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Elastica project: dynamic relaxation for post-formed elastic gridshells

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Abstract

An elastic gridshell is an experimental structural typology truly effective in terms of material consumption by span ratio. Because classical beam theory is not valid for large deflections, the form-finding, the dimensioning and the optimization of elastic gridshells are usually done with dynamic relaxation method. The apparent complexity of this method makes that typology a subject area restricted to researchers or engineers and still too inaccessible for architects.

The goal of this year's project is to make those concepts understandable for a group of architecture students and to re-build, from nothing, the set of necessary tools for the form-finding, dimensioning, fabrication and assembly of a full-scale elastic gridshell in post-formed bending. This reconstitution is historical as well as technical and practical, and produces all the necessary theoretical notions for the execution of this project. Euler's elastica, which gave its name to the project, is the starting point of this adventure that lets us to recreate – partly – almost two millenniums of history of active bending technics. This approach strongly interlinks theoretical and practical fields in order to recontextualize modern methods of calculation. In order to popularize those notions, we engaged in three different works:

- The development of a complete, generic, open-source and ergonomic algorithm, usable by all, for form-finding, dimensioning and optimization of elastic gridshells using dynamic relaxation.
- The publication in La Villette editions, of a collective book that compiles our study and the historical, scientific and conceptual knowledge over this subject in a pedagogic way, co-written with the master cycle students.
- The practical application of our study by constructing a full-scale public pavilion, in partnership with the municipality of Paris on the belvedere of the Butte du Chapeau Rouge park (Paris XIX).

Keywords: gridshell, timber, dynamic relaxation, grasshopper, form finding, teaching, pedagogy.

1. Introduction

Elastic gridshells have come a long way from the primitive hut described by Eugène Viollet-le-Duc in its *Histoire de l'habitation humaine* and the vernacular pygmies lobembes to the post-formed gridshell of the Manheim Multihalle of Frei Otto (arch.) and Happold and Liddell (ing.). Yet this millennial typology remains anecdotal, except numerous educational pavilions and private self-building installations, in modern architecture. The Multihalle in Mannheim, built in 1975, remains almost the only large-scale architectural object meeting a defined program and need.

Why such reluctance? The scientific literature, which is becoming increasingly rich on this subject, emphasizes their capacity to overcome large spans with less material. The material used, often wood, is part of the current problem of limiting anthropogenic carbon emissions. If it is not generalized, the use of elastic gridshells should be much more prominent in the covers of large projects.

Our goal is to propose solutions on to this issue. The main problem we identified – apart from the issues of acoustics, thermal insulation and standardization of the finishing work which would merit their own study – is the complexity for the architect to grasp the form of a post-formed elastic gridshell.

As for funicular domes, elastic gridshells take their shape in a natural way, according to the laws of physics: a rod of a given length takes one and only one natural shape when bent. This does not restrict architectural freedom: the assembly of elements of different lengths in various positions makes it possible to create free and varied shapes. However, the natural form is unknown and it must be found: the process that leads to it, the form-finding, uses complex tools such as dynamic relaxation that the architect does not possess (with a few exceptions). Some tools, such as Kangaroo, already allow an easy simulation of active bending but operate in a "black box": the detail of the performed calculation is not accessible to the designer.

To overcome this difficulty in being able to know or predict the real shape of the work, we wanted to popularize the current scientific knowledge on this typology and to gather it in a simple and ergonomic design tool: an algorithm called ELASTICA. This tool also allows anyone to access the detailed data of the calculation and to conduct a complete study of dimensioning and verification of the structure. Finally, it has been applied to the construction of a post-formed elastic gridshell at human scale called "Elastica" on the belvedere of the Parc de la Butte du Chapeau in the 19th district of Paris.

2. State of the art

2.1. What is an elastic gridshell?

In architecture, a shell is a continuous thin structure with a curved surface. Its rigidity is related to its curvature (shape resistance). Thus, a gridshell is a structural lattice of bars forming a curved surface.



Figure 1: Schematic typological definition of the gridshell [Jean-Maurice Michaud / Sofia Colabella]

Labbé [9] classifies gridshells into two main groups:

- "(...) those with pre-calculated members, both in their curvature and in their geometrical resolution but also in their "inactive-bending" fixings,
- and those known as "active-bending" which start from an initially flat grid, which connections are not fixed until after their assembling, once the structure is established in its architectural form".

The first category works in compression and is not the subject of this study. The elements in the second one, called elastic gridshells, are working in flexion and compression and have two main characteristics:

- They are in active bending; the shape is given by the bending of straight elements maintained fixed. This condition is necessary to qualify a gridshell as elastic.
- They are post-formed which means that the grid is manufactured flat, not braced. The thin and hinged elements form a deformable unit that is then flexed during the erection. This condition is not necessary to be part of elastic gridshells, however our study will be placed in this framework.

The natural shape of an elastic gridshell depends on the initial grid and the displacements imposed on its support points. Let us take the simplest of them as an example: a simple flexible rod on the ends of which one pushes laterally. Initially the rod is in compression. Very slender, its equilibrium in compression by shortening quickly gives way to an unstable equilibrium in flexion: this is buckling. This can be generalized by describing a post-formed elastic gridshell as the post-buckling shape of a flat grid subjected to imposed displacements of its supports.

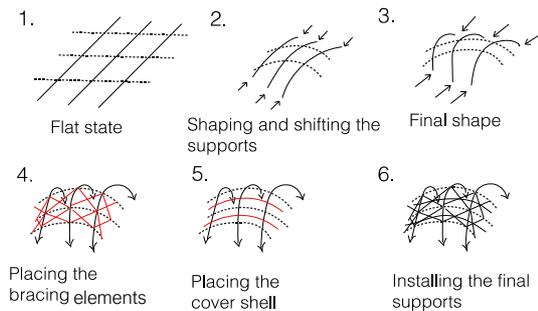


Figure 2: Schematic typological definition of elastic gridshells.

Once the ends of the bars are in their final position, the bent gridshell, which is by nature very deformable, must be stabilized and rigidified by adding bracing to limit the deformation of the mesh and possibly by adding shear blocks. The final grid is very rigid and can cover a large span without intermediate supports, and this with very little material.

The design of an elastic gridshell must consider many constraints: material properties and section, types of connections between elements, initial grid layout, support conditions, deployment, use and context, etc. It is therefore not the result of a simple artistic reflection; on the contrary, it is the result, through modeling and/or calculation, of a long process of research into the natural shape of the desired shell.

2.2. A typology very present in vernacular and primitive architecture

According to the anthropologist Reclus [14], who studied different ethnic groups, vernacular architectures, by using local materials, are a singular continuity of the environment in which they are established. The flexible and resistant natural materials that constitute the main resources of certain communities explain why the use of elastic gridshells took place really early in history: they could date back to the Neolithic period.

Vitruvius, in his treatise *De architectura*, already foreshadowed the active bending potential of certain materials: “*The elm tree and the ash contain much water and but little air and fire, with a moderate portion of earth. They are therefore pliant, and being so full of water, and from want of stiffness, soon bend under a superincumbent weight.*”



Figure 3: Primitive hut in active bending according to Viollet-le-Duc.

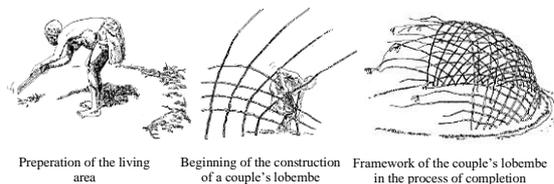


Figure 4: Steps to build a lobembe according to Philippart de Foy [*Les Pygmées d'Afrique centrale*. Éditions Parenthèse, Paris, 1984].

