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# Distribution-free mixed GWMA-CUSUM and CUSUM-GWMA Mann-Whitney charts to monitor unknown shifts in the process location

K. Mabude<sup>1</sup>, J.-C. Malela-Majika<sup>1</sup>, P. Castagliola<sup>2</sup> and S.C. Shongwe<sup>1\*</sup>

## Abstract

The Mann-Whitney (MW) test is one of the most important nonparametric tests used in the comparison of the location parameters of two populations. Unlike the  $t$ -test, the MW test can be used when the assumption of normality fails to hold. In this paper, the MW U statistic is used to construct two efficient distribution-free monitoring schemes, namely the mixed generally weighted moving average-cumulative sum (GWMA-CUSUM) MW U scheme (denoted as U-MGC) as well as its reversed version, i.e. the CUSUM-GWMA MW U scheme (denoted as U-MCG). The performances of the proposed schemes are investigated using the average run-length ( $ARL$ ) and average extra quadratic loss ( $AEQL$ ) values through extensive simulations. The newly proposed charts are found to be superior in small shifts detection than their competing (existing and others that are briefly introduced here) distribution-free Shewhart, EWMA, CUSUM, mixed EWMA-CUSUM, mixed CUSUM-EWMA and GWMA MW U charts in many situations. A real-life example is used to demonstrate the design and implementation of the new schemes.

**Keywords:** Asymptotic control limits; Distribution-free; Exact control limits; Mann-Whitney U statistic; Mixed GWMA-CUSUM scheme; Mixed CUSUM-GWMA scheme; Phase I; Phase II.

## 1. Introduction

Shewhart control charts (or monitoring schemes) are well-known and appreciated because of their simplicity and quick detection of large shifts in the process parameters. However, they are relatively slow in detecting small and moderate shifts in the process. To solve this problem, alternative monitoring schemes such as the cumulative sum (CUSUM) and the exponentially weighted moving average (EWMA) are recommended (see Page, 1954; Roberts, 1959). The CUSUM and EWMA monitoring schemes are efficient in detecting small shifts in the process parameters. Many authors have contributed to the improvement of these monitoring schemes; see for example, Lucas (1982), Shamma and Shamma (1992), Abbas et al. (2013) and Zaman et al. (2015). Lucas (1982) proposed a composite Shewhart-CUSUM (CSCUSUM) monitoring scheme that combines the strengths of both Shewhart and CUSUM schemes. To improve the sensitivity of the EWMA scheme toward large shifts, Lucas and Saccucci (1990) designed a

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composite Shewhart-EWMA (CSEWMA) monitoring scheme. The Lucas (1982)'s and Lucas and Saccucci (1990)'s schemes were observed to perform better for both small and large mean shifts. Capizzi and Masarotto (2010) developed a CSEWMA scheme when the process parameters are estimated. Shamma and Shamma (1992) extended the EWMA monitoring scheme to a double EWMA (DEWMA) scheme. Zhang and Chen (2005) compared the zero-state performance of the EWMA and DEWMA schemes designed using the same smoothing parameters. Mahmoud and Woodall (2010) compared the performance of the EWMA and DEWMA schemes in terms of the zero-state and worst-case average run-length (ARL).

In order to increase the sensitivity of the EWMA and CUSUM schemes in detecting small shifts in the process mean, Abbas et al. (2013) proposed a mixed EWMA-CUSUM (denoted as MEC) scheme which is a combination of the EWMA and CUSUM schemes where the CUSUM statistic is used as input in the EWMA scheme. Later on, Zaman et al. (2015) developed the reversed version of Abbas et al. (2013)'s scheme, the mixed CUSUM-EWMA (denoted as MCE) scheme where the EWMA statistic is used as input in the CUSUM scheme. They showed that the MEC and MCE schemes outperform the CUSUM and EWMA monitoring schemes in detecting small shifts. Moreover, Osei-Aning et al. (2017) confirmed Abbas et al. (2013)'s and Zaman et al. (2015)'s findings using autocorrelated observations for a first order autoregressive process, respectively. Abbasi et al. (2018) also proposed a nonparametric MEC scheme based on the arc sign transformation to efficiently detect moderate to large shifts. Malela-Majika and Rapoo (2017) proposed the MCE and MEC monitoring schemes based on the Wilcoxon rank-sum statistic (WRS), denoted as W-MCE and W-MEC schemes, for monitoring the process location. To improve the performance of the EWMA monitoring scheme, Sheu and Lin (2003) proposed an extension of the EWMA scheme called the generally weighted moving average (GWMA) monitoring scheme and showed that the GWMA scheme performs better than the EWMA scheme in detecting small shifts in the process mean. Mabude et al. (2020) proposed the GWMA scheme based on the WRS and showed that the GWMA WRS scheme outperforms the EWMA WRS scheme by Li et al. (2010) in many situations.

In an effort to increase the detection ability of the parametric GWMA and CUSUM schemes in monitoring small shifts, Lu (2017) proposed the mixed GWMA-CUSUM scheme and its reversed version, i.e. CUSUM-GWMA scheme for individual observations. Lu (2017) and Ali and Haq (2018a) were the first to propose the GWMA-CUSUM monitoring scheme for monitoring samples of size  $n \geq 1$  (including the case of individual observations). However, Lu (2017) is the first to introduce the CUSUM-GWMA scheme for individual observations. Next,

Ali and Haq (2018b) designed GWMA-CUSUM schemes for monitoring the process dispersion for subgroup observations. They showed that the GWMA-CUSUM schemes outperform the CUSUM, EWMA and EWMA-CUSUM in detecting very small shifts. More recently, Huang et al. (2020) studied the GWMA-CUSUM scheme and its reversed version for individual observations. It should be noted that the use of control charts for individual observations (instead of sub-grouped observations) is a common situation in industrial applications as soon as the cost in sampling and/or inspection is high, or if the sampling itself implies a destructive testing. The aforementioned GWMA-CUSUM and CUSUM-GWMA monitoring schemes are parametric ones since they are designed under the assumption of normality or under a specific distribution. When the underlying process distribution departs from normality or when it is different from the specified distribution, the performance of parametric schemes under both small and large shifts is known to degrade considerably. To overcome this problem, nonparametric test statistics can be used to design robust and more efficient monitoring schemes. For some recent contributions on nonparametric monitoring schemes, see Chen et al. (2020), Shongwe (2020), Wu et al. (2020), Chong et al. (2020), Malela-Majika (2020), Song et al. (2020), Alevizakos et al. (2020), Aslam et al. (2020), Shafqat et al. (2020) and finally, the chapter-contributed book edited by Koutras and Triantafyllou (2020).

So far, mixed GWMA-CUSUM and CUSUM-GWMA monitoring schemes designed for any kind of continuous distributions (and not only limited to the normal distribution) are currently unavailable in statistical process monitoring (SPM) literature. Therefore, the design of robust monitoring schemes that are able to monitor various types of quality processes without any distributional assumption are very important and needed in the SPM literature. Because the MW test is one of the most powerful nonparametric tests, in this paper, we develop the mixed GWMA-CUSUM (MGC) monitoring scheme and its reversed version, i.e. the mixed CUSUM-GWMA (MCG) monitoring scheme based on the MW U statistic. These schemes are denoted as U-MGC and U-MCG schemes, respectively.

The rest of this paper is organised as follows: in Section 2, we present the characteristics of the proposed monitoring schemes using the MW U statistic. Section 3 investigates the in-control (IC) and out-of-control (OOC) run-length profiles as well as the effect of the Phase I and/or Phase II sample size on the performance of the proposed schemes. Moreover, the proposed schemes are compared to some well-known monitoring schemes under various distributions. An illustrative example is given in Section 4 to demonstrate the application of the proposed schemes. A summary and recommendations are provided in Section 5.

## 2. Design of the proposed control charts

### 2.1 The MW U statistic

Assume that  $X = \{x_i, i = 1, 2, 3, \dots, m\}$  represents the IC Phase I (or reference) sample with unknown continuous cumulative density function (c.d.f.)  $F_X(x)$  and  $Y = \{y_{tj}, j = 1, 2, \dots, n; t = 1, 2, 3, \dots\}$  represents the Phase II (or test) sample with c.d.f.  $F_Y(y)$ . The test samples at time  $t$  ( $t = 1, 2, 3, \dots$ ) are assumed to be independent and identically distributed (i.i.d.), from each other and, from the reference sample. Let us assume that  $F_Y(y) = F_X(y - \delta)$ , where  $\delta$  is the shift in the location. The process is considered as IC if  $\delta = 0$ , i.e. if  $F_Y(y) = F_X(y)$ .

The MW U statistic at time  $t = 1, 2, 3, \dots$  (denoted as  $U_t$  statistic) represents the total number of pairs  $(x_i, y_{tj})$  for which the  $y_{tj}$  (Phase II sample) are strictly greater than the  $x_i$  (Phase I sample), i.e.

$$U_t = \sum_{i=1}^m \sum_{j=1}^n I(d_{tij}), \quad (1)$$

where  $t = 1, 2, 3, \dots$  and  $I(d_{tij})$  is an indicator function defined as follows:

$$d_{tij} = y_{tj} - x_i \quad (2)$$

and

$$I(d_{tij}) = \begin{cases} 1 & \text{if } d_{tij} > 0 \\ 0 & \text{if } d_{tij} \leq 0 \end{cases} \quad (3)$$

Note that there are  $mn$  pairs of  $(x_i, y_{tj})$  for each Phase II sample; therefore,  $0 \leq U_t \leq mn$ . For the two most extreme orderings every  $x_i$  precedes every  $y_{tj}$  (so that  $U_t = 0$ ) and every  $y_{tj}$  precedes every  $x_i$  (so that  $U_t = mn$ ), respectively. Assuming that no tie is observed, the mean  $\mu_{U_t}$  and variance  $\sigma_{U_t}^2$  of the statistic  $U_t$  are known to be

$$\mu_{U_t} = \frac{mn}{2} \quad (4)$$

and

$$\sigma_{U_t}^2 = \frac{mn(m+n+1)}{12},$$

respectively. To keep the notations simple, in the rest of the manuscript  $\mu_{U_t}$  and  $\sigma_{U_t}^2$  will be denoted as  $\mu_U$  and  $\sigma_U^2$ , respectively. For more details on the MW monitoring scheme, readers are referred to Chakraborti and Van de Wiel (2008) and Malela-Majika et al (2016).

### 2.2 The CUSUM MW U scheme

Page (1954) introduced the CUSUM monitoring scheme for a quick detection of small and moderate shifts in the process mean. The plotting statistics of this chart consider an equal weighted combination of the current and past observations. In this section, we introduce a similar plotting statistic using the MW U statistic. Thus, the plotting statistics of the CUSUM chart based on the MW U statistic are given by

$$C_t^+ = \max[0, (U_t - \mu_U - K_C) + C_{t-1}^+]$$

and

$$C_t^- = \max[0, (\mu_U - K_C - U_t) + C_{t-1}^-]$$

(5)

for  $t = 1, 2, 3, \dots$ , where  $K_C = k_C \sigma_U$  is the reference parameter (or coefficient) of the CUSUM chart. The starting values  $C_0^+$  and  $C_0^-$  are typically initialized to zero. The charting statistics are plotted against the upper control limit  $H_C > 0$ , with  $H_C = h_C \sigma_U$ , where  $h_C > 0$  is the control limit coefficient of the CUSUM scheme. The process is considered to be OOC if the charting statistic (i.e.  $C_t^+$  or  $C_t^-$ ) falls on or above the control limit  $H_C$  for any value of  $t$ , that is  $C_t^+ \geq H_C$  (or  $C_t^- \geq H_C$ ), otherwise, the process is considered to be IC. Note that in this paper, the CUSUM monitoring scheme using the MW U statistic is denoted as U-CUSUM chart.

### 2.3 The GWMA MW U scheme

Following Sheu and Lin (2003)'s idea, the charting statistics of the U-GWMA scheme, denoted as  $G_t$ , is given by

$$G_t = \sum_{i=1}^t (q^{(i-1)\alpha} - q^{i\alpha}) U_{t-i+1} + q^{t\alpha} U_0, \quad t = 1, 2, 3, \dots \quad (6)$$

where  $q \in [0, 1)$  and  $\alpha > 0$  are two parameters to be fixed. The starting value of the  $U$  statistic at  $t = 0$  is considered to be equal to  $\mu_U$  (i.e.  $U_0 = \mu_U$ ). For more details on the GWMA plotting statistic, readers are referred to Sheu and Lin (2003).

In particular, the expected value of (6) is given by

$$E(G_t) = \mu_{G_t} = E \left[ \sum_{i=1}^t (q^{(i-1)\alpha} - q^{i\alpha}) U_{t-i+1} + q^{t\alpha} \mu_U \right] = \mu_U, \quad (7)$$

and the variance of (6) is then defined by

$$Var(G_t) = \sigma_{G_t}^2 = \sigma_U^2 Q_t, \quad (8)$$

where

$$Q_t = \sum_{i=1}^t (q^{(i-1)\alpha} - q^{i\alpha})^2.$$

Therefore, the time varying control limits of the U-GWMA monitoring scheme can be calculated as

$$UCL_{G_t} / LCL_{G_t} = \mu_{G_t} \pm L\sigma_U\sqrt{Q_t} \quad (9)$$

where  $L$  ( $L > 0$ ) is the coefficient representing the distance from the center line to the control limits. This coefficient is used to fix the predefined nominal IC  $ARL$  ( $ARL_0$ ) value.

Note that when  $\alpha = 1$ , it can be shown that

$$Var(G_t) = \sigma_{G_t}^2 = (1 - q) \frac{1 - q^{2t}}{1 + q} \sigma_U^2. \quad (10)$$

Therefore, when the process has been running for a long time, that is, when  $t \rightarrow \infty$ , the variance of the U-GWMA scheme is given by

$$Var(G_t) = \sigma_G^2 = mn \left( \frac{m + n + 1}{12} \right) \left( \frac{1 - q}{1 + q} \right). \quad (11)$$

In Equation (11), when  $1 - q = \lambda$  (i.e.,  $q = 1 - \lambda$ ), the U-GWMA scheme is equivalent to the U-EWMA monitoring scheme. The U-GWMA scheme gives a signal at the sampling time  $t$  if the U-GWMA scheme plotting statistic,  $G_t$ , plots outside of the control limits defined in Equation (9).

