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From particular polynomials to rational solutions to the PII equation

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Abstract

The Painlevé equations were derived by Painlevé and Gambier in the years 1895 – 1910. Given a rational function R in w, w' and analytic in z , they searched what were the second order ordinary differential equations of the form $w'' = R(z, w, w')$ with the properties that the singularities other than poles of any solution or this equation depend on the equation only and not of the constants of integration. They proved that there are fifty equations of this type, and the Painlevé II is one of these.

Here, we construct solutions to the Painlevé II equation (PII) from particular polynomials. We obtain rational solutions written as a derivative with respect to the variable x of a logarithm of a quotient of a determinant of order $n + 1$ by a determinant of order n . We obtain an infinite hierarchy of rational solutions to the PII equation.

We give explicitly the expressions of these solutions solution for the first orders.

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Keywords:

Painlevé equation II; rational solutions; determinants;

1 Introduction

We consider the Painlevé II equation (PII) which can be written in the form

$$u_{xx} - 2u^3 + 4xu - 4(n + 1) = 0, \quad (1)$$

where the subscript x denote derivative and n is an arbitrary integer.

This equation is one of the six famous equations discovered by Painlevé [1] at

the beginning of twentieth century. Many studies have realized and we refer for example to the works of Vorob'ev and Yablonskii [2], Airault [3], Murata [4] or Umemura [5].

Different types of solutions were given in terms of special polynomials. PII equation have solutions which can be expressed in terms of classical special functions, more precisely in terms of Airy functions. Recent works can be quoted as for example the following papers, [6], [7], [8], [9].

In the following, we consider particular polynomials and we construct rational solutions to the PII equation by using the Hirota bilinear method. Then we obtain rational solutions to the PII equation as a derivative of a logarithm of a quotient of a determinant of order $N + 1$ by a determinant of order N . That provides an effective method to construct an infinite hierarchy of rational solutions of order N . We present rational solutions for the first simplest orders.

2 Rational solutions to the PII equation

We consider the polynomials p_n defined by

$$\left\{ \begin{array}{l} p_n(x) = \sum_{k=0}^n x^k \frac{1}{3} \frac{\binom{n-k}{3}}{\binom{n-k}{3}!} \times \left(1 - \left[\frac{1}{2} \left(n - k + 1 - 3 \left[\frac{n-k}{3} \right] \right) \right] \right), \text{ for } n \geq 0, \\ p_n(x) = 0, \text{ for } n < 0. \end{array} \right. \quad (2)$$

In the previous definition of p_n , $[x]$ means the largest integer less than or equal to x . We denote A_n the determinant defined by

$$A_n = \det(p_{n+1-2i+j}(x))_{\{1 \leq i \leq n, 1 \leq j \leq n\}} \quad (3)$$

With these notations we have the following result

Theorem 2.1 *The function v_n defined by*

$$v_n(x) = \partial_x \left(\ln \frac{A_{n+1}}{A_n} \right) \quad (4)$$

is a rational to the PII equation (1)

$$u_{xx} - 2u^3 + 4xu - 4(n+1) = 0.$$

3 Explicit rational solutions to the PII equation for the first orders

In the following, we will call

$$v_k(x, t) = \partial_x \left(\ln \frac{A_{k+1}(x, t)}{A_k(x, t)} \right)$$

a rational solution to the PII equation of order k :

$$u_{xx} - 2u^3 + 4xu - 4(k+1) = 0.$$

In this section we will give some explicit examples of rational solutions to the PII equation.

3.1 Rational solutions of order 1 to the PII equation

Example 3.1 *The function $v_k(x, t)$ defined by*

$$\frac{2x^3 + 1}{x(x^3 - 1)} \tag{5}$$

is a rational to the PII equation with $k = 1$.

3.2 Rational solutions of order 2 to the PII equation

Example 3.2 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = 3 \frac{x^2 (x^6 - 2x^3 + 10)}{(x^6 - 5x^3 - 5)(x^3 - 1)} \tag{6}$$

is a rational to the PII equation with $k = 2$.

3.3 Rational solutions of order 3 to the PII equation

Example 3.3 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{4x^{15} - 50x^{12} + 250x^9 + 1400x^6 - 1750x^3 + 875}{(x^9 - 15x^6 - 175)(x^6 - 5x^3 - 5)x} \tag{7}$$

is a rational to the PII equation with $k = 3$.