The way of life of the pygmies of the great equatorial forest (Congo and Central African Republic) has remained traditional. Being a nomadic civilization, their huts, the lobembes, built by single women or men, are temporary habitats. The mongulu is a more solid variant for a life period of two weeks to a month. They are domes made of latticework of thin branches bent and anchored in the ground. Leaves are used to form a primitive form of waterproofing barrier.

This construction method is very similar to that of the huts of the Khoikhoi, Kraals or Nama of southern Africa (structures made in fire-tanned tamarisk branches) and to the seasonal wigwams or wickiup, of some native Americans tribes, such as the Apaches or the Ojibwe (structures made in green wood from young trees forming a circle of 10 to 16 feet of diameter).

These examples of gridshells are in active bending but not post-formed. To complete this non-exhaustive list, let us mention the Mongolian yurt whose circumference is a post-formed mesh in flexion around the central vertical axis blocked by straps and anchored to the ground.

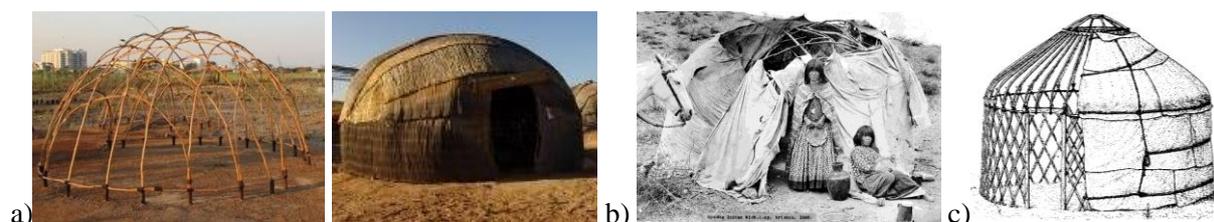


Figure 5: Huts of the Haru Oms, Nama people (a), Apache wickiup (b) and Mongolian yurt (c) [Exploring Africa / Richtersveld.net / Wikivillage.co.za and Rose, Noah H. / Smith Archive & Alamy Stock Photo].

Elastic gridshells are therefore used in primitive and vernacular architecture. One could deduce that it is therefore a very accessible typology. However, its use in contemporary architecture is rare and the studies that allowed the understanding of this typology only began in the 1970s.

2.3. A typology rarely used in contemporary architecture

In 1897 the Russian engineer Vladimir Shukhov built a production workshop for a steel company in Vyksa. It is the first double-curved grid structure working in compression. Contrary to this example, an elastic gridshell is built flat, before being set up by bending it.

It was not until 1962 that this typology was highlighted by the work of Frei Otto who, using a study he had been carrying out since the late 1950s on lightweight shells from suspended net models, built a first trial model of an elastic gridshell during a visit to the University of Berkeley (California). This structure is designed from a square grid of 52 m² round steel rods anchored to the ground at its four corners.

Later in the same year, he built a wooden trial gridshell at the German Building Exhibition in Deubau, Essen. Developed with Bernd Friedrich Romberg using suspended models, this 198 m² structure has a super-elliptical plan (squirrelle), sort of intersection between a 15 m × 15 m square and a circle in plan.

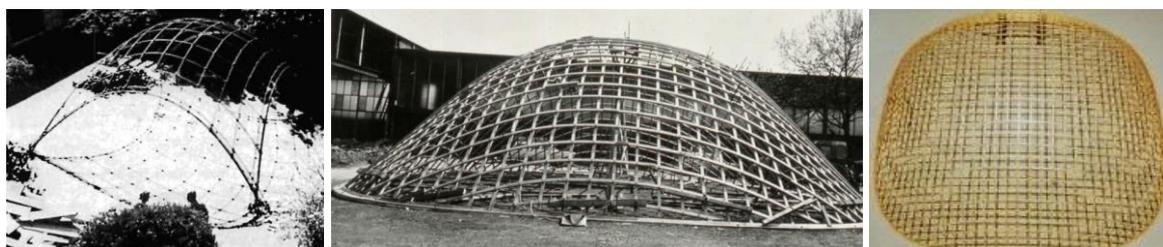


Figure 6: From left to right: gridshell in Berkeley, trial gridshell in Essen and its model [from [7] and [12]].

Its maximum diagonal span is 16.82 m, with a center height of 4.85 m. Two orthogonal layers (lattice is composed with doubled laths in one direction and single laths in the other) of 60 mm x 13 mm pine elements are assembled to the floor by bolting at the knots, forming a square grid with a mesh size of 482 mm per side. The grid, composed of 1,140 linear metres of laths, was then lifted using a mobile crane and wooden stilts on the perimeter before being fixed to a rim beam driven into the ground.



Figure 7: Construction of the Essen Gridshell [ibid].

However, the first architectural project answering to a program, and one of the few of this scale, is undoubtedly the Multihalle in the Herzogenried Park in Mannheim (Germany), built in 1975 for the Bundesgartenschau. The winning architects of the competition, Carlfried Mutschler, Winfried Langner, and Heinz Eckebrecht, encountered difficulties in developing their idea of a free-form, airy and light structure: their proposal for large parasols suspended by helium balloons was rejected by the authorities. They then asked Frei Otto to help them, who became their engineering consultant.

The project was designed using the suspended net method, to which we will come back later, then numerically calculated and tested. The grid built on the ground is composed of two interlaced orthogonal networks, each composed by a double layer of laths 55 mm wide, forming a square mesh of 500 mm side. The knots are held by initially loose bolts to allow their rotation during erection, which was carried out using height-adjustable scaffold towers. The curved grid, still flexible at the time, is then blocked at the ends and braced to stiffen it. The western hemlock timber was shaped green and not dried because the flexibility of the wood increases with its humidity level.



Figure 8: The Mannheim Multihalle, exterior and interior views [Verein Multihalle Mannheim, Daniel Lukac].

This project will make the concept of elastic gridshells popular among architects and engineers around the world. However, the complexity of its constructive approach, highlighted by the work published by Happold and Liddell [7], probably explains why similar projects are exceptionally rare to this day.

3. ELASTICA algorithm design method

3.1. Study of bending behavior

Designing an elastic gridshell requires a good understanding of bending. We know from the work of Galileo, later corrected by Huygens, Leibniz and Bernoulli, that the natural shape of a chain suspended in pure traction is a hyperbolic cosine. Similarly, to understand an elastic gridshell, it is fundamental to question the natural shape of the flexible rod in pure flexion.

In the framework of classical beam theory, which is based on the Euler-Bernoulli hypotheses (plane sections remain plane and normal to the axis when it deforms), bending moments induce deflection and not the opposite which implies that the supports, including the sliding supports, remain immobile on the drawing. The deformation thus shows an elongation of the neutral axis, whereas it is supposed to remain constant in length: this theory is therefore valid only for small deflections and rotations, where this inconsistency is considered insignificant. This theoretical framework, which is the one in which engineers and architects usually evolve, is sufficient for most structures.

In fact, when the initially flat grid is shaped by the bending resulting from its buckling, the hypothesis of small displacements and fixed supports is fundamentally questioned since it is precisely the displacement of the mesh supports that will give the shape to the gridshell.