#### 2.4 Design of the mixed GWMA-CUSUM MW U monitoring scheme

The U-MGC scheme is constructed by combining the GWMA and CUSUM charts using the MW U statistic. That is, the charting statistic of the U-GWMA scheme in Equation (6) is used as input of the charting statistics of the U-CUSUM scheme in Equation (5). Hence, at the sampling time  $t$ , the charting statistics of the proposed U-MGC monitoring scheme (denoted as  $MGC_{U_t}^+$  and  $MGC_{U_t}^-$ ) are then defined by

$$MGC_{U_t}^+ = \max[0, (G_t - \mu_U - K_t) + MGC_{U_{t-1}}^+] \quad (12)$$

and

$$MGC_{U_t}^- = \max[0, (\mu_U - K_t - G_t) + MGC_{U_{t-1}}^-]$$

where  $K_t$  is a time-varying reference value. The starting values  $MGC_{U_0}^+$  and  $MGC_{U_0}^-$  are typically taken to be equal to 0 and  $K_t$  is given by

$$K_t = k (Var(G_t))^{\frac{1}{2}} = k \sigma_U \sqrt{Q_t}, \quad (13)$$

where  $k$  is the reference parameter (or coefficient) of the GWMA scheme.

When  $\alpha = 1$ , Equation (13) simplifies to

$$K_t = k \sigma_U \left( (1 - q) \frac{1 - q^{2t}}{1 + q} \right)^{\frac{1}{2}}. \quad (14)$$

Therefore, the exact (or time varying) upper control limit (i.e. Case E) is given by

$$H_t = h(\text{Var}(G_t))^{\frac{1}{2}} = h \sigma_U \sqrt{Q_t}, \quad (15)$$

where  $h > 0$  is the control limit coefficient used to fix the predefined nominal IC  $ARL_0$ .

When  $\alpha = 1$ , as  $t \rightarrow \infty$ , the exact and asymptotic (i.e. Case E and Case A) upper control limits of the proposed U-MGC scheme are given by

$$H_t = h\sigma_U \left( (1-q) \frac{1-q^{2t}}{1+q} \right)^{\frac{1}{2}} \quad (16)$$

and

$$H = h\sigma_U \left( \frac{1-q}{1+q} \right)^{\frac{1}{2}}, \quad (17)$$

respectively. The charting statistic of the U-MGC scheme are plotted against the upper control limit  $H_t > 0$  (or  $H > 0$ ) in Case E (or Case A) and the process is said to be OOC if the charting statistic  $MGC_{U_t}$  (i.e.  $MGC_{U_t}^+$  or  $MGC_{U_t}^-$ ) falls on or above the upper control limit for any value of  $t$ , that is,  $MGC_{U_t} \geq H_t$  (or  $MGC_{U_t} \geq H$ ).

## 2.5 Design of the mixed CUSUM-GWMA MW U chart

Unlike the U-MGC chart presented in the previous sub-section, the U-MCG chart is constructed by combining the CUSUM and GWMA charts using the MW U statistic. That is, the charting statistic of the U-CUSUM chart in Equation (5) is used as input of the charting statistic of the U-GWMA chart in Equation (6). Hence, the two charting statistics of the proposed U-MCG chart are defined by

$$MCG_{U_t}^+ = \sum_{i=1}^t (q^{(i-1)\alpha} - q^{i\alpha}) C_{t-i+1}^+ + q^{t\alpha} MCG_{U_0}^+, \quad t = 1, 2, 3, \dots$$

and

$$MCG_{U_t}^- = \sum_{i=1}^t (q^{(i-1)\alpha} - q^{i\alpha}) C_{t-i+1}^- + q^{t\alpha} MCG_{U_0}^-, \quad t = 1, 2, 3, \dots \quad (18)$$

where the initial (or starting) values of  $MCG_{U_t}^+$  and  $MCG_{U_t}^-$  charting statistics are equal to  $\mu_U$  as defined in Equation (7), respectively; that is,  $MCG_{U_0}^+ = MCG_{U_0}^- = \mu_{C_t}$ . The mean and variance of the  $MCG_{U_t}^+$  and  $MCG_{U_t}^-$  charting statistics are given by

$$E(MCG_{U_t}^+) = E(MCG_{U_t}^-) = \mu_{C_t}$$

and

$$\text{Var}(MCG_{U_t}^+) = \text{Var}(MCG_{U_t}^-) = Q_t \sigma_{C_t}^2. \quad (19)$$



The control limits of the proposed U-MCG chart are

$$UCL/LCL = \mu_{C_t} \pm L \cdot \sigma_{C_t} \sqrt{Q_t}, \quad (20)$$

where  $L$  is the width coefficient and  $\mu_{C_t}$  and  $\sigma_{C_t}$  are the time-varying mean and variance of the charting statistics of the U-CUSUM scheme in the IC situation up to specific time  $t$  and when  $t \rightarrow \infty$ , they both become constant.

The U-MCG chart is constructed by plotting the charting statistics  $MCG_{U_t}^+$  and  $MCG_{U_t}^-$  against the sample number or sampling time  $t$ . The process is said to be OOC if the charting statistic  $MCG_{U_t}$  (i.e.  $MCG_{U_t}^+$  or  $MCG_{U_t}^-$ ) falls beyond the control limits defined in Equation (20), that is, for any value of  $t$ ,  $MCG_{U_t} \geq UCL$  (or  $MCG_{U_t} \leq LCL$ ).

### 3. Performance study of the proposed control charts

#### 3.1 Performance measures

The performance of a monitoring scheme for specific shifts in the process parameter is usually measured using the average run-length ( $ARL$ ) values. The  $ARL$  is defined as the number of rational subgroups to be plotted before the control chart signals for the first time. In practice, to avoid many false alarms, one must ensure that when the process is IC (i.e.  $\delta = 0$ , where  $\delta$  is the shift in the location as defined in sub-section 2.1), the IC  $ARL$  (denoted as  $ARL_0$ ) value is set to be equal (or close) to some high desired values such as 370 and 500. When the process is OOC (i.e.  $\delta \neq 0$ ), small OOC  $ARL$  ( $ARL_1$ ) values reveal that the chart performs better for that specific shifts. When the actual shift delta is unknown and the only available information is an interval  $[\delta_{min}, \delta_{max}]$  in which this shift is likely to occur, the literature recommends the use of some overall performance measure (see for example, Ou et al. (2012), Sanusi et al. (2017) and Malela-Majika and Rapoo (2017)) like, for instance, the average extra quadratic loss ( $AEQL$ ) defined as

$$AEQL = \frac{1}{\Delta} \sum_{\delta_{min}}^{\delta_{max}} \delta^2 \times ARL(\delta), \quad (21)$$

where  $\Delta$  is the number of increment between  $\delta_{min}$  and  $\delta_{max}$ . The  $AEQL$  metric is recommended when the magnitude of the shift is more important (i.e. the quality practitioner is interested in shifts according to their magnitude). When comparing several schemes or sets of parameters, the one that yields the minimum  $AEQL$  value is considered to be the best. In other words, the smaller the  $AEQL$  value, the more efficient the chart is in detecting shifts in the process parameter.

## 3.2 Performance analysis

### 3.2.1 Determination of the optimal design parameters

A short description on how the optimal values are obtained for the control limits coefficients (i.e.  $h$  and  $L$ ) of the U-MGC and U-MCG schemes that are used to fix a specified nominal  $ARL_0$  value are provided in this section. In this paper,  $h$  and  $L$  are determined as follows: (i) the values of  $h$  (or  $L$ ) that yield attained  $ARL_0$  values as close as possible to the nominal  $ARL_0$  value of 500 are computed using the  $N(0,1)$  distribution for Cases A and E; (ii) the OOC  $ARL$ s and the corresponding  $AEQL$  values are computed separately for Case A and for Case E. The  $h$  (or  $L$ ) that provides the minimum value of the  $AEQL$  is considered to be the optimal design parameter. This is also checked under other probability distributions to make sure that the attained  $ARL_0$  remains closer to the nominal value of 500.

### 3.2.2 IC design and robustness of the proposed MCG and MGC monitoring schemes

The IC robustness is one of the most important keys in the design and implementation of monitoring schemes. If the IC characteristics of a monitoring scheme, such as the IC average and median of the run-length ( $ARL_0$ ,  $MRL_0$ , etc.) are the same (or almost the same) across all continuous distributions, the scheme under consideration is said to be IC robust. To demonstrate the IC robustness of the proposed monitoring schemes, a Monte Carlo simulation was conducted to compute the IC characteristics of the proposed monitoring schemes for some symmetrical and skewed distributions. The following distributions are considered in this paper:

- (i) Standard normal distribution, denoted as  $N(0,1)$ , to study the effect of symmetrical distributions,
- (ii) Gamma distribution with the parameter  $\omega = 3$  and  $\beta = 1$ , denoted as  $GAM(3, 1)$ , to study the effect of skewed distributions,
- (iii) Student's  $t$  distribution with degrees of freedom  $\nu = 3$ , denoted as  $t(3)$ , to study the effect of heavy-tailed distributions.

The above distributions have been transformed to have a unit variance and a mean equal to 0 for a fair comparison of the U-MGC and U-MCG schemes under different probability distributions.

**Table 1:** Attained  $ARL_0$  values and optimal parameters of the U-MGC and U-MCG monitoring schemes for a nominal  $ARL_0$  of 500 when  $k \in \{0.1, 0.5\}$ ,  $\alpha = 1$ ,  $q \in \{0.1, 0.2, \dots, 0.8\}$  and  $(m, n) = (100, 5)$  under different distributions

Case	Chart	$k$	Distribution	$q$							
				0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Case A	U-MGC	0.1	$N(0,1)$	501.55	500.72	499.73	500.30	502.68	502.41	498.77	499.59
			$GAM(3,1)$	511.05	509.61	513.42	509.46	509.23	487.33	513.24	501.54
			$t(3)$	502.30	493.47	501.25	510.11	503.47	499.40	507.82	512.64
			$h$	22.868	26.719	31.367	36.684	44.589	55.194	69.497	94.289
		0.5	$N(0,1)$	500.56	499.18	500.24	505.73	500.55	499.81	501.76	506.84
			$GAM(3,1)$	498.77	507.06	513.54	501.47	508.26	500.06	510.11	509.55
	$t(3)$		509.34	496.44	504.00	498.99	503.21	502.01	504.20	501.49	
	$h$		6.199	7.332	8.733	10.524	12.922	16.394	21.672	31.454	
	U-MCG	0.1	$N(0,1)$	500.52	501.59	500.05	502.11	501.76	498.01	509.48	499.05
			$GAM(3,1)$	494.22	490.27	489.26	512.65	498.30	509.01	494.35	514.81
			$t(3)$	493.05	500.24	499.64	510.12	491.20	495.92	508.22	497.24
			$L$	21.384	23.634	26.431	29.362	32.582	37.162	44.313	54.313
0.5		$N(0,1)$	499.02	502.20	500.61	500.10	503.41	501.37	501.35	504.53	
		$GAM(3,1)$	494.29	496.54	498.20	500.73	501.81	510.03	509.79	503.53	
	$t(3)$	509.07	509.83	509.82	512.67	503.25	500.15	498.29	496.42		
	$L$	5.740	6.234	6.784	7.384	8.088	8.898	9.8283	11.029		
Case E	U-MGC	0.1	$N(0,1)$	502.19	500.96	505.90	501.85	500.26	500.71	503.10	501.86
			$GAM(3,1)$	521.47	510.49	502.54	492.97	506.48	504.49	501.02	505.33
			$t(3)$	493.23	507.81	501.29	493.81	500.12	497.08	496.28	506.18
			$h$	22.909	26.931	31.673	36.869	44.362	55.340	70.240	95.241
		0.5	$N(0,1)$	500.48	500.99	499.19	498.95	508.45	500.54	502.21	504.73
			$GAM(3,1)$	491.32	500.05	495.97	489.44	491.09	498.23	509.91	509.90
	$t(3)$		501.77	490.00	516.01	483.76	520.18	506.92	493.82	496.20	
	$h$		6.203	7.298	8.755	10.505	12.981	16.491	21.631	31.592	
	U-MCG	0.1	$N(0,1)$	500.06	498.81	499.58	510.67	504.65	509.79	501.92	502.10
			$GAM(3,1)$	504.71	507.03	508.08	493.52	510.50	509.50	506.76	511.01
			$t(3)$	504.39	521.43	498.09	498.91	506.11	517.94	495.47	518.89
			$L$	19.484	19.478	19.315	19.163	19.092	18.931	18.420	18.001
0.5		$N(0,1)$	500.59	499.76	500.94	502.84	505.76	504.57	505.96	504.44	
		$GAM(3,1)$	511.02	500.25	493.41	509.73	501.30	495.31	500.42	510.62	
	$t(3)$	509.51	496.24	516.14	494.61	514.44	489.48	512.07	509.29		
	$L$	5.210	5.060	4.952	4.831	4.651	4.418	4.161	3.6933		

**Table 2:** Attained  $ARL_0$  values and optimal parameters of the U-MGC and U-MCG monitoring schemes for a nominal  $ARL_0$  of 500 when  $k \in \{0.1, 0.5\}$ ,  $q \in \{0.1, 0.5, 0.7\}$ ,  $\alpha \in \{0.1, 0.8, 1.5, 2.5\}$  and  $(m, n) = (100, 5)$  under different distributions

