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Generalized Keisler Theorems for Atomic and Molecular Logics

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Abstract

The Keisler theorems dealing with the definability in first-order logic of classes of structures are adapted to atomic and molecular logics. These logics are based on Dunn’s gaggle theory and generalize modal logics. We show how notions of bisimulation can be automatically defined from the truth conditions of the connectives of any atomic or molecular logic. Then, we adapt the basic notion of ultraproduct of first-order logic to our atomic and molecular logics. This allows us to state our generalized Keisler theorems. The connectives of molecular logics should be in a specific format called ‘normality’ for our theorems to hold. We consider modal logic, the Lambek calculus, (modal) intuitionistic logic, weakly aggregative modal logic and temporal logic as case study. In a companion article [3], we use and extend these results to first-order logic and protologics, *i.e.* logics such that the truth conditions of their connectives are expressible by first-order formulas.

1 Introduction

The model theory of non-classical logics is often developed on a case by case basis by considering each logic independently, and similar theorems are proved for each non-classical logic by frequently adapting the same proof method. For example, for modal logic, temporal logic and XPath, a number of model-theoretical results dealing with the definability of classes of models by means of a set of formulas or a single formula have been proved [8, 23, 1] by adapting the Keisler theorems of first-order logic (FOL for short) [9]. This type of theorems provides conditions of definability of classes of models in a specific logic. More precisely, they state that a class of models is definable in a given logic if this class of models and its complement are both closed under a specific construction called ultraproduct and under a specific notion of bisimulation associated to the logic. Similar adaptations to other logics have been made for the van Benthem characterization theorem of modal logic [27, 23, 5, 20, 10]. A natural question that comes up to mind is to wonder whether the Keisler theorems of FOL transfer to an arbitrary logic. We shall see in that paper that the Keisler theorems do transfer to any atomic and molecular logics if their connectives are ‘normal’.

The framework of atomic and molecular logics is based on Dunn’s gaggle theory [12, 13]. Atomic logics generalize “gaggle logics” by adding types to formulas. Gaggle logics were introduced in [2] where it is also shown that a very large number of non-classical logics are actually

gaggle logics and therefore also atomic logics (see Figures 6-11 in that article). Atomic and molecular logics are a generalization of gaggle logics which behave as ‘normal form’ logics. We indeed show that every non-classical logic such that the truth conditions of its connectives are expressible in first-order logic is as expressive as an atomic or molecular logic [4, 3].

As it turns out, an appropriate notion of bisimulation can be automatically associated to any atomic or molecular logic from the truth conditions of its connectives. We will rediscover the already known invariance notions of modal logic, the Lambek calculus, (modal) intuitionistic logic, weakly aggregative modal logic and temporal logic as specific instances of our generic notion of C-bisimulation. This will allow us to state our generalized Keisler theorems for atomic and molecular logics.

Structure of the article We start in Section 2 by recalling modal logic, the Lambek calculus, temporal logic and modal intuitionistic logic. In Section 3 we introduce atomic logics, in Section 4 molecular logics and in Section 5 their Boolean versions with Boolean connectives. In Section 7, we show how notions of bisimulations can be automatically defined from the truth conditions of the connectives of atomic and molecular logics, after some preliminaries in Section 6 where we introduce universal and existential connectives. We illustrate our general definitions by rediscovering the bisimulation notions of modal logic, the Lambek calculus, temporal logic, (modal) intuitionistic logic and weakly aggregative modal logic. In Section 8, we adapt the notions of ultraproducts and ultrapowers to atomic and molecular logics and state in Section 9 our main theorems for atomic and molecular logics. We conclude in Section 10. All the proofs of this paper are in the appendix.

2 Non-Classical Logics

In this section, we recall modal logic, the Lambek calculus and temporal logics. The generic results of the article will be applied to these logics in the sequel. Logics will always be semantically presented by following a tri-partite representation: language, class of models, satisfaction relation.

In this section, \mathbb{A} is a set of *propositional letters* which can be finite or infinite.

2.1 Modal Logic

The set \mathbb{I} is a set of indices which can be finite or infinite. The *multi-modal language* \mathcal{L}_{ML} is defined inductively by the following grammar in BNF:

$$\mathcal{L}_{\text{ML}} : \varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \diamond_j \varphi \mid \square_j \varphi$$

where $p \in \mathbb{A}$ and $j \in \mathbb{I}$.

We present the so-called *possible world semantics* of modal logic. A *Kripke model* M is a tuple $M \triangleq (W, \{R_1, \dots, R_m, \dots, P_1, \dots, P_n, \dots\})$ where

- W is a non-empty set whose elements are called *possible worlds*;
- $R_1, \dots, R_m, \dots \subseteq W \times W$, $m \in \mathbb{I}$ are binary relations over W called *accessibility relations*;
- $P_1, \dots, P_n, \dots \subseteq W$ are unary relations interpreting the propositional letters of \mathbb{A} .

We write $w \in M$ for $w \in W$ by abuse and the pair (M, w) is called a *pointed Kripke model*. The class of all pointed Kripke models is denoted \mathcal{E}_{ML} .

We define the *satisfaction relation* $\models_{\text{ML}} \subseteq \mathcal{E}_{\text{ML}} \times \mathcal{L}_{\text{ML}}$ inductively by the following *truth conditions*. Below, we write $(M, w) \models \varphi$ for $((M, w), \varphi) \in \models_{\text{ML}}$. For all $(M, w) \in \mathcal{E}_{\text{ML}}$, all $\varphi, \psi \in \mathcal{L}_{\text{ML}}$, all $p_i \in \mathbb{A}$ and all $j \in \mathbb{I}$,

$$\begin{aligned} (M, w) \models p_i & \quad \text{iff } P_i(w) \text{ holds;} \\ (M, w) \models \neg p_i & \quad \text{iff } P_i(w) \text{ does not hold;} \\ (M, w) \models (\varphi \wedge \psi) & \quad \text{iff } (M, w) \models \varphi \text{ and } (M, w) \models \psi; \\ (M, w) \models (\varphi \vee \psi) & \quad \text{iff } (M, w) \models \varphi \text{ or } (M, w) \models \psi; \\ (M, w) \models \diamond_j \varphi & \quad \text{iff there exists } v \in W \text{ such that } R_j w v \text{ and } (M, v) \models \varphi; \\ (M, w) \models \square_j \varphi & \quad \text{iff for all } v \in W \text{ such that } R_j w v, (M, v) \models \varphi. \end{aligned}$$

The triple $(\mathcal{L}_{\text{ML}}, \mathcal{E}_{\text{ML}}, \models_{\text{ML}})$ forms a logic, that we call *modal logic*. Bisimulations for modal logic can be found in [8].

2.2 Lambek Calculus

The *Lambek language* \mathcal{L}_{LC} is the set of formulas defined inductively by the following grammar in BNF:

$$\mathcal{L}_{\text{LC}} : \varphi ::= p \mid (\varphi \otimes \varphi) \mid (\varphi \subset \varphi) \mid (\varphi \supset \varphi)$$

where $p \in \mathbb{P}$. A *Lambek model* is a tuple $M = (W, \{R, P_1, \dots, P_n, \dots\})$ where:

- W is a non-empty set;
- $R \subseteq W \times W \times W$ is a ternary relation over W ;
- $P_1, \dots, P_n, \dots \subseteq W$ are unary relations over W .

We write $w \in M$ for $w \in W$ by abuse and (M, w) is called a *pointed Lambek model*. The class of all pointed Lambek models is denoted \mathcal{E}_{LC} . We define the *satisfaction relation* $\models_{\text{Int}} \subseteq \mathcal{E}_{\text{LC}} \times \mathcal{L}_{\text{LC}}$ by the following *truth conditions*. Below, we write $(M, w) \models \varphi$ for $((M, w), \varphi) \in \models_{\text{LC}}$. For all Lambek models $M = (W, \{R, P_1, \dots, P_n, \dots\})$, all $w \in M$, all $\varphi, \psi \in \mathcal{L}_{\text{LC}}$ and all $p_i \in \mathbb{P}$,

$$\begin{aligned} (M, w) \models p_i & \quad \text{iff } P_i(w) \text{ holds;} \\ (M, w) \models (\varphi \otimes \psi) & \quad \text{iff there are } v, u \in W \text{ such that } Rvwu, \\ & \quad (M, v) \models \varphi \text{ and } (M, u) \models \psi; \\ (M, w) \models (\varphi \supset \psi) & \quad \text{iff for all } v, u \in W \text{ such that } Rvwu, \\ & \quad \text{if } (M, v) \models \varphi \text{ then } (M, u) \models \psi; \\ (M, w) \models (\psi \subset \varphi) & \quad \text{iff for all } v, u \in W \text{ such that } Rvwu, \\ & \quad \text{if } (M, v) \models \varphi \text{ then } (M, u) \models \psi. \end{aligned}$$

The triple $(\mathcal{L}_{\text{LC}}, \mathcal{E}_{\text{LC}}, \models_{\text{LC}})$ forms a logic, that we call the *Lambek calculus*. Bisimulations for the Lambek calculus, called *directed bisimulations*, can be found in [29].

2.3 Temporal Logic

The *temporal language* \mathcal{L}_{TL} is defined inductively by the following grammar in BNF:

$$\mathcal{L}_{\text{TL}} : \varphi ::= \top \mid \perp \mid p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid U(\varphi, \varphi) \mid S(\varphi, \varphi)$$

where $p \in \mathbb{A}$. A *temporal model* is a tuple $M = (W, \{<, P_1, \dots, P_n, \dots\})$ where:

- W is a non-empty set;
- $<\subseteq W \times W$ is a binary relation over W ;
- $P_1, \dots, P_n, \dots \subseteq W$ are unary relations over W .

We write $w \in M$ for $w \in W$ by abuse and the pair (M, w) is called a *pointed temporal model*. The class of all pointed temporal models is denoted \mathcal{E}_{TL} . We define the *satisfaction relation* $\models_{\text{TL}} \subseteq \mathcal{E}_{\text{TL}} \times \mathcal{L}_{\text{TL}}$ by the following *truth conditions*. Below, we write $(M, w) \models \varphi$ for $((M, w), \varphi) \in \models_{\text{TL}}$. For all temporal models $M = (W, \{<, P_1, \dots, P_n, \dots\})$, all $w \in M$, all $\varphi, \psi \in \mathcal{L}_{\text{TL}}$ and all $p_i \in \mathbb{P}$,

$(M, w) \models \top$	always;
$(M, w) \models \perp$	never;
$(M, w) \models p_i$	iff $P_i(w)$ holds;
$(M, w) \models \neg p_i$	iff $P_i(w)$ does not hold;
$(M, w) \models (\varphi \wedge \psi)$	iff $(M, w) \models \varphi$ and $(M, w) \models \psi$;
$(M, w) \models (\varphi \vee \psi)$	iff $(M, w) \models \varphi$ or $(M, w) \models \psi$;
$(M, w) \models U(\varphi, \psi)$	iff there is $v \in W$ such that $w < v$ and $(M, v) \models \varphi$ and for all $u \in W$ such that $w < u < v$, $(M, u) \models \psi$;
$(M, w) \models S(\varphi, \psi)$	iff there is $v \in W$ such that $v < w$ and $(M, v) \models \varphi$ and for all $u \in W$ such that $v < u < w$, $(M, u) \models \psi$.

The triple $(\mathcal{L}_{\text{TL}}, \mathcal{E}_{\text{TL}}, \models_{\text{TL}})$ forms a logic, that we call *temporal logic*. Bisimulations for temporal logic can be found in [23].

2.4 Modal Intuitionistic Logic

The *modal intuitionistic language* \mathcal{L}_{Int} is defined inductively by the following grammar in BNF:

$$\mathcal{L}_{\text{Int}} : \varphi ::= \top \mid \perp \mid p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \Rightarrow \varphi) \mid \diamond\varphi \mid \square\varphi$$

where $p \in \mathbb{A}$. A *modal intuitionistic model* is a tuple $M = (W, \{R, R_\diamond, P_1, \dots, P_n, \dots\})$ where:

- W is a non-empty set;
- $R \subseteq W \times W$ is a binary relation over W which is reflexive and transitive (R is *reflexive* if for all $w \in W$ Rww and *transitive* if for all $u, v, w \in W$, Ruv and Rvw imply Ruw);
- $R_\diamond \subseteq W \times W$ is a binary relation over W ;
- $P_1, \dots, P_n, \dots \subseteq W$ are unary relations over W such that for all $v, w \in W$, if Rvw and $P_n(v)$ then $P_n(w)$.

We write $w \in M$ for $w \in W$ by abuse and the pair (M, w) is called a *pointed modal intuitionistic model*. The class of all pointed modal intuitionistic models is denoted \mathcal{E}_{Int} . We define the *satisfaction relation* $\models_{\text{Int}} \subseteq \mathcal{E}_{\text{Int}} \times \mathcal{L}_{\text{Int}}$ by the following *truth conditions*. Below, we write $(M, w) \models \varphi$ for $((M, w), \varphi) \in \models_{\text{Int}}$. For all modal intuitionistic models $M = (W, \{R, R_\diamond, P_1, \dots, P_n, \dots\})$,

all $w \in M$, all $\varphi, \psi \in \mathcal{L}_{\text{Int}}$ and all $p_i \in \mathbb{P}$,

$(M, w) \models \top$		always;
$(M, w) \models \perp$		never;
$(M, w) \models p_i$	iff	$P_i(w)$ holds;
$(M, w) \models (\varphi \wedge \psi)$	iff	$(M, w) \models \varphi$ and $(M, w) \models \psi$;
$(M, w) \models (\varphi \vee \psi)$	iff	$(M, w) \models \varphi$ or $(M, w) \models \psi$;
$(M, w) \models (\varphi \Rightarrow \psi)$	iff	for all $v \in W$ such that Rwv , if $(M, v) \models \varphi$ then $(M, v) \models \psi$;
$(M, w) \models \Box\varphi$	iff	for all $v \in W$ such that Rwv , for all $u \in W$ such that $R_\diamond vu$, $(M, u) \models \varphi$;
$(M, w) \models \Diamond\varphi$	iff	for all $v \in W$ such that Rwv , there is $u \in W$ such that $R_\diamond vu$ and $(M, u) \models \varphi$.

The triple $(\mathcal{L}_{\text{Int}}, \mathcal{E}_{\text{Int}}, \models_{\text{Int}})$ forms a logic, that we call *modal intuitionistic logic*. Bisimulations for (modal) intuitionistic logic can be found in [26, 27].

2.5 Common Logical Notions

In the present section, we define a number of notions which are common to all logics and in particular to the logics introduced beforehand. The way we define logics is different from many proposals considered in universal logic [6] such as pairs of Suzsko’s abstract logics, Tarski’s consequence operators or logical structures. Often a logic is viewed as a pair of a language together with a consequence relation on this language. Our approach to defining logics is somehow more ‘semantic’ in that respect than the usual proposals. It corresponds in fact to the “abstract logics” of García-Matos & Väänänen [17] or to the “rooms” of Mossakowski et al. [25].

A *logic* is a triple $\mathbf{L} \triangleq (\mathcal{L}, \mathcal{E}, \models)$ where

- \mathcal{L} is a *logical language* defined as a set of well-formed expressions built from a set of *connectives* \mathbf{C} and a set of *propositional letters* \mathbf{A} ;
- \mathcal{E} is a *class of pointed models*;
- \models is a *satisfaction relation* which relates in a compositional manner elements of \mathcal{L} to models of \mathcal{E} by means of so-called *truth conditions*.

Let $\mathbf{L} = (\mathcal{L}, \mathcal{E}, \models)$ be a logic and let $\Gamma \subseteq \mathcal{L}$, $\varphi \in \mathcal{L}$ and $M \in \mathcal{E}$. We write $M \models \Gamma$ when for all $\psi \in \Gamma$, we have $M \models \psi$. Then, we say that

- φ is *true (satisfied)* at M or M is a *model* of φ when $M \models \varphi$;
- φ is a *logical consequence* of Γ , written $\Gamma \models_{\mathbf{L}} \varphi$, when for all $M \in \mathcal{E}$, if $M \models \Gamma$ then $M \models \varphi$;
- φ is *valid*, written $\models_{\mathbf{L}} \varphi$, when for all models $M \in \mathcal{E}$, we have $M \models \varphi$;
- φ is *satisfiable* when there is a model $M \in \mathcal{E}$ such that $M \models \varphi$.

If Γ is a singleton $\Gamma = \{\psi\}$, we also write by abuse $\psi \models \varphi$ for $\{\psi\} \models \varphi$.

A set of formulas of \mathcal{L} is called a *theory*. A set Δ of formulas of \mathcal{L} is said to be a *set of axioms* for a theory Γ iff Γ and Δ have the same logical consequences. A theory is called *finitely axiomatizable* iff it has a finite set of axioms. A logic \mathbf{L} is *axiomatizable* if its set of validities is finitely axiomatizable.

3 Atomic Logics

Atomic logics are logics such that the truth conditions of their connectives are defined by first-order formulas of the form $\forall x_1 \dots x_n (\pm_1 \mathbf{Q}_1 x_1 \vee \dots \vee \pm_n \mathbf{Q}_n x_n \vee \pm \mathbf{R} x_1 \dots x_n x)$ or $\exists x_1 \dots x_n (\pm_1 \mathbf{Q}_1 x_1 \wedge \dots \wedge \pm_n \mathbf{Q}_n x_n \wedge \pm \mathbf{R} x_1 \dots x_n x)$ where the \pm_i s and \pm are either empty or \neg . Likewise, propositional letters are defined by first-order formulas of the form $\pm \mathbf{R} x$. We will represent the structure of these formulas by means of so-called *skeletons* whose various arguments capture the different features and patterns from which they can be redefined completely. Atomic logics are also generalizations of our gaggle logics [2] with types associated to formulas.

We recall that \mathbb{N}^* denotes the set of natural numbers without 0 and that for all $n \in \mathbb{N}^*$, \mathfrak{S}_n denotes the group of permutations over the set $\{1, \dots, n\}$. Permutations are generally denoted σ, τ , the identity permutation Id is sometimes denoted 1 as the neutral element of every permutation group and σ^- stands for the inverse permutation of the permutation σ . For example, the permutation $\sigma = (3, 1, 2)$ is the permutation that maps 1 to 3, 2 to 1 and 3 to 2 (see for instance [30] for more details).

Definition 1 (Atomic skeletons and connectives). The sets of *atomic skeletons* \mathbb{P} and \mathbb{C} are defined as follows:

$$\begin{aligned} \mathbb{P} &\triangleq \mathfrak{S}_1 \times \{+, -\} \times \{\forall, \exists\} \times \mathbb{N}^* \\ \mathbb{C} &\triangleq \mathbb{P} \cup \bigcup_{n \in \mathbb{N}^*} \left\{ \mathfrak{S}_{n+1} \times \{+, -\} \times \{\forall, \exists\} \times \mathbb{N}^{*n+1} \times \{+, -\}^n \right\}. \end{aligned}$$

\mathbb{P} is called the set of *propositional letter skeletons* and \mathbb{C} is called the set of *connective skeletons*. They can be represented by tuples $(\sigma, \pm, \mathbb{A}, \bar{k}, \Xi_j)$ or $(\sigma, \pm, \mathbb{A}, k)$ if it is a propositional letter skeleton, where $\mathbb{A} \in \{\forall, \exists\}$ is called the *quantification signature* of the skeleton, $\bar{k} = (k, k_1, \dots, k_n) \in \mathbb{N}^{*n+1}$ is called the *type signature* of the skeleton and $\Xi_j = (\pm_1, \dots, \pm_n) \in \{+, -\}^n$ is called the *tonicity signature* of the skeleton; $(\mathbb{A}, \bar{k}, \Xi_j)$ is called the *signature* of the skeleton. The *arity* of a propositional letter skeleton is 0 and its *type* is k . The *arity* of a skeleton $\star \in \mathbb{C}$ is n , its *input types* are k_1, \dots, k_n and its *output type* is k .

A (*atomic*) *connective* or *propositional letter* is a symbol generally denoted \star or p to which is associated a (atomic) skeleton. Its arity, signature, quantification signature, type signature, tonicity signature, input and output types are the same as its skeleton. By abuse, we sometimes identify a connective with its skeleton. If \mathbb{C} is a set of atomic connectives, its set of propositional letters is denoted $\mathbb{P}(\mathbb{C})$. Propositional letters are denoted p, p_1, p_2 , etc. and connectives \star, \star_1, \star_2 , etc. \square

We need to distinguish between connectives and skeletons because in general we need a countable number of propositional letters or connectives of the same skeleton, like in some modal logics, where we need multiple modalities of the same (similarity) type/skeleton.

Definition 2 (Atomic language). Let \mathbb{C} be a set of atomic connectives. The (*typed*) *atomic language* $\mathcal{L}_{\mathbb{C}}$ associated to \mathbb{C} is the smallest set that contains the propositional letters and that is closed under the atomic connectives. That is,

- $\mathbb{P}(\mathbb{C}) \subseteq \mathcal{L}_{\mathbb{C}}$;
- for all $\star \in \mathbb{C}$ of arity $n > 0$ and of type signature (k, k_1, \dots, k_n) and for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathbb{C}}$ of types k_1, \dots, k_n respectively, we have that $\star(\varphi_1, \dots, \varphi_n) \in \mathcal{L}_{\mathbb{C}}$ and $\star(\varphi_1, \dots, \varphi_n)$ is of type k .

Elements of \mathcal{L}_C are called *atomic formulas* and are denoted $\varphi, \psi, \alpha, \dots$. The *type of a formula* $\varphi \in \mathcal{L}_C$ is denoted $k(\varphi)$.

The *skeleton syntactic tree* of a formula $\varphi \in \mathcal{L}_C$ is the syntactic tree of the formula φ in which the nodes labeled with subformulas of φ are replaced by the skeleton of their outermost connective.

A set of atomic connectives C is *plain* if for all $\star \in C$ of skeleton $(\sigma, \pm, \mathbb{A}, (k, k_1, \dots, k_n), (\pm_1, \dots, \pm_n))$ there are atoms $p_1, \dots, p_n \in \mathbb{P}$ of types k_1, \dots, k_n respectively. *In the sequel, we assume that all sets of connectives C are plain.* \square

Our assumption that all sets of connectives C considered are plain makes sense. Indeed, we want all connectives of C to appear in some formula of \mathcal{L}_C . If C was not plain then there would be a connective of C (with input type k) which would be necessarily composed with another connective of C (of output type k), if we want such a connective to appear in a formula of \mathcal{L}_C . Yet, in that case, we should instead view C as a set of *molecular* connectives (introduced in the next section).

