



**HAL**  
open science

# Time asymptotics of structured populations with diffusion and dynamic boundary conditions

Mustapha Mokhtar-Kharroubi, Quentin Richard

► **To cite this version:**

Mustapha Mokhtar-Kharroubi, Quentin Richard. Time asymptotics of structured populations with diffusion and dynamic boundary conditions. *Discrete and Continuous Dynamical Systems - Series B*, 2018, 23 (10), pp.4087-4116. 10.3934/dcdsb.2018127 . hal-03881400

**HAL Id: hal-03881400**

**<https://hal.science/hal-03881400>**

Submitted on 1 Dec 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1 **TIME ASYMPTOTICS OF STRUCTURED POPULATIONS WITH**  
2 **DIFFUSION AND DYNAMIC BOUNDARY CONDITIONS**

MUSTAPHA MOKHTAR-KHARROUBI

UMR 6626 Laboratoire de Mathématiques de Besançon  
Université Bourgogne Franche-Comté  
Besançon, 25000, FRANCE

QUENTIN RICHARD

UMR 6626 Laboratoire de Mathématiques de Besançon  
Université Bourgogne Franche-Comté  
Besançon, 25000, FRANCE

(Communicated by the associate editor name)

ABSTRACT. This work revisits and extends in various directions a work by J.Z. Farkas and P. Hinow (Math. Biosci. Eng., 8 (2011) 503-513) on structured populations models (with bounded sizes) with diffusion and generalized Wentzell boundary conditions. In particular, we provide first a self-contained  $L^1$  generation theory making explicit the domain of the generator. By using Hopf maximum principle, we show that the semigroup is always irreducible regardless of the reproduction function. By using weak compactness arguments, we show first a stability result of the essential type and then deduce that the semigroup has a spectral gap and consequently the asynchronous exponential growth property. Finally, we show how to extend this theory to models with arbitrary sizes and point out an open problem pertaining to this extension.

3 **1. Introduction.** Structured population models are widely discussed in the lit-  
4 erature on population dynamics (see e.g. [17, 19]). A model with size-structure  
5 appeared in a work by J.W. Sinko and W. Streifer [30] (see also [36] and the ref-  
6 erences therein). The introduction of spatial diffusion in population biology goes  
7 back to A. Kolmogorov I. Petrovskii and N. Piscunov [16] and J.G. Skellam [31]. We  
8 refer to the book by J.D. Murray [23] for a survey of reaction-diffusion equations  
9 in biology. Later, R. Waldstätter, K.P. Hadeler and G. Greiner [34] introduced  
10 diffusion in structure variable other than space. In [15], K.P. Hadeler introduced  
11 diffusion in a size-structured model where the main concern is the understanding  
12 of relevant boundary conditions for realistic models. In this context, some special  
13 cases of general Robin boundary condition were considered. Other developments  
14 for more general boundary conditions are due to J.Z. Farkas and P. Hinow [11], J.Z.  
15 Farkas and A. Calsina [6, 7] and A. Bartłomiejczyk and H. Leszczyński [3, 4].

The goal of the present work is to provide a systematic spectral analysis of the diffusive and linear structured population model considered by J.Z. Farkas and P.

---

2010 *Mathematics Subject Classification.* 47D06, 92D25, 35B40, 35B50.

*Key words and phrases.* Structured populations, Wentzell boundary condition, diffusion, Hopf's maximum principle, weak compactness, essential type, asynchronous exponential growth.

Hinow [11]

$$u_t(s, t) + (\gamma(s)u(s, t))_s = (d(s)u_s(s, t))_s - \mu(s)u(s, t) + \int_0^m \beta(s, y)u(y, t)dy, \quad (1)$$

with generalized Wentzell-Robin (or dynamic) boundary conditions

$$[(d(s)u_s(s, t))_s]_{s=0} - b_0u_s(0, t) + c_0u(0, t) = 0, \quad (2)$$

$$[(d(s)u_s(s, t))_s]_{s=m} + b_mu_s(m, t) + c_mu(m, t) = 0, \quad (3)$$

1 and

$$b_0 - \gamma(0) > 0, \quad b_m + \gamma(m) > 0, \quad (4)$$

2 The different parameters will be defined thereafter. We note that there exists an  
3 important literature on second order equations with Wentzell boundary conditions  
4 which goes back to W. Feller [13] and A.D. Wentzell [38] (see e.g. A. Favini G.R.  
5 Goldstein J.A. Goldstein and S. Romanelli [12] and the references therein). We refer  
6 to [4] and to the book by A. Bobrowski [5] Chapter 3 for a biological interpretation  
7 of such boundary conditions.

8 Here  $u(s, t)$  denotes the density of individuals of size  $s \in [0, m]$  at time  $t \geq 0$ .  
9 The function  $d$  stands for the size-specific diffusion coefficient while  $\mu, \gamma$  denote  
10 respectively the mortality and growth rate of the individuals. Furthermore the  
11 non-local integral term in (1) represents the recruitment of individuals into the  
12 population. More precisely,  $\beta(s, y)$  is the rate at which individuals of size  $y$  produce  
13 individuals of size  $s$ .

14 *The object of this work is to improve and extend [11] in various directions.*

15 In [11] the authors write (1)-(2)-(3) in the matrix form

$$\begin{cases} U'(t) &= \mathcal{A}U(t), \\ U(0) &= (u^0, u_0^0, u_m^0) \in \mathcal{X}, \end{cases} \quad (5)$$

where

$$\begin{aligned} \mathcal{A} \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} &= A \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} + K \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} \\ &= \begin{pmatrix} (du')' - (\gamma u)' - \mu u \\ (b_0 - \gamma(0))u'(0) - \rho_0 u_0 \\ -(b_m + \gamma(m))u'(m) - \rho_m u_m \end{pmatrix} + \begin{pmatrix} \int_0^m \beta(\cdot, y)u(y)dy \\ \int_0^m \beta_0(y)u(y)dy \\ \int_0^m \beta_m(y)u(y)dy \end{pmatrix}, \end{aligned}$$

and show their well-posedness in the sense of semigroup theory in the space

$$\mathcal{X} = (L^1(0, m) \times \mathbb{R}^2, \|\cdot\|_{\mathcal{X}})$$

endowed with the norm

$$\|(x, x_0, x_m)\|_{\mathcal{X}} = \|x\|_{L^1(0, m)} + c_1|x_0| + c_2|x_m|$$

where

$$c_1 = \frac{d(0)}{b_0 - \gamma(0)}, \quad c_2 = \frac{d(m)}{b_m + \gamma(m)}.$$

Actually, to deal with well-posedness of the Cauchy problem, the term

$$K \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} := \begin{pmatrix} \int_0^m \beta(\cdot, y)u(y)dy \\ \int_0^m \beta_0(y)u(y)dy \\ \int_0^m \beta_m(y)u(y)dy \end{pmatrix}$$

1 can be ignored since it can be treated by elementary (bounded) perturbation argu-  
 2 ments. In [11], the authors define first  $A$  on *smooth* functions

$$A_s : D(A_s) \rightarrow \mathcal{X} \quad (6)$$

where

$$D(A_s) = \{(u, u_0, u_m) \in C^2 [0, m] \times \mathbb{R}^2 : u(0) = u_0, u(m) = u_m\}$$

and show the *dissipativity* of  $A_s$ . Then they refer to  $C^\alpha$ -theory of elliptic equations ([14] Theorem 6.31) for the proof that the *closure* of  $A_s$  denoted by  $A$ , is a generator. A priori such an argument gives *no* information on the domain of  $A$  apart from the fact that

$$D(A) \supset D(A_s).$$

3 The authors claim that the generator  $A$  is resolvent compact because the embedding  
 4 of  $W^{1,1}[0, m]$  into  $L^1(0, m)$  is compact but they do *not* prove that the domain of  
 5  $A$  is embedded in  $W^{1,1}[0, m]$ . Thus, there is a priori a *gap* in their proof that  $A$  is  
 6 resolvent compact.

Here we define  $A$  on an *explicit domain*

$$D(A) = \{(u, u_0, u_m) \in W^{2,1}(0, m) \times \mathbb{R}^2 : u(0) = u_0, u(m) = u_m\}$$

7 where  $W^{2,1}(0, m)$  is the usual Sobolev space of functions in  $L^1(0, m)$  having the first  
 8 two distributional derivatives in  $L^1(0, m)$ . Indeed, besides dissipativity arguments  
 9 following [11], we show here directly that the operator is closed, densely defined  
 10 and satisfies the rank condition. Thus, a *self-contained generation theory with an*  
 11 *explicit generator* is given. (In particular, the knowledge of  $D(A)$  allows to assert  
 12 that  $A$  is resolvent compact.) This is the *first* contribution of this work.

In [11], the authors show that  $(e^{tA})_{t \geq 0}$  is irreducible under the assumption that  $\beta$  is continuous on  $[0, m]^2$  and

$$\beta(\cdot, \cdot) > 0.$$

We show here that this strict positivity assumption is *unnecessary*. Indeed,

$$e^{tA} \geq e^{tA}$$

13 and we show that  $(e^{tA})_{t \geq 0}$  is irreducible by using Hopf's maximum principle. In  
 14 particular,  $(e^{tA})_{t \geq 0}$  is irreducible even if  $\beta = 0$ . This is our *second* contribution.

15 We show the *existence* of an algebraically simple leading real eigenvalue of  $\mathcal{A}$ .  
 16 This is our *third* contribution.

We deal also with a much more important issue. Indeed, in [11] the authors "deduce" from the fact that  $\mathcal{A}$  is resolvent compact and  $(e^{tA})_{t \geq 0}$  is irreducible that  $(e^{tA})_{t \geq 0}$  converges (in operator norm) exponentially to the spectral projection  $P$  associated to the leading eigenvalue  $\hat{\lambda}$  of  $\mathcal{A}$

$$e^{-t\hat{\lambda}} e^{tA} \rightarrow P \quad (t \rightarrow \infty).$$

A priori, such a proof is *not* complete. Indeed, such a conclusion can be reached only if we know that the semigroup  $(e^{tA})_{t \geq 0}$  has a *spectral gap* (i.e. its essential type is *strictly* less than its type) which is *not* at all a consequence of the resolvent compactness of  $\mathcal{A}$  and the irreducibility of  $e^{tA}$ . In fact, we need to study the spectrum of the semigroup  $(e^{tA})_{t \geq 0}$  *itself*. We can show this property by using tools developed in the context of Transport theory [20, 22]. Indeed, by using weak

compactness arguments (we assume that  $K$  is weakly compact), we show first that the semigroups  $(e^{tA})_{t \geq 0}$  and  $(e^{t\mathcal{A}})_{t \geq 0}$  have the same essential type

$$\omega_{ess}((e^{tA})_{t \geq 0}) = \omega_{ess}((e^{t\mathcal{A}})_{t \geq 0});$$

(the weak compactness of  $K$  is insured e.g. if there exists  $\tilde{\beta} \in L^1(0, m)$  such that

$$\beta(s, y) \leq \tilde{\beta}(s);$$

in particular, it is trivially satisfied if  $\beta$  is continuous on  $[0, m]^2$ ). It follows that the essential type of  $(e^{tA})_{t \geq 0}$  is less than or equal to the spectral bound of  $A$

$$\omega_{ess}((e^{tA})_{t \geq 0}) \leq s(A) := \sup \{\Re(\lambda); \lambda \in \sigma(A)\}.$$

Secondly, by exploiting the fact that  $A$  is resolvent compact and Marek's results [18], we show that the spectral bound of  $A$  is strictly less than that of  $\mathcal{A}$

$$s(A) < s(\mathcal{A}) := \sup \{\Re(\lambda); \lambda \in \sigma(\mathcal{A})\}$$

once

$$K \neq 0$$

i.e. once  $\beta(., .)$  is *not equal to zero* almost everywhere. This implies that  $(e^{t\mathcal{A}})_{t \geq 0}$  exhibits a spectral gap

$$\omega_{ess}((e^{t\mathcal{A}})_{t \geq 0}) < \omega((e^{t\mathcal{A}})_{t \geq 0})$$

where  $\omega((e^{t\mathcal{A}})_{t \geq 0})$  is the type of  $(e^{t\mathcal{A}})_{t \geq 0}$  or equivalently the spectral bound of its generator  $\mathcal{A}$ , i.e.

$$\omega_{ess}((e^{t\mathcal{A}})_{t \geq 0}) < s(\mathcal{A})$$

(the type of a positive semigroup in  $L^p$  spaces coincides with the spectral bound of its generator [10] and is an element of the spectrum [20]). The fact that

$$e^{-ts(A)} e^{t\mathcal{A}} \rightarrow P \quad (t \rightarrow \infty)$$

- 1 exponentially is then just a consequence of standard functional analytic results (see  
2 e.g. [35] Proposition 2.3). This is our *fourth* (key) contribution.

A *fifth* contribution is the generalization of this theory to the case

$$m = \infty$$

allowing *arbitrary* sizes, i.e. we study also the model

$$u_t(s, t) + (\gamma(s)u(s, t))_s = (d(s)u_s(s, t))_s - \mu(s)u(s, t) + \int_0^\infty \beta(s, y)u(y, t)dy, \quad (7)$$

$$[(d(s)u_s(s, t))_s]_{s=0} - b_0u_s(0, t) + c_0u(0, t) = 0. \quad (8)$$

To our knowledge, the spectral analysis of this model appears here for the first time. The generation theory in

$$\mathcal{X} = (L^1(0, +\infty) \times \mathbb{R}, \|\cdot\|_{\mathcal{X}})$$

turns out to be much more involved. Indeed, the domain of the generator turns out to be much more tricky since it consists of those  $(u, u_0) \in L^1(\mathbb{R}_+) \times \mathbb{R}$  such that

$$u \in W^{2,1}(0, c) \quad \forall c > 0, \quad u(0) = u_0$$

$$(du')' - (\gamma u)' \in L^1(\mathbb{R}_+) \quad \text{and} \quad \lim_{s \rightarrow +\infty} d(s)u'(s) - \gamma(s)u(s) = 0.$$

A priori the domain of the generator is larger than the space

$$\{(u, u_0) \in W^{2,1}(\mathbb{R}_+) \times \mathbb{R}; u(0) = u_0\}$$

1 but we show that this space is a *core* of the domain generator.