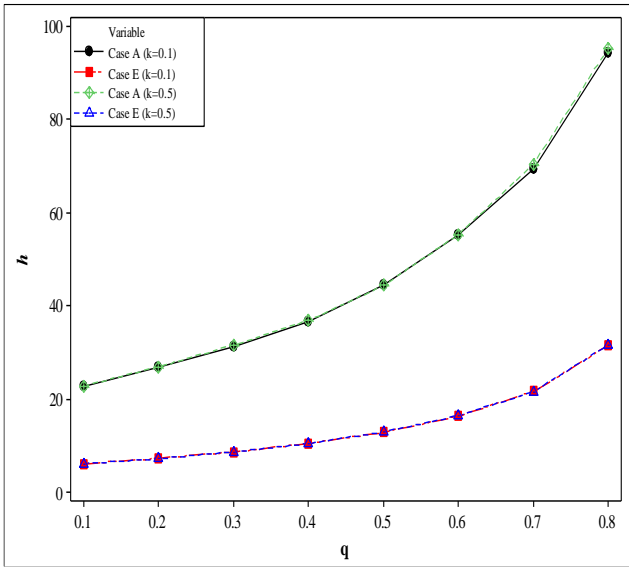
Case	Chart	$k$	$q$	0.1				0.5				0.7			
			$\alpha$	0.1	0.8	1.5	2.5	0.1	0.8	1.5	2.5	0.1	0.8	1.5	2.5
Case A	U-MGC	0.1	$N(0,1)$	500.45	499.54	502.48	501.39	500.78	498.51	500.76	502.91	501.00	500.43	500.11	506.38
			$GAM(3,1)$	511.33	509.36	505.53	505.19	503.20	507.59	504.16	510.82	502.04	498.32	499.77	504.60
			$t(3)$	504.01	501.44	499.29	503.32	504.18	502.44	501.24	501.55	505.39	504.21	503.01	501.11
			$h$	21.493	22.718	22.849	22.826	28.256	46.794	40.084	34.337	31.343	76.963	54.601	37.676
		0.5	$N(0,1)$	502.04	502.04	499.81	498.78	504.04	501.99	502.22	500.92	501.9	500.1	500.6	501.55
			$GAM(3,1)$	510.93	504.34	500.43	502.44	503.18	504.27	511.35	503.41	501.74	502.31	504.06	503.12
	$t(3)$		501.33	501.96	503.46	505.46	500.23	500.33	501.67	500.04	503.11	501.64	500.46	501.66	
	$h$		5.632	6.226	6.229	6.223	6.616	13.429	11.648	9.787	7.029	24.045	16.378	10.848	
	U-MCG	0.1	$N(0,1)$	501.54	502.05	500.16	501.16	499.51	502.18	497.88	498.11	501.51	500.31	501.44	500.25
			$GAM(3,1)$	506.23	500.01	504.45	503.66	500.95	510.06	501.39	505.18	500.46	508.22	502.26	501.05
			$t(3)$	500.85	503.88	501.49	502.23	502.51	504.05	501.77	502.22	509.34	501.56	500.49	503.21
			$L$	20.886	21.462	21.496	21.539	26.836	34.036	30.386	27.726	32.936	47.053	37.374	29.289
0.5		$N(0,1)$	501.68	499.98	500.16	498.33	503.56	501.29	500.1	506.04	500.52	502.03	498.84	503.06	
		$GAM(3,1)$	503.26	502.33	502.05	505.18	501.2	499.26	501.2	501.46	507.29	512.07	503.15	497.46	
	$t(3)$	505.54	500.16	504.11	500.46	502.42	500.19	503.34	500.79	501.65	502	500.39	502.69		
	$L$	5.534	5.721	5.736	5.742	8.127	8.079	7.709	7.186	12.897	9.857	9.031	7.476		
Case E	U-MGC	0.1	$N(0,1)$	499.42	498.90	501.51	507.35	495.90	503.80	495.92	502.03	501.75	501.48	506.48	508.28
			$GAM(3,1)$	500.95	507.08	500.23	512.73	510.13	504.79	495.41	500.99	514.41	509.26	512.70	497.24
			$t(3)$	509.03	501.34	502.19	499.65	492.73	501.07	500.55	504.42	492.89	494.73	512.12	496.48
			$h$	21.349	22.728	22.890	23.032	28.252	47.060	39.871	34.674	31.203	78.034	54.604	37.501
		0.5	$N(0,1)$	500.51	501.47	499.87	494.86	500.16	502.77	498.22	498.40	503.54	515.71	498.07	508.03
			$GAM(3,1)$	496.60	503.44	491.19	500.57	508.17	500.08	494.64	501.38	499.14	498.28	501.17	493.94
	$t(3)$		506.77	494.06	501.45	511.04	501.24	507.10	501.14	499.13	510.41	499.20	506.56	496.05	
	$h$		5.626	6.206	6.213	6.216	6.642	13.367	11.515	9.780	7.103	23.826	16.503	10.755	
	U-MCG	0.1	$N(0,1)$	509.35	503.24	509.75	502.37	498.94	499.06	500.74	503.89	496.12	511.32	506.48	508.28
			$GAM(3,1)$	503.76	498.30	494.24	510.24	491.01	493.45	501.94	504.24	504.72	503.68	514.10	498.79
			$t(3)$	516.42	493.75	499.96	490.53	496.31	491.46	498.72	509.28	511.44	497.77	499.03	493.93
			$L$	20.876	21.359	21.532	21.542	26.856	34.040	30.397	27.754	32.952	47.067	37.389	29.298
0.5		$N(0,1)$	502.12	499.77	500.43	499.78	497.22	500.21	501.63	500.33	509.09	503.70	500.35	490.07	
		$GAM(3,1)$	501.55	497.07	500.49	506.43	498.61	503.97	500.92	499.81	518.19	503.85	499.01	493.61	
	$t(3)$	499.32	505.01	502.20	510.19	498.64	499.72	505.37	503.32	494.66	504.11	512.92	505.08		
	$L$	5.589	5.803	5.832	5.846	8.207	8.122	7.949	7.287	13.112	9.974	9.152	7.606		

Tables 1 and 2 display the actual  $ARL_0$  values and design parameters of the proposed U-MCG and U-MGC schemes when  $(m, n) = (100, 5)$  for a nominal  $ARL_0$  of 500. For instance, when  $\alpha = 1$ ,  $q = 0.5$  and  $k = 0.1$ , then it is found that the design parameter  $h = 44.589$  is such that the U-MGC scheme in Case A yields  $ARL_0$  values of 502.68, 509.23 and 503.47 under the  $N(0,1)$ ,  $GAM(3,1)$  and  $t(3)$  distributions, respectively. It is important to note that when  $\alpha = 1$  (with  $q = 1 - \lambda$ ), then the GWMA scheme reduces to the EWMA scheme; thus, this implies that the results in Table 1 correspond to the mixed EWMA-CUSUM and mixed CUSUM-EWMA MW U (denoted as U-MEC and U-MCE) schemes. Stated differently, when  $\alpha = 1$  (with  $q = 1 - \lambda$ ), then the U-MGC scheme reduces to the U-MEC scheme and similarly, the U-MCG scheme reduces to the U-MCE scheme. The results in Tables 1 and 2 show that for both Cases A and E, the  $ARL_0$  values of the U-MGC and U-MCG schemes are close enough to the nominal value of 500 across all the distributions considered in this study. These findings confirm that the U-MGC and U-MCG schemes are IC robust.

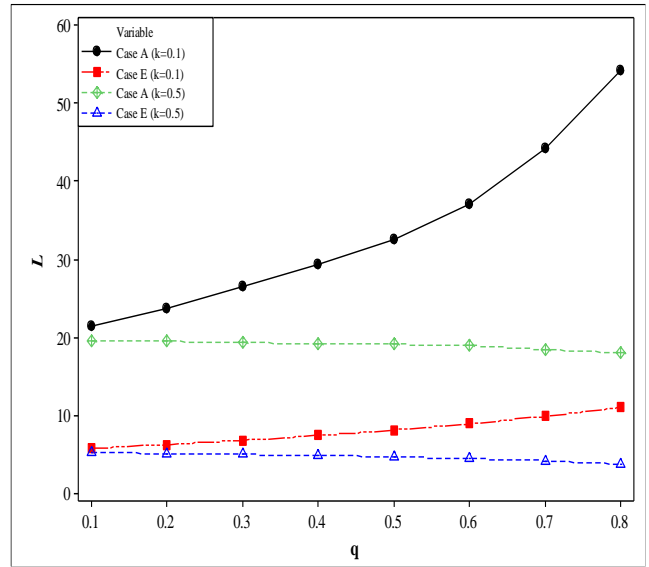
Moreover, the findings in Tables 1 and 2 as well as in Figures 1 and 2 can be summarized as follows:

- Regardless of Case A or E, the control limit coefficient (i.e.  $h$ ) is an increasing function of  $q$ ; i.e. the larger (smaller) the value of  $q$ , the wider (narrower) the control limit of the U-MGC scheme (see Figure 1(a)).
- For small reference constants (i.e.  $k$  values), the  $h$  values increases rapidly for both Cases E and A. However, for large  $k$  values,  $h$  increases at a small rate for both Cases E and A. Moreover, the smaller (larger) the magnitude of  $k$ , the wider (narrower) the control limit of the U-MGC scheme.
- The Case A's  $h$  values of the U-MGC scheme are slightly different from the Case E's  $h$  values (see Figure 1(a)).
- For the U-MCG scheme, in Case A, the control limit constant (i.e.  $L$ ) is an increasing function of  $q$ . However, in Case E,  $L$  is a decreasing function of  $q$  at a small rate (see Figure 1(b)). Note that in Case A, for small  $k$  values,  $L$  increases at a high rate (see Figure 1(b)).
- For both Case A and Case U, regardless of the value of  $\alpha$ , the smaller (larger) the magnitude of  $k$ , the wider (narrower) the control limit (see Figure 2). Moreover, when  $k$  is kept fixed the larger (smaller) the value of  $q$ , the wider (narrower) the control limit of the U-MGC scheme (see Figures 2(a)-(d)).

- A thorough examination of Figure 2 shows that for very large values of  $\alpha$ , the control limits converge toward a unique value regardless of the value  $q$ .

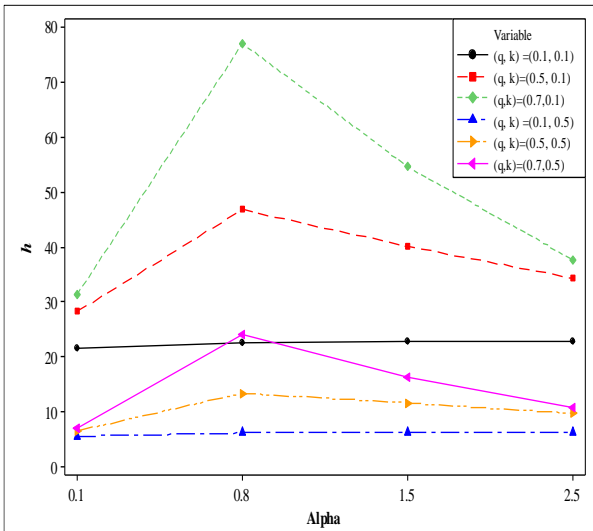


(a) U-MGC scheme

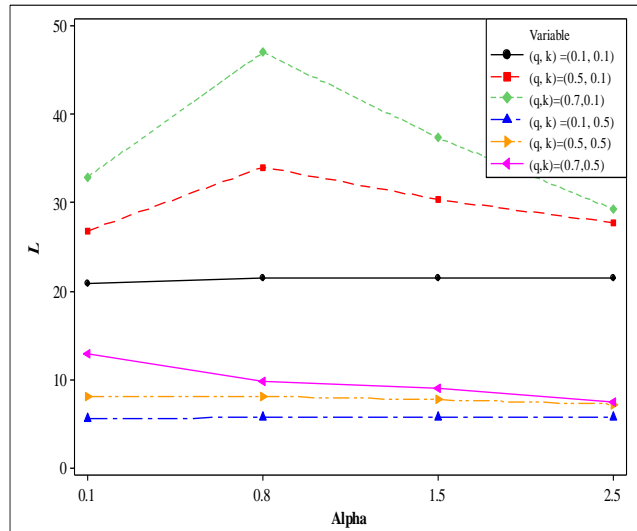


(b) U-MCG scheme

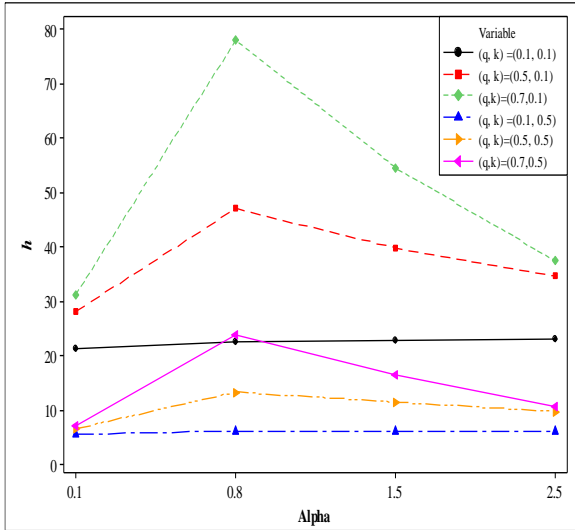
**Figure 1.** Optimal parameters of the proposed monitoring schemes when  $\alpha = 1$ ,  $k \in \{0.1, 0.5\}$  and  $(m, n) = (100, 5)$  for a nominal  $ARL_0$  of 500



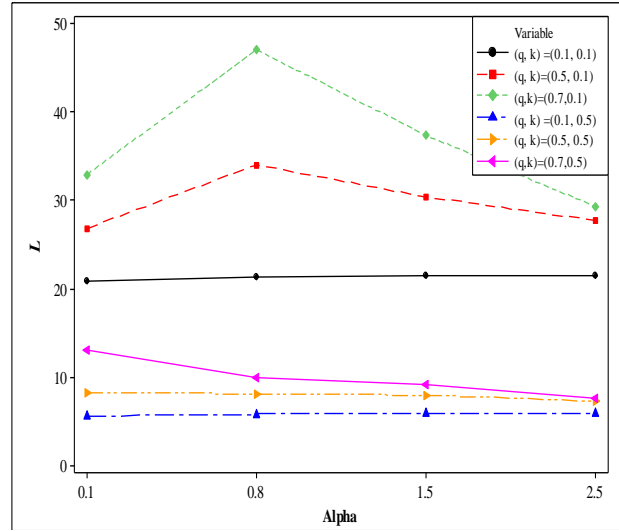
(a) Case A U-MGC scheme



(b) Case A U-MCG scheme



(c) Case E U-MGC scheme



(d) Case E U-MCG scheme

**Figure 2.** Optimal parameters of the proposed monitoring schemes when  $(m, n) = (100, 5)$ ,  $\alpha \in \{0.1, 0.8, 1.5, 2.5\}$ ,  $k \in \{0.1, 0.5\}$  and  $(m, n) = (100, 5)$  for a nominal  $ARL_0$  of 500

### 3.2.3 OOC Performance of the U-MCG and U-MGC monitoring schemes

This section investigates the OOC performance of the proposed U-MCG and U-MGC schemes for specific shifts as well as for an overall performance. The performance of the new schemes is first investigated when  $\delta_{min} = 0$ ,  $\delta_{max} = 2.5$ ,  $(q, \alpha, k) = (0.1, 0.1, 0.1)$ ,  $m \in \{50, 100, 400\}$  and  $n \in \{5, 10\}$  for a nominal  $ARL_0$  value of 500; and secondly, when  $\delta_{min} = 0$ ,  $\delta_{max} = 2.5$ ,  $(m, n) = (100, 5)$  and  $k \in \{0.1, 0.5\}$  for different values of  $\alpha$  and  $q$  under the  $N(0,1)$ ,  $GAM(3,1)$  and  $t(3)$  distributions.

