



HAL
open science

From particular polynomials to rational solutions to the PII equation

Pierre Gaillard

► **To cite this version:**

Pierre Gaillard. From particular polynomials to rational solutions to the PII equation. 2022. hal-03882963

HAL Id: hal-03882963

<https://hal.science/hal-03882963>

Preprint submitted on 2 Dec 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

From particular polynomials to rational solutions to the PII equation

Pierre Gaillard,

Université de Bourgogne Franche Comté,
Institut de mathématiques de Bourgogne,
9 avenue Alain Savary BP 47870
21078 Dijon Cedex, France :
E-mail : Pierre.Gaillard@u-bourgogne.fr

Abstract

The Painlevé equations were derived by Painlevé and Gambier in the years 1895 – 1910. Given a rational function R in w, w' and analytic in z , they searched what were the second order ordinary differential equations of the form $w'' = R(z, w, w')$ with the properties that the singularities other than poles of any solution or this equation depend on the equation only and not of the constants of integration. They proved that there are fifty equations of this type, and the Painlevé II is one of these.

Here, we construct solutions to the Painlevé II equation (PII) from particular polynomials. We obtain rational solutions written as a derivative with respect to the variable x of a logarithm of a quotient of a determinant of order $n + 1$ by a determinant of order n . We obtain an infinite hierarchy of rational solutions to the PII equation.

We give explicitly the expressions of these solutions solution for the first orders.

PACS numbers :
33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

Keywords:

Painlevé equation II; rational solutions; determinants;

1 Introduction

We consider the Painlevé II equation (PII) which can be written in the form

$$u_{xx} - 2u^3 + 4xu - 4(n + 1) = 0, \quad (1)$$

where the subscript x denote derivative and n is an arbitrary integer.

This equation is one of the six famous equations discovered by Painlevé [1] at

the beginning of twentieth century. Many studies have realized and we refer for example to the works of Vorob'ev and Yablonskii [2], Airault [3], Murata [4] or Umemura [5].

Different types of solutions were given in terms of special polynomials. PII equation have solutions which can be expressed in terms of classical special functions, more precisely in terms of Airy functions. Recent works can be quoted as for example the following papers, [6], [7], [8], [9].

In the following, we consider particular polynomials and we construct rational solutions to the PII equation by using the Hirota bilinear method. Then we obtain rational solutions to the PII equation as a derivative of a logarithm of a quotient of a determinant of order $N + 1$ by a determinant of order N . That provides an effective method to construct an infinite hierarchy of rational solutions of order N . We present rational solutions for the first simplest orders.

2 Rational solutions to the PII equation

We consider the polynomials p_n defined by

$$\left\{ \begin{array}{l} p_n(x) = \sum_{k=0}^n x^k \frac{1}{3} \frac{\binom{n-k}{3}}{\binom{n-k}{3}!} \times \left(1 - \left[\frac{1}{2} \left(n-k+1 - 3 \left[\frac{n-k}{3} \right] \right) \right] \right), \text{ for } n \geq 0, \\ p_n(x) = 0, \text{ for } n < 0. \end{array} \right. \quad (2)$$

In the previous definition of p_n , $[x]$ means the largest integer less than or equal to x . We denote A_n the determinant defined by

$$A_n = \det(p_{n+1-2i+j}(x))_{\{1 \leq i \leq n, 1 \leq j \leq n\}} \quad (3)$$

With these notations we have the following result

Theorem 2.1 *The function v_n defined by*

$$v_n(x) = \partial_x \left(\ln \frac{A_{n+1}}{A_n} \right) \quad (4)$$

is a rational to the PII equation (1)

$$u_{xx} - 2u^3 + 4xu - 4(n+1) = 0.$$

3 Explicit rational solutions to the PII equation for the first orders

In the following, we will call

$$v_k(x, t) = \partial_x \left(\ln \frac{A_{k+1}(x, t)}{A_k(x, t)} \right)$$

a rational solution to the PII equation of order k :

$$u_{xx} - 2u^3 + 4xu - 4(k+1) = 0.$$

In this section we will give some explicit examples of rational solutions to the PII equation.

