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Abstract

We propose a general framework for simultaneously calculating the threshold value for population growth and determining the sign of the growth bound of the evolution family generated by the problem below

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \mathsf{A}v(t) + \mathsf{F}(t)v(t) - \mathsf{V}(t)v(t),$$

where $\mathsf{A} : D(\mathsf{A}) \subset X \to X$ is a Hille-Yosida linear operator (possibly unbounded, non-densely defined) on a Banach space $(X, \|\cdot\|)$, and the maps $t \in \mathbb{R} \mapsto \mathsf{V}(t) \in \mathcal{L}(X_0, X)$, $t \in \mathbb{R} \mapsto \mathsf{F}(t) \in \mathcal{L}(X_0, X)$ are p-periodic in time and continuous in the operator norm topology. We give applications of our approach for two general examples of an age-structured model, and a delay differential system. Other examples concern the dynamics of a nonlocal problem arising in population genetics and the dynamics of a structured human-vector malaria model.

Keywords: Reproduction number; Growth bound; Threshold dynamics; evolutionary systems
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1 Introduction

The basic reproductive number (universally denoted by the symbol $\mathcal{R}_0$) is an important quantity in epidemiology and life-history theory. The general definition of the $\mathcal{R}_0$ in life-history theory is *the average lifetime offspring number in a given environment.* In epidemiology, the basic reproduction number is an important quantity defined as *the average number of secondary infections produced by a single infective host in an entirely uninfected host population* [23, 10, 1, 2, 8, 16, 9, 18, 19]. Usually, the $\mathcal{R}_0$ allows for expressing the relatively long-term threshold phenomenon for some epidemic models, *i.e.* the disease dies out if the basic reproduction number is less than unity and the disease persists in the population if it is greater than unity. Within a context of multi-strain interactions, $\mathcal{R}_0$ is a function of the pathogen traits or label, x, and so is denoted by $\mathcal{R}_0(x)$. Furthermore, the emphasis on "entirely uninfected host population", illustrated by the index "0" in $\mathcal{R}_0$ is fundamental because $\mathcal{R}_0$ is not only a function of the pathogen x but also of the environment E experienced by the pathogens. Therefore, the basic reproduction number of a pathogen trait x in the environment E is thus written as $\mathcal{R}(x, E)$. In a pathogen-free environment $E = E_0$, note that $\mathcal{R}(x, E_0) = \mathcal{R}_0(x)$. Finally, the quantity $\mathcal{R}(x, E)$ measures the pathogen's "fitness", *i.e.* the pathogens' competitive process in the adaptive dynamics (*e.g.*, see [21]).

The main purpose of this work is to propose a simple and general framework for the calculation of the threshold value for population growth for the below evolution problem

$$\begin{cases} \dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = \mathsf{A}v(t) + \mathsf{F}(t)v(t) - \mathsf{V}(t)v(t),\ t > t_0, \\ v(t_0) = x \in X_0 = \overline{D(\mathsf{A})}, \end{cases} \tag{1.1}$$

where $\mathsf{A} : D(\mathsf{A}) \subset X \to X$ is a possibly unbounded non-densely defined Hille-Yosida linear operator, $(X, \|\cdot\|)$ a Banach space, and the maps $t \in \mathbb{R} \mapsto \mathsf{V}(t) \in \mathcal{L}(X_0, X)$, $t \in \mathbb{R} \mapsto \mathsf{F}(t) \in \mathcal{L}(X_0, X)$ are p-periodic in time and continuous in the operator norm topology. Here, we recall that A is a Hille-Yosida operator if there exists $\omega_0 \in \mathbb{R}$ and $M_0 \geq 1$ such that $(\omega_0, +\infty) \subset \rho(\mathsf{A})$ –the resolvent set of A– and $R_\lambda(\mathsf{A}) = (\lambda I - \mathsf{A})^{-1}$ satisfies

$$\|R_\lambda(\mathsf{A})^n\|_{\mathcal{L}(X)} \leq \frac{M_0}{(\lambda - \omega_0)^n},\ \forall n \geq 1,\ \lambda > \omega_0. \tag{1.2}$$

Note that the threshold value we will obtain for System (1.1) does not necessarily match the basic reproduction number of such a system. However, both quantities will definitely serve as threshold values for the global extinction of the population of concern. Determining the threshold value for population growth for Problem (1.1) is not an easy task in general, and in particular, not yet addressed in the literature with only information that A is a Hille-Yosida linear operator on a given Banach space $(X, \|\cdot\|)$.

Indeed, in the context of both finite-dimensional (*i.e.*, $\mathsf{A} \equiv 0$) and autonomous systems (*i.e.*, F and V are independent of time t), general approaches for the calculations of reproduction numbers are well known (see [8, 31]). Moreover, still being in the context of finite-dimensional systems, but with time-periodic properties (*i.e.*, F and V are p-periodic in time t), computation formulas of the reproduction number are well established for a large class of compartmental epidemic models, see for instance [5, 6, 4, 32]. Furthermore, in [18, 19], the author proposed a very general approach for the computation of the reproduction number in heterogeneous environments, *i.e.* for nonautonomous systems. Such an approach is based on the generation evolution operator, with a clear biological meaning (the GEO approach), and can be applied to a large class of mathematical models including particular cases of problems of type (1.1) when the Banach space is $X = L^1$.

The approach proposed in this work does not necessarily require specifying the Banach space X. Such an approach is based on determining whether the spectral radius of a certain linear operator derived from the evolution family generated by Problem (1.1) is less or greater than the unity. Within this context of infinite-dimensional population structure and time heterogeneity, we can mention the work in [30] where the same idea is developed for problems of type (1.1). However, the results presented here generalized the ones proposed in [30] to the cases where the domain of the operator A is non-dense (*i.e.* $\overline{D(\mathsf{A})} \neq X$).

The advantage in emphasizing the spectral radius of a linear operator derived from the evolution family generated by Problem (1.1) is twice. First, in comparison to results in [18, 19] (when the Banach space $X = L^1$) and in [30] (for any given Banach space X, but with a densely defined operator A), the approach proposed here will allow determining the threshold value for population growth of a large class of nonautonomous systems. The second advantage of such an approach consists in showing at the same time that the sign of the growth bound of the evolution family generated by Problem (1.1) is equal to the sign of the spectral radius of our linear operator minus the unity. Consequently, this allows determining the global convergence to zero of the evolution family generated by Problem (1.1). Finally, we will show that the approach developed here can be applied to time-delay differential systems. Within this context of time-delay differential equations, note that similar results have been obtained in [33, 20].

More precisely, two key assumptions are made to derive our main result. The first assumption concerns the exponential stability of the evolution family $\{U_{\mathsf{V}}(t,t_0)\}_{t\geq t_0}\subset\mathcal{L}(X_0)$ generated by the unperturbed Cauchy problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t}=\mathsf{A}u(t)-\mathsf{V}(t)u(t),\ t>t_0,\ u(t_0)=x\in X_0. \tag{1.3}$$

The existence of such an evolution family follows from the assumption on A and the periodic perturbations (see [24, Proposition 4.1]). Such assumption reads,

Assumption 1.1 *The evolution family $\{U_{\mathsf{V}}(t,t_0)\}_{t\geq t_0}\subset\mathcal{L}(X_0)$ generated by (1.3) is exponentially stable i.e., there exists $M_{\mathsf{V}}\geq 1$ and $\omega_{\mathsf{V}}<0$ such that*

$$\|U_{\mathsf{V}}(t,t_0)x\|\leq M_{\mathsf{V}}e^{\omega_{\mathsf{V}}(t-t_0)}\|x\|,\ \forall t\geq t_0,\ x\in X_0.$$

Next, since we are concerned with the threshold for population dynamics models, the second assumption guarantee that any solution of (1.1) with an initial condition with a nonnegative initial condition remains nonnegative, in forwarding time. Throughout the paper, if otherwise stated, we assume that the Banach space X has a positive cone X_+ that is normal and generating. We recall that X_+ is normal if there exists an equivalent norm $\|\cdot\|_1$ such that

$$y-x\in X_+\Longrightarrow\|x\|_1\leq\|y\|_1.$$

The cone X_+ is called generating if $X=X_+-X_-$. Hence, setting $X_0=\overline{D(\mathsf{A})}$, it is clear that $X_{0+}=X_0\cap X_+$ is a positive cone of X_0 that is normal and generating. Recall that the exponential bound of an evolution family $\{U(t,t_0)\}_{t\geq t_0}\subset\mathcal{L}(X_0)$ is defined by

$$\omega(U)=\inf\left\{\omega\in\mathbb{R}:\text{it exits } M_\omega\geq 1,\ \text{such that } \|U(t+s,s)\|_{\mathcal{L}(X_0)}\leq M_\omega e^{\omega t},\ \forall t\geq 0,\ \forall s\in\mathbb{R}\right\}.$$

Assumption 1.1 for the exponential stability of the evolution family $\{U_{\mathsf{V}}(t,t_0)\}_{t\geq t_0}\subset\mathcal{L}(X_0)$ is classical in obtaining a threshold value even in finite dimension autonomous cases, *eg.*, see [30, 33]. Our next assumption reads as

Assumption 1.2 *i) $\mathsf{A}:D(\mathsf{A})\subset X_0\to X$ is resolvent positive i.e., there exists $\lambda_0>\omega_0$ such that $R_\lambda(\mathsf{A})X_+\subset X_{0+}$ for all $\lambda\geq\lambda_0$, where ω_0 is the constant stated in (1.2).*

ii) There exists $\lambda_1>\omega_0$ such that $\lambda_1 x-\mathsf{V}(t)x\in X_+$ for all $x\in X_{0+}$ and $t\in\mathbb{R}$.

iii) For each $t\in\mathbb{R}$ and each $x\in X_{0+}$ we have $\mathsf{F}(t)x\in X_+$.

Before stating our main result, we make a short comment on condition ii) of Assumption 1.2. In fact, it can be replaced by the following more general condition

$$\lim_{h\to 0^+}\frac{1}{h}\mathrm{dist}(x-h\mathsf{V}(t)x,X_+)=0,\ \forall x\in X_{0+},\ t\in\mathbb{R}. \tag{1.4}$$

where $\mathrm{dist}(\cdot)$ is the distance to a set. We refer to [11, Theorem 4.4] for equivalent characterization of (1.4). Therefore, the main result of this work is given by the following theorem

Theorem 1.3 *Let Assumptions 1.1 and 1.2 be satisfied. Then the linear operator $\mathcal{C}$ defined by*

$$\mathcal{C}[f](t)=\lim_{\lambda\to+\infty}\int_{-\infty}^{t}U_{\mathsf{V}}(t,s)\lambda R_\lambda(\mathsf{A})\mathsf{F}(s)f(s)\mathrm{d}s,\ t\in\mathbb{R},\ f\in C_p(\mathbb{R},X_0), \tag{1.5}$$

is bounded and maps $C_p(\mathbb{R}, X_0)$ –resp. $C_p(\mathbb{R}, X_{0+})$– into $C_p(\mathbb{R}, X_0)$ –resp. $C_p(\mathbb{R}, X_{0+})$. Moreover,

$$\operatorname{sign}\left(\mathsf{r}(\mathcal{C}) - 1\right) = \operatorname{sign}\left(\mathsf{r}(U_{\mathsf{F}}(p,0)) - 1\right), \tag{1.6}$$

where $\{U_{\mathsf{F}}(t,t_0)\}_{t\geq t_0} \subset \mathcal{L}(X_0)$ is the p-periodic evolution family generates by (1.1), and $\mathsf{r}(\mathcal{C})$, $\mathsf{r}(U_{\mathsf{F}}(p,0))$ are the spectral radius of $\mathcal{C}$ and $U_{\mathsf{F}}(p,0)$.

Note that, the identity (1.6) between the spectral radius $\mathsf{r}(\mathcal{C})$ of the linear operator $\mathcal{C}$ derived from the evolution family $\{U_{\mathsf{F}}(t,t_0)\}_{t\geq t_0}$ which generated by (1.1) allows characterizing a threshold value for the population growth of some class of nonautonomous systems on a given Banach space X. Such a result generalized the one in [18, 19] when $X = L^1$, and in [30], for any given Banach space X with a densely defined operator A. Importantly, the above theorem also allows determining the global convergence to zero of the evolution family generated by Problem (1.1). Finally, we think that further technical development is necessary to have a similar result as in Theorem 1.3 as a base of the linearized stability principle of the nonautonomous nonlinear problems in general. However, such a linearized stability principle is well established in the case of densely defined linear operators on $X = L^1$ [19].