**Table 3:** Case E *ARL* and *AEQL* values of the U-MGC and U-MCG schemes along with their design parameters for different Phase I and Phase II sample sizes when  $(q, \alpha, k) = (0.1, 0.1, 0.1)$  for a nominal  $ARL_0$  value of 500

Distribution	Scheme	MGC						MCG					
	<i>m</i>	50		100		400		50		100		400	
	Shift	<i>n</i> = 5	<i>n</i> = 10	<i>n</i> = 5	<i>n</i> = 10	<i>n</i> = 5	<i>n</i> = 10	<i>n</i> = 5	<i>n</i> = 10	<i>n</i> = 5	<i>n</i> = 10	<i>n</i> = 5	<i>n</i> = 10
<i>N</i> (0,1)	0.00	508.10	495.75	499.42	505.52	507.43	490.67	502.56	495.73	504.38	496.29	499.76	505.02
	0.25	154.64	131.88	84.66	71.58	43.77	31.04	146.09	145.40	83.82	66.73	41.18	30.50
	0.50	28.80	20.74	22.32	17.86	19.83	14.59	29.15	20.16	20.66	17.19	18.20	14.16
	0.75	17.04	12.56	13.74	11.76	13.29	9.88	16.31	11.98	12.91	11.29	11.82	9.59
	1.00	12.86	9.61	10.06	9.07	9.00	7.72	12.36	9.12	9.45	8.72	8.67	7.44
	1.25	10.63	8.03	8.03	7.61	7.62	6.49	10.25	7.60	7.51	7.31	7.15	6.28
	1.50	9.31	7.32	7.03	6.70	6.58	5.74	8.94	6.68	6.54	6.42	6.15	5.53
	1.75	8.45	6.69	6.30	6.12	5.91	5.17	8.12	6.10	6.18	5.90	5.38	5.07
	2.00	7.87	6.33	5.74	5.66	5.40	5.00	7.55	5.69	5.68	5.41	4.95	4.87
	2.25	7.43	6.08	5.71	5.43	5.10	4.86	7.19	5.31	5.44	5.09	4.66	4.49
	2.50	7.15	5.73	5.41	5.36	5.01	4.50	7.02	5.07	5.34	5.01	4.47	4.13
	<i>AEQL</i>	21.65	17.15	16.20	15.40	14.77	13.05	20.94	15.69	15.61	14.62	13.53	12.41
<i>t</i> (3)	0.00	492.08	499.97	509.03	493.39	502.18	494.66	507.38	512.85	509.88	500.61	495.43	499.53
	0.25	80.17	69.32	37.48	30.65	29.23	21.07	73.28	62.44	36.52	28.44	26.85	20.51
	0.50	18.50	13.77	14.97	11.67	13.37	10.60	17.88	13.05	14.11	11.35	12.71	10.33
	0.75	12.65	9.52	9.82	7.95	9.12	7.59	12.17	9.05	9.24	7.80	8.72	7.37
	1.00	10.18	7.69	7.61	6.58	7.27	6.24	9.81	7.30	7.11	6.19	6.89	6.03
	1.25	8.92	6.96	6.72	5.71	6.26	5.57	8.58	6.40	5.96	5.57	5.94	5.30
	1.50	8.16	6.51	6.02	5.24	5.66	5.07	7.83	5.90	5.62	5.27	5.37	4.98
	1.75	7.69	6.21	5.60	4.89	5.26	4.94	7.39	5.54	5.42	4.93	5.08	4.70
	2.00	7.37	5.97	5.38	4.58	5.10	4.71	7.15	5.27	5.20	4.62	4.91	4.39
	2.25	7.20	5.74	5.33	4.41	5.03	4.46	6.99	5.12	5.17	4.52	4.67	4.17
	2.50	7.09	5.57	5.30	4.34	5.01	4.26	6.87	5.05	5.11	4.35	4.45	4.08
	<i>AEQL</i>	19.30	15.31	14.20	11.93	13.34	11.78	18.63	13.84	13.56	11.94	12.47	11.21
<i>GAM</i> (3,1)	0.00	503.32	505.15	500.95	506.72	516.96	493.35	498.21	497.01	497.19	502.10	499.27	496.58
	0.25	164.70	172.15	79.28	64.96	42.58	30.71	178.87	145.00	65.19	52.88	34.95	25.89
	0.50	30.57	21.28	19.06	17.86	18.89	14.40	26.11	17.79	18.79	14.92	15.75	12.40
	0.75	16.95	12.66	12.14	11.69	12.13	9.74	14.70	10.74	12.23	10.01	10.44	8.53
	1.00	12.84	9.64	9.08	9.06	9.12	7.51	11.40	8.43	9.29	7.85	8.06	6.78
	1.25	10.64	8.03	7.44	7.61	7.42	6.29	9.65	7.10	7.75	6.70	6.75	5.84
	1.50	9.31	7.05	6.44	6.68	6.36	5.53	8.59	6.40	6.80	5.99	5.97	5.13
	1.75	8.45	6.41	5.79	6.12	5.71	4.98	7.92	5.94	6.16	5.56	5.36	4.90
	2.00	7.87	6.04	5.33	5.76	5.21	4.80	7.44	5.58	5.80	5.13	4.83	4.82
	2.25	7.43	5.79	5.07	5.34	5.04	4.65	7.18	5.29	5.54	4.87	4.80	4.36
	2.50	7.15	5.44	5.02	5.07	5.01	4.30	7.02	5.10	5.22	4.81	4.74	4.05
	<i>AEQL</i>	21.75	16.84	14.78	15.16	14.44	12.56	20.60	15.26	15.51	13.67	13.37	11.86
Design parameters		22.549	22.879	21.909	22.767	19.094	19.972	21.839	21.863	20.876	22.156	18.441	19.642

Table 3 displays the Case E performance of the U-MGC and U-MCG monitoring schemes in terms of the *ARL* and *AEQL* values when  $(q, \alpha, k) = (0.1, 0.1, 0.1)$ ,  $m \in \{50, 100, 400\}$  and  $n \in \{5, 10\}$  for a nominal  $ARL_0$  value of 500. For  $m \leq 50$ , in terms of the *AEQL* values, it can be observed that the overall sensitivity of the U-MGC and U-MCG schemes increases as the Phase I and/or Phase II sample size(s) increase(s). In terms of the *ARL* values, the U-MGC and U-MCG schemes perform better under heavy-tailed distributions regardless of the size of the shift in the location parameter. However, under small and moderate shifts, they perform better under symmetrical distributions compared to the skewed ones. Note though, they are similarly sensitive under symmetrical and skewed distributions for large shifts. When  $m > 50$ , in terms of the *ARL* and *AEQL* values, the sensitivity of the U-MGC and U-MCG schemes increase as the Phase I and/or Phase II sample size(s) increase(s). They both perform better under non-normal distributions. Moreover, when the Phase I is kept fixed, the larger (smaller) the Phase II sample size, the larger (smaller) the design parameter. However, when the Phase II sample size is kept fixed, the larger (smaller) the Phase I sample size, the smaller (larger) the design parameter. Note that similar findings are also observed for the Case A performance of the U-



MGC and U-MCG schemes. To preserve writing space, the rest of this paper will focus on the investigation of the performance of the U-MGC and U-MCG scheme when  $m = 100$  and  $n = 5$ .

**Table 4:** Case A OOC ARL values of the U-MGC and U-MCG monitoring schemes for a nominal  $ARL_0$  value of 500 when  $k \in \{0.1, 0.5\}$ ,  $\alpha = 1$ ,  $q \in \{0.1, 0.3, 0.5, 0.7\}$  and  $(m, n) = (100, 5)$

$k$	Distribution	$\frac{q}{\delta}$	MGC chart				MCG chart			
			0.1	0.3	0.5	0.7	0.1	0.3	0.5	0.7
0.1	N(0,1)	0.25	80.53	79.32	78.07	77.41	79.18	81.82	77.96	84.67
		0.50	24.02	21.73	18.44	14.94	22.71	23.09	23.15	23.97
		1.00	12.67	9.81	8.44	7.72	11.26	11.51	11.00	12.87
		1.50	8.76	7.14	6.51	4.79	8.18	8.52	8.84	9.89
		2.00	7.04	6.07	4.93	4.07	6.95	7.26	7.61	8.63
		2.50	6.30	6.01	5.22	3.43	6.23	6.79	7.04	8.04
	t(3)	0.25	37.19	33.58	30.40	30.22	38.34	38.52	38.78	38.74
		0.50	16.21	13.71	12.13	9.17	15.91	16.16	16.34	17.38
		1.00	8.88	8.00	7.12	5.21	8.93	9.27	9.61	10.63
		1.50	7.41	6.17	5.31	4.34	7.18	7.51	7.87	8.89
		2.00	6.82	5.89	4.65	3.82	6.47	6.91	7.20	8.20
		2.50	6.28	5.43	4.61	3.29	6.14	6.52	7.03	8.03
	GAM(3,1)	0.25	65.66	64.76	63.99	64.76	66.30	76.04	67.25	63.34
		0.50	19.74	17.52	14.67	12.94	19.80	20.08	20.09	21.00
		1.00	10.22	9.02	7.04	6.16	10.31	10.67	10.91	11.95
		1.50	7.99	6.73	5.49	4.73	7.82	8.18	8.51	9.55
		2.00	6.73	5.99	4.88	3.91	6.89	7.15	7.51	8.52
		2.50	6.25	5.47	4.79	4.00	6.23	6.86	7.03	8.03
0.5	N(0,1)	0.25	99.52	114.52	136.11	204.14	84.23	88.42	93.50	84.90
		0.50	13.24	12.85	18.23	37.42	11.79	11.97	12.21	12.08
		1.00	5.18	4.20	3.87	4.23	4.39	4.47	4.66	4.57
		1.50	2.97	2.97	2.34	1.80	3.07	3.20	3.27	3.12
		2.00	2.42	2.56	2.10	1.21	2.50	2.70	2.78	2.47
		2.50	2.15	2.22	1.65	1.09	2.13	2.25	2.36	2.10
	t(3)	0.25	35.87	45.33	60.81	128.79	35.33	31.26	32.42	31.72
		0.50	7.54	6.95	7.04	13.05	6.80	6.84	7.02	7.21
		1.00	4.00	3.04	2.67	2.32	3.37	3.46	3.59	3.47
		1.50	2.69	2.49	2.00	1.42	2.60	2.80	2.90	2.58
		2.00	2.32	2.20	1.84	1.19	2.23	2.35	2.45	2.21
		2.50	2.14	2.10	1.29	1.11	2.11	2.15	2.18	2.09
	GAM(3,1)	0.25	97.05	137.00	178.90	335.23	102.81	93.09	94.59	92.02
		0.50	10.15	13.35	17.14	43.77	9.71	9.81	10.19	10.01
		1.00	4.06	3.78	3.03	3.99	3.90	4.06	4.18	4.12
		1.50	3.03	2.69	2.31	2.00	2.98	3.06	3.12	2.99
		2.00	2.19	2.14	2.16	1.31	2.49	2.76	2.87	2.41
		2.50	2.07	2.03	1.93	1.20	2.22	2.23	2.39	2.04

**Table 5:** Case E OOC ARL values of the U-MGC and U-MCG monitoring schemes for a nominal  $ARL_0$  value of 500 when  $k \in \{0.1, 0.5\}$ ,  $\alpha = 1$ ,  $q \in \{0.1, 0.3, 0.5, 0.7\}$  and  $(m, n) = (100, 5)$

