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Enumeration of walks with small steps avoiding a quadrant

Andrew Elvey Price ^{*1}

¹ CNRS, Institut Denis Poisson, Université de Tours, France

Abstract. We address the enumeration of walks with weighted small steps avoiding a quadrant. In particular we give an exact, integral-expression solution for the generating function $C(x, y; t)$ counting these walks by length and end-point. Moreover, we determine precisely when this generating function is algebraic, D-finite or D-algebraic with respect to x , showing that this complexity is the same as for walks in the quarter-plane with the same starting point, as long as the starting point (p, q) of the walks lies in the quarter plane then. Finally, we give an integral-free expression for the solution in the cases where (p, q) lies just outside the quarter plane, that is $p = 0$ or $q = 0$ with our convention, proving a conjecture of Raschel and Trotignon.

Keywords: Lattice path, elliptic function, cone

1 Introduction

The systematic study of walks with small steps in the quarter plane was initiated by Bousquet-Mélou and Mishna in 2010 [5], and since then there has been great progress on the model [2, 14, 19, 18, 17, 1, 16, 8, 9]. The model is defined as follows: given a step set $S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, determine the generating function

$$Q(x, y; t) = \sum_{n=0}^{\infty} \sum_{i, j \geq 1} q(i, j; n) t^n x^i y^j,$$

where $q(i, j; n)$ is the number of walks of length n , starting at $(1, 1)$, and ending at (i, j) using steps in S and staying in the strictly positive quadrant.¹ A priori, there are 256 distinct step sets S , but after removing duplicates and cases that are equivalent to half-plane models, Bousquet-Mélou and Mishna indentified 79 non-trivial and combinatorially distinct models. The study of these models is now in some sense complete as it is known for each S precisely where the generating function fits into the hierarchy

Algebraic \subset D-finite \subset D-Algebraic.

*andrew.elvey@univ-tours.fr.

¹Note that most of the literature has considered the equivalent question of walks starting at $(0, 0)$ and staying in the non-strictly positive quadrant, for which the resulting generating function is $\frac{1}{xy}Q(x, y; t)$.

Recall that a generating function $G(x)$ is called Algebraic with respect to the variable x if it satisfies some non-trivial polynomial equation $P(G(x), x) = 0$, whose coefficients do not depend on that variable (but may depend on other variables), and it is called D-finite (resp. D-algebraic) if it satisfies a linear (resp. polynomial) differential equation with respect to x whose coefficients are polynomials in x . For a multivariate series to be algebraic (resp. D-finite, D-algebraic) it must be algebraic (resp. D-finite, D-algebraic) with respect to each variable.

Of the 79 models proposed by Mishna and Bousquet-Mélou, 4 models admit an algebraic generating function [5, 2], 19 further models admit a D-finite generating function [5, 14], 9 further models admit a D-algebraic generating function [1, 16] and the remaining 47 models admit a generating function which is not D-algebraic [8, 9]. Moreover, in the 74 cases known as *non-singular*, an exact integral expression is known for the generating function [19], while other exact expressions are known in the 5 singular cases [18, 17]. In recent years a number of articles have focused on the equivalent question for walks avoiding a quadrant [4, 20, 6, 11, 3], that is determining the generating function $C(x, y; t)$ which counts walks starting at $(1, 1)$ whose intermediate points are required to lie in the three-quadrant cone

$$\mathcal{C} = \{(i, j) : i > 0 \text{ or } j > 0\}.$$

Between them these 5 articles classify 10 models into the complexity hierarchy (algebraic, D-finite, D-algebraic), while excursions have been enumerated for 4 further models [7, 12]. Remarkably the generating function has the same nature in each case as in the quarter plane, a fact that led Dreyfus and Trotignon to conjecture that the nature is the same for any of the 74 non-singular step-sets S [9]. We give exact integral expression solutions for $C(x, y; t)$ in each of these 74 cases, analogous to those of Raschel in the quarter-plane [19]. We then prove that the nature of $C(x, y; t)$ as a function of x (or y) is the same as that of $Q(x, y; t)$, and we conjecture that these series also have the same nature as functions of t . In fact we do this in the more general setting of walks with *weighted* steps and starting at any point in the positive quadrant or on the positive x -axis.