The rest of this work is organized as follows. Section 2 is devoted to preliminary results and some general remarks. In Section 3, we give detailed proof of our main result (Theorem 1.3) in several steps. Finally, in Section 4 we give some applications of our main result presented in Theorem 1.3. This includes two general examples of where our results can be applied. The first example is about an age-structured model, and the second concerns a delay differential system. The next two examples concern the dynamics of a nonlocal problem arising in population genetics and the dynamics of a structured human-vector malaria model.

2 Preliminaries and comments on Theorem 1.3

We recall that the part of A in $X_0 = \overline{D(\mathsf{A})}$ is the linear operator $\mathsf{A}_0 : D(\mathsf{A}_0) \subset X_0 \to X$ given by

$$D(\mathsf{A}_0) = \{x \in D(\mathsf{A}) : \mathsf{A}x \in X_0\}, \ \mathsf{A}_0 x = \mathsf{A}x, \ \forall x \in D(\mathsf{A}_0).$$

Since A is a Hille-Yosida linear operator, see (1.2), the resolvent set $\rho(\mathsf{A})$ of A is such that, $\rho(\mathsf{A}) = \rho(\mathsf{A}_0)$, $\overline{D(\mathsf{A}_0)} = \overline{D(\mathsf{A})} = X_0$ and A_0 is a Hille-Yosida linear operator with dense domain [24]. Consequently, A_0 generates a C_0-semigroup $\{T_{\mathsf{A}_0}(t)\}_{t\geq 0} \subset \mathcal{L}(X_0)$ satisfying

$$\|T_{\mathsf{A}_0}(t)\|_{\mathcal{L}(X_0)} \leq M_0 e^{\omega_0 t}, \ \forall t \geq 0. \tag{2.1}$$

Furthermore, noting that $t \in \mathbb{R} \mapsto \mathsf{V}(t)$ is uniformly bounded, the Cauchy problem (1.3) generates an exponentially bounded evolution family $\{U_{\mathsf{V}}(t,t_0)\}_{t\geq t_0} \subset \mathcal{L}(X_0)$ [24, Proposition 4.1]. Moreover, for each $x \in X_0$ and $t_0 \in \mathbb{R}$, the map defined by

$$u(t) = U_{\mathsf{V}}(t,t_0)x, \ t \geq t_0$$

is the unique mild solution of (1.3) with initial condition x at time $t = t_0$ and

$$U_{\mathsf{V}}(t,t_0)x = T_{\mathsf{A}_0}(t-t_0)x + \lim_{\lambda\to+\infty} \int_{t_0}^{t} T_{\mathsf{A}_0}(t-s)\lambda R_\lambda(\mathsf{A})\mathsf{V}(s)U_{\mathsf{V}}(s,t_0)x\mathrm{d}s, \ \forall x \in X_0. \tag{2.2}$$

In a similar manner, using the fact that $t \mapsto \mathsf{F}(t)$ is uniformly bounded in $\mathbb{R}$, one can conclude from [26, Theorem 1.6] that Problem (1.1) generates an exponentially bounded evolution family $\{U_{\mathsf{F}}(t,t_0)\}_{t\geq t_0} \subset$

$\mathcal{L}(X_0)$. Moreover, using the results in [26, Theorem 1.6] one knows that for each $x \in X_0$ and $t_0 \in \mathbb{R}$, the map

$$u(t) = U_{\mathsf{F}}(t, t_0)x, \ t \geq t_0$$

is the unique solution of (1.1) with initial condition x at time $t = t_0$ with

$$U_{\mathsf{F}}(t, t_0)x = U_{\mathsf{V}}(t, t_0)x + \lim_{\lambda \to +\infty} \int_{t_0}^{t} U_{\mathsf{V}}(t, s)\lambda R_\lambda(\mathsf{A})\mathsf{F}(s)U_{\mathsf{F}}(s, t_0)x\mathrm{d}s, \ \forall x \in X_0. \tag{2.3}$$

It is now clear from (2.2) and (2.3) that the evolution families $\{U_{\mathsf{V}}(t, t_0)\}_{t \geq t_0} \subset \mathcal{L}(X_0)$ and $\{U_{\mathsf{F}}(t, t_0)\}_{t \geq t_0} \subset \mathcal{L}(X_0))$ are p-periodic, that is

$$U_{\mathsf{V}}(t + p, t_0 + p) = U_{\mathsf{V}}(t, t_0), \ \forall t \geq t_0, \ \text{ and } \ U_{\mathsf{F}}(t + p, t_0 + p) = U_{\mathsf{F}}(t, t_0), \ \forall t \geq t_0.$$

Remark 2.1 *When $\mathsf{V}(t) \equiv 0$ we have $U_{\mathsf{V}}(t, t_0) = T_{\mathsf{A}_0}(t - t_0)$ so that Assumption 1.1 is translated to $\omega_0 < 0$, where ω_0 is the constant stated in (2.1).*

The proof of Theorem 1.3 will be given in Section 3. However, we complete this section by adding some remarks that may be useful for applications of our main result. Let us first note that the linear operator $\mathcal{C}$ in Theorem 1.3 can be expressed as follow

$$\mathcal{C}[f](t) = \lim_{t_0 \to -\infty} \lim_{\lambda \to +\infty} \int_{t_0}^{t} U_{\mathsf{V}}(t, s)\lambda R_\lambda(\mathsf{A})\mathsf{F}(s)f(s)\mathrm{d}s, \ t \in \mathbb{R}, \ f \in C_p(\mathbb{R}, X_0),$$

and for the case $\mathsf{V}(t) \equiv 0$, it takes the following form

$$\mathcal{C}[f](t) = \lim_{t_0 \to -\infty} \lim_{\lambda \to +\infty} \int_{t_0}^{t} T_{\mathsf{A}_0}(t - s)\lambda R_\lambda(\mathsf{A})\mathsf{F}(s)f(s)\mathrm{d}s, \ t \in \mathbb{R}, \ f \in C_p(\mathbb{R}, X_0).$$

Since the evolution family $\{U_{\mathsf{F}}(t, t_0)\}_{t \geq t_0}$ is exponentially bounded and periodic, we infer from [17, Theorem 7.2.2] that the spectrum $\sigma(U_{\mathsf{F}}(p + s, s))$, $s \in \mathbb{R}$ satisfies

$$\sigma(U_{\mathsf{F}}(p + s, s)) \setminus \{0\} = \sigma(U_{\mathsf{F}}(p, 0)) \setminus \{0\}, \ \forall s \in \mathbb{R}$$

and from [30, Proposition 5.5] that

$$\omega(U_{\mathsf{F}}) = \frac{\ln\left(\mathsf{r}(U_{\mathsf{F}}(p + s, s))\right)}{p} = \lim_{t \to +\infty} \frac{\ln \|U_{\mathsf{F}}(t + s, s))\|}{t}, \ \forall s \in \mathbb{R}.$$

Moreover, we also have from [30, Theorem 5.7] that

$$\operatorname{sign}\left(\omega(U_{\mathsf{F}})\right) = \operatorname{sign}\left(\mathsf{r}(U_{\mathsf{F}}(s + p, s)) - 1\right), \ \forall s \in \mathbb{R}. \tag{2.4}$$

From where, using Theorem 1.3 together with (2.4) we obtain

$$\operatorname{sign}\left(\omega(U_{\mathsf{F}})\right) = \operatorname{sign}\left(\mathsf{r}(\mathcal{C}) - 1\right) = \operatorname{sign}\left(\mathsf{r}(U_{\mathsf{F}}(s + p, s)) - 1\right), \ \forall s \in \mathbb{R}.$$

Finally, by setting

$$\mathcal{F}[f](t) = \mathsf{F}(t)f(t), \ \forall t \in \mathbb{R}, \ f \in C_p(\mathbb{R}, X_0) \tag{2.5}$$

and

$$\mathcal{R}[f](t) \ = \ \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U_{\mathsf{V}}(t, s)\lambda R_\lambda(\mathsf{A})f(s)\mathrm{d}s, \ t \in \mathbb{R}, \ f \in C_p(\mathbb{R}, X)$$

it comes

$$\mathcal{C} = \mathcal{R} \circ \mathcal{F},$$

and since $\mathsf{r}(\mathcal{F} \circ \mathcal{R}) = \mathsf{r}(\mathcal{R} \circ \mathcal{F})$ we deduce that $\mathsf{r}(\mathcal{C}) = \mathsf{r}(\mathcal{F} \circ \mathcal{R})$ where

$$(\mathcal{F} \circ \mathcal{R})[f](t) = \mathsf{F}(t) \lim_{t_0 \to -\infty} \lim_{\lambda \to +\infty} \int_{t_0}^{t} U_{\mathsf{V}}(t,s) \lambda R_\lambda(\mathsf{A}) f(s) \mathrm{d}s, \forall t \in \mathbb{R},\ f \in C_p(\mathbb{R}, X). \tag{2.6}$$

The latter gives an alternative way to compute $\mathsf{r}(\mathcal{C})$ and is sometimes easier to handle. In many applications, the positive perturbation $\mathsf{F}(t)$ maps X_0 into X_1, with X_1 a closed subspace of X so that the linear operator $\mathcal{F} \circ \mathcal{R}$ maps $C_p(\mathbb{R}, X_1)$ into $C_p(\mathbb{R}, X_1)$. The following lemma will allow simplifying the computation of $\mathsf{r}(\mathcal{C})$ in our applications.

Lemma 2.2 *Let Assumptions 1.1 and 1.2 be satisfied. Assume that there exists a closed subspace $X_1 \subset X$ such that $\mathsf{F}(t)X_0 \subset X_1$ for all $t \in \mathbb{R}$. Then $\mathsf{r}(\mathcal{C}) = \mathsf{r}(\mathcal{F} \circ \mathcal{R}) = \mathsf{r}(\mathcal{C}_1)$ with $\mathcal{C}_1$ the restriction of $\mathcal{F} \circ \mathcal{R}$ to $C_p(\mathbb{R}, X_1)$.*

The lemma can be easily proved by using Gelfand's formula and the proof is thus omitted.

Remark 2.3 *In practice, we may combine the fact that $\mathsf{r}(\mathcal{C}) = \mathsf{r}(\mathcal{F} \circ \mathcal{R})$ together with Lemma 2.2 to simplify the determination of the linear operator $\mathcal{F} \circ \mathcal{R}$. Indeed, under the condition of Theorem 1.3, and $\mathsf{F}(t)X_0 \subset X_1$, the map*

$$\lim_{t_0 \to -\infty} \lim_{\lambda \to +\infty} \int_{t_0}^{t} U_{\mathsf{V}}(t,s) \lambda R_\lambda(\mathsf{A}) f(s) \mathrm{d}s, \forall t \in \mathbb{R},\ f \in C_p(\mathbb{R}, X_1)$$

is the unique entire solution of

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \mathsf{A}u(t) - \mathsf{V}(t)u(t) + f(t),\ t \in \mathbb{R}.$$

3 Proof of Theorem 1.3

The main point in the proof of our main result lies in the fact that we will be able to go from a nonautonomous problem to an autonomous problem by using the evolution semigroup approach. The results obtained in the autonomous case are then transferred to the nonautonomous case. Let us mention that such an approach has been used in [15] in the context of extrapolated semigroup and functional differential equations. We also refer to [19] where the evolution semigroups have been used successfully in defining the basic reproduction number for nonautonomous problems. In what follows, it is not necessary that X has a normal and generating cone X_+ nor that the Assumption 1.2 be satisfied. However the linear operator $\mathsf{A} : D(\mathsf{A}) \subset X \to X$ is still assumed to be Hille-Yosida while the maps $t \to \mathsf{V}(t)$, and $t \to \mathsf{F}(t)$ are assumed to be p-periodic and continuous in the operator norm topology.