As previously, the irreducibility of the semigroup is shown by using Hopf's maximum principle. Similarly, if

$$L^1(\mathbb{R}_+) \ni u \rightarrow \int_0^\infty \beta(\cdot, y)u(y)dy \in L^1(\mathbb{R}_+)$$

is weakly compact, (e.g. if there exists  $\tilde{\beta} \in L^1(0, \infty)$  such that

$$\beta(s, y) \leq \tilde{\beta}(s) \text{ ),}$$

then the semigroups  $(e^{tA})_{t \geq 0}$  and  $(e^{t\mathcal{A}})_{t \geq 0}$  have the same essential type. On the other hand, we *cannot* appeal to Marek's arguments [18] to infer the existence of a spectral gap because  $\mathcal{A}$  is *not* a priori resolvent compact. In this case, we show that the spectral gap property

$$\omega_{ess}((e^{tA})_{t \geq 0}) < s(\mathcal{A})$$

holds if  $\beta_0(\cdot) \neq 0$ , if there exists a measurable set  $I \subset \mathbb{R}_+$  with positive measure such that

$$u \in L^1(\mathbb{R}_+), u(y) > 0 \text{ a.e.} \implies \int_0^\infty \beta(s, y)u(y)dy > 0 \text{ a.e. } s \in I.$$

2 and if

$$\lim_{\lambda \rightarrow s(A)} r_\sigma(K(\lambda - A)^{-1}) > 1 \tag{9}$$

3 where  $r_\sigma$  refers to a spectral radius. We do not know whether (9) is always satisfied.

4 In particular, we do not know whether

$$\lim_{\lambda \rightarrow s(A)} r_\sigma(K(\lambda - A)^{-1}) = +\infty \tag{10}$$

always holds. Note that if

$$\eta := \lim_{\lambda \rightarrow s(A)} r_\sigma(K(\lambda - A)^{-1}) < +\infty$$

then the semigroup generated by

$$A + cK$$

has a spectral gap once

$$c > \eta^{-1}.$$

5 If  $\beta$  is bounded below by a separable kernel

$$\beta(x, y) \geq \beta_1(x)\beta_2(y) \tag{11}$$

then we show that

$$r_\sigma(K(\lambda - A)^{-1}) \geq \left\| \beta_2 \left( (\lambda - A)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right) \right\|_1 \Big|_{L^1(\mathbb{R}_+)}$$

where  $(U)_1$  refers to the first component of  $U \in \mathcal{X}$ . In particular (9) is satisfied if

$$\lim_{\lambda \rightarrow s(A)} \left\| \beta_2 \left( (\lambda - A)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right) \right\|_1 \Big|_{L^1(\mathbb{R}_+)} > 1.$$

6 Note that (11) holds e.g. if  $\beta$  is continuous at some point  $(\bar{x}, \bar{y})$  with  $\beta(\bar{x}, \bar{y}) > 0$ .

7 Whether (10) is a general property of such biological models is an open problem.

8 The authors are indebted to the referees for their constructive remarks and sugges-  
9 tions.

10 **2. Models with bounded sizes.**

1 **2.1. Framework and hypotheses.**

2 In order to analyze the problem described by (1)-(2)-(3), following [11] we rewrite  
 3 the boundary conditions (2)-(3). We substitute the diffusion term in (2)-(3), by  
 4 the remainder of (1) evaluated in 0 and  $m$  respectively. We thus get the following  
 5 dynamic equations

$$u_t(0, t) = -u(0, t)\rho_0 + u_s(0, t)(b_0 - \gamma(0)) + \int_0^m \beta_0(y)u(y, t)dy, \quad (12)$$

$$u_t(m, t) = -u(m, t)\rho_m - u_s(m, t)(b_m + \gamma(m)) + \int_0^m \beta_m(y)u(y, t)dy, \quad (13)$$

where

$$\rho_0 = \gamma'(0) + \mu(0) + c_0,$$

$$\rho_m = \gamma'(m) + \mu(m) + c_m$$

and

$$\beta_0 = \beta(0, \cdot), \quad \beta_m = \beta(m, \cdot).$$

Following [11], the Banach space

$$\mathcal{X} = (L^1(0, m) \times \mathbb{R}^2, \|\cdot\|_{\mathcal{X}})$$

is endowed with the norm

$$\|(x, x_0, x_m)\|_{\mathcal{X}} = \|x\|_{L^1(0, m)} + c_1|x_0| + c_2|x_m|,$$

where

$$c_1 = \frac{d(0)}{b_0 - \gamma(0)}, \quad c_2 = \frac{d(m)}{b_m + \gamma(m)}.$$

6 We denote by  $\mathcal{X}_+$  the nonnegative cone of  $\mathcal{X}$ . We introduce some hypotheses on  
 7 the different parameters:

- 8 1.  $\gamma, d \in W^{1, \infty}(0, m)$  and  $\mu, \beta_0, \beta_m \in L^\infty(0, m)$ ,
- 9 2. the functions  $\mu, \gamma'$  and  $s \mapsto \beta(s, y)$  are continuous at  $s = 0$  and  $s = m$  for  
 10 every  $y \in [0, m]$ ,
3. the operator

$$L^1(0, m) \ni u \rightarrow \int_0^m \beta(\cdot, y)u(y)dy \in L^1(0, m)$$

11 is weakly compact,

- 12 4.  $b_0, b_m > 0$ ,  $c_0, c_m \geq 0$ ,  $\beta, \mu \geq 0$  and  $d(s) \geq d_0 > 0$  for all  $s \in [0, m]$ .

**Remark 1.** According to the general criterion of weak compactness (see e.g. Section 4 in [37]), the third hypothesis amounts to

$$\sup_{y \in [0, m]} \int_0^m \beta(s, y)ds < \infty \text{ and } \lim_{|E| \rightarrow 0} \sup_{y \in [0, m]} \int_E \beta(s, y)ds = 0$$

13 and is satisfied as soon as there exists  $\tilde{\beta} \in L^1(0, m)$  such that  $\beta(s, y) \leq \tilde{\beta}(s)$  a.e.  
 14  $(s, y) \in [0, m]^2$ . This is the case for example if  $\beta$  is continuous on  $[0, m]^2$ .

Using (1)-(12)-(13), we define the operator  $\mathcal{A}$  by:

$$\begin{aligned} \mathcal{A} \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} &= A \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} + K \begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} \\ &= \begin{pmatrix} (du)' - (\gamma u)' - \mu u \\ (b_0 - \gamma(0))u'(0) - \rho_0 u_0 \\ -(b_m + \gamma(m))u'(m) - \rho_m u_m \end{pmatrix} + \begin{pmatrix} \int_0^m \beta(\cdot, y)u(y)dy \\ \int_0^m \beta_0(y)u(y)dy \\ \int_0^m \beta_m(y)u(y)dy \end{pmatrix}, \end{aligned}$$

where the domain of  $\mathcal{A}$  is given by

$$D(\mathcal{A}) = \{(u, u_0, u_m) \in W^{2,1}(0, m) \times \mathbb{R}^2 : u(0) = u_0, u(m) = u_m\}.$$

We are then concerned with the following Cauchy problem

$$\begin{cases} U'(t) &= \mathcal{A}U(t), \\ U(0) &= (u^0, u_0^0, u_m^0) \in \mathcal{X} \end{cases}$$

where

$$U(t) = (u(t), u_0(t), u_m(t))^T.$$

## 2.2. Semigroup generation.

We show here that  $\mathcal{A}$  is the generator of a  $C_0$ -semigroup. The dissipativity arguments are essentially those in [11] but we prove directly that  $\mathcal{A}$  is closed, densely defined and satisfies the rank condition.

**Theorem 2.1.** *Let Assumption (4) be satisfied. Then  $\mathcal{A}$  is the infinitesimal generator of a quasi-contractive  $C_0$ -semigroup  $\{U(t)\}_{t \geq 0}$  on  $\mathcal{X}$ .*

*Proof.* We may restrict ourselves to the operator  $A$ ; straightforward (bounded) perturbation arguments will end the proof.

- Let us show that  $\overline{D(A)} = \mathcal{X}$ . Let  $(u, u_0, u_m)^T \in \mathcal{X}$ . Let  $(u^j)_j$  be  $C^\infty$  functions with compact supports such that  $u^j \rightarrow u$  in  $L^1(0, m)$  and

$$\text{support } (u^j) \subset [j^{-1}, m - j^{-1}]$$

We look for a parabola

$$f_0^j(s) = as^2 + bs + c \quad (s \in [0, j^{-1}])$$

such that

$$f_0^j(0) = u_0, \quad f_0^j(j^{-1}) = 0, \quad \frac{d}{ds} f_0^j(j^{-1}) = 0.$$

This amounts to  $c = u_0$  and

$$\begin{aligned} aj^{-2} + bj^{-1} + u_0 &= 0 \\ 2aj^{-1} + b &= 0. \end{aligned}$$

We find

$$f_0^j(s) = j^2 u_0 s^2 - 2j u_0 s + u_0 = u_0 (js - 1)^2.$$

Similarly, we look for a parabola

$$f_m^j(s) = as^2 + bs + c \quad (s \in [m - j^{-1}, m])$$

such that

$$f_m^j(m) = u_m, \quad f_m^j(m - j^{-1}) = 0, \quad \frac{d}{ds} f_m^j(m - j^{-1}) = 0.$$



We find

$$f_m^j(s) = u_m j^2 s^2 - 2u_m j^2 s(m - j^{-1}) + u_m j^2 (m - j^{-1})^2 = u_m j^2 (s - m + j^{-1})^2.$$

Define

$$v^j(s) = \begin{cases} f_0^j(s) & \text{if } s \in [0, j^{-1}] \\ u^j(s) & \text{if } s \in [j^{-1}, m - j^{-1}] \\ f_m^j(s) & \text{if } s \in [m - j^{-1}, m]. \end{cases}$$

Then  $v^j \in W^{2,1}(0, m)$ ,  $v^j(0) = u_0$  and  $v^j(m) = u_m$ , i.e.

$$(v^j, v^j(0), v^j(m))^T \in D(A).$$

Let us show that  $v^j \rightarrow u$  in  $L^1(0, m)$ . It suffices to show that

$$\int_0^{j^{-1}} |f_0^j(s)| ds + \int_{m-j^{-1}}^m |f_m^j(s)| ds \rightarrow 0 \quad (j \rightarrow +\infty).$$

1 We have

$$\begin{aligned} \int_0^{j^{-1}} |f_0^j(s)| ds &= |u_0| \int_0^{j^{-1}} (js - 1)^2 ds \\ &= j^2 |u_0| \int_0^{j^{-1}} (s - j^{-1})^2 ds \\ &= \frac{|u_0|}{3j} \rightarrow 0 \quad (j \rightarrow +\infty). \end{aligned}$$

Similarly

$$\int_{m-j^{-1}}^m |f_m^j(s)| ds = \frac{|u_m|}{3j} \rightarrow 0 \quad (j \rightarrow +\infty).$$

Finally

$$(v^j, v^j(0), v^j(m))^T \rightarrow (u, u_0, u_m)^T \text{ in } \mathcal{X}$$

2 and  $\overline{D(A)} = \mathcal{X}$ .

2. Let us show that for  $\omega$  large enough  $A - \omega$  is a dissipative operator. Let  $\lambda > 0$ ,  $U = (u, u_0, u_m)^T \in D(A)$  and  $H = ((\lambda + \omega)I - A)U$ .