Note that by our definition, steps directly between $(1, 0)$ and $(0, 1)$ are allowed, whereas they are forbidden in [6], for example. We do not expect this to affect the nature of the generating function.

2 Functional equations for walks avoiding a quadrant

We start with a step-set $S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, a weight $w_s > 0$ for each $s \in S$ and a starting point (p, q) with $p > 0$, $q \geq 0$. We will determine the generating function $C(x, y; t)$ counting walks starting at (p, q) , taking steps from S with all intermediate

points lying in the three-quadrant cone \mathcal{C} and with the weight of the walk being the product of the weights w_s of the steps. Note that the standard starting point is $(p, q) = (1, 1)$ and in the unweighted case $w_s = 1$ for each $s \in S$.

The following lemma results from considering the final step of a walk counted by $C(x, y; t)$:

Lemma 1. *Define the single step generating function $P(x, y)$ by*

$$P(x, y) = \sum_{(\alpha, \beta) \in S} w_{(\alpha, \beta)} x^\alpha y^\beta$$

Then there are series $A_H(\frac{1}{x}; t) \in \frac{t}{x} \mathbb{Z}[\frac{1}{x}][[t]]$, $A_V(\frac{1}{y}; t) \in \frac{t}{y} \mathbb{Z}[\frac{1}{y}][[t]]$ and $B(t) \in t\mathbb{Z}[[t]]$ which satisfy

$$C(x, y; t) = x^p y^q + tP(x, y)C(x, y; t) - B(t) - A_H\left(\frac{1}{x}; t\right) - A_V\left(\frac{1}{y}; t\right). \quad (2.1)$$

Moreover, this equation together with the fact that $c(i, j; n) = 0$ for $i, j \leq 0$, characterises the generating function

$$C(x, y; t) = \sum_{t \geq 0} \sum_{i, j \in \mathbb{Z}} c(i, j; n) x^i y^j t^n,$$

as well as the series A_H , A_V and B .

The series A_H , A_V and B in the lemma above can be understood combinatorially: They count walks starting at (p, q) and ending just outside \mathcal{C} whose intermediate points all lie within \mathcal{C} . More precisely, $A_H(\frac{1}{x}; t)$ counts those walks ending on the negative x -axis, $A_V(\frac{1}{y}; t)$ counts those walks ending on the negative y -axis, and $B(t)$ counts those walks ending at $(0, 0)$.

The unusual condition that the coefficients $c(i, j; n)$ of $C(x, y; t)$ vanish for $i, j \leq 0$ makes this equation difficult to solve directly, so we partition \mathcal{C} into three quadrants $\mathcal{Q}_{-1} = \{(i, j) : i > 0, j < 0\}$, $\mathcal{Q}_0 = \{(i, j) : i > 0, j \geq 0\}$ and $\mathcal{Q}_1 = \{(i, j) : i \leq 0, j > 0\}$, as shown in figure 1. A similar decomposition was used in [4, 6], but we have shifted the quadrants \mathcal{Q}_{-1} , \mathcal{Q}_0 down one space compared to those articles so that it is impossible to step directly between \mathcal{Q}_{-1} and \mathcal{Q}_1 and so that our condition on the starting point (p, q) is now that $(p, q) \in \mathcal{Q}_0$.

Now, for $j = -1, 0, 1$, we define $Q_j(x, y; t)$ to be the generating function counting walks in \mathcal{C} , starting at (p, q) and ending in \mathcal{Q}_j , so

$$C(x, y; t) = Q_{-1}(x, y; t) + Q_0(x, y; t) + Q_1(x, y; t),$$

and $Q_{-1} \in \frac{x}{y} \mathbb{Z}\left[x, \frac{1}{y}\right][[t]]$, $Q_0 \in x\mathbb{Z}[x, y][[t]]$ and $Q_1 \in y\mathbb{Z}\left[\frac{1}{x}, y\right][[t]]$. The following lemma rewrites (2.1) as three equations characterising $Q_{-1}(x, y; t)$, $Q_0(x, y; t)$ and $Q_1(x, y; t)$.