3.1 Step 1: From the nonautonomous to an autonomous problem

We first mention some known results which will be of importance later on. Consider the map

$$f \in C_p(\mathbb{R}, X_0) \mapsto U_{\mathsf{V}}(\cdot, \cdot - t) f(\cdot - t) \in C_p(\mathbb{R}, X_0), \tag{3.1}$$

with

$$(U_{\mathsf{V}}(\cdot, \cdot - t) f(\cdot - t))(s) = U_{\mathsf{V}}(s, s - t) f(s - t),\ \forall s \in \mathbb{R}.$$

It is well known that the map (3.1) defines a C_0-semigroup on $C_p(\mathbb{R}, X_0)$ (see for example [7] and [30, Appendix B]). We denote its generator by $-\mathcal{V}_0 : D(\mathcal{V}_0) \subset C_p(\mathbb{R}, X_0) \to C_p(\mathbb{R}, X_0)$ and we set

$$T_{-\mathcal{V}_0}(t)[f] = U_{\mathsf{V}}(\cdot, \cdot - t)f(\cdot - t),\ \forall f \in C_p(\mathbb{R}, X_0),\ t \geq 0.$$

Our arguments for the proof of Theorem 1.3 as well as the main result of this section strongly rely on the linear operator

$$\mathcal{R}_\lambda[f](t) := \lim_{\mu \to +\infty} \int_{-\infty}^{t} e^{-\lambda(t-s)} U_{\mathsf{V}}(t,s)\mu R_\mu(\mathsf{A})f(s)\mathrm{d}s,\ t \in \mathbb{R},\ f \in C_p(\mathbb{R}, X),\ \lambda > \omega_{\mathsf{V}}, \tag{3.2}$$

for which we will first investigate its properties. Before proceeding, we first recall some results obtained in [26] in the context of the integrated semigroup. A similar result can be found in [14] where extrapolated semigroups approach is used. Denote by $BUC_{\mathsf{r}}(\mathbb{R}, X)$ the Banach space of bounded and uniformly continuous functions with relatively compact range. Note that we always have the inclusions $C_p(\mathbb{R}, X_0) \subset C_p(\mathbb{R}, X) \subset BUC_r(\mathbb{R}, X)$ where all the Banach spaces are endowed with the supremum norm. For more compactness in the notations, we set

$$\mathcal{X} := C_p(\mathbb{R}, X) \ \text{ and } \ \mathcal{X}_0 := C_p(\mathbb{R}, X_0)$$

with the norm in $\mathcal{X}$ and $\mathcal{X}_0$ defined by

$$\|f\|_\infty := \sup_{t \in \mathbb{R}} \|f(t)\|,\ \forall f \in \mathcal{X}.$$

The next result can be obtained from [26, Theorem 1.11].

Lemma 3.1 *Let Assumption 1.1 be satisfied. Then the following properties hold true:*

i) For each $f \in BUC_r(\mathbb{R}, X)$ there exists a unique entire mild solution $u_f \in C(\mathbb{R}, X_0)$ of

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \mathsf{A}u(t) - \mathsf{V}(t)u(t) + f(t),\ t > t_0,\ u(t_0) = x_0 \in X_0,$$

given by

$$u_f(t) = \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U_{\mathsf{V}}(t,s)\lambda R_\lambda(\mathsf{A})f(s)\mathrm{d}s,\ \forall t \in \mathbb{R}, \tag{3.3}$$

where the limit is uniform with respect to t in $\mathbb{R}$. Moreover, for each $f \in \mathcal{X}_0$, the above limit (3.3) takes the form

$$u_f(t) = \int_{-\infty}^{t} U_{\mathsf{V}}(t,s)f(s)\mathrm{d}s,\ \forall t \in \mathbb{R}.$$

ii) There exists a constant $C_D > 0$ such that for each $f \in BUC_r(\mathbb{R}, X)$ the solution $u_f \in C(\mathbb{R}, X_0)$ satisfies

$$\|u_f\|_\infty \leq C_D \|f\|_\infty.$$

The Lemma 3.1 ensures that the linear operator $\mathcal{R}_\lambda$ defined in (3.2) is continuous from $\mathcal{X}$ into $\mathcal{X}_0$ with $\mathcal{R}_\lambda \mathcal{X} \subset \mathcal{X}_0$. However, to be able to perform our study, we need to obtain a more refined estimate on the operator norm of $\mathcal{R}_\lambda$. It reads as

Lemma 3.2 *Let Assumption 1.1 be satisfied. For each $\lambda > \omega_{\mathsf{V}}$, the linear operator $f \in \mathcal{X} \mapsto \mathcal{R}_\lambda[f]$ is continuous in $\mathcal{X}$ with*

$$\|\mathcal{R}_\lambda[f]\|_\infty \leq \frac{\widehat{M}_{\mathsf{V}}}{\lambda - \omega_{\mathsf{V}}} \|f\|_\infty, \ \forall f \in \mathcal{X}, \tag{3.4}$$

with $\widehat{M}_{\mathsf{V}} = M_0 M_{\mathsf{V}}$.

Proof. Let $f \in \mathcal{X}$ be given and fixed. Note that for each $\mu > \max(\omega_0, \omega_{\mathsf{V}})$, the linear operator

$$\mathcal{R}^\mu_\lambda[f](t) := \int_{-\infty}^t e^{-\lambda(t-s)} U_{\mathsf{V}}(t,s) \mu R_\mu(\mathsf{A}) f(s) \mathrm{d}s, \ t \in \mathbb{R}, \ f \in \mathcal{X}$$

satisfies

$$\|\mathcal{R}^\mu_\lambda[f]\|_\infty \leq \frac{M_{\mathsf{V}}}{\lambda - \omega_{\mathsf{V}}} \frac{\mu M_0}{\mu - \omega_0} \|f\|_\infty, \ \forall \lambda > \omega_0. \tag{3.5}$$

and the result follows by taking the limit when $\mu \to +\infty$ in (3.5). ∎

Theorem 3.3 *Let Assumption 1.1 be satisfied. There exists a Hille-Yosida linear operator $-\mathcal{V} : D(\mathcal{V}) \subset \mathcal{X} \to \mathcal{X}$ on $\mathcal{X}$ such that $-\mathcal{V}_0$ is the part of $-\mathcal{V}$ in $\mathcal{X}_0$. Moreover, $(\omega_{\mathsf{V}}, +\infty) \subset \rho(-\mathcal{V})$ and for each $\lambda > \omega_{\mathsf{V}}$ we have*

$$(\lambda + \mathcal{V})^{-1}[f] := R_\lambda(-\mathcal{V})[f] = \mathcal{R}_\lambda[f], \ \forall f \in \mathcal{X},$$

and

$$\|R_\lambda(-\mathcal{V})^n[f]\|_\infty \leq \frac{\widehat{M}_0}{(\lambda - \omega_{\mathsf{V}})^n} \|f\|_\infty, \ \forall n \geq 1,$$

with $\widehat{M}_0 = M_{\mathsf{V}} \widehat{M}_{\mathsf{V}}$.

Proof. Thanks to Lemma 3.1, $f \mapsto \mathcal{R}_\lambda[f]$ maps $\mathcal{X}$ into $\mathcal{X}_0$ *i.e.,*

$$\mathcal{R}_\lambda[f] \in C_p(\mathbb{R}, X_0), \ \forall f \in C_p(\mathbb{R}, X) \tag{3.6}$$

and

$$\mathcal{R}_\lambda[f] = 0 \Longleftrightarrow f = 0. \tag{3.7}$$

Moreover, using Lemma 3.1 (with A replaced respectively by $\mathsf{A} - \lambda$ and $\mathsf{A} - \delta$) and (3.6) it follows that for each $f \in C_p(\mathbb{R}, X)$, $\lambda > \omega_{\mathsf{V}}$ and $\delta > \omega_{\mathsf{V}}$ we have

$$\mathcal{R}_\lambda \circ \mathcal{R}_\delta[f](t) := \int_{-\infty}^t e^{-\lambda(t-s)} U_{\mathsf{V}}(t,s) \mathcal{R}_\delta[f](s) \mathrm{d}s, \ \forall t \in \mathbb{R}. \tag{3.8}$$

We claim that for each $n \geq 2$ and $f \in C_p(\mathbb{R}, X)$

$$\mathcal{R}^n_\lambda[f](t) := \int_{-\infty}^t \frac{(t-s)^{n-2}}{(n-2)!} e^{-\lambda(t-s)} U_{\mathsf{V}}(t,s) \mathcal{R}_\lambda[f](s) \mathrm{d}s, \ \forall t \in \mathbb{R}. \tag{3.9}$$

To prove the above claim, we argue by recurrence in $n \geq 2$. From (3.8), one can see that the property is clearly satisfied when $n = 2$. Assume that (3.9) is satisfied for some $n \geq 2$. Note that for each $n \geq 2$ and $t \in \mathbb{R}$

$$\begin{aligned}
\mathcal{R}^{n+1}_\lambda[f](t) &= \int_{-\infty}^t e^{-\lambda(t-s)} U_{\mathsf{V}}(t,s) \mathcal{R}^n_\lambda[f](s) \mathrm{d}s \\
&= \int_{-\infty}^t \int_{-\infty}^s \frac{(s-l)^{n-2}}{(n-2)!} e^{-\lambda(t-l)} U_{\mathsf{V}}(t,l) \mathcal{R}_\lambda[f](l) \mathrm{d}l \mathrm{d}s \\
&= \int_{-\infty}^t \int_l^t \frac{(s-l)^{n-2}}{(n-2)!} e^{-\lambda(t-l)} U_{\mathsf{V}}(t,l) \mathcal{R}_\lambda[f](l) \mathrm{d}s \mathrm{d}l \\
&= \int_{-\infty}^t \frac{(t-l)^{n-1}}{(n-1)!} e^{-\lambda(t-l)} U_{\mathsf{V}}(t,l) \mathcal{R}_\lambda[f](l) \mathrm{d}l
\end{aligned}$$

which proves the claim. Therefore, using the above formula with n instead of $n+1$ combined with (3.4) we obtain

$$\|\mathcal{R}_\lambda^n[f]\|_\infty \leq \frac{M_{\mathsf{V}}}{(\lambda-\omega_{\mathsf{V}})^{n-1}}\|\mathcal{R}_\lambda[f]\|_\infty \leq \frac{M_{\mathsf{V}}\widehat{M}_{\mathsf{V}}}{(\lambda-\omega_{\mathsf{V}})^n}\|f\|_\infty,\ \forall \lambda > \omega_{\mathsf{V}}.$$

Next, we prove that $\mathcal{R}_\lambda$ satisfies the resolvent identity. Let $\lambda > \omega_{\mathsf{V}}$ and $\delta > \omega_{\mathsf{V}}$ be given and fixed such that $\lambda \neq \delta$. Then using (3.8) one has

$$\mathcal{R}_\lambda \circ \mathcal{R}_\delta[f](t) = \int_{-\infty}^t e^{-\lambda(t-s)}U_{\mathsf{V}}(t,s)\lim_{\mu\to+\infty}\left(\int_{-\infty}^t e^{-\delta(s-l)}U_{\mathsf{V}}(s,l)\mu R_\mu(\mathsf{A})f(l)\mathrm{d}l\right)\mathrm{d}s.$$

Since $f \in \mathcal{X} = C_p(\mathbb{R}, X)$, and $\delta > \omega_{\mathsf{V}}$ it follows that the limit

$$\lim_{\mu\to+\infty}\left(\int_{-\infty}^t e^{-\delta(s-l)}U_{\mathsf{V}}(s,l)\mu R_\mu(\mathsf{A})f(s)\right)\mathrm{d}s$$

exists uniformly for $t \in \mathbb{R}$ (see Lemma 3.1 with A replaced by $A-\delta$). From where we obtain

$$\begin{aligned}\mathcal{R}_\lambda \circ \mathcal{R}_\delta[f](t) &= \lim_{\mu\to+\infty}\int_{-\infty}^t\left(\int_{-\infty}^t e^{-\delta(s-l)}e^{-\lambda(t-s)}U_{\mathsf{V}}(t,l)\mu R_\mu(\mathsf{A})f(l)\mathrm{d}l\right)\mathrm{d}s\\ &= \lim_{\mu\to+\infty}\int_{-\infty}^t\left(\int_l^t e^{-\delta(s-l)}e^{-\lambda(t-s)}U_{\mathsf{V}}(t,l)\mu R_\mu(\mathsf{A})f(l)\mathrm{d}s\right)\mathrm{d}l\end{aligned}$$

which gives after integration

$$\mathcal{R}_\lambda \circ \mathcal{R}_\delta[f](t) = \frac{1}{\lambda-\delta}\left(\mathcal{R}_\delta[f](t) - \mathcal{R}_\lambda[f](t)\right),\ \forall t \in \mathbb{R}.$$

Because $\mathcal{R}_\lambda$, $\lambda > \omega_{\mathsf{V}}$ is a family of bounded linear operators that satisfies (3.7) and the resolvent formula in $(\omega_{\mathsf{V}}, +\infty)$ we deduce from [3, Proposition B.6] there exists a closed linear operator $-\mathcal{V}$ such that $\mathcal{R}_\lambda = R_\lambda(-\mathcal{V})$.

