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# Enumeration of walks with small steps avoiding a quadrant

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**Abstract.** We address the enumeration of walks with weighted small steps avoiding a quadrant. In particular we give an exact, integral-expression solution for the generating function C(x, y; t) counting these walks by length and end-point. Moreover, we determine precisely when this generating function is algebraic, D-finite or D-algebraic with respect to x, showing that this complexity is the same as for walks in the quarter-plane with the same starting point, as long as the starting point (p,q) of the walks lies in the quarter plane then. Finally, we give an integral-free expression for the solution in the cases where (p,q) lies just outside the quarter plane, that is p=0 or q=0 with our convention, proving a conjecture of Raschel and Trotignon.

**Keywords:** Lattice path, elliptic function, cone

## 1 Introduction

The systematic study of walks with small steps in the quarter plane was initiated by Bousquet-Mélou and Mishna in 2010 [5], and since then there has been great progress on the model [2, 14, 19, 18, 17, 1, 16, 8, 9]. The model is defined as follows: given a step set  $S \subset \{-1,0,1\}^2 \setminus \{(0,0)\}$ , determine the generating function

$$Q(x,y;t) = \sum_{n=0}^{\infty} \sum_{i,j\geq 1} q(i,j;n)t^n x^i y^j,$$

where q(i, j; n) is the number of walks of length n, starting at (1, 1), and ending at (i, j) using steps in S and staying in the strictly positive quadrant. A priori, there are 256 distinct step sets S, but after removing duplicates and cases that are equivalent to half-plane models, Bousquet-Mélou and Mishna indentified 79 non-trivial and combinatorially distinct models. The study of these models is now in some sense complete as it is known for each S precisely where the generating function fits into the hierarchy

Algebraic 
$$\subset$$
 D-finite  $\subset$  D-Algebraic.

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<sup>&</sup>lt;sup>1</sup>Note that most of the literature has considered the equivalent question of walks starting at (0,0) and staying in the non-strictly positive quadrant, for which the resulting generating function is  $\frac{1}{xy}Q(x,y;t)$ .

Recall that a generating function G(x) is called Algebraic with respect to the variable x if it satisfies some non-trivial polynomial equation P(G(x),x)=0, whose coefficients do not depend on that variable (but may depend on other variables), and it is called D-finite (resp. D-algebraic) if it satisfies a linear (resp. polynomial) differential equation with respect to x whose coefficients are polynomials in x. For a multivariate series to be algebraic (resp. D-finite, D-algebraic) it must be algebraic (resp. D-finite, D-algebraic) with respect to each variable.

Of the 79 models proposed by Mishna and Bousquet-Mélou, 4 models admit an algebraic generating function [5, 2], 19 further models admit a D-finite generating function [5, 14], 9 further models admit a D-algebraic generating function [1, 16] and the remaining 47 models admit a generating function which is not D-algebraic [8, 9]. Moreover, in the 74 cases known as *non-singular*, an exact integral expression is known for the generating function [19], while other exact expressions are known in the 5 singular cases [18, 17]. In recent years a number of articles have focused on the equivalent question for walks avoiding a quadrant [4, 20, 6, 11, 3], that is determining the generating function C(x, y; t) which counts walks starting at (1, 1) whose intermediate points are required to lie in the three-quadrant cone

$$C = \{(i, j) : i > 0 \text{ or } j > 0\}.$$

Between them these 5 articles classify 10 models into the complexity heirarchy (algebraic, D-finite, D-algberaic), while excursions have been enumerated for 4 further models [7, 12]. Remarkably the generating function has the same nature in each case as in the quarter plane, a fact that led Dreyfus and Trotignon to conjecture that the nature is the same for any of the 74 non-singular step-sets S [9]. We give exact integral expression solutions for C(x,y;t) in each of these 74 cases, analogous to those of Raschel in the quarter-plane [19]. We then prove that the nature of C(x,y;t) as a function of x (or y) is the same as that of Q(x,y;t), and we conjecture that these series also have the same nature as functions of t. In fact we do this in the more general setting of walks with weighted steps and starting at any point in the positive quadrant or on the positive x-axis.

Note that by our definition, steps directly between (1,0) and (0,1) are allowed, whereas they are forbidden in [6], for example. We do not expect this to affect the nature of the generating function.