Let  $H = (h, h_0, h_m)^T$ . We have to prove that

$$\|H\|_{\mathcal{X}} \geq \lambda \|U\|_{\mathcal{X}}.$$

By definition of  $H$ , we have

$$(\lambda + \widehat{\mu}(s))u(s) + (\gamma u)'(s) - (du')'(s) = h(s), \quad s \in (0, m), \quad (14)$$

$$(\lambda + \widehat{\rho}_0)u_0 - (b_0 - \gamma(0))u'(0) = h_0, \quad (15)$$

$$(\lambda + \widehat{\rho}_m)u_m + (b_m + \gamma(m))u'(m) = h_m \quad (16)$$

where

$$\widehat{\mu}(s) := \omega + \mu(s), \quad \widehat{\rho}_0 := \omega + \rho_0, \quad \widehat{\rho}_m := \omega + \rho_m.$$

We multiply (14) by  $\text{sign}(u(s))$ , integrate between 0 and  $m$  and then multiply (15) and (16) respectively by  $\text{sign}(u_0)$  and  $\text{sign}(u_m)$ . We get

$$\lambda \|u\|_{L^1} + \int_0^m \widehat{\mu}|u| - \int_0^m (du')' \text{sign}(u) + \int_0^m (\gamma u)' \text{sign}(u) = \int_0^m h \text{sign}(u),$$

$$(\lambda + \widehat{\rho}_0)|u_0| - (b_0 - \gamma(0))u'(0) \text{sign}(u(0)) = h_0 \text{sign}(u(0)),$$

$$(\lambda + \widehat{\rho}_m)|u_m| + (b_m + \gamma(m))u'(m) \text{sign}(u(m)) = h_m \text{sign}(u(m))$$

which is equivalent to

$$\lambda \|u\|_{L^1} + \int_0^m \widehat{\mu}|u| - \int_0^m (du')' \text{sign}(u) + \int_0^m (\gamma u)' \text{sign}(u) = \int_0^m h \text{sign}(u), \quad (17)$$

$$u'(0) \text{sign}(u(0)) = \frac{(\lambda + \widehat{\rho}_0)|u_0|}{b_0 - \gamma(0)} - \frac{h_0 \text{sign}(u(0))}{b_0 - \gamma(0)}, \quad (18)$$

$$u'(m) \text{sign}(u(m)) = -\frac{(\lambda + \widehat{\rho}_m)|u_m|}{b_m + \gamma(m)} + \frac{h_m \text{sign}(u(m))}{b_m + \gamma(m)}. \quad (19)$$

(a) Any nonempty open set of the real line is a finite or countable union of *disjoints* open intervals (see [2] Theorem 3.11, p. 51) so

$$\begin{aligned} \{u > 0\} &= \{s \in (0, m) : u(s) > 0\} = \bigcup_{i \in \mathbb{N}} (a_{i,1}, a_{i,2}), \\ \{u < 0\} &= \{s \in (0, m) : u(s) < 0\} = \bigcup_{i \in \mathbb{N}} (b_{i,1}, b_{i,2}). \end{aligned}$$

1 Since  $u \in W^{1,1}(0, m) \hookrightarrow C([0, m])$  then  $\forall i, j \in \mathbb{N} : u(a_{i,1}) = 0, u(a_{i,2}) =$   
2  $0, u(b_{j,1}) = 0$  and  $u(b_{j,2}) = 0$  (except possibly at 0 and  $m$ ). Thus

$$\begin{aligned} &\int_0^m (\gamma u)' \text{sign}(u) = \int_{\{u>0\}} (\gamma u)' - \int_{\{u<0\}} (\gamma u)' \\ &= \sum_{i \in \mathbb{N}} [\gamma(a_{i,2})u(a_{i,2}) - \gamma(a_{i,1})u(a_{i,1})] - \sum_{j \in \mathbb{N}} [\gamma(b_{j,2})u(b_{j,2}) - \gamma(b_{j,1})u(b_{j,1})] \\ &= \gamma(m) |u(m)| - \gamma(0) |u(0)|. \end{aligned} \quad (20)$$

3 (b) Consider  $\int_0^m (du')'(s) \text{sign}(u(s)) ds$ . Since  $u' \in W^{1,1}(0, m) \hookrightarrow C([0, m])$   
4 we have  $\forall i, j \in \mathbb{N} : u'(a_{i,2}) \leq 0, u'(a_{i,1}) \geq 0, u'(b_{j,2}) \geq 0$  and  $u'(b_{j,1}) \leq 0$   
5 (except possibly at 0 and  $m$ ). We have

$$\begin{aligned} &\int_0^m (du')' \text{sign}(u) = \int_{\{u>0\}} (du')' - \int_{\{u<0\}} (du')' \\ &= \sum_{i \in \mathbb{N}} [d(a_{i,2})u'(a_{i,2}) - d(a_{i,1})u'(a_{i,1})] - \sum_{j \in \mathbb{N}} [d(b_{j,2})u'(b_{j,2}) - d(b_{j,1})u'(b_{j,1})] \\ &\leq d(m)u'(m) \text{sign}(u(m)) - d(0)u'(0) \text{sign}(u(0)). \end{aligned}$$

6 Hence

$$\begin{aligned} &\lambda \|u\|_{L^1} + \int \widehat{\mu}|u| + \gamma(m)|u(m)| - \gamma(0)|u(0)| \\ &\leq d(m)u'(m) \text{sign}(u(m)) - d(0)u'(0) \text{sign}(u(0)) + \int h \text{sign}(u). \end{aligned}$$

Since

$$d(0)u'(0) \text{sign}(u(0)) = \frac{d(0)(\lambda + \widehat{\rho}_0)|u_0|}{b_0 - \gamma(0)} - \frac{d(0)h_0 \text{sign}(u(0))}{b_0 - \gamma(0)}$$

and

$$d(m)u'(m) \text{sign}(u(m)) = -\frac{d(m)(\lambda + \widehat{\rho}_m)|u_m|}{b_m + \gamma(m)} + \frac{d(m)h_m \text{sign}(u(m))}{b_m + \gamma(m)}$$

then

$$\begin{aligned} & \lambda \|u\|_{L^1} + \left[ \gamma(m) + \frac{d(m)(\lambda + \widehat{\rho}_m)}{b_m + \gamma(m)} \right] |u(m)| + \left[ -\gamma(0) + \frac{d(0)(\lambda + \widehat{\rho}_0)}{b_0 - \gamma(0)} \right] |u(0)| \\ & + \int \widehat{\mu}|u| \\ & \leq \frac{d(m)h_m \text{sign}(u(m))}{b_m + \gamma(m)} + \frac{d(0)h_0 \text{sign}(u(0))}{b_0 - \gamma(0)} + \int h \text{sign}(u) \\ & \leq \frac{d(m)|h_m|}{b_m + \gamma(m)} + \frac{d(0)|h_0|}{b_0 - \gamma(0)} + \|h\|_{L^1} \end{aligned}$$

or

$$\begin{aligned} & \lambda \|u\|_{L^1} + \int \widehat{\mu}|u| + \left[ \frac{\gamma(m)}{c_2} + (\lambda + \widehat{\rho}_m) \right] c_2 |u(m)| + \left[ -\frac{\gamma(0)}{c_1} + (\lambda + \widehat{\rho}_0) \right] c_1 |u(0)| \\ & \leq c_2 |h_m| + c_1 |h_0| + \|h\|_{L^1}. \end{aligned}$$

Note that if

$$\frac{\gamma(m)}{c_2} + \widehat{\rho}_m \geq 0 \quad \text{and} \quad -\frac{\gamma(0)}{c_1} + \widehat{\rho}_0 \geq 0$$

then

$$\lambda \|u\|_{L^1} + \int \widehat{\mu}|u| + \lambda c_2 |u(m)| + \lambda c_1 |u(0)| \leq c_2 |h_m| + c_1 |h_0| + \|h\|_{L^1}.$$

But

$$\frac{\gamma(m)}{c_2} + \widehat{\rho}_m = \frac{\gamma(m)(b_m + \gamma(m))}{d(m)} + \gamma'(m) + \mu(m) + c_m + \omega$$

and

$$-\frac{\gamma(0)}{c_1} + \widehat{\rho}_0 = -\frac{\gamma(0)(b_0 - \gamma(0))}{d(0)} + \gamma'(0) + \mu(0) + c_0 + \omega$$

are *nonnegative* for  $\omega$  large enough. Hence

$$\lambda \|u\|_{L^1} + \int (\mu + \omega)|u| + \lambda c_2 |u(m)| + \lambda c_1 |u(0)| \leq c_2 |h_m| + c_1 |h_0| + \|h\|_{L^1}$$

and

$$\lambda \|U\|_{\mathcal{X}} \leq \|H\|_{\mathcal{X}}$$

1 for  $\omega$  large enough..This ends the proof of the dissipativity of  $A - \omega$ .

3. Let us prove that  $(A, D(A))$  is a *closed* operator.

Let  $(U^n)_{n \in \mathbb{N}} := (u^n, u_0^n, u_m^n)_{n \in \mathbb{N}} \subset D(A)$  and let  $U := (u, u_0, u_m) \in \mathcal{X}$  and  $G := (g, g_0, g_m) \in \mathcal{X}$  such that  $\lim_{n \rightarrow \infty} \|U^n - U\|_{\mathcal{X}} = 0$  and  $\lim_{n \rightarrow \infty} \|AU^n - G\|_{\mathcal{X}} = 0$ . Note that

$$u^n(0) = u_0^n \rightarrow u_0 \quad \text{and} \quad u^n(m) = u_m^n \rightarrow u_m.$$

Since

$$(b_0 - \gamma(0))(u^n)'(0) - \rho_0 u^n(0) \rightarrow g_0$$

then

$$(u^n)'(0) \rightarrow h_0 := \frac{g_0 + \rho_0 u_0}{b_0 - \gamma(0)}.$$

Similarly

$$-(b_m + \gamma(m))(u^n)'(m) - \rho_m u^n(m) \rightarrow g_m$$

and

$$(u^n)'(m) \rightarrow h_m := -\frac{g_m + \rho_m u_m}{b_m + \gamma(m)}.$$

Let

$$f_n := d(u^n)' - \gamma u^n.$$

Since

$$(d(u^n)')' - (\gamma u^n)' - \mu u^n \rightarrow g$$

then

$$f_n' \rightarrow g + \mu u$$

( $L^1$  convergence) while

$$f_n(0) = d(0)(u^n)'(0) - \gamma(0)u^n(0) \rightarrow d(0)h_0 - \gamma(0)u_0$$

so

$$f_n(x) = f_n(0) + \int_0^x f_n'(s)ds \rightarrow z(x) := d(0)h_0 - \gamma(0)u_0 + \int_0^x (g + \mu u)(s)ds$$

( $L^1$  convergence). It follows that

$$(u^n)' \rightarrow \frac{z + \gamma u}{d}$$

( $L^1$  convergence) so  $u \in W^{1,1}(0, m)$  and  $u^n \rightarrow u$  in  $W^{1,1}(0, m)$ . In particular

$$u(0) = \lim_{n \rightarrow \infty} u^n(0) = \lim_{n \rightarrow \infty} u_0^n = u_0$$

and

$$u(m) = \lim_{n \rightarrow \infty} u^n(m) = \lim_{n \rightarrow \infty} u_m^n = u_m.$$

Knowing that  $u^n \rightarrow u$  in  $W^{1,1}(0, m)$ , the fact that

$$(d(u^n)')' - (\gamma u^n)' - \mu u^n \rightarrow g$$

1 implies that  $(u^n)''$  converges in  $L^1(0, m)$  so that  $u \in W^{2,1}(0, m)$  and  $u^n \rightarrow u$  in  
 2  $W^{2,1}(0, m)$ . Finally  $U \in D(A)$ ,  $G = AU$ . This ends the proof of the closedness  
 3 of  $A$ .

4. Let us prove that  $(\lambda I - A) : D(A) \rightarrow \mathcal{X}$  is a surjective operator for  $\lambda > 0$  large enough.

We consider first a *particular case*

$$H = (h, h_0, h_m)^T \in L^2(0, m) \times \mathbb{R}^2.$$

We look for  $U := (u, u_0, u_m)^T \in D(A)$  such that  $(\lambda I - A)U = H$ , i.e.

$$(\lambda + \mu)u - (du')' + (\gamma u)' = h \text{ in } [0, m], \quad (21)$$

$$(\lambda + \rho_0)u_0 - (b_0 - \gamma(0))u'(0) = h_0, \quad (22)$$

$$(\lambda + \rho_m)u_m + (b_m + \gamma(m))u'(m) = h_m. \quad (23)$$

We multiply (21) by  $v \in H^1(0, m)$  and integrate between 0 and  $m$  to get

$$\lambda \int_0^m uv + \int_0^m \mu uv - \int_0^m (du')'v + \int_0^m (\gamma u)'v = \int_0^m hv.$$

An integration by parts, with (22)-(23) leads to

$$\begin{aligned} \lambda \int_0^m uv + \int_0^m \mu uv + \int_0^m du'v' - \int_0^m \gamma uv' + K_0 u(0)v(0) + K_m u(m)v(m) \\ = \int_0^m hv + c_1 h_0 v(0) + c_2 h_m v(m), \end{aligned} \quad (24)$$

where  $K_0 = c_1(\lambda + \rho_0) - \gamma(0)$  and  $K_m = c_2(\lambda + \rho_m) + \gamma(m)$ .  
We define the bilinear form

$$a : H^1(0, m) \times H^1(0, m) \rightarrow \mathbb{R}$$

by the left hand side and a linear form  $L : H^1(0, m) \rightarrow \mathbb{R}$  by the right hand side of (24), to get

$$a(u, v) = L(v).$$

Let us check the conditions of Lax-Milgram Theorem. The continuity of  $a$  and  $L$  are easily obtained by using the trace theory. The inequality

$$2ab \leq \frac{a^2}{\varepsilon^2} + (\varepsilon b)^2 \quad (\forall \varepsilon > 0)$$

implies

$$\int_0^m \gamma uu' \leq \|\gamma\|_{L^\infty} \|u\|_{L^2} \|u'\|_{L^2} \leq \|\gamma\|_{L^\infty} \left( \frac{\|u\|_{L^2}^2}{2\varepsilon^2} + \frac{\varepsilon^2 \|u'\|_{L^2}^2}{2} \right)$$

and consequently

$$|a(u, u)| \geq \left( \lambda - \frac{\|\gamma\|_{L^\infty}}{2\varepsilon^2} \right) \|u\|_{L^2}^2 + \left( d_0 - \frac{\|\gamma\|_{L^\infty} \varepsilon^2}{2} \right) \|u'\|_{L^2}^2 + K_0 u(0)^2 + K_m u(m)^2.$$

Taking first  $\varepsilon > 0$  small enough and then  $\lambda$  large enough, we finally get a coercivity estimate  $|a(u, u)| \geq K \|u\|_{H^1}^2$  where  $K > 0$  is a constant. By Lax-Milgram Theorem, for every  $H \in L^2(0, m) \times \mathbb{R}^2$ , there exists a unique  $u \in H^1(0, m)$  such that  $a(u, v) = L(v)$  for every  $v \in H^1(0, m)$ . Now, we need to verify that  $U$  belongs to  $D(A)$ , where  $U$  is defined by  $U := (u, u(0), u(m)) = (u, u_0, u_m)$ . For this, we use (24) with  $v \in C_c^\infty([0, m])$ . Then

$$\left| \int_0^m du'v' \right| \leq (|\lambda| + \|\mu\|_{L^\infty}) \|u\|_{L^2} \|v\|_{L^2} + \|\gamma\|_{L^\infty} \left| \int_0^m uv' \right| + \|h\|_{L^2} \|v\|_{L^2}.$$

Since  $u \in H^1(0, m)$  then  $|\int_0^m uv'| \leq C \|v\|_{L^2}$ . Consequently

$$\left| \int_0^m du'v' \right| \leq [ (|\lambda| + \|\mu\|_{L^\infty}) \|u\|_{L^2} + C \|\gamma\|_{L^\infty} + \|h\|_{L^2} ] \|v\|_{L^2} \leq K \|v\|_{L^2}.$$

- 1 Thus  $du' \in H^1(0, m)$  and  $u \in H^2(0, m) \subset W^{2,1}(0, m)$  so  $U \in D(A)$ .  
 2 Now we prove that  $(\lambda I - A)U = H$  i.e. (21)-(22)-(23) are satisfied. An in-  
 3 tegration by parts of (24) with  $v \in C_c^\infty(0, m)$  implies (21). Moreover, an  
 4 integration by parts of (24) with  $v \in C^\infty(0, m)$  and  $v(0) = 1, v(m) = 0$   
 5 (respectively  $v(0) = 0, v(m) = 1$ ) gives us (22) (resp. (23)).