To complete the proof, it remains to prove that $-\mathcal{V}_0$ is the part of $-\mathcal{V}$ in $\mathcal{X}_0 = C_p(\mathbb{R}, X_0)$. To do so, let $f \in \mathcal{X}_0$ be given. Then for each $t \in \mathbb{R}$ we have

$$\begin{aligned}R_\lambda(-\mathcal{V})[f](t) &= \lim_{\mu\to+\infty}\int_{-\infty}^t e^{-\lambda(t-s)}U_{\mathsf{V}}(t,s)\mu R_\mu(\mathsf{A})f(s)\mathrm{d}s\\ &= \int_{-\infty}^t e^{-\lambda(t-s)}U_{\mathsf{V}}(t,s)f(s)\mathrm{d}s\\ &= \int_0^{+\infty} e^{-\lambda s}U_{\mathsf{V}}(t,t-s)f(t-s)\mathrm{d}s\\ &= \int_0^{+\infty} e^{-\lambda s}\mathcal{T}_{-\mathcal{V}_0}(s)[f](t)\mathrm{d}s\\ &= R_\lambda(-\mathcal{V}_0)[f](t)\end{aligned}$$

providing that

$$R_\lambda(-\mathcal{V})[f] = R_\lambda(-\mathcal{V}_0)[f],\ \forall f \in \mathcal{X}_0. \tag{3.10}$$

The equality (3.10) has the following immediate consequence

$$D(\mathcal{V}_0) \subset D(\mathcal{V}) \text{ and } -\mathcal{V}_0[f] = -\mathcal{V}[f],\ \forall f \in D(\mathcal{V}_0).$$

Let $f \in D(\mathcal{V})$ be given such that $-\mathcal{V}[f] \in \mathcal{X}_0$. Note that (3.6) ensures that $D(\mathcal{V}) \subset \mathcal{X}_0$. Then using the equality (3.10) we have

$$f = R_\lambda(-\mathcal{V})(\lambda + \mathcal{V})[f] = \lambda R_\lambda(-\mathcal{V}_0)[f] + R_\lambda(-\mathcal{V}_0)\mathcal{V}[f] \in D(\mathcal{V}_0).$$

The proof is completed. ∎

Noting that the linear operator $\mathcal{F}$ defined in (2.5) is continuous from $\mathcal{X}_0$ to $\mathcal{X}$, Theorem 3.3 ensures that the part $(-\mathcal{V}+\mathcal{F})_0$ of $-\mathcal{V}+\mathcal{F}$ generates a strongly continuous C_0-semigroup $\{T_{(-\mathcal{V}+\mathcal{F})_0}(t)\}_{t\geq 0}$ on $\mathcal{X}_0$. Moreover, it is uniquely determined by

$$T_{(-\mathcal{V}+\mathcal{F})_0}(t)[f] = T_{-\mathcal{V}_0}(t)[f] + \lim_{\lambda\to\infty}\int_0^t T_{-\mathcal{V}_0}(t-l)\lambda R_\lambda(-\mathcal{V})\mathcal{F}T_{(-\mathcal{V}+\mathcal{F})_0}(l)[f]\mathrm{d}l,\ \forall t \geq 0,\ \forall f \in \mathcal{X}_0. \tag{3.11}$$

Consider the C_0-semigroup on $\mathcal{X}_0$ defined by

$$\mathcal{T}_0(t)[f](s) = U_{\mathsf{F}}(s, s-t)f(s-t),\ \forall s \in \mathbb{R},\ t \geq 0.$$

In the following, we will prove that the semigroups $\{T_{(-\mathcal{V}+\mathcal{F})_0}(t)\}_{t\geq 0}$ and $\{\mathcal{T}_0(t)\}_{t\geq 0}$ coincides in $\mathcal{X}_0$. To do so, we will prove that $\{\mathcal{T}_0(t)\}_{t\geq 0}$ satisfies (3.11) and conclude by the uniqueness of the solution to (3.11). The following result will be crucial in our arguments.

Proposition 3.4 *Let Assumption 1.1 be satisfied. Then for each $f \in \mathcal{X}$ and each $t \in \mathbb{R}$, the map $s \mapsto g(t,s)$ defined by*

$$g(t,s) := \lim_{\mu\to+\infty}\int_0^t U_{\mathsf{V}}(s, s-t+l)\mu R_\mu(\mathsf{A})\mathsf{F}(s-t+l)\mathcal{T}_0(l)[f](s-t+l)\mathrm{d}l,\ \forall s \in \mathbb{R}$$

satisfies $g(t,\cdot) \in \mathcal{X}_0$ and

$$\lambda R_\lambda(-\mathcal{V})[g(t,\cdot)](s) = \int_0^t T_{-\mathcal{V}_0}(t-l)\lambda R_\lambda(-\mathcal{V})\mathcal{F}\mathcal{T}_0(l)[f](s)\mathrm{d}l,\ \forall s \in \mathbb{R}, \tag{3.12}$$

as well as the identity

$$g(t,s) = \lim_{\lambda\to+\infty}\int_0^t T_{-\mathcal{V}_0}(t-l)\lambda R_\lambda(-\mathcal{V})\mathcal{F}\mathcal{T}_0(l)[f](s)\mathrm{d}l,\ \forall s \in \mathbb{R}. \tag{3.13}$$

Proof. By definition, the map $g(t,\cdot)$ belongs in $\mathcal{X}_0$. Thanks to Lemma 3.1 with A replaced with $\mathsf{A}-\lambda$, all the limits below exist uniformly for $t \in \mathbb{R}$. Since $g(t,\cdot)$ belongs in $\mathcal{X}_0$ we have

$$Z_t^s := R_\lambda(-\mathcal{V})[g(t,\cdot)](s) = \int_{-\infty}^s e^{-\lambda(s-r)}U_{\mathsf{V}}(s,r)g(t,r)\mathrm{d}r,\ \forall s \in \mathbb{R},$$

so that

$$\begin{aligned}
Z_t^s &= \lim_{\mu\to+\infty}\int_{-\infty}^s\int_0^t e^{-\lambda(s-r)}U_{\mathsf{V}}(s,r-t+l)\mu R_\mu(\mathsf{A})\mathsf{F}(r-t+l)\mathcal{T}_0(l)[f](r-t+l)\mathrm{d}l\mathrm{d}r\\
&= \lim_{\mu\to+\infty}\int_0^t\int_{-\infty}^s e^{-\lambda(s-r)}U_{\mathsf{V}}(s,r-t+l)\mu R_\mu(\mathsf{A})\mathsf{F}(r-t+l)\mathcal{T}_0(l)[f](r-t+l)\mathrm{d}r\mathrm{d}l\\
&= \lim_{\mu\to+\infty}\int_0^t\int_{-\infty}^{s-t+l} e^{-\lambda(s-t+l-r)}U_{\mathsf{V}}(s,r)\mu R_\mu(\mathsf{A})\mathsf{F}(r)\mathcal{T}_0(l)[f](r)\mathrm{d}r\mathrm{d}l\\
&= \lim_{\mu\to+\infty}\int_0^t U_{\mathsf{V}}(s,s-t+l)\int_{-\infty}^{s-t+l} e^{-\lambda(s-t+l-r)}U_{\mathsf{V}}(s-t+l,r)\mu R_\mu(\mathsf{A})\mathsf{F}(r)\mathcal{T}_0(l)[f](r)\mathrm{d}r\mathrm{d}l\\
&= \int_0^t U_{\mathsf{V}}(s,s-t+l)R_\lambda(-\mathcal{V})(\mathcal{F}\mathcal{T}_0(l)[f])(s-t+l)\mathrm{d}l\\
&= \int_0^t T_{-\mathcal{V}_0}(t-l)R_\lambda(-\mathcal{V})(\mathcal{F}\mathcal{T}_0(l)[f])(s)\mathrm{d}l.
\end{aligned}$$

Since $-\mathcal{V}$ is a Hille-Yosida linear operator on $\mathcal{X}$, the limit (3.13) follows from (3.12) (See for example [25, 29]). ∎

Proposition 3.5 *Let Assumption 1.1 be satisfied. The C_0-semigroups $\{T_{(-\mathcal{V}+\mathcal{F})_0}(t)\}_{t\geq 0}$ and $\{\mathcal{T}_0(t)\}_{t\geq 0}$ coincide in $\mathcal{X}_0$ that is for all $t \geq 0$ and $f \in \mathcal{X}_0$ we have*

$$\mathcal{T}_0(t)[f](s) = T_{(-\mathcal{V}+\mathcal{F})_0}(t)[f](s) = U_{\mathsf{F}}(s, s-t)f(s-t),\ \forall s \in \mathbb{R}.$$

Proof. Let $f \in \mathcal{X}_0$ be given and fixed. Recall that for each $t \geq 0$, $x \in X_0$, and $s \in \mathbb{R}$ we have from (2.3)

$$U_{\mathsf{F}}(s, s-t)x = U_{\mathsf{V}}(s, s-t)x + \lim_{\mu\to+\infty} \int_{s-t}^{s} U_{\mathsf{V}}(s, l)\mu R_\mu(\mathsf{A})\mathsf{F}(l)U_{\mathsf{F}}(l, s-t)x\mathrm{d}l. \tag{3.14}$$

In particular for each $f \in \mathcal{X}_0 = C_p(\mathbb{R}, X_0)$, replacing x by $f(s-t)$ in (3.14) gives for each $t \geq 0$ and $s \in \mathbb{R}$

$$\begin{aligned}
\mathcal{T}_0(t)[f](s) &= T_{-\mathcal{V}_0}(t)[f](s) + \lim_{\mu\to+\infty} \int_{s-t}^{s} U_{\mathsf{V}}(s, l)\mu R_\mu(\mathsf{A})\mathsf{F}(l)U_{\mathsf{F}}(l, s-t)f(s-t)\mathrm{d}l \\
&= T_{-\mathcal{V}_0}(t)[f](s) \\
&\quad + \lim_{\mu\to+\infty} \int_{0}^{t} U_{\mathsf{V}}(s, s-t+l)\mu R_\mu(\mathsf{A})\mathsf{F}(s-t+l)U_{\mathsf{F}}(s-t+l, s-t)f(s-t)\mathrm{d}l \\
&= T_{-\mathcal{V}_0}(t)[f](s) + \lim_{\mu\to+\infty} \int_{0}^{t} U_{\mathsf{V}}(s, s-t+l)\mu R_\mu(\mathsf{A})\mathsf{F}(s-t+l)\mathcal{T}_0(l)[f](s-t+l)\mathrm{d}l.
\end{aligned}$$

Thus, using (3.13) we obtain for each $t \geq 0$

$$\mathcal{T}_0(t)[f](s) = T_{-\mathcal{V}_0}(t)[f](s) + \lim_{\lambda\to+\infty} \int_{0}^{t} T_{-\mathcal{V}_0}(t-l)(\lambda R_\lambda(-\mathcal{V})\mathcal{F}\mathcal{T}_0(l)[f])(s)\mathrm{d}l,\ \forall s \in \mathbb{R}.$$

The proof is completed since the semigroup satisfying (3.11) is uniquely determined. ∎