## 2 Functional equations for walks avoiding a quadrant

We start with a step-set  $S \subset \{-1,0,1\}^2 \setminus \{(0,0)\}$ , a weight  $w_s > 0$  for each  $s \in S$  and a starting point (p,q) with p > 0,  $q \ge 0$ . We will determine the generating function C(x,y;t) counting walks starting at (p,q), taking steps from S with all intermediate

points lying in the three-quadrant cone C and with the weight of the walk being the product of the weights  $w_s$  of the steps. Note that the standard starting point is (p,q) = (1,1) and in the unweighted case  $w_s = 1$  for each  $s \in S$ .

The following lemma results from considering the final step of a walk counted by C(x, y; t):

**Lemma 1.** Define the single step generating function P(x,y) by

$$\mathsf{P}(x,y) = \sum_{(lpha,eta) \in S} w_{(lpha,eta)} x^lpha y^eta$$

Then there are series  $A_H(\frac{1}{x};t) \in \frac{t}{x}\mathbb{Z}[\frac{1}{x}][[t]]$ ,  $A_V(\frac{1}{y};t) \in \frac{t}{y}\mathbb{Z}[\frac{1}{y}][[t]]$  and  $B(t) \in t\mathbb{Z}[[t]]$  which satisfy

$$C(x,y;t) = x^p y^q + tP(x,y)C(x,y;t) - B(t) - A_H\left(\frac{1}{x};t\right) - A_V\left(\frac{1}{y};t\right). \tag{2.1}$$

Moreover, this equation together with the fact that c(i,j;n) = 0 for  $i,j \leq 0$ , characterises the generating function

$$C(x,y;t) = \sum_{t\geq 0} \sum_{i,j\in\mathbb{Z}} c(i,j;n) x^i y^j t^n,$$

as well as the series  $A_H$ ,  $A_V$  and B.

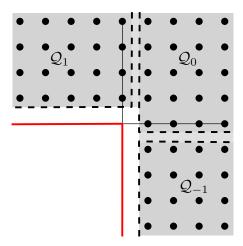
The series  $A_H$ ,  $A_V$  and B in the lemma above can be understood combinatorially: They count walks starting at (p,q) and ending just outside C whose intermediate points all lie within C. More precisely,  $A_H(\frac{1}{x};t)$  counts those walks ending on the negative x-axis,  $A_V(\frac{1}{y};t)$  counts those walks ending on the negative y-axis, and B(t) counts those walks ending at (0,0).

The unusual condition that the coefficients c(i,j;n) of C(x,y;t) vanish for  $i,j \leq 0$  makes this equation difficult to solve directly, so we partition  $\mathcal{C}$  into three quadrants  $\mathcal{Q}_{-1} = \{(i,j): i > 0, \ j < 0\}, \ \mathcal{Q}_0 = \{(i,j): i > 0, \ j \geq 0\}$  and  $\mathcal{Q}_1 = \{(i,j): i \leq 0, \ j > 0\}$ , as shown in figure 1. A similar decomposition was used in [4, 6], but we have shifted the quadrants  $\mathcal{Q}_{-1}$ ,  $\mathcal{Q}_0$  down one space compared to those articles so that it is impossible to step directly between  $\mathcal{Q}_{-1}$  and  $\mathcal{Q}_1$  and so that our condition on the starting point (p,q) is now that  $(p,q) \in \mathcal{Q}_0$ .

Now, for j = -1, 0, 1, we define  $Q_j(x, y; t)$  to be the generating function counting walks in C, starting at (p, q) and ending in  $Q_j$ , so

$$C(x,y;t) = Q_{-1}(x,y;t) + Q_0(x,y;t) + Q_1(x,y;t),$$

and  $Q_{-1} \in \frac{x}{y}\mathbb{Z}\left[x,\frac{1}{y}\right][[t]]$ ,  $Q_0 \in x\mathbb{Z}\left[x,y\right][[t]]$  and  $Q_1 \in y\mathbb{Z}\left[\frac{1}{x},y\right][[t]]$ . The following lemma rewrites (2.1) as three equations characterising  $Q_{-1}(x,y;t)$ ,  $Q_0(x,y;t)$  and  $Q_1(x,y;t)$ .