Definition 3 (C -models). Let C be a set of atomic connectives. A C -model is a tuple $M = (W, \mathcal{R})$ where W is a non-empty set and \mathcal{R} is a set of relations over W such that each n -ary connective $\star \in C$ of type signature (k, k_1, \dots, k_n) is associated to a $k_1 + \dots + k_n + k$ -ary relation $R_\star \in \mathcal{R}$.

An *assignment* is a tuple $(w_1, \dots, w_k) \in W^k$ for some $k \in \mathbb{N}^*$, generally denoted \bar{w} . The set of assignments of a C -model M is denoted $\bar{w}(M, C)$. A *pointed C -model* (M, \bar{w}) is a C -model M together with an assignment \bar{w} . In that case, we say that (M, \bar{w}) is of *type k* . The class of all pointed C -models is denoted \mathcal{M}_C . \square

Definition 4 (Atomic logics). Let C be a set of atomic connectives and let $M = (W, \mathcal{R})$ be a C -model. We define the *interpretation function of \mathcal{L}_C in M* , denoted $\llbracket \cdot \rrbracket^M : \mathcal{L}_C \rightarrow \bigcup_{k \in \mathbb{N}^*} W^k$, inductively as follows: for all propositional letters $p \in C$ of type k , all connectives $\star \in C$ of skeleton $(\sigma, \pm, \mathbb{A}, (k, k_1, \dots, k_n), (\pm_1, \dots, \pm_n))$ of arity $n > 0$, for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$,

$$\begin{aligned} \llbracket p \rrbracket^M &\triangleq \begin{cases} R_p & \text{if } \pm = + \\ W^k - R_p & \text{if } \pm = - \end{cases} \\ \llbracket \star(\varphi_1, \dots, \varphi_n) \rrbracket^M &\triangleq f_\star(\llbracket \varphi_1 \rrbracket^M, \dots, \llbracket \varphi_n \rrbracket^M) \end{aligned}$$

where the function f_\star is defined as follows: for all $W_1 \in \mathcal{P}(W^{k_1}), \dots, W_n \in \mathcal{P}(W^{k_n})$, $f_\star(W_1, \dots, W_n) \triangleq \{\bar{w}_{n+1} \in W^k \mid \mathcal{C}^\star(W_1, \dots, W_n, \bar{w}_{n+1})\}$ where $\mathcal{C}^\star(W_1, \dots, W_n, \bar{w}_{n+1})$ is called the *truth condition* of \star and is defined as follows:

- if $\mathbb{A} = \forall$: “ $\forall \bar{w}_1 \in W^{k_1} \dots \bar{w}_n \in W^{k_n} (\bar{w}_1 \uparrow_1 W_1 \vee \dots \vee \bar{w}_n \uparrow_n W_n \vee R_\star^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}_{n+1})$ ”;
- if $\mathbb{A} = \exists$: “ $\exists \bar{w}_1 \in W^{k_1} \dots \bar{w}_n \in W^{k_n} (\bar{w}_1 \uparrow_1 W_1 \wedge \dots \wedge \bar{w}_n \uparrow_n W_n \wedge R_\star^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}_{n+1})$ ”;

where, for all $j \in \llbracket 1; n \rrbracket$, $\bar{w}_j \uparrow_j W_j \triangleq \begin{cases} \bar{w}_j \in W_j & \text{if } \pm_j = + \\ \bar{w}_j \notin W_j & \text{if } \pm_j = - \end{cases}$ and

$R_\star^{\pm\sigma} \bar{w}_1 \dots \bar{w}_{n+1}$ holds iff $\pm R_\star \bar{w}_{\sigma-(1)} \dots \bar{w}_{\sigma-(n+1)}$ holds, with the notations $+R_\star \triangleq R_\star$ and $-R_\star \triangleq W^{k+k_1+\dots+k_n} - R_\star$. If \mathcal{E}_C is a class of pointed C -models, the *satisfaction relation* $\Vdash \subseteq \mathcal{E}_C \times \mathcal{L}_C$ is defined as follows: for all $\varphi \in \mathcal{L}_C$ and all $(M, \bar{w}) \in \mathcal{E}_C$, $((M, \bar{w}), \varphi) \in \Vdash$ iff $\bar{w} \in \llbracket \varphi \rrbracket^M$. We usually write $(M, \bar{w}) \Vdash \varphi$ instead of $((M, \bar{w}), \varphi) \in \Vdash$ and we say that φ is *true* in (M, \bar{w}) .

The logic $(\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ is the *atomic logic associated to \mathcal{E}_C and C* . The logics of the form $(\mathcal{L}_C, \mathcal{M}_C, \Vdash)$ are called *basic atomic logics*. \square

Permutations of \mathfrak{S}_2	unary signatures
$\tau_1 = (1, 2)$	$t_1 = (\exists, (1, 1), +)$
$\tau_2 = (2, 1)$	$t_2 = (\forall, (1, 1), +)$
	$t_3 = (\forall, (1, 1), -)$
	$t_4 = (\exists, (1, 1), -)$
Permutations of \mathfrak{S}_3	binary signatures
$\sigma_1 = (1, 2, 3)$	$s_1 = (\exists, (1, 1, 1), (+, +))$
$\sigma_2 = (3, 2, 1)$	$s_2 = (\forall, (1, 1, 1), (+, -))$
$\sigma_3 = (3, 1, 2)$	$s_3 = (\forall, (1, 1, 1), (-, +))$
$\sigma_4 = (2, 1, 3)$	$s_4 = (\forall, (1, 1, 1), (+, +))$
$\sigma_5 = (2, 3, 1)$	$s_5 = (\exists, (1, 1, 1), (+, -))$
$\sigma_6 = (1, 3, 2)$	$s_6 = (\exists, (1, 1, 1), (-, +))$
	$s_7 = (\exists, (1, 1, 1), (-, -))$
	$s_8 = (\forall, (1, 1, 1), (-, -))$

Figure 1: Permutations of \mathfrak{S}_2 and \mathfrak{S}_3 and ‘families’ of unary and binary signatures

The \pm sign in $R_*^{\pm\sigma}$ is the \pm sign in $(\sigma, \pm, \mathcal{A}, (k, k_1, \dots, k_n), (\pm_1, \dots, \pm_n))$.

Example 1 (Lambek calculus, modal logic). The Lambek calculus, where $\mathbf{C} = \{p, \otimes, \supset, \subset\}$ is defined in Section 2, is an example of atomic logic. Here $\otimes, \supset, \subset$ are the connectives of skeletons $(\sigma_1, +, s_1), (\sigma_5, -, s_3), (\sigma_3, -, s_2)$. Another example of atomic logic is modal logic where $\mathbf{C} = \{p, \top, \perp, \wedge, \vee, \diamond_j, \square_j \mid j \in \mathbb{I}\}$ is such that

- \top, \perp are connectives of skeletons $(\text{Id}, +, \exists, 1)$ and $(\text{Id}, -, \forall, 1)$ respectively;
- $\wedge, \vee, \diamond_j, \square_j$ are connectives of skeletons $(\sigma_1, +, s_1), (\sigma_1, -, s_4), (\tau_2, +, t_1)$ and $(\tau_2, -, t_2)$ respectively;
- the \mathbf{C} -models $M = (W, \mathcal{R}) \in \mathcal{E}_{\mathbf{C}}$ are such that $R_{\wedge} = R_{\vee} = \{(w, w, w) \mid w \in W\}$, $R_{\diamond_j} = R_{\square_j}$ and $R_{\top} = R_{\perp} = W$.

With these conditions on the \mathbf{C} -models of $\mathcal{E}_{\mathbf{C}}$, for all $(M, w) \in \mathcal{E}_{\mathbf{C}}$,

$$\begin{array}{ll}
w \in \llbracket \diamond_j \varphi \rrbracket^M & \text{iff } \exists v (v \in \llbracket \varphi \rrbracket^M \wedge R_{\diamond_j} wv) \\
w \in \llbracket \square_j \varphi \rrbracket^M & \text{iff } \forall v (v \in \llbracket \varphi \rrbracket^M \vee -R_{\square_j} wv) \\
w \in \llbracket \wedge(\varphi, \psi) \rrbracket^M & \text{iff } \exists vu (v \in \llbracket \varphi \rrbracket^M \wedge u \in \llbracket \psi \rrbracket^M \wedge R_{\wedge} vuw) \\
& \text{iff } w \in \llbracket \varphi \rrbracket^M \wedge w \in \llbracket \psi \rrbracket^M \\
w \in \llbracket \vee(\varphi, \psi) \rrbracket^M & \text{iff } \forall vu (v \in \llbracket \varphi \rrbracket^M \vee u \in \llbracket \psi \rrbracket^M \vee -R_{\vee} vuw) \\
& \text{iff } w \in \llbracket \varphi \rrbracket^M \vee w \in \llbracket \psi \rrbracket^M
\end{array}$$

Other examples are given in Figure 2 as well as in [2, 4].

4 Molecular Logics

Molecular logics are basically logics whose primitive connectives are compositions of atomic connectives in which it is possible to repeat the same argument at different places in the connective. That is why we call them ‘molecular’, just as molecules are compositions of atoms in chemistry.

Atomic Connective	Truth condition	Non-classical connective in the literature
The existentially positive orbit		
$(\tau_1, +, t_1) \varphi$	$\exists v (v \in \llbracket \varphi \rrbracket \wedge Rvw)$	$\diamond^- \varphi$ [28] \diamond_\downarrow [12]
$(\tau_2, -, t_2) \varphi$	$\forall v (v \in \llbracket \varphi \rrbracket \vee -Rvw)$	$\square \varphi$ [21]
The universally positive orbit		
$(\tau_1, +, t_2) \varphi$	$\forall v (v \in \llbracket \varphi \rrbracket \vee Rvw)$	$+\downarrow \varphi$ [12] [15, p. 401]
$(\tau_2, -, t_1) \varphi$	$\exists v (v \in \llbracket \varphi \rrbracket \wedge -Rvw)$	[12]
The existentially negative orbit		
$(\tau_1, +, t_4) \varphi$	$\exists v (v \notin \llbracket \varphi \rrbracket \wedge Rvw)$	$?\varphi$ [12][15, p. 402] $\exists_1 \varphi$ [12][7, Def. 10.7.7]
$(\tau_2, +, t_4) \varphi$	$\exists v (v \notin \llbracket \varphi \rrbracket \wedge Rvw)$	$?\downarrow \varphi$ [12][16] [15, p. 402] $\exists_2 \varphi$ [7, Def. 10.7.7]
The universally negative orbit		
$(\tau_1, +, t_3) \varphi$	$\forall v (v \notin \llbracket \varphi \rrbracket \vee Rvw)$	φ^\perp [12, 14] φ° [19] $\diamond_1^- \varphi$ [7, Def. 10.7.2]
$(\tau_2, +, t_3) \varphi$	$\forall v (v \notin \llbracket \varphi \rrbracket \vee Rvw)$	$\sim \varphi$ [18] ${}^\perp \varphi$ [12, 14] ${}^\circ \varphi$ [19] $\diamond_2^- \varphi$ [7, Def. 10.7.2]
The symmetrical existentially positive orbit		
$(\tau_1, -, t_1) \varphi$	$\exists v (v \in \llbracket \varphi \rrbracket \wedge -Rvw)$	[12]
$(\tau_2, +, t_2) \varphi$	$\forall v (v \in \llbracket \varphi \rrbracket \vee Rvw)$	$+\varphi$ [12] [15, p. 402] φ^* [7, Def. 7.1.19]
The symmetrical universally positive orbit		
$(\tau_1, -, t_2) \varphi$	$\forall v (v \in \llbracket \varphi \rrbracket \vee -Rvw)$	$\square^- \varphi$ [28] \square_\downarrow [12]
$(\tau_2, +, t_1) \varphi$	$\exists v (v \in \llbracket \varphi \rrbracket \wedge Rvw)$	$\diamond \varphi$ [21]
The symmetrical existentially negative orbit		
$(\tau_1, -, t_4) \varphi$	$\exists v (v \notin \llbracket \varphi \rrbracket \wedge -Rvw)$	$?\varphi$ [12][7, Ex. 1.4.5] φ^1 [19]
$(\tau_2, -, t_4) \varphi$	$\exists v (v \notin \llbracket \varphi \rrbracket \wedge -Rvw)$	$?\downarrow \varphi$ [12] [7, Ex. 1.4.5] ${}^1 \varphi$ [19]
The symmetrical universally negative orbit		
$(\tau_1, -, t_3) \varphi$	$\forall v (v \notin \llbracket \varphi \rrbracket \vee -Rvw)$	[12]
$(\tau_2, -, t_3) \varphi$	$\forall v (v \notin \llbracket \varphi \rrbracket \vee -Rvw)$	$\neg_i \varphi$ [22, 29] $\perp \varphi$ [16]

Figure 2: The unary connectives of atomic logics of type (1, 1)

Definition 5 (Molecular skeleton and connective). The class \mathbb{C}^* of *molecular skeletons* is the smallest set such that:

- $\mathbb{P} \subseteq \mathbb{C}^*$ and \mathbb{C}^* contains as well, for each $k, l \in \mathbb{N}^*$, a symbol id_k^l of *type signature* (k, k) , *output type* k and *arity* 1;
- for all atomic skeleton $\star \in \mathbb{C}$ of type signature (k, k_1^0, \dots, k_n^0) and all $c_1, \dots, c_n \in \mathbb{C}^*$ of output types or types (if they are propositional letters) k_1^0, \dots, k_n^0 respectively, $c \triangleq \star(c_1, \dots, c_n)$ is a molecular skeleton of \mathbb{C}^* of *output type* k .

If $c \in \mathbb{C}^*$, we define its *decomposition tree* as follows. If $c = p \in \mathbb{P}$ or $c = id_k^l$, then its decomposition tree T_c is the tree consisting of a single node labeled with p or id_k^l respectively. If $c = \star(c_1, \dots, c_n) \in \mathbb{C}^*$ then its decomposition tree T_c is the tree defined inductively as follows: the root of T_c is c and it is labeled with \star and one sets edges between that root and the roots c_1, \dots, c_n of the decomposition trees T_{c_1}, \dots, T_{c_n} respectively.

If $c \triangleq \star(c_1, \dots, c_n)$ is a molecular skeleton with output type k and k_1, \dots, k_m are the k s of the different id_k^l s which appear in c_1, \dots, c_n (in an order which follows the first appearance of the id_k^l s in the inorder traversal of the decomposition trees of c_1, \dots, c_n), then the *type signature* of c is (k, k_1, \dots, k_m) and its *arity* is m . We also define the *quantification signature* $\mathbb{A}(c)$ of $c = \star(c_1, \dots, c_n)$ by $\mathbb{A}(c) \triangleq \mathbb{A}(\star)$.

A *molecular connective* is a symbol to which is associated a molecular skeleton. Its arity, type signature, output type, quantification signature and decomposition tree are the same as its skeleton. The set of *atomic connectives associated to a set C of molecular connectives* is the set of labels different from id_k^l of the decomposition trees of the molecular connectives of C . \square

Note that the same label (atomic connective) may appear several times in a decomposition tree. Note also that the vertices of a decomposition tree are molecular connectives.

Every atomic connective \star of type signature (k, k_1, \dots, k_n) can be seen as the (specific) molecular connectives $c \triangleq \star(id_{k_1}^1, \dots, id_{k_n}^n)$. One needs to introduce the connective id_k^l in order to deal with molecular connectives whose skeletons are for example of the form $\star(p, id_k^l)$ where $p \in \mathbb{P}$ or molecular connectives in which the same argument(s) appear at different places, like for example in $\star(id_k^1, \dots, id_k^1)$ which is of arity 1.

Example 2 (Modal intuitionistic logic). Let us consider the skeletons $c, c', \star_1, \star_2, \star_3$ defined by the following first-order formulas.

$$\begin{aligned} c(x) &\triangleq \forall y (Rxy \rightarrow \forall z (R_\diamond yz \rightarrow Q(z))) \\ c'(x) &\triangleq \forall y (Rxy \rightarrow \exists z (R_\diamond yz \wedge Q(z))) \\ \star_1(x) &\triangleq \forall y (Rxy \rightarrow Q(y)) \\ \star_2(x) &\triangleq \forall z (R_\diamond xz \rightarrow Q(z)) \\ \star_3(x) &\triangleq \exists z (R_\diamond xz \wedge Q(z)) \end{aligned}$$

These first-order formulas can be naturally represented by molecular skeletons. Then, $\star_1, \star_2, \star_3$ are atomic skeletons and the connectives associated to c, c' are molecular connectives. Indeed, c is the composition of \star_1 and \star_2 , $c = \star_1(\star_2)$, and c' is the composition of \star_1 and \star_3 , $c' = \star_1(\star_3)$. To be more precise, c and c' will have the same semantics as $\star_1(\star_2(id_1^1))$ and $\star_1(\star_3(id_1^1))$. The connective associated to c corresponds to the connective \Box of modal intuitionistic logic and the connective associated to c' corresponds to the connective \Diamond of modal intuitionistic logic [27] defined in Section 2.4. \square

Definition 6 (Molecular language). Let \mathbf{C} be a set of molecular connectives. The (*typed*) *molecular language* $\mathcal{L}_{\mathbf{C}}$ associated to \mathbf{C} is the smallest set that contains the propositional letters and that is closed under the molecular connectives while respecting the type constraints. That is,

- the propositional letters of \mathbf{C} belong to $\mathcal{L}_{\mathbf{C}}$;
- for all $c \in \mathbf{C}$ of type signature (k, k_1, \dots, k_m) and for all $\varphi_1, \dots, \varphi_m \in \mathcal{L}_{\mathbf{C}}$ of types k_1, \dots, k_m respectively, we have that $c(\varphi_1, \dots, \varphi_m) \in \mathcal{L}_{\mathbf{C}}$ and $c(\varphi_1, \dots, \varphi_m)$ is of *type* k .

Elements of $\mathcal{L}_{\mathbf{C}}$ are called *molecular formulas* and are denoted $\varphi, \psi, \alpha, \dots$. The *type* of a *formula* $\varphi \in \mathcal{L}_{\mathbf{C}}$ is denoted $k(\varphi)$. We use the same abbreviations as for the atomic language. \square

Definition 7 (Molecular logic). If \mathbf{C} is a set of molecular connectives, then a \mathbf{C} -*model* M is a \mathbf{C}' -*model* M where \mathbf{C}' is the set of atomic connectives associated to \mathbf{C} . The class of all pointed \mathbf{C} -models is also denoted $\mathcal{M}_{\mathbf{C}}$, like for atomic logics.

The truth conditions for molecular connectives are defined naturally from the truth conditions of atomic connectives. We define the *interpretation function of $\mathcal{L}_{\mathbf{C}}$ in M* , denoted $\llbracket \cdot \rrbracket^M : \mathcal{L}_{\mathbf{C}} \rightarrow \bigcup_{k \in \mathbb{N}^*} W^k$, inductively as follows: for all propositional letters $p \in \mathbf{C}$ of skeleton $(\sigma, \pm, \mathbb{A}, k)$, all molecular connectives $\star(c_1, \dots, c_n) \in \mathbf{C}$ of arity $m > 0$ and all $k, l \in \mathbb{N}^*$, for all $\varphi, \varphi_1, \dots, \varphi_m \in \mathcal{L}_{\mathbf{C}}$,

$$\begin{aligned} \llbracket p \rrbracket^M &\triangleq \pm R_p \\ \llbracket id_k^l(\varphi) \rrbracket^M &\triangleq \llbracket \varphi \rrbracket^M \\ \llbracket \star(c_1, \dots, c_n)(\varphi_1, \dots, \varphi_m) \rrbracket^M &\triangleq f_{\star}(\llbracket c_1(\varphi_1^1, \dots, \varphi_{i_1}^1) \rrbracket^M, \dots, \llbracket c_n(\varphi_1^n, \dots, \varphi_{i_n}^n) \rrbracket^M) \end{aligned}$$

where for all $j \in \{1, \dots, n\}$, the formulas $\varphi_1^j, \dots, \varphi_{i_j}^j$ are those $\varphi_1, \dots, \varphi_m$ for which there is a corresponding id_k^l in c_j (the φ_i^j 's appear in the same order as their corresponding id_k^l 's in c_j).

If $\mathcal{E}_{\mathbf{C}}$ is a class of pointed \mathbf{C} -models, the triple $(\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \llbracket - \rrbracket)$ is a logic called the *molecular logic associated to $\mathcal{E}_{\mathbf{C}}$ and \mathbf{C}* . \square

As one can easily notice, every atomic logic can be canonically mapped to an equi-expressive molecular logic: each atomic connective \star of type signature (k, k_1, \dots, k_n) of the given atomic logic has to be transformed into the molecular connective of skeleton $\star(id_{k_1}^1, \dots, id_{k_n}^n)$. Note that the id_k^l are in fact specific atomic connectives whose associated relations are the identity relations.

Example 3 (Temporal logic). Temporal logic is an example of logic in which one needs to resort both to our types and to combine basic connectives in order to be able to express the truth conditions of its connectives. Let us consider the skeletons defined by the following first-order formulas:

$$\begin{aligned} \star_1(x) &\triangleq \exists y z z' (P y \wedge Q z z' \wedge R y z z' x) \\ \star'_1(x) &\triangleq \exists y z z' (P y \wedge Q z z' \wedge R x z z' y) \\ \star_2(x, x') &\triangleq \forall y (P y \vee \neg S y x x') \end{aligned}$$

\star_1, \star'_1 and \star_2 are atomic skeletons (this is independent from the definitions of R and S). The connectives of skeletons $c = \star_1(id_1^1, \star_2(id_1^2))$ and $c' = \star'_1(id_1^1, \star_2(id_1^2))$ are molecular connectives. Together with some propositional letters, truth constants, Boolean conjunction and disjunction, they form a set of molecular connectives $\mathbf{C} = \{p, \neg p, \top, \perp, \wedge, \vee, c, c'\}$. If we choose the class of \mathbf{C} -models appropriately, the molecular connectives c and c' then correspond to the connectives

‘until’ U and ‘since’ S of temporal logic respectively. Let us be a bit more precise. Let $M = (W, \{<, P\})$ be a temporal model. We represent this temporal model by the C-model $M^{U,S} = (W, \{R, S, P\})$ such that for all $y, z, z', x \in W$,

$$Ryzz'x \text{ iff } x < y, x = z \text{ and } y = z' \quad (1)$$

$$Syyx' \text{ iff } x < y < x'. \quad (2)$$

One can show that for all $\varphi \in \mathcal{L}_{\text{TL}}$ and all $w \in W$, $(M, w) \models \varphi$ iff $(M^{U,S}, w) \models T(\varphi)$ where $T : \mathcal{L}_{\text{TL}} \rightarrow \mathcal{L}_{\text{C}}$ is defined inductively on the formulas of \mathcal{L}_{TL} in such a way that $T(U(\varphi, \psi)) \triangleq c(T(\varphi), T(\psi))$ and $T(S(\varphi, \psi)) \triangleq c'(T(\varphi), T(\psi))$. Indeed, the standard translation of the until and since operators are:

$$\begin{aligned} ST_x(U(\varphi, \psi)) &= \exists y(x < y \wedge ST_y(\varphi) \wedge \forall z(x < z < y \rightarrow ST_z(\psi))) \\ ST_x(S(\varphi, \psi)) &= \exists y(y < x \wedge ST_y(\varphi) \wedge \forall z(y < z < x \rightarrow ST_z(\psi))) \quad \square \end{aligned}$$

Example 4 (Weakly aggregative modal logic). The modal connectives of weakly aggregative modal logics [11, 24] are expressed by first-order formulas of the following form:

$$\star(x) \triangleq \exists x_1 \dots x_n (Px_1 \wedge \dots \wedge Px_n \wedge Rx_1 \dots x_n x)$$

The arity of such connectives is 1, their argument is just repeated at different places of the atomic connective \star . So, they are in fact molecular connectives, their molecular skeleton is of the form $\star(id_1^1, \dots, id_1^1)$. \square

There are infinitely more examples of molecular logics since we proved in [4] that any logic such that the truth conditions of its connectives are expressible in first-order logic is as expressive as a molecular logic.