3.1 Rational solutions of order 1 to the PII equation

Example 3.1 *The function $v_k(x, t)$ defined by*

$$\frac{2x^3 + 1}{x(x^3 - 1)} \tag{5}$$

is a rational to the PII equation with $k = 1$.

3.2 Rational solutions of order 2 to the PII equation

Example 3.2 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = 3 \frac{x^2 (x^6 - 2x^3 + 10)}{(x^6 - 5x^3 - 5)(x^3 - 1)} \tag{6}$$

is a rational to the PII equation with $k = 2$.

3.3 Rational solutions of order 3 to the PII equation

Example 3.3 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{4x^{15} - 50x^{12} + 250x^9 + 1400x^6 - 1750x^3 + 875}{(x^9 - 15x^6 - 175)(x^6 - 5x^3 - 5)x} \tag{7}$$

is a rational to the PII equation with $k = 3$.

3.4 Rational solutions of order 4 to the PII equation

Example 3.4 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \tag{8}$$

with

$$n(x, t) = 5x^{24} - 190x^{21} + 2450x^{18} - 2800x^{15} + 134750x^{12} - 1041250x^9 +$$

$$1715000x^6 + 4287500x^3 + 1071875$$

and

$d(x, t) = (x^{15} - 35x^{12} + 175x^9 - 1225x^6 - 12250x^3 + 6125)(x^9 - 15x^6 - 175)x$
is a rational to the PII equation with $k = 4$.

3.5 Rational solutions of order 5 to the PII equation

Example 3.5 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (9)$$

with

$$n(x, t) = 3(2x^{33} - 175x^{30} + 5600x^{27} - 73500x^{24} + 808500x^{21} + 257250x^{18} - 153063750x^{15} + 1209075000x^{12} + 90037500x^9 + 20633593750x^6 - 11554812500x^3 + 57774062500)x^2$$

and

$$d(x, t) = (x^{21} - 70x^{18} + 1155x^{15} - 9800x^{12} - 67375x^9 - 1414875x^6 + 4716250x^3 + 2358125)(x^{15} - 35x^{12} + 175x^9 - 1225x^6 - 12250x^3 + 6125)$$

is a rational to the PII equation with $k = 5$.

3.6 Rational solutions of order 6 to the PII equation

Example 3.6 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (10)$$

with

$$n(x, t) = 7x^{48} - 1204x^{45} + 81480x^{42} - 2773400x^{39} + 57420650x^{36} - 596048250x^{33} + 2961805000x^{30} - 201383875000x^{27} + 5418794390625x^{24} - 38100013393750x^{21} + 313753601218750x^{18} + 771769036500000x^{15} - 10091655480156250x^{12} + 58843893702343750x^9 + 133591542459375000x^6 - 111326285382812500x^3 + 27831571345703125$$

and

$$d(x, t) = (x^{27} - 126x^{24} + 4725x^{21} - 80850x^{18} + 242550x^{15} - 12733875x^{12} - 202327125x^9 + 3034906875x^6 + 11802415625)(x^{21} - 70x^{18} + 1155x^{15} - 9800x^{12} - 67375x^9 - 1414875x^6 + 4716250x^3 + 2358125)x$$

is a rational to the PII equation with $k = 6$.

3.7 Rational solutions of order 7 to the PII equation

Example 3.7 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (11)$$

with

$$n(x, t) = 8x^{63} - 2436x^{60} + 307860x^{57} - 21236600x^{54} + 907622100x^{51} - 24766537950x^{48} + 459897761400x^{45} -$$

$7268461893000 x^{42} + 17185043452500 x^{39} + 5545487726528750 x^{36} - 155690409080456250 x^{33} + 2380328650825125000 x^{30} -$
 $11581750581025625000 x^{27} + 30516125198934375000 x^{24} - 1609613959883304375000 x^{21} + 30155017717413651562500 x^{18} +$
 $6390151046230514062500 x^{15} + 384626916672776121093750 x^{12} - 2908243070510727167968750 x^9 +$
 $3346471478395905234375000 x^6 + 4183089347994881542968750 x^3 + 697181557999146923828125$
and

$d(x, t) = (x^{36} - 210 x^{33} + 15015 x^{30} - 512050 x^{27} + 7882875 x^{24} - 121396275 x^{21} - 1618617000 x^{18} -$
 $24886236375 x^{15} + 1933235679375 x^{12} - 6750981737500 x^9 + 35442654121875 x^6 + 177213270609375 x^3 -$
 $59071090203125)(x^{27} - 126 x^{24} + 4725 x^{21} - 80850 x^{18} + 242550 x^{15} - 12733875 x^{12} - 202327125 x^9 +$
 $3034906875 x^6 + 11802415625)x$

is a rational to the PII equation with $k = 7$.