We deal now with the *surjectivity* of  $(\lambda I - A)$ . Let

$$H = (h, h_0, h_m) \in L^1(0, m) \times \mathbb{R}^2.$$

There exists a sequence  $(H_n)_{n \geq 0} = (h^n, h_0, h_m) \in L^2(0, m) \times \mathbb{R}^2$  such that  $\lim_{n \rightarrow \infty} \|H_n - H\|_{\mathcal{X}} = 0$ .

We know that  $\forall n \geq 0, \exists! U_n \in D(A) : (\lambda I - A)U_n = H_n$ . In particular  $\forall n, m \geq 0, (\lambda I - A)(U_n - U_m) = H_n - H_m$ . Using the dissipativity result shown before, we get

$$\|U_n - U_m\|_{\mathcal{X}} \leq C \|H_n - H_m\|_{\mathcal{X}}.$$

- 6 It follows that  $(U_n)_{n \geq 0}$  is a Cauchy sequence in  $\mathcal{X}$ . Let  $U$  be its limit. Since  
 7  $AU_n = -H_n + \lambda U_n$  then  $AU_n$  converges to  $-H + \lambda U$ . The *closedness* of  $A$

1 implies that  $U \in D(A)$  and  $(\lambda I - A)U = H$  and this ends the proof of the  
 2 surjectivity.

3 Thus  $A$  generates a  $C_0$ -semigroup  $\{T(t)\}_{t \geq 0}$  by Lumer-Phillips Theorem (see [26]  
 4 Theorem 4.3, p. 14). Finally, as a bounded perturbation of  $A$ ,  $\mathcal{A}$  generates also a  
 5 quasi-contraction  $C_0$ -semigroup  $\{U(t)\}_{t \geq 0}$ .  $\square$

### 6 2.3. On irreducibility.

7 To understand time asymptotics of  $\{U(t)\}_{t \geq 0}$ , we need to prove a key result re-  
 8 lated to positivity. We remind first some definitions and results about positive and  
 9 irreducible operators. We denote by  $\langle \cdot, \cdot \rangle$  the duality pairing between  $\mathcal{X}$  and  $\mathcal{X}'$ .

#### 10 Definition 2.2.

- 11 1. For  $x \in \mathcal{X}$ , the notation  $x > 0$  means  $x \in \mathcal{X}_+$  and  $x \neq 0$ .
- 12 2. An operator  $O \in L(\mathcal{X})$  is said to be positive if it leaves the positive cone  $\mathcal{X}_+$   
 13 invariant. We note this by  $O \geq 0$ .
- 14 3. A  $C_0$ -semigroup  $\{Z(t), t \geq 0\}$  on  $\mathcal{X}$  is said to be positive if each operator  $Z(t)$   
 15 is positive.
- 16 4. A positive operator  $O \in L(\mathcal{X})$  is said to be positivity improving if for any  
 17  $x > 0$  and  $x' > 0$ , we have  $\langle Ox, x' \rangle > 0$ .
- 18 5. A positive operator  $O \in L(\mathcal{X})$  is said to be irreducible if for any  $x > 0$  and  
 19  $x' > 0$  there exists an integer  $n$  such that  $\langle O^n x, x' \rangle > 0$ .
- 20 6. A  $C_0$ -semigroup  $\{Z(t), t \geq 0\}$  on  $\mathcal{X}$  is said to be irreducible if for any  $x > 0$   
 21 and  $x' > 0$  there exists  $t > 0$  such that  $\langle Z(t)x, x' \rangle > 0$ .

22 We recall that a  $C_0$ -semigroup  $\{Z(t), t \geq 0\}$  on  $\mathcal{X}$  with generator  $B$  is positive  
 23 if and only if, for  $\lambda$  large enough, the resolvent operator  $(\lambda I - B)^{-1}$  is positive. We  
 24 recall also that a  $C_0$ -semigroup  $\{Z(t), t \geq 0\}$  on  $\mathcal{X}$  with generator  $B$  is irreducible  
 25 if, for  $\lambda$  large enough, the resolvent operator  $(\lambda I - B)^{-1}$  is positivity improving,  
 26 (see e.g. [8] p. 165).

27 **Definition 2.3.** For a closed operator  $B : D(B) \subset \mathcal{X} \rightarrow \mathcal{X}$ , we denote by  $\sigma(B)$  its  
 28 spectrum and by  $s(B)$  its spectral bound defined by

$$s(B) := \begin{cases} \sup \{\Re(\lambda); \lambda \in \sigma(B)\} & \text{if } \sigma(B) \neq \emptyset, \\ -\infty & \text{if } \sigma(B) = \emptyset. \end{cases}$$

29 The main result of this subsection is:

30 **Theorem 2.4.** *The  $C_0$ -semigroup  $\{U(t)\}_{t \geq 0}$  is irreducible.*

31 *Proof.* We have to show that the resolvent  $(\lambda I - \mathcal{A})^{-1}$  is positivity improving for  
 32 large  $\lambda$ . It is easy to see that for large  $\lambda$

$$\begin{aligned} (\lambda I - \mathcal{A})^{-1} &= (\lambda I - A - K)^{-1} = (\lambda I - A)^{-1} \sum_{n=0}^{\infty} (K(\lambda I - A)^{-1})^n \\ &= (\lambda I - A)^{-1} + (\lambda I - A)^{-1} \sum_{n=1}^{\infty} (K(\lambda I - A)^{-1})^n. \end{aligned}$$

It follows that if  $(\lambda I - A)^{-1} \geq 0$  then

$$(\lambda I - \mathcal{A})^{-1} \geq (\lambda I - A)^{-1}$$

33 because  $K$  is a positive operator. Hence it suffices to prove that  $(\lambda I - A)^{-1}$  is  
 34 positivity improving.

Let us show *first* that

$$(\lambda I - A)^{-1} \geq 0.$$

Let  $U = (\lambda I - A)^{-1}H$  with  $H = (h, h_0, h_m) \in \mathcal{X}_+$ . Since  $C^+([0, m])$  is dense in  $L_+^1(0, m)$ , we may assume without loss of generality that

$$h \in C^+([0, m]).$$

Thus

$$\begin{aligned} (\lambda + \mu(s))u(s) + (\gamma u)'(s) - (du')'(s) &= h(s), s \in (0, m), \\ (\lambda + \rho_0)u_0 - (b_0 - \gamma(0))u'(0) &= h_0 \\ (\lambda + \rho_m)u_m + (b_m + \gamma(m))u'(m) &= h_m. \end{aligned}$$

The first equation is

$$-u'' + \rho_1 u' + \rho_2 u = \rho_3$$

where  $\rho_1 = -(d' - \gamma)/d$ ,

$$\rho_2(s) = (\lambda + \mu(s) + \gamma'(s))/d(s) > 0 \quad \forall s \text{ for } \lambda \text{ large enough}$$

and

$$\rho_3 = h/d \geq 0.$$

The *absolute minimum* of  $u$  is achieved at some  $\bar{s} \in [0, m]$ . Let us show that  $u(\bar{s}) \geq 0$ . If not, i.e. if  $u(\bar{s}) < 0$  then  $\bar{s} \notin (0, m)$ . Indeed, this would imply that

$$0 \geq -u''(\bar{s}) = -\rho_2(\bar{s})u(\bar{s}) + \rho_3(\bar{s}) \geq -\rho_2(\bar{s})u(\bar{s}) > 0$$

which is contradictory. Hence  $\bar{s} = 0$  or  $\bar{s} = m$ . If  $\bar{s} = 0$  since

$$(\lambda + \rho_0)u(0) - (b_0 - \gamma(0))u'(0) = h_0$$

then

$$-(b_0 - \gamma(0))u'(0) = -(\lambda + \rho_0)u(0) + h_0 \geq -(\lambda + \rho_0)u(0) > 0.$$

- 1 It follows that  $u'(0) < 0$  and then  $u'(s) < 0$  in the neighborhood of  $s = 0$  which  
 2 contradicts the fact that the absolute minimum is achieved at 0. We argue similarly  
 3 if  $\bar{s} = m$ . Finally,  $u \geq 0$ .

Let us show now that  $(\lambda I - A)^{-1}$  is *positivity improving*.

We note first that for any  $\mu > \lambda$ , the resolvent identity

$$(\lambda I - A)^{-1} = (\mu I - A)^{-1} + (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1}$$

shows that

$$(\lambda I - A)^{-1} \geq (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1}$$

so

$$(\lambda I - A)^{-1}H \geq (\lambda I - A)^{-1}G$$

where

$$G := (\mu - \lambda)(\mu I - A)^{-1}H \in \mathcal{X}_+$$

has the peculiarity of belonging to

$$D(A) \subset W^{2,1}(0, m) \times \mathbb{R}^2 \subset C([0, m]) \times \mathbb{R}^2.$$

Hence, without loss of generality, we may assume that  $H = (h, h_0, h_m) \in \mathcal{X}_+$  is such that  $h \in C^+([0, m])$ . Let us show that

$$u(s) > 0 \text{ a.e.}, u(0) > 0, u(m) > 0$$

once

$$H = (h, h_0, h_m) \in \mathcal{X}_+ - \{0\}.$$

1 Let us show by contradiction that  $\min u > 0$ .

The *absolute minimum* of  $u$  is achieved at some  $\bar{s} \in [0, m]$ . Suppose  $u(\bar{s}) = 0$ . Then

$$v := -u$$

satisfies the equation

$$v'' - \rho_1 v' + \tilde{\rho}_2 v = h/d \geq 0$$

where  $\tilde{\rho}_2 \leq 0$ . Note that

$$\max v = -\min u \geq 0.$$

If  $u$  reaches its minimum in  $(0, m)$  then  $v$  reaches its maximum in  $(0, m)$ . By the maximum principle (see [27] Theorem 3, p. 6),  $v$  must be constant and then  $u$  is equal to the constant  $u(\bar{s}) = 0$ . It follows that

$$0 = h_0, \quad 0 = h_m, \quad 0 = h$$

which is contradictory. Hence

$$u(s) > 0 \quad \forall s \in (0, m)$$

and  $u(0) = 0$  or  $u(m) = 0$ . Thus  $v$  reaches its maximum (equal to zero) at  $\bar{s} = 0$  or  $\bar{s} = m$ . If  $\bar{s} = 0$  then  $v'(0) < 0$  by Hopf's maximum principle (see [27] Theorem 4, p. 7); since

$$(b_0 - \gamma(0))v'(0) = h_0 \geq 0$$

we get a contradiction. If  $\bar{s} = m$  then  $v'(m) > 0$  by Hopf's maximum principle; since

$$-(b_m + \gamma(m))u'(m) = h_m$$

2 we get also a contradiction. Finally  $\min u > 0$ . □

### 3 2.4. On the spectral bound of the generator.

4 Let  $s(\mathcal{A})$  be the spectral bound of  $\mathcal{A}$ . We have:

5 **Theorem 2.5.** *The spectral bound of  $\mathcal{A}$  is finite, i.e.  $s(\mathcal{A}) > -\infty$ .*

6 *Proof.* According to Theorem 2.4, for  $\lambda > s(\mathcal{A})$ ,  $(\lambda - \mathcal{A})^{-1}$  is positivity improving  
7 and therefore irreducible. Since  $(\lambda - \mathcal{A})^{-1}$  is also compact then

$$r_\sigma((\lambda - \mathcal{A})^{-1}) > 0,$$

(see [25] Theorem 3), where

$$r_\sigma(O) = \sup\{|\lambda| : \lambda \in \sigma(O)\}$$

8 is the *spectral radius* of  $O$  a bounded operator. On the other hand

$$r_\sigma((\lambda - \mathcal{A})^{-1}) = \frac{1}{\lambda - s(\mathcal{A})}$$

9 (see [24] Proposition 2.5, p. 67) whence  $s(\mathcal{A}) > -\infty$ . □

10 **Remark 2.** Theorem 2.5 provides us with the existence of a real leading eigenvalue  
11 since  $s(\mathcal{A}) \in \sigma(\mathcal{A})$  (see e.g. [20] Theorem 5.2, p. 102).



1 **2.5. On asynchronous exponential growth.**

2 Let us remind some definitions and results about *asynchronous exponential growth*  
 3 (see [10], [24] and [35] for the details).

