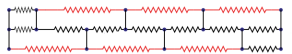


Stability of paths depends on the values of parameters  $K_p$ ,  $M_p$  and  $N_p$

commensurate mode :

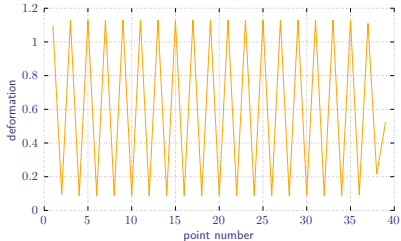
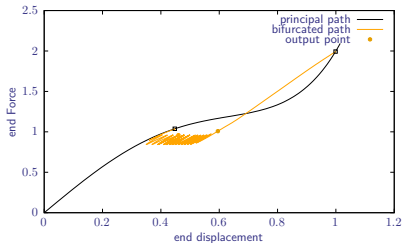


Macro mode :

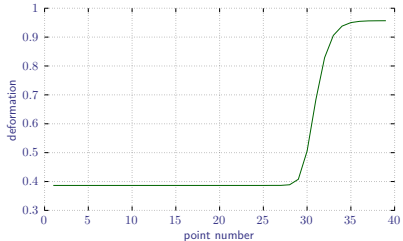
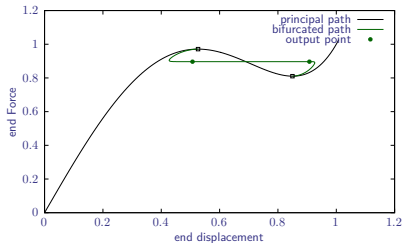


# Mesososcopic study

commensurate mode :



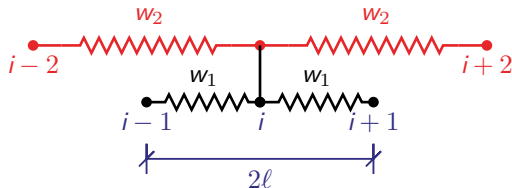
Macro mode :



# Equivalent media

## Micromorphic

from mesoscopic study : commensurate wavelength mode  $\lambda = 2\ell$



Cauchy-Born kinematics :

$$u(\vec{x}) = \mathbf{E} \cdot \vec{x} + \mathbf{s}(\vec{x})$$

point position

macroscopic strain

internal shift  
 $\approx$  bif. mode

# Equivalent media

## Micromorphic

Micromorphic energy :  $u(X)$  macroscopic displ. &  $s(X)$  internal shift

$$\mathcal{W}^m(u, s) = \int_0^L w_1(u, x) + 2w_2(u, x) + G_1(u, x) \left(\frac{s}{\ell}\right)^2 + k_1(u, x) \left(\frac{s}{\ell}\right)^4 + \frac{N_1}{6} \left(\frac{s}{\ell}\right)^6 dX$$

Equilibrium equations

$$\frac{\partial \mathcal{W}^m}{\partial u} \quad \text{and} \quad \frac{\partial \mathcal{W}^m}{\partial s}$$

Macro

internal

Stability condition

$$\left[ \begin{array}{cc} \frac{\partial^2 \mathcal{W}^s}{\partial u^2} & \frac{\partial^2 \mathcal{W}^s}{\partial u \partial s} \\ \frac{\partial^2 \mathcal{W}^s}{\partial u \partial s} & \frac{\partial^2 \mathcal{W}^s}{\partial s^2} \end{array} \right] \Big|_{s=0}$$

def positive

# Equivalent media

## Micromorphic

### Stability condition

$$\det \begin{bmatrix} \frac{\partial^2 W^s}{\partial u^2} & \frac{\partial^2 W^s}{\partial u \partial s} \\ \frac{\partial^2 W^s}{\partial u \partial s} & \frac{\partial^2 W^s}{\partial s^2} \end{bmatrix} \Bigg|_{s=0} = 0 \quad \Leftrightarrow \quad g_1(u,x) (g_1(u,x) + 2g_2(u,x)) = 0$$

Exact same 2 conditions as for the mesoscopic case:

- $g_1(u,x) + 2g_2(u,x) = 0$
- $g_1(u,x) = 0$

Micromorphic medium able to capture both instabilities

≈ vibration where micromorphic media can capture optical branches

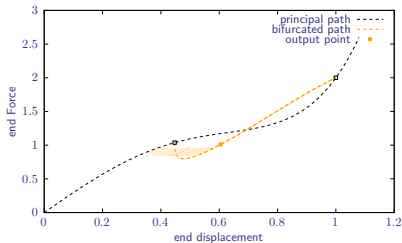


# Equivalent media

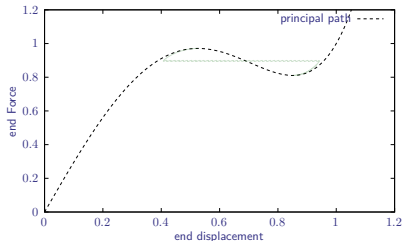
## Micromorphic

Numerical solutions : discrete vs micromorphic

commensurate mode :



Macro mode :



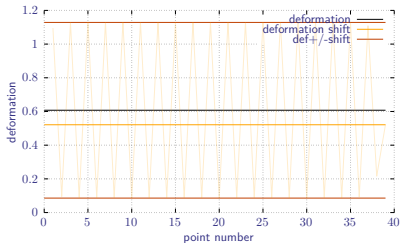
# Equivalent media

## Micromorphic

Numerical solutions : discrete vs micromorphic

commensurate mode :

Macro mode :



# Equivalent media

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## Second gradient

[Triantafyllidis & Bardenhagen 1994]

Taylor-series expansion of displacement in strain expression

$$\epsilon_n^{p+} = \frac{u_{n+p} - u_n}{pl} = u_{,x} + \frac{1}{2}pl u_{,xx} + \frac{1}{6}(pl)^2 u_{,xxx} + \dots$$

$p = 1, 2$

$$\epsilon_n^{p-} = \frac{u_n - u_{n-p}}{pl} = u_{,x} - \frac{1}{2}pl u_{,xx} + \frac{1}{6}(pl)^2 u_{,xxx} - \dots$$

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Second gradient energy density:

$$\mathcal{W}^s(u_{,x}) = w_1(u_{,x}) + 2w_2(u_{,x}) + (u_{,xx})^2 \frac{\ell^2}{2} h(u_{,x})$$

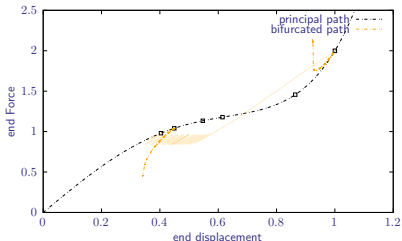


# Equivalent media

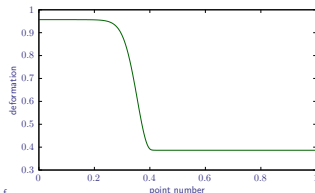
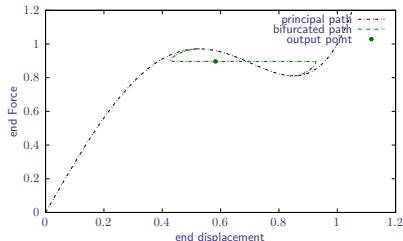
## Second gradient

Numerical solutions : discrete vs second gradient (AUTO)

commensurate mode :

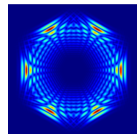


Macro mode :





C. Combescure, *Selecting generalized continuum theories for nonlinear periodic solids based on the instabilities of the underlying microstructure*, 2022, J. of El



Coss&Vita

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