5 Boolean Atomic and Molecular Logics

Atomic and molecular logics do not include Boolean connectives as primitive connectives. In fact, they can be defined in terms of specific atomic connectives, as follows.

Definition 8 (Boolean connectives). The *Boolean connectives* called *conjunctions*, *disjunctions*, *negations* and *Boolean constants* (of type k) are the atomic connectives denoted, respectively:

$$\mathbb{B} \triangleq \{\wedge_k, \vee_k, \neg_k, \top_k, \perp_k \mid k \in \mathbb{N}^*\}$$

The skeleton of \wedge_k is $(\text{Id}, +, \exists, (k, k, k), (+, +))$, the skeleton of \vee_k is $(\text{Id}, -, \forall, (k, k, k), (+, +))$, the skeleton of \neg_k is $(\text{Id}, +, \exists, (k, k), -)$, the skeleton of \top_k is $(\text{Id}, +, \exists, k)$ and the skeleton of \perp_k is $(\text{Id}, -, \forall, k)$.

In any C-model $M = (W, \mathcal{R})$ containing Boolean connectives, the associated relation of any \vee_k or \wedge_k is $R_{\wedge_k} = R_{\vee_k} \triangleq \{(\bar{w}, \bar{w}, \bar{w}) \mid \bar{w} \in W^k\}$, the associated relation of any \neg_k is $R_{\neg_k} \triangleq \{(\bar{w}, \bar{w}) \mid \bar{w} \in W^k\}$ and the associated relation of any \top_k or \perp_k is $R_{\perp_k} = R_{\top_k} \triangleq W^k$.

Atomic or molecular logics containing Boolean connectives are called *Boolean atomic or molecular logics*. We say that a set of atomic connectives \mathcal{C} is *complete for conjunctions and disjunctions* when it contains all conjunctions, disjunctions and constants $\wedge_k, \vee_k, \top_k, \perp_k$, for k ranging over all input types and output types of the atomic connectives of \mathcal{C} . We say that a set of atomic connectives \mathcal{C} is *complete for Boolean connectives* when it contains all conjunctions, disjunctions, constants as well as negations $\wedge_k, \vee_k, \top_k, \perp_k, \neg_k$, for k ranging over all input types and output types of the atomic connectives of \mathcal{C} . The *completion* of a set of atomic connectives \mathcal{C} with conjunctions and disjunctions is the smallest set of connectives including \mathcal{C} which is complete for conjunctions and disjunction. \square

Proposition 1. *Let \mathcal{C} be a set of atomic connectives containing Boolean connectives. and let $M = (W, \mathcal{R})$ be a \mathcal{C} -model. Then, for all $k \in \mathbb{N}^*$, all $\varphi, \psi \in \mathcal{L}_{\mathcal{C}}$, if $k(\varphi) = k(\psi) = k$, then*

$$\begin{aligned} \llbracket \top_k \rrbracket^M &\triangleq W^k \\ \llbracket \perp_k \rrbracket^M &\triangleq \emptyset \\ \llbracket \neg_k \varphi \rrbracket^M &\triangleq W^k - \llbracket \varphi \rrbracket^M \\ \llbracket (\varphi \wedge_k \psi) \rrbracket^M &\triangleq \llbracket \varphi \rrbracket^M \cap \llbracket \psi \rrbracket^M \\ \llbracket (\varphi \vee_k \psi) \rrbracket^M &\triangleq \llbracket \varphi \rrbracket^M \cup \llbracket \psi \rrbracket^M. \end{aligned}$$

It turns out that Boolean negation can also be simulated systematically at the level of atomic connectives by applying a transformation on them. The Boolean negation of a formula then boils down to taking the Boolean negation of the outermost connective of the formula. This transformation is defined as follows.

Definition 9 (Boolean negation). Let \star be a n -ary connective of skeleton $(\sigma, \pm, \mathcal{A}, \bar{k}, \pm_1, \dots, \pm_n)$. The *Boolean negation* of \star is the connective $-\star$ of skeleton $(\sigma, -\pm, -\mathcal{A}, \bar{k}, -\pm_1, \dots, -\pm_n)$ where $-\mathcal{A} \triangleq \exists$ if $\mathcal{A} = \forall$ and $-\mathcal{A} \triangleq \forall$ otherwise, which is associated in any \mathcal{C} -model to the same relation as \star . If $\varphi = \star(\varphi_1, \dots, \varphi_n)$ is an atomic formula, the *Boolean negation* of φ is the formula $-\varphi \triangleq -\star(\varphi_1, \dots, \varphi_n)$. \square

Proposition 2 ([4]). *Let \mathcal{C} be a set of atomic connectives such that $-\star \in \mathcal{C}$ for all $\star \in \mathcal{C}$. Let $\varphi \in \mathcal{L}_{\mathcal{C}}$ of type k and let $M = (W, \mathcal{R})$ be a \mathcal{C} -model. Then, for all $\bar{w} \in W^k$, $\bar{w} \in \llbracket -\varphi \rrbracket^M$ iff $\bar{w} \notin \llbracket \varphi \rrbracket^M$.*

6 Universal and Existential Molecular Connectives

Universal and existential molecular connectives are essentially molecular connectives such that the quantification patterns of the quantification signatures of their successive atomic connectives are of the form $\forall \dots \forall$ or $\exists \dots \exists$ respectively. They essentially behave as ‘macroscopic’ atomic connectives of quantification signatures \forall or \exists .

Definition 10 (Universal and existential molecular connective). A *universal (resp. existential) molecular skeleton* is a molecular skeleton c different from any id_k^l for any $k, l \in \mathbb{N}^*$ such that $\mathcal{A}(c) = \forall$ (resp. $\mathcal{A}(c) = \exists$) and such that for each node of its decomposition tree labeled with $\star = (\sigma, \pm, \mathcal{A}, \bar{k}, (\pm_1, \dots, \pm_n))$ and each of its j th children labeled with some $\star_j \in \mathcal{C}$ such that the subtree generated by this j th children contains at least one id_k^l , we have that $\mathcal{A}(\star_j) = \pm_j \mathcal{A}$. A *universal (resp. existential) molecular connective* is a molecular connective with a universal (resp. existential) skeleton. \square

Example 5. On the one hand, the molecular connective $\star(p, id_k^l)$ is a universal (resp. existential) molecular connective if $\mathcal{A}(\star) = \forall$ (resp. $\mathcal{A}(\star) = \exists$). Likewise, $\supset (id_1^1, \square id_1^2)$ and $\otimes (\diamond id_1^1, p)$ are universal and existential molecular connectives respectively. On the other hand, the molecular connectives $\square \diamond^- id_1^1$ and $\supset (\square id_1^1, \square id_1^2)$ are neither universal nor existential molecular connectives. \square

Just as we have tonicity signatures for atomic connectives, we can also define an adaptation of this notion for universal and existential molecular connectives, which, we repeat, are some sort of ‘macroscopic’ atomic connectives.

Definition 11 (Tonicity signature of a molecular connective). Let c be a molecular connective and let c' be a molecular subconnective of c . We define the *tonicity* of c' w.r.t. c , denoted

$tn(c', c)$ inductively as follows. If $c = c'$ then $tn(c', c) = +$. Otherwise, if $c = \star(c_1, \dots, c_n)$ with $\star = (\sigma, \pm, \mathbb{E}, \bar{k}, (\pm_1, \dots, \pm_n))$ and c' appears in c_j then $tn(c', c) = \pm_j tn(c', c_j)$. The *tonicity signature* of a molecular connective is the tuple (\pm_1, \dots, \pm_l) of the tonicities $tn(id_k^i, c)$ of the connectives labeling the leafs of the decomposition tree of c of the form id_k^i (possibly with repetition). \square

A molecular connective c is generally represented as $\star(c_1, \dots, c_n)$ where \star is an *atomic* connective, but it can also be represented as $c_0(c'_1, \dots, c'_n)$ where c_0 is a *molecular* connective such that the id_k^i s which appear in c_0 are all distincts. This representation and decomposition in terms of molecular connectives is equivalent to the original one. For example the modal connective of weakly agregative modal logic $\star(id_1^1, \dots, id_1^1)$ can be represented in terms of molecular connectives with distinct id_k^i s as follows: $\star(id_1^1, \dots, id_1^n)(id_1^{n+1}, \dots, id_1^{n+1})$. We now formalize this idea of decomposition into molecular connectives.

Definition 12 (Decomposition of a molecular connective). Let c be a molecular connective and let T_c be its decomposition tree. A *decomposition* of c is an expression of the form $c_0(c_1, \dots, c_n)$ where c_0 is a molecular connective whose decomposition tree is a subtree of T_c with root c and whose leafs, corresponding to the nodes c_1, \dots, c_n of T_c , have been replaced by distinct id_k^i s and where c_1, \dots, c_n are molecular connectives whose decomposition trees are the subtrees of T_c generated by the nodes c_1, \dots, c_n of T_c . \square

Example 6. For any molecular connective $\star(id_1^1, \dots, id_1^n)$ corresponding to an atomic connective \star of arity n , a decomposition is $\star(id_1^{n+1}, \dots, id_1^{2n})(id_1^1, \dots, id_1^n)$. Likewise, a decomposition of $\star(p, id_k^l)$ is $\star(id_1^1, id_k^2)(p, id_k^l)$ (if p is of type 1). \square

Based on this observation, we can decompose molecular connectives into an alternation of universal and existential molecular subconnectives. That is what the following definition captures, at least at the first level of alternation depth. In that definition, if c_0 is universal and the tonicity of the j^{th} molecular connective c_j is positive for example, then c_j will have an existential quantification signature (and its 'head' will behave as an existential molecular connective).

Definition 13 (Maximal decomposition). A decomposition $c = c_0(c_1, \dots, c_n)$ of a molecular connective c is *maximal* when c_0 is a universal or existential molecular connective of tonicity signature (\pm_1, \dots, \pm_n) with the id_k^i s all distinct, and we have that $\mathbb{E}(c_j) = -\pm_j \mathbb{E}(c_0)$ for all $j \in \llbracket 1; n \rrbracket$ such that c_j is not of the form id_k^l for some $k, l \in \mathbb{N}^*$. \square

Example 7. The decomposition of $\star(p, id_k^l)$ in $\star(id_1^1, id_k^2)(p, id_k^l)$ (if p is of type 1) is not maximal because $\star(p, id_k^l)$ is already a universal or existential molecular connective. On the other hand, the decomposition of $\star(p, id_k^l)$ in $\star(p, id_k^l)(id_k^l)$ is maximal. \square

The alternation of universal and existential molecular subconnectives inherent to any molecular connective is fully captured by the following notion of *quantified decomposition tree*. It is an abstraction of the notion of decomposition tree of Definition 5 which considers as first-class citizens universal and existential molecular (sub)connectives.

Definition 14 (Quantified decomposition tree). If $c \in \mathbb{C}^*$ is a molecular skeleton, we define its *quantified decomposition tree* T_c' inductively as follows. If c is a propositional letter or id_k^l for some $k, l \in \mathbb{N}^*$ then its quantified decomposition tree T_c' is the tree consisting of a single node labeled with c . Otherwise, c admits a maximal decomposition $c = c_0(c_1, \dots, c_n)$. Then, its quantified decomposition tree T_c' is the tree defined inductively as follows: the root of T_c' is c and it is labeled with c_0 and one sets edges between that root and the roots c_1, \dots, c_n of the quantified decomposition trees $T_{c_1}', \dots, T_{c_n}'$ respectively. The quantified decomposition tree of a molecular connective is the quantified decomposition tree of its skeleton. \square

Note that a propositional letter p can occur as the label of a node only if the quantified decomposition tree in which it appears consists of this single node only (Example 7 illustrates this phenomenon).

Definition 15 (Relation associated to a molecular connective). Let \mathcal{C} be a set of molecular connectives and let $c = \star(c_1, \dots, c_n) \in \mathcal{C}$ be a molecular connective with \star of skeleton $(\sigma, \pm, \mathcal{A}, (k^*, k_1^*, \dots, k_n^*), (\pm_1^*, \dots, \pm_n^*))$. Assume that the decomposition tree of c has l different leaves labeled with id_k^i s of output types k_1, \dots, k_l , in that order. Then, the arity of c is l .

Let $M = (W, \mathcal{R})$ be a \mathcal{C} -model. We define the relation R_c inductively as follows. First, we set $R_{-\star} = R_\star$ if \star is an atomic connective and if $\pm \in \{+, -\}$,

$$\pm p \triangleq \begin{cases} p & \text{if } \pm = + \\ -p & \text{if } \pm = - \end{cases} \text{ and } \pm \star(c_1, \dots, c_n) = \begin{cases} \star(c_1, \dots, c_n) & \text{if } \pm = + \\ -\star(c_1, \dots, c_n) & \text{if } \pm = - \end{cases}.$$

- If the arity l of c is 0 then $R_c = \llbracket c \rrbracket^M$;
- If the arity l of c is greater than 0 and $\mathcal{A} = \exists$ then for all $\bar{w} \in W^k, \bar{w}_1 \in W^{k_1}, \dots, \bar{w}_l \in W^{k_l}$, we set

$$R_c \bar{w}_1 \dots \bar{w}_l \bar{w} \text{ iff } \exists \bar{v}_1 \in W^{k_1^*}, \dots, \bar{v}_n \in W^{k_n^*} \\ (R_{\star}^{\pm \sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \wedge R_{\pm_1^* c_1} \bar{w}_1^1 \dots \bar{w}_{i_1}^1 \bar{v}_1 \wedge \dots \wedge R_{\pm_n^* c_n} \bar{w}_1^n \dots \bar{w}_{i_n}^n \bar{v}_n)$$

where, for all $j \in \llbracket 1; n \rrbracket$, the tuples $\bar{w}_1^j, \dots, \bar{w}_{i_j}^j$ are those associated to the leaves of the decomposition tree of c which are also leaves in the decomposition (sub)tree of c_j (the \bar{w}_i^j s appear in the same order as their corresponding leaves in the decomposition tree of c_j). Moreover, we set $R_{id_k^i} \bar{v} \bar{w}$ iff $R_{-id_k^i} \bar{v} \bar{w}$ iff $\bar{v} = \bar{w}$.

- If the arity l of c is greater than 0 and $\mathcal{A} = \forall$ then for all $\bar{w} \in W^k, \bar{w}_1 \in W^{k_1}, \dots, \bar{w}_l \in W^{k_l}$, we set

$$R_c \bar{w}_1 \dots \bar{w}_l \bar{w} \text{ iff } \forall \bar{v}_1 \in W^{k_1^*}, \dots, \bar{v}_n \in W^{k_n^*} \\ (R_{\star}^{\pm \sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \vee R_{\pm_1^* c_1} \bar{w}_1^1 \dots \bar{w}_{i_1}^1 \bar{v}_1 \vee \dots \vee R_{\pm_n^* c_n} \bar{w}_1^n \dots \bar{w}_{i_n}^n \bar{v}_n)$$

where, for all $j \in \llbracket 1; n \rrbracket$, the tuples $\bar{w}_1^j, \dots, \bar{w}_{i_j}^j$ are defined as above. However, for that universal case, we set $R_{id_k^i} \bar{v} \bar{w}$ iff $R_{-id_k^i} \bar{v} \bar{w}$ iff $\bar{v} \neq \bar{w}$. \square

Unsurprisingly, the semantics of universal and existential molecular connectives is similar to the semantics of atomic connectives of quantification signature \forall and \exists respectively. That is what the following proposition shows. In this proposition and the definition above, the molecular connective $c = \star(c_1, \dots, c_n)$ yields three different values: n, m and l . They can be all different in general. The value m is the arity of c , n is the arity of \star and l is the number of leaves of the decomposition tree T_c of c . The value m is also the number of different labels id_k^i of the leaves of the decomposition tree T_c . Hence, the size l of the tuple of the tonicity signature of a molecular connective c is larger than the arity m of c because in the former case we take into account repetitions of the same id_k^i s.

Proposition 3. *Let \mathcal{C} be a set of molecular connectives and let $c = \star(c_1, \dots, c_n) \in \mathcal{C}$ be a molecular connective of type signature (k, k_1, \dots, k_m) and tonicity signature (\pm_1, \dots, \pm_l) with $\star = (\sigma, \pm, \mathcal{A}, \bar{k}, (\pm_1^*, \dots, \pm_n^*))$. Assume that the decomposition tree of c has $l > 0$ different leaves labeled by some id_k^i . Let $M = (W, \mathcal{R})$ be a \mathcal{C} -model and let $\bar{w} \in W^k$.*

- If c is an existential molecular connective then

$$\begin{aligned} \bar{w} \in \llbracket c(\varphi_1, \dots, \varphi_m) \rrbracket^M \text{ iff } \exists \bar{w}_1 \dots \bar{w}_l \\ (\bar{w}_1 \uparrow_1 \llbracket \varphi_{i_1} \rrbracket^M \wedge \dots \wedge \bar{w}_l \uparrow_l \llbracket \varphi_{i_l} \rrbracket^M \wedge R_c \bar{w}_1 \dots \bar{w}_l \bar{w}) \end{aligned}$$

- If c is a universal molecular connective then

$$\begin{aligned} \bar{w} \in \llbracket c(\varphi_1, \dots, \varphi_m) \rrbracket^M \text{ iff } \forall \bar{w}_1 \dots \bar{w}_l \\ (\bar{w}_1 \uparrow_1 \llbracket \varphi_{i_1} \rrbracket^M \vee \dots \vee \bar{w}_l \uparrow_l \llbracket \varphi_{i_l} \rrbracket^M \vee R_c \bar{w}_1 \dots \bar{w}_l \bar{w}) \end{aligned}$$

where the R_c s are defined in Definition 15 and for all $j \in \llbracket 1; l \rrbracket$,

$$\bar{w}_j \uparrow_j \llbracket \varphi_{i_j} \rrbracket^M \triangleq \begin{cases} \bar{w}_j \in \llbracket \varphi_{i_j} \rrbracket^M & \text{if } \pm_j = + \\ \bar{w}_j \notin \llbracket \varphi_{i_j} \rrbracket^M & \text{if } \pm_j = - \end{cases} \text{ and } i_1, \dots, i_l \in \llbracket 1; m \rrbracket \text{ are those indices corre-}$$

sponding to the m different id_k^i s appearing in c (we basically map the l leafs of the decomposition tree of c to their labels id_k^i s in this tree).

7 Automatic Bisimulations for Atomic and Molecular Logics

In this section, we are going to see that notions of bisimulations can be automatically defined for atomic and molecular logics on the basis of the definition of the truth conditions of their connectives, not only for plain atomic logics but also for molecular logics. These notions are such that they preserve the truth of the formulas of the atomic logic considered between models. We will illustrate these results on modal logic, the Lambek calculus, (modal) intuitionistic logic and temporal logic of Section 2. The bisimulation notions that we will find out by applying our generic definitions on these atomic logics will correspond to the bisimulation notions introduced in the literature for these logics.

7.1 Atomic Logics

Definition 16 (C -bisimulation). Let C be a set of atomic connectives, let $\star \in C$ and let $M_1 = (W_1, \mathcal{R}_1)$ and $M_2 = (W_2, \mathcal{R}_2)$ be two C -models. A binary relation $Z \subseteq \bigcup_{k \in \mathbb{N}^*} (W_1^k \times W_2^k) \cup$

$(W_2^k \times W_1^k)$ is a C -bisimulation between M_1 and M_2 when it is non-empty and for all $\star \in C$, if $\{M, M'\} = \{M_1, M_2\}$, then for all $\bar{w}_1, \dots, \bar{w}_n, \bar{w}'_1, \dots, \bar{w}'_n, \bar{w}, \bar{w}' \in \bar{w}(M, C) \cup \bar{w}(M', C)$,

1. if \star is a propositional letter p then, if $\bar{w}Z\bar{w}'$ and $\bar{w} \in \llbracket p \rrbracket$ then $\bar{w}' \in \llbracket p \rrbracket$;
2. if \star has skeleton $(\sigma, \pm, \exists, \bar{k}, (\pm_1, \dots, \pm_n))$ and we have $\bar{w}Z\bar{w}'$ and $R_\star^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}$, then $\exists \bar{w}'_1, \dots, \bar{w}'_n (\bar{w}_1 \bowtie \bar{w}'_1 \wedge \bar{w}_2 \bowtie \bar{w}'_2 \wedge \dots \wedge \bar{w}_n \bowtie \bar{w}'_n \wedge R_\star^{\pm\sigma} \bar{w}'_1 \dots \bar{w}'_n \bar{w}')$;
3. if \star has skeleton $(\sigma, \pm, \forall, \bar{k}, (\pm_1, \dots, \pm_n))$ and we have $\bar{w}Z\bar{w}'$ and $-R_\star^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}$, then $\exists \bar{w}_1, \dots, \bar{w}_n (\bar{w}_1 \bowtie \bar{w}'_1 \wedge \bar{w}_2 \bowtie \bar{w}'_2 \wedge \dots \wedge \bar{w}_n \bowtie \bar{w}'_n \wedge -R_\star^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w})$;

where, for all $j \in \llbracket 1; n \rrbracket$, we define $\bar{w}_j \bowtie \bar{w}'_j \triangleq \begin{cases} \bar{w}_j Z \bar{w}'_j & \text{if } \pm_j = + \\ \bar{w}'_j Z \bar{w}_j & \text{if } \pm_j = - \end{cases}$.