3.4 Rational solutions of order 4 to the PII equation

Example 3.4 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \tag{8}$$

with

$$n(x, t) = 5x^{24} - 190x^{21} + 2450x^{18} - 2800x^{15} + 134750x^{12} - 1041250x^9 +$$

$$1715000x^6 + 4287500x^3 + 1071875$$

and

$d(x, t) = (x^{15} - 35x^{12} + 175x^9 - 1225x^6 - 12250x^3 + 6125)(x^9 - 15x^6 - 175)x$
is a rational to the PII equation with $k = 4$.

3.5 Rational solutions of order 5 to the PII equation

Example 3.5 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (9)$$

with

$$n(x, t) = 3(2x^{33} - 175x^{30} + 5600x^{27} - 73500x^{24} + 808500x^{21} + 257250x^{18} - 153063750x^{15} + 1209075000x^{12} + 90037500x^9 + 20633593750x^6 - 11554812500x^3 + 57774062500)x^2$$

and

$$d(x, t) = (x^{21} - 70x^{18} + 1155x^{15} - 9800x^{12} - 67375x^9 - 1414875x^6 + 4716250x^3 + 2358125)(x^{15} - 35x^{12} + 175x^9 - 1225x^6 - 12250x^3 + 6125)$$

is a rational to the PII equation with $k = 5$.

3.6 Rational solutions of order 6 to the PII equation

Example 3.6 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (10)$$

with

$$n(x, t) = 7x^{48} - 1204x^{45} + 81480x^{42} - 2773400x^{39} + 57420650x^{36} - 596048250x^{33} + 2961805000x^{30} - 201383875000x^{27} + 5418794390625x^{24} - 38100013393750x^{21} + 313753601218750x^{18} + 771769036500000x^{15} - 10091655480156250x^{12} + 58843893702343750x^9 + 133591542459375000x^6 - 111326285382812500x^3 + 27831571345703125$$

and

$$d(x, t) = (x^{27} - 126x^{24} + 4725x^{21} - 80850x^{18} + 242550x^{15} - 12733875x^{12} - 202327125x^9 + 3034906875x^6 + 11802415625)(x^{21} - 70x^{18} + 1155x^{15} - 9800x^{12} - 67375x^9 - 1414875x^6 + 4716250x^3 + 2358125)x$$

is a rational to the PII equation with $k = 6$.

3.7 Rational solutions of order 7 to the PII equation

Example 3.7 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (11)$$

with

$$n(x, t) = 8x^{63} - 2436x^{60} + 307860x^{57} - 21236600x^{54} + 907622100x^{51} - 24766537950x^{48} + 459897761400x^{45} -$$

$7268461893000 x^{42} + 17185043452500 x^{39} + 5545487726528750 x^{36} - 155690409080456250 x^{33} + 2380328650825125000 x^{30} -$
 $11581750581025625000 x^{27} + 30516125198934375000 x^{24} - 1609613959883304375000 x^{21} + 30155017717413651562500 x^{18} +$
 $6390151046230514062500 x^{15} + 384626916672776121093750 x^{12} - 2908243070510727167968750 x^9 +$
 $3346471478395905234375000 x^6 + 4183089347994881542968750 x^3 + 697181557999146923828125$
and

$d(x, t) = (x^{36} - 210 x^{33} + 15015 x^{30} - 512050 x^{27} + 7882875 x^{24} - 121396275 x^{21} - 1618617000 x^{18} -$
 $24886236375 x^{15} + 1933235679375 x^{12} - 6750981737500 x^9 + 35442654121875 x^6 + 177213270609375 x^3 -$
 $59071090203125)(x^{27} - 126 x^{24} + 4725 x^{21} - 80850 x^{18} + 242550 x^{15} - 12733875 x^{12} - 202327125 x^9 +$
 $3034906875 x^6 + 11802415625)x$

is a rational to the PII equation with $k = 7$.

3.8 Rational solutions of order 8 to the PII equation

Example 3.8 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (12)$$

with

$n(x, t) = 9(x^{78} - 500 x^{75} + 107800 x^{72} - 13290200 x^{69} + 1054149250 x^{66} - 56954270450 x^{63} +$
 $2182435255000 x^{60} - 61218792635000 x^{57} + 1184599351560625 x^{54} - 15202431913418750 x^{51} + 680722416635511250 x^{48} -$
 $44511247889858750000 x^{45} + 1618864610751294156250 x^{42} - 30760665217171483343750 x^{39} + 366271140702488271250000 x^{36} -$
 $4205302302646655773062500 x^{33} + 16923129031360190701562500 x^{30} + 1588127001713110471112500000 x^{27} -$
 $15772764663338734621882812500 x^{24} + 158138338361678866550734375000 x^{21} - 827373130590858340520281250000 x^{18} -$
 $9439946497886250816235390625000 x^{15} + 74627563317580621197850000000000 x^{12} + 2467699223074068506178906250000 x^9 +$
 $466549384362441076949449462890625 x^6 - 118758025110439546859859863281250 x^3 + 593790125552197734299299316406250)x^2$