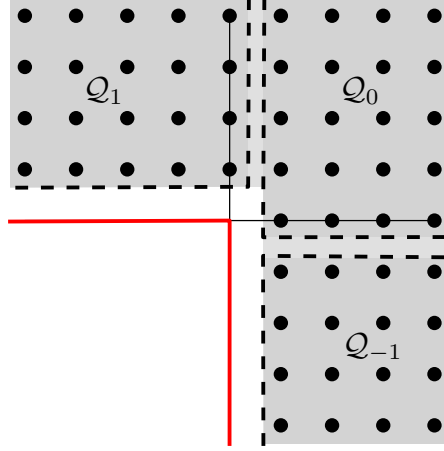


Figure 1: The three-quadrant cone \mathcal{C} partitioned into three quadrants \mathcal{Q}_1 , \mathcal{Q}_0 and \mathcal{Q}_{-1} .

Lemma 2. Define the kernel $K(x, y; t) = tP(x, y) - 1$. There are series $V_1(y; t), V_2(y; t) \in \mathbb{Z}[y][[t]]$ and $H_1(x; t), H_2(x; t) \in \mathbb{Z}[x][[t]]$ satisfying the three equations

$$K(x, y; t)Q_{-1}(x, y; t) = A_V \left(\frac{1}{y}; t \right) + H_1(x; t) + \frac{1}{y}H_2(x; t) \quad (2.2)$$

$$K(x, y; t)Q_0(x, y; t) = -xy + B(t) - V_1(y; t) - xV_2(y; t) - H_1(x; t) - \frac{1}{y}H_2(x; t) \quad (2.3)$$

$$K(x, y; t)Q_1(x, y; t) = A_H \left(\frac{1}{x}; t \right) + V_1(y; t) + xV_2(y; t). \quad (2.4)$$

Moreover, these three equations characterise the series $Q_{-1}(x, y; t)$, $Q_0(x, y; t)$, $Q_1(x, y; t)$, $V_1(y; t)$, $V_2(y; t)$, $H_1(x; t)$, $H_2(x; t)$, $A_H \left(\frac{1}{x}; t \right)$, $A_V \left(\frac{1}{y}; t \right)$ and $B(t)$.

Combinatorially, the series $(V_1(0; t) - V_1(y; t))$ (resp. $xV_2(y; t)$, $(H_1(x; t) - H_1(0, t))$, $\frac{1}{y}(H_2(0; t) - H_2(x; t))$) counts walks whose final step is from \mathcal{Q}_0 (resp. \mathcal{Q}_1 , \mathcal{Q}_{-1} , \mathcal{Q}_0) to \mathcal{Q}_1 (resp. \mathcal{Q}_0 , \mathcal{Q}_0 , \mathcal{Q}_{-1}).

2.1 Parameterisation of the kernel curve

Following the method used in the quarter plane pioneered by Fayolle and Raschel [13, 19] we start by fixing $t \in \left(0, \frac{1}{\sum_{s \in S} w_s}\right)$ and then we consider the curve $\mathcal{W} = \{(x, y) : K(x, y; t) = 0\}$. From now on, we will make the following assumption:

Assumption: S is a non-singular step-set. That is, for any line ℓ through the origin, at least one element of S lies on each side of ℓ .

As explained in [6], if S did not have this property, the generating function $C(x, y; t)$ would be algebraic.

Under this assumption, the curve \mathcal{W} is known to have genus 1, so we will be able to parameterise it using elliptic functions $X(z)$ and $Y(z)$. More precisely the following lemma follows from [10, Proposition 2.1, Lemma 2.6].