**Definition 2.6.** Let  $\mathcal{L}(\mathcal{X})$  be the space of bounded linear operators on  $\mathcal{X}$  and let  $\mathcal{K}(\mathcal{X})$  be the subspace of compact operators on  $\mathcal{X}$ . The essential norm  $\|L\|_{ess}$  of  $L \in \mathcal{L}(\mathcal{X})$  is given by

$$\|L\|_{ess} = \inf_{K \in \mathcal{K}(\mathcal{X})} \|L - K\|_X.$$

Let  $\{Z(t); t \geq 0\}$  be a  $C_0$ -semigroup on  $\mathcal{X}$  with generator  $B : D(B) \subset \mathcal{X} \rightarrow \mathcal{X}$ . The growth bound (or type) of  $\{Z(t); t \geq 0\}$  is given by

$$\omega_0(B) = \lim_{t \rightarrow \infty} \frac{\ln(\|Z(t)\|_X)}{t},$$

and the essential growth bound (or essential type) of  $\{Z(t); t \geq 0\}$  is given by

$$\omega_{ess}(B) = \lim_{t \rightarrow \infty} \frac{\ln(\|Z(t)\|_{ess})}{t}.$$

4 **Definition 2.7** (Asynchronous Exponential Growth). [35, Definition 2.2]

5 Let  $\{Z(t)\}_{t \geq 0}$  be a  $C_0$ -semigroup with infinitesimal generator  $B$  in the Banach  
 6 space  $X$ . We say that  $\{Z(t)\}_{t \geq 0}$  has asynchronous exponential growth with intrinsic  
 7 growth constant  $\lambda_0 \in \mathbb{R}$  if there exists a nonzero finite rank operator  $P_0$  in  $X$  such  
 8 that  $\lim_{t \rightarrow \infty} e^{-\lambda_0 t} Z(t) = P_0$ .

9 We recall the following standard result (see e.g. [8] Theorem 9.11, p. 224).

**Theorem 2.8.** *Let  $X$  be a Banach lattice and let  $\{Z(t)\}_{t \geq 0}$  be a positive  $C_0$ -semigroup on  $X$  with infinitesimal generator  $B$ . If  $\{Z(t)\}_{t \geq 0}$  is irreducible and if*

$$\omega_{ess}(B) < \omega_0(B)$$

10 *then  $\{Z(t)\}_{t \geq 0}$  has asynchronous exponential growth with intrinsic growth constant*  
 11  *$\lambda_0 = \omega_0(B)$  and one-rank spectral projection  $P_0$ .*

12 **Remark 3.** Note that  $\mathcal{A}$  has a compact resolvent (and consequently the spectrum  
 13 of  $\mathcal{A}$  is composed (at most) of isolated eigenvalues with finite algebraic multiplic-  
 14 ity). This follows from the fact that the canonical injection  $i : (D(A), \|\cdot\|_{D(A)}) \rightarrow$   
 15  $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$  is compact (by Rellich Kondrachov's Theorem) and  $D(A) = D(\mathcal{A})$  since  
 16  $K \in \mathcal{L}(\mathcal{X})$  (see e.g. [10] Proposition II.4.25, p. 117).

17 We are ready to give the main result of this subsection.

18 **Theorem 2.9.** *If  $K \neq 0$  then the semigroup  $\{U(t)\}_{t \geq 0}$  generated by  $\mathcal{A}$  has asyn-*  
 19 *chronous exponential growth.*

*Proof.* The semigroups  $\{U(t)\}_{t \geq 0}$  and  $\{T(t)\}_{t \geq 0}$  are related by the Duhamel equation

$$U(t) = T(t) + \int_0^t T(t-s)KU(s)ds.$$

Since  $K$  is a weakly compact operator then so is  $T(t-s)KU(s)$  for all  $s \geq 0$ . It follows that the *strong* integral

$$\int_0^t T(t-s)KU(s)ds$$

is a weakly compact operator (see [21] Theorem 1 or [29] Theorem 2.2). Hence  $U(t) - T(t)$  is a weakly compact operator and consequently (see [21] Theorem 2.10)  $\{U(t)\}_{t \geq 0}$  and  $\{T(t)\}_{t \geq 0}$  have the *same* essential type

$$\omega_{ess}(\mathcal{A}) = \omega_{ess}(A),$$

in particular

$$\omega_{ess}(\mathcal{A}) \leq \omega_0(A).$$

Let  $\lambda > s(\mathcal{A}) \geq s(A)$ . The positivity improving compact operators  $O_1 := (\lambda - A)^{-1}$  and  $O_2 := (\lambda - \mathcal{A})^{-1}$  are such that

$$O_2 \geq O_1 \geq 0 \text{ and } O_2 \neq O_1$$

since  $K \neq 0$ . It follows from ([18] Theorem 4.3) that

$$r_\sigma(O_1) < r_\sigma(O_2).$$

In addition, according to ([24] Proposition 2.5, p. 67),

$$r_\sigma [(\lambda - A)^{-1}] = \frac{1}{\lambda - s(A)} \text{ and } r_\sigma [(\lambda - \mathcal{A})^{-1}] = \frac{1}{\lambda - s(\mathcal{A})}$$

so

$$s(A) < s(\mathcal{A}).$$

Note that  $s(A) = \omega_0(A)$  and  $s(\mathcal{A}) = \omega_0(\mathcal{A})$  since  $\{U(t)\}_{t \geq 0}$  and  $\{T(t)\}_{t \geq 0}$  are positive semigroups on  $L^1$  spaces (see e.g. [10] Theorem VI.1.15, p. 358) so  $\omega_0(A) < \omega_0(\mathcal{A})$  and

$$\omega_{ess}(\mathcal{A}) < \omega_0(\mathcal{A}).$$

1 By combining this last result and the irreducibility of  $\{U(t)\}_{t \geq 0}$ , Theorem 2.8 ends  
2 the proof. □

3 **Remark 4.** Note that in Theorem 2.9, the requirement  $K \neq 0$  amounts to the fact  
4 that the function  $\beta$  is not identically zero.

5 **3. Models with unbounded sizes.**

6 From now on, we consider the general model, described by (7)-(8).

7 **3.1. Framework and hypotheses.**

8 The boundary condition (8) can be rewritten into the following dynamic form

$$u_t(0, t) = -u(0, t)\rho_0 + u_s(0, t)(b_0 - \gamma(0)) + \int_0^\infty \beta_0(y)u(y, t)dy. \quad (25)$$

Let

$$\mathcal{X}_\infty = (L^1(0, \infty) \times \mathbb{R}, \|\cdot\|_{\mathcal{X}_\infty})$$

with norm

$$\|(x, x_0)\|_{\mathcal{X}_\infty} = \|x\|_{L^1(0, \infty)} + c_1|x_0|.$$

9 We assume that

$$b_0 - \gamma(0) > 0 \quad (26)$$

10 and denote by  $\mathcal{X}_{\infty,+}$  the nonnegative cone of  $\mathcal{X}_\infty$ . We now introduce some hypothe-  
11 ses on the different parameters:

- 12 1.  $\gamma, d \in W^{1,\infty}(0, \infty)$  and  $\mu, \beta_0 \in L^\infty(0, \infty)$ ,  
13 2. the functions  $\mu, \gamma'$  and  $s \mapsto \beta(s, y)$  are continuous at  $s = 0$ , for every  $y \geq 0$ ,

3. the operator

$$L^1(0, \infty) \ni u \rightarrow \int_0^\infty \beta(\cdot, y)u(y)dy \in L^1(0, \infty)$$

1 is weakly compact,

2 4.  $b_0 > 0$ ,  $c_0 \geq 0$ ,  $\beta, \mu \geq 0$  and  $d(s) \geq d_0 > 0$  a.e.  $s \geq 0$ .

**Remark 5.** According to the general criterion of weak compactness, the third hypothesis amounts to

$$\begin{aligned} \sup_{y \in [0, \infty)} \int_0^\infty \beta(s, y)ds < \infty, \quad \lim_{c \rightarrow +\infty} \sup_{y \in [0, \infty)} \int_c^\infty \beta(s, y)ds = 0, \\ \lim_{|E| \rightarrow 0} \sup_{y \in [0, \infty)} \int_E \beta(s, y)ds = 0. \end{aligned}$$

Define

$$W_{loc}^{2,1}(\mathbb{R}_+) := \{u \in L_{loc}^1(\mathbb{R}_+); u \in W^{2,1}(0, c) \forall c > 0\}.$$

3 By means of (7)-(25), we define the operator  $\mathcal{A}_\infty$  by

$$\begin{aligned} \mathcal{A}_\infty \begin{pmatrix} u \\ u_0 \end{pmatrix} &= A_\infty \begin{pmatrix} u \\ u_0 \end{pmatrix} + K_\infty \begin{pmatrix} u \\ u_0 \end{pmatrix} \\ &= \begin{pmatrix} (du')' - (\gamma u)' - \mu u \\ (b_0 - \gamma(0))u'(0) - \rho_0 u_0 \end{pmatrix} + \begin{pmatrix} \int_0^\infty \beta(\cdot, y)u(y)dy \\ \int_0^\infty \beta_0(y)u(y)dy \end{pmatrix} \end{aligned}$$

with domain  $D(\mathcal{A}_\infty)$  given by

$$\begin{aligned} \{(u, u_0) \in \mathcal{X}_\infty; u \in W_{loc}^{2,1}(\mathbb{R}_+), u(0) = u_0, (du')' - (\gamma u)' \in L^1(\mathbb{R}_+) \\ \text{and } \lim_{s \rightarrow +\infty} d(s)u'(s) - \gamma(s)u(s) = 0\}. \end{aligned}$$

Note that

$$d(s)u'(s) - \gamma(s)u(s) = d(0)u'(0) - \gamma(0)u(0) + \int_0^s z(\tau)d\tau$$

where

$$z := (du')' - (\gamma u)' \in L^1(\mathbb{R}_+)$$

shows that  $\lim_{s \rightarrow +\infty} d(s)u'(s) - \gamma(s)u(s)$  exists.

As previously, we are concerned with the Cauchy problem

$$\begin{cases} U'(t) &= \mathcal{A}_\infty U(t), \\ U(0) &= (u^0, u_0^0) \in \mathcal{X}_\infty \end{cases}$$

where

$$U(t) = (u(t), u_0(t))^T.$$

#### 4 3.2. Semigroup generation.

5 The main result of this subsection is:

6 **Theorem 3.1.** *Let Assumption (26) be satisfied. Then  $\mathcal{A}_\infty$  is the infinitesimal*  
7 *generator of a quasi-contractive  $C_0$ -semigroup  $\{U_\infty(t)\}_{t \geq 0}$  on  $\mathcal{X}_\infty$ .*

8 *Proof.* As previously, we restrict ourselves to  $A_\infty$  since  $K_\infty$  is bounded.

1. Let us show that  $\overline{D(A_\infty)} = \mathcal{X}_\infty$ . Let  $(u, u_0)^T \in \mathcal{X}_\infty$ . Let  $(u^j)_j$  be  $C^\infty$  functions with compact supports such that  $u^j \rightarrow u$  in  $L^1(0, \infty)$  and

$$\text{support}(u^j) \subset [j^{-1}, +\infty).$$

As in the finite case, we introduce the functions

$$v^j(s) = \begin{cases} f_0^j(s) & \text{if } s \in [0, j^{-1}] \\ u^j(s) & \text{if } s \geq j^{-1}, \end{cases}$$

where

$$f_0^j(s) = j^2 u_0 s^2 - 2j u_0 s + u_0 = u_0 (j s - 1)^2$$

and we verify that

$$D(A_\infty) \ni (v^j, v^j(0))^T \rightarrow (u, u_0)^T \in \mathcal{X}_\infty$$

1 so  $\overline{D(A_\infty)} = \mathcal{X}_\infty$ .

2. Let us prove that  $(A_\infty, D(A_\infty))$  is a closed operator. We argue as previously. Let  $(U^n)_{n \in \mathbb{N}} := (u^n, u_0^n)_{n \in \mathbb{N}} \subset D(A_\infty)$  then let  $U := (u, u_0) \in \mathcal{X}_\infty$  and  $G := (g, g_0) \in \mathcal{X}_\infty$  such that  $\lim_{n \rightarrow \infty} \|U^n - U\|_{\mathcal{X}_\infty} = 0$  and  $\lim_{n \rightarrow \infty} \|A_\infty U^n - G\|_{\mathcal{X}_\infty} = 0$ . Let

$$f_n := d(u^n)' - \gamma u^n.$$

2 Note that by assumption

$$\lim_{s \rightarrow +\infty} f_n(s) = 0. \quad (27)$$

Since

$$(d(u^n)')' - (\gamma u^n)' - \mu u^n \rightarrow g$$

( $L^1(0, \infty)$  convergence) and

$$(b_0 - \gamma(0))(u^n)'(0) - \rho_0 u^n(0) \rightarrow g_0$$

then

$$f_n' \rightarrow g + \mu u$$

( $L^1(0, \infty)$  convergence) while

$$f_n(0) = d(0)(u^n)'(0) - \gamma(0)u^n(0) \rightarrow d(0)h_0 - \gamma(0)u_0$$

where

$$h_0 := \frac{g_0 + \rho_0 u_0}{b_0 - \gamma(0)}.$$

3 Hence

$$f_n(s) = f_n(0) + \int_0^s f_n'(\tau) d\tau \rightarrow z(s) := d(0)h_0 - \gamma(0)u_0 + \int_0^s (g + \mu u)(\tau) d\tau \quad (28)$$

in  $L^1(0, c)$  for any finite  $c$ . It follows that

$$(u^n)' \rightarrow \frac{z + \gamma u}{d}$$

in  $L^1(0, c)$  for any finite  $c$  so  $u' \in L^1(0, c)$  and  $u^n \rightarrow u$  in  $W^{1,1}(0, c)$  for any finite  $c$ . In particular

$$u(0) = \lim_{n \rightarrow \infty} u^n(0) = \lim_{n \rightarrow \infty} u_0^n = u_0.$$

Finally

$$f_n' - \mu u^n = (d(u^n)')' - (\gamma u^n)' - \mu u^n \rightarrow g$$

( $L^1(0, \infty)$  convergence) implies that  $(u^n)''$  converges in  $L^1(0, c)$  for any finite  $c$  so that  $u \in W^{2,1}(0, c)$  for any finite  $c$  and

$$(d(u)')' - (\gamma u)' - \mu u = g.$$

Note that (28) shows that

$$|f_n(s) - z(s)| \leq |f_n(0) - (d(0)h_0 - \gamma(0)u_0)| + \int_0^{+\infty} |f'_n(\tau) - (g(\tau) + \mu(\tau)u(\tau))| d\tau \rightarrow 0$$

so

$$f_n(s) \rightarrow z(s) = d(s)u'(s) - \gamma(s)u(s) \text{ uniformly on } \mathbb{R}_+$$

and (27) implies

$$\lim_{s \rightarrow +\infty} d(s)u'(s) - \gamma(s)u(s) = 0.$$

1

Thus  $U \in D(A_\infty)$  and  $G = A_\infty U$ .