**Figure 1:** The three-quadrant cone C partitioned into three quadrants  $Q_1$ ,  $Q_0$  and  $Q_{-1}$ .

**Lemma 2.** Define the kernel K(x,y;t) = tP(x,y) - 1. There are series  $V_1(y;t), V_2(y;t) \in \mathbb{Z}[y][[t]]$  and  $H_1(x;t), H_2(x;t) \in \mathbb{Z}[x][[t]]$  satisfying the three equations

$$K(x,y;t)Q_{-1}(x,y;t) = A_V\left(\frac{1}{y};t\right) + H_1(x;t) + \frac{1}{y}H_2(x;t)$$
(2.2)

$$K(x,y;t)Q_0(x,y;t) = -xy + B(t) - V_1(y;t) - xV_2(y;t) - H_1(x;t) - \frac{1}{y}H_2(x;t)$$
 (2.3)

$$K(x,y;t)Q_1(x,y;t) = A_H\left(\frac{1}{x};t\right) + V_1(y;t) + xV_2(y;t).$$
(2.4)

Moreover, these three equations characterise the series  $Q_{-1}(x,y;t)$ ,  $Q_0(x,y;t)$ ,  $Q_1(x,y;t)$ ,  $V_1(y;t)$ ,  $V_2(y;t)$ ,  $H_1(x;t)$ ,  $H_2(x;t)$ ,  $A_H\left(\frac{1}{x};t\right)$ ,  $A_V\left(\frac{1}{y};t\right)$  and B(t).

Combinatorially, the series  $(V_1(0;t) - V_1(y;t))$  (resp.  $xV_2(y;t)$ ,  $(H_1(x;t) - H_1(0,t))$ ,  $\frac{1}{y}(H_2(0;t) - H_2(x;t))$ ) counts walks whose final step is from  $Q_0$  (resp.  $Q_1$ ,  $Q_{-1}$ ,  $Q_0$ ) to  $Q_1$  (resp.  $Q_0$ ,  $Q_0$ ,  $Q_{-1}$ ).

#### 2.1 Parameterisation of the kernel curve

Following the method used in the quarter plane pioneered by Fayolle and Raschel [13, 19] we start by fixing  $t \in \left(0, \frac{1}{\sum_{s \in S} w_s}\right)$  and then we consider the curve  $\mathcal{W} = \{(x,y) : K(x,y;t) = 0\}$ . From now on, we will make the following assumption:

**Assumption:** *S* is a non-singular step-set. That is, for any line  $\ell$  through the origin, at least one element of *S* lies on each side of  $\ell$ .

As explained in [6], if *S* did not have this property, the generating function C(x, y; t) would be algebraic.

Under this assumption, the curve W is known to have genus 1, so we will be able to parameterise it using elliptic functions X(z) and Y(z). More precisely the following lemma follows from [10, Proposition 2.1, Lemma 2.6].

**Lemma 3.** There are meromorphic functions  $X(z), Y(z) : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$  and numbers  $\gamma, \tau \in i\mathbb{R}$  with  $\Im(\pi\tau) > \Im(2\gamma) > 0$  satisfying the following conditions

- K(X(z), Y(z)) = 0
- $X(z) = X(z + \pi) = X(z + \pi\tau) = X(-\gamma z)$
- $Y(z) = Y(z + \pi) = X(z + \pi \tau) = Y(\gamma z)$
- $|X(-\frac{\gamma}{2})|, |Y(\frac{\gamma}{2})| < 1$
- Counting with multiplicity, the functions X(z) and Y(z) each contain two poles and two roots in each fundamental domain  $\{z_c + r_1\pi + r_2\pi\tau : r_1, r_2 \in [0,1)\}$ .

Moreover, X(z) and Y(z) are differentially algebraic with respect to z.

We intend to substitue  $x \to X(z)$  and  $y \to Y(z)$  into (2.2), (2.3) and (2.4), however we can only do this as long as the series in these equations converge, which occurs as long as  $|x| \le 1 \le |y|$  for (2.2),  $|x|, |y| \le 1$  for (2.3) and  $|y| \le 1 \le |x|$  for (2.4). So to substitute  $x \to X(z)$  and  $y \to Y(z)$ , we need to understand how |X(z)| and |Y(z)| compare to 1, for which we use the following lemma, which follows from [10, Lemma 2.9].