Let us take the case of flexible rod, with quadratic moment I and modulus of elasticity E , on two sliding supports, subjected to a following force (a load on the support that continues to act at a point even if it moves). The deflection y is related to the curve's abscissa x by the relation $d^2y/dx^2 = -M(x)/EI$. The curvature corresponds to the radius of curvature R evaluated at the studied point. It is related to the bending moment by the relation $M = EI/R$. The equilibrium of moments leads to the differential equation relating the deflection $y(x)$ to the force F , and solutions shape was found by Euler:

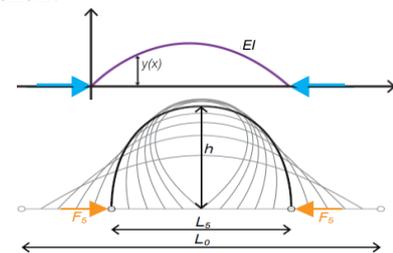


Figure 9: buckling of the flexible rod and elastica graph at various stages.

$$EI * \frac{d^2y}{dx^2} + F * y(x) = 0 \rightarrow y(x) = A * \cos(\omega x) + B * \sin(\omega x) = 0 \quad (1)$$

Where $\omega^2 = F/EI$. A and B can be determined by the boundary conditions (no deflection at the ends):

$$y(0) = 0 \rightarrow A = 0 \quad \text{and} \quad y(L) = 0 \rightarrow B * \sin(\omega L) = 0 \quad (2)$$

As long as $\omega L < \pi$, the only solution is $A = 0$ and $B = 0$: this is the stable compression equilibrium (shortening of the rod). When $\omega L = \pi$, the equation is verified for every value of B and the system becomes unstable: the rod buckles as soon as the force exceeds the Euler's critical load $F_{cr} = \pi^2 EI/L$. Once the rod's buckling is initiated, the hypothesis of small deflections is no longer valid. The curvature can no longer be written d^2y/dx^2 but is still by definition the inverse of the radius of curvature ($1/R$):

$$\frac{EI}{R} + F * y(x) = 0 \quad (3)$$

The EI/R moment at any point, and so the curvature $1/R$, are thus proportional to the distance from the point to the axis $y = 0$, the guiding line: this is the geometrical definition of singular curve called elastica.

Watzky and Bueno [15] explained that Jacques Bernoulli questioned in 1691 the form taken by a vertical elastic lamina embedded at the bottom end and pulled downwards on the other end. The resolution of this simple problem is so complicated that Huygens and Leibniz gave up but Bernoulli “obtains the equations of the rectangle elastica [in 1692], then those of the general case in 1694.” Fifty years later, Euler [1] characterizes its various forms. More than a century later, the values, tabulated, of a closed form of the solutions are approached and Born measures them precisely from photographs of bent blades in 1906. He wrote in his memoirs on this subject: “I felt for the first time the pleasure of finding a theory in agreement with the measurements – one of the most pleasant experiences I know.” In line with this reasoning, we filmed the buckling of a flexible rod to digitally obtain the kinematics of the curve with an image recognition algorithm, measured the radius of curvature at any point and deduced the bending moment. The experimental results, which converge towards elastica, have been confronted with numerical calculations made with the ELASTICA algorithm which confirmed them.

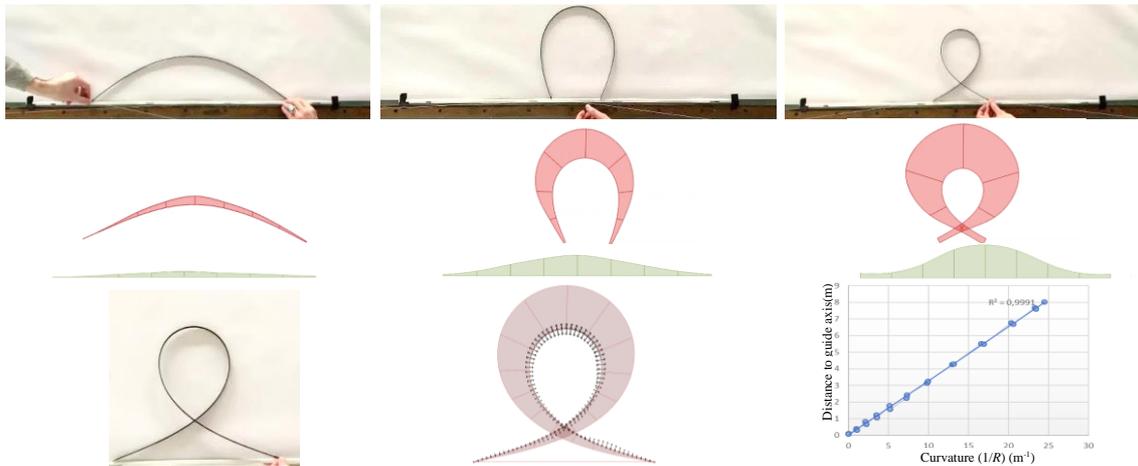


Figure 10: From top to bottom: extracts from the video of the experiment; deflection and bending moment of the flexible rod by image recognition processing; post-buckling shape obtained by the experiment and by the numerical dynamic relaxation algorithm ELASTICA.

3.2. Form finding studies

Form finding can be carried out either experimentally, by means of hanging chain nets or active bending models, or numerically, for example through dynamic relaxation. The work of Frei Otto, ARUP and Happold & Liddell, particularly on the gridshells of Essen and Mannheim, has been tested by all three methods and provides valuable data for assessing the relevance of each method.

3.2.1. Form finding with a model

Hanging chain nets model: it is simple to realize, although it requires sliding links to make sure that all the cables are tight. Its use can be surprising, since the notion of an antifunicular – and therefore pure compression – is used to model an object in flexion and compression. To confirm its relevance, in 1973, Linkwitz digitally modeled the Mannheim model using photogrammetry. The calculations conducted by Happold took into account the bending and led to results similar to those of the hanging chain nets.

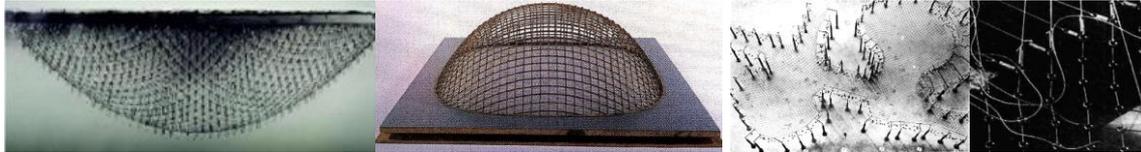


Figure 11: Hanging chain net and active-bending models of the trial gridshell in Essen; and hanging chain net model of the Mannheim Multihalle and zoom on the links [from [7] and [12]].

The shape of a hanging chain (hyperbolic cosine) is determined only by its axial stiffness and a flexible rod (elastica) is determined both by its axial stiffness and by its bending stiffness. To claim that one is close to the other is therefore equivalent to saying that the bending stiffness of the flexible rod is negligible compared to its axial stiffness. This is what Douthe [5] wanted to verify, arguing that this hypothesis was based on the fact that “*the stiffness of the shell comes from its shape and that this shape has been specially studied to take up the dead weight loads by membrane stress, i.e. only by axial forces in the gridshell elements. However, (...) the material resists well to axial forces (much better at least than to bending forces), the dimensions of the gridshell elements can therefore be reduced and, consequently, the total weight of the structure as well. This reduction in cross-sections is accompanied by a decrease in inertia so that the bending stiffness of the elements is low, which legitimizes retrospectively the initial hypothesis and the form finding using a net model.*”

Nevertheless, as the inertia of the grid is not zero, Douthe studies the differences between the funicular and the elastica shape according to the attack angle α at the base and the loading rate p . He carries out this study on a simple beam, a rectangular grid and a free-form grid. He concludes that the shape of the gridshell is almost funicular if the angle of attack α is less than 65° (optimum at 57.5°), which corresponds to a pL^3/EI ratio below 65, confirming *a posteriori* the modelling of Mannheim by Frei Otto. Like Happold and Liddel [7], we can therefore conclude that “*a funicular shape is an advantage but is not essential.*”

About the active bending model. In 1973, as Happold [7] reminded us, “*there was no previous engineering experience in this field.*” To overcome this, his team first studied a simpler example by loading a PMMA model at the 1/16th scale of the trial gridshell in Essen and comparing the results with the data collected by the Warmbronn Workshop on the actual project. The tests were conducted with pinned or rigidly glued nodes and with or without bracing. The team found, and retained for the Mannheim project, that the addition of bracing on the diagonals of the lattice reduced deflection and increased the maximum nodal load causing buckling of the shell, but that the collapse was more sudden.