When such a C -bisimulation Z exists and $\bar{w}Z\bar{w}'$, we say that (M, \bar{w}) and (M', \bar{w}') are C -bisimilar and we write it $(M, \bar{w}) \rightarrow_C (M', \bar{w}')$. \square

Importantly, note that the clause(s) defining a C-bisimulation allow us to define back the atomic connective(s) that led to their definition(s): we can recover the skeleton of the connectives considered (tonicity signature, quantification signature, type signature, *etc*) from the mere expression of the clauses of a given C-bisimulation. Hence, in that sense, a C-bisimulation completely characterizes a given (atomic) logic. Note also that case 1. is a particular instance of cases 2. and 3. with $n = 0$.

Example 8 (Modal logic). Let us consider the connectives of modal logic: $\mathbf{C} = \{p, \neg p, \wedge, \vee, \diamond, \square\}$ where p has skeleton $(\text{Id}, +, \exists, 1)$, $\neg p$ has skeleton $(\text{Id}, -, \forall, 1)$, \diamond has skeleton $(\tau_2, +, t_1)$ and \square has skeleton $(\tau_2, -, t_2)$. Let $M_1 = (W_1, \{R_1, P_1\})$ and $M_2 = (W_2, \{R_2, P_2\})$ be two Kripke models (they are also C-models). A binary relation Z between M_1 and M_2 is a C-bisimulation between M_1 and M_2 when for all $M, M' \in \{M_1, M_2\}$ with $M = (W, \{R, P\})$ and $M' = (W', \{R', P'\})$, all $w, v, u \in M$ and all $w', v', u' \in M'$,

- if wZw' and $w \in \llbracket p \rrbracket$ then $w' \in \llbracket p \rrbracket$ (condition for p);
- if wZw' and $w' \in \llbracket p \rrbracket$ then $w \in \llbracket p \rrbracket$ (condition for $\neg p$);
- if wZw' and Rwv then there is $v' \in W'$ such that vZv' and $R'w'v'$ (condition for $\diamond = (\tau_2, +, t_1)$);
- if wZw' and $R'w'v'$ then there is $v \in W$ such that vZv' and Rwv (condition for $\square = (\tau_2, -, t_2)$).

Note that every C-bisimulation can be canonically extended into a *symmetric* C-bisimulation: one sets $w'Zw$ when wZw' already holds. \square

Note that the conditions for the Boolean connectives \wedge and \vee always trivially hold and that is why we never mention them (it suffices to take $v' = u' = w'$ and $v = u = w$ below, since $w = v = u$ and $w' = v' = u'$ respectively):

- if wZw' and $R_\wedge wvu$ then there are $v', u' \in W'$ such that vZv', uZu' and $R_\wedge w'v'u'$ (condition for \wedge);
- if wZw' and $R_\vee w'v'u'$ then there are $v, u \in W$ such that vZv', uZu' and $R_\vee wvu$ (condition for \vee).

Example 9 (Lambek calculus). Let us consider the connectives of the Lambek calculus: $\mathbf{C} = \{p, \otimes, \supset, \subset\}$ where p has skeleton $(\text{Id}, +, \exists, 1)$, \otimes_3 has skeleton $(\sigma_1, +, s_1)$, \supset has skeleton $(\sigma_5, -, s_3)$ and \subset has skeleton $(\sigma_3, -, s_2)$. Let $M_1 = (W_1, \{R_1, P_1\})$ and $M_2 = (W_2, \{R_2, P_2\})$ be two Lambek models (they are also C-models). A binary relation Z between M_1 and M_2 is a C-bisimulation between M_1 and M_2 when for all $M, M' \in \{M_1, M_2\}$ with $M = (W, \{R, P\})$ and $M' = (W', \{R', P'\})$, all $w, v, u \in M$ and all $w', v', u' \in M'$,

- if wZw' and $w \in \llbracket p \rrbracket$ then $w' \in \llbracket p \rrbracket$ (condition for p);
- if wZw' and $Rvwu$ then there are $v', u' \in W'$ such that vZv', uZu' and $Rv'u'w'$ (condition for \otimes);
- if vZv' and $Rv'u'w'$ then there are $u, w \in W$ such that $u'Zu, wZw'$ and $Rvuw$ (condition for $\supset = (\sigma_5, -, s_3)$);
- if uZu' and $Rv'u'w'$ then there are $v, w \in W$ such that $v'Zv, wZw'$ and $Rvuw$ (condition for $\subset = (\sigma_3, -, s_2)$). \square

The following proposition shows that the notions of \mathbf{C} -bisimulation for the Lambek calculus and directed bisimulation coincide (directed bisimulations are defined for example in [29, Definition 13.2]) and likewise for modal logic.

Proposition 4. • *Let $\mathbf{C} = \{p, \neg p, \wedge, \vee, \diamond_j, \square_j \mid j \in \mathbb{I}\}$ be the connectives of Example 8 and let M and M' be two \mathbf{C} -models. Then, a \mathbf{C} -bisimulation between M and M' is a modal bisimulation between M and M' and vice versa.*

- *Let $\mathbf{C} = \{p, \otimes, \supset, \subset\}$ be the connectives of Example 9 and let M and M' be two \mathbf{C} -models. Then, a \mathbf{C} -bisimulation between M and M' is a directed bisimulation between M and M' and vice versa.*

Example 10 (Intuitionistic logic). Let us consider the connectives of intuitionistic logic: $\mathbf{C} = \{p, \perp, \top, \wedge, \vee, \Rightarrow\}$ where p has skeleton $(\text{Id}, +, \exists, 1)$, \top has skeleton $(\text{Id}, +, \exists, 1)$, \perp has skeleton $(\text{Id}, -, \forall, 1)$, \wedge and \vee are Boolean connectives and \Rightarrow has skeleton $(\sigma_5, -, s_3)$ (here, \top and \perp are represented by specific propositional letters of respective signatures $(\text{Id}, +, \exists, 1)$ and $(\text{Id}, -, \forall, 1)$). Let $M_1 = (W_1, R_1, P)$ and $M_2 = (W_2, R_2, P)$ be two intuitionistic models. We represent these intuitionistic models by the \mathbf{C} -models $M_1^{\Rightarrow} = (W_1, R_1, \Rightarrow, P)$ and $M_2^{\Rightarrow} = (W_2, R_2, \Rightarrow, P)$ respectively such that for all $u_1, v_1, w_1 \in W_1$ and all $u_2, v_2, w_2 \in W_2$,

$$R_{1, \Rightarrow} u_1 v_1 w_1 \text{ iff } R_1 u_1 w_1 \text{ and } v_1 = w_1 \quad (3)$$

$$R_{2, \Rightarrow} u_2 v_2 w_2 \text{ iff } R_2 u_2 w_2 \text{ and } v_2 = w_2 \quad (4)$$

One can show that for all $\varphi \in \mathcal{L}_{\mathbf{C}}$ and all $w_1 \in W_1$, $M_1, w_1 \models \varphi$ iff $M_1^{\Rightarrow}, w_1 \models \varphi$ (and likewise for M_2 and M_2^{\Rightarrow}). Now, a binary relation Z between M_1^{\Rightarrow} and M_2^{\Rightarrow} is a \mathbf{C} -bisimulation between M_1^{\Rightarrow} and M_2^{\Rightarrow} iff for all $M, M' \in \{M_1^{\Rightarrow}, M_2^{\Rightarrow}\}$, all $w, w', v', u' \in \overline{w}(M, \mathbf{C}) \cup \overline{w}(M', \mathbf{C})$ and all $p \in \mathbb{P}$,

- if wZw' and $w \in \llbracket p \rrbracket$ then $w' \in \llbracket p \rrbracket$ (condition for p);
- if vZv' and $R'_{\Rightarrow} v' u' w'$ then there are $u, w \in W$ such that $u'Zu$, wZw' and $R_{\Rightarrow} v u w$ (*) (condition for \Rightarrow);
- conditions for \top and \perp trivially hold because of their semantics.

Using Expressions (3) and (4), one can easily show that condition (*) is equivalent to the following condition:

- if vZv' and $R'v'w'$ then there is $w \in W$ such that $w'Zw$, wZw' and Rvw (**).

That condition (**) is Olkhovikov's condition "step" of [27, Definition 1] of his "basic asimulation". \square

Definition 17. Let \mathbf{C} be a set of atomic connectives. Let (M, \overline{w}) and (M', \overline{w}') be two pointed \mathbf{C} -models. We write $(M, \overline{w}) \rightsquigarrow_{\mathbf{C}} (M', \overline{w}')$ when for all $\varphi \in \mathcal{L}_{\mathbf{C}}$, $(M, \overline{w}) \models \varphi$ implies $(M', \overline{w}') \models \varphi$. \square

Proposition 5 (Truth preservation between \mathbf{C} -bisimilar pointed \mathbf{C} -models). *Let \mathbf{C} be a set of atomic connectives and let $M_1 = (W_1, \mathcal{R}_1)$ and $M_2 = (W_2, \mathcal{R}_2)$ be two \mathbf{C} -models. Let Z be a \mathbf{C} -bisimulation between M_1 and M_2 . Then, if $\{M, M'\} = \{M_1, M_2\}$ then for all $\overline{w} \in \overline{w}(M, \mathbf{C})$, all $\overline{w}' \in \overline{w}(M', \mathbf{C})$, if $\overline{w}Z\overline{w}'$ then $(M, \overline{w}) \rightsquigarrow_{\mathbf{C}} (M', \overline{w}')$.*

7.2 Molecular Logics

Definition 18 (*C*-bisimulation for molecular connectives). Let \mathbf{C} be a set of molecular connectives and let $M_1 = (W_1, \mathcal{R}_1)$ and $M_2 = (W_2, \mathcal{R}_2)$ be two *C*-models. For all $c_0 \in \mathbf{C}$, let V'_{c_0} be the vertices of the quantified decomposition tree of c_0 . We associate to each vertex $c \in V'_{c_0}$ of output type or type (if it is a propositional letter) k a relation $Z_c \subseteq (W_1^k \times W_2^k) \cup (W_2^k \times W_1^k)$ such that

- i. $Z \triangleq \bigcup \{Z_{id_k^i} \mid id_k^i \text{ appears in some } c \in \mathbf{C}\}$ is non-empty;
- ii. for all \bar{w}, \bar{w}' of size k such that $\bar{w}Z\bar{w}'$, we also have that for all $c \in \mathbf{C}$ of output type or type (if it is a propositional letter) k that $\bar{w}Z_c\bar{w}'$.

We say that this set of relations $\{Z_c \mid c \in V'_{c_0}, c_0 \in \mathbf{C}\}$ is a *C*-bisimulation between M_1 and M_2 when for all $c_0 \in \mathbf{C}$, all vertice $c \in V'_{c_0}$ whose label is different from any id_k^l , if $\{M, M'\} = \{M_1, M_2\}$ then for all $\bar{w}_1, \dots, \bar{w}_n, \bar{w}'_1, \dots, \bar{w}'_n, \bar{w}, \bar{w}' \in \bar{w}(M, \mathbf{C}) \cup \bar{w}(M', \mathbf{C})$,

1. if c is of arity 0 then, $\bar{w}Z_c\bar{w}'$ and $\bar{w} \in \llbracket c \rrbracket$ imply $\bar{w}' \in \llbracket c \rrbracket$;
2. if c is of arity greater than 0 and is maximally decomposed into $c = c_0(c_1, \dots, c_n)$, with c_0 an existential molecular connective of tonicity signature (\pm_1, \dots, \pm_n) , and we have that $\bar{w}Z_c\bar{w}'$ and $R_{c_0}\bar{w}_1 \dots \bar{w}_n\bar{w}$, then there are $\bar{w}'_1\bar{w}'_2 \dots \bar{w}'_n$ such that $R_{c_0}\bar{w}'_1 \dots \bar{w}'_n\bar{w}'$ and
 - i. for all $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = +$, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $\bar{w}_iZ_{c_j}\bar{w}'_j$;
 - ii. for all $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = -$, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $\bar{w}'_jZ_{c_j}\bar{w}_i$.
3. if c is of arity greater than 0 and is maximally decomposed into $c = c_0(c_1, \dots, c_n)$, with c_0 a universal molecular connective of tonicity signature (\pm_1, \dots, \pm_n) , and we have that $\bar{w}Z_c\bar{w}'$ and $-R_{c_0}\bar{w}'_1 \dots \bar{w}'_n\bar{w}'$, then there are $\bar{w}_1\bar{w}_2 \dots \bar{w}_n$ such that $-R_{c_0}\bar{w}_1 \dots \bar{w}_n\bar{w}$ and
 - i. for all $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = +$, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $\bar{w}_jZ_{c_j}\bar{w}'_i$;
 - ii. for all $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = -$, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $\bar{w}'_iZ_{c_j}\bar{w}_j$.

When we have that $\bar{w}Z\bar{w}'$, we say that (M, \bar{w}) and (M', \bar{w}') are *C*-bisimilar and we write it $(M, \bar{w}) \rightarrow_{\mathbf{C}} (M', \bar{w}')$. \square

Note that if we consider molecular connectives of the form $c \triangleq \star(id_{k_1}^1, \dots, id_{k_n}^n)$, representing the atomic connective \star of type signature (k, k_1, \dots, k_n) , then the definition of a *C*-bisimulation for these kind of molecular connectives is identical to the definition of a *C*-bisimulation for the associated atomic connectives. Hence, our definition of *C*-bisimulation for molecular connectives is a genuine generalization of our definition of *C*-bisimulation for atomic connectives. The bisimulation relation Z in the atomic case corresponds in the molecular case to $\bigcup \{Z_{id_k^i} \mid id_k^i \text{ appears in some } c \in \mathbf{C}\}$.

Fact 1. Let \mathbf{C} be a set of molecular connectives and let $M_1 = (W_1, \mathcal{R}_1)$ and $M_2 = (W_2, \mathcal{R}_2)$ be two *C*-models. Let $\{Z_c \mid c \in V'_{c_0}, c_0 \in \mathbf{C}\}$ be a set of relations between M_1 and M_2 . We set $Z \triangleq \bigcup \{Z_{id_k^i} \mid id_k^i \text{ appears in some } c \in \mathbf{C}\}$. Then, $\{Z_c \mid c \in V'_{c_0}, c_0 \in \mathbf{C}\}$ is a *C*-bisimulation if, and only if, Z is non-empty and conditions 1, 2, 3 of Definition 18 hold, in which all Z_c and all $Z_{id_k^i}$ for $c \in \mathbf{C}$ and id_k^i appearing in some $c \in \mathbf{C}$ are replaced by Z .

Example 11 (Temporal logic). Let us consider the skeletons defined by the following first-order formulas:

$$\begin{aligned}\star_1(w) &\triangleq \exists vuu' (Pv \wedge Quu' \wedge Rvuu'w) \\ \star'_1(w) &\triangleq \exists vuu' (Pv \wedge Quu' \wedge Rwu'v) \\ \star_2(w, w') &\triangleq \forall v (Pv \vee \neg Svww')\end{aligned}$$

\star_1 , \star'_1 and \star_2 are atomic skeletons (this is independent from the definitions of R and S). The connectives of skeletons $c = \star_1(id_1^1, \star_2(id_1^2))$ and $c' = \star'_1(id_1^1, \star_2(id_1^2))$ are normal molecular connectives standing for the ‘until’ and the ‘since’ temporal operators U and S . Let $M_1 = (W_1, \{<_1, P\})$ and $M_2 = (W_2, \{<_2, P\})$ be two temporal models. We represent these temporal models by the C-models $M_1^{U,S} = (W_1, \{R_1, S_1, P\})$ and $M_2^{U,S} = (W_2, \{R_2, S_2, P\})$ respectively such that for all $v_1, u_1, u'_1, w_1, w'_1 \in W_1$,

$$R_1v_1u_1u'_1w_1 \text{ iff } w_1 <_1 v_1, w_1 = u_1 \text{ and } v_1 = u'_1 \quad (5)$$

$$S_1v_1w_1w'_1 \text{ iff } w_1 <_1 v_1 <_1 w'_1 \quad (6)$$

and likewise for R_2 and S_2 of M_2 . One can show that for all $\varphi \in \mathcal{L}_C$ and all $w_1 \in W_1$, $(M_1, w_1) \models \varphi$ iff $(M_1^{U,S}, w_1) \Vdash \varphi$ (and likewise for M_2 and $M_2^{U,S}$).

Let $Z_c, Z_{c'}, Z_p, Z_{\neg p}, Z_{id_1^1}, Z_{id_1^2} \subseteq (W_1 \times W_2) \cup (W_2 \times W_1)$ and $Z_{\star_2} \subseteq ((W_1 \times W_1) \times (W_2 \times W_2)) \cup ((W_2 \times W_2) \times (W_1 \times W_1))$. Then, by Definition 18, $\{Z_c, Z_{c'}, Z_p, Z_{\neg p}, Z_{id_1^1}, Z_{id_1^2}, Z_{\star_2}\}$ is a C-bisimulation iff for all $w, v \in W$, all $w', v' \in W'$ and all $p \in \mathbb{P}(C)$,

- if wZ_pw' and $w \in \llbracket p \rrbracket$ then $w' \in \llbracket p \rrbracket$ (condition for p);
- if $wZ_{\neg p}w'$ and $w' \in \llbracket p \rrbracket$ then $w \in \llbracket p \rrbracket$ (condition for $\neg p$);
- if $w < v$ and wZ_cw' then there is $v' \in W'$ such that $w' < v'$ and $vZ_{id_1^1}v'$ and $(w, v)Z_{\star_2}(w', v')$,
if $(w, v)Z_{\star_2}(w', v')$ and $w' < u' < v'$ then there is $u \in W$ such that $uZ_{id_1^2}u'$ and $w < u < v$ (condition for Until c);
- if $v < w$ and $wZ_{c'}w'$ then there is $v' \in W'$ such that $v' < w'$ and $vZ_{id_1^1}v'$ and $(v, w)Z_{\star_2}(v', w')$,
if $(v, w)Z_{\star_2}(v', w')$ and $v' < u' < w'$ then there is $u \in W$ such that $uZ_{id_1^2}u'$ and $v < u < w$ (condition for Since c').

Then, by setting $Z \triangleq Z_{id_1^1} \cup Z_{id_1^2}$, this entails by Fact 1 that the above definition is equivalent to the following one, in which Z is non-empty:

- if wZw' and $w \in \llbracket p \rrbracket$ then $w' \in \llbracket p \rrbracket$;
- if wZw' and $w' \in \llbracket p \rrbracket$ then $w \in \llbracket p \rrbracket$;
- if $w < v$ and wZw' then there is $v' \in W'$ such that $w' < v'$ and vZv' and $(w, v)Z_{\star_2}(w', v')$,
if $(w, v)Z_{\star_2}(w', v')$ and $w' < u' < v'$ then there is $u \in W$ such that uZu' and $w < u < v$;
- if $v < w$ and wZw' then there is $v' \in W'$ such that $v' < w'$ and vZv' and $(v, w)Z_{\star_2}(v', w')$,
if $(v, w)Z_{\star_2}(v', w')$ and $v' < u' < w'$ then there is $u \in W$ such that uZu' and $v < u < w$.

We have rediscovered the notion of bisimulation for temporal logic introduced by Kurtonina & de Rijke [23]. The relation Z_{\star_2} is presented differently in [23], it is split up into two relations $Z_1 \subseteq ((W_1 \times W_1) \times (W_2 \times W_2))$ and $Z_2 \subseteq ((W_2 \times W_2) \times (W_1 \times W_1))$ but the two formal definitions boil down to the same. \square

Example 12 (Modal intuitionistic logic). Let $\mathbf{C} = \{p, \top, \perp, \wedge, \vee, \Rightarrow, c, c'\}$ where $c, c' \in \mathbb{C}^*$ are the molecular connectives of Example 2 and where $\{p, \top, \perp, \wedge, \vee, \Rightarrow\}$ are defined in Example 10.

Let $M_1 = (W_1, \{R_1, R_{1,\diamond}, P\})$ and $M_2 = (W_2, \{R_2, R_{2,\diamond}, P\})$ be two modal intuitionistic models. The set of binary relations $\{Z_{id_1^1}, Z_{id_1^2}, Z_p, Z_{\Rightarrow}, Z_c, Z_{c'}, Z_{\star_3}\}$ is a \mathbf{C} -bisimulation iff for all $M, M' \in \{M_1, M_2\}$ with $M = (W, \{R, R_\diamond, P\})$ and $M' = (W', \{R', R'_\diamond, P\})$, all $w, v, u, w', v', u' \in \bar{w}(M, \mathbf{C}) \cup \bar{w}(M', \mathbf{C})$ and all $p \in \mathbb{P}$,

- if wZ_pw' and $w \in \llbracket p \rrbracket$ then $w' \in \llbracket p \rrbracket$
(condition for p , like in Example 10);
- if $vZ_{\Rightarrow}v'$ and $R'v'w'$ then there is $w \in W$ such that $w'Z_{id_1^1}w$, $wZ_{id_1^2}w'$ and Rvw
(condition for \Rightarrow , like in Example 10);
- if wZ_cw' and $R'_c w'v'$ then there is $v \in W$ such that $vZ_{id_1^1}v'$ and $R_c wv$;
(condition for $c = \star_1(\star_2)$);
- if $wZ_{c'}w'$ and $R'w'v'$ then there is $v \in W$ such that $vZ_{\star_3}v'$ and Rwv ,
if $wZ_{\star_3}w'$ and $R_\diamond wv$ then there is $v' \in W'$ such that $vZ_{id_1^1}v'$ and $R'_\diamond w'v'$
(condition for $c' = \star_1(\star_3)$).

Then, by setting $Z \triangleq Z_{id_1^1} \cup Z_{id_1^2}$, this entails by Fact 1 that the above definition is equivalent to the following one, in which Z is non-empty:

- if wZw' and $w \in \llbracket p \rrbracket$ then $w' \in \llbracket p \rrbracket$
(condition for p , like in Example 10);
- if vZv' and $R'v'w'$ then there is $w \in W$ such that $w'Zw$, wZw' and Rvw
(condition for \Rightarrow , like in Example 10);
- if wZw' and $R'w'u'$ and $R_\diamond u'v'$ then there are $v, u \in W$ such that vZv' and Rwu and $R_\diamond wv$
(condition for $\star = \star_1(\star_2)$);
- if wZw' and $R'w'v'$ then there is $v \in W$ such that $vZ_{\star_3}v'$ and Rwv ,
if $wZ_{\star_3}w'$ and $R_\diamond wv$ then there is $v' \in W'$ such that vZv' and $R'_\diamond w'v'$
(condition for $\star' = \star_1(\star_3)$).