and

$d(x, t) = (x^{45} - 330 x^{42} + 40425 x^{39} - 2502500 x^{36} + 84084000 x^{33} - 1832505675 x^{30} + 11262876625 x^{27} -$
 $512899261875 x^{24} + 7377858613125 x^{21} + 1213488967315625 x^{18} - 17576518502668125 x^{15} + 133158051535884375 x^{12} +$
 $574407281135187500 x^9 + 6031276451919468750 x^6 - 15078191129798671875 x^3 - 5026063709932890625)(x^{36} -$
 $210 x^{33} + 15015 x^{30} - 512050 x^{27} + 7882875 x^{24} - 121396275 x^{21} - 1618617000 x^{18} - 24886236375 x^{15} +$
 $1933235679375 x^{12} - 6750981737500 x^9 + 35442654121875 x^6 + 177213270609375 x^3 - 59071090203125)$

is a rational to the PII equation with $k = 8$.

3.9 Rational solutions of order 9 to the PII equation

Example 3.9 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (13)$$

with

$n(x, t) = 10 x^{99} - 7755 x^{96} + 2666400 x^{93} - 540640100 x^{90} + 72656083500 x^{87} - 6885043415250 x^{84} + 478283191135750 x^{81} -$
 $24994039735665000 x^{78} + 995443251650347500 x^{75} - 30774522400428056250 x^{72} + 789241902945708712500 x^{69} -$
 $14132497027598524575000 x^{66} - 393571820961782463000000 x^{63} + 57511589969121983194500000 x^{60} - 2854771266271166837595000000 x^{57} +$
 $84652533666392628524713125000 x^{54} - 1738764748269819504005959687500 x^{51} + 22638787834126423159915558593750 x^{48} -$
 $61928167008387784364885625000000 x^{45} + 5460097506593342274778937132812500 x^{42} - 256859730759859813814707877835937500 x^{39} +$

$$\begin{aligned}
& 6006042255440231821441520571457031250 x^{36} - 71587944916670660974192241056230468750 x^{33} + 30463503029660443902979658432343750000 x^{30} - \\
& 1396403982432912014390426264854687500000 x^{27} + 134645528786293109228906905198024218750000 x^{24} \\
& - 738305471222290011644262679497430078125000 x^{21} + 5071842391337685049803906177804245117187500 x^{18} \\
& + 2979908030285868256083103840168751953125000 x^{15} - 45568646174733860329832593365104018554687500 x^{12} \\
& + 275363606772833103855319861209986108398437500 x^9 + 392043101168101368200794378671844628906250000 x^6 \\
& - 245026938230063355125496486669902893066406250 x^3 + 40837823038343892520916081111650482177734375
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & (x^{54} - 495 x^{51} + 96525 x^{48} - 9984975 x^{45} + 606981375 x^{42} - 23516192700 x^{39} + 543710792625 x^{36} - \\
& 11165422393125 x^{33} + 66400727518125 x^{30} + 9869935300225000 x^{27} + 683470066202673750 x^{24} - 32427901962268312500 x^{21} + \\
& 545830518898711921875 x^{18} - 834900983129995031250 x^{15} + 43832301614324739140625 x^{12} + 278578628037708342093750 x^9 - \\
& 4178679420565625131406250 x^6 - 8125209984433159977734375)(x^{45} - 330 x^{42} + 40425 x^{39} - 2502500 x^{36} + 84084000 x^{33} - \\
& 1832505675 x^{30} + 11262876625 x^{27} - 512899261875 x^{24} + 7377858613125 x^{21} + 1213488967315625 x^{18} - 17576518502668125 x^{15} + \\
& 133158051535884375 x^{12} + 574407281135187500 x^9 + 6031276451919468750 x^6 - 15078191129798671875 x^3 - 5026063709932890625)x
\end{aligned}$$

is a rational to the PII equation with $k = 9$.

We could go on and present more explicit rational solutions, but they become very complicated. For example, in the case of order 10 the numerator includes 59 terms and the degree of the polynomial in x is equal to 175; the denominator contains 59 terms and its degree in x is equal to 176. It will be relevant to study in detail the structure of these polynomials.

4 Conclusion

Rational solutions to the Painlevé equation have been given in terms of determinants involving explicit particular polynomials. This gives a very efficient method to construct rational solutions of this equation. We have constructed some examples of these rational solutions for the first orders.

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