3.8 Rational solutions of order 8 to the PII equation

Example 3.8 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (12)$$

with

$n(x, t) = 9(x^{78} - 500 x^{75} + 107800 x^{72} - 13290200 x^{69} + 1054149250 x^{66} - 56954270450 x^{63} +$
 $2182435255000 x^{60} - 61218792635000 x^{57} + 1184599351560625 x^{54} - 15202431913418750 x^{51} + 680722416635511250 x^{48} -$
 $44511247889858750000 x^{45} + 1618864610751294156250 x^{42} - 30760665217171483343750 x^{39} + 366271140702488271250000 x^{36} -$
 $4205302302646655773062500 x^{33} + 16923129031360190701562500 x^{30} + 1588127001713110471112500000 x^{27} -$
 $15772764663338734621882812500 x^{24} + 158138338361678866550734375000 x^{21} - 827373130590858340520281250000 x^{18} -$
 $9439946497886250816235390625000 x^{15} + 74627563317580621197850000000000 x^{12} + 2467699223074068506178906250000 x^9 +$
 $466549384362441076949449462890625 x^6 - 118758025110439546859859863281250 x^3 + 593790125552197734299299316406250)x^2$

and

$d(x, t) = (x^{45} - 330 x^{42} + 40425 x^{39} - 2502500 x^{36} + 84084000 x^{33} - 1832505675 x^{30} + 11262876625 x^{27} -$
 $512899261875 x^{24} + 7377858613125 x^{21} + 1213488967315625 x^{18} - 17576518502668125 x^{15} + 133158051535884375 x^{12} +$
 $574407281135187500 x^9 + 6031276451919468750 x^6 - 15078191129798671875 x^3 - 5026063709932890625)(x^{36} -$
 $210 x^{33} + 15015 x^{30} - 512050 x^{27} + 7882875 x^{24} - 121396275 x^{21} - 1618617000 x^{18} - 24886236375 x^{15} +$
 $1933235679375 x^{12} - 6750981737500 x^9 + 35442654121875 x^6 + 177213270609375 x^3 - 59071090203125)$

is a rational to the PII equation with $k = 8$.

3.9 Rational solutions of order 9 to the PII equation

Example 3.9 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (13)$$

with

$n(x, t) = 10 x^{99} - 7755 x^{96} + 2666400 x^{93} - 540640100 x^{90} + 72656083500 x^{87} - 6885043415250 x^{84} + 478283191135750 x^{81} -$
 $24994039735665000 x^{78} + 995443251650347500 x^{75} - 30774522400428056250 x^{72} + 789241902945708712500 x^{69} -$
 $14132497027598524575000 x^{66} - 393571820961782463000000 x^{63} + 57511589969121983194500000 x^{60} - 2854771266271166837595000000 x^{57} +$
 $84652533666392628524713125000 x^{54} - 1738764748269819504005959687500 x^{51} + 22638787834126423159915558593750 x^{48} -$
 $61928167008387784364885625000000 x^{45} + 5460097506593342274778937132812500 x^{42} - 256859730759859813814707877835937500 x^{39} +$