3.2 Step 2: proof of Theorem 1.3

By making an autonomous reformulation of the problem (1.1) in terms of evolution semigroups in Section 3.1, we now have all the elements necessary for the proof of Theorem 1.3. In this section, we will always assume that Assumption 1.2 and Assumption 1.1 are satisfied. The Banach space X has a positive cone that is normal and generating. Using Theorem 3.3, one knows that there exists a Hille-Yosida linear operator $-\mathcal{V} : D(\mathcal{V}) \subset C_p(\mathbb{R}, X_0) \to C_p(\mathbb{R}, X)$ such that the semigroup $\{T_{(-\mathcal{V}+\mathcal{F})_0}(t)\}$ generated by $(-\mathcal{V}+\mathcal{F})_0$, the part of $-\mathcal{V}+\mathcal{F}$ in $\overline{D(\mathcal{V})} = C_p(\mathbb{R}, X_0)$ is given by

$$T_{(-\mathcal{V}+\mathcal{F})_0}(t)[f](s) = U_{\mathsf{F}}(s, s-t)f(s-t),\ \forall s \in \mathbb{R},\ t \geq 0,\ f \in C_p(\mathbb{R}, X_0). \tag{3.15}$$

Using [30, Lemma 5.8], it follows that the spectral bound of $(-\mathcal{V}+\mathcal{F})_0$ that is $\mathsf{s}((-\mathcal{V}+\mathcal{F})_0)$ and the growth bound $\omega((-\mathcal{V}+\mathcal{F})_0)$ of $\{T_{(-\mathcal{V}+\mathcal{F})_0}(t)\}_{t\geq 0}$ are equal *i.e.*,

$$\mathsf{s}((-\mathcal{V}+\mathcal{F})_0) = \omega((-\mathcal{V}+\mathcal{F})_0).$$

Moreover, using the equality

$$R_\lambda(-\mathcal{V})[f](t) = \lim_{\lambda\to+\infty} \int_{-\infty}^{t} U_{\mathsf{V}}(t, s)\lambda R_\lambda(\mathsf{A})f(s)\mathrm{d}s,\ t \in \mathbb{R}, \lambda > \omega_0$$

it is clear that $-\mathcal{V}$ is resolvent positive and Theorem 3.3 implies $\mathsf{s}(-\mathcal{V}) \leq \omega_{\mathsf{V}} < 0$. Since $\mathcal{F}$ is a positive operator, the positive cone $C_p(\mathbb{R}, X_+)$ is normal and generating, we deduce from [30, Lemma 5.8] that

$$\operatorname{sign}\left(\mathsf{r}(\mathcal{V}^{-1}\mathcal{F}) - 1\right) = \operatorname{sign}\left(\mathsf{s}(-\mathcal{V} + \mathcal{F})\right).$$

Recalling that $(-\mathcal{V} + \mathcal{F})_0$ and $(-\mathcal{V} + \mathcal{F})$ have the same spectrum (see [24, Lemma 2.1 and Lemma 2.2]) we deduce that $\mathsf{s}((-\mathcal{V} + \mathcal{F})_0) = \mathsf{s}(-\mathcal{V} + \mathcal{F})$ that is

$$\operatorname{sign}\left(\mathsf{r}(\mathcal{V}^{-1}\mathcal{F}) - 1\right) = \operatorname{sign}\left(\mathsf{s}((-\mathcal{V} + \mathcal{F})_0)\right) = \operatorname{sign}\left(\omega((-\mathcal{V} + \mathcal{F})_0)\right).$$

By [30, Lemma B.1, Proposition 5.6] and (3.15) we have $\omega((-\mathcal{V} + \mathcal{F})_0) = \omega(U_{\mathsf{F}})$ and hence

$$\operatorname{sign}\left(\mathsf{r}(\mathcal{V}^{-1}\mathcal{F}) - 1\right) = \operatorname{sign}\left(\omega(U_{\mathsf{F}})\right) \Longleftrightarrow \operatorname{sign}\left(\mathsf{r}(\mathcal{V}^{-1}\mathcal{F}) - 1\right) = \operatorname{sign}\left(\mathsf{r}(U_{\mathsf{F}}(p, 0)) - 1\right).$$

The other properties of Theorem 1.3 are now completed by noting that

$$\mathcal{V}^{-1}\mathcal{F}[f](t) = \mathcal{R}_0(-\mathcal{V})\mathcal{F}[f](t) = \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U_{\mathsf{V}}(t, s)\lambda R_\lambda(\mathsf{A})\mathsf{F}(s)f(s)\mathrm{d}s, \ t \in \mathbb{R}.$$

To complete the proof, we show that formula (1.5) takes the form (2.6). To this end, we note that for each $t_0 < t$ and each $f \in C_p(\mathbb{R}, X)$ we have

$$\begin{aligned}
\mathcal{C}[f](t) &= \lim_{\lambda \to +\infty} \int_{-\infty}^{t_0} U_{\mathsf{V}}(t, s)\lambda R_\lambda(\mathsf{A})f(s)\mathrm{d}s + \lim_{\lambda \to +\infty} \int_{t_0}^{t} U_{\mathsf{V}}(t, s)\lambda R_\lambda(\mathsf{A})f(s)\mathrm{d}s \\
&= U_{\mathsf{V}}(t, t_0) \lim_{\lambda \to +\infty} \int_{-\infty}^{t_0} U_{\mathsf{V}}(t_0, s)\lambda R_\lambda(\mathsf{A})f(s)\mathrm{d}s + \lim_{\lambda \to +\infty} \int_{t_0}^{t} U_{\mathsf{V}}(t, s)\lambda R_\lambda(\mathsf{A})f(s)\mathrm{d}s \\
&= U_{\mathsf{V}}(t, t_0)\mathcal{C}[f](t_0) + \lim_{\lambda \to +\infty} \int_{t_0}^{t} U_{\mathsf{V}}(t, s)\lambda R_\lambda(\mathsf{A})f(s)\mathrm{d}s
\end{aligned}$$

Recalling that $\omega(U_{\mathsf{V}}) < 0$ and $t \mapsto \mathcal{C}[f](t)$ is uniformly bounded, it follows that

$$\lim_{t_0 \to -\infty} U_{\mathsf{V}}(t, t_0)\mathcal{C}[f](t_0) = 0.$$

Hence, taking the limit when $t_0 \to -\infty$ it comes

$$\mathcal{R}[f](t) = \lim_{t_0 \to -\infty} \lim_{\lambda \to +\infty} \int_{t_0}^{t} U_{\mathsf{V}}(t, s)\lambda R_\lambda(\mathsf{A})f(s)\mathrm{d}s, \ \forall t \in \mathbb{R},$$

and the result follows using the equality $\mathcal{C} = \mathcal{R} \circ \mathcal{F}$.

4 Applications of Theorem 1.3

In this section, we introduce some applications of our main result presented in Theorem 1.3. We start by introducing two general examples of where our results can be applied. The first example is about an age-structured model, and the second concerns a delay differential system. The next two examples are devoted to the dynamics of a nonlocal problem arising in population genetics, and the dynamics of a structured human-vector malaria model.

4.1 Application to an age-structured model

Let us consider the following age-structured model describing the dynamics of a structured population $u(t,a)$ at time t, and with a structural variable a:

$$\begin{cases} (\partial_t + \partial_a)n(t,a) = -(\mu_0 + d(t,a))n(t,a) \\ n(t,0) = \int_0^\infty \beta(t,a)n(t,a)\mathrm{d}a, \\ n(t_0,\cdot) = n_0 \in L^1_+((0,+\infty),\mathbb{R}), \end{cases} \tag{4.1}$$

and $\mu_0 > 0$, the maps $(t,a) \mapsto d(t,a)$ and $(t,a) \mapsto \beta(t,a)$ are p-periodic in time with

$$d(t,\cdot), \beta(t,\cdot) \in L^\infty_+((0,+\infty),\mathbb{R}),\ \forall t \in \mathbb{R}$$

and are continuous from $\mathbb{R}$ into $L^\infty((0,+\infty),\mathbb{R})$. To apply our results to (4.1), we first bring the system in our abstract framework. To do this, we introduce the Banach spaces

$$X := \mathbb{R} \times L^1((0,+\infty),\mathbb{R}),\ X_0 := \{0_{\mathbb{R}}\} \times L^1((0,+\infty),\mathbb{R}) \text{ and } X_1 := \mathbb{R} \times \{0_{L^1}\}.$$

Consider the linear operator $\mathsf{A} : D(\mathsf{A}) \subset X_0 \to X$ defined by

$$\mathsf{A}\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\phi(0) \\ -\phi' - \mu_0\phi \end{pmatrix} \text{ with } D(\mathsf{A}) = \{0_{\mathbb{R}}\} \times W^{1,1}((0,+\infty),\mathbb{R}).$$

We also define for each $t \in \mathbb{R}$, the linear operators $\mathsf{V}(t) : X_0 \to X$ and $\mathsf{F}(t) : X_0 \to X$ by

$$\mathsf{V}(t)\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ d(t,\cdot)\phi \end{pmatrix} \text{ and } \mathsf{F}(t)\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \int_0^\infty \beta(t,a)\phi(a)\mathrm{d}a \\ 0_{L^1} \end{pmatrix}. \tag{4.2}$$

Thus, making the identification $v(t) := \begin{pmatrix} 0 \\ n(t,\cdot) \end{pmatrix}$ and $v_0 := \begin{pmatrix} 0 \\ n_0 \end{pmatrix}$ the system (4.1) can be rewritten as the following abstract Cauchy problem

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \mathsf{A}v(t) - \mathsf{V}(t)v(t) + \mathsf{F}(t)v(t),\ t > t_0,\ v(t_0) = v_0 \in X_0. \tag{4.3}$$

The Cauchy problem (4.3) is associated with the following unperturbed inhomogeneous system

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \mathsf{A}u(t) - \mathsf{V}(t)u(t) + f(t),\ t > t_0,\ u(t_0) = u_0 \in X_0 \tag{4.4}$$

with $f \in C_p(\mathbb{R}, X)$. To determine the threshold for the age-structured model (4.3), we will make use of Theorem 1.3, Lemma 2.2 and Remark 2.3. It is of course classical to show that Assumptions 1.2 is satisfied. We refer to [25] and the references therein where such verification is done in many different situations. In what follows, we only give the steps for deriving the linear operator that gives the threshold dynamics of (4.1). Note that by definition, we have $\mathsf{F}(t)X_0 \subset X_1$. Thanks to Remark 2.3 and Lemma 2.2, this can be done by determining firstly the entire solution of (4.4) and secondly using the explicit form of the linear operator $\mathsf{F}(t)$ for $t \in \mathbb{R}$. To this end, let $f = \begin{pmatrix} m \\ 0 \end{pmatrix} \in C_p(\mathbb{R}, X_1)$ be given. Thus the abstract Cauchy problem (4.4) is given explicitly by

$$\begin{cases} (\partial_t + \partial_a)n(t,a) = -d(t,a)n(t,a) \\ n(t,0) = m(t), \\ n(t_0,\cdot) = \phi \in L^1((0,+\infty),\mathbb{R}). \end{cases} \tag{4.5}$$

Next, solving (4.5) along the characteristics gives the following formula

$$n(t,a)=\begin{cases} e^{-\mu_0(t-t_0)}e^{-\int_{a-t+t_0}^{a} d(l-a+t_0,l)\mathrm{d}l}\phi(a-t+t_0) & \text{if} \quad 0\leq t-t_0\leq a \\ e^{-\mu_0 a}e^{-\int_0^a d(l+t-a,l)\mathrm{d}l}m(t-a) & \text{if} \quad t-t_0>a. \end{cases} \tag{4.6}$$

From where we deduce that the evolution family generated by $\mathsf{A}-\mathsf{V}$ in (4.4) is given by

$$U_{\mathsf{V}}(t,t_0)\begin{pmatrix}0\\ \phi\end{pmatrix}=\begin{pmatrix}0\\ \widehat{U}_V(t,t_0)\phi\end{pmatrix},\ t\geq t_0$$

with

$$(\widehat{U}_V(t,t_0)\phi)(a)=\begin{cases} e^{-\mu_0(t-t_0)}e^{-\int_{a-t+t_0}^{a} d(l-a+t_0,l)\mathrm{d}l}\phi(a-t+t_0) & \text{if} \quad 0\leq t-t_0\leq a \\ 0 & \text{if} \quad t-t_0>a. \end{cases} \tag{4.7}$$