3. We consider now the dissipativity of  $(A_\infty - \omega I)$  for  $\omega$  large enough. Let  $\lambda > 0, U = (u, u_0)^T \in D(A_\infty)$  and  $H = ((\lambda + \omega)I - A_\infty)U$ .

Let  $H = (h, h_0)^T$ . We have to prove that

$$\|H\|_{\mathcal{X}_\infty} \geq \lambda \|U\|_{\mathcal{X}_\infty}.$$

By definition of  $H$ , we have

$$\begin{aligned} (\lambda + \widehat{\mu}(s))u(s) + (\gamma u)'(s) - (du')'(s) &= h(s), s \in (0, \infty), \\ (\lambda + \widehat{\rho}_0)u_0 - (b_0 - \gamma(0))u'(0) &= h_0, \end{aligned}$$

where

$$\widehat{\mu}(s) := \omega + \mu(s), \quad \widehat{\rho}_0 := \omega + \rho_0.$$

By integration

$$\begin{aligned} \lambda \|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u| - \int_0^\infty (du')' \text{sign}(u) + \int_0^\infty (\gamma u)' \text{sign}(u) &= \int_0^\infty h \text{sign}(u), \\ u'(0) \text{sign}(u(0)) &= \frac{(\lambda + \widehat{\rho}_0)|u_0|}{b_0 - \gamma(0)} - \frac{h_0 \text{sign}(u(0))}{b_0 - \gamma(0)}. \end{aligned}$$

Since  $u \in W_{loc}^{2,1}(\mathbb{R}_+) \subset C^1(0, \infty)$ , we get, for every finite  $m > 0$

$$\int_0^m (du')' \text{sign}(u) \leq d(m)u'(m) \text{sign}(u(m)) - d(0)u'(0) \text{sign}(u(0))$$

and

$$\int_0^m (\gamma u)' \text{sign}(u) = \gamma(m)|u(m)| - \gamma(0)|u(0)|.$$

Consequently

$$\int_0^m (du')' \text{sign}(u) - \int_0^m (\gamma u)' \text{sign}(u) \leq (d(m)u'(m) - \gamma(m)u(m)) \text{sign}(u(m)) + l_0$$

where  $l_0 = -d(0)u'(0) \text{sign}(u(0)) + \gamma(0)|u(0)|$ . Since

$$\lim_{m \rightarrow +\infty} d(m)u'(m) - \gamma(m)u(m) = 0$$

then

$$\begin{aligned} & \int_0^\infty (du')' \text{sign}(u) - \int_0^\infty (\gamma u)' \text{sign}(u) \\ &= \lim_{m \rightarrow +\infty} \int_0^m (du')' \text{sign}(u) - \int_0^m (\gamma u)' \text{sign}(u) \leq l_0. \end{aligned}$$

Hence

$$\lambda \|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u| \leq l_0 + \int_0^\infty h \text{sign}(u)$$

so

$$\lambda \|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u| - \gamma(0)|u(0)| \leq -d(0)u'(0)\text{sign}(u(0)) + \int_0^\infty h\text{sign}(u).$$

Since

$$d(0)u'(0)\text{sign}(u(0)) = \frac{d(0)(\lambda + \widehat{\rho}_0)|u_0|}{b_0 - \gamma(0)} - \frac{d(0)h_0\text{sign}(u(0))}{b_0 - \gamma(0)}$$

then

$$\lambda \|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u|ds + \left[ -\gamma(0) + \frac{d(0)(\lambda + \widehat{\rho}_0)}{b_0 - \gamma(0)} \right] |u(0)| \leq \frac{d(0)|h_0|}{b_0 - \gamma(0)} + \|h\|_{L^1}$$

or

$$\lambda \|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u|ds + \left[ -\frac{\gamma(0)}{c_1} + (\lambda + \widehat{\rho}_0) \right] c_1 |u(0)| \leq \|h\|_{L^1} + c_1 |h_0|.$$

Note that if

$$-\frac{\gamma(0)}{c_1} + \widehat{\rho}_0 \geq 0$$

then

$$\lambda \|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u|ds + \lambda c_1 |u(0)| \leq \|h\|_{L^1} + c_1 |h_0|.$$

Since

$$-\frac{\gamma(0)}{c_1} + \widehat{\rho}_0 = -\frac{\gamma(0)(b_0 - \gamma(0))}{d(0)} + \gamma'(0) + \mu(0) + c_0 + \omega$$

is *nonnegative* for  $\omega$  large enough then

$$\lambda \|u\|_{L^1} + \int_0^\infty (\mu + \omega)|u|ds + \lambda c_1 |u(0)| \leq c_1 |h_0| + \|h\|_{L^1}$$

and

$$\lambda \|U\|_{\mathcal{X}_\infty} \leq \|H\|_{\mathcal{X}_\infty}$$

1 for  $\omega$  large enough. Finally  $A_\infty - \omega I$  is dissipative.

4. Let us prove that  $(\lambda I - A_\infty) : D(A_\infty) \rightarrow \mathcal{X}_\infty$  is a surjective operator for  $\lambda > 0$  large enough. We consider first a particular case

$$H = (h, h_0)^T \in L^1(0, \infty) \cap L^2(0, \infty) \times \mathbb{R}$$

We look for  $U = (u, u_0)^T \in D(A_\infty)$  such that  $(\lambda I - A_\infty)U = H$ , i.e.

$$(\lambda + \mu)u - (du')' + (\gamma u)' = h \text{ in } \mathbb{R}_+, \quad (29)$$

$$(\lambda + \rho_0)u_0 - (b_0 - \gamma(0))u'(0) = h_0. \quad (30)$$

Multiply (29) by  $v \in H^1(0, \infty)$  and integrate to get

$$\lambda \int_0^\infty uv + \int_0^\infty \mu uv - \int_0^\infty (du')'v + \int_0^\infty (\gamma u)'v = \int_0^\infty hv.$$

2 An integration by parts and (30) lead to

$$\lambda \int_0^\infty uv + \int_0^\infty \mu uv + \int_0^\infty du'v' - \int_0^\infty \gamma uv' + K_0 u(0)v(0) = \int_0^\infty hv + c_1 h_0 v(0). \quad (31)$$

One can show that the bilinear form defined by the left hand of (31) is coercive. By Lax-Milgram's Theorem, there exists a unique  $u \in H^1(\mathbb{R}_+)$  satisfying (31) for all  $v \in H^1(\mathbb{R}_+)$ . It follows easily that  $u \in H^2(\mathbb{R}_+)$ . One sees that

$U = (u, u(0))$  satisfies (29)-(30). Since  $u \in H^2(\mathbb{R}_+)$  then  $u \in W^{2,1}(0, c)$  for every  $c > 0$  and

$$\lim_{m \rightarrow \infty} u(m) = 0, \quad \lim_{m \rightarrow \infty} u'(m) = 0.$$

Since  $\gamma, d \in L^\infty(\mathbb{R}_+)$  then

$$\lim_{m \rightarrow +\infty} d(m)u'(m) - \gamma(m)u(m) = 0.$$

Let us prove that  $u \in L^1(\mathbb{R}_+)$ . Consider  $\lambda := \tilde{\lambda} + \omega$ , with  $\tilde{\lambda}, \omega > 0$ . Since

$$\begin{aligned} (\tilde{\lambda} + \hat{\mu}(s))u(s) + (\gamma u)'(s) - (du')'(s) &= h(s), \quad s \in (0, \infty), \\ (\lambda + \hat{\rho}_0)u_0 - (b_0 - \gamma(0))u'(0) &= h_0, \end{aligned}$$

then

$$\int_0^m (\tilde{\lambda} + \hat{\mu}(s))|u(s)| ds = \int_0^m h \operatorname{sign}(u) + \int_0^m (du')' \operatorname{sign}(u) - \int_0^m (\gamma u)' \operatorname{sign}(u)$$

and

$$u'(0) \operatorname{sign}(u(0)) = \frac{(\tilde{\lambda} + \hat{\rho}_0)|u_0|}{b_0 - \gamma(0)} - \frac{h_0 \operatorname{sign}(u(0))}{b_0 - \gamma(0)}$$

so, using the above estimates,

$$\begin{aligned} &(\tilde{\lambda} + \omega) \int_0^m |u(s)| ds \\ &\leq \int_0^m h \operatorname{sign}(u) + (d(m)u'(m) - \gamma(m)u(m)) \operatorname{sign}(u(m)) - d(0)u'(0) \operatorname{sign}(u(0)) \\ &\quad + \gamma(0)|u(0)|. \end{aligned}$$

The fact that

$$\lim_{m \rightarrow +\infty} d(m)u'(m) - \gamma(m)u(m) = 0$$

gives

$$(\tilde{\lambda} + \omega) \int_0^{+\infty} |u(s)| ds \leq \int_0^{+\infty} h \operatorname{sign}(u) - d(0)u'(0) \operatorname{sign}(u(0)) + \gamma(0)|u(0)| < +\infty$$

and  $u \in L^1(\mathbb{R}_+)$ . Equation (29) shows that  $(du')' - (\gamma u)' \in L^1(0, \infty)$ . As for the previous finite case, by exploiting the *closedness* of  $A_\infty$ , we get the surjectivity of

$$(\lambda I - A_\infty) : D(A_\infty) \rightarrow \mathcal{X}_\infty.$$

- 1 Finally  $A_\infty$  generates a  $C_0$ -semigroup  $\{T_\infty(t)\}_{t \geq 0}$  by Lumer-Phillips' Theorem.  $\square$

Note that a priori the domain of the generator is *not*

$$\{(u, u_0) \in W^{2,1}(0, \infty) \times \mathbb{R} : u(0) = u_0\}$$

- 2 but this subspace turns out to be a core of  $D(A_\infty)$ . Indeed, we have:

**Proposition 1.** *Let  $B : D(B) \subset \mathcal{X}_\infty \rightarrow \mathcal{X}_\infty$ , be the restriction of  $A_\infty$  to*

$$\{(u, u_0) \in W^{2,1}(0, \infty) \times \mathbb{R} : u(0) = u_0\}.$$

- 3 *Then  $B$  is closable with closure  $A_\infty$ .*

*Proof.* Note first that  $A_\infty$  is closed and

$$B \subset A_\infty$$

1 (in the sense of graphs) so  $\overline{B} \subset A_\infty$  and  $\overline{B}$  is a graph, i.e.  $B$  is closable.

To show that  $\overline{B} = A_\infty$ , it suffices to show that for any  $U = (u, u(0)) \in D(A_\infty)$  there exists a sequence

$$U_n := (u^n, u^n(0)) \in D(B)$$

such that  $u^n(0) \rightarrow u(0)$ ,  $(u^n)'(0) \rightarrow u'(0)$ ,

$$u^n \rightarrow u \text{ in } L^1(\mathbb{R}_+)$$

2 and

$$(d(u^n)')' - (\gamma u^n)' \rightarrow (du')' - (\gamma u)' \text{ in } L^1(\mathbb{R}_+). \quad (32)$$

Let

$$\sigma : \mathbb{R} \rightarrow \mathbb{R}$$

be a  $C^2$  function such that

$$\sigma(s) = \begin{cases} 1 & \text{for } s \leq 0 \\ 0 & \text{for } s \geq 1. \end{cases}$$

Let

$$\sigma_n(s) := \sigma(s - n).$$

Note that

$$\sigma_n(s) = \begin{cases} 1 & \text{for } s \leq n \\ 0 & \text{for } s \geq n + 1. \end{cases}$$

Let  $U = (u, u(0)) \in D(A_\infty)$  and

$$u^n(s) := \sigma_n(s)u(s) \quad (s \geq 0).$$

Note that  $u^n \in W^{2,1}(0, \infty)$  and  $u^n = u$  on  $[0, n]$ . In particular  $u^n(0) = u(0)$  and  $(u^n)'(0) = u'(0)$ . Since  $\sigma_n(s) \leq 1$  and

$$\lim_{n \rightarrow +\infty} \sigma_n(s) = 1 \quad \forall s \geq 0$$

then

$$u^n \rightarrow u \text{ in } L^1(\mathbb{R}_+)$$

by the dominated convergence theorem. It suffices to show (32).