$$\begin{aligned}
& 6006042255440231821441520571457031250 x^{36} - 71587944916670660974192241056230468750 x^{33} + 30463503029660443902979658432343750000 x^{30} - \\
& 1396403982432912014390426264854687500000 x^{27} + 134645528786293109228906905198024218750000 x^{24} \\
& - 738305471222290011644262679497430078125000 x^{21} + 5071842391337685049803906177804245117187500 x^{18} \\
& + 2979908030285868256083103840168751953125000 x^{15} - 45568646174733860329832593365104018554687500 x^{12} \\
& + 275363606772833103855319861209986108398437500 x^9 + 392043101168101368200794378671844628906250000 x^6 \\
& - 245026938230063355125496486669902893066406250 x^3 + 40837823038343892520916081111650482177734375
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & (x^{54} - 495 x^{51} + 96525 x^{48} - 9984975 x^{45} + 606981375 x^{42} - 23516192700 x^{39} + 543710792625 x^{36} - \\
& 11165422393125 x^{33} + 66400727518125 x^{30} + 9869935300225000 x^{27} + 683470066202673750 x^{24} - 32427901962268312500 x^{21} + \\
& 545830518898711921875 x^{18} - 834900983129995031250 x^{15} + 43832301614324739140625 x^{12} + 278578628037708342093750 x^9 - \\
& 4178679420565625131406250 x^6 - 8125209984433159977734375)(x^{45} - 330 x^{42} + 40425 x^{39} - 2502500 x^{36} + 84084000 x^{33} - \\
& 1832505675 x^{30} + 11262876625 x^{27} - 512899261875 x^{24} + 7377858613125 x^{21} + 1213488967315625 x^{18} - 17576518502668125 x^{15} + \\
& 133158051535884375 x^{12} + 574407281135187500 x^9 + 6031276451919468750 x^6 - 15078191129798671875 x^3 - 5026063709932890625)x
\end{aligned}$$

is a rational to the PII equation with $k = 9$.

We could go on and present more explicit rational solutions, but they become very complicated. For example, in the case of order 10 the numerator includes 59 terms and the degree of the polynomial in x is equal to 175; the denominator contains 59 terms and its degree in x is equal to 176. It will be relevant to study in detail the structure of these polynomials.

4 Conclusion

Rational solutions to the Painlevé equation have been given in terms of determinants involving explicit particular polynomials. This gives a very efficient method to construct rational solutions of this equation. We have constructed some examples of these rational solutions for the first orders.

References

- [1] P. Painlevé, H. Segur Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme, Acta Matg., V. 25, 1-85, 1900
- [2] A.P. Vorob'ev, On the rational solutions of the second Painlevé equation, Differ. Uravn., V. 1, N.1, 79-81, 1965
- [3] H. Airault, Rational solutions of the Painlevé equations, SIAM, V. 61, 31-53, 1979
- [4] Y. Murata, Rational Solutions of the Second and the fourth Painlevé equation, Funk. Ekv., V. 28, 1-32, 1985
- [5] H. Umemura, H. Watanabe, Solutions of the second and fourth Painlevé equations I, Nagoya Math. J., V. 148, 141-198, 1997

- [6] V.V. Tsegel'nik, Properties of solutions of two second-order differential equations with the Painlevé property, *Teoret. Mat. Fiz.*, V. 206, N. 3, 361-367, 2021
- [7] I. Mahmood, M. Waseem, Lax representation and Darboux solutions of the classical Painlevé second equation, *Adv. Math. Phys.*, V. 2021, 8851043-1-5, 2021
- [8] V.I. Gromak, Analytic properties of solutions to equations in the generalized hierarchy of the second Painlevé equation, *Diff Equat* , V.56, N. 8, 993-1009, 2020
- [9] N.A. Kudryashov, Rational solutions of equations associated with the Second Painlevé Equation, *Regul. Chaot. Dyn.*, V. 25, 273-280, 2020
- [10] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, *Max-Planck-Institut für Mathematik*, MPI 02-31, V. **161**, 2002
- [11] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, *Lett. Math. Phys.*, V. **68**, 77-90, 2004
- [12] P. Gaillard, V.B. Matveev, New formulas for the eigenfunctions of the two-particle Calogero-Moser system, *Lett. Math. Phys.*, V. **89**, 1-12, 2009
- [13] P. Gaillard, V.B. Matveev, Wronskian and Casorai determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions, *J. Phys A : Math. Theor.*, V. **42**, 404409-1-16, 2009
- [14] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spe. Top.*, V. **185**, 247-258, 2010
- [15] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A : Meth. Theor.*, V. **44**, 435204-1-15, 2011
- [16] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, *J. Math. Sciences : Adv. Appl.*, V. **13**, N. 2, 71-153, 2012
- [17] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, *J. Math. Phys.*, V. **54**, 013504-1-32, 2013
- [18] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, *J. Mod. Phys.*, V. **4**, N. 4, 246-266, 2013
- [19] P. Gaillard, V.B. Matveev, Wronskian addition formula and Darboux-Pöschl-Teller potentials, *J. Math.*, V. **2013**, ID 645752, 1-10, 2013