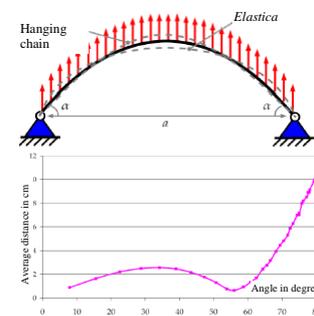


Figure 12: Diagram of the problem studied by Douthe, and evolution of the distance to the hanging chain form with the angle α .

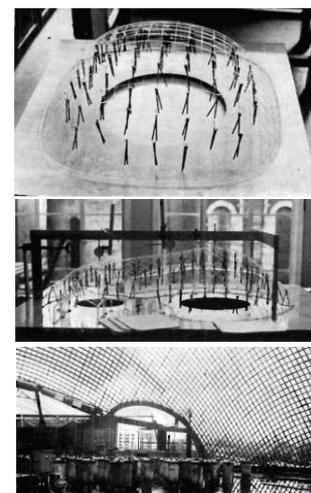


Figure 13: Load test by adding nails to the nodes of the Essen and Mannheim models and loading tests with water-filled garbage cans [from [7]].

An active bending model of the Multihalle was then fabricated in PMMA at 1/60th scale and tested in the same way. The buckling collapse load of the model was measured at 2.8 kg/m² without bracing and at 12.5 kg/m² with bracing. Happold and Liddell demonstrate that an extrapolation of the critical load is possible from a model to a real project by multiplying it by the ratio of EI_{xx} / aS^3 of the project and the model (EI_{xx} is the out-of-plane bending stiffness, a the spacing of laths and S the gridshell span).

From these studies and our experiences on the essential question of the extrapolation of the results from the model to the real project, we conclude that:

- The shape of a funicular and the shape of an elastic gridshell can be transposed from the model to the real project, regardless of the stiffness, the section and length of the material used.
- The buckling force is transposable but subject to several measurement biases.
- The shear and node stiffness are difficult to transpose, and this may reduce the relevance of the results of the previous point.

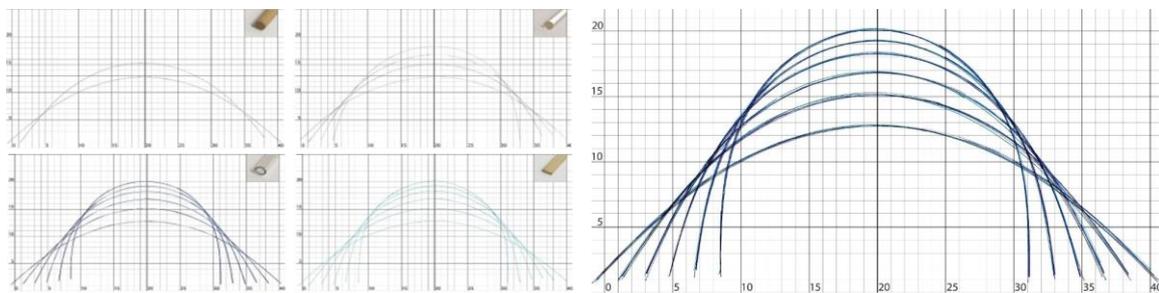


Figure 14: Scalability tests on the section and stiffness of the material, and on the length of the element.

3.2.3. The numerical method: discretization of the Chebyshev lattice surface

To overcome the inaccuracies of a form finding with a model, a numerical method is generally necessary. The shape resulting from the initial phase of intention, which we will now call “architect’s shape”, is not the real shape of the project, which must respect the rules of physics (especially bending). The form finding consists in determining, from the architect’s shape, what the real shape is going to be.

The first step consists in dividing the architect’s shape into a network of two layers of bars forming equilateral meshes (a necessary condition for flat fabrication). The division of any surface into equilateral parallelograms is called a Chebyshev lattice, named after the mathematician who, in 1878, having a rather modest salary, accepted a contract to optimize the cutting of military uniforms. The human body is indeed a complex shape that it is complicated to cover with a flat fabric without folds. Until the end of the 19th century, clothes were made by eye and therefore not fitted. Custom tailoring was a time-consuming and expensive art. Chebyshev [4] therefore devised a method to create a piece of clothing adapted to the human anatomy, in large quantities, quickly and at low cost.

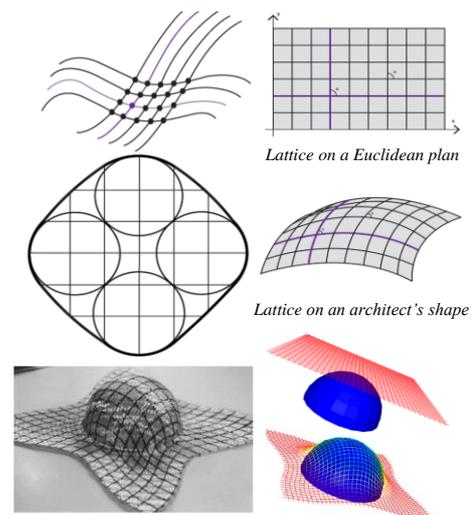


Figure 15: Discretization of a fabric and Chebyshev pattern for dressing a half-sphere; Chebyshev lattice: principle, by model and by dynamic relaxation [BOISSE, P. and BOUHAYA, L.].

The problem formulated by Ghys [6] highlights the link with gridshells: “A flattened fabric is formed by two networks of interwoven straight threads (...) which form small squares. (...) The initial small squares can become deformed: their sides do not change in length but the angle between the threads is no longer necessarily straight.” Thus, the change of angle between the threads allows them to envelop

a double-curved surface without any fold. To achieve a Chebyshev lattice on any surface, one can go for a dynamic relaxation method, or by the geometric method, called "the compass", used by Frei Otto.

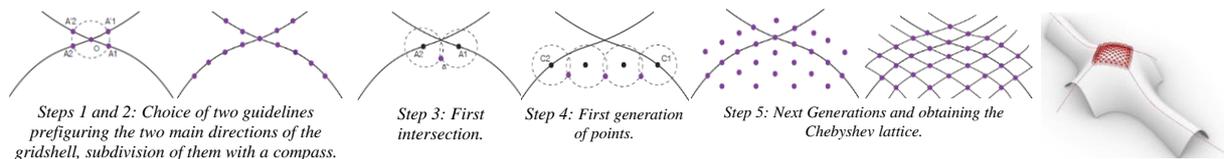


Figure 16: Compass method and application by the ELASTICA algorithm.

3.2.2. Dynamic relaxation

This form finding method, even though simple since it is based on the laws of Newtonian physics, is iterative and heavy by its quantity of calculations: its development had to wait until the end of the 20th century and computer-assisted numerical modeling. According to Bouhaya [3], “Southwell has developed a relaxation method for the solution of partial differential equations by finite difference approximations. The concept of the method itself was established much earlier by Rayleigh (...). The development of this method is attributed to Otter in 1964 for the study of pre-stressed concrete pressure vessels. One year later, the dynamic relaxation method was developed by Day for the same type of problems”. It was then further developed and applied, among others, to tensioned structures in the 1970s and to inflatable structures in 1977 (Barnes), as well as to shells (Otter 1964), prestressed cables (Barnes 1975), folds and creases (Zhang 1989), and tensioned and inflatable structures (Barnes 1977 and 1999 [2]).