Note that

$$\begin{aligned} (u^n)' &= \sigma_n' u + \sigma_n u' \\ (\gamma u^n)' &= (\gamma \sigma_n u)' = (\gamma u) \sigma_n' + \sigma_n (\gamma u)' \\ d(u^n)' &= d\sigma_n' u + d\sigma_n u' \end{aligned}$$

and

$$(d(u^n)')' = \sigma_n''(du) + \sigma_n'(du)' + \sigma_n'(du') + \sigma_n(du')'$$

3 so

$$\begin{aligned} & (d(u^n)')' - (\gamma u^n)' \\ &= \sigma_n''(du) + \sigma_n'(du)' + \sigma_n'(du') + \sigma_n(du')' - (\gamma u) \sigma_n' - \sigma_n (\gamma u)' \\ &= \sigma_n \left[ (du')' - (\gamma u)' \right] + [\sigma_n''(du) + \sigma_n'(du') - (\gamma u) \sigma_n'] + 2\sigma_n'(du'). \end{aligned}$$

Since  $(du')' - (\gamma u)' \in L^1(\mathbb{R}_+)$  then

$$\sigma_n \left[ (du')' - (\gamma u)' \right] \rightarrow (du')' - (\gamma u)'$$



in  $L^1(\mathbb{R}_+)$  by the dominated convergence theorem. Note that

$$\begin{aligned}\sup_s |\sigma'_n(s)| &= \sup_s |\sigma'(s)| < +\infty \\ \sup_s |\sigma''_n(s)| &= \sup_s |\sigma''(s)| < +\infty\end{aligned}$$

and the supports of  $\sigma'_n$  and  $\sigma''_n$  are included in  $[n, n+1]$  so

$$\sigma''_n(du) + \sigma'_n(d'u) - (\gamma u) \sigma'_n \rightarrow 0$$

in  $L^1(\mathbb{R}_+)$  in  $L^1(\mathbb{R}_+)$  by the dominated convergence theorem because  $du$ ,  $d'u$  and  $\gamma u$  belong to  $L^1(\mathbb{R}_+)$ . The most tricky term is

$$\sigma'_n(du').$$

Since

$$\lim_{s \rightarrow +\infty} d(s)u'(s) - \gamma(s)u(s) = 0,$$

for any  $\varepsilon > 0$  there exists  $\bar{s} > 0$  such that

$$|d(s)u'(s) - \gamma(s)u(s)| \leq \varepsilon \quad (s \geq \bar{s}).$$

Then

$$|d(s)u'(s)| \leq \varepsilon + |\gamma(s)u(s)| \quad (s \geq \bar{s})$$

1 and

$$\begin{aligned}\int_{\mathbb{R}_+} |\sigma'_n(s)d(s)u'(s)| ds &= \int_n^{n+1} |\sigma'_n(s)d(s)u'(s)| ds \\ &\leq \sup_s |\sigma'(s)| \int_n^{n+1} |d(s)u'(s)| ds \\ &\leq \varepsilon \sup_s |\sigma'(s)| + \sup_s |\sigma'(s)| \int_n^{n+1} |\gamma(s)u(s)| ds\end{aligned}$$

(for  $n$  large enough) so

$$\limsup_{n \rightarrow +\infty} \int_{\mathbb{R}_+} |\sigma'_n(s)d(s)u'(s)| ds \leq \varepsilon \sup_s |\sigma'(s)|$$

2 since  $\gamma u \in L^1(\mathbb{R}_+)$ . Hence  $\sigma'_n(du') \rightarrow 0$  in  $L^1(\mathbb{R}_+)$  since  $\varepsilon$  is arbitrary. This ends  
3 the proof.  $\square$

### 4 3.3. On irreducibility.

5 The main result of this subsection is:

6 **Proposition 2.** *The  $C_0$ -semigroup  $\{U_\infty(t)\}_{t \geq 0}$  is irreducible.*

*Proof.* As for the previous finite case, it suffices to prove that  $(\lambda I - A_\infty)^{-1}$  is positivity improving. Let us show first that

$$(\lambda I - A_\infty)^{-1} \geq 0.$$

Let  $U := (u, u_0) = (\lambda I - A_\infty)^{-1}H$  with  $H = (h, h_0) \in \mathcal{X}_{\infty,+}$  and denote by  $C_c^+([0, \infty])$  the set of nonnegative continuous functions with compact support in  $[0, \infty[$ . Since  $C_c^+([0, \infty])$  is dense in  $L_+^1(0, \infty)$  we may assume without loss of generality that

$$h \in C_c^+([0, \infty]).$$

Since  $h \in (L^2 \cap L^1) \times \mathbb{R}$  then  $u \in H^2(0, \infty)$ . Now

$$\begin{aligned} (\lambda + \mu(s))u(s) + (\gamma u)'(s) - (du')'(s) &= h(s), s \in (0, \infty), \\ (\lambda + \rho_0)u_0 - (b_0 - \gamma(0))u'(0) &= h_0 \end{aligned}$$

shows that  $u'' \in C(0, \infty)$ . We write

$$-u'' + \rho_1 u' + \rho_2 u = \rho_3$$

where  $\rho_1 = -(d' - \gamma)/d$ ,

$$\rho_2(s) = (\lambda + \mu(s) + \gamma'(s))/d(s) > 0 \quad \forall s \text{ for } \lambda \text{ large enough}$$

and

$$\rho_3 = h/d \geq 0.$$

We want to show that  $\inf u \geq 0$ . If  $\inf u < 0$  then the *absolute minimum* of  $u$  is achieved at some  $\bar{s} \in [0, +\infty)$  since  $\lim_{s \rightarrow +\infty} u(s) = 0$ . This implies that  $\bar{s} = 0$  otherwise

$$0 \geq -u''(\bar{s}) = -\rho_2(\bar{s})u(\bar{s}) + \rho_3(\bar{s}) \geq -\rho_2(\bar{s})u(\bar{s}) > 0$$

would lead to a contradiction. But if  $\bar{s} = 0$  then  $u(0) < 0$  and the boundary condition

$$(\lambda + \rho_0)u(0) - (b_0 - \gamma(0))u'(0) = h_0$$

gives

$$-(b_0 - \gamma(0))u'(0) = -(\lambda + \rho_0)u(0) + h_0 \geq -(\lambda + \rho_0)u(0) > 0$$

so  $u'(0) < 0$  and then  $u'(s) < 0$  in the neighborhood of  $s = 0$  which contradicts the fact that the absolute minimum is achieved at 0. Hence

$$\inf u \geq 0.$$

Let us show now that  $(\lambda I - A_\infty)^{-1}$  is *positivity improving*. As for the previous finite case, by using the resolvent identity, we may assume, without loss of generality, that

$$H \in D(A_\infty) \cap \mathcal{X}_+.$$

In particular  $u'' \in C(0, \infty)$ . Let us show that

$$u(s) > 0 \text{ a.e. and } u(0) > 0$$

once

$$H = (h, h_0) \in \mathcal{X}_{\infty,+} - \{0\}.$$

- 1 Let us show by contradiction that  $u > 0$  everywhere. If the absolute minimum of
- 2  $u$  is *not* achieved, then  $u > 0$  since  $u \geq 0$ . Consequently we only need to deal with
- 3 the case where it is achieved at some  $\bar{s} \in [0, \infty)$ .

Suppose  $u(\bar{s}) = 0$ . Since  $H \neq \{0\}$  then either  $h_0 > 0$  or  $\int_0^\infty h(s)ds > 0$ . In any case, let  $\bar{c} > \bar{s}$  such that ( $h_0 > 0$  or  $\int_0^{\bar{c}} h(s)ds > 0$ ). Note that the  $C^2$  function

$$v := -u$$

satisfies the equation

$$v'' - \rho_1 v' + \tilde{\rho}_2 v = h/d \geq 0$$

on  $[0, \bar{c}]$ , where  $\tilde{\rho}_2 \leq 0$ . Note also that

$$\max_{[0, \bar{c}]} v = -\min_{[0, \bar{c}]} u \geq 0.$$

If  $u$  reaches its minimum in  $(0, \bar{c})$  then  $v$  reaches its maximum in  $(0, \bar{c})$ . By the maximum principle (see [27] Theorem 3, p. 6),  $v$  must be constant and then  $u$  is equal to the constant  $u(\bar{s}) = 0$ . It follows that

$$h_0 = 0, \quad h = 0 \text{ on } [0, \bar{c}]$$

which is contradictory.

If  $v$  reaches its maximum (equal to zero) at  $\bar{s} = 0$  then  $v'(0) < 0$  by Hopf's maximum principle (see [27] Theorem 4, p. 7) which is contradictory since

$$(b_0 - \gamma(0))v'(0) = h_0 \geq 0.$$

1 Finally  $u > 0$  everywhere. □

### 2 3.4. Asynchronous exponential growth.

3 The main result of this subsection is:

4 **Theorem 3.2.** *We assume that  $\beta_0(\cdot) \neq 0$ . Let there exist a measurable subset*  
5  *$I \subset \mathbb{R}_+$  with positive measure such that*

$$u \in L^1(\mathbb{R}_+), \quad u(y) > 0 \text{ a.e.} \implies \int_0^\infty \beta(s, y)u(y)dy > 0 \text{ a.e. } s \in I. \quad (33)$$

6 *If*

$$\lim_{\lambda \rightarrow s(A_\infty)} r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) > 1 \quad (34)$$

7 *then the semigroup  $\{U_\infty(t)\}_{t \geq 0}$  generated by  $A_\infty$  has asynchronous exponential*  
8 *growth.*

*Proof.* Since  $A_\infty$  is resolvent positive and  $K_\infty \geq 0$  then

$$K_\infty(\lambda - A_\infty)^{-1} \leq K_\infty(\mu - A_\infty)^{-1} \quad (\lambda > \mu)$$

9 and

$$(s(A_\infty), +\infty) \ni \lambda \mapsto r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) \quad (35)$$

is nonincreasing. Since  $K_\infty(\lambda - A_\infty)^{-1}$  is weakly compact then  $(K_\infty(\lambda - A_\infty)^{-1})^2$  is compact (see e.g. [9] Corollary VI.13, p. 510). Note that

$$(s(A_\infty), +\infty) \ni \lambda \mapsto r_\sigma(K_\infty(\lambda - A_\infty)^{-1})$$

10 is convex and therefore continuous (see [20] p. 107). Assume *momentarily* that

$$r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) > 0 \quad (\lambda > s(A_\infty)). \quad (36)$$

Then

$$(s(A_\infty), +\infty) \ni \lambda \mapsto r_\sigma(K_\infty(\lambda - A_\infty)^{-1})$$

is *strictly decreasing* (see [20] p. 106). If

$$\lim_{\lambda \rightarrow s(A_\infty)} r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) > 1$$

then there exists a unique

$$\bar{\lambda} > s(A_\infty)$$

such that

$$r_\sigma(K_\infty(\bar{\lambda} - A_\infty)^{-1}) = 1.$$

Since  $K_\infty(\bar{\lambda} - A_\infty)^{-1}$  is positive and power compact then

$$1 = r_\sigma(K_\infty(\bar{\lambda} - A_\infty)^{-1})$$

is an isolated eigenvalue of  $K_\infty(\bar{\lambda} - A_\infty)^{-1}$  associated to a nonnegative eigenfunction  $U$  so

$$K_\infty(\bar{\lambda} - A_\infty)^{-1}U = U.$$

Let

$$V := (\bar{\lambda} - A_\infty)^{-1}U.$$

Then  $V \neq 0$  and

$$K_\infty V = K_\infty(\bar{\lambda} - A_\infty)^{-1}U = U = (\bar{\lambda} - A_\infty)V$$

so

$$A_\infty V = \bar{\lambda}V.$$

As for the previous finite case, the weak compactness of  $K_\infty$  implies that  $\{U_\infty(t)\}_{t \geq 0}$  and  $\{T_\infty(t)\}_{t \geq 0}$  have the same essential type

$$\omega_{ess}(\mathcal{A}_\infty) = \omega_{ess}(A_\infty).$$

Since

$$\omega_{ess}(A_\infty) \leq s(A_\infty)$$

then

$$\omega_{ess}(\mathcal{A}_\infty) \leq s(A_\infty) < \bar{\lambda} = s(\mathcal{A}_\infty).$$

Thus  $\{U_\infty(t)\}_{t \geq 0}$  exhibits a spectral gap and consequently  $\{U_\infty(t)\}_{t \geq 0}$  has asynchronous exponential growth since it is irreducible. Finally, we have just to check (36). To this end, let  $\bar{K} \in \mathcal{L}(\mathcal{X}_\infty)$  be defined by

$$\bar{K} \begin{pmatrix} u \\ u_0 \end{pmatrix} = \begin{pmatrix} \chi_I(s) \int_0^\infty \beta(s, y) u(y) dy \\ \int_0^\infty \beta_0(y) u(y) dy \end{pmatrix}.$$

where  $\chi_I$  is the indicator function of  $I$ . Then

$$K(\lambda - A_\infty)^{-1} \geq \bar{K}(\lambda - A_\infty)^{-1}.$$

We identify  $L^1(I)$  to the closed subspace of  $L^1(\mathbb{R}_+)$  of functions vanishing a.e. outside  $I$ . Let

$$\mathcal{X}_\infty^I := L^1(I) \times \mathbb{R} \subset \mathcal{X}_\infty.$$

Since

$$\bar{K}(\lambda - A_\infty)^{-1} : \mathcal{X}_\infty \rightarrow \mathcal{X}_\infty^I$$

then

$$\bar{K}(\lambda - A_\infty)|_{\mathcal{X}_\infty^I}^{-1} : \mathcal{X}_\infty^I \rightarrow \mathcal{X}_\infty^I$$

and

$$K(\lambda - A_\infty)^{-1} \geq \bar{K}(\lambda - A_\infty)|_{\mathcal{X}_\infty^I}^{-1}$$

so

$$r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) \geq r_\sigma(\bar{K}(\lambda - A_\infty)|_{\mathcal{X}_\infty^I}^{-1}).$$