$k$	Distribution	$\frac{q}{\delta}$	MGC scheme				MCG scheme			
			0.1	0.3	0.5	0.7	0.1	0.3	0.5	0.7
0.1	N(0,1)	0.25	83.54	86.06	90.90	93.78	82.98	83.73	83.95	80.65
		0.50	23.87	26.78	29.26	34.32	22.80	22.89	23.40	23.88
		1.00	11.90	13.44	15.21	18.51	11.29	11.51	11.95	12.86
		1.50	8.71	9.94	11.34	14.10	8.24	8.48	8.97	9.89
		2.00	7.35	8.45	9.76	12.27	7.00	7.23	7.73	8.64
		2.50	6.87	7.90	9.05	11.34	7.00	6.75	7.08	8.05
	t(3)	0.25	40.69	44.84	47.12	53.06	37.64	38.88	39.72	39.25
		0.50	16.92	18.82	21.04	25.08	16.00	16.07	16.60	17.30
		1.00	9.49	10.80	12.04	15.19	9.00	9.22	9.72	10.66
		1.50	7.61	8.74	10.09	12.63	7.24	7.49	7.97	8.90
		2.00	7.00	8.03	9.23	11.65	6.53	6.88	7.27	8.20
		2.50	6.61	7.65	8.98	11.18	6.18	6.49	7.05	8.03
	GAM(3,1)	0.25	70.33	81.09	82.49	85.97	68.42	69.53	61.94	66.77
		0.50	20.85	23.56	25.85	30.54	19.92	20.18	20.50	21.19
		1.00	10.95	12.48	14.10	17.22	10.39	10.60	11.08	11.98
1.50		8.35	9.54	10.93	13.63	7.89	8.17	8.63	9.56	
2.00		7.23	8.32	9.63	12.15	6.93	7.14	7.64	8.53	
2.50		6.92	7.93	9.05	11.39	6.30	6.82	7.06	8.04	
0.5	N(0,1)	0.25	86.81	82.53	81.64	80.29	87.61	88.89	89.64	89.86
		0.50	11.80	12.21	13.40	15.68	11.71	11.61	12.14	12.07
		1.00	4.56	5.22	6.28	8.20	4.35	4.47	4.60	4.61
		1.50	3.22	3.80	4.72	6.39	3.08	3.19	3.26	3.13
		2.00	2.80	3.16	4.09	5.59	2.54	2.68	2.78	2.51
	2.50	2.37	3.01	4.08	5.07	2.13	2.24	2.34	2.12	
	t(3)	0.25	31.13	30.42	27.09	28.84	32.24	31.42	32.70	31.28
		0.50	6.90	7.65	8.97	11.08	6.82	6.88	7.08	7.18
		1.00	3.54	4.16	5.11	6.82	3.38	3.48	3.59	3.51
		1.50	2.89	3.31	4.21	5.76	2.62	2.79	2.88	2.63
		2.00	2.44	3.06	4.02	5.23	2.22	2.33	2.41	2.24
	2.50	2.20	3.01	4.00	5.06	2.11	2.15	2.18	2.11	
	GAM(3,1)	0.25	95.04	72.90	71.58	60.21	111.58	99.79	86.32	87.85
		0.50	9.89	10.10	11.36	13.56	9.82	9.69	9.87	10.03
		1.00	4.11	4.78	5.76	7.64	3.92	4.04	4.18	4.13
1.50		3.08	3.66	4.47	5.17	2.99	3.05	3.11	3.02	
2.00		2.88	3.08	4.02	5.52	2.51	2.74	2.00	2.49	
2.50	2.42	3.00	4.00	5.04	2.06	2.21	2.00	2.06		

**Table 6:** Case E OOC ARL values of the U-MGC and U-MCG monitoring schemes for a nominal  $ARL_0$  value of 500 when  $k = 0.1$ ,  $q \in \{0.1, 0.5, 0.7\}$ ,  $\alpha \in \{0.1, 0.8, 1.5, 2.5\}$  and  $(m, n) = (100, 5)$

Distribution	$q$	$\alpha$ $\delta$	MGC scheme				MCG scheme			
			0.1	0.8	1.5	2.5	0.1	0.8	1.5	2.5
$N(0,1)$	0.1	0.25	84.66	80.04	87.48	83.77	83.68	92.47	92.13	88.61
		0.50	22.32	16.63	14.63	12.60	20.66	13.56	12.05	12.00
		1.00	10.06	7.42	4.56	4.55	9.45	6.34	4.35	4.37
		1.50	7.03	5.21	4.31	3.94	6.54	4.73	3.44	3.21
		2.00	5.74	3.80	3.51	3.43	5.68	3.17	3.03	2.84
	2.50	5.41	3.39	3.32	3.30	5.34	3.01	2.73	2.54	
	0.5	0.25	81.37	77.29	75.95	80.27	78.99	87.54	88.22	92.00
		0.50	15.31	13.46	12.80	12.46	12.82	11.79	11.85	12.06
		1.00	7.29	6.52	5.81	4.45	6.36	4.576	4.64	4.61
		1.50	5.47	4.92	4.32	4.03	4.95	4.22	3.37	3.34
		2.00	4.94	4.17	3.88	3.39	4.23	3.49	3.00	3.02
	2.50	4.49	4.00	3.41	3.05	4.04	3.11	2.80	3.00	
	0.7	0.25	90.08	105.05	97.33	90.12	84.13	78.27	78.83	80.10
		0.50	18.86	26.25	21.66	18.12	14.26	14.75	13.35	13.36
		1.00	8.65	10.49	8.99	7.98	6.94	8.04	6.63	6.51
1.50		7.17	9.42	7.43	10.81	5.18	7.43	5.16	5.86	
2.00		6.55	8.44	6.70	9.26	4.82	6.14	4.98	4.58	
2.50	5.87	6.89	5.50	8.54	3.69	5.45	4.12	4.44		
$t(3)$	0.1	0.25	37.48	30.64	29.08	29.87	36.52	33.10	32.56	31.14
		0.50	14.97	9.12	7.64	7.19	14.11	7.66	7.01	6.80
		1.00	7.61	5.51	4.37	4.36	7.11	4.27	3.98	3.53
		1.50	6.02	4.92	4.14	3.99	5.62	3.52	3.13	2.99
		2.00	5.38	3.54	3.39	3.11	5.20	3.11	2.97	2.74
	2.50	5.30	3.18	3.10	3.02	5.11	3.00	2.49	2.40	
	0.5	0.25	29.40	26.89	29.87	27.91	25.19	30.22	34.08	32.65
		0.50	7.42	9.12	8.39	8.02	5.70	6.93	7.06	7.01
		1.00	3.79	5.30	4.70	4.36	3.13	3.44	3.68	3.64
		1.50	3.11	4.32	3.97	3.52	2.75	2.84	3.07	3.08
		2.00	3.00	4.05	3.54	3.17	2.54	2.66	2.89	3.01
	2.50	2.92	4.01	3.23	3.06	2.34	2.42	2.62	3.00	
	0.7	0.25	45.32	55.93	51.71	47.14	39.93	41.49	39.93	39.00
		0.50	10.89	16.87	12.77	10.11	7.99	8.18	6.59	6.53
		1.00	6.68	10.06	7.97	7.23	5.81	6.23	5.49	5.06
1.50		5.86	9.82	6.24	6.06	4.81	5.40	5.21	4.82	
2.00		4.21	8.80	5.67	5.34	4.34	4.70	4.51	4.19	
2.50	4.01	7.27	5.22	4.81	3.57	3.78	3.68	3.53		
$GAM(3,1)$	0.1	0.25	86.02	82.09	83.01	80.72	91.05	96.36	90.01	94.69
		0.50	19.06	10.50	9.84	9.93	18.72	9.86	10.00	9.94
		1.00	9.08	6.11	4.70	4.52	8.78	4.10	4.17	3.97
		1.50	6.74	5.01	4.33	4.12	5.84	3.76	3.21	3.03
		2.00	5.63	3.60	3.47	3.15	5.23	3.15	3.00	2.81
	2.50	5.41	3.26	3.18	3.08	5.01	3.04	2.57	2.46	
	0.5	0.25	77.23	67.08	68.94	78.44	105.63	82.19	89.70	99.84
		0.50	10.05	11.68	10.97	10.63	8.31	9.91	10.00	9.95
		1.00	4.45	6.02	5.34	4.98	3.29	4.10	4.24	4.20
		1.50	3.24	4.73	4.15	3.93	2.89	3.11	3.17	3.15
		2.00	3.00	4.09	3.92	3.28	2.52	2.73	3.00	3.00
	2.50	3.00	4.04	3.47	3.07	2.40	2.57	2.94	3.00	
	0.7	0.25	75.30	82.57	82.04	75.94	72.79	76.25	68.42	68.38
		0.50	15.50	16.33	17.98	14.88	11.59	11.87	10.42	10.30
		1.00	8.49	10.24	8.81	7.90	7.01	7.12	6.75	6.40
1.50		7.77	9.89	7.98	6.41	6.81	6.46	7.33	5.49	
2.00		6.41	8.31	6.58	5.16	5.17	5.05	5.94	4.69	
2.50	5.90	7.54	5.02	4.60	4.30	4.42	4.29	3.72		

The findings in Tables 4 to 6 can be summarized as follows:

(i) When  $\alpha = 1$  (see Tables 4 and 5),

- for small reference constants, in Case A, the sensitivity of the U-MGC scheme increases as  $q$  increases regardless of the nature of the underlying process distribution. However, the sensitivity of the U-MCG scheme is almost the same regardless of the value of  $q$  except for very small shifts ( $0 < \delta \leq 0.25$ ) where the sensitivity of the U-MCG scheme decreases in the intervals  $0.1 \leq q \leq 0.3$  and  $0.5 \leq q \leq 0.7$ ,

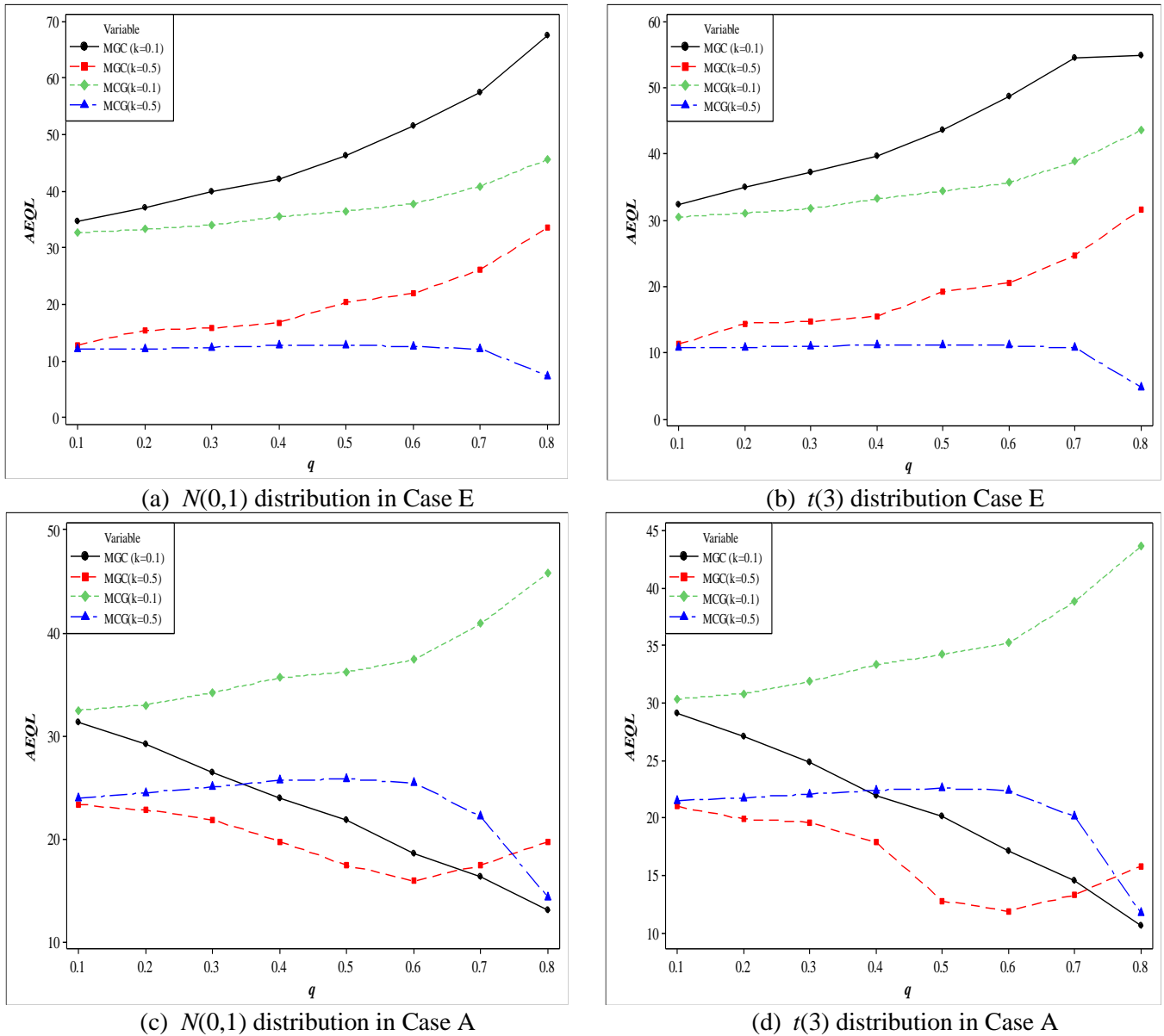
- in Case E, when  $k$  is small, the sensitivity of the U-MGC scheme decreases as  $q$  increases regardless of the nature of the underlying process distribution,
- in Case A, when  $k$  is large, the sensitivity of the U-MGC scheme decreases as  $q$  increases for small shifts in the location parameter, and it performs the worst for large values of  $q$ . However, for moderate and large shifts in the location parameter, the sensitivity of the U-MGC scheme increases as  $q$  increases. For the U-MCG scheme, under the  $N(0,1)$  distribution, for small shifts, when  $0 < q \leq 0.5$ , the sensitivity of the U-MCG scheme decrease as  $q$  increases. When  $0.5 < q < 1$ , the sensitivity of the U-MCG scheme increases as  $q$  increases. For moderate and large shifts in the location parameter, the sensitivity of the U-MCG scheme is almost the same for all  $q$  values. Under non-normal distribution, the U-MCG scheme performs the worst for small values of  $q$ . Its performance tends to be the same as  $q$  increases. For moderate and small shifts, the sensitivity of the U-MCG scheme is almost the same for all  $q$  values, and
- in Case E, when  $k$  is large, regardless of the nature of the underlying process distribution, the sensitivity of the U-MGC scheme decreases as  $q$  increases except for very small shifts where the performance of the U-MGC scheme increases as  $q$  increases. From small-to-large shifts, the sensitivity of the U-MCG scheme is almost the same for all values of  $q$  regardless of the nature of the distribution, except for very small shift under skewed distribution, where the sensitivity of the U-MCG scheme increases as  $q$  increases.