Lemma 3. *There are meromorphic functions $X(z), Y(z) : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ and numbers $\gamma, \tau \in i\mathbb{R}$ with $\Im(\pi\tau) > \Im(2\gamma) > 0$ satisfying the following conditions*

- $K(X(z), Y(z)) = 0$
- $X(z) = X(z + \pi) = X(z + \pi\tau) = X(-\gamma - z)$
- $Y(z) = Y(z + \pi) = X(z + \pi\tau) = Y(\gamma - z)$
- $|X(-\frac{\gamma}{2})|, |Y(\frac{\gamma}{2})| < 1$
- *Counting with multiplicity, the functions $X(z)$ and $Y(z)$ each contain two poles and two roots in each fundamental domain $\{z_c + r_1\pi + r_2\pi\tau : r_1, r_2 \in [0, 1)\}$.*

Moreover, $X(z)$ and $Y(z)$ are differentially algebraic with respect to z .

We intend to substitute $x \rightarrow X(z)$ and $y \rightarrow Y(z)$ into (2.2), (2.3) and (2.4), however we can only do this as long as the series in these equations converge, which occurs as long as $|x| \leq 1 \leq |y|$ for (2.2), $|x|, |y| \leq 1$ for (2.3) and $|y| \leq 1 \leq |x|$ for (2.4). So to substitute $x \rightarrow X(z)$ and $y \rightarrow Y(z)$, we need to understand how $|X(z)|$ and $|Y(z)|$ compare to 1, for which we use the following lemma, which follows from [10, Lemma 2.9].

Lemma 4. *The complex plane can be partitioned into simply connected regions $\{\Omega_s\}_{s \in \mathbb{Z}}$ (see Figure 2) satisfying*

$$\bigcup_{s \in \mathbb{Z}} \Omega_{4s} \cup \Omega_{4s+1} = \{z \in \mathbb{C} : |Y(z)| < 1\},$$

$$\bigcup_{s \in \mathbb{Z}} \Omega_{4s} \cup \Omega_{4s-1} = \{z \in \mathbb{C} : |X(z)| < 1\}$$

and for $s \in \mathbb{Z}$,

$$\begin{aligned} \pi + \Omega_s &= \Omega_s, \\ s\pi\tau + \gamma - \Omega_{2s} \cup \Omega_{2s+1} &= \Omega_{2s} \cup \Omega_{2s+1}, \\ s\pi\tau - \gamma - \Omega_{2s} \cup \Omega_{2s-1} &= \Omega_{2s} \cup \Omega_{2s-1}. \end{aligned}$$

2.2 Analytic reformulation of functional equations

Using the results in the previous section, we can substitute $x = X(z)$ and $y = Y(z)$ into (2.2), (2.3) and (2.4) for z in the regions Ω_{-1} , Ω_0 and Ω_1 , respectively, yielding (2.9), (2.10) and (2.11) in the following proposition:

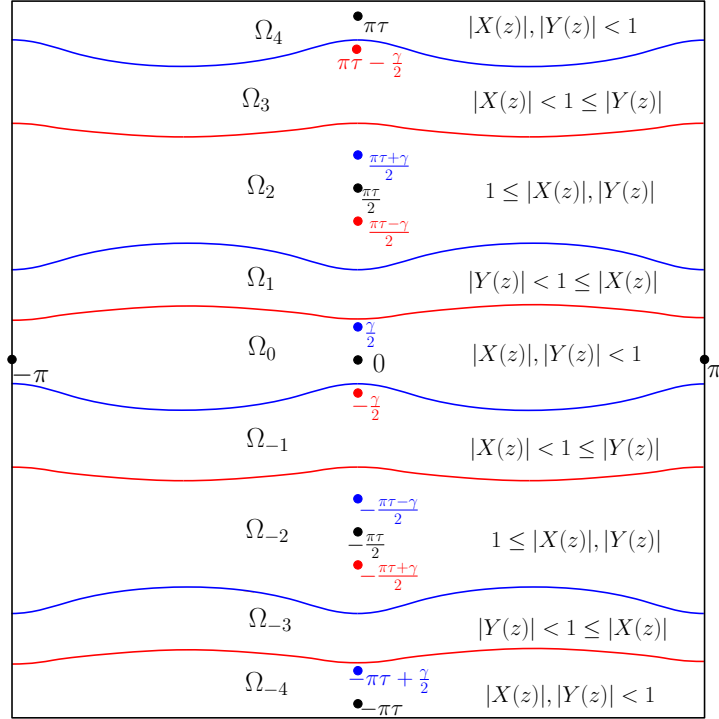


Figure 2: The complex plane partitioned into regions Ω_j . For z on the blue lines, $|Y(z)| = 1$, while on the red lines $|X(z)| = 1$.