- [20] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, *J. Theor. Appl. Phys.*, V. **7**, N. **45**, 1-6, 2013
- [21] P. Gaillard, Deformations of third order Peregrine breather solutions of the NLS equation with four parameters, *Phys. Rev. E*, V. **88**, 042903-1-9, 2013
- [22] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, *J. Math.*, V. **2013**, 1-111, 2013
- [23] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, *J. Math. Phys.*, V. **54**, 073519-1-22, 2013
- [24] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, *Commun. Theor. Phys.*, V. **61**, 365-369, 2014
- [25] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, *Phys. Scripta*, V. **89**, 015004-1-7, 2014
- [26] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, *J. Of Phys. : Conf. Ser.*, V. **482**, 012016-1-7, 2014
- [27] P. Gaillard, M. Gastineau, Eighteen parameter deformations of the Peregrine breather of order ten solutions of the NLS equation, *Int. J. Mod. Phys. C*, V. **26**, N. 2, 1550016-1-14, 2014
- [28] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, *J. Math. Phys.*, V. **5**, 093506-1-12, 2014
- [29] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, *J. Phys. : Conf. Ser.*, V. **574**, 012031-1-5, 2015
- [30] P. Gaillard, Tenth Peregrine breather solution of the NLS, *Ann. Phys.*, V. **355**, 293-298, 2015
- [31] P. Gaillard, M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions of the NLS equation *Phys. Lett. A*, V. **379**, 1309-1313, 2015
- [32] P. Gaillard, Other $2N-2$ parameters solutions to the NLS equation and $2N+1$ highest amplitude of the modulus of the N -th order AP breather, *J. Phys. A: Math. Theor.*, V. **48**, 145203-1-23, 2015
- [33] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves, *Adv. Res.*, V. **4**, N. **5**, 346-364, 2015

- [34] P. Gaillard, Higher order Peregrine breathers solutions to the NLS equation, *Jour. Phys. : Conf. Ser.*, V. **633**, 012106-1-6, 2016
- [35] P. Gaillard, M. Gastineau Patterns of deformations of Peregrine breather of order 3 and 4, solutions to the NLS equation with multi-parameters, *Journal of Theoretical and Applied Physics*, V. **10**,1-7, 2016
- [36] P. Gaillard, M. Gastineau Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, *Commun. Theor. Phys*, V. **65**, N. **2**, 136-144, 2016
- [37] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, *Annals Of Physics*, V. **367**, 1-5, 2016
- [38] P. Gaillard, M. Gastineau Twenty two parameters deformations of the twelfth Peregrine breather solutions to the NLS equation, *Adv. Res.*, V. **10**, 83-89, 2016
- [39] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, *Theor. And Math. Phys.*, V. **189**, N. **1**, 1440-1449, 2016
- [40] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, *Jour. of Math. Phys.*, V. **57**, 063505-1-13, doi: 10.1063/1.4953383, 2016
- [41] P. Gaillard, M. Gastineau Families of deformations of the thirteenth Peregrine breather solutions to the NLS equation depending on twenty four parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. **3**, 130-139, 2017
- [42] P. Gaillard, From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on $2N^2$ parameters, *Int. Jour. of Appl. Sci. And Math.*, V. **4**, N. **3**, 60-70, 2017
- [43] P. Gaillard, Families of rational solutions of order 5 to the KPI equation depending on 8 parameters, *New Hor. in Math. Phys.*, V. **1**, N. **1**, 26-31, 2017
- [44] P. Gaillard, 6-th order rational solutions to the KPI Equation depending on 10 parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. **2**, 92-98, 2017
- [45] P. Gaillard, N -Order rational solutions to the Johnson equation depending on $2N - 2$ parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. **9**, 19-37, 2017
- [46] P. Gaillard, Families of rational solutions to the KPI equation of order 7 depending on 12 parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. **11**, 24-30, 2017
- [47] P. Gaillard, Rational solutions to the Johnson equation and rogue waves, *Int. Jour. of Inn. In Sci. and Math.*, V. **6**, N. **1**, 14-19, 2018