(ii) When  $\alpha \neq 1$  (see Table 6),

- for small values of  $q$  (i.e.  $0 < q < 0.5$ ), the sensitivity of both the U-MGC and U-MCG schemes increases as  $\alpha$  increases regardless of the nature of the distribution,
- for  $0.5 \leq q < 0.7$ , regardless of the nature of the distribution, the sensitivity of the U-MGC scheme increases in the interval  $0 < \alpha \leq 1.5$  and remains almost the same for  $\alpha > 1.5$ . However, the sensitivity of the U-MCG scheme increases when  $0 < \alpha \leq 1$  and remains almost the same when  $\alpha > 1$ , and
- in Case E, for large values of  $q$  (i.e.  $0.7 \leq q < 1$ ), regardless of the nature of the distribution, the sensitivity of the U-MGC scheme decreases in the interval  $0 < \alpha < 1$  and increases when  $\alpha > 1$ . For small and moderate shifts, the sensitivity of the U-MCG scheme remains almost the same under skewed and heavy-tailed distributions when  $0 < \alpha < 1$ . Under symmetrical distributions, the sensitivity of the U-MCG

scheme decreases when  $0 < \alpha < 1$ . For large shifts and  $\alpha > 1$ , under symmetrical and heavy-tailed distributions, the sensitivity of the U-MCG scheme is almost the same.

Note that the Case A of the above discussion yields the same findings. Therefore, to preserve writing space, the results are not displayed in this study.



**Figure 3.** AEQL values of the U-MGC and U-MCG schemes when  $\alpha = 1$ ,  $k \in \{0.1, 0.5\}$  and  $(m, n) = (100, 5)$  for a nominal  $ARL_0$  of 500 under the normal and non-normal distribution

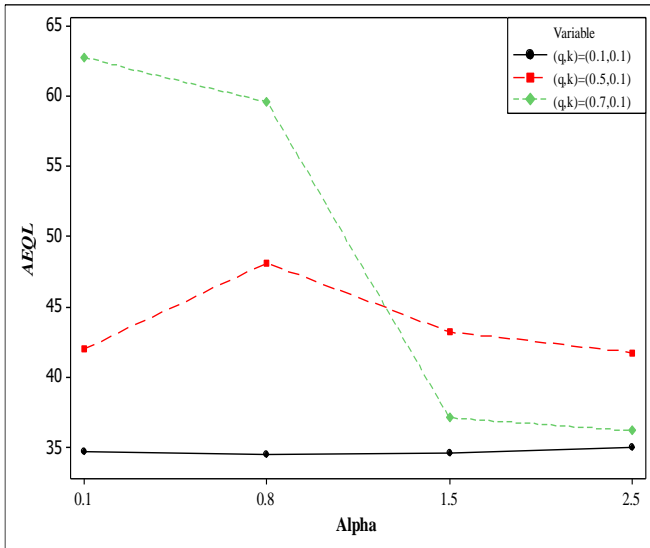
Figure 3 displays the overall performance of the proposed schemes when  $\delta_{max} = 2.5$ ,  $(m, n) = (100, 5)$ ,  $k \in \{0.1, 0.5\}$  and  $\alpha = 1$  for different values of  $q$  under normal and non-normal distributions. From Figure 3, the following is observed in terms the overall performance (i.e. AEQL values):

(i) In Case E (see Figures 3(a)-(b)),

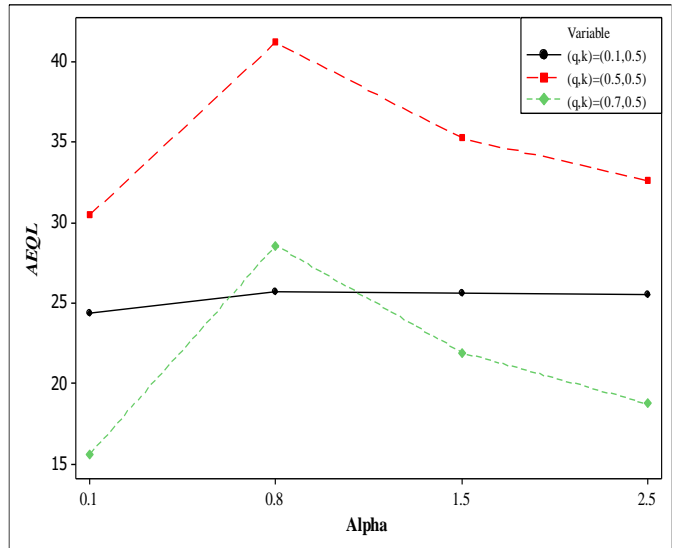
- the overall performance of the proposed U-MGC and U-MCG schemes deteriorates as  $q$  increases except for the  $0.7 \leq q < 1$  where the sensitivity of the U-MCG scheme for  $k = 0.5$  increases,
- the U-MGC and U-MCG schemes perform better for large values of the reference constants (i.e.  $k$ ) regardless of the underlying process distribution, and
- the U-MCG scheme is superior to the U-MGC scheme for all values of  $q$  regardless of the underlying process distribution.

(ii) In Case A (see Figures 3(c)-(d)),

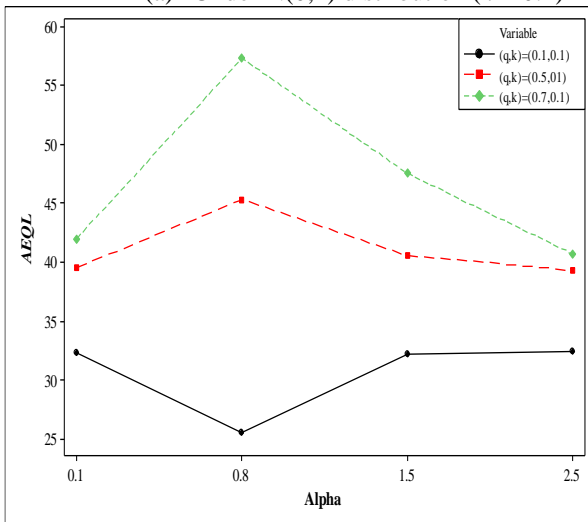
- regardless of the nature of the distribution, the sensitivity of the U-MCG scheme deteriorates as  $q$  increases when  $k = 0.1$ ; whereas, the sensitivity of the U-MGC scheme for a reference constant of 0.1 increases as  $q$  increases. However, when  $k = 0.5$ , the sensitivity of the U-MGC scheme increases in the interval  $0 < q \leq 0.6$  and decreases for  $0.6 < q < 1$ ; whereas, the sensitivity of the U-MCG scheme deteriorates in the interval  $0 < q \leq 0.6$  and increases for  $0.6 < q < 1$ .
- under the normal distribution, the U-MGC scheme with a reference constant of 0.5 is superior, except for large values of  $q$  for which the U-MGC scheme with small values of  $k$  outperforms all other designs followed by the U-MCG scheme with  $k = 0.5$ . However, under non-normal distributions, the U-MGC scheme with a reference constant of 0.5 is superior except for large values of  $q$  for which the U-MGC scheme with small values of  $k$  outperforms all other designs followed by the U-MCG scheme with  $k = 0.5$ .
- under both normal and non-normal distributions, the U-MCG scheme with  $k = 0.1$  is inferior compared to other designs.



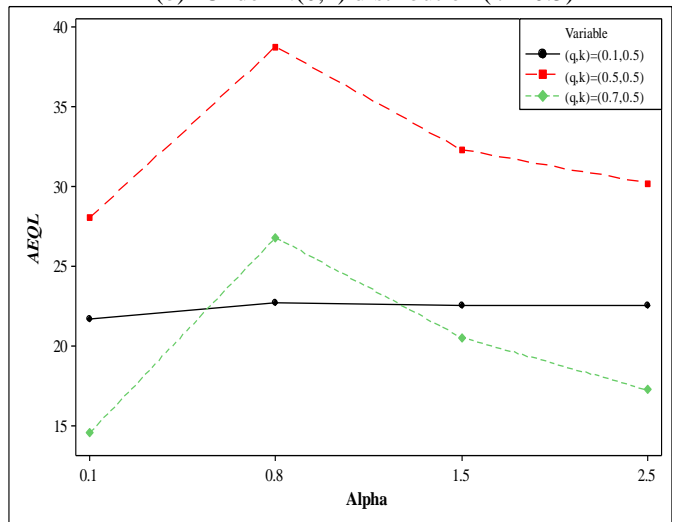
(a) Under N(0,1) distribution ( $k = 0.1$ )



(b) Under N(0,1) distribution ( $k = 0.5$ )

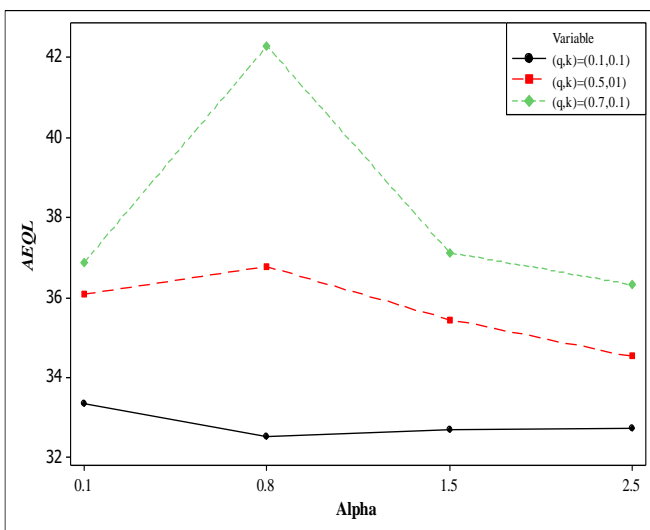


(c) Under  $t(3)$  distribution ( $k = 0.1$ )

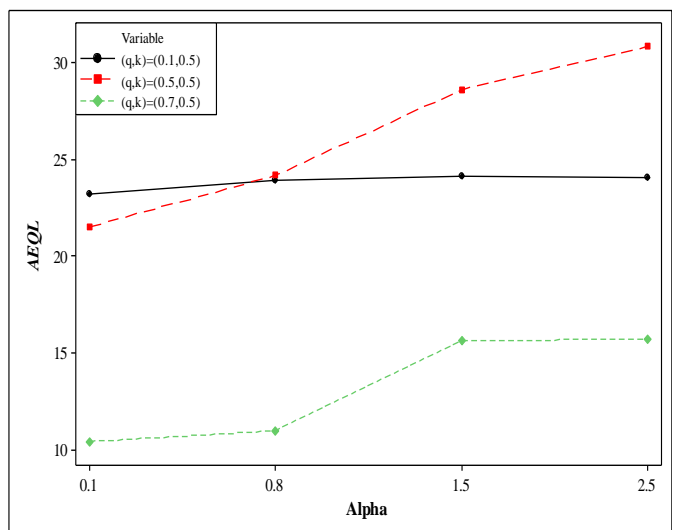


(d) Under  $t(3)$  distribution ( $k = 0.5$ )

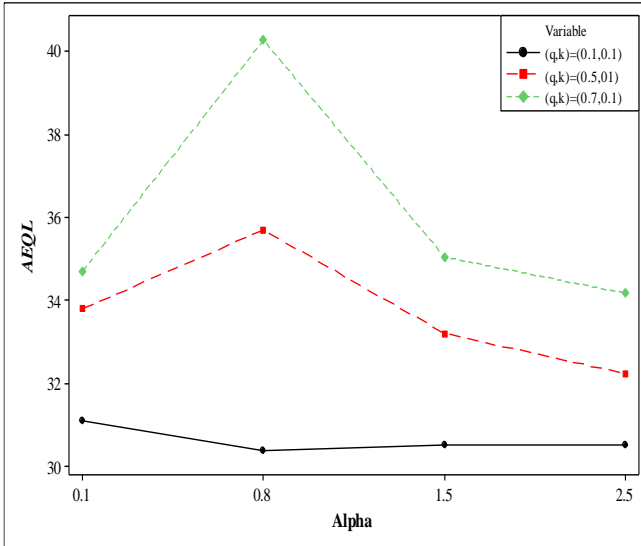
**Panel 1: U-MGC scheme**



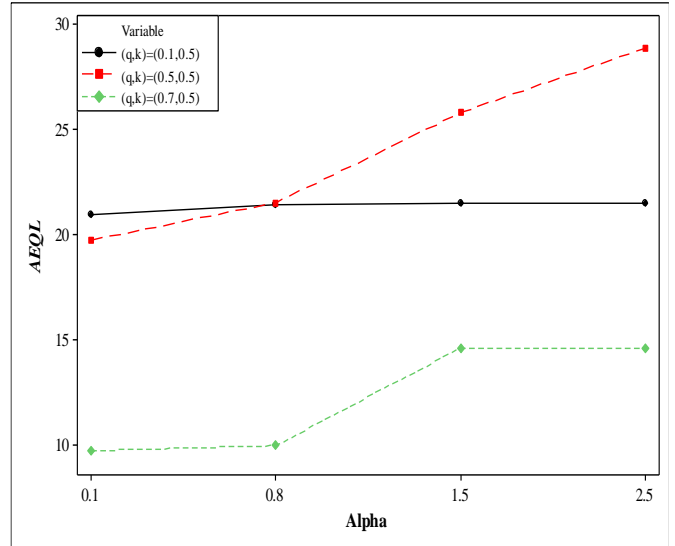
(a) Under N(0,1) distribution ( $k = 0.1$ )



(b) Under N(0,1) distribution ( $k = 0.5$ )