From (4.7), it easily follows that the evolution family $\{U_{\mathsf{V}}(t,t_0)\}_{t\geq t_0}$ is exponentially stable with

$$\|U_{\mathsf{V}}(t,t_0)\|_{\mathcal{L}(X_0)}\leq e^{-\mu_0(t-t_0)},\ \forall t\geq t_0.$$

Next, we infer from (4.7) that letting t_0 goes to $-\infty$, the entire solution of (4.5) is given by

$$n(t,a)=e^{-\mu_0 a}e^{-\int_0^a d(l+t-a,l)\mathrm{d}l}m(t-a),\ \forall t\in\mathbb{R},\ a\geq 0.$$

Therefore, recalling the definition of the linear operator $\mathsf{F}(t)$ in (4.2), we deduce that

$$\begin{aligned}\mathsf{F}(t)\begin{pmatrix}0\\ n(t,\cdot)\end{pmatrix} &= \begin{pmatrix}\int_0^{+\infty}\beta(t,a)n(t,a)\mathrm{d}a\\ 0_{L^1}\end{pmatrix}\\ &= \begin{pmatrix}\int_0^{+\infty}\beta(t,a)e^{-\mu_0 a}e^{-\int_0^a d(l+t-a,l)\mathrm{d}l}m(t-a)\mathrm{d}a\\ 0_{L^1}\end{pmatrix},\ \forall t\geq t_0.\end{aligned}$$

As a consequence, the threshold dynamics of the age-structured model is given by the spectral radius of the following linear operator

$$\mathcal{C}[m](t):=\int_0^{+\infty}\beta(t,a)e^{-\mu_0 a}e^{-\int_0^a d(l+t-a,l)\mathrm{d}l}m(t-a)\mathrm{d}a,\ \forall t\in\mathbb{R}.$$

4.2 Application to a delay differential equation

We will show that our result can be applied to derive the threshold value for population growth (or equivalently the basic reproductive number) for a system of delay differential equations. A similar result can be also found in [20, 33]. Let $\tau\geq 0$ be given and fixed and set $E=C([-\tau,0],\mathbb{R}^n)$. Let $\mathsf{V}:\mathbb{R}\to M_n(\mathbb{R})$ and $\mathsf{F}:\mathbb{R}\to\mathcal{L}(E,\mathbb{R}^n)$ be p-periodic continuous maps. For any given $t_0\in\mathbb{R}$ and a function function $x\in C([t_0-\tau,\delta),\mathbb{R}^n)$, with $\delta>0$, we define for $t\in[t_0-\tau,\delta)$ the map $x_t\in E$ by setting

$$x_t(\theta):=x(t+\theta),\ \forall\theta\in[-\tau,0].$$

Consider the following system of delay differential equation

$$\begin{cases}\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}=\widehat{\mathsf{F}}(t)x_t-\widehat{\mathsf{V}}(t)x(t),\ t\geq t_0\\ x(t_0)=\varphi\in C([-\tau,0],\mathbb{R}^n).\end{cases} \tag{4.8}$$

Denote by $\{\Phi(t,t_0)\}_{t\geq t_0}$ the evolution family generated by

$$\begin{cases} \dfrac{\mathrm{d}q(t)}{\mathrm{d}t} = -\widehat{\mathsf{V}}(t)q(t),\ t\geq t_0 \\ q(t_0)=q_0\in\mathbb{R}^n. \end{cases}$$

Let E_+ and $\mathbb{R}^n_+$ denote the positive cones of E and $\mathbb{R}^n$, respectively. As in [33] we make the following assumption

(DH1) For each $t\in\mathbb{R}$, we have $\mathsf{F}(t)E_+\subset\mathbb{R}^n_+$

(DH2) For each $t\in\mathbb{R}$, the matrix $-\mathsf{V}(t)$ is quasi-positive and $\omega(\Phi)<0$.

Let us note that condition **(DH2)** ensures that for each $\alpha\in(0,-\omega(\Phi))$ there exists $M_0\geq 1$ such that

$$\|\Phi(t,t_0)q_0\|\leq M_0e^{-\alpha(t-t_0)}\|q_0\|,\ \forall t\geq t_0,\ q_0\in\mathbb{R}^n.$$

In order to apply our result to (4.8) we reformulate it first as a partial differential equation and secondly as an abstract Cauchy problem. To this end, we proceed formally by setting

$$z(t,\theta)=x(t+\theta)$$

so that when x is C^1 we obtain

$$\begin{cases} \dfrac{\partial z(t,\theta)}{\partial t}-\dfrac{\partial z(t,\theta)}{\partial\theta}=0, \\ \dfrac{\partial z(t,0)}{\partial\theta}=\widehat{\mathsf{F}}(t)[z(t,\cdot)]-\widehat{\mathsf{V}}(t)z(t,0) \\ z(t_0,\cdot)=\varphi\in E=C([-\tau,0],\mathbb{R}^n). \end{cases} \tag{4.9}$$

Next, we reformulate (4.9) as an abstract non-densely defined Cauchy problem. Consider the Banach space $X=\mathbb{R}^n\times E$. Let $\mathsf{A}:D(\mathsf{A})\subset X\to X$ be the linear operator

$$\mathsf{A}\begin{pmatrix}0_{\mathbb{R}^n}\\ \phi\end{pmatrix}=\begin{pmatrix}-\phi'(0)\\ \phi'\end{pmatrix},\ \forall\begin{pmatrix}0_{\mathbb{R}^n}\\ \phi\end{pmatrix}\in D(\mathsf{A})=\{0_{\mathbb{R}^n}\}\times C^1([-\tau,0],\mathbb{R}^n).$$

Define for each $t\in\mathbb{R}$, the linear operators $\mathsf{F}(t):X\to X$ and $\mathsf{V}(t):X\to X$ by

$$\mathsf{F}(t)\begin{pmatrix}0_{\mathbb{R}^n}\\ \phi\end{pmatrix}=\begin{pmatrix}\widehat{\mathsf{F}}(t)[\phi]\\ 0_E\end{pmatrix}\text{ and }\mathsf{V}(t)\begin{pmatrix}0_{\mathbb{R}^n}\\ \phi\end{pmatrix}=\begin{pmatrix}\widehat{\mathsf{V}}(t)\phi(0)\\ 0_E\end{pmatrix}.$$

Therefore, setting

$$v(t)=\begin{pmatrix}0_{\mathbb{R}^n}\\ z(t,\cdot)\end{pmatrix},\ t\geq t_0\ \text{ and }\ v(t_0)=\begin{pmatrix}0_{\mathbb{R}^n}\\ \psi\end{pmatrix}$$

system (4.9) becomes

$$\begin{cases} \dfrac{\mathrm{d}v(t)}{\mathrm{d}t}=\mathsf{A}v(t)+\mathsf{F}(t)v(t)-\mathsf{V}(t)v(t) \\ u(0)=u_0\in X_0. \end{cases} \tag{4.10}$$

It is well known (see [22, 27]) that the linear operator A is Hille-Yosida with $\rho(\mathsf{A})=\mathbb{C}\setminus\{0\}$. Moreover, for each $\lambda>0$ we have

$$(\lambda I-\mathsf{A})^{-1}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=\begin{pmatrix}0\\ \phi\end{pmatrix}\Longleftrightarrow\phi(\theta)=e^{\lambda\theta}\frac{\varphi(0)+\alpha}{\lambda}+\int_\theta^0 e^{\lambda(\theta-s)}\varphi(s)\mathrm{d}s,\ \theta\in[-\tau,0]. \tag{4.11}$$

Thus, using (4.11) together with conditions **(DH1)-(DH2)**, one can see that Assumption 1.2 is trivially satisfied. Hence, to apply our results, we have to show that Assumption 1.1 is also satisfied. Before proceeding, we note that the part A_0 of A in $\overline{D(\mathsf{A})} = \{0_{\mathbb{R}^n}\} \times E$ generates a C_0-semigroup given by

$$T_{\mathsf{A}_0}(t)\begin{pmatrix} 0_Y \\ \phi \end{pmatrix} = \begin{pmatrix} 0_Y \\ \widehat{T}_{\mathsf{A}_0}(t)\phi \end{pmatrix}, \ \forall t \geq 0$$

with

$$(\widehat{T}_{\mathsf{A}_0}(t)\phi)(\theta) = \begin{cases} \phi(t+\theta) & \text{if} \quad -\tau \leq t+\theta \leq 0 \\ \phi(0) & \text{if} \quad t+\theta \geq 0. \end{cases}$$

Let us also note that the unperturbed system corresponding to (4.10) is given by

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = \mathsf{A}u(t) - \mathsf{V}(t)u(t) \\ u(0) = u_0 \in X_0. \end{cases} \tag{4.12}$$

The following lemma can be found in [27, Theorem 3.2].

Lemma 4.1 *There exists a unique evolution family $\{U_{\mathsf{V}}(t,t_0)\}_{t\geq t_0}$ on $X_0 = \{0_{\mathbb{R}^n}\} \times E$ satisfying for all $t \geq t_0$*

$$U_{\mathsf{V}}(t,t_0)\begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} = T_{\mathsf{A}_0}(t-t_0)\begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} - \lim_{\lambda\to+\infty}\int_{t_0}^{t} T_{\mathsf{A}_0}(t-s)\lambda R_\lambda(\mathsf{A})\mathsf{V}(s)U_{\mathsf{V}}(s,t_0)\begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix}\mathrm{d}s,$$

with

$$U_{\mathsf{V}}(t,t_0)\begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{U}_{\mathsf{V}}(t,t_0)\psi \end{pmatrix} \tag{4.13}$$

and

$$(\widehat{U}_{\mathsf{V}}(t,t_0)\psi)(\theta) = \begin{cases} \psi(t-t_0+\theta) & \text{if} \quad -\tau \leq t-t_0+\theta \leq 0 \\ \Phi(t+\theta,t_0)\psi(0) & \text{if} \quad t-t_0+\theta \geq 0. \end{cases} \tag{4.14}$$

To show that Assumption 1.1 is satisfied, observe that for $t \geq t_0$ we have

$$-\tau \leq t-t_0+\theta \leq 0 \ \text{ and } \theta \in [-\tau,0] \Rightarrow t-t_0 \in [0,\tau]$$

and from Lemma 4.1 it follows that

$$\|(\widehat{U}_{\mathsf{V}}(t,t_0)\psi)(\theta)\| \leq \begin{cases} e^{\alpha\tau}e^{-\alpha(t-t_0)}\|\psi\|_E & \text{if} \quad -\tau \leq t-t_0+\theta \leq 0 \\ M_0e^{\alpha\tau}e^{-\alpha(t-t_0)}\|\psi(0)\| & \text{if} \quad t-t_0+\theta \geq 0. \end{cases}$$

From where do we obtain

$$\|\widehat{U}_{\mathsf{V}}(t,t_0)\psi\|_E \leq M_0e^{\alpha\tau}e^{-\alpha(t-t_0)}\|\psi\|_E, \ \forall t \geq t_0,$$

so that Assumption 1.1 is satisfied for the unperturbed system (4.12). Thus observing that $\mathsf{F}(t)X_0 \subset X_1$ with $X_1 = \mathbb{R}^n \times \{0_E\}$ one can define (see Lemma 2.2 and Remark 2.3) the threshold for (4.10) as the spectral radius of the operator

$$\mathcal{C}\begin{pmatrix} f \\ 0_E \end{pmatrix}(t) := \mathsf{F}(t)\lim_{\lambda\to+\infty}\int_{-\infty}^{t} U_{\mathsf{V}}(t,s)\lambda R_\lambda(\mathsf{A})\begin{pmatrix} f(s) \\ 0_E \end{pmatrix}\mathrm{d}s, \ \forall t \in \mathbb{R}, \ f \in C_p(\mathbb{R},\mathbb{R}^n). \tag{4.15}$$