**Lemma 4.** The complex plane can be partitioned into simply connected regions  $\{\Omega_s\}_{s\in\mathbb{Z}}$  (see Figure 2) satisfying

$$\bigcup_{s\in\mathbb{Z}}\Omega_{4s}\cup\Omega_{4s+1}=\{z\in\mathbb{C}:|Y(z)|<1\},$$

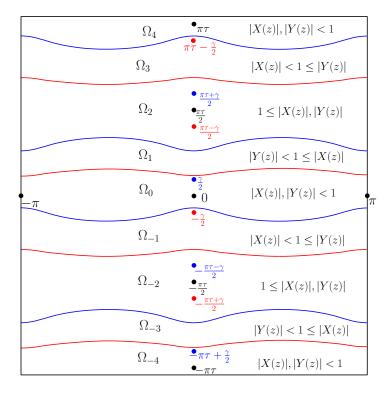
$$\bigcup_{s\in\mathbb{Z}}\Omega_{4s}\cup\Omega_{4s-1}=\{z\in\mathbb{C}:|X(z)|<1\}$$

and for  $s \in \mathbb{Z}$ ,

$$egin{aligned} \pi+\Omega_s&=\Omega_s,\ s\pi au+\gamma-\Omega_{2s}\cup\Omega_{2s+1}&=\Omega_{2s}\cup\Omega_{2s+1},\ s\pi au-\gamma-\Omega_{2s}\cup\Omega_{2s-1}&=\Omega_{2s}\cup\Omega_{2s-1}. \end{aligned}$$

## 2.2 Analytic reformulation of functional equations

Using the results in the previous section, we can substitute x = X(z) and y = Y(z) into (2.2), (2.3) and (2.4) for z in the regions  $\Omega_{-1}$ ,  $\Omega_0$  and  $\Omega_1$ , respectively, yielding (2.9), (2.10) and (2.11) in the following proposition:



**Figure 2:** The complex plane partitioned into regions  $\Omega_j$ . For z on the blue lines, |Y(z)| = 1, while on the red lines |X(z)| = 1.

#### **Proposition 1.** *The functions*

$$L_H(z) := H_1(X(z);t) + \frac{1}{Y(z)}H_2(X(z);t),$$
 for  $z \in \Omega_0 \cup \Omega_{-1}$ , (2.5)

$$L_V(z) := V_1(Y(z);t) + X(z)V_2(Y(z);t),$$
 for  $z \in \Omega_0 \cup \Omega_1$ , (2.6)

$$P_V(z) := A_V\left(\frac{1}{Y(z)}; t\right),$$
 for  $z \in \Omega_{-1} \cup \Omega_{-2},$  (2.7)

$$P_H(z) := A_H\left(\frac{1}{X(z)}; t\right),$$
 for  $z \in \Omega_1 \cup \Omega_2.$  (2.8)

are well defined and satisfy the equations

$$0 = P_V(z) + L_H(z) \qquad \qquad \text{for } z \in \Omega_{-1}, \tag{2.9}$$

$$0 = -X(z)^{p}Y(z)^{q} + B(t) - L_{V}(z) - L_{H}(z) \qquad \text{for } z \in \Omega_{0},$$
 (2.10)

$$0 = P_H(z) + L_V(z) \qquad \qquad \text{for } z \in \Omega_1$$
 (2.11)

$$P_H(z) = P_H(\pi \tau - \gamma - z) = P_H(z + \pi)$$
 (2.12)

$$P_V(z) = P_V(-\pi\tau + \gamma - z) = P_V(z + \pi)$$
 (2.13)

While these equations are apriori defined on different sets, they can be used to show that the functions extend meromorphically to all of  $\mathbb{C}$ , and so the equations hold on all of  $\mathbb{C}$ . Simply taking the sum of the three equations (2.9), (2.10) and (2.11) yields (2.14) in the theorem below.

**Theorem 1.** The functions  $P_H(z)$  and  $P_V(z)$  extend to meromorphic functions on  $\mathbb{C}$  which, along with the constant B(t), are uniquely defined by the equation

$$X(z)^{p}Y(z)^{q} = P_{V}(z) + B(t) + P_{H}(z),$$
(2.14)

along with (2.12), (2.13) and the conditions

- $P_H(z)$  has no poles in  $\Omega_0 \cup \Omega_1 \cup \Omega_2$ ,
- the poles of X(z) for  $z \in \Omega_1 \cup \Omega_2$  are roots of  $P_H(z)$ ,
- $P_V(z)$  has no poles in  $\Omega_0 \cup \Omega_{-1} \cup \Omega_{-2}$ ,
- the poles of Y(z) for  $z \in \Omega_{-1} \cup \Omega_{-2}$  are roots of  $P_Y(z)$ .