(c) Under  $t(3)$  distribution ( $k = 0.1$ )



(d) Under  $t(3)$  distribution ( $k = 0.5$ )

**Panel 2: U-MGC scheme**

**Figure 4.** *AEQL* values of the U-MGC and U-MCG schemes when  $\alpha = \{0.1, 0.8, 1.5, 2.5\}$ ,  $k \in \{0.1, 0.5\}$  and  $(m, n) = (100, 5)$  for a nominal  $ARL_0$  of 500 under the normal and non-normal distribution

Figure 4 displays the overall performance of the proposed schemes when  $\delta_{max} = 2.5$ ,  $(m, n) = (100, 5)$  and  $k \in \{0.1, 0.5\}$  for different values of  $\alpha$  and  $q$  under normal and non-normal distributions. The findings in Figure 4 can be summarized as follows:

- Under the  $N(0,1)$  distribution and small reference coefficients, the U-MGC scheme performs better for small values of  $q$  for all possible values of  $\alpha$ . Moreover, the U-MGC scheme designed using  $k = 0.5$  is superior to the one with  $k = 0.7$  when  $0 < \alpha \leq 1.25$ ; however, the converse is observed when  $\alpha > 1.25$ . For large reference coefficients, the U-MGC scheme with large values of  $q$  performs better, except when  $0.68 < \alpha \leq 1.15$ , as the U-MGC scheme with  $k = 0.1$  performs better in that scenario. Under non-normal distributions, for small reference coefficients, the U-MGC scheme performs better for small values of  $q$ , for all possible values of  $\alpha$ . However, for large reference coefficients, the U-MGC scheme with large values of  $q$  performs better except when  $0.59 < \alpha \leq 1.31$ , as the U-MGC scheme with  $k = 0.1$  performs better in this scenario (see Figure 4-Panel 1).
- Under both normal and non-normal distributions, for small reference coefficients, the U-MCG scheme performs better for small values of  $q$ , for all possible values of  $\alpha$ . For large reference coefficients, the U-MCG scheme performs better for large values of  $q$ , for all possible values of  $\alpha$ . However, the U-MCG scheme designed using  $k = 0.5$  is



superior to the one with  $k = 0.1$  when  $0 < \alpha \leq 0.75$ ; whereas, the converse is observed when  $\alpha > 0.75$  (see Figure 4-Panel 2).

### 3.2.3 Performance comparison

In this section, the proposed U-MCG and U-MGC schemes are compared to a number of MW U-type monitoring schemes: (i) Shewhart (denoted MW U), (ii) EWMA (denoted U-EWMA), (iii) CUSUM (denoted U-CUSUM), (iv) mixed EWMA-CUSUM (denoted U-MEC), (v) mixed CUSUM-EWMA (denoted U-MCE) and, (vi) GWMA (denoted U-GWMA). The MW U scheme was first discussed in Chakraborti and Van de Wiel (2008), the U-CUSUM and U-EWMA schemes are equivalent to the existing CUSUM and EWMA WRS schemes discussed in Li et al. (2010), the U-GWMA scheme is introduced in Section 2.3 here and the U-MEC and U-MCE schemes are introduced in Table 1 here. The comparison is conducted when  $(m,n) = (100,5)$ ,  $k = 0.5$ ,  $(q, \alpha, \lambda) = (0.9, 1.5, 0.1)$  and  $\delta_{max} = 2.5$  for a nominal  $ARL_0$  value of 500 under the  $N(0,1)$ ,  $t(3)$  and  $GAM(3,1)$  distributions. Tables 7 to 9 present the IC and OOC performances of the competing schemes for specific shifts as well as their overall performances and their corresponding design parameters. The latter are determined such that the actual  $ARL_0$  value is closer or equal to the nominal  $ARL_0$  value. The schemes that perform better are shaded in grey. When two columns are shaded, the schemes under consideration are similar (or almost similar) in performance. To study the overall performance for small, moderate and large shifts, the  $AEQL$  values were calculated using Equation (21) where  $AEQL_1$ ,  $AEQL_2$  and  $AEQL_3$  correspond to small, moderate and large shifts with  $\delta_{max} = 0.75, 1.5$  and  $2.5$ , respectively; whereas,  $AEQL_4$  and  $AEQL_5$  correspond to small-to-moderate (i.e.,  $0 < \delta \leq 1.5$ ) and small-to-large shifts (i.e.,  $0 < \delta \leq 2.5$ ), respectively.

**Table 7:** Case E performance comparison of eight monitoring schemes under the  $N(0, 1)$  distribution when  $(m, n) = (100, 5)$ ,  $(q, \alpha, \lambda) = (0.9, 1.5, 0.1)$  for a nominal  $ARL_0$  value of 500

Shift ( $\delta$ )	MW U	U-CUSUM	U-GWMA	U-EWMA	U-MCE	U-MEC	U-MGC	U-MCG
0.00	499.76	503.91	501.22	502.05	505.40	503.51	498.82	501.45
0.25	249.04	343.37	220.59	335.62	262.28	255.36	87.17	87.40
0.50	62.17	123.24	46.26	114.22	122.73	129.74	14.28	11.72
0.75	18.73	37.37	13.55	28.61	35.17	35.98	12.10	6.14
1.00	7.24	15.00	5.57	12.84	15.04	14.97	9.06	4.37
1.25	3.56	9.59	2.89	7.93	9.44	9.54	8.27	3.53
1.50	2.12	7.09	1.90	5.65	7.06	7.15	6.76	3.08
1.75	1.52	5.80	1.45	4.35	5.70	5.73	5.34	2.78
2.00	1.24	4.86	1.21	3.52	4.88	4.89	5.08	2.53
2.25	1.11	4.28	1.10	2.97	4.28	4.30	4.01	2.29
2.50	1.04	3.86	1.04	2.54	3.88	3.87	4.00	2.13
<b><math>AEQL_1</math></b>	13.88	24.43	10.99	21.87	22.29	22.88	5.27	3.95
<b><math>AEQL_2</math></b>	5.86	15.31	4.79	12.65	15.23	15.32	12.40	5.61
<b><math>AEQL_3</math></b>	5.43	20.75	5.34	14.58	20.72	20.77	20.49	10.88
<b><math>AEQL_4</math></b>	9.87	19.87	7.89	17.26	18.76	19.10	8.84	4.78
<b><math>AEQL_5</math></b>	8.10	20.22	6.87	16.19	19.54	19.77	13.50	7.22
<b>Parameter</b>	$k = 2.795$	$h_c = 5.298$	$L = 2.665$	$L = 2.991$	$L = 338$	$H = 338.4$	$h = 31.293$	$L = 5.195$

$AEQL_1$  is the AEQL for small shifts only ( $0 < \delta \leq 0.75$ )

$AEQL_2$  is the AEQL for moderate shifts only ( $0.75 < \delta \leq 1.5$ )  
 $AEQL_3$  is the AEQL for large shifts only ( $1.5 < \delta \leq 2.5$ )  
 $AEQL_4$  is the AEQL for small-to-moderate shifts ( $0 < \delta \leq 1.5$ )  
 $AEQL_5$  is the AEQL for small-to-large shifts ( $0 < \delta \leq 2.5$ )

**Table 8:** Case E performance comparison of eight monitoring schemes under the  $t(3)$  distribution when  $(m, n) = (100, 5)$ ,  $(q, \alpha, \lambda) = (0.9, 1.5, 0.1)$  for a nominal  $ARL_0$  value of 500

Shift ( $\delta$ )	MW U	U-CUSUM	U-GWMA	U-EWMA	U-MCE	U-MEC	U-MGC	U-MCG
0.00	500.69	502.60	512.71	504.33	495.64	510.75	496.84	503.69
0.25	185.41	253.74	147.37	241.17	221.82	229.14	42.06	42.26
0.50	31.72	45.87	19.66	42.24	38.60	37.21	13.42	6.81
0.75	7.21	13.64	5.04	11.52	12.43	11.99	10.16	4.24
1.00	2.81	7.98	2.36	6.83	11.04	11.43	8.78	3.36
1.25	1.69	5.83	1.58	4.94	7.77	7.69	8.04	2.93
1.50	1.31	4.72	1.30	3.79	5.96	6.01	6.55	2.62
1.75	1.18	4.07	1.17	2.96	4.93	4.99	5.24	2.38
2.00	1.11	3.62	1.11	2.43	4.30	4.31	5.09	2.24
2.25	1.07	3.30	1.08	2.12	3.84	3.85	4.04	2.16
2.50	1.05	3.05	1.06	2.08	3.52	3.53	4.01	2.10
$AEQL_1$	7.86	11.67	5.65	10.70	10.17	10.12	3.90	2.24
$AEQL_2$	2.80	9.24	2.58	7.69	12.20	12.32	12.48	4.61
$AEQL_3$	5.01	15.68	5.03	10.63	18.43	18.52	20.48	10.08
$AEQL_4$	5.33	10.45	4.12	9.20	11.18	11.22	7.96	3.43
$AEQL_5$	5.20	12.54	4.48	9.77	14.08	14.14	12.97	6.09
Parameters	$k = 2.795$	$h_c = 5.298$	$L = 2.665$	$L = 2.991$	$L = 336$	$H = 338.4$	$h = 31.293$	$L = 5.195$

**Tables 9:** Case E performance comparison of eight monitoring schemes under the  $GAM(3,1)$  distribution when  $(m, n) = (100, 5)$ ,  $(q, \alpha, \lambda) = (0.9, 1.5, 0.1)$  for a nominal  $ARL_0$  value of 500

Shift ( $\delta$ )	MW U	U-CUSUM	U-GWMA	U-EWMA	U-MCE	U-MEC	U-MGC	U-MCG
0.00	501.79	504.46	492.77	511.47	501.49	506.61	497.39	514.48
0.25	392.71	405.86	313.61	364.14	311.63	315.83	62.21	94.33
0.50	113.51	100.22	64.44	91.05	59.99	57.64	17.09	9.94
0.75	34.18	19.43	19.85	15.77	14.66	14.91	12.47	5.30
1.00	11.27	12.62	6.76	10.28	9.44	9.79	9.84	3.88
1.25	4.54	7.97	2.95	6.54	6.99	7.01	8.69	3.50
1.50	2.17	6.11	1.76	4.70	5.08	5.04	8.01	3.26
1.75	1.34	5.03	1.28	3.72	4.01	3.99	5.60	3.07
2.00	1.08	4.32	1.08	3.11	3.34	3.33	5.24	2.83
2.25	1.01	3.85	1.01	2.79	2.84	2.87	4.01	2.51
2.50	1.00	3.52	1.00	2.48	2.52	2.52	4.00	2.33
$AEQL_1$	24.05	20.45	15.63	18.13	14.24	14.18	5.06	3.79
$AEQL_2$	7.75	12.94	5.11	10.36	10.60	10.69	13.81	5.56
$AEQL_3$	4.95	18.54	4.90	13.36	13.94	13.95	20.85	12.00
$AEQL_4$	15.90	16.70	10.37	14.24	12.42	12.44	9.44	4.67
$AEQL_5$	11.52	17.43	8.18	13.89	13.03	13.04	14.00	7.60
Parameters	$k = 2.795$	$h_c = 5.298$	$L = 2.665$	$L = 2.991$	$L = 336$	$H = 338.4$	$h = 31.293$	$L = 5.195$

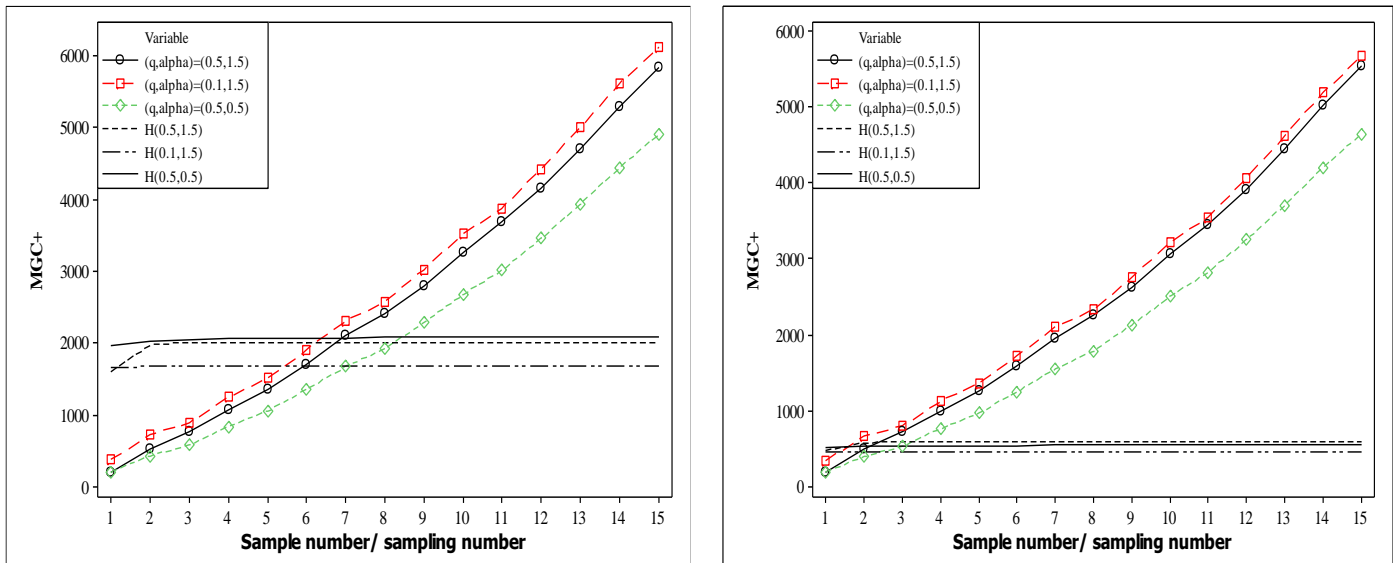
From Table 7, it can be seen that under the  $N(0,1)$  distribution, the U-MCG scheme performs better for small shifts in the location parameter; whereas, the U-CUSUM, U-EWMA, U-MEC and U-MCE have the worst performances for small shifts (see the  $AEQL_1$  values). However, the U-GWMA and MW U schemes perform better than all competing schemes considered in this study under moderate and large shifts (see the  $AEQL_2$  and  $AEQL_3$  values). Moreover, for small-to-moderate shifts, the U-MCG scheme outperform all competing schemes (see the  $AEQL_4$  values); while the U-GWMA scheme performs better for small-to-large shifts (see the  $AEQL_5$  values). From Tables 8 and 9, it can be observed that under the  $t(3)$  and  $GAM(3,1)$  distributions, the U-MCG scheme outperforms the competing schemes under small shifts (see the  $AEQL_1$  values). Under moderate shifts, the U-GWMA scheme is superior compared to all

other competing schemes (see the  $AEQL_2$  values). For large shifts, the Shewhart MW U scheme performs better (see the  $AEQL_3$  values). In addition, under the  $t(3)$  distribution, for small-to-moderate shifts in the location parameter, the U-MCG scheme is superior to all competing schemes (see the  $AEQL_4$  values); while the U-GWMA scheme performs better for small-to-large shifts (see the  $AEQL_5$  values). Under the  $GAM(3,1)$  distribution, for both small-to-moderate and small-to-large shifts in the location parameter, the U-MCG scheme is superior to all competing schemes considered in this study (see the  $AEQL_4$  and  $AEQL_5$  values). In the latter comparison, the U-CUSUM, U-EWMA, U-MEC and U-MCE schemes do not have a range of shift values where they yield a better performance for the considered  $AEQL$  situations. It can also be seen that the proposed U-MCG scheme with  $(q, \alpha) = (0.9, 1.5)$  has very interesting run-length characteristics regardless of the size of the shift. However, for these parameters, the U-MGC schemes performs relatively worst especially for moderate and large shifts. This poor performance of the U-MGC scheme is due to the large value of  $q$ . In some other situations, the U-MGC scheme presents very interesting run-length characteristics (see for instance, Tables 4 to 6).