Proposition 1. *The functions*

$$L_H(z) := H_1(X(z); t) + \frac{1}{Y(z)} H_2(X(z); t), \quad \text{for } z \in \Omega_0 \cup \Omega_{-1}, \quad (2.5)$$

$$L_V(z) := V_1(Y(z); t) + X(z) V_2(Y(z); t), \quad \text{for } z \in \Omega_0 \cup \Omega_1, \quad (2.6)$$

$$P_V(z) := A_V \left(\frac{1}{Y(z)}; t \right), \quad \text{for } z \in \Omega_{-1} \cup \Omega_{-2}, \quad (2.7)$$

$$P_H(z) := A_H \left(\frac{1}{X(z)}; t \right), \quad \text{for } z \in \Omega_1 \cup \Omega_2. \quad (2.8)$$

are well defined and satisfy the equations

$$0 = P_V(z) + L_H(z) \quad \text{for } z \in \Omega_{-1}, \quad (2.9)$$

$$0 = -X(z)^p Y(z)^q + B(t) - L_V(z) - L_H(z) \quad \text{for } z \in \Omega_0, \quad (2.10)$$

$$0 = P_H(z) + L_V(z) \quad \text{for } z \in \Omega_1 \quad (2.11)$$

$$P_H(z) = P_H(\pi\tau - \gamma - z) = P_H(z + \pi) \quad (2.12)$$

$$P_V(z) = P_V(-\pi\tau + \gamma - z) = P_V(z + \pi) \quad (2.13)$$

While these equations are a priori defined on different sets, they can be used to show that the functions extend meromorphically to all of \mathbb{C} , and so the equations hold on all of \mathbb{C} . Simply taking the sum of the three equations (2.9), (2.10) and (2.11) yields (2.14) in the theorem below.

Theorem 1. *The functions $P_H(z)$ and $P_V(z)$ extend to meromorphic functions on \mathbb{C} which, along with the constant $B(t)$, are uniquely defined by the equation*

$$X(z)^p Y(z)^q = P_V(z) + B(t) + P_H(z), \quad (2.14)$$

along with (2.12), (2.13) and the conditions

- $P_H(z)$ has no poles in $\Omega_0 \cup \Omega_1 \cup \Omega_2$,
- the poles of $X(z)$ for $z \in \Omega_1 \cup \Omega_2$ are roots of $P_H(z)$,
- $P_V(z)$ has no poles in $\Omega_0 \cup \Omega_{-1} \cup \Omega_{-2}$,
- the poles of $Y(z)$ for $z \in \Omega_{-1} \cup \Omega_{-2}$ are roots of $P_V(z)$.

Note that combining (2.14), (2.12), (2.13) yields

$$P_H(2\pi\tau - 2\gamma + z) - P_H(z) = W(z), \quad (2.15)$$

where $W(z)$ is an elliptic function with periods π and $\pi\tau$ given by

$$W(z) := (X(z - 2\gamma)^p - X(z)^p) Y(z)^q.$$

3 Solving the functional equation

In the previous section we reduced the problem to finding the unique meromorphic functions $P_V, P_H : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ and constant $B(t)$ characterised by Theorem 1 (for each t), as these determine $A_H(\frac{1}{x}, t)$ and $A_V(\frac{1}{y}, t)$ using 2.8 and 2.7, respectively, after which $C(x, y; t)$ is determined by (2.1).

an equation analogous to (2.14) was found by Raschel for walks in the quarter plane [19], the only difference being that the transformations $z \rightarrow \pi\tau - \gamma - z$ and $z \rightarrow -\pi\tau