Note that combining (2.14), (2.12), (2.13) yields

$$P_H(2\pi\tau - 2\gamma + z) - P_H(z) = W(z),$$
 (2.15)

where W(z) is an elliptic function with periods  $\pi$  and  $\pi\tau$  given by

$$W(z) := (X(z-2\gamma)^p - X(z)^p) Y(z)^q.$$

## 3 Solving the functional equation

In the previous section we reduced the problem to finding the unique meromorphic functions  $P_V, P_H : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$  and constant B(t) characterised by Theorem 1 (for each t), as these determine  $A_H(\frac{1}{x}, t)$  and  $A_V(\frac{1}{y}, t)$  using 2.8 and 2.7, respectively, after which  $\mathbb{C}(x, y; t)$  is determined by (2.1).

an equation analogous to (2.14) was found by Raschel for walks in the quarter plane [19], the only difference being that the transformations  $z \to \pi\tau - \gamma - z$  and  $z \to -\pi\tau + \gamma - z$  here are  $z \to \gamma - z$  and  $z \to -\gamma - z$  in the quarter plane. Raschel used this equation to derive an integral-expression solution determining Q(x, y; t), and the equation has since been used to determine precisely when Q(x, y; t) is differentially algebraic [1, 8, 15] and to determine when it is algebraic or D-finite with respect to x or y [14, 16].

Due to this striking similarity we were able to use these methods to prove the same results for C(x,y;t), in particular showing that it is D-algebraic in exactly the same cases as Q(x,y;t) and that it is algebraic or D-finite with respect to x (or y) in the same cases as Q(x,y;t). We note that Fayolle and Raschel also showed that the unweighted models that are algebraic or D-finite with respect to t have the same nature with respect to t, however these results relied on the precise ratios  $\frac{\pi\tau}{\gamma}$  that could occur in these cases, so they do not apply so readily to our equation. Nonetheless, we expect that the same result holds for C(x,y;t).

#### 3.1 Integral expression

In this section we give integral expressions analogous to those of Raschel [19] which determine  $P_V(z)$  and  $P_H(z)$  exactly. In order to write these expressions, we define an auxiliary elliptic function  $\omega(z)$  which satisfies

$$\omega(z) = \omega(\pi\tau - \gamma - z) = \omega(-\pi\tau + \gamma - z) = \omega(z + \pi)$$

and shares the poles of X(z) in  $\Omega_1 \cup \Omega_2$ , while  $1/\omega(z)$  has the same poles as Y(z) in  $\Omega_{-1} \cup \Omega_{-2}$ . Finally,  $\omega(z)/X(z)$  converges to 1 at the poles of X(z) in  $\Omega_1 \cup \Omega_2$ .

**Theorem 2.** Let  $z_0 \in \Omega_0$  and let  $\mathcal{L}$  be a path from  $z_0$  to  $z_0 + \pi$  contained in the closure  $\overline{\Omega_0}$  of  $\Omega_0$ . Then  $P_V(z)$ ,  $P_H(z)$  and B(t) are given by the integrals

$$P_H(u) = \frac{1}{2\pi i t} \int_{\mathcal{L}} X(z)^p Y(z)^q \frac{\omega'(z)}{\omega(z) - \omega(u)} dz \qquad \text{for } u \in \Omega_1 \cup \Omega_2$$
 (3.1)

$$P_{V}(u) = -\frac{1}{2\pi i t} \int_{\mathcal{L}} X(z)^{p} Y(z)^{q} \frac{\omega(u)}{\omega(z)} \frac{\omega'(z)}{\omega(z) - \omega(u)} dz \qquad \text{for } u \in \Omega_{-1} \cup \Omega_{-2}$$
 (3.2)

$$B(t) = -\frac{1}{2\pi i t} \int_{\mathcal{L}} X(z)^p Y(z)^q \frac{\omega'(z)}{\omega(z)} dz$$
(3.3)

The proof of this theorem involves checking that these expressions satisfy the conditions in Theorem 1.

## 3.2 Classification of C(x, y; t) into complexity hierarchy

In this section we very briefly describe the properties of the step set S, or equivalently X(z) and Y(z), which determine the nature of the generating function C(x,y;t).

In certain cases, called *finite group* cases,  $\gamma = \frac{M}{N}\pi\tau$  for some positive  $M, N \in \mathbb{Z}$  independent of t. Then applying (2.15) N times yields

$$P_H(2\pi\tau(N-M)+z) - P_H(z) = P_H((2\pi\tau-2\gamma)N+z) - P_H(z) = E(z),$$

where E(z) is called the orbit sum of the model. We can use this equation to solve for  $P_H(z)$ , yielding an expression for C(x,y;t) which is D-finite in x. in the cases where E(z) = 0, we can even prove that C(x,y;t) is algebraic in x.