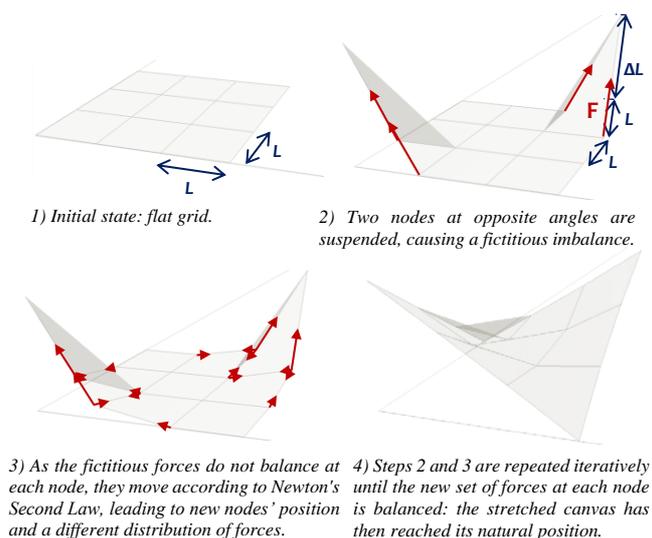


Figure 17: General principle of dynamic relaxation applied to an example of the form finding of a stretched canvas [10].

Dynamic relaxation allows to solve static equilibrium problems by a fictitious dynamic calculation. It is valid for large deflections. Bouhaya [3] explains “it is an iterative method that describes the movement of the structure from the moment of loading to its equilibrium (...)”

Theory from an analytical point of view: According to Barnes [2], “the basis of the method is to trace step-by-step for small time increments, Δt , the motion of each node of a structure (from an initial disturbed instant) until, due to artificial damping, the structure comes to rest in static equilibrium.”

Indeed, the architect’s shape is not the natural form of the project, so it is not at rest. It wants to move to its natural position: it needs to relax. Thus, the fictitious motion of a structure modeled by a discrete mesh of bars (for a gridshell, this comes from a Chebyshev lattice, it therefore represents the real physical elements of the structure), at the intersection of which are located the nodes subjected to forces, must be calculated. Indeed, according to Newton's second law $\sum \vec{F} = m * \vec{a}$, if the forces at each node do not balance, then the nodes (to which we attribute a mass, real or fictitious) experience a fictitious acceleration $\vec{a} = \sum \vec{F} / m$, and therefore move at a velocity that varies with time. This lets us calculate at each iteration, the position of each node at the next instant.

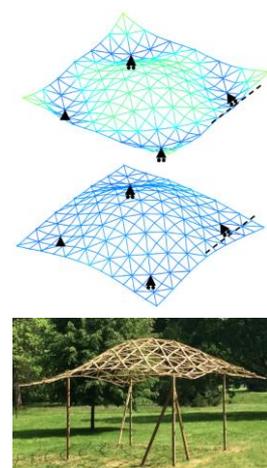


Figure 18: trial gridshell by Rombouts [13]: mesh of the architect's form, form finding after RD and built project.

In the case of a gridshell, there are (at least) three forces acting at the nodes: the nodal dead weight ($\vec{F} = m * \vec{g}$), the force induced by the bending of the elements as described by Barnes, and the Hooke force in each element, proportional to the stiffness and deformation ($\vec{F} = ES * \vec{\Delta L}$), which ensures the equilibrium of each node.

To calculate the value of the bending forces, let us start from the bending moment M which causes the bending of the elements. Since the moment is the product of the force by its lever arm at the point considered, the force field applied to the nodes is deduced from the relation $\vec{M} = \vec{OA} \wedge \vec{F}$. The algebraic value of the moment being $M = \frac{EI}{R}$ and, by definition, $R = \frac{L_{i-1,i+1}}{2 \sin(\alpha_i)}$, we obtain $M = \frac{2EI \sin(\alpha_i)}{L_{i-1,i+1}}$. We deduce that in a system composed of curved beams, each trio of consecutive nodes admits on the ends of each of the two segments formed two opposite forces of the same values, $F_{i-1,i} = \frac{2EI \sin(\alpha_i)}{L_{i-1,i} * L_{i-1,i+1}}$ for the first segment and $F_{i,i+1} = \frac{2EI \sin(\alpha_i)}{L_{i,i+1} * L_{i-1,i+1}}$ for the second.

Once the acceleration \vec{a}_t at a fictitious instant t has been calculated and knowing the initial velocities \vec{v}_{t+dt} at the same instant, we deduce nodal velocities at the following instant $t + dt$: $\vec{v}_{t+dt/2} = \vec{a}_t * dt/2 + \vec{v}_t$. In the same way, it is easily demonstrated [10] that the body moves on average at the velocity it has at average time $t + dt/2$. The displacement is then $D = \vec{v}_{t+dt/2} * dt = \vec{a}_t * \frac{dt^2}{2} + \vec{v}_t * dt = 0$.

We then obtain the positions of each node at time $t + dt$. The operation is repeated until an equilibrium of forces is reached at each node: the structure is then at its natural position. This can only be done by adding a damping in the system, which is explained below.

General theory from an energetic point of view: at the initial instant, the deviation between the initial architect's shape and the equilibrium shape being maximum, the potential energy (which we define by simplification as being related to the forces resulting from a deviation between the position at instant t and the equilibrium position) of the system is maximum, the initial nodal velocities being zero, the kinetic energy $\sum_{nodes} \frac{1}{2} * m * v^2$ of the system is as well. The relaxation then causes the nodes to move. At convergence, equilibrium position is obtained when the potential energy is zero (the forces are in equilibrium) and the kinetic energy is zero (the nodes no longer move: their velocities is zero).

In the intermediate stages, an energy transfer takes place between potential and kinetic energies. When the system passes through its equilibrium position, the forces balance and the potential energy becomes zero. But nodal velocities, which is then maximum (and therefore the kinetic energy as well), causes a continuation of the movement in the opposite direction: the system oscillates. On the other hand, when the deformation reaches a maximum, the velocities and thus the kinetic energy are zero and the potential energy regains a local maximum.

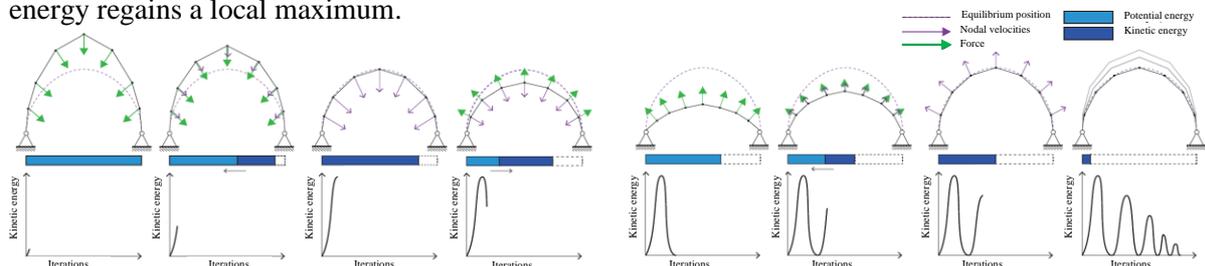


Figure 21: Energy transfer during the dynamic relaxation of a gridshell.

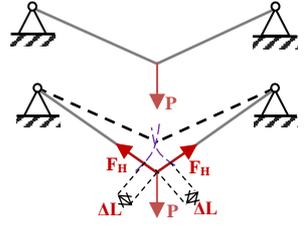


Figure 19: Equilibrium of a node by the spring forces from Hooke's law on the example of a loaded cable.

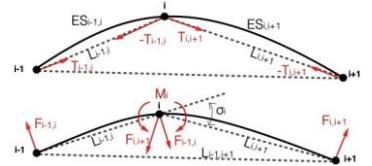


Figure 20: Spring forces from Hooke's law and forces due to bending in the case of a gridshell.