- [48] P. Gaillard, Multiparametric families of solutions of the KPI equation, the structure of their rational representations and multi-rogue waves, *Theo. And Mat. Phys.*, V. **196**, N. 2, 1174-1199, 2018
- [49] P. Gaillard, The Johnson equation, Fredholm and wronskian representations of solutions and the case of order three, *Adv. In Math. Phys.*, V. **2018**, 1-18, 2018
- [50] P. Gaillard, Families of Solutions of Order 5 to the Johnson Equation Depending on 8 Parameters, *NHIMP*, V. **2**, N. 4, 53-61, 2018
- [51] P. Gaillard, Multiparametric families of solutions to the Johnson equation, *J. Phys. : Conf. Series*, V. **1141**, 012102-1-10, 2018
- [52] P. Gaillard, Rational solutions to the Boussinesq equation, *Fund. Jour. Of Math. And Appl.*, V. , 109-112, 2019
- [53] P. Gaillard, Differential relations for the solutions to the NLS equation and their different representations, *Comm. In Adv. Math. Sci.*, V. **2**, N. 4, 1-4, 2019
- [54] P. Gaillard, Multi-parametric families of solutions of order N to the Boussinesq and KP equations and the degenerate rational case, *UJMA*, V. **3**, N. 2, 44-52, 2020
- [55] P. Gaillard, The mKdV equation and multi-parameters rational solutions, *Wave Motion*, V. **100**, 102667-1-9, 2021
- [56] P. Gaillard, Degenerate Riemann theta functions, Fredholm and wronskian representations of the solutions to the KdV equation and the degenerate rational case, *Jour. Of Geom. And Phys.*, V. **161**, 104059-1-12, 2021
- [57] P. Gaillard, 2N parameter solutions to the Burgers equation, *Jour. Of Appl. Nonlin. Dyn.*, V. **1**, N. **1**, 1-6, 2021
- [58] P. Gaillard, Multi-Parametric rational Solutions to the KdV equation, *Asi. Jour. Of Res. and Rev. In Phys.*, V. **3**, N. 3, 14-21, 2021
- [59] P. Gaillard, Multiparametric solutions to the Gardner equation and the degenerate rational case, *Jour. Of Appl. Ana. And Appl.*, V. **11**, N. 4, 2102-2113, 2021
- [60] P. Gaillard, Other families of rational solutions to the KPI equation, *Asi. Res. Jour Of Math.*, V. **17**, N. 6, 27-34, 2021
- [61] P. Gaillard, Rational solutions to the mKdV equation associated to particular polynomials, *Wave Motion*, V. **107**, 102824-1-11, 2021
- [62] P. Gaillard, Multiparametric Rational Solutions of Order N to the KPI Equation and the Explicit Case of Order 3, *Arch. Of cur. Res. Int.*, V. **21**, N . 6, 58-71, 2021

- [63] P. Gaillard, Rational solutions to the KPI equation from particular polynomials, *Wave Motion*, V. **108**, 102828-1-9, 2021
- [64] P. Gaillard, Rogue Waves of the Lakshmanan Porsezian Daniel Equation Depending on Multi-parameters, *As. Jour. Of Adv. Res. And Rep.*, V 16, N. 3, 32-40, 2022
- [65] P. Gaillard, Quasi-rational and rational solutions to the defocusing nonlinear Schrödinger equation, *Jour. Of Math. Sci. And Modelling*, V. 5, N. 1, 24-34, 2022
- [66] P. Gaillard, Rational solutions to the KdV equation in terms of particular polynomials, *Jour. Of Nonlin. Mod. and Ana.*, V. 4, N. 4, 615-626, 2022