Since  $(\lambda - A_\infty)^{-1} : \mathcal{X}_\infty \rightarrow \mathcal{X}_\infty$  is positivity improving then our assumptions on  $\beta_0$  and  $\beta$  imply that

$$\bar{K}(\lambda - A_\infty)|_{\mathcal{X}_\infty^I}^{-1} : \mathcal{X}_\infty^I \rightarrow \mathcal{X}_\infty^I$$

is positivity improving too. Since  $\bar{K}(\lambda - A_\infty)|_{\mathcal{X}_\infty^I}^{-1}$  is weakly compact then

$\left(\bar{K}(\lambda - A_\infty)|_{\mathcal{X}_\infty^I}^{-1}\right)^2$  is compact (see e.g. [9] Corollary VI.13, p. 510) and irreducible so

$$r_\sigma \left[ \left( \bar{K}(\lambda - A_\infty)|_{\mathcal{X}_\infty^I}^{-1} \right)^2 \right] > 0$$

(see e.g. [25] Theorem 3) and finally

$$r_\sigma \left[ \overline{K}(\lambda - A_\infty)_{|\mathcal{X}_\infty^I}^{-1} \right] > 0.$$

1 This shows (36) and ends the proof.

2

□

**Remark 6.** Note that if

$$\lim_{\lambda \rightarrow s(A_\infty)} r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) \leq 1$$

then  $r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) < 1$  ( $\lambda > s(A_\infty)$ ) and

$$(\lambda I - \mathcal{A}_\infty)^{-1} = (\lambda I - A_\infty - K_\infty)^{-1} = (\lambda I - A_\infty)^{-1} \sum_{n=0}^{\infty} (K_\infty(\lambda I - A_\infty)^{-1})^n$$

3 ( $\forall \lambda > s(A_\infty)$ ) shows that  $s(\mathcal{A}_\infty) \leq s(A_\infty)$ . In fact  $s(\mathcal{A}_\infty) = s(A_\infty)$  since

4  $s(\mathcal{A}_\infty) \geq s(A_\infty)$  due to  $K_\infty \geq 0$ .

5 **Remark 7.** Roughly speaking Theorem 3.2 expresses that  $\{U_\infty(t)\}_{t \geq 0}$  has asyn-  
6 chronous exponential growth once  $s(\mathcal{A}_\infty) > s(A_\infty)$ . We mention that the spectral  
7 bound of generators of perturbed positive semigroups is characterized in [33] (see  
8 also [32]). Note that  $s(A_\infty)$  is not known explicitly. In case  $s(A_\infty) = 0$ , then (34)  
9 could be interpreted in terms of the basic reproduction number  $\mathcal{R}_0$  (see [32]), we  
10 thank one of the referees for drawing our attention to this fact.

**Remark 8.** Note that  $K_\infty(\lambda - A_\infty)^{-1}$  and  $(\lambda - A_\infty)^{-1}K_\infty$  have the same non-zero spectrum (see e.g. [1] p. 196) and consequently

$$r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) = r_\sigma((\lambda - A_\infty)^{-1}K_\infty).$$

On the other hand,  $(\lambda - A_\infty)^{-1}K_\infty$  is never positivity improving since

$$K_\infty \begin{pmatrix} 0 \\ u_0 \end{pmatrix} = 0 \quad \forall u_0 \in \mathbb{R}.$$

11 We end this subsection by a useful criterion to estimate a spectral radius.

**Lemma 3.3.** *Let*

$$\beta(x, y) = \beta_1(x)\beta_2(y)$$

where  $\beta_1 \in L^1(0, \infty)$  and  $\beta_2 \in L^\infty(0, \infty)$ . We assume that  $\beta_1$  is continuous at 0. Then for every  $\lambda > s(A_\infty)$

$$r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) = \left\| \beta_2 \left( (\lambda - A_\infty)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right) \right\|_{L^1(\mathbb{R}_+)}.$$

*Proof.* We know that

$$\begin{aligned} K_\infty(\lambda - A_\infty)^{-1} \begin{pmatrix} f \\ f_0 \end{pmatrix} &= \begin{pmatrix} \beta_1(\cdot) \left\| \beta_2 \left( (\lambda - A_\infty)^{-1} \begin{pmatrix} f \\ f_0 \end{pmatrix} \right) \right\|_{L^1} \\ \beta_1(0) \left\| \beta_2 \left( (\lambda - A_\infty)^{-1} \begin{pmatrix} f \\ f_0 \end{pmatrix} \right) \right\|_{L^1} \end{pmatrix} \\ &= \left\| \beta_2 \left( (\lambda - A_\infty)^{-1} \begin{pmatrix} f \\ f_0 \end{pmatrix} \right) \right\|_{L^1} \begin{pmatrix} \beta_1(\cdot) \\ \beta_1(0) \end{pmatrix} \end{aligned}$$

so  $K_\infty(\lambda - A_\infty)^{-1}$  is a one-rank operator with a single non-zero eigenvalue

$$\left\| \beta_2 \left( (\lambda - A_\infty)^{-1} \begin{pmatrix} \beta_1(\cdot) \\ \beta_1(0) \end{pmatrix} \right) \right\|_{L^1}$$

associated to eigenvector

$$\begin{pmatrix} \beta_1(\cdot) \\ \beta_1(0) \end{pmatrix}.$$

Hence

$$r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) = \left\| \beta_2 \left( (\lambda - A_\infty)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right) \right\|_{L^1(\mathbb{R}_+)}.$$

1

□

**Remark 9.** Note that if the kernel  $\beta$  is not separable but is bounded below by a separable kernel, i.e.

$$\beta(x, y) \geq \beta_1(x)\beta_2(y),$$

then a simple domination argument shows

$$r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) \geq \left\| \beta_2 \left( (\lambda - A_\infty)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right) \right\|_{L^1(\mathbb{R}_+)}.$$

Simplified models (with constant coefficients) are dealt with in [28] to check the property

$$\lim_{\lambda \rightarrow s(A_\infty)} \left\| \beta_2 \left( (\lambda - A_\infty)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right) \right\|_{L^1(\mathbb{R}_+)} = +\infty.$$

2

Received xxxx 20xx; revised xxxx 20xx.

3

## REFERENCES

- 4 [1] Y. A. Abramovich and C. D. Aliprantis, *Problems in Operator Theory*, vol. 51 of Graduate  
5 Studies in Mathematics, American Mathematical Society, Providence, RI, 2002.
- 6 [2] T. M. Apostol, *Mathematical Analysis, Second Edition*, Reading, Addison-Wesley Publishing  
7 Co., 1974.
- 8 [3] A. Bartłomiejczyk and H. Leszczyński, Method of lines for physiologically structured models  
9 with diffusion, *Appl. Numer. Math.*, **94** (2015), 140–148.
- 10 [4] A. Bartłomiejczyk and H. Leszczyński, Structured populations with diffusion and Feller con-  
11 ditions, *Math. Biosci. Eng.*, **13** (2016), 261–279.
- 12 [5] A. Bobrowski, *Convergence of One-Parameter Operator Semigroups*, vol. 30 of New Mathe-  
13 matical Monographs, Cambridge University Press, Cambridge, 2016.
- 14 [6] A. Calsina and J. Z. Farkas, Steady states in a structured epidemic model with Wentzell  
15 boundary condition, *J. Evol. Equ.*, **12** (2012), 495–512.
- 16 [7] A. Calsina and J. Z. Farkas, On a strain-structured epidemic model, *Nonlinear Anal. Real  
17 World Appl.*, **31** (2016), 325–342.
- 18 [8] P. Clément, H. Heijmans, S. Angenent, C. J. van Duijn and B. de Pagter, *One-Parameter  
19 Semigroups*, vol. 5, North-Holland Publishing Co.(Amsterdam), 1987.
- 20 [9] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, Pure and Applied  
21 Mathematics, Vol. 7, Interscience Publishers, Inc., New York; Interscience Publishers, Ltd.,  
22 London, 1958.
- 23 [10] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, vol.  
24 63, Springer-Verlag, 2000.
- 25 [11] J. Z. Farkas and P. Hinow, Physiologically structured populations with diffusion and dynamic  
26 boundary conditions, *Mathematical Biosciences and Engineering*, **8** (2011), 503–513.
- 27 [12] A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli,  $C_0$ -semigroups generated by  
28 second order differential operators with general Wentzell boundary conditions, *Proc. Amer.  
29 Math. Soc.*, **128** (2000), 1981–1989.
- 30 [13] W. Feller, Diffusion processes in one dimension, *Trans. Amer. Math. Soc.*, **77** (1954), 1–31.
- 31 [14] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*,  
32 Springer Berlin, Germany, 1983.

- 1 [15] K. P. Hadeler, Structured populations with diffusion in state space, *Mathematical Biosciences*  
2 *and Engineering*, **7** (2010), 37–49.
- 3 [16] A. Kolmogorov, I. Petrovskii and N. Piscunov, A study of the equation of diffusion with  
4 increase in the quantity of matter, and its application to a biological problem, *Byul.*  
5 *Moskovskogo Gos. Univ.*, **1** (1937), 1–25. Available from: [https://biomath.usu.edu/files/](https://biomath.usu.edu/files/2pd.pdf)  
6 [2pd.pdf](https://biomath.usu.edu/files/2pd.pdf).
- 7 [17] P. Magal and S. Ruan, *Structured Population Models in Biology and Epidemiology*, Lecture  
8 Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2008.
- 9 [18] I. Marek, Frobenius theory of positive operators: Comparison theorems and applications,  
10 *SIAM Journal on Applied Mathematics*, **19** (1970), 607–628.
- 11 [19] J. A. Metz and O. Diekmann, *The Dynamics of Physiologically Structured Populations*, Lec-  
12 ture Notes in Biomathematics, Springer Berlin Heidelberg, 1986.
- 13 [20] M. Mokhtar-Kharroubi, *Mathematical Topics in Neutron Transport Theory: New Aspects*,  
14 vol. 46, World Scientific, 1997.
- 15 [21] M. Mokhtar-Kharroubi, On the convex compactness property for the strong operator topology  
16 and related topics, *Mathematical methods in the applied sciences*, **27**, (2004), 687–701.
- 17 [22] M. Mokhtar-Kharroubi, Spectral theory for neutron transport, in *Evolutionary Equations with*  
18 *Applications in Natural Sciences* (eds. J. Banasiak and M. Mokhtar-Kharroubi), Springer,  
19 2015, 319–386.
- 20 [23] J. D. Murray, *Mathematical Biology*, Biomathematics, Springer Verlag, Heiderberg, 1989.
- 21 [24] R. Nagel, W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas,  
22 F. Neubrander and U. Schlotterbeck, *One-Parameter Semigroups of Positive Operators*, vol.  
23 1184, Springer-Verlag Berlin, 1986.
- 24 [25] B. de Pagter, Irreducible compact operators, *Mathematische Zeitschrift*, **192** (1986), 149–153.
- 25 [26] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*,  
26 Applied Mathematical Sciences, Springer New York, 1983.
- 27 [27] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer  
28 Verlag, 1984.
- 29 [28] Q. Richard, Work in progress.
- 30 [29] G. Schlichtermann, On weakly compact operators, *Mathematische Annalen*, **292** (1992),  
31 263–266.
- 32 [30] J. W. Sinko and W. Streifer, A new model for age-size structure of a population, *Ecology*, **48**  
33 (1967), 910–918.
- 34 [31] J. G. Skellam, The formulation and interpretation of mathematical models of diffusional  
35 processes in population biology, in *The mathematical theory of the dynamics of biological*  
36 *populations*, New York: Academic press, 1973, 63–85.
- 37 [32] H. R. Thieme, Spectral bound and reproduction number for infinite-dimensional population  
38 structure and time heterogeneity, *SIAM Journal on Applied Mathematics*, **70** (2009), 188–  
39 211.
- 40 [33] J. Voigt On resolvent positive operators and positive  $C_0$ -semigroups on AL-spaces, *Semigroup*  
41 *Forum*, **38** (1989), 263–266.
- 42 [34] R. Waldstätter, K. P. Hadeler and G. Greiner, A Lotka-McKendrick model for a population  
43 structured by the level of parasitic infection, *SIAM J. Math. Anal.*, **19** (1988), 1108–1118.
- 44 [35] G. F. Webb, An operator-theoretic formulation of asynchronous exponential growth, *Trans-*  
45 *actions of the American Mathematical Society*, **303** (1987), 751–763.
- 46 [36] G. F. Webb, Population models structured by age, size, and spatial position, in *Struc-*  
47 *tured Population Models in Biology and Epidemiology*, vol. 1936 of Lecture Notes in Math.,  
48 Springer, Berlin, 2008, 1–49.
- 49 [37] L. W. Weis, A generalization of the Vidav-Jorgens perturbation theorem for semigroups and  
50 its application to transport theory, *Journal of Mathematical Analysis and Application*, **129**  
51 (1988), 6–23.
- 52 [38] A. D. Wentzell, On boundary conditions for multi-dimensional diffusion processes, *Theor.*  
53 *Probability Appl.*, **4** (1959), 164–177.
- 54 *E-mail address:* [mustapha.mokhtar-kharroubi@univ-fcomte.fr](mailto:mustapha.mokhtar-kharroubi@univ-fcomte.fr)
- 55 *E-mail address:* [quentin.richard@univ-fcomte.fr](mailto:quentin.richard@univ-fcomte.fr)