This fictitious oscillation around the equilibrium position would be endless without the addition of damping, i.e. a way to fully dissipate the total energy of the system.

Viscous damping: it is the most intuitive method, because it is based on physical notions. Indeed, in the real world, a ball rolling in a valley is slowed down by the friction of the air and ends up stopping. In the same way, the roller coaster summits must be descending. In dynamic relaxation, viscous damping is modeled by adding to each node a force proportional to a damping coefficient c and the opposite of the nodal velocities: $\vec{F}_a = -c * \vec{v}$.

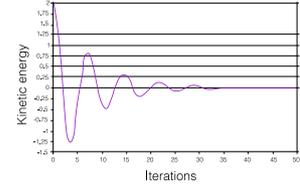


Figure 22: Kinetic energy of dynamic relaxation with viscous damping.

The coefficient c is at the choice of the user because it does not change the nature of the equilibrium, only the number of oscillations required before it. However, a too high damping coefficient may cause the system to diverge. In the general case, the damping of the system is done in a pseudo periodic regime (sufficiently low damping, therefore the convergence is slow).

Kinetic damping: More abstract, this method does not require setting new parameters and often allows for faster convergence. Barnes [2] explains that kinetic damping “is an artificial damping (...). In this procedure the undamped motion of the structure is traced and when a local peak in the total kinetic energy of the system is detected, all velocity components are set to zero. The process is then restarted from the current geometry and repeated through further (generally decreasing) peaks until the energy of all modes of vibration has been dissipated and static equilibrium is achieved.”

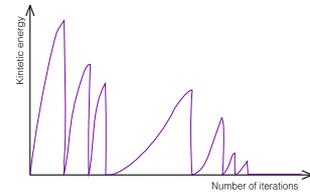


Figure 23: Kinetic energy of dynamic relaxation with kinetic damping.

3.2.4. The ELASTICA algorithm

This algorithm is the concrete application of the above, usable for any type of elastic gridshell, and available in open source on the website www.construire-l-architecture.com. We have chosen a discretization by the compass method and kinetic damping.

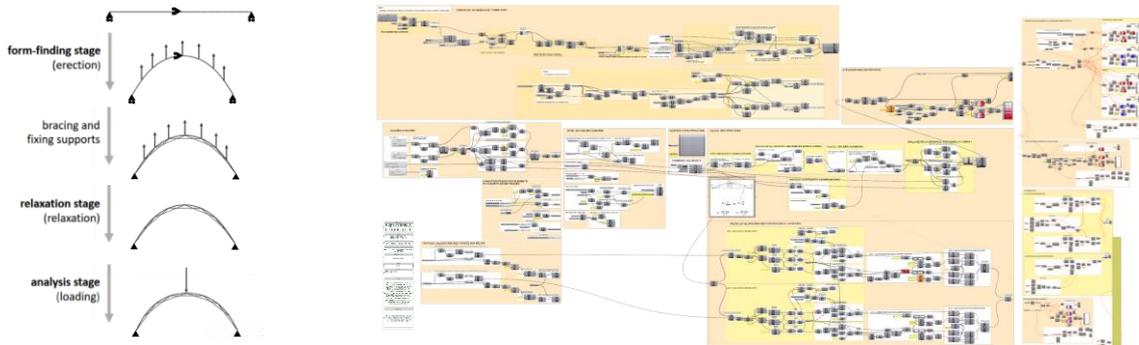


Figure 24: Chronology of form finding stages and stability control using dynamic relaxation by Rombouts [13], and extract of the ELASTICA algorithm.

Table 1: Parameters of the ELASTICA algorithm for the Elastica project.

Input data	Symbol	Value	Unit	Mechanical parameters	Symbol	Formula	Unit	
Lath width	b	0.045	m	Surface	S	bh	m ²	
Lath height	h	0.012	m			$2bh$		
Initial mesh length	L_0	0.5	m		m	$V\rho + m_a$	kg	
Mass of the connecting element	m_a	0.4	kg	Inertia	I	$bh^3/12$	m ⁴	
Timber density	ρ	500	kg/m ³			$2bh^3/12$		
Modulus of elasticity of wood	E	11 500	MPa			$b(3h)^3/12 - bh^3/12 = 26bh^3/12$		
					Axial stiffness	R_a	ES/L_0	MN/m
					Inflectional stiffness	R_f	$2EI/L_0^3$	MN/m

Calculation of the time interval dt of the iterations. A time interval too short or a nodal mass too high can lead to a divergence. Commonly, we choose a time interval dt and deduce from it the fictitious nodal masses – different from the real nodal masses of the project – able to ensure the convergence of the algorithm by the Barnes-Han-Lee formula [3]: $m = \frac{dt^2}{2} * (\sum_{\mu} R_a + \sum_{\mu} R_f)$ where μ is the number of bars connected to each node (4 without bracing and 6 with). For convenience, we have set the nodal mass m equal to the real mass and deducted dt . This simple formula does not always ensure convergence: it is advisable to divide it by a safety factor (2 has been chosen in our case after several tests).

4. Elastica project: application to a pavilion at human scale

4.1. Site and architectural design

The park of the Butte du Chapeau Rouge, developed from 1938 by Léon Azéma and then by his son Jean, is located between the Mouzaïa and America quarters, in the 19th district of Paris. Perched on a hill and overlooking the plains of Saint-Denis, it is representative of the 1930s architecture. The project has been designed after various tests comparing numerical and model studies.

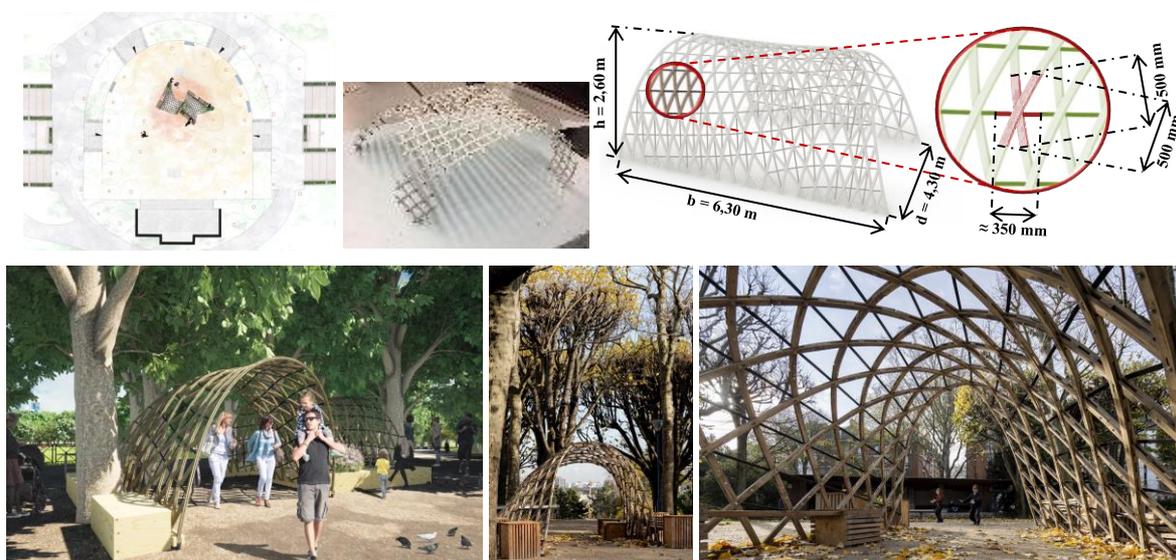


Figure 25: Masterplan, model, renderings, and photos of the Elastica project [photo Salem Mostefaoui].

4.2. Choice of the gridshell's material

Based on the Douthe method [5], we have chosen the material after a weighted multi-criteria analysis.