It turns out that the conditions for $\star' = \star_1(\star_3)$ are the conditions (diam-2(1)) and (diam-2(2)) of Olkhovikov [27, Definition 9] and that the condition for $\star = \star_1(\star_2)$ is the condition (box-2) of Olkhovikov [27, Definition 5], as expected. \square

Example 13 (Weakly aggregative modal logic). Let c be a molecular connective of skeleton $\star(id_1^1, \dots, id_1^1)$ with $\star = (\text{Id}, \pm, \exists, \bar{k}, (+, \dots, +))$ corresponding to the ‘diamond’ modality of a weakly aggregative modal logic. Then, if we spell out the condition of \mathbf{C} -bisimulation of Definition 18 for this connective, we obtain the following:

- if $\bar{w}Z_{c'}\bar{w}'$ and $R_{\star}^{\pm\sigma}\bar{w}_1\dots\bar{w}_n\bar{w}$, then there are $\bar{w}'_1\bar{w}'_2\dots\bar{w}'_n$ such that $R_{\star}^{\pm\sigma}\bar{w}'_1\dots\bar{w}'_n\bar{w}'$ and for all $j \in \llbracket 1; n \rrbracket$, there is $i \in \llbracket 1; n \rrbracket$ such that $\bar{w}_i Z_{id_1^i} \bar{w}'_j$.

For the molecular connective c' of skeleton $\star'(id_1^1, \dots, id_1^1)$ with $\star' = (\text{Id}, \pm, \forall, \bar{k}, (+, \dots, +))$ corresponding to the ‘box’ modality of a weakly aggregative modal logic, we obtain:

- if $\bar{w}Z_{c'}\bar{w}'$ and $-R_{\star}^{\pm\sigma}\bar{w}'_1\dots\bar{w}'_n\bar{w}'$, then there are $\bar{w}_1\bar{w}_2\dots\bar{w}_n$ such that $-R_{\star}^{\pm\sigma}\bar{w}_1\dots\bar{w}_n\bar{w}$ and for all $j \in \llbracket 1; n \rrbracket$, there is $i \in \llbracket 1; n \rrbracket$ such that $\bar{w}_j Z_{id_1^i} \bar{w}'_i$.

These two conditions are the conditions “Forth” and “Back” of a wa^n -bisimulation for weakly aggregative modal logics [24, Definition 4]. \square

Definition 19. Let \mathbf{C} be a set of molecular connectives. For all $c_0 \in \mathbf{C}$ and all vertex c of the quantified decomposition tree T_{c_0} , we define the language $\mathcal{L}_{c\mathbf{C}}$ as follows:

$$\mathcal{L}_{c\mathbf{C}} \triangleq \begin{cases} \{c(\varphi_1, \dots, \varphi_n) \mid \varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathbf{C}}\} & \text{if } c \text{ is of arity } n > 0 \\ \{c\} & \text{if } c \text{ is of arity } 0 \end{cases}$$

Let (M, \bar{w}) and (M', \bar{w}') be two pointed \mathbf{C} -models. We write $(M, \bar{w}) \rightsquigarrow_{c\mathbf{C}} (M', \bar{w}')$ when for all $\varphi \in \mathcal{L}_{c\mathbf{C}}$, $(M, \bar{w}) \Vdash \varphi$ implies $(M', \bar{w}') \Vdash \varphi$. We also write $(M, \bar{w}) \rightsquigarrow_{\mathbf{C}} (M', \bar{w}')$ when for all $\varphi \in \mathcal{L}_{\mathbf{C}}$, $(M, \bar{w}) \Vdash \varphi$ implies $(M', \bar{w}') \Vdash \varphi$. \square

Proposition 6 (Truth preservation between \mathbf{C} -bisimilar pointed \mathbf{C} -models). *Let \mathbf{C} be a set of molecular connectives and let $M_1 = (W_1, \mathcal{R}_1)$ and $M_2 = (W_2, \mathcal{R}_2)$ be two \mathbf{C} -models. Let $\mathbf{C}_0 \subseteq \mathbf{C}$ and for all $c \in \mathbf{C}_0$, let D_c be the vertices of the quantified decomposition tree T_c . Let $\{Z_c \mid c \in V'_{c_0}, c_0 \in \mathbf{C}_0\}$ be a \mathbf{C}_0 -bisimulation between M_1 and M_2 . If $\{M, M'\} = \{M_1, M_2\}$ then for all $c_0 \in \mathbf{C}_0$ and all $c \in V'_{c_0}$, for all $\bar{w} \in \bar{w}(M, \mathbf{C})$ and all $\bar{w}' \in \bar{w}(M', \mathbf{C})$, if $\bar{w}Z_c\bar{w}'$ then $(M, \bar{w}) \rightsquigarrow_{c\mathbf{C}_0} (M', \bar{w}')$. In particular, if $\bar{w}Z\bar{w}'$ then $(M, \bar{w}) \rightsquigarrow_{\mathbf{C}_0} (M', \bar{w}')$.*

Definition 20 (Normal connectives). A molecular connective is *normal* when its molecular skeleton can be decomposed maximally into a molecular skeleton $c_0(c_1, \dots, c_n)$ such that for all $i \in \llbracket 1; n \rrbracket$,

1. c_i is either id_k^i or $c'_i(id_k^i, \dots, id_k^i)$ with c'_i either universal or existential, and in that case we note $id(c_i)$ the unique id_k^i appearing in c_i ;
2. if $id(c_i) = id(c_j)$ then $c_i = c_j$. \square

In the above definition, since the decomposition $c_0(c_1, \dots, c_n)$ is maximal, we have in particular, by definition, that if c_0 is of tonicity signature (\pm_1, \dots, \pm_n) , then $\mathcal{A}(c_i) = -\pm_i \mathcal{A}(c_0)$.

Example 14. The molecular connectives of modal intuitionistic logic, weakly aggregative modal logic and temporal logic are normal. \square

8 Ultrafilters, Ultraproducts and Ultrapowers

In this section, we are going to recall and generalize to molecular logics a number of key notions and results of model theory, such as ultrafilter, ultraproducts and the Łoś theorem. Our definitions and proofs are basically the same as those of FOL [9].

Definition 21 (Filter and ultrafilter). Let I be a non-empty set. A *filter* F over I is a set $F \subseteq \mathcal{P}(I)$ such that $I \in F$; if $X, Y \in F$ then $X \cap Y \in F$; if $X \in F$ and $X \subseteq Z \subseteq I$ then $Z \in F$. A filter is called *proper* if it is distinct from $\mathcal{P}(I)$. An *ultrafilter* over I is a proper filter U such that for all $X \in \mathcal{P}(I)$, $X \in U$ iff $I - X \notin U$. A *countably incomplete ultrafilter* is an ultrafilter which is not closed under countable intersections. \square

In the rest of this section, I is a non-empty set and U is an ultrafilter over I .

Definition 22 (Ultraproduct of sets). For each $i \in I$, let W_i be a non-empty set. For all $(w_i)_{i \in I}, (v_i)_{i \in I} \in \prod_{i \in I} W_i$, we say that $(w_i)_{i \in I}$ and $(v_i)_{i \in I}$ are *U -equivalent*, written $(w_i)_{i \in I} \sim_U (v_i)_{i \in I}$, if $\{i \in I \mid w_i = v_i\} \in U$. Note that \sim_U is an equivalence relation on $\prod_{i \in I} W_i$. The equivalence class of $(w_i)_{i \in I}$ under \sim_U is denoted $\prod_U w_i \triangleq \left\{ (v_i)_{i \in I} \in \prod_{i \in I} W_i \mid (v_i)_{i \in I} \sim_U (w_i)_{i \in I} \right\}$.

The *ultraproduct* of $(W_i)_{i \in I}$ modulo U is $\prod_U W_i \triangleq \left\{ \prod_U w_i \mid (w_i)_{i \in I} \in \prod_{i \in I} W_i \right\}$. When $W_i = W$ for all $i \in I$, the ultraproduct is called the *ultrapower* of W modulo U , written $\prod_U W$. \square

Definition 23 (Ultraproduct and ultrapower). Let C be a set of molecular connectives and let $(M_i, \bar{w}_i)_{i \in I}$ be a family of pointed C -models. The *ultraproduct* $\prod_U (M_i, \bar{w}_i)$ of (M_i, \bar{w}_i) modulo U is the pointed C -model $\left(\prod_U M_i, \prod_U \bar{w}_i \right)$ where $\prod_U M_i = (W_U, \mathcal{R}_U)$ and $\prod_U \bar{w}_i$ are defined by:

- $W_U = \prod_U W_i$;
- for all $n + 1$ -ary relations R_\star^i of M_i , the $n + 1$ -ary relation $\prod_U R_\star \in \mathcal{R}_U$ is defined for all $\prod_U w_i^1, \dots, \prod_U w_i^{n+1} \in W_U$ by $\prod_U R_\star \prod_U w_i^1 \dots \prod_U w_i^{n+1}$ iff $\{i \in I \mid R_\star^i w_i^1 \dots w_i^{n+1}\} \in U$;
- $\prod_U \bar{w}_i \triangleq \left(\prod_U w_i^1, \dots, \prod_U w_i^k \right)$ if $(\bar{w}_i)_{i \in I} = (w_i^1, \dots, w_i^k)_{i \in I}$.

If $(M_i, s_i)_{i \in I}$ is a family of pointed structures, the *ultraproduct* $\prod_U (M_i, s_i)$ is the pointed structure $\left(\prod_U M_i, \prod_U s_i \right)$ where $\prod_U M_i$ is defined as above (the M_i are viewed as C -models) and

$\prod_U s_i : \mathcal{V} \rightarrow \prod_U W_i$ is the assignment such that for all $x \in \mathcal{V}$, $\left(\prod_U s_i \right)(x) = \prod_U s_i(x)$.

If for all $i \in I$, $(M_i, \bar{w}_i) = (M, \bar{w})$ (and $(M_i, s_i) = (M, s)$) then $\prod_U (M_i, \bar{w}_i)$ is also called an *ultrapower* of (M, \bar{w}) (resp. (M, s)) modulo U , also denoted $\prod_U (M, \bar{w})$ (resp. $\prod_U (M, s)$). \square

Proposition 7. Let C be a set of molecular connectives and let $(M_i, \bar{w}_i)_{i \in I}$ be a family of C -models. Let $\prod_U (M_i, \bar{w}_i)$ be an ultraproduct of $(M_i, \bar{w}_i)_{i \in I}$. Then, for all $\varphi \in \mathcal{L}_C$, $\prod_U (M_i, \bar{w}_i) \Vdash \varphi$ iff $\{i \in I \mid (M_i, \bar{w}_i) \Vdash \varphi\} \in U$.

9 Keisler Theorems for Molecular Logics

We are going to adapt the Keisler theorems for FOL [9, Theorem 6.1.15 & 4.1.12] to molecular logics.

Theorem 1. Let C be a set of normal connectives complete for conjunction and disjunction and let (M, \bar{w}) and (M', \bar{w}') be pointed C -models. Then the following are equivalent:

1. $(M, \bar{w}) \rightsquigarrow_C (M', \bar{w}')$;
2. there exists a countably incomplete ultrafilter U over \mathbb{N} such that $\prod_U (M, \bar{w}) \rightarrow_C \prod_U (M', \bar{w}')$.

Definition 24 (Definability, closure under C -bisimulation and ultraproducts). Let C be a set of molecular connectives, let (M, \bar{w}) be a pointed C -model of type k and let $T \subseteq \mathcal{L}_C$. We write $(M, \bar{w}) \Vdash T$ when for all $\varphi \in T$ of type k , it holds that $(M, \bar{w}) \Vdash \varphi$.

Let K and K' be classes of pointed C -models. We say that K is *definable by a set of formulas in K'* of \mathcal{L}_C when there is a set T of formulas of \mathcal{L}_C such that K is the set of pointed C -models (M, \bar{w}) of K' such that $(M, \bar{w}) \Vdash T$. The *types of K* , denoted $k(K)$, is the set of all types of the pointed C -models of K and we define $K' - K \triangleq \{(M, \bar{w}) \mid (M, \bar{w}) \text{ is a pointed } C\text{-model of } K' \text{ not in } K \text{ of type in } k(K)\}$.

We say that K is *closed under C -bisimulations in K'* when for all $(M, \bar{w}) \in K$ and all pointed C -models (M', \bar{w}') of K' , if $(M, \bar{w}) \rightarrow_C (M', \bar{w}')$ then $(M', \bar{w}') \in K$. We say that K is *closed under ultraproducts (ultrapowers)* when for all non-empty sets I , if for all $i \in I$ $(M_i, \bar{w}_i) \in K$ (resp. $(M, \bar{w}) \in K$) then $\prod_U (M_i, \bar{w}_i) \in K$ (resp. $\prod_U (M, \bar{w}) \in K$) for all ultraproducts U over I . □

Theorem 2. Let C be a set of normal connectives complete for conjunction and disjunction and let K and K' be classes of pointed C -models such that K' is closed under ultraproducts. Then, the following are equivalent:

1. K is definable in K' by a set of formulas of \mathcal{L}_C ;
2. K is closed under C -bisimulations in K' and closed under ultraproducts, and $K' - K$ is closed under ultraproducts.

Theorem 3. Let C be a set of normal connectives complete for conjunction and disjunction and let K and K' be classes of pointed C -models of the same type such that K' is closed under ultraproducts. Then, the following are equivalent:

1. K is definable in K' by a single formula of \mathcal{L}_C ;
2. K is closed under C -bisimulation in K' , closed under ultraproducts and $K' - K$ is closed under ultraproducts.

Corollary 1. *Let $L = (\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ be a molecular logic whose set of connectives C is normal and complete for conjunction and disjunction and whose class \mathcal{E}_C of C -models are all of the same type. Then, L is axiomatizable iff \mathcal{E}_C is closed under C -bisimulation and ultraproducts and $\mathcal{M}_C - \mathcal{E}_C$ is closed under ultraproducts.*

These theorems can be instantiated for example to the Lambek calculus, (modal) intuitionistic logic, temporal logic or modal logic and in that case we (re)discover the existing results for temporal [23] and modal [8] logics; those for the Lambek calculus and (modal) intuitionistic logic are novel. Yet, our generic results apply in fact to an infinite number of logics, in particular to all (Boolean) atomic logics.

Moreover, unlike existing results, our theorems provide conditions of definability w.r.t. a given class of models K' and not w.r.t. the class of all models. As such, they generalize those existing for modal and temporal logics. No such theorems exist for the Lambek calculus and many other logics. Note also that unlike modal logic [8, Theorem 2.76] (and first-order logic [9, Theorem 4.1.12]), our Theorem 3 does not require $K' - K$ to be closed under C -bisimulation (resp. isomorphism).

10 Conclusion

We have introduced a generic method which allows us to find out an appropriate notion of bisimulation for atomic and molecular logics. This bisimulation notion comes as well with a number of associated model-theoretical results for the logic considered (Theorems 1, 2 and 3). We have applied this method to modal logic, temporal logic, (modal) intuitionistic logic, weakly aggregative modal logic and the Lambek calculus. In doing so, we have rediscovered the definitions of bisimulation of the literature of these logics. We have also rediscovered some of the associated model-theoretical results of modal logic, temporal logic as well as novel results regarding the Lambek calculus and (modal) intuitionistic logic.

These generalizations and new versions of existing theorems confirm, together with the rediscovery of numerous existing results, the soundness and generic character of our overall approach. Our method is applicable to a much wider class of logics than the examples of logics that we have dealt with in the article, in fact an infinite number of logics. Other examples include obviously all the atomic logics listed in Figure 2 as well as all gaggle logics [2], some of them having been already well-studied. In the companion article [3], we show, using the results of this article, that our approach in fact extends to first-order logic as well as all logics such that the truth conditions of their connectives are defined by first-order formulas, *i.e.* any (normal) protologics.

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A Proofs of Theorems 1, 2, 3 and Corollary 1

This appendix contains notations and notions related to first-order logic such as “pointed structures”, “assignments”, the first-order language with function symbols “ $\mathcal{L}_{\text{FOL}}^{\mathcal{P}\mathcal{F}}$ ” and the first-order language without function symbols “ $\mathcal{L}_{\text{FOL}}^{\mathcal{P}}$ ”. These notions and notations are properly defined in the companion article [3]. We also introduce in this appendix the ‘standard’ translation $ST_{\bar{x}}$ from molecular logics to first-order logic, which is not defined in the companion article.

Definition 25 (Translation from atomic and molecular logics to FOL). Let \mathcal{C} be a set of atomic connectives possibly containing Boolean connectives.

Syntax. For all $k \in \mathbb{N}^*$ and all $\bar{x} \in (\mathcal{V} \cup \mathcal{C})^k$, we define the mappings $ST_{\bar{x}} : \mathcal{L}_{\mathcal{C}}^k \rightarrow \mathcal{L}_{\text{FOL}}^{\mathcal{P}}(\bar{x})$, where $\mathcal{L}_{\mathcal{C}}^k$ is the set of formulas of $\mathcal{L}_{\mathcal{C}}$ of type k , as follows: for all $p \in \mathcal{C}$, all $\star \in \mathcal{C}$ of skeleton $(\sigma, \pm, \mathbb{A}, (k, k_1, \dots, k_n), (\pm_1, \dots, \pm_n))$ and all $\varphi_1, \dots, \varphi_n \in$

$$\begin{aligned} \mathcal{L}_{\mathcal{C}}, \quad & ST_{\bar{w}}(p) \triangleq \mathbb{P}(\bar{w}) \\ & ST_{\bar{w}}(\varphi \wedge_k \psi) \triangleq ST_{\bar{w}}(\varphi) \wedge ST_{\bar{w}}(\psi) \\ & ST_{\bar{w}}(\varphi \vee_k \psi) \triangleq ST_{\bar{w}}(\varphi) \vee ST_{\bar{w}}(\psi) \\ & ST_{\bar{w}}(\star(\varphi_1, \dots, \varphi_n)) \triangleq \mathbb{A}\bar{w}_1 \dots \bar{w}_n \\ & \quad \quad \quad (*_1 ST_{\bar{w}_1}(\varphi_1) \times \dots \times *_n ST_{\bar{w}_n}(\varphi_n) \times \mathbb{R}_{\star}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}) \end{aligned} \quad \text{where}$$

$\bar{x}_1, \dots, \bar{x}_n$ are tuples of free variables of size k_1, \dots, k_n ,

$$\times = \begin{cases} \wedge & \text{if } \mathbb{A} = \exists \\ \vee & \text{if } \mathbb{A} = \forall \end{cases} \text{ and for all } j \in \llbracket 1; n \rrbracket, *_{j} = \begin{cases} \neg & \text{if } \pm_j = - \\ \text{empty} & \text{if } \pm_j = + \end{cases}$$

Semantics. Let (M, \bar{w}) be a pointed \mathcal{C} -model of type k with $\bar{w} = (w_1, \dots, w_k)$. Let $\bar{x} = (x_1, \dots, x_k) \in (\mathcal{V} \cup \mathcal{C})^k$. A *pointed structure associated to (M, \bar{w}) and \bar{x}* , denoted $ST_{\bar{x}}(M, \bar{w})$, is

a pointed structure $(M, s_{\bar{x}}^{\bar{w}})$ (the set of predicates \mathcal{P} considered are a copy of the relations of M) where the assignment $s_{\bar{x}}^{\bar{w}}$ is such that $s_{\bar{x}}^{\bar{w}}(x_1) = w_1, \dots, s_{\bar{x}}^{\bar{w}}(x_k) = w_k$.

The above translations canonically extend to molecular logics. Indeed, if \mathbf{C} is a set of molecular connectives, every molecular formula of $\mathcal{L}_{\mathbf{C}}$ can be viewed as a formula of $\mathcal{L}_{\mathbf{C}'}$, where \mathbf{C}' is the set of atomic connectives associated to \mathbf{C} . Likewise, any pointed \mathbf{C} -model can also be viewed as a pointed \mathbf{C}' -model. Then, we apply the above translations to obtain the translation of molecular formulas or \mathbf{C} -models into FOL.

The following proposition follows straightforwardly from the truth conditions of Definition 4.

Proposition 8. *Let \mathbf{C} be a set of molecular connectives, let (M, \bar{w}) be a pointed \mathbf{C} -model of type k , let $\varphi \in \mathcal{L}_{\mathbf{C}}$ of type k and let $\bar{x} \in \mathcal{V}^k$. Then, $(M, \bar{w}) \Vdash \varphi$ iff $(M, s_{\bar{x}}^{\bar{w}}) \Vdash ST_{\bar{x}}(\varphi)$.*

Proposition 9. *Let \mathbf{C} be a set of molecular connectives and let M be a \mathbf{C} -model. Let $\prod_U M$ be an ultrapower of M . Then, for all $\varphi \in \mathcal{L}_{\mathbf{C}}$, we have $(M, \bar{w}) \Vdash \varphi$ iff $\prod_U (M, \bar{w}) \Vdash \varphi$. Likewise, if (M, s) is a pointed structure then for all $\varphi \in \mathcal{L}_{\text{FOL}}^{\mathcal{P}\mathcal{F}}$, we have $(M, s) \models \varphi$ iff $\prod_U (M, s) \models \varphi$.*

Proof: $\prod_U (M, \bar{w}) \Vdash \varphi$

iff $\{i \in I \mid (M, \bar{w}) \Vdash \varphi\} \in U$ by Proposition 7

iff $(M, \bar{w}) \Vdash \varphi$ because $\emptyset \notin U$ since $I \in U$ by definition and because U is a proper filter.

The proof for first-order logic is similar. \square

Proposition 10 (An ultraproduct version of the compactness theorem). *Let Σ be a set of formulas of $\mathcal{L}_{\text{FOL}}^{\mathcal{P}\mathcal{F}}$, all with the same number of free variables or constants, let I be the set of all finite subsets of Σ , and for each $i \in I$, let (M_i, s_i) be a model of i . Then there exists an ultrafilter U over I such that $\prod_U (M_i, s_i)$ is a model of Σ .*

Proof: The original formulation of this result [9, Corollary 4.1.11] is for sentences of $\mathcal{L}_{\text{FOL}}^{\mathcal{P}\mathcal{F}}$, but the proof can easily be adapted to our setting with free variables using item (ii) of the fundamental theorem of ultraproducts [9, Theorem 4.1.9]. \square

We recall the definition of ω -saturation. In that definition, the \mathbf{C} -model is simply viewed as a structure. We state it in its general form for tuples of variables (x_1, \dots, x_k) . By [9, Proposition 2.3.6] this definition is equivalent to its usual formulation for a single variable x .