In all other cases we have  $\frac{\gamma}{\pi\tau} \notin \mathbb{Q}$  for generic t. The model is then said to decouple if there are rational functions  $R_1$ ,  $R_2$  satisfying  $X(z)^p Y(z)^q = R_1(X(z)) + R_2(Y(z))$ . These cases can be solved (with integral-free expressions) as, using (2.14), the function

$$f(z) := R_1(X(z)) - P_H(z) = -R_2(Y(z)) + B(t) + P_v(z)$$

is an elliptic function which can be determined exactly. In these cases every function used is D-algebraic in all of its variables.

Finally in the infinite group cases which do not decouple, the generating function can be shown to be non-D-algebraic in x using galois theory of q-difference equations, as in [8, 15] for Q(x, y; t).

#### 3.3 Special case: walks starting on x-axis

We will now study the special cases where the walk starts at some point (p,0) for p > 0. Trotignon and Raschel conjectured that with this starting point all finite group models admit algebraic generating functions [20]. Indeed, if q = 0 it is easy to see that the orbit sum E(z) defined is section 3.2 is equal to 0, so this follows from our more general results. Moreover, if q = 0, the model trivially decouples, so even in the infinite group case the generating function C(x, y; t) is D-algebraic.

The following lemma follows directly from Theorem 1, where  $\omega(z)$  is defined as in Section 3.1.

**Lemma 5.** If the starting point of the walks is (p,0) for some  $p \ge 1$ , then there is a degree p polynomial H satisfying

$$P_V(z) = H(\omega(z)) - H(0),$$
 (3.4)

$$B(t) = H(0), \tag{3.5}$$

$$P_H(z) = X(z)^p - H(\omega(z)). \tag{3.6}$$

Moreover, this polynomial is uniquely determined by the fact that the right hand side of (3.6) has a root at  $z = \delta$ .

In fact, we can convert these directly to formulae for  $A_H$ ,  $A_V$  and B using series  $W_1\left(\frac{1}{x};t\right)\in x\mathbb{Z}[\frac{1}{x}][[t]]$  and  $W_2\left(\frac{1}{y};t\right)\in \frac{1}{y}\mathbb{Z}[\frac{1}{y}][[t]]$  satisfying  $W_1\left(\frac{1}{X(z)};t\right)=\omega(z)$  for  $z\in\Omega_{-1}\cup\Omega_{-2}$  and  $W_2\left(\frac{1}{Y(z)};t\right)=\omega(z)$  for  $z\in\Omega_1\cup\Omega_2$ . Note that these depend on the step-set but not the starting point (p,q) of the walk. For general p, we can rewrite Lemma 5 as the following theorem:

**Theorem 3.** If the starting point of the walks is (p,0) for some  $p \ge 1$ , then there is a degree p polynomial H satisfying

$$A_V\left(\frac{1}{y}\right) = H\left(W_2\left(\frac{1}{y};t\right)\right) - H(0),\tag{3.7}$$

$$B(t) = H(0), \tag{3.8}$$

$$A_H\left(\frac{1}{x}\right) = x^p - H\left(W_1\left(\frac{1}{x};t\right)\right). \tag{3.9}$$

Moreover, this polynomial is uniquely determined by the fact that the right hand side of (3.9) is a series in  $\frac{1}{r}\mathbb{Z}[\frac{1}{r}][[t]]$ .

In the p=1 case, we have  $W_2\left(\frac{1}{y};t\right)=A_V\left(\frac{1}{y}\right)$  and  $W_1\left(\frac{1}{x};t\right)=x-B(t)-A_H\left(\frac{1}{x}\right)$ , which can be used as alternative definitions for  $W_1$  and  $W_2$ .