1) The performance criterion $\mathbf{M1} = \sigma_y / E$ is to maximize in order to have the material with the best ratio strength (high elastic limit σ_y) – flexibility (low E) in order to achieve relatively small bending radii.

2) The criterion $\mathbf{M2} = E$. Counter-intuitively, according to the above, it should be maximized because too much flexibility is harmful for a gridshell in the service phase. Indeed, stress variations under loads are not decisive for the structural analysis of the gridshell (higher inertia in service phase and load moment generally opposite to the bending moment) contrary to the phenomenon of surface buckling. The more flexible a structure is, the more likely it is to buckle. However, it is easy to stiffen the gridshell by adding bracing and shear blocks.

3) The criteria $\mathbf{M3} = K_{IC}/E$. K_{IC} [$\text{MPa}\cdot\text{m}^{1/2}$] is the tenacity of the material (its ability to resist crack propagation). During erection, stress concentrations can locally occur at the pseudo-punctual contact between the bolt and the wood at the holes, the risk of cracking is important (it is common to have to change laths during the installation).

4) The economic criterion $M4 = E^{1/2} / C_p$. The cost of the structure is $C = C_p * S * L_{tot}$ where C_p is the volume cost of the material, $S = b * h$ the lath section and L_{tot} their total length. The critical buckling force being proportional to EI/R^3 and I , in first approach, proportional to S^2 , we can write $S \propto \sqrt{\frac{F_{cr} R^3}{E}}$. Thus, maximizing the stability/cost ratio means maximizing **M4**.

Other criteria proposed by Douthe related to environmental impact are not considered in our study because of our pavilion is small, and the materials' environmental data is not easily accessible.

Table 2: Mechanical and economic material parameters and multi-criteria evaluation.

Material	Density [kg/m ³]	Elastic modulus E [Gpa]	Yield strength σ_y [MPa]	Flexural resistanc e [MPa]	Tenacity K_{rc} [MPa.m ^{1/2}]	Volume cost C_p [€/m ³]	M1	M2	M3	M4	Global Score
							Coef. 4	[MPa] Coef. 1	[MPa/m ^{3/2}] Coef. 3	[MN ^{0.5} *m ^{0.5} /€] Coef. 5	
Oak	380 - 480	13 300	40	105	0.6	1 150	3.01×10^{-3}	13 300	4.51×10^{-5}	100.28×10^{-3}	3.96/5
Pine	300 - 400	10 200	24	97	0.5	850	2.35×10^{-3}	10 200	4.90×10^{-5}	118.82×10^{-3}	3.60/5
Ash	600 - 750	12 900	40	113	0.75	1 200	3.10×10^{-3}	12 900	5.81×10^{-5}	94.65×10^{-3}	4.08/5
Black locust	670 - 770	16 900	40	126	0.9	1 500	2.37×10^{-3}	16 900	5.33×10^{-5}	86.67×10^{-3}	3.60/5
Niangon	620 - 780	14 430	42	103	1	2 000	2.91×10^{-3}	14 430	6.93×10^{-5}	60.06×10^{-3}	3.62/5
Steel	7 850	210 000	275	600	85	23 550	1.31×10^{-3}	210 000	40.48×10^{-5}	19.46×10^{-3}	2.58/4
Carbon Fiber Reinforced Plastic	1 800	90 000	2500	1200	45	175 000	27.78×10^{-3}	90 000	50.00×10^{-5}	1.71×10^{-3}	2.85/4
Glass Fiber Reinforced Plastic	2 600	26 000	3200	1000	65	14 500	123.08×10^{-3}	26 000	250.00×10^{-5}	11.12×10^{-3}	3.77/4

This shows that ash is the most suitable material for our project. We also decided to increase the maximum curvature allowed by the wood by maintaining its moisture content at the fibre saturation point, soaking it before erection in order to make it more flexible.

4.2. Verification and adjustment of details

Nodal load assumptions according to the Eurocodes:

- Dead loads: $G = 1.26 \text{ kg}$
- Live loads due to wind $W = 3.42 \text{ kg}$
- Live loads due to snow: $S = 2.15 \text{ kg}$ maximum
- Accidental loads: people with a mass of 100 kg hanging from various gridshell's nodes.

Load combinations:

- Service Limit State: for deflections' calculation. However, as the pavilion is temporary, no limit is imposed on this serviceability criterion and creep in timber is neglected.

$$1.00 G = 1.26 \text{ kg} / 1.00 G + 1.00 W + 0.60 S = 5.97 \text{ kg}$$

- Ultimate Limit State: a safety criterion used for stress and surface's buckling verifications.

$$1.35 G = 1.70 \text{ kg} / 1.35 G + 1.50 W + 1.05 S = 9.09 \text{ kg}$$

Buckling limit load of the surface: predominant ruin mode for gridshells.

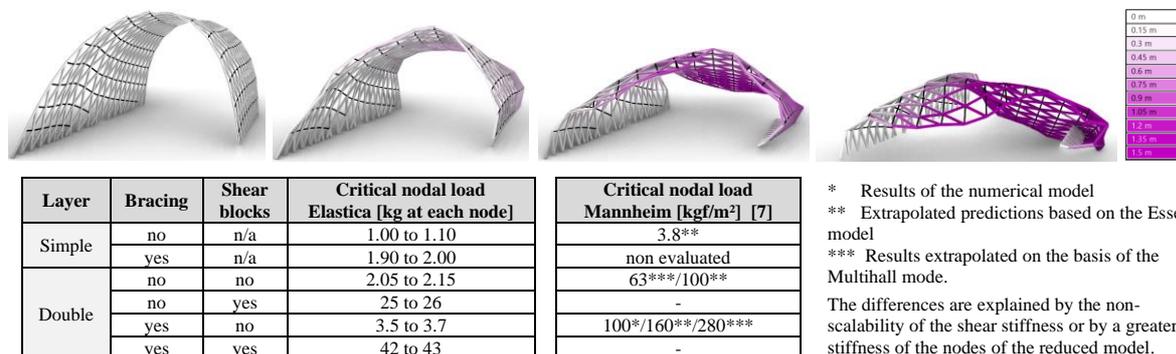


Figure 26: ELASTICA's kinematics of gridshell's global buckling when the nodal load exceeds the critical load, and critical buckling loads for Elastica and Mannheim gridshells.

According to Happold and Liddell, all other things being equal, the use of bracing on double lathing increases the critical load by a factor of between 1.60 and 4.44. Furthermore, we can predict that the addition of shear blocks will increase the critical buckling load by a factor of about 13, determined by the ratio of inertias with $(26bh^3/12)$ and without $(2bh^3/12)$ these blocks. As for the results of our modeling of the Elastica project, we can conclude that, all other things being equal, buckling resistance is increasing:

- By a factor of 1.97 to 2 by designing a double layer grid.
- By a factor of 1.67 to 1.86 adding bracing.
- By a factor of 11.03 to 12.14 adding shear blocks.

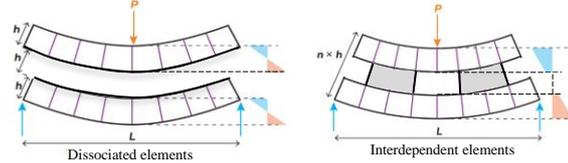


Figure 27: Principle of increasing inertia by connecting the layers with shear blocks.

These results correlate with our theoretical predictions and with Happold and Liddell's analyses.

We also wished to compare them with the formula proposed by Douthe [5] who believes that “in order to obtain an expression of the critical pressure p_{cr} that will cause the shell to collapse, it is assumed that this load is close to that which causes the instability of an equivalent cylindrical shell subjected to hydrostatic loading, i.e. of the type: $p_{cr} = 3EI/R^3$ ” (I is here the inertia per unit of length). The proximity to the results on the three designs tested confirm this theory.

Table 3: Critical buckling loads by the ELASTICA algorithm and by the formula of cylindrical shells.