Next, recall that from (4.11) we have

$$\lambda R_\lambda(\mathsf{A}) \begin{pmatrix} f(s) \\ 0_E \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ e^{\lambda \cdot} f(s) \end{pmatrix}, \ \forall s \in \mathbb{R}$$

and using (4.13) we obtain for each $\lambda > 0$

$$\int_{-\infty}^{t} U_{\mathsf{V}}(t,s)\lambda R_\lambda(\mathsf{A}) \begin{pmatrix} f(s) \\ 0_E \end{pmatrix} \mathrm{d}s = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \int_{-\infty}^{t} \widehat{U}_{\mathsf{V}}(t,s)(e^{\lambda \cdot} f(s))\mathrm{d}s \end{pmatrix}, \ \forall t \in \mathbb{R}$$

that is

$$\mathsf{F}(t) \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U_{\mathsf{V}}(t,s)\lambda R_\lambda(\mathsf{A}) \begin{pmatrix} f(s) \\ 0_E \end{pmatrix} \mathrm{d}s = \begin{pmatrix} \widehat{\mathsf{F}}(t) \lim\limits_{\lambda \to +\infty} \int_{-\infty}^{t} \widehat{U}_{\mathsf{V}}(t,s)(e^{\lambda \cdot} f(s))\mathrm{d}s \\ 0_E \end{pmatrix}, \ \forall t \in \mathbb{R}. \tag{4.16}$$

From the above formulas (4.15) and (4.16) we obtain that the spectral radius of $\mathcal{C}$ is given by the spectral radius of the linear operator $\widehat{\mathcal{C}} : C_p(\mathbb{R}, \mathbb{R}^n) \to C_p(\mathbb{R}, \mathbb{R}^n)$

$$\widehat{\mathcal{C}}[f](t) := \widehat{\mathsf{F}}(t) \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \widehat{U}_{\mathsf{V}}(t,s)(e^{\lambda \cdot} f(s))\mathrm{d}s, \ \forall t \in \mathbb{R}. \tag{4.17}$$

In the following, we show that the linear operator $\widehat{\mathcal{C}}$ corresponds to the one given in [33]. In fact, using (4.14) and (4.17) we obtain

$$\begin{aligned} \left(\int_{-\infty}^{t} \widehat{U}_{\mathsf{V}}(t,s)(e^{\lambda \cdot} f(s))\mathrm{d}s \right)(\theta) &= \int_{-\infty}^{t} \widehat{U}_{\mathsf{V}}(t,s)(e^{\lambda \cdot} f(s))(\theta)\mathrm{d}s \\ &= \int_{-\infty}^{t+\theta} \widehat{U}_{\mathsf{V}}(t,s)(e^{\lambda \cdot} f(s))(\theta)\mathrm{d}s + \int_{t+\theta}^{t} \widehat{U}_{\mathsf{V}}(t,s)(e^{\lambda \cdot} f(s))(\theta)\mathrm{d}s \\ &= \int_{-\infty}^{t+\theta} \Phi(t+\theta,s) f(s)\mathrm{d}s + \int_{t+\theta}^{t} e^{\lambda(t-s+\theta)} f(s)\mathrm{d}s \end{aligned}$$

and since for all $f \in C_p(\mathbb{R}, \mathbb{R}^n)$ we have

$$\lim_{\lambda \to +\infty} \int_{t+\cdot}^{t} e^{\lambda(t-s+\cdot)} f(s)\mathrm{d}s = 0_E \ \text{ in } E$$

it follows that

$$\begin{aligned} \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \widehat{U}_{\mathsf{V}}(t,s)(e^{\lambda \cdot} f(s))\mathrm{d}s &= \int_{-\infty}^{t+\cdot} \Phi(t+\cdot,s) f(s)\mathrm{d}s \\ &= \int_{0}^{+\infty} \Phi(t+\cdot, t-s+\cdot) f(t-s+\cdot)\mathrm{d}s. \end{aligned} \tag{4.18}$$

Therefore, we deduce from (4.17) and (4.18) that for all $f \in C_p(\mathbb{R}, \mathbb{R}^n)$

$$\widehat{\mathcal{C}}[f](t) = \widehat{\mathsf{F}}(t) \int_{0}^{+\infty} \Phi(t+\cdot, t-s+\cdot) f(t-s+\cdot)\mathrm{d}s, \ \forall t \in \mathbb{R}.$$

4.3 Application to an evolutionary epidemiological model

Here we introduce a nonlocal system of equations structured by both the age of infection a and a phenotypic trait $x \in \mathbb{R}^N$ (a label of the pathogen strain) with $N \geq 1$. At time t, the state variables $S = S(t)$, $i = i(t, a, x)$, and $M = M(t, x)$ denote the density of healthy area, infected area since a-time unit by a pathogen phenotype x, and spores respectively. Infected area exits at rate $\mu_0 > 0$, and spores become unviable at rate $\delta > 0$. The healthy area is infected at rate $\beta(t, x)$. An infected area produces spores at rate $r(a, \cdot)$, a-time since infected. The evolution in the space of phenotypic values is modelled by an integral operator with kernel $K(x - y)$ describing mutations from a pathogen strain with phenotypic value $y \in \mathbb{R}^N$ to another one with phenotypic value $x \in \mathbb{R}^N$. We then have the linearisation of the model ([12]) around the disease-free $(\bar{S}(t), 0, 0)$

$$\begin{cases} (\partial_t + \partial_a) i(t, a, x) = -\mu_0 i(t, a, x), \\ i(t, 0, x) = \beta(t, x) \bar{S}(t) M(t, x), \\ \partial_t M(t, x) = \displaystyle\int_{\mathbb{R}^N} \int_0^\infty K(x - y) r(a, y) i(t, a, y) \mathrm{d}a\mathrm{d}y - \delta M(t, x) \end{cases} \tag{4.19}$$

with initial conditions in $L^1((0, +\infty) \times \mathbb{R}^N, \mathbb{R})$ and $L^1(\mathbb{R}^N, \mathbb{R})$, respectively. In order to apply our result we first reformulate the above system in our abstract framework. To this end, we introduce the Banach spaces

$$Y := L^1(\mathbb{R}^N, \mathbb{R}) \text{ and } Z := L^1((0, +\infty) \times \mathbb{R}^N, \mathbb{R})$$

as well as

$$\begin{cases} X := Y \times Y \times Z \\ X_0 := Y \times \{0_Y\} \times Z \\ X_1 := Y \times Y \times \{0_Z\}. \end{cases}$$

Consider the linear operator $\mathsf{A} : D(\mathsf{A}) \subset X_0 \to X$ defined by

$$\mathsf{A} \begin{pmatrix} p \\ 0 \\ \phi \end{pmatrix} = \begin{pmatrix} -\delta p \\ -\phi(0) \\ -\phi' - \mu_0 \phi \end{pmatrix},$$

with

$$D(\mathsf{A}) = \left\{ \begin{pmatrix} p \\ 0 \\ \phi \end{pmatrix} \in Y \times \{0_Y\} \times Z : \phi(\cdot, x) \in W^{1,1}((0, +\infty), \mathbb{R}),\ \forall x \in \mathbb{R}^N \right\}.$$

It is classical to show that the linear operator A is Hille-Yosida and A_0, the part of A in $\overline{D(\mathsf{A})} = X_0$ generates a C_0-semigroup. Next, we define for each $t \in \mathbb{R}$, the linear operator $\mathsf{F}(t) : X_0 \to X$ by

$$\mathsf{F}(t) \begin{pmatrix} p \\ 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \int_{\mathbb{R}^N} \int_0^\infty K(x - y) r(a, y) \phi(a, y) \mathrm{d}a\mathrm{d}y \\ \beta(t, \cdot) \bar{S}(t) p \\ 0 \end{pmatrix}. \tag{4.20}$$

We can define the cone X_+ as $X_+ = Y_+ \times Y_+ \times Z_+$ where $Y_+ = L^1_+(\mathbb{R}^N, \mathbb{R})$ and $Z_+ = L^1_+((0, +\infty) \times \mathbb{R}^N, \mathbb{R})$ are the classical normal and generating cones associated to Y and Z, respectively. Next, making the identification $v(t) := \begin{pmatrix} M(t, \cdot) \\ 0 \\ i(t, \cdot, \cdot) \end{pmatrix}$ the system (4.19) can be rewritten as the following abstract Cauchy problem

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \mathsf{A}v(t) + \mathsf{F}(t)v(t),\ t > t_0,\ v(t_0) = v_0 \in X_0.$$

Noting that $\mathsf{F}(t)X_0 \subset X_1$ and using Lemma 2.2 together with Remark 2.3, we can obtain the threshold for (4.19) by considering the following unperturbed problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \mathsf{A}u(t) + f(t),\ t > t_0,\ u(t_0) = u_0 \in X_0, \tag{4.21}$$

where $f(t) \in X_1$ for all $t \geq t_0$. More precisely, setting $f = \begin{pmatrix} f_1 \\ f_2 \\ 0 \end{pmatrix} \in C_p(\mathbb{R}, X_1)$, we are reduce to solve the following problem along the characteristics

$$\begin{cases} (\partial_t + \partial_a) i(t,a,x) = -\mu_0 i(t,a,x),\ t > t_0 \\ i(t,0,x) = f_2(t,x),\ t > t_0 \\ \partial_t M(t,x) = f_1(t,x) - \delta M(t,x),\ t > t_0 \end{cases} \tag{4.22}$$

with initial conditions at time $t = t_0$ given by

$$i(t_0,\cdot,\cdot) = i_0 \in Z, \ \text{ and } \ M(t_0,\cdot) = M \in Y.$$

This reads as

$$i(t,a,x) = \begin{cases} e^{-\mu_0(t-t_0)} i_0(a - t + t_0) \ \text{ if } \ 0 \leq t - t_0 \leq a \\ e^{-\mu_0 a} f_2(t-a,x) \ \text{ if } \ t - t_0 > a \end{cases} \tag{4.23}$$

and

$$M(t,x) = e^{-\delta(t-t_0)} M_0(x) + \int_{t_0}^{t} e^{-\delta(t-s)} f_1(s,x)\mathrm{d}s,\ t \geq t_0. \tag{4.24}$$

The semigroup generated by A_0 the part of A in X_0 is obtained from (4.23) and (4.24) by setting $f_1 = f_2 \equiv 0$. From where it is straightforward that the semigroup generated by A_0 is exponentially stable. As t_0 goes to $-\infty$ in (4.23) provides that the unique entire solution to (4.21) (equivalently (4.22)) is given by

$$\begin{cases} i(t,a,x) = e^{-\mu_0 a} f_2(t-a,x)\ \forall t \in \mathbb{R} \\ M(t,x) = \displaystyle\int_{-\infty}^{t} e^{-\delta(t-s)} f_1(s,x)\mathrm{d}s,\ \forall t \in \mathbb{R}. \end{cases}$$

Next, using (4.24) and the definition of $\mathsf{F}(t)$ in (4.20) we obtain

$$\mathsf{F}(t) \begin{pmatrix} M(t,\cdot) \\ 0 \\ i(t,\cdot,\cdot) \end{pmatrix} = \begin{pmatrix} \int_{\mathbb{R}^N} \int_0^\infty K(\cdot - y) r(a,y) e^{-\mu_0 a} f_2(t-a,y)\mathrm{d}a\mathrm{d}y \\ \beta(t,\cdot) S(t) \int_{-\infty}^{t} e^{-\delta(t-s)} f_1(s,\cdot)\mathrm{d}s \\ 0 \end{pmatrix},\ \forall t \in \mathbb{R}.$$

Thus, the threshold dynamics of (4.19) is given by the spactral radius of the linear operator $\mathcal{C}$ defined by

$$\mathcal{C} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t) = \begin{pmatrix} \mathcal{C}_{12}[f_2](t) \\ \mathcal{C}_{21}[f_1](t) \end{pmatrix}$$

where we have set for each $f_i \in C_p(\mathbb{R}, Y)$, $i = 1, 2$

$$\begin{cases} \mathcal{C}_{12}[f_2](t) := \displaystyle\int_{\mathbb{R}^N} \int_0^\infty K(\cdot - y) r(a,y) e^{-\mu_0 a} f_2(t-a,y)\mathrm{d}a\mathrm{d}y,\ \forall t \in \mathbb{R} \\ C_{21}[f_1](t) := \beta(t,\cdot) S(t) \displaystyle\int_{-\infty}^{t} e^{-\delta(t-s)} f_1(s,\cdot)\mathrm{d}s,\ \forall t \in \mathbb{R}. \end{cases}$$

Using similar arguments as in [13], we conclude that $\text{sign}(\mathsf{r}(\mathcal{C})-1)=\text{sign}(\mathsf{r}(\mathcal{C}_{12}\circ\mathcal{C}_{21})-1)$ where the linear operator $\mathcal{C}_{12}\circ\mathcal{C}_{21}$ is given by

$$(\mathcal{C}_{12}\circ\mathcal{C}_{21})[f](t)=\int_{\mathbb{R}^N}\int_0^\infty K(\cdot-y)r(a,y)e^{-\mu_0 a}\left(\beta(t-a,y)S(t-a)\int_{-\infty}^{t-a}e^{-\delta(t-a-s)}f_1(s,y)\mathrm{d}s\right)\mathrm{d}a\mathrm{d}y,$$

for every $f\in C_p(\mathbb{R},Y)$ and $t\in\mathbb{R}$.