#### 3.4 Special case: simple walks

We now describe the case of simple walks, that is, unweighted walks with step-set  $S = \{(0,1), (1,0), (0,-1), (-1,0)\}$ . In this case X(z) and Y(z) can be written in terms of the Jacobi theta function

$$\vartheta(z,\tau) := \sum_{n=0}^{\infty} e^{i\pi\tau n(n+1)} \left( e^{(2n+1)iz} - e^{-(2n+1)iz} \right).$$

as

$$X(z) = e^{-i\gamma} \frac{\vartheta(z,\tau)\vartheta(z+\gamma,\tau)}{\vartheta(z-\gamma,\tau)\vartheta(z+2\gamma,\tau)} \quad \text{and} \quad Y(z) = e^{-i\gamma} \frac{\vartheta(z,\tau)\vartheta(z-\gamma,\tau)}{\vartheta(z+\gamma,\tau)\vartheta(z-2\gamma,\tau)},$$

where  $\gamma = \frac{\pi \tau}{4}$  and is related to t by

$$e^{-i\gamma}\frac{\vartheta\left(\frac{\gamma}{2},\tau\right)^2}{\vartheta\left(\frac{3\gamma}{2},\tau\right)^2} = \frac{1+2t-\sqrt{1+4t}}{2t}.$$

Moreover, the function  $\omega(z)$  defined in Section 3.1 is given by

$$\omega(z) = e^{-3i\gamma} \frac{\vartheta(2\gamma,\tau)\vartheta'(0,\frac{3\tau}{2})\vartheta(\gamma,\frac{3\tau}{2})}{\vartheta'(0,\tau)\vartheta(2\gamma,\frac{3\tau}{2})\vartheta(3\gamma,\frac{3\tau}{2})} \cdot \frac{\vartheta(z+\gamma,\frac{3\tau}{2})\vartheta(z+2\gamma,\frac{3\tau}{2})}{\vartheta(z-\gamma,\frac{3\tau}{2})\vartheta(z+4\gamma,\frac{3\tau}{2})},$$

which has  $\pi$  and  $\frac{3\pi\tau}{2}$  as periods. Since X(z) and  $\omega(z)$  share the periods  $\pi$  and  $3\pi\tau$ , they are related by a polynomial equation. One such equation is

$$\frac{1}{2t} - Y(z) - \frac{1}{Y(z)} = X(z) + \frac{1}{X(z)} - \frac{1}{2t} = \frac{\omega(z) + c_1}{\omega(z) - c_1} \left( \omega(z) + \frac{c_1^2}{\omega(z)} + c_2 \right),$$

where  $c_1$  and  $c_2$  are given by

$$c_{1} = -e^{-i\gamma} \frac{\vartheta(2\gamma, \tau)\vartheta'(0, \frac{3\tau}{2})\vartheta(\gamma, \frac{3\tau}{2})}{\vartheta'(0, \tau)\vartheta(2\gamma, \frac{3\tau}{2})\vartheta(3\gamma, \frac{3\tau}{2})},$$

$$c_{2} = \frac{1+4t}{2t} \cdot \frac{1+c_{3}}{1-c_{3}} + c_{1}c_{3} + \frac{c_{1}}{c_{3}}, \quad \text{where} \qquad c_{3} = -e^{i\gamma} \frac{\vartheta(\frac{5\gamma}{2}, \frac{3\pi\tau}{2})}{\vartheta(\frac{\gamma}{2}, \frac{3\pi\tau}{2})}.$$

So the series  $W_1\left(\frac{1}{x};t\right) \in x\mathbb{Z}\left[\frac{1}{x}\right]$  and  $W_2\left(\frac{1}{y};t\right) \in \frac{1}{y}\mathbb{Z}\left[\frac{1}{y}\right]$  defined in Section 3.3 satisfy

$$-\frac{1}{2t} + x + \frac{1}{x} = \frac{W_1\left(\frac{1}{x};t\right) + c_1}{W_1\left(\frac{1}{x};t\right) - c_1} \left(W_1\left(\frac{1}{x};t\right) + \frac{c_1^2}{W_1\left(\frac{1}{x};t\right)} + c_2\right),$$

$$\frac{1}{2t} - y - \frac{1}{y} = \frac{W_2\left(\frac{1}{y};t\right) + c_1}{W_2\left(\frac{1}{y};t\right) - c_1} \left(W_2\left(\frac{1}{y};t\right) + \frac{c_1^2}{W_2\left(\frac{1}{y};t\right)} + c_2\right).$$

Then for any starting point (p,0), the generating functions  $A_H$ ,  $A_V$ , B and hence C(x,y;t) can be determined by Theorem 3. Moreover,  $c_1$ ,  $c_2$  and t can be shown to be modular functions of  $\tau$ , so they are all algebraically related. Hence in these cases the generating function C(x,y;t) is algebraic in t as well as the other variables.