Layer	Bracing	Shear blocks	Critical nodal load by modeling [kg]	Critical nodal load by Douthe's formula [kg]
Simple	no	n/a	≈ 1.05	1.02
Double	no	no	≈ 2.10	2.05
	no	yes	≈ 25.50	26.71

Before concluding, let us recall that any structure must be dimensioned with a safety factor on the results of the calculations. Happold and Liddell evaluate it by considering various parameters: variations in modulus of elasticity E (between the different laths, due to moisture and due to creep), accuracy of shape of shell, variations in loading, accuracy of computer model and assumptions, nature and significance of buckling collapse and consequences of failure. Their global safety coefficient was 2.85, and ours is 4.85. Since the variability in loading is already taken into account in the ULS combination (9.09 kg), we apply an overall safety factor of $4.85/1.40 = 3.46$ (31.45 kg) to this load. The results of our study show that the expected loads on the Elastica gridshell require a double lath design with bracing and shear blocks.

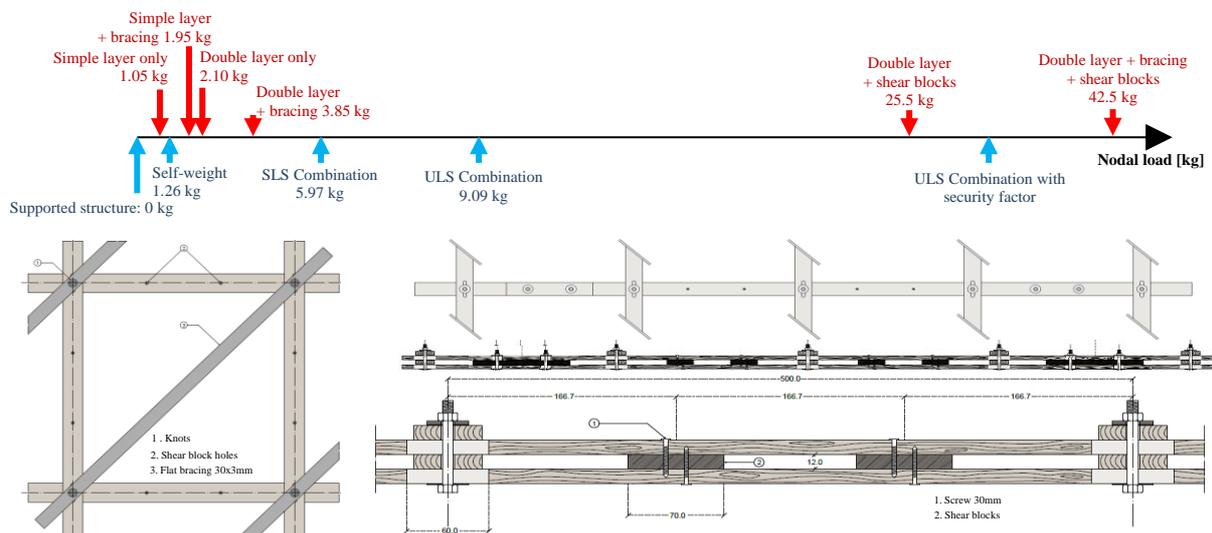


Figure 28: Synthesis of the project loads and critical buckling loads according to the different possible designs, and details of the Elastica gridshell: double layer grid, bracing and shear blocks.

Accidental loads: According to our tests, the critical position of punctual loads is in the middle of the vault on the lateral plane (top), and very close to an edge (entrance arch). 24 people, each weighing 100 kg, can hang simultaneously in this critical area.

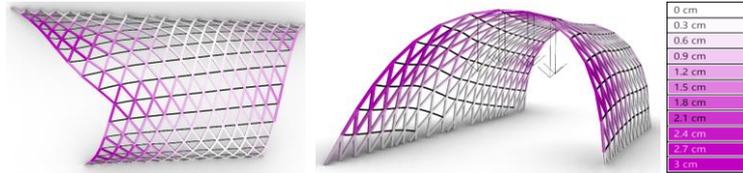


Figure 29: Gridshell deflection under the action of its self weight and of 7 persons of 100 kg suspended at 7 nodes in the critical zone.

Stress in timber elements: they are due to its bending during the assembly phase and then, in the service phase, to the loading (self-weight and variable loads). Stress is equal to $\sigma = \frac{M*y}{I}$ where y is the relative position (distance to the neutral axis). However, shear blocks are added after erection, increasing the inertia of the members from $2bh^3/12$ in the erection stage to $26bh^3/12$ in service phase. In addition, during erection stage the laths work independently (maximum stress at $y_{max} = h/2$), while in the service phase, double lathing works as a single Vierendeel truss ($y_{max} = 3h/2$). As $M = EI/R$, the maximal stress is equal to $\sigma_{max} = \frac{E*h}{2*R}$ for the part of the stress received in the erection stage and $\sigma_{max} = \frac{3E*h}{2*R}$ for the part of the stress received in the service phase. We have therefore simulated and combined three stress cartographies to obtain the resulting stresses in timber elements in service stage.

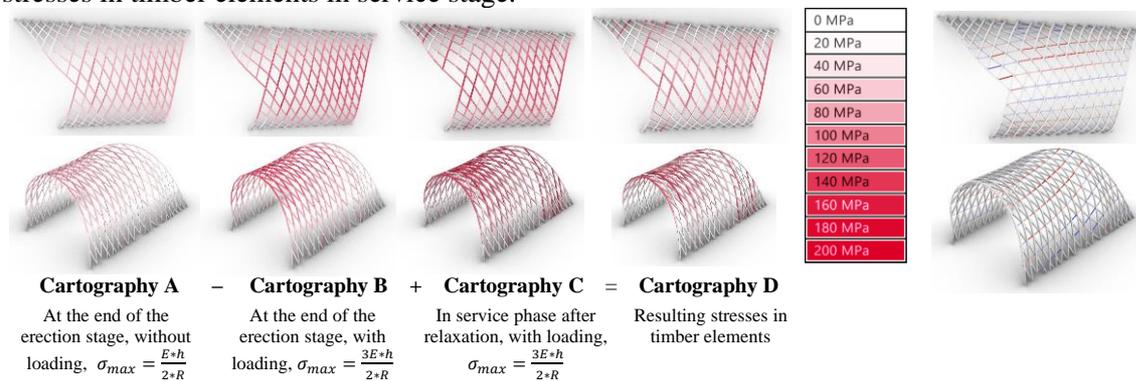


Figure 30: Stress cartography in timber elements, and forces in bracing, under SLU combination.

38% of the elements remain in the elastic domain during the erection stage and 39% in the service phase, all remaining below the breaking point (respectively 49% and 50% with timber soaking, reducing the elastic modulus E of about 27% at fibre saturation point during the erection). The local plastification of laths is allowed by the load transfer on the parts remaining in the elastic domain.

Bracing verification. Flat sections in steel S235 (width $b = 30$ mm and height $h = 3$ mm) in traction are dimensioned by the intrinsic resistance of their section $N_{max} = f_s.b.h = 2156$ kg and those in compression by their critical buckling force $N_{cr} = \pi^2 EI / l_f^2 = 89$ kg for a 40 cm long element.

Horizontal reactions: their average value at each lath end is 37.5 kg under self-weight and 261 kg at the SLU. The gridshell's base is anchored by steel rods driven 25 cm into the ground. Benches and plant containers act as buttresses. The whole system can resist a 270 kg horizontal reaction.

5. Conclusions

The outcomes achieved so far are:

- Popularization work of elastic gridshells
- Calculation and verification algorithm
- User's manual

Outcomes expected during next steps:

- User-friendly interface
- Algorithm's adaptation to other typologies like braiding (BAYA study [11]).

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