4.4 Application to a structured human-vector malaria model

The below system describes the transmission dynamics human-vector malaria, where both human (subscript h) and mosquitoes (subscript m) populations are structured by the chronological age a and time since infection τ (*eg.* [28]). The density of susceptible human and mosquitoes, at the disease-free, aged a is given respectively by $\bar{S}_h(t,a)$ and $\bar{S}_m(t,a)$ at time t. Natural death rate of humans aged a is $\mu_h(a)$, and if infected since time τ, the disease induced mortality is $\nu_h(t,a,\tau)$. The force of infection from mosquitoes to humans at time t is given by $\int_0^\infty\int_0^\infty\beta_m(t,s,\tau)I_m(t,s,\tau)\mathrm{d}s\mathrm{d}\tau$. Humans aged a and infected since time τ recover from the disease at rate $\gamma_h(t,a,\tau)$. Natural death rate of mosquitoes aged a is $\mu_m(a)$, and if infected since time τ, the disease induced mortality is $\nu_m(t,a,\tau)$. The force on infection from humans to mosquitoes at time t is given by $\int_0^\infty\int_0^\infty\beta_h(t,s,\tau)I_h(t,s,\tau)\mathrm{d}s\mathrm{d}\tau$. We then have the following model

$$\begin{cases}(\partial_t+\partial_a+\partial_\tau)I_h(t,a,\tau)=-(\mu_h(a)+\nu_h(t,a,\tau)+\gamma_h(t,a,\tau))I_h(t,a,\tau)\\(\partial_t+\partial_a+\partial_\tau)I_m(t,a,\tau)=-(\mu_m(a)+\nu_m(t,a,\tau))I_m(t,a,\tau)\\I_h(t,a,0)=\bar{S}_h(t,a)\displaystyle\int_0^\infty\int_0^\infty\beta_m(t,s,\tau)I_m(t,s,\tau)\mathrm{d}s\mathrm{d}\tau,\\I_m(t,a,0)=\bar{S}_m(t,a)\displaystyle\int_0^\infty\int_0^\infty\beta_h(t,s,\tau)I_h(t,s,\tau)\mathrm{d}s\mathrm{d}\tau,\\I_m(t,0,\tau)=0=I_h(t,0,\tau)\end{cases}\tag{4.25}$$

with initial conditions in $L^1((0,+\infty)^2,\mathbb{R}^2)$. For the parameters, we assume that there exists $\mu_0>0$ such that $\mu_k(a)\geq\mu_0$, $k\in\{m,h\}$ for almost every $a\in(0,+\infty)$ and $\mu_k\in L^\infty_+((0,+\infty),\mathbb{R})$. We also assume that the maps $t\mapsto\beta_k(t,\cdot,\cdot)\in L^\infty_+((0,+\infty)^2,\mathbb{R})$ and $t\mapsto\nu_k(t,\cdot,\cdot)\in L^\infty_+((0,+\infty)^2,\mathbb{R})$, $k\in\{m,h\}$, are continuous and p-periodic. Next, we proceed as in the preceding sections. We first rewrite (4.25) in a more convenient form. To do this, we define

$$I:=\begin{pmatrix}I_h\\I_m\end{pmatrix}\quad\text{and}\quad\vartheta(t,\cdot,\cdot):=\begin{pmatrix}\mu_h(\cdot)+\nu_h(t,\cdot,\cdot)+\gamma_h(t,\cdot,\cdot)&0\\0&\mu_m(\cdot)+\nu_m(t,\cdot,\cdot)\end{pmatrix}$$

and the periodic transmission matrix

$$\beta(t,\cdot,\cdot):=\begin{pmatrix}0&\beta_m(t,\cdot,\cdot)\\\beta_h(t,\cdot,\cdot)&0\end{pmatrix},\ \forall t\in\mathbb{R}.$$

Thus, setting $\bar{S}=\begin{pmatrix}\bar{S}_h\\\bar{S}_m\end{pmatrix}$ the system (4.25) rewrites as the following more compact form

$$\begin{cases}(\partial_t+\partial_a+\partial_\tau)I(t,a,\tau)=-\vartheta(t,a,\tau)I(t,a,\tau)\\I(t,a,0)=\text{diag}(\bar{S}(t,a))\displaystyle\int_0^\infty\int_0^\infty\beta(t,s,\tau)I(t,s,\tau)\mathrm{d}s\mathrm{d}\tau,\\I(t,0,\tau)=0\\I(0,\cdot,\cdot)=\varphi\in L^1(\mathbb{R}^2_+,\mathbb{R}^2),\end{cases}\tag{4.26}$$

Consider the Banach spaces

$$Y := L^1((0,+\infty),\mathbb{R}^2),\ Z = L^1((0,+\infty)^2,\mathbb{R}^2)$$

as well has

$$\begin{cases} X := Y \times Y \times Z, \\ X_0 = \{0_Y\} \times \{0_Y\} \times Z, \\ X_1 = Y \times \{0_Y\} \times \{0_Z\}. \end{cases}$$

Let $\mathsf{A} : D(\mathsf{A}) \subset X_0 \to X$ be the linear operator defined by

$$\mathsf{A}\begin{pmatrix} 0_Y \\ 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0,\cdot) \\ -\varphi(\cdot,0) \\ -(\partial_a + \partial_\tau)\varphi - \vartheta(\cdot,\cdot)\varphi \end{pmatrix}$$

with domain

$$D(\mathsf{A}) = \{0_Y\} \times \{0_Y\} \times W^{1,1}(\mathbb{R}^2_+,\mathbb{R}^2).$$

Next, we define for each $t \in \mathbb{R}$ the linear operator $\mathsf{F}(t) : X_0 \to X$ as

$$\mathsf{F}(t)\begin{pmatrix} 0_Y \\ 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} \mathrm{diag}(\bar{S}(t,\cdot)) \int_0^\infty \int_0^\infty \beta(t,s,\tau)\varphi(s,\tau)\mathrm{d}s\mathrm{d}\tau \\ 0_Y \\ 0 \end{pmatrix}$$

Thus by setting $v(t) := \begin{pmatrix} 0_Y \\ 0_Y \\ I(t,\cdot,\cdot) \end{pmatrix}$ we rewrite (4.26) as the following abstract Cauchy problem

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \mathsf{A}v(t) + \mathsf{F}(t)v(t),\ t > 0,\ v(0) = v_0.$$

Noting that $\mathsf{F}(t)X_0 \subset X_1$, Lemma 2.2 and Remark 2.3 ensure that we can obtain the threshold dynamics of (4.26) by determining the entire solution of the following unperturbed inhomogeneous problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \mathsf{A}u(t) + f(t),\ t > t_0,\ v(t_0) = v_0,$$

where $f = \begin{pmatrix} w \\ 0_Y \\ 0_Z \end{pmatrix}$ with $w \in C_p(\mathbb{R},Y)$. Thus, using the notation $w(t)(a) = w(t,a)$, the foregoing problem (4.26) has the following explicit form

$$\begin{cases} (\partial_t + \partial_a + \partial_\tau)I(t,a,\tau) = -\vartheta(a,\tau)I(t,a,\tau) \\ I(t,a,0) = w(t,a) \\ I(t,0,\tau) = 0 \\ I(t_0,\cdot,\cdot) = \varphi \in L^1(\mathbb{R}^2_+,\mathbb{R}^2). \end{cases} \tag{4.27}$$

Integrating (4.27) along the characteristics gives

$$I(t,a,\tau) = \begin{cases} e^{-\int_{a-t+t_0}^a \vartheta(l-a+t,l,l-a+\tau)\mathrm{d}l}\varphi(a-t+t_0,\tau-t+t_0) & \text{if} \quad a \geq t-t_0,\ t-t_0 \leq \tau \\ e^{-\int_{a-\tau}^a \vartheta(l-a+t,l,l-a+\tau)\mathrm{d}l}w(t-\tau,a-\tau) & \text{if} \quad t-t_0 > \tau,\ a > \tau \\ 0 & \text{if} \quad t-t_0 > a,\ \tau \geq a \end{cases} \tag{4.28}$$

The semigroup generated by A_0, the part of A in $\overline{D(\mathsf{A})}$, is obtained from (4.28) by setting $\varphi \equiv 0$. It is exponentially stable so that we can obtain the unique entire mild solution to (4.26) by letting $t_0 \to -\infty$ in (4.28). Thus, we deduce that the entire solution takes the following form

$$I(t,a,\tau) = \begin{cases} e^{-\int_{a-\tau}^{a} \vartheta(l-a+t,l,l-a+\tau)\mathrm{d}l} w(t-\tau,a-\tau) & \text{if} \quad t \in \mathbb{R},\ a > \tau \\ 0 & \text{if} \quad t \in \mathbb{R},\ \tau \geq a \end{cases}$$

so that

$$\mathsf{F}(t) \begin{pmatrix} 0_Y \\ 0_Y \\ I(t,\cdot,\cdot) \end{pmatrix} = \begin{pmatrix} \mathrm{diag}(\bar{S}(t,\cdot)) \int_0^{+\infty} \int_\tau^\infty \beta(t,s,\tau) \left(e^{-\int_{s-\tau}^{s} \vartheta(l-s+t,l,l-s+\tau)\mathrm{d}l} w(t-\tau,s-\tau) \right) \mathrm{d}s\mathrm{d}\tau \\ 0_Y \\ 0 \end{pmatrix}.$$

Hence, the threshold dynamics of (4.25) is given by the spectral radius of the following linear operator

$$(\mathcal{C}[w](t))(a) = \mathrm{diag}(\bar{S}(t,a)) \int_0^\infty \int_\tau^\infty \beta(t,s,\tau) e^{-\int_{s-\tau}^{s} \vartheta(l-s+t,l,l-s+\tau)\mathrm{d}l} w(t-\tau,s-\tau)\mathrm{d}s\mathrm{d}\tau,$$

for all $t \in \mathbb{R}$, $m \in C_p(\mathbb{R},Y)$ with the notation $w(t)(a) = w(t,a)$. Setting $w = \begin{pmatrix} w_h \\ w_m \end{pmatrix}$ the linear operator $\mathcal{C}$ takes the following form

$$\mathcal{C}[w] = \begin{pmatrix} \mathcal{C}_m[w_m] \\ \mathcal{C}_h[w_h] \end{pmatrix}$$

where we have set for $k = h, m$

$$(\mathcal{C}_k[w_k](t))(a) := \bar{S}_k(t,a) \int_0^\infty \int_\tau^\infty \beta_k(t,s,\tau) e^{-\int_{s-\tau}^{s} \vartheta_k(l-s+t,l,l-s+\tau)\mathrm{d}l} w_k(t-\tau,s-\tau)\mathrm{d}s\mathrm{d}\tau.$$

Using similar arguments in [13], we deduce that $\mathsf{r}(\mathcal{C}) - 1$, $\mathsf{r}(\mathcal{C}_h \circ \mathcal{C}_m) - 1$ and $\mathsf{r}(\mathcal{C}_m \circ \mathcal{C}_h) - 1$ have the same sign. Note that $\mathsf{r}(\mathcal{C}_h \circ \mathcal{C}_m) = \mathsf{r}(\mathcal{C}_m \circ \mathcal{C}_h)$ is actually the basic reproduction number of (4.25); see [28].

Declarations

Data Availability. Not applicable.

Ethical Approval. Not applicable.

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