#### 4. Illustrative example

In this section, the implementation and application of the proposed U-MGC and U-MCG monitoring schemes is illustrated using the data on the inside diameters of piston rings manufactured by a forging process from Montgomery (2005, page 223). The first set of data contains hundred and twenty five Phase I observations ( $m = 125$ ) collected when the process was considered to be IC. These data are used as the Phase I data for which a goodness of fit test for normality is not rejected. The second set of data contains fifteen test samples each of size  $n = 5$  which are considered to be the Phase II data. The proposed monitoring schemes are designed in Case E when  $k \in \{0.1, 0.5\}$ ,  $\alpha \in \{0.5, 1.5\}$  and  $q \in \{0.1, 0.5\}$  for a nominal  $ARL_0$  value of 500. When  $k = 0.1$ ,  $(q, \alpha) = (0.5, 1.5)$ ,  $(0.1, 1.5)$  and  $(0.5, 0.5)$ , we have  $h = 38.862$ ,  $22.362$  and  $47.764$  which are the control limit coefficients of the U-MGC scheme with  $ARL_0$  values of 499.58, 500.52 and 501.57, respectively. However, when  $k = 0.5$ , we have  $h = 11.423$ ,  $6.224$  and  $12.569$  so that the corresponding  $ARL_0$  values are 499.58, 500.52 and 501.57, respectively. The plot of the U-MGC schemes' plotting statistics for  $k = 0.1$  and  $0.5$  are shown in Figure 5. It is observed that when  $k = 0.1$ ,  $(q, \alpha) = (0.5, 1.5)$ ,  $(0.1, 1.5)$  and  $(0.5, 0.5)$ , the proposed U-MGC scheme gives a signal on the seventh, sixth and ninth Phase II subgroups, respectively (see Figure 5(a)). However, when  $k = 0.5$ ,  $(q, \alpha) = (0.5, 1.5)$ ,  $(0.1, 1.5)$

and (0.5, 0.5), the proposed U-MGC scheme gives a signal on the third, second and fourth Phase II subgroups, respectively (see Figure 5(b)).

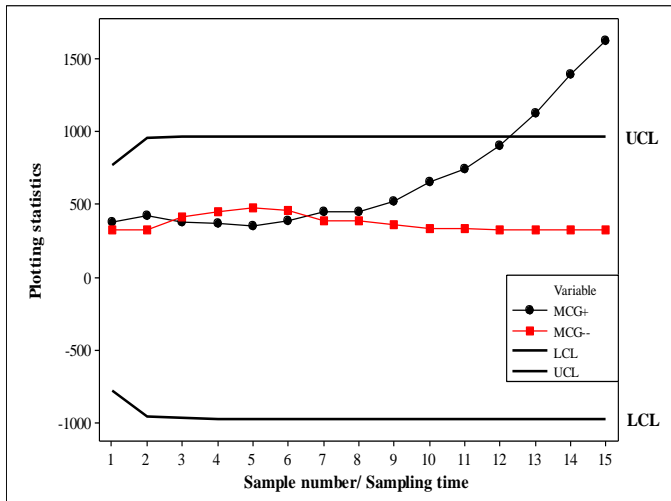


(a) MGC scheme with  $k = 0.1$

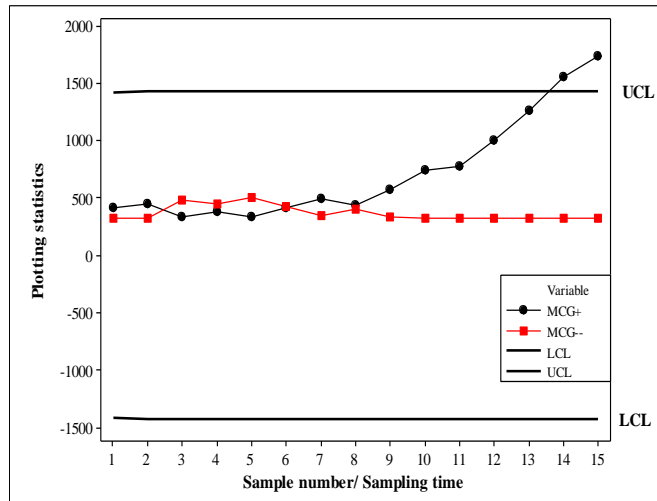
(b) MGC scheme with  $k = 0.5$

**Figure 5.** Implementation of the U-MGC scheme for the Montgomery (2005)'s piston ring data

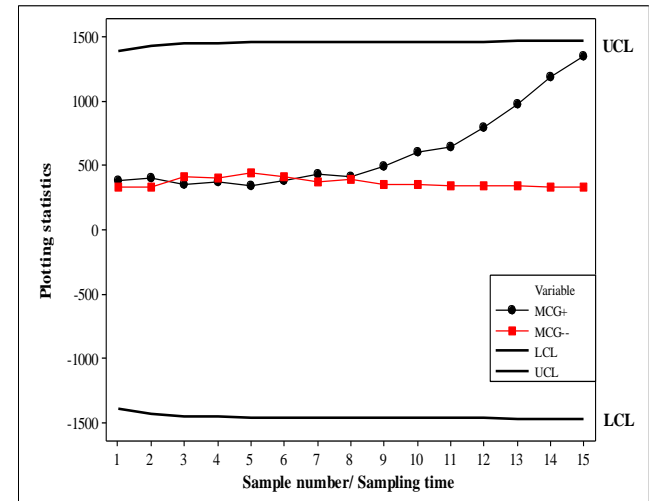
For the U-MCG scheme, when  $k = 0.1$ ,  $(q, \alpha) = (0.5, 1.5)$ ,  $(0.1, 1.5)$  and  $(0.5, 0.5)$ , we have  $L = 18.606$ ,  $19.076$  and  $33.642$  with  $ARL_0$  values equal to  $500.78$ ,  $498.38$  and  $499.46$ , respectively. However, when  $k = 0.5$ , we have  $L = 4.829$ ,  $5.743$  and  $7.646$  with  $ARL_0$  values of  $503.01$ ,  $499.44$  and  $500.57$ , respectively. The plot of the U-MCG scheme plotting statistics for  $k = 0.1$  and  $0.5$  are shown in Figure 6. It is observed that when  $k = 0.1$ ,  $(q, \alpha) = (0.5, 1.5)$  and  $(0.1, 1.5)$ , the proposed U-MGC scheme gives a signal on the twelfth and fourteenth Phase II subgroups, respectively (see Figures 6(a)-(b)). When  $k = 0.1$  and  $(q, \alpha) = (0.5, 0.5)$ , the proposed U-MGC scheme does not give a signal in Phase II (see Figure 6(c)). However, when  $k = 0.5$ ,  $(q, \alpha) = (0.5, 1.5)$ ,  $(0.1, 1.5)$  and  $(0.5, 0.5)$ , the U-MGC scheme gives a signal on the fifth, third and second Phase II subgroups, respectively (see Figure 6(d)-(f)). Therefore, in these conditions, the U-MGC scheme is more sensitive compared to the U-MCG scheme since it gives a signal sooner.



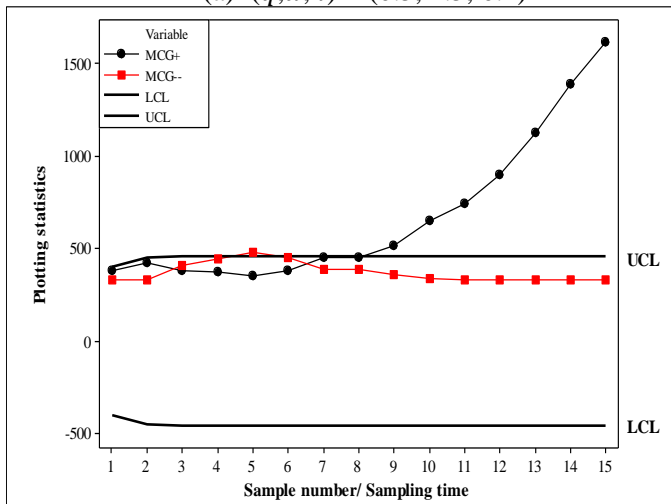
(a)  $(q, \alpha, k) = (0.5, 1.5, 0.1)$



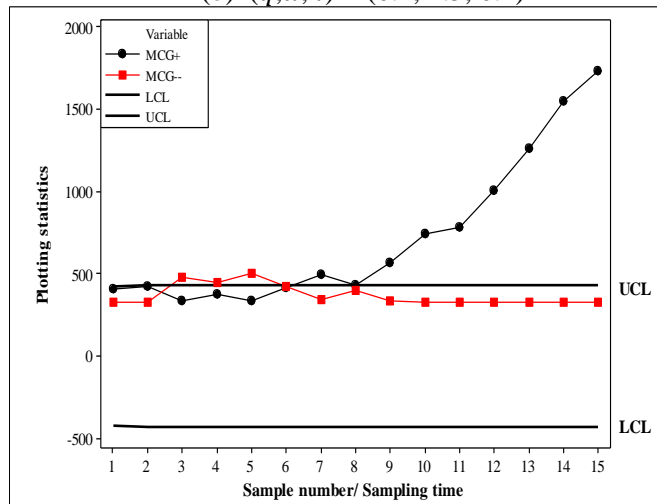
(b)  $(q, \alpha, k) = (0.1, 1.5, 0.1)$



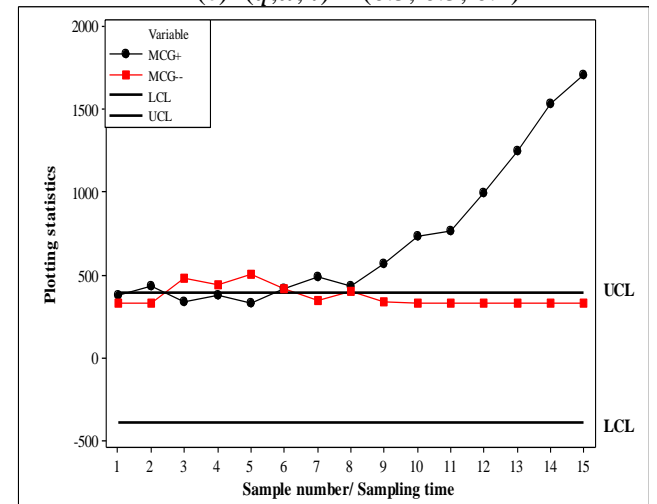
(c)  $(q, \alpha, k) = (0.5, 0.5, 0.1)$



(d)  $(q, \alpha, k) = (0.5, 1.5, 0.5)$



(e)  $(q, \alpha, k) = (0.1, 1.5, 0.5)$



(f)  $(q, \alpha, k) = (0.5, 0.5, 0.5)$

**Figure 6.** Implementation of the U-MCG scheme for the Montgomery (2005)'s piston ring data

## 5. Conclusion and recommendations

In an effort to efficiently monitor small location shifts in a two-sample distribution-free scenario, in this paper, the mixed GWMA-CUSUM and CUSUM-GWMA MW U monitoring schemes (denoted as U-MGC and U-MCG schemes) are proposed. The newly proposed schemes are revealed to be more efficient than the numerous MW-type schemes in monitoring small shifts in the process location. Thus, practitioners are recommended to use the proposed U-MCG monitoring scheme to efficiently detect small shifts. Note though, the U-GWMA scheme discussed herein is recommended for moderate shifts and both the Shewhart MW U and U-GWMA schemes are recommended when monitoring large shifts in the process location. In terms of the overall performance, the U-MCG scheme is recommended for monitoring small-to-moderate shifts and the U-GWMA scheme is preferred for monitoring small-to-large shifts. Since the proposed U-MGC scheme is relatively insensitive to large shifts, the design of an improved version is needed. Therefore, in future we intend to investigate the performance of the combined Shewhart-MGC MW U schemes as well as the Shewhart-MCG MW U schemes in order to further improve the sensitivity of the proposed schemes in monitoring moderate to large shifts in the location parameter.

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