Definition 26 (ω -saturated \mathbf{C} -model). Let \mathbf{C} be a set of molecular connectives. A \mathbf{C} -model $M = (W, \mathcal{R})$ is ω -saturated when for all finite $Y = \{w_1, \dots, w_n\} \subseteq W$ given, if all finite subsets $\Gamma' \subseteq \Gamma(c'_1, \dots, c'_n, x_1, \dots, x_k) \subseteq \mathcal{L}_{\text{FOL}}^{\mathcal{P}\mathcal{F}}$ with new constants c'_1, \dots, c'_n and free variables x_1, \dots, x_k are realized in M by assignments $s : \mathcal{V} \cup \mathcal{C} \cup \{c'_1, \dots, c'_n\} \rightarrow W$ such that $s(c'_1) = w_1, \dots, s(c'_n) = w_n$ ($\mathcal{L}_{\text{FOL}}^{\mathcal{P}\mathcal{F}}$ is also defined with appropriate relations and function symbols), then $\Gamma(c'_1, \dots, c'_n, x_1, \dots, x_k)$ is realized in M (by an assignment s such that $s(c'_1) = w_1, \dots, s(c'_n) = w_n$ as well).

Proposition 11. *Let $\mathcal{L}_{\text{FOL}}^{\mathcal{P}}$ be countable, U a countably incomplete ultrafilter over I and M a structure. Then, the ultrapower $\prod_U M$ is ω -saturated.*

Proof: See [9, Theorem 6.1.1]. In fact [9, Theorem 6.1.1] proves that M is ω_1 -saturated, which implies that it is ω -saturated. \square

Lemma 1. Let \mathcal{C} be a set of normal connectives and let $M_1 = (W_1, \mathcal{R}_1)$ and $M_2 = (W_2, \mathcal{R}_2)$ be two ω -saturated \mathcal{C} -models. For all $c \in \mathcal{C}$, let V_c be the vertices of the quantified decomposition tree T_c . Let $\mathcal{C}^+ \triangleq \bigcup_{c \in \mathcal{C}} V_c$ be the set of vertices of all the quantified decomposition trees associated

to the connectives of \mathcal{C} . For all $c = \star(c_1, \dots, c_n) \in \mathcal{C}^+$, we define the binary relation $Z_c \subseteq (W_1^k \times W_2^k) \cup (W_2^k \times W_1^k)$ (where k is the output type of \star) as follows: if $\{M, M'\} = \{M_1, M_2\}$ then for all $\bar{w} \in \bar{w}(M, \mathcal{C})$, all $\bar{w}' \in \bar{w}(M', \mathcal{C})$, $\bar{w}Z_c\bar{w}'$ iff $(M, \bar{w}) \rightsquigarrow_{c\mathcal{C}} (M', \bar{w}')$. Then, the set of binary relations $\{Z_c \mid c \in \mathcal{C}^+\}$ is a \mathcal{C} -bisimulation between M_1 and M_2 .

Proof: (Here we identify connectives with their skeletons.) We define $\mathcal{L}_{c_j\mathcal{C}}^\vee$ (resp. $\mathcal{L}_{c_j\mathcal{C}}^\wedge$) to be the language consisting of formulas which belong to $\mathcal{L}_{c_j\mathcal{C}}$ (resp. $\mathcal{L}_{c_j\mathcal{C}}$) or which are disjunctions (resp. conjunctions) of such formulas. Now, we prove the three conditions of Definition 18.

Condition 1. It holds trivially.

Condition 2. Assume that $c = c_0(c_1, \dots, c_n)$ is maximally decomposed, with c_0 which is an existential molecular connective of type signature (\pm_1, \dots, \pm_n) . Before proceeding further, note that by definition of normal connectives we have that if $\mathcal{A}(c_0) = \exists$ then $\mathcal{A}(c_i) = \forall$ iff $\pm_i = +$, if $\mathcal{A}(c_0) = \forall$ then $\mathcal{A}(c_i) = \forall$ iff $\pm_i = -$ and if $c_i = c_j$ then $\pm_i = \pm_j$.

Let $\bar{w}_1, \dots, \bar{w}_n, \bar{w} \in \bar{w}(M, \mathcal{C})$ and $\bar{w}' \in \bar{w}(M', \mathcal{C})$ be such that $R_{c_0}\bar{w}_1 \dots \bar{w}_n\bar{w}$ and $\bar{w}Z_c\bar{w}'$. For all $j \in \{1, \dots, n\}$, let us define

$$\begin{aligned} \Pi_+(M, \bar{w}_j) &\triangleq \left\{ ST_{\bar{x}_j}(\varphi) \mid \varphi \in \mathcal{L}_{c_j\mathcal{C}}^\vee \text{ and } (M, \bar{w}_i) \Vdash \varphi \text{ for all } i \in \llbracket 1; n \rrbracket \text{ such that } c_i = c_j \right\} \\ \Pi_-(M, \bar{w}_j) &\triangleq \left\{ \neg ST_{\bar{x}_j}(\varphi) \mid \varphi \in \mathcal{L}_{c_j\mathcal{C}}^\wedge \text{ and not } (M, \bar{w}_i) \Vdash \varphi \text{ for all } i \in \llbracket 1; n \rrbracket \text{ such that } c_i = c_j \right\}. \end{aligned}$$

Let $I_- \triangleq \{i \in \{1, \dots, n\} \mid \pm_i = -\}$ and $I_+ \triangleq \{i \in \{1, \dots, n\} \mid \pm_i = +\}$. Let $\Gamma(\bar{x}_1, \dots, \bar{x}_n) \triangleq \bigcup_{i \in I_+} \Pi_+(M, \bar{w}_i) \cup \bigcup_{i \in I_-} \Pi_-(M, \bar{w}_i)$ and let $\Gamma \subseteq \Gamma(\bar{x}_1, \dots, \bar{x}_n)$ be finite. Then, $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_n$ where all $\Gamma_i \subseteq \Pi_{\pm_i}(M, \bar{w}_i)$ are finite.

For all $i \in \llbracket 1; n \rrbracket$, let us define

$$\varphi_i \triangleq \begin{cases} \bigwedge \{\varphi \mid ST_{\bar{x}_i}(\varphi) \in \Gamma_i\} & \text{if } \pm_i = + \text{ (i.e. } \mathcal{A}(c_i) = \forall) \\ \bigvee \{\varphi \mid \neg ST_{\bar{x}_i}(\varphi) \in \Gamma_i\} & \text{if } \pm_i = - \text{ (i.e. } \mathcal{A}(c_i) = \exists) \end{cases}$$

and then

$$\psi_i \triangleq \begin{cases} \bigwedge \{\varphi_j \mid j \in \llbracket 1; n \rrbracket \text{ and } id(c_j) = id(c_i)\} & \text{if } \pm_i = + \text{ (i.e. } \mathcal{A}(c_i) = \forall) \\ \bigvee \{\varphi_j \mid j \in \llbracket 1; n \rrbracket \text{ and } id(c_j) = id(c_i)\} & \text{if } \pm_i = - \text{ (i.e. } \mathcal{A}(c_i) = \exists) \end{cases}$$

and let us define $\psi \triangleq c_0(\psi_1, \dots, \psi_n)$ (note that $id(c_j) = id(c_i)$ is in fact equivalent to $c_j = c_i$, because of Condition 3 of Definition 20). This formula does not necessarily belong to $\mathcal{L}_{\mathcal{C}}$, but in any case we have that $(M, \bar{w}) \Vdash \psi$ (in an extended language). The formulas ψ_i are either conjunctions of disjunctions of formulas of $\mathcal{L}_{c_j\mathcal{C}}$ (if $\pm_i = +$) or disjunctions of conjunctions of formulas of $\mathcal{L}_{c_j\mathcal{C}}$ (if $\pm_i = -$). Using the distributivity of \vee and \wedge over their dual connective, they can be transformed equivalently into disjunctions of conjunctions of formulas of $\mathcal{L}_{c_j\mathcal{C}}$ (if $\pm_i = +$) or into conjunctions of disjunctions of formulas of $\mathcal{L}_{c_j\mathcal{C}}$ (if $\pm_i = -$). So, by Definition 20 and by our construction of the ψ_i s in two steps, there are $\chi_1, \dots, \chi_m \in \mathcal{L}_{\mathcal{C}}$ such that $\chi_1 \vee \dots \vee \chi_m$ and ψ are true in the same pointed models (*). So, $(M, \bar{w}) \Vdash \chi_1 \vee \dots \vee \chi_m$. Therefore,

$(M', \overline{w'}) \Vdash \chi_1 \vee \dots \vee \chi_m$ because $(M, \overline{w}) \rightsquigarrow_{cC} (M', \overline{w'})$ by assumption (because $\overline{w}Z_c\overline{w'}$). Hence, $(M', \overline{w'}) \Vdash \psi$, again because of (*). Then, by definition of c_0 , there are $\overline{w'_1}, \dots, \overline{w'_n} \in \overline{w}(M', C)$ such that $\overline{w'_1} \uparrow_1 \llbracket \psi_1 \rrbracket$ and \dots and $\overline{w'_n} \uparrow_n \llbracket \psi_n \rrbracket$ and $R'_{c_0} \overline{w'_1} \dots \overline{w'_n} \overline{w'}$. Therefore, for all $i \in \llbracket 1; n \rrbracket$, $(M', \overline{w'_i}) \Vdash \Gamma_i$ and $R'_{c_0} \overline{w'_1} \overline{w'_2} \dots \overline{w'_n} \overline{w'}$. So, if $\Gamma \triangleq \Gamma(\overline{x}_1, \dots, \overline{x}_n) \cup \{R'_{c_0} \overline{x}_1 \dots \overline{x}_n \overline{c}\}$ where the tuple of new constants \overline{c} is interpreted by the distinguished elements $\overline{w'}$, then M' realizes every finite subset of Γ , namely in some states related to $\overline{w'}$. Thus, by ω -saturation of M' , there are $\overline{w'_1}, \dots, \overline{w'_n} \in \overline{w}(M', C)$ such that for all $i \in \llbracket 1; n \rrbracket$, we have that $(M', \overline{w'_i}) \Vdash \Pi_{\pm_i}(M, \overline{w}_i)$ (**) and $R'_{c_0} \overline{w'_1} \dots \overline{w'_n} \overline{w'}$.

Now, let us take $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = +$ and let us assume towards a contradiction that for all $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ it is not the case that $(M, \overline{w}_i) \rightsquigarrow_{c_j C} (M', \overline{w'_j})$. Then, for all $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$, there is $\varphi_i \in \mathcal{L}_{c_j C}$ such that $(M, \overline{w}_i) \Vdash \varphi_i$ and not $(M', \overline{w'_j}) \Vdash \varphi_i$. Let us define

$$\psi_j \triangleq \bigvee \{ \varphi_i \mid i \in \llbracket 1; n \rrbracket \text{ and } c_j = c_i \}. \quad (7)$$

Then, $\psi_j \in \mathcal{L}_{c_j C}^\vee$ and for all $i \in \llbracket 1; n \rrbracket$ such that $c_i = c_j$, $(M, \overline{w}_i) \Vdash \psi_j$. So, $ST_{\overline{w}_j}(\psi_j) \in \Pi_+(M, \overline{w}_j)$. Therefore, by (**), we have that $(M', \overline{w'_j}) \Vdash ST_{\overline{w}_j}(\psi_j)$, so $(M', \overline{w'_j}) \Vdash \psi_j$. That is, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $(M', \overline{w'_j}) \Vdash \varphi_i$, which is impossible. Therefore, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $(M, \overline{w}_i) \rightsquigarrow_{c_j C} (M', \overline{w'_j})$. This proves Clause 2.(i) of Definition 18 of a C-bisimulation for molecular connectives.

Now, let us take $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = -$ and let us assume towards a contradiction that for all $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ it is not the case that $(M', \overline{w'_j}) \rightsquigarrow_{c_j C} (M, \overline{w}_i)$. Then, for all $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$, there is $\varphi_i \in \mathcal{L}_{c_j C}$ such that $(M', \overline{w'_j}) \Vdash \varphi_i$ and not $(M, \overline{w}_i) \Vdash \varphi_i$. Let us define

$$\psi_j \triangleq \bigwedge \{ \varphi_i \mid i \in \llbracket 1; n \rrbracket \text{ and } c_j = c_i \}.$$

Then, $\psi_j \in \mathcal{L}_{c_j C}^\wedge$ and for all $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$, not $(M, \overline{w}_i) \Vdash \psi_j$. So, $\neg ST_{\overline{w}_j}(\psi_j) \in \Pi_-(M, \overline{w}_j)$. Therefore, by (**), $(M', \overline{w'_j}) \Vdash \neg ST_{\overline{w}_j}(\psi_j)$. That is, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and not $(M', \overline{w'_j}) \Vdash \varphi_i$, which is impossible. Therefore, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $(M', \overline{w'_j}) \rightsquigarrow_{c_j C} (M, \overline{w}_i)$. This proves Clause 2.(ii) of Definition 18 of a C-bisimulation for molecular connectives.

Condition 3. It is proved similarly to Condition 2, with dual definitions of (sets of) formulas where conjunctions and replaced by disjunctions and vice versa. Assume that $c = c_0(c_1, \dots, c_n)$ is maximally decomposed, with c_0 which is a universal molecular connective of type signature (\pm_1, \dots, \pm_n) . Let $\overline{w'_1}, \dots, \overline{w'_n}, \overline{w'} \in \overline{w}(M', C)$ and $\overline{w} \in \overline{w}(M, C)$ be such that $\neg R'_{c_0} \overline{w'_1} \dots \overline{w'_n} \overline{w'}$ and $\overline{w}Z_c\overline{w'}$. Then we define for all $j \in \{1, \dots, n\}$,

$$\begin{aligned} \Pi_+(M', \overline{w'_j}) &\triangleq \left\{ \neg ST_{\overline{w}_j}(\varphi) \mid \varphi \in \mathcal{L}_{c_j C}^\wedge \text{ and not } (M', \overline{w'_j}) \Vdash \varphi \text{ for all } i \in \llbracket 1; n \rrbracket \text{ such that } c_i = c_j \right\} \\ \Pi_-(M', \overline{w'_j}) &\triangleq \left\{ ST_{\overline{w}_j}(\varphi) \mid \varphi \in \mathcal{L}_{c_j C}^\vee \text{ and } (M', \overline{w'_j}) \Vdash \varphi \text{ for all } i \in \llbracket 1; n \rrbracket \text{ such that } c_i = c_j \right\}. \end{aligned}$$

Let $I_- \triangleq \{i \in \{1, \dots, n\} \mid \pm_i = -\}$ and $I_+ \triangleq \{i \in \{1, \dots, n\} \mid \pm_i = +\}$. Let $\Gamma(\overline{x}_1, \dots, \overline{x}_n) \triangleq \bigcup_{i \in I_+} \Pi_+(M, \overline{w}_i) \cup \bigcup_{i \in I_-} \Pi_-(M, \overline{w}_i)$ and let $\Gamma \subseteq \Gamma(\overline{x}_1, \dots, \overline{x}_n)$ be finite. Then, $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_n$

where all $\Gamma_i \subseteq \Pi_{\pm_i}(M, \bar{w}_i)$ are finite. For all $i \in \llbracket 1; n \rrbracket$, let us define

$$\varphi_i \triangleq \begin{cases} \bigvee \{\varphi \mid \neg ST_{\bar{x}_i}(\varphi) \in \Gamma_i\} & \text{if } \pm_i = + \text{ (i.e. } \mathcal{A}(c_i) = \exists) \\ \bigwedge \{\varphi \mid ST_{\bar{x}_i}(\varphi) \in \Gamma_i\} & \text{if } \pm_i = - \text{ (i.e. } \mathcal{A}(c_i) = \forall) \end{cases}$$

and then

$$\psi_i \triangleq \begin{cases} \bigvee \{\varphi_j \mid j \in \llbracket 1; n \rrbracket \text{ and } id(c_j) = id(c_i)\} & \text{if } \pm_i = + \text{ (i.e. } \mathcal{A}(c_i) = \exists) \\ \bigwedge \{\varphi_j \mid j \in \llbracket 1; n \rrbracket \text{ and } id(c_j) = id(c_i)\} & \text{if } \pm_i = - \text{ (i.e. } \mathcal{A}(c_i) = \forall) \end{cases}$$

and let us define $\psi \triangleq c_0(\psi_1, \dots, \psi_n)$ (note that $id(c_j) = id(c_i)$ is in fact equivalent to $c_j = c_i$, because of Condition 3 of Definition 20). This formula ψ does not necessarily belong to $\mathcal{L}_{\mathcal{C}}$, but in any case we have that it is not the case that $(M', \bar{w}') \Vdash \psi$ (in an extended language). The formulas ψ_i are either disjunctions of conjunctions of formulas of $\mathcal{L}_{c_j\mathcal{C}}$ (if $\pm_i = +$) or conjunctions of disjunctions of formulas of $\mathcal{L}_{c_j\mathcal{C}}$ (if $\pm_i = -$). Using the distributivity of \vee and \wedge over their dual connective, they can be transformed equivalently into conjunctions of disjunctions of formulas of $\mathcal{L}_{c_j\mathcal{C}}$ (if $\pm_i = +$) or into disjunctions of conjunctions of formulas of $\mathcal{L}_{c_j\mathcal{C}}$ (if $\pm_i = -$). By our construction of the ψ_i s in two steps and by definition of normal connectives, there are $\chi_1, \dots, \chi_m \in \mathcal{L}_{\mathcal{C}\mathcal{C}}$ such that $\chi_1 \wedge \dots \wedge \chi_m$ and ψ are true in the same pointed models (*). So, it is not the case that $(M', \bar{w}') \Vdash \chi_1 \wedge \dots \wedge \chi_m$. Therefore, it is not the case that $(M, \bar{w}) \Vdash \chi_1 \wedge \dots \wedge \chi_m$ because $(M, \bar{w}) \rightsquigarrow_{c\mathcal{C}} (M', \bar{w}')$ by assumption (because $\bar{w}Z_c\bar{w}'$). Hence, it is not the case that $(M, \bar{w}) \Vdash \psi$, again because of (*). Then, by definition of c_0 , there are $\bar{w}_1, \dots, \bar{w}_n \in \bar{w}(M, \mathcal{C})$ such that not $\bar{w}_1 \uparrow_1 \llbracket \psi_1 \rrbracket$ and ... and not $\bar{w}_n \uparrow_n \llbracket \psi_n \rrbracket$ and $-R_{c_0}\bar{w}_1 \dots \bar{w}_n\bar{w}$. Therefore, for all $i \in \llbracket 1; n \rrbracket$, $(M, \bar{w}_i) \models \Gamma_i$ and $-R_{c_0}\bar{w}_1\bar{w}_2 \dots \bar{w}_n\bar{w}$. So, if $\Gamma \triangleq \Gamma(\bar{x}_1, \dots, \bar{x}_n) \cup \{\mathbf{R}_{c_0}\bar{x}_1 \dots \bar{x}_n\bar{c}\}$ where the tuple of new constants \bar{c} is interpreted by the distinguished elements \bar{w}' , then M realizes every finite subset of Γ , namely in some states related to \bar{w} . Thus, by ω -saturation of M , there are $\bar{w}_1, \dots, \bar{w}_n \in \bar{w}(M, \mathcal{C})$ such that for all $i \in \llbracket 1; n \rrbracket$, we have that $(M, \bar{w}_i) \models \Pi_{\pm_i}(M', \bar{w}'_i)$ (***) and $-R_{c_0}\bar{w}_1 \dots \bar{w}_n\bar{w}$.

Now, let us take $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = +$ and let us assume towards a contradiction that for all $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ it is not the case that $(M, \bar{w}_j) \rightsquigarrow_{c_j\mathcal{C}} (M', \bar{w}'_i)$. Then, for all $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$, there is $\varphi_i \in \mathcal{L}_{c_j\mathcal{C}}$ such that $(M, \bar{w}_j) \Vdash \varphi_i$ and not $(M', \bar{w}'_i) \Vdash \varphi_i$. Let us define

$$\psi_j \triangleq \bigwedge \{\varphi_i \mid i \in \llbracket 1; n \rrbracket \text{ and } c_j = c_i\}. \quad (8)$$

Then, $\psi_j \in \mathcal{L}_{c_j\mathcal{C}}^\wedge$ and for all $i \in \llbracket 1; n \rrbracket$ such that $c_i = c_j$, not $(M', \bar{w}'_i) \Vdash \psi_j$. So, $\neg ST_{\bar{x}_j}(\psi_j) \in \Pi_+(M', \bar{w}'_j)$. Therefore, by (***), we have that $(M, \bar{w}_j) \models \neg ST_{\bar{x}_j}(\psi_j)$, so not $(M, \bar{w}_j) \Vdash \psi_j$. That is, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and not $(M, \bar{w}_j) \Vdash \varphi_i$, which is impossible. Therefore, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $(M, \bar{w}_i) \rightsquigarrow_{c_j\mathcal{C}} (M', \bar{w}'_i)$. This proves Clause 3.(i) of Definition 18 of a \mathcal{C} -bisimulation for molecular connectives.

Now, let us take $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = -$ and let us assume towards a contradiction that for all $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ it is not the case that $(M', \bar{w}'_i) \rightsquigarrow_{c_j\mathcal{C}} (M, \bar{w}_j)$. Then, for all $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$, there is $\varphi_i \in \mathcal{L}_{c_j\mathcal{C}}$ such that $(M', \bar{w}'_i) \Vdash \varphi_i$ and not $(M, \bar{w}_j) \Vdash \varphi_i$. Let us define

$$\psi_j \triangleq \bigvee \{\varphi_i \mid i \in \llbracket 1; n \rrbracket \text{ and } c_j = c_i\}.$$

Then, $\psi_j \in \mathcal{L}_{c_j\mathcal{C}}^\vee$ and for all $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$, $(M', \bar{w}'_i) \Vdash \psi_j$. So, $ST_{\bar{x}_j}(\psi_j) \in \Pi_-(M', \bar{w}'_j)$. Therefore, by (***), $(M, \bar{w}_j) \models ST_{\bar{x}_j}(\psi_j)$. That is, there is $i \in \llbracket 1; n \rrbracket$ such that

$c_j = c_i$ and $(M, \bar{w}_j) \Vdash \varphi_i$, which is impossible. Therefore, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $(M', \bar{w}'_i) \rightsquigarrow_{c_j C} (M, \bar{w}_j)$. This proves Clause 3.(ii) of Definition 18 of a C -bisimulation for molecular connectives. \square

Theorem 1. *Let C be a set of normal connectives complete for conjunction and disjunction and let (M, \bar{w}) and (M', \bar{w}') be pointed C -models. Then the following are equivalent:*

1. $(M, \bar{w}) \rightsquigarrow_C (M', \bar{w}')$;
2. *there exists a countably incomplete ultrafilter U over \mathbb{N} such that $\prod_U (M, \bar{w}) \rightarrow_C \prod_U (M', \bar{w}')$.*

Proof: We first prove that 2. implies 1. By Proposition 9, $(M, \bar{w}) \Vdash \varphi$ iff $\prod_U (M, \bar{w}) \Vdash \varphi$. By assumption and Proposition 6, this implies that $\prod_U (M', \bar{w}') \Vdash \varphi$ and, again by Proposition 9, the latter is equivalent to $(M', \bar{w}') \Vdash \varphi$.