## 4 Nature of C(x, y; t) with respect to t

The main remaining problem is to prove that the generating function C(x, y; t) has the same nature (algebraic, D-finite, D-algebraic) as a function of t as it does as a function of t and t. Even for C(x, y; t), which counts walks confined to a quadrant, this has not been proven for weighted models, so it is not surprising that we have so far been unable to prove it for C(x, y; t). However, we expect that if this is proven for C(x, y; t), the result for C(x, y; t) will follow using the same method applied to Theorem 1.

## References

- [1] Olivier Bernardi, Mireille Bousquet-Mélou, and Kilian Raschel. "Counting quadrant walks via Tutte's invariant method". In: *Combinatorial Theory* (To appear).
- [2] Alin Bostan and Manuel Kauers. "The complete generating function for Gessel walks is algebraic". In: *Proceedings of the American Mathematical Society* 138.9 (2010), pp. 3063–3078.
- [3] Mireille Bousquet-Mélou. "Enumeration of three-quadrant walks via invariants: some diagonally symmetric models". In: (in preparation).
- [4] Mireille Bousquet-Mélou. "Square lattice walks avoiding a quadrant". In: *Journal of Combinatorial Theory, Series A* 144 (2016), pp. 37–79.
- [5] Mireille Bousquet-Mélou and Marni Mishna. "Walks with small steps in the quarter plane". In: *Contemp. Math* 520 (2010), pp. 1–40.
- [6] Mireille Bousquet-Mélou and Michael Wallner. "More models of walks avoiding a quadrant". In: *arXiv preprint arXiv:2110.07633* (2021).

[7] Timothy Budd. "Winding of simple walks on the square lattice". In: *J. Combin. Theory Ser. A* 172 (2020), pp. 105191, 59.

- [8] Thomas Dreyfus, Charlotte Hardouin, Julien Roques, and Michael Singer. "On the nature of the generating series of walks in the quarter plane". In: *Inventiones mathematicae* 213.1 (2018), pp. 139–203.
- [9] Thomas Dreyfus, Charlotte Hardouin, Julien Roques, and Michael F Singer. "Walks in the quarter plane: Genus zero case". In: *Journal of Combinatorial Theory, Series A* 174 (2020), p. 105251.
- [10] Thomas Dreyfus and Kilian Raschel. "Differential transcendence & algebraicity criteria for the series counting weighted quadrant walks". In: *Publications Mathématiques de Besançon-Algèbre et Théorie des Nombres* 1 (2019), pp. 41–80.
- [11] Thomas Dreyfus and Amélie Trotignon. "On the nature of four models of symmetric walks avoiding a quadrant". In: *Annals of Combinatorics* (2021), pp. 1–28.
- [12] Andrew Elvey Price. "Counting lattice walks by winding angle". In: *Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics*. 2020.
- [13] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev. *Random walks in the quarter-plane*. Vol. 40. Applications of Mathematics (New York). Springer-Verlag, Berlin, 1999.
- [14] Guy Fayolle and Kilian Raschel. "On the Holonomy or Algebraicity of Generating Functions Counting Lattice Walks in the Quarter-Plane". In: *Markov Processes and Related Fields* 16.3 (2010), pp. 485–496.
- [15] Charlotte Hardouin and Michael F Singer. "On Differentially Algebraic Generating Series for Walks in the Quarter Plane". In: *Selecta Mathematica* 27.5 (2021), pp. 1–49.
- [16] Irina Kurkova and Kilian Raschel. "On the functions counting walks with small steps in the quarter plane". In: *Publications mathématiques de l'IHÉS* 116.1 (2012), pp. 69–114.
- [17] Stephen Melczer and Marni Mishna. "Singularity analysis via the iterated kernel method". In: *Combinatorics, Probability and Computing* 23.5 (2014), pp. 861–888.
- [18] Marni Mishna and Andrew Rechnitzer. "Two non-holonomic lattice walks in the quarter plane". In: *Theoretical Computer Science* 410.38-40 (2009), pp. 3616–3630.
- [19] Kilian Raschel. "Counting walks in a quadrant: a unified approach via boundary value problems". In: *Journal of the European Mathematical Society* 14.3 (2012), pp. 749–777.
- [20] Kilian Raschel and Amélie Trotignon. "On Walks Avoiding a Quadrant". In: *Electron. J. Combin.* 26.3 (2019), Paper 3, 31. ISSN: 1077-8926.