Now, we prove that 1. implies 2. Let U be a countably incomplete ultrafilter over \mathbb{N} (it exists by [8, Example 2.72]). By Proposition 11, the ultrapowers $\prod_U M$ and $\prod_U M'$ are ω -saturated. Now, for all $\varphi \in \mathcal{L}_C$, $\prod_U (M, \bar{w}) \Vdash \varphi$ implies $\prod_U (M', \bar{w}') \Vdash \varphi$. This claim follows from 1. and Proposition 9. Then, we apply Lemma 1 since $\prod_U M$ and $\prod_U M'$ are ω -saturated and we obtain the result. \square

Lemma 2. *Let C be a set of molecular connectives and let $(M_i, \bar{w}_i)_{i \in I}$ be a family of pointed C -models of type k . Then, for all $\varphi \in \mathcal{L}_C$ of type k and all tuples of free variables \bar{x} of size k , $\prod_U (M_i, \bar{w}_i) \Vdash \varphi$ iff $\prod_U (M_i, s_{\bar{x}}^{\bar{w}_i}) \Vdash ST_{\bar{x}}(\varphi)$.*

Proof: We have that $\prod_U (M_i, \bar{w}_i) = \left(\prod_U M_i, \prod_U \bar{w}_i \right)$ where $\prod_U \bar{w}_i = \left(\prod_U w_i^1, \dots, \prod_U w_i^k \right)$ if $\bar{w}_i = (w_i^1, \dots, w_i^k)$ and $\prod_U (M_i, s_{\bar{x}}^{\bar{w}_i}) = \left(\prod_U M_i, \prod_U s_{\bar{x}}^{\bar{w}_i} \right)$. So, it suffices to prove that $s_{\bar{x}}^{\prod_U \bar{w}_i}$ and $\prod_U s_{\bar{x}}^{\bar{w}_i}$ coincide on the variables x_j of the tuple $\bar{x} = (x_1, \dots, x_k)$ to obtain the result. And it turns out that for all $x_j \in \{x_1, \dots, x_k\}$, $s_{\bar{x}}^{\prod_U \bar{w}_i}(x_j) = \prod_U w_i^j = \prod_U s_{\bar{x}}^{\bar{w}_i}(x_j) = \left(\prod_U s_{\bar{x}}^{\bar{w}_i} \right)(x_j)$. \square

Theorem 2. *Let C be a set of normal connectives complete for conjunction and disjunction and let K and K' be classes of pointed C -models such that K' is closed under ultraproducts. Then, the following are equivalent:*

1. K is definable in K' by a set of formulas of \mathcal{L}_C ;
2. K is closed under C -bisimulations in K' and closed under ultraproducts, and $K' - K$ is closed under ultrapowers.

Proof: The implication from 1. to 2. follows from Propositions 6, 7 and 9. For the converse, assume that K and $K' - K$ satisfy the closure conditions. Define T as the set of formulas of \mathcal{L}_C holding in K :

$$T \triangleq \bigcup_{k \in \mathbb{N}^*} \{\varphi \in \mathcal{L}_C \mid k(\varphi) = k \text{ and, for all } (M, \bar{w}) \in K \text{ of type } k, (M, \bar{w}) \Vdash \varphi\} \quad (9)$$

We will show that T defines the class K in K' . Let (M, \bar{w}) be a pointed C -model of K' of type k and assume that $(M, \bar{w}) \Vdash T$. To complete the proof of the theorem, we show that (M, \bar{w}) must be in K .

Define Σ to be the theory of FOL of (M, \bar{w}) , with \bar{x} a tuple of variables of the same size as \bar{w} :

$$\Sigma \triangleq \{\neg ST_{\bar{x}}(\varphi) \mid \varphi \in \mathcal{L}_C \text{ of type } k \text{ such that not } (M, \bar{w}) \Vdash \varphi\}.$$

Σ is finitely satisfiable in $ST_{\bar{x}}(K) \triangleq \{(M, s_{\bar{x}}^{\bar{w}}) \mid (M, \bar{w}) \in K\}$. Indeed, assume towards a contradiction that the set $\{\neg ST_{\bar{x}}(\psi_1), \dots, \neg ST_{\bar{x}}(\psi_n)\} \subseteq \Sigma$ is not satisfiable in $ST_{\bar{x}}(K)$. Then the formula $(\psi_1 \vee \dots \vee \psi_n)$ of type k would be true on all pointed models in K of type k . Moreover, it belongs to \mathcal{L}_C because C is complete for disjunction. So $(\psi_1 \vee \dots \vee \psi_n)$ would belong to T . However, it would be false in (M, \bar{w}) , which is impossible. But then Proposition 10, adapted from [9, Corollary 4.1.11], shows that Σ is satisfiable in an ultraproduct $\prod_U (N_i, s_{\bar{x}}^{\bar{v}_i})$ of pointed models of $ST_{\bar{x}}(K)$.

Let us take $(N, \bar{v}) \triangleq \prod_U (N_i, \bar{v}_i)$, which belongs to K by closure of K under taking ultraproducts. Then, by Lemma 2, we have that for all $\varphi \in \mathcal{L}_C$ of type k , $\prod_U (N_i, \bar{v}_i) \Vdash \varphi$ iff $\prod_U (N_i, s_{\bar{x}}^{\bar{v}_i}) \Vdash ST_{\bar{x}}(\varphi)$. So, we have that $(N, \bar{v}) \rightsquigarrow_C (M, \bar{w})$ because for all $\varphi \in \mathcal{L}_C$, not $(M, \bar{w}) \Vdash \varphi$ implies not $(N, \bar{v}) \Vdash \varphi$ is equivalent to, for all $\varphi \in \mathcal{L}_C$, $(N, \bar{v}) \Vdash \varphi$ implies $(M, \bar{w}) \Vdash \varphi$. So, by Theorem 1, there exists an ultrafilter U' such that $\prod_{U'} (N, \bar{v}) \rightarrow_C \prod_{U'} (M, \bar{w})$.

By closure under ultraproducts, the pointed model $\prod_{U'} (N, \bar{v})$ belongs to K . Moreover, $\prod_{U'} (M, \bar{w})$ is in K' by the assumption of closure of K' under ultraproduct. Hence, by closure under C -bisimulations, $\prod_{U'} (M, \bar{w})$ is in K as well. By closure of $K' - K$ under ultrapowers, it follows that (M, \bar{w}) is in K . This completes the proof. \square

Theorem 3. *Let C be a set of normal connectives complete for conjunction and disjunction and let K and K' be classes of pointed C -models of the same type such that K' is closed under ultraproducts. Then, the following are equivalent:*

1. K is definable in K' by a single formula of \mathcal{L}_C ;
2. K is closed under C -bisimulation in K' , closed under ultraproducts and $K' - K$ is closed under ultraproducts.

Proof: The direction from 1. to 2. follows from Propositions 6 and 7. For the converse, we assume that K and $K' - K$ satisfy the stated closure conditions and that their pointed C -models are all of type k . Then, by Theorem 2, there is a set T of formulas of \mathcal{L}_C of type k defining K in K' .

Let k be the type of the \mathbf{C} -models of K and $K' - K$ and let \bar{x} be a tuple of k variables. Let us define:

$$T_2 \triangleq \{\neg ST_{\bar{x}}(\varphi) \mid \varphi \in \mathcal{L}_{\mathbf{C}} \text{ of type } k \text{ and for all } (M, \bar{w}) \in K' - K, \text{ not } (M, \bar{w}) \Vdash \varphi\}.$$

We are going to show that T_2 defines $ST_{\bar{x}}(K' - K) \triangleq \{ST_{\bar{x}}(M, \bar{w}) \mid (M, \bar{w}) \in K' - K\}$ in $ST_{\bar{x}}(K')$. Let $(M, \bar{w}) \in K'$ of type k be such that $ST_{\bar{x}}(M, \bar{w}) \models T_2$ and let us define

$$\Sigma \triangleq \{\varphi \in \mathcal{L}_{\mathbf{C}} \mid (M, \bar{w}) \Vdash \varphi\}.$$

Then, Σ is finitely satisfiable in $K' - K$. Indeed, assume towards a contradiction that there are $\varphi_1, \dots, \varphi_n \in \Sigma$ which are not satisfiable in $K' - K$. Then, $\neg ST_{\bar{x}}(\varphi_1 \wedge \dots \wedge \varphi_n) \in T_2$. Therefore, $ST_{\bar{x}}(M, \bar{w}) \models \neg ST_{\bar{x}}(\varphi_1) \vee \dots \vee \neg ST_{\bar{x}}(\varphi_n)$. However, $(M, \bar{w}) \Vdash (\varphi_1 \wedge \dots \wedge \varphi_n)$. So, we reach a contradiction and Σ is finitely satisfiable in $K' - K$. Thus, by compactness, we have that Σ is satisfiable in an ultraproduct (N, \bar{v}) of \mathbf{C} -models of $K' - K$ (the proof of Proposition 10 can easily be adapted to molecular logics using Proposition 7). By assumption, this ultraproduct (N, \bar{v}) belongs to $K' - K$. Then, by definition of Σ , we have that $(M, \bar{w}) \rightsquigarrow_{\mathbf{C}} (N, \bar{v})$. Therefore, by Theorem 1, there exists a countably incomplete ultrafilter U over \mathbb{N} such that $\prod_U (M, \bar{w}) \rightarrow_{\mathbf{C}}$

$\prod_U (N, \bar{v})$. But $\prod_U (N, \bar{v})$ belongs to $K' - K$ by closure of $K' - K$ under ultraproduct (and

therefore also to K'). Now, if (M, \bar{w}) belonged to K then $\prod_U (M, \bar{w})$ would also belong to K by

closure of K under ultraproduct. Then, we would have by closure under \mathbf{C} -bisimulation of K in K' that $\prod_U (N, \bar{v})$ would belong to K as well, which is impossible. Therefore, (M, \bar{w}) does not

belong to K . Moreover, (M, \bar{w}) belongs to K' by assumption. This entails that (M, \bar{w}) belongs to $K' - K$. Hence, $ST_{\bar{x}}(M, \bar{w}) \in ST_{\bar{x}}(K' - K)$ and T_2 does define $ST_{\bar{x}}(K' - K)$ in $ST_{\bar{x}}(K')$.

Now, let us define

$$T_1 \triangleq \{ST_{\bar{x}}(\varphi) \mid \varphi \in T\}.$$

Then, because T defines K in K' , we have that $T_1 \cup T_2$ is unsatisfiable in K' . That is, there is no pointed \mathbf{C} -model (M, \bar{w}) of K' of type k such that $ST_{\bar{x}}(M, \bar{w}) \Vdash T_1 \cup T_2$. So then, by compactness, there exist $ST_{\bar{x}}(\varphi_1), \dots, ST_{\bar{x}}(\varphi_n) \in T_1$ and $\neg ST_{\bar{x}}(\psi_1), \dots, \neg ST_{\bar{x}}(\psi_m) \in T_2$ such that for all pointed \mathbf{C} -models (M, \bar{w}) of K' , it is not the case that $ST_{\bar{x}}(M, \bar{w}) \Vdash ST_{\bar{x}}(\varphi_1) \wedge \dots \wedge ST_{\bar{x}}(\varphi_n) \wedge \neg ST_{\bar{x}}(\psi_1) \wedge \dots \wedge \neg ST_{\bar{x}}(\psi_m)$ (*). To complete the proof, we show that K is in fact defined in K' by the conjunction $(\varphi_1 \wedge \dots \wedge \varphi_n)$. By definition, for any (M, \bar{w}) in K we have $(M, \bar{w}) \Vdash (\varphi_1 \wedge \dots \wedge \varphi_n)$. Conversely, for any (M, \bar{w}) of K' , if $(M, \bar{w}) \Vdash (\varphi_1 \wedge \dots \wedge \varphi_n)$ then there must be $i \in \llbracket 1; m \rrbracket$ such that $ST_{\bar{x}}(M, \bar{w}) \Vdash \neg ST_{\bar{x}}(\psi_i)$ does not hold. Indeed, otherwise, we would have that $ST_{\bar{x}}(M, \bar{w}) \Vdash ST_{\bar{x}}(\varphi_1) \wedge \dots \wedge ST_{\bar{x}}(\varphi_n) \wedge \neg ST_{\bar{x}}(\psi_1) \wedge \dots \wedge \neg ST_{\bar{x}}(\psi_m)$, which is impossible by (*). Therefore, there must be $i \in \llbracket 1; m \rrbracket$ such that $(M, \bar{w}) \Vdash \psi_i$. Hence, by definition of T_2 , this entails that (M, \bar{w}) does not belong to $K' - K$, whence (M, \bar{w}) belongs to K . \square

Corollary 1. *Let $L = (\mathcal{L}_{\mathbf{C}}, \mathcal{E}_{\mathbf{C}}, \Vdash)$ be a molecular logic whose set of connectives \mathcal{C} is normal and complete for conjunction and disjunction and whose class $\mathcal{E}_{\mathbf{C}}$ of \mathbf{C} -models are all of the same type. Then, L is axiomatizable iff $\mathcal{E}_{\mathbf{C}}$ is closed under \mathbf{C} -bisimulation and ultraproducts and $\mathcal{M}_{\mathbf{C}} - \mathcal{E}_{\mathbf{C}}$ is closed under ultraproducts.*

Proof: It follows straightforwardly from Theorem 3 and the definition of axiomatizability recalled in Section 2.5. \square

B Proofs of Propositions 1, 2, 3, 4, 5, 6, 7

Proposition 1. *Let C be a set of atomic connectives containing Boolean connectives. and let $M = (W, \mathcal{R})$ be a C -model. Then, for all $k \in \mathbb{N}^*$, all $\varphi, \psi \in \mathcal{L}_C$, if $k(\varphi) = k(\psi) = k$, then*

$$\begin{aligned} \llbracket \top_k \rrbracket^M &\triangleq W^k \\ \llbracket \perp_k \rrbracket^M &\triangleq \emptyset \\ \llbracket \neg_k \varphi \rrbracket^M &\triangleq W^k - \llbracket \varphi \rrbracket^M \\ \llbracket (\varphi \wedge_k \psi) \rrbracket^M &\triangleq \llbracket \varphi \rrbracket^M \cap \llbracket \psi \rrbracket^M \\ \llbracket (\varphi \vee_k \psi) \rrbracket^M &\triangleq \llbracket \varphi \rrbracket^M \cup \llbracket \psi \rrbracket^M. \end{aligned}$$

Proof: The proof is without particular difficulty. See Example 1 for the cases of conjunction and disjunction. \square

Proposition 2. *Let C be a set of atomic connectives such that $\neg \star \in C$ for all $\star \in C$. Let $\varphi \in \mathcal{L}_C$ and M be a C -model. Then, for all $\bar{w} \in \bar{w}(M, C)$, $\bar{w} \in \llbracket \neg \varphi \rrbracket^M$ iff $\bar{w} \notin \llbracket \varphi \rrbracket^M$.*

Proof: It is without particular difficulty, it suffices to check the definitions. \square

Proposition 3. *Let C be a set of molecular connectives and let $c = \star(c_1, \dots, c_n) \in C$ be a molecular connective of type signature (k, k_1, \dots, k_m) and tonicity signature (\pm_1, \dots, \pm_l) with $\star = (\sigma, \pm, \bar{E}, \bar{k}, (\pm_1^*, \dots, \pm_n^*))$. Assume that the decomposition tree of c has $l > 0$ different leaves labeled by some id_k^i . Let $M = (W, \mathcal{R})$ be a C -model and let $\bar{w} \in W^k$.*

- *If c is an existential molecular connective then*

$$\begin{aligned} \bar{w} \in \llbracket c(\varphi_1, \dots, \varphi_m) \rrbracket^M &\text{ iff } \exists \bar{w}_1 \dots \bar{w}_l \\ (\bar{w}_1 \uparrow_1 \llbracket \varphi_{i_1} \rrbracket^M \wedge \dots \wedge \bar{w}_l \uparrow_l \llbracket \varphi_{i_l} \rrbracket^M \wedge R_c \bar{w}_1 \dots \bar{w}_l \bar{w}) \end{aligned}$$

- *If c is a universal molecular connective then*

$$\begin{aligned} \bar{w} \in \llbracket c(\varphi_1, \dots, \varphi_m) \rrbracket^M &\text{ iff } \forall \bar{w}_1 \dots \bar{w}_l \\ (\bar{w}_1 \uparrow_1 \llbracket \varphi_{i_1} \rrbracket^M \vee \dots \vee \bar{w}_l \uparrow_l \llbracket \varphi_{i_l} \rrbracket^M \vee R_c \bar{w}_1 \dots \bar{w}_l \bar{w}) \end{aligned}$$

where the R_c s are defined in Definition 15 and for all $j \in \llbracket 1; l \rrbracket$,

$$\bar{w}_j \uparrow_j \llbracket \varphi_{i_j} \rrbracket^M \triangleq \begin{cases} \bar{w}_j \in \llbracket \varphi_{i_j} \rrbracket^M & \text{if } \pm_j = + \\ \bar{w}_j \notin \llbracket \varphi_{i_j} \rrbracket^M & \text{if } \pm_j = - \end{cases} \text{ and } i_1, \dots, i_l \in \llbracket 1; m \rrbracket \text{ are those indices corre-}$$

sponding to the m different id_k^i s appearing in c (we basically map the l leafs of the decomposition tree of c to their labels id_k^i s in this tree).

Proof: We only prove the existential case, the proof for the universal case is similar. The proof is by induction on c .

Assume that $c = \star(id_{k_1}^1, \dots, id_{k_n}^n)$ is an existential molecular connective with \star of skeleton $(\sigma, \pm, \exists, \bar{k}, (\pm_1^*, \dots, \pm_n^*))$. Then, $l = n$ and we have that $R_c \bar{w}_1 \dots \bar{w}_n \bar{w}$ holds iff there are $\bar{v}_1 \dots \bar{v}_n$ such that $R_{\star}^{\pm \sigma} \bar{v}_1 \dots \bar{v}_n \bar{w}$ and $\bar{w}_1 = \bar{v}_1$ and \dots and $\bar{w}_n = \bar{v}_n$. Then, we have that $R_c \bar{w}_1 \dots \bar{w}_n \bar{w}$ iff $R_{\star}^{\pm \sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}$. However, by definition, we have that $\bar{w} \in \llbracket c(\varphi_1, \dots, \varphi_m) \rrbracket^M$ iff $\exists \bar{w}_1 \dots \bar{w}_n (\bar{w}_1 \uparrow_1 \llbracket \varphi_{i_1} \rrbracket^M \wedge \dots \wedge \bar{w}_n \uparrow_n \llbracket \varphi_{i_n} \rrbracket^M \wedge R_c \bar{w}_1 \dots \bar{w}_n \bar{w})$. We obtain the result by replacing $R_{\star}^{\pm \sigma}$ by R_c .

Now, we prove the induction step. If $c = \star(c_1, \dots, c_n)$ is an existential molecular connective with \star of skeleton $(\sigma, \pm, \exists, \bar{k}, (\pm_1^*, \dots, \pm_n^*))$ then we have that $\bar{w} \in \llbracket c(\varphi_1, \dots, \varphi_m) \rrbracket^M$

iff $\exists \bar{v}_1 \dots \bar{v}_n (R_{\star}^{\pm\sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \wedge \bar{v}_1 \triangleright_1 \llbracket c_1(\varphi_1^1, \dots, \varphi_{i_1}^1) \rrbracket^M \wedge \dots \wedge \bar{v}_n \triangleright_n \llbracket c_n(\varphi_1^n, \dots, \varphi_{i_n}^n) \rrbracket^M)$ by the truth condition for \star

iff $\exists \bar{v}_1 \dots \bar{v}_n (R_{\star}^{\pm\sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \wedge \bar{v}_1 \in \llbracket \pm_1^{\star} c_1(\varphi_1^1, \dots, \varphi_{i_1}^1) \rrbracket^M \wedge \dots \wedge \bar{v}_n \in \llbracket \pm_n^{\star} c_n(\varphi_1^n, \dots, \varphi_{i_n}^n) \rrbracket^M)$

iff $\exists \bar{v}_1 \dots \bar{v}_n (R_{\star}^{\pm\sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \wedge \exists \bar{w}_1^1 \dots \bar{w}_{i_1}^1 (\bar{w}_1^1 \triangleright_1^1 \llbracket \varphi_1^1 \rrbracket^M \wedge \dots \wedge \bar{w}_{i_1}^1 \triangleright_{i_1}^1 \llbracket \varphi_{i_1}^1 \rrbracket^M \wedge R_{\pm_1^{\star} c_1} \bar{w}_1^1 \dots \bar{w}_{i_1}^1 \bar{v}_1) \wedge \dots \wedge \exists \bar{w}_1^n \dots \bar{w}_{i_n}^n (\bar{w}_1^n \triangleright_1^n \llbracket \varphi_1^n \rrbracket^M \wedge \dots \wedge \bar{w}_{i_n}^n \triangleright_{i_n}^n \llbracket \varphi_{i_n}^n \rrbracket^M \wedge R_{\pm_n^{\star} c_n} \bar{w}_1^n \dots \bar{w}_{i_n}^n \bar{v}_n))$

iff $\exists \bar{w}_1^1 \dots \bar{w}_{i_1}^1 \dots \bar{w}_1^n \dots \bar{w}_{i_n}^n (\bar{w}_1^1 \triangleright_1^1 \llbracket \varphi_1^1 \rrbracket^M \wedge \dots \wedge \bar{w}_{i_1}^1 \triangleright_{i_1}^1 \llbracket \varphi_{i_1}^1 \rrbracket^M \wedge \dots \wedge \bar{w}_1^n \triangleright_1^n \llbracket \varphi_1^n \rrbracket^M \wedge \dots \wedge \bar{w}_{i_n}^n \triangleright_{i_n}^n \llbracket \varphi_{i_n}^n \rrbracket^M \wedge \exists \bar{v}_1 \dots \bar{v}_n (R_{\star}^{\pm\sigma} \bar{v}_1 \dots \bar{v}_n \bar{w} \wedge R_{\pm_1^{\star} c_1} \bar{w}_1^1 \dots \bar{w}_{i_1}^1 \bar{v}_1 \wedge \dots \wedge R_{\pm_n^{\star} c_n} \bar{w}_1^n \dots \bar{w}_{i_n}^n \bar{v}_n))$

iff $\exists \bar{w}_1 \dots \bar{w}_l (\bar{w}_1 \triangleright_1 \llbracket \varphi_{i_1} \rrbracket^M \wedge \dots \wedge \bar{w}_l \triangleright_l \llbracket \varphi_{i_l} \rrbracket^M \wedge R_c \bar{w}_1 \dots \bar{w}_l \bar{w})$ because the union of the sets of leaves of T_{c_j} for each c_j , which corresponds to the tuples $\bar{w}_1^1 \dots \bar{w}_{i_1}^1 \dots \bar{w}_1^n \dots \bar{w}_{i_n}^n$, is the set of leaves of T_c , which is denoted here $\bar{w}_1 \dots \bar{w}_l$. This proves the induction step. \square

Proposition 4. • Let $C = \{p, \neg, \wedge, \vee, \diamond, \square\}$ be the connectives of Example 8 and let M and M' be two C -models. Then, a C -bisimulation between M and M' is a modal bisimulation between M and M' and vice versa.

- Let $C = \{p, \otimes, \supset, \subset\}$ be the connectives of Example 9 and let M and M' be two C -models. Then, a C -bisimulation between M and M' is a directed bisimulation between M and M' and vice versa.

Proof: It suffices to compare the conditions of Examples 8 and 9 with the definitions of modal bisimulation [8, Def. 2.16] and directed bisimulation. This said, we obtain a notion of bisimulation which is slightly more general than the usual definition of modal bisimulation. They are in fact equivalent because both definitions of bisimulation can be extended to obtain symmetric relations. \square

Proposition 5. Let C be a set of atomic connectives and let $M_1 = (W_1, \mathcal{R}_1)$ and $M_2 = (W_2, \mathcal{R}_2)$ be two C -models. Let Z be a C -bisimulation between M_1 and M_2 . Then, if $\{M, M'\} = \{M_1, M_2\}$ then for all $\bar{w} \in \bar{w}(M, C)$, all $\bar{w}' \in \bar{w}(M', C)$, if $\bar{w}Z\bar{w}'$ then $(M, \bar{w}) \rightsquigarrow_C (M', \bar{w}')$.

Proof: We prove it by induction on a formula φ . If $\varphi = p$ is a propositional letter then the result holds by definition of Z . Assume that $\varphi = \star(\varphi_1, \dots, \varphi_n)$.

If \star has skeleton $(\sigma, \pm, \exists, (k, k_1, \dots, k_n), (\pm_1, \dots, \pm_n))$, then assume that $(M, \bar{w}) \models \varphi$. Then, there are $\bar{w}_1, \dots, \bar{w}_n$ such that $\bar{w}_1 \triangleright_1 \llbracket \varphi_1 \rrbracket$ and \dots and $\bar{w}_n \triangleright_n \llbracket \varphi_n \rrbracket$ and $R_{\star}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}$. Now, $\bar{w}Z\bar{w}'$, so by definition of Z , there are $\bar{w}'_1, \bar{w}'_2, \dots, \bar{w}'_n$ such that $\bar{w}_1 \triangleright_1 \bar{w}'_1$ and \dots and $\bar{w}_n \triangleright_n \bar{w}'_n$ and $R_{\star}^{\pm\sigma} \bar{w}'_1 \dots \bar{w}'_n \bar{w}'$ where $\bar{w}_j \triangleright_j \bar{w}'_j \triangleq \begin{cases} \bar{w}_j Z \bar{w}'_j & \text{if } \pm_j = + \\ \bar{w}'_j Z \bar{w}_j & \text{if } \pm_j = - \end{cases}$. Now, by Induction

Hypothesis, for all $i \in \{1, \dots, n\}$, if $\pm_i = +$ then $\bar{w}_i \in \llbracket \varphi_i \rrbracket$ and $\bar{w}_i Z \bar{w}'_i$, therefore $\bar{w}'_i \in \llbracket \varphi_i \rrbracket$ and if $\pm_i = -$ then $\bar{w}_i \notin \llbracket \varphi_i \rrbracket$ and $\bar{w}_i Z \bar{w}'_i$, therefore $\bar{w}'_i \notin \llbracket \varphi_i \rrbracket$. So, in all cases, $\bar{w}'_i \triangleright_i \llbracket \varphi_i \rrbracket$. Moreover, $R_{\star}^{\pm\sigma} \bar{w}'_1 \dots \bar{w}'_n \bar{w}'$. Thus, there are $\bar{w}'_1, \dots, \bar{w}'_n$ such that $\bar{w}'_1 \triangleright_1 \llbracket \varphi_1 \rrbracket$ and \dots and $\bar{w}'_n \triangleright_n \llbracket \varphi_n \rrbracket$ and $R_{\star}^{\pm\sigma} \bar{w}'_1 \dots \bar{w}'_n \bar{w}'$. Hence, $(M', \bar{w}') \models \varphi$.

If \star has skeleton $(\sigma, \pm, \forall, (k, k_1, \dots, k_n), (\pm_1, \dots, \pm_n))$, then assume by contraposition that it is not the case that $(M', \bar{w}') \models \varphi$. Then, there are $\bar{w}'_1, \dots, \bar{w}'_n$ such that not $\bar{w}'_1 \triangleright_1 \llbracket \varphi_1 \rrbracket$ and \dots and not $\bar{w}'_n \triangleright_n \llbracket \varphi_n \rrbracket$ and $-R_{\star}^{\pm\sigma} \bar{w}'_1 \dots \bar{w}'_n \bar{w}'$. Now, $\bar{w}Z\bar{w}'$, so by definition of Z , there are $\bar{w}_1, \dots, \bar{w}_n$ such that $\bar{w}_1 \triangleright_1 \bar{w}'_1$ and \dots and $\bar{w}_n \triangleright_n \bar{w}'_n$ and $-R_{\star}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}$ where

for all $i \in \{1, \dots, n\}$, $\bar{w}_i \triangleright_i \bar{w}'_i \triangleq \begin{cases} \bar{w}_i Z \bar{w}'_i & \text{if } \pm_i = + \\ \bar{w}'_i Z \bar{w}_i & \text{if } \pm_i = - \end{cases}$. Let $i \in \{1, \dots, n\}$. If $\pm_i = +$ then

$\bar{w}'_i \notin \llbracket \varphi_i \rrbracket$ and $\bar{w}_i Z \bar{w}'_i$. Therefore, by Induction Hypothesis, $\bar{w}_i \notin \llbracket \varphi_i \rrbracket$. If $\pm_i = -$, then $\bar{w}'_i \in \llbracket \varphi_i \rrbracket$ and $\bar{w}'_i Z \bar{w}_i$. Therefore, by Induction Hypothesis, $\bar{w}_i \in \llbracket \varphi_i \rrbracket$. So, in all cases, not $\bar{w}_i \triangleright_i \llbracket \varphi_i \rrbracket$. Thus, there are $\bar{w}_1, \dots, \bar{w}_n$ such that not $\bar{w}_1 \triangleright_1 \llbracket \varphi_1 \rrbracket$ and \dots and not $\bar{w}_n \triangleright_n \llbracket \varphi_n \rrbracket$.

and $-R_{\star}^{\pm\sigma}w_1\dots w_nw$. That is, not $(M, \bar{w}) \Vdash \varphi$. So, we have proved that if not $(M', \bar{w}') \Vdash \varphi$ and $\bar{w}Z\bar{w}'$ then not $(M, \bar{w}) \Vdash \varphi$. Hence we obtain the result by contraposition. \square

Proposition 6. *Let C be a set of molecular connectives and let $M_1 = (W_1, \mathcal{R}_1)$ and $M_2 = (W_2, \mathcal{R}_2)$ be two C -models. Let $C_0 \subseteq C$ and for all $c \in C_0$, let D_c be the vertices of the quantified decomposition tree T_c . Let $\{Z_c \mid c \in V'_{c_0}, c_0 \in C_0\}$ be a C_0 -bisimulation between M_1 and M_2 . If $\{M, M'\} = \{M_1, M_2\}$ then for all $c_0 \in C_0$ and all $c \in V'_{c_0}$, for all $\bar{w} \in \bar{w}(M, C)$ and all $\bar{w}' \in \bar{w}(M', C)$, if $\bar{w}Z_c\bar{w}'$ then $(M, \bar{w}) \rightsquigarrow_{cC_0} (M', \bar{w}')$. In particular, if $\bar{w}Z\bar{w}'$ then $(M, \bar{w}) \rightsquigarrow_{C_0} (M', \bar{w}')$.*

Proof: (Here we confuse connectives with their skeletons to ease the presentation of the ideas.) We prove by induction on c that for all vertices c of the decomposition tree of a molecular connective $c_0 \in C^*$, if $\bar{w}Z_c\bar{w}'$ then $(M, \bar{w}) \rightsquigarrow_{cC} (M', \bar{w}')$.

If c is of arity 0 then it follows straightforwardly from condition 1 of Definition 18. Assume now that $\bar{w}Z_c\bar{w}'$ with c neither a propositional letter nor a symbol id_k^i . We prove by induction on $\varphi \in \mathcal{L}_{cC}$ that $(M, \bar{w}) \Vdash \varphi$ implies $(M', \bar{w}') \Vdash \varphi$. Let $\varphi = c(\varphi_1, \dots, \varphi_m)$ and let $c_0(c_1, \dots, c_n)$ be a maximal decomposition of the skeleton of c , where $c_1, \dots, c_n \in C^*$ are of arity k_1, \dots, k_n respectively. Then $\varphi = c(\varphi_1, \dots, \varphi_m) = c_0(c_1(\varphi_1^1, \dots, \varphi_1^{k_1}), \dots, c_n(\varphi_n^1, \dots, \varphi_n^{k_n}))$ where for all $j \in \{1, \dots, n\}$, the formulas $\varphi_j^1, \dots, \varphi_j^{k_j}$ are those $\varphi_1, \dots, \varphi_m$ for which there is a corresponding id_k^i in c_j (the φ_j^i s appear in the same order as their corresponding id_k^i s in c_j).

1. Assume that c_0 is an existential molecular connective of tonicity signature (\pm_1, \dots, \pm_n) . Assume that $(M, \bar{w}) \Vdash c(\varphi_1, \dots, \varphi_m)$. Then, $(M, \bar{w}) \Vdash c_0(c_1(\varphi_1^1, \dots, \varphi_1^{k_1}), \dots, c_n(\varphi_n^1, \dots, \varphi_n^{k_n}))$. Then, by Proposition 3, there are $\bar{w}_1, \dots, \bar{w}_n$ such that $R_{c_0}\bar{w}_1\dots\bar{w}_n\bar{w}$ and $\bar{w}_1 \Vdash_1 \llbracket c_1(\varphi_1^1, \dots, \varphi_1^{k_1}) \rrbracket$ and \dots and $\bar{w}_n \Vdash_n \llbracket c_n(\varphi_n^1, \dots, \varphi_n^{k_n}) \rrbracket$ (1). Then, by definition of Z_c , there are $\bar{w}'_1, \dots, \bar{w}'_n$ such that $R_{c_0}\bar{w}'_1\dots\bar{w}'_n\bar{w}'$ (4) and

- (i) for all $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = +$, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $\bar{w}_i Z_{c_j} \bar{w}'_j$;
- (ii) for all $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = -$, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $\bar{w}'_j Z_{c_j} \bar{w}_i$ (2).

Since $c_1(\varphi_1^1, \dots, \varphi_1^{k_1}) \in \mathcal{L}_{c_1C}, \dots, c_n(\varphi_n^1, \dots, \varphi_n^{k_n}) \in \mathcal{L}_{c_nC}$, by Induction Hypothesis, for all $\bar{w} \in \bar{w}(M, C)$, all $\bar{w}' \in \bar{w}(M', C)$ and all $j \in \llbracket 1; n \rrbracket$, if $(M, \bar{w}_j) \Vdash c_j(\varphi_j^1, \dots, \varphi_j^{k_j})$ and $\bar{w}_j Z_{c_j} \bar{w}'_j$ then $(M', \bar{w}'_j) \Vdash c_j(\varphi_j^1, \dots, \varphi_j^{k_j})$; and if $(M', \bar{w}'_j) \Vdash c_j(\varphi_j^1, \dots, \varphi_j^{k_j})$ and $\bar{w}'_j Z_{c_j} \bar{w}_j$ then $(M, \bar{w}_j) \Vdash c_j(\varphi_j^1, \dots, \varphi_j^{k_j})$ (3).

From (1) and (2), we derive by means of (3) that $\bar{w}'_1 \Vdash_1 \llbracket c_1(\varphi_1^1, \dots, \varphi_1^{k_1}) \rrbracket$ and \dots and $\bar{w}'_n \Vdash_n \llbracket c_n(\varphi_n^1, \dots, \varphi_n^{k_n}) \rrbracket$. Then, with (4), it follows that $(M', \bar{w}') \Vdash c_0(c_1(\varphi_1^1, \dots, \varphi_1^{k_1}), \dots, c_n(\varphi_n^1, \dots, \varphi_n^{k_n}))$. That is, $(M', \bar{w}') \Vdash c(\varphi_1, \dots, \varphi_m)$.

2. Assume that c_0 is a universal molecular connective of tonicity signature (\pm_1, \dots, \pm_n) . By contraposition, assume that it is not the case that $(M', \bar{w}') \Vdash c(\varphi_1, \dots, \varphi_m)$. That is, it is not the case that $(M', \bar{w}') \Vdash c_0(c_1(\varphi_1^1, \dots, \varphi_1^{k_1}), \dots, c_n(\varphi_n^1, \dots, \varphi_n^{k_n}))$. Then, by Proposition 3, there are $\bar{w}'_1, \dots, \bar{w}'_n \in \bar{w}(M, C)$ such that $-R_{c_0}\bar{w}'_1\dots\bar{w}'_n\bar{w}'$ and not $\bar{w}'_1 \Vdash_1 \llbracket c_1(\varphi_1^1, \dots, \varphi_1^{k_1}) \rrbracket$ and \dots and not $\bar{w}'_n \Vdash_n \llbracket c_n(\varphi_n^1, \dots, \varphi_n^{k_n}) \rrbracket$ (1). Now, by definition of Z_c and since $\bar{w}Z_c\bar{w}'$ there are $\bar{w}_1, \dots, \bar{w}_n$ such that $-R_{c_0}\bar{w}_1\dots\bar{w}_n\bar{w}$ (2) and

- (i) for all $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = +$, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $\bar{w}_j Z_{c_j} \bar{w}'_i$;
- (ii) for all $j \in \llbracket 1; n \rrbracket$ such that $\pm_j = -$, there is $i \in \llbracket 1; n \rrbracket$ such that $c_j = c_i$ and $\bar{w}'_i Z_{c_j} \bar{w}_j$ (3).

Like for the previous case, combining (1) and (3) with the Induction Hypothesis, we obtain that it holds that $\text{not } \bar{w}_1 \uparrow_1 \llbracket c_1(\varphi_1^1, \dots, \varphi_1^{k_1}) \rrbracket$ and ... and $\text{not } \bar{w}_n \uparrow_n \llbracket c_n(\varphi_n^1, \dots, \varphi_n^{k_n}) \rrbracket$. Together with (2), we obtain that it is not the case that $(M, \bar{w}) \Vdash c_0(c_1(\varphi_1^1, \dots, \varphi_1^{k_1}), \dots, c_n(\varphi_n^1, \dots, \varphi_n^{k_n}))$. That is, it is not the case that $(M, \bar{w}) \Vdash c(\varphi_1, \dots, \varphi_m)$.

Finally, assume that $\bar{w}Z_c\bar{w}'$ with $c = id_k^i$. Then, since by definition of C-bisimulation, we have that for all $c \in \mathbf{C}$ of output type or type (if it is a propositional letter) k that $\bar{w}Z_c\bar{w}'$, we obtain that $(M, \bar{w}) \Vdash \varphi$ implies $(M', \bar{w}') \Vdash \varphi$ holds for all formulas φ of the form $c(\varphi_1, \dots, \varphi_m)$. This holds not only for connectives c of types or output types k but in fact for any connective c of any (output) type since the antecedent of the implication is false in the case where these (output) types are different from k . So, the result holds for all formulas φ of the language \mathcal{L}_C : $(M, \bar{w}) \rightsquigarrow_C (M', \bar{w}')$. \square

Proposition 7. *Let \mathbf{C} be a set of molecular connectives, let $(M_i, \bar{w}_i)_{i \in I}$ be a family of \mathbf{C} -models, let $\prod_U (M_i, \bar{w}_i)$ be an ultraproduct of $(M_i, \bar{w}_i)_{i \in I}$. Then, for all $\varphi \in \mathcal{L}_C$, $\prod_U (M_i, \bar{w}_i) \Vdash \varphi$ iff $\{i \in I \mid (M_i, \bar{w}_i) \Vdash \varphi\} \in U$.*

Let $(M_i, s_i)_{i \in I}$ be a family of pointed structures and let $\prod_U (M_i, s_i)$ be an ultraproduct of $(M_i, s_i)_{i \in I}$. Then, for all $\varphi \in \mathcal{L}_{\text{FOL}}^{\text{PF}}$, $\prod_U (M_i, s_i) \models \varphi$ iff $\{i \in I \mid (M_i, s_i) \models \varphi\} \in U$.

Proof: We consider the completion of \mathbf{C} under Boolean negation: \mathbf{C} can always be extended to include the Boolean negation of each connective (defined in Definition 9). Now, we prove it for this extended language by induction on φ .

1) The base case $\varphi = p$ holds by definition.

2) a) If $\varphi = \star(\varphi_1, \dots, \varphi_n)$ with $\star = (\sigma, \pm, \exists, (k, k_1, \dots, k_n), (\pm_1, \dots, \pm_n))$, then

$$\prod_U (M_i, \bar{w}_i) \Vdash \star(\varphi_1, \dots, \varphi_n)$$

iff there are $\prod_U \bar{w}_i^1, \dots, \prod_U \bar{w}_i^n$ such that $\prod_U R_{\star}^{\pm\sigma} \prod_U \bar{w}_i^1 \dots \prod_U \bar{w}_i^n \prod_U \bar{w}_i$ and $\prod_U \bar{w}_i^1 \uparrow_1 \llbracket \varphi_1 \rrbracket$ and ... and $\prod_U \bar{w}_i^n \uparrow_n \llbracket \varphi_n \rrbracket$

iff there are $\prod_U \bar{w}_i^1, \dots, \prod_U \bar{w}_i^n$ such that $\{j \in I \mid R_{\star}^{\pm\sigma} \bar{w}_j^1 \dots \bar{w}_j^n \bar{w}_j\} \in U$ and $\{j \in I \mid \bar{w}_j^1 \uparrow_1 \llbracket \varphi_1 \rrbracket\} \in U$ and ... and $\{j \in I \mid \bar{w}_j^n \uparrow_n \llbracket \varphi_n \rrbracket\} \in U$ by Induction Hypothesis

$$\text{iff there are } \prod_U \bar{w}_i^1, \dots, \prod_U \bar{w}_i^n \text{ such that } \{j \in I \mid R_{\star}^{\pm\sigma} \bar{w}_j^1 \dots \bar{w}_j^n \bar{w}_j \text{ and } \bar{w}_j^1 \uparrow_1 \llbracket \varphi_1 \rrbracket \text{ and } \bar{w}_j^n \uparrow_n \llbracket \varphi_n \rrbracket\}$$

$\in U$ by closure under intersection of the filter definition (from left to right) and by closure by superset of the filter definition (from right to left)

iff $\{j \in I \mid \text{there are } \bar{w}_j^1, \dots, \bar{w}_j^n \text{ such that } R_{\star}^{\pm\sigma} \bar{w}_j^1 \dots \bar{w}_j^n \bar{w}_j \text{ and } \bar{w}_j^1 \uparrow_1 \llbracket \varphi_1 \rrbracket \text{ and } \dots \text{ and } \bar{w}_j^n \uparrow_n \llbracket \varphi_n \rrbracket\} \in U$

$$\text{iff } \{j \in I \mid (M_j, \bar{w}_j) \Vdash \star(\varphi_1, \dots, \varphi_n)\} \in U$$

b) If $\varphi = \star(\varphi_1, \dots, \varphi_n)$ with $\star = (\sigma, \pm, \forall, (t, t_1, \dots, t_n), (\pm_1, \dots, \pm_n))$, then

$$\prod_U (M_i, \bar{w}_i) \Vdash \varphi \text{ iff } \{i \in I \mid (M_i, \bar{w}_i) \Vdash \varphi\} \in U$$

$$\text{iff } \left(\text{not } \prod_U (M_i, \bar{w}_i) \Vdash \varphi \right) \text{ iff } \{i \in I \mid (M_i, \bar{w}_i) \Vdash \varphi\} \notin U$$

iff $\prod_U (M_i, \bar{w}_i) \Vdash \neg \varphi$ iff $\{i \in I \mid \text{not } (M_i, \bar{w}_i) \Vdash \varphi\} \in U$ by Proposition 2 and because U is an *ultrafilter* (we recall that $\neg \varphi$ is defined in Definition 9)

iff $\prod_U (M_i, \bar{w}_i) \Vdash \neg \varphi$ iff $\{i \in I \mid (M_i, \bar{w}_i) \Vdash \neg \varphi\} \in U$ by Proposition 2.

This case thus boils down to the case 2) a).

c) If $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi = \varphi_1 \vee \varphi_2$ then the result follows directly from the definition of an ultrafilter.

As for the second part of the proposition, for pointed structures, the proof is similar and follows the main lines of the proof of [9, Theorem 4.1.9]. \square

C Proof of Fact 1

Fact 1. *Let \mathcal{C} be a set of molecular connectives and let $M_1 = (W_1, \mathcal{R}_1)$ and $M_2 = (W_2, \mathcal{R}_2)$ be two \mathcal{C} -models. Let $\{Z_c \mid c \in V'_{c_0}, c_0 \in \mathcal{C}\}$ be a set of relations between M_1 and M_2 . We set $Z \triangleq \bigcup \{Z_{id_k^i} \mid id_k^i \text{ appears in some } c \in \mathcal{C}\}$. Then, $\{Z_c \mid c \in V'_{c_0}, c_0 \in \mathcal{C}\}$ is a \mathcal{C} -bisimulation if, and only if, Z is non-empty and conditions 1, 2, 3 of Definition 18 hold, in which all Z_c and all $Z_{id_k^i}$ for $c \in \mathcal{C}$ and id_k^i appearing in some $c \in \mathcal{C}$ are replaced by Z .*

Proof: For the left to right direction, we use conditions *i* and *ii* of Definition 18 to obtain the result. For the right to left direction, we set for all id_k^i and all $c \in \mathcal{C}$ of output type k that $Z_c = Z_{id_k^i} \triangleq Z \cap (W_1^k \times W_2^k) \cup (W_2^k \times W_1^k)$ and we thus obtain conditions *i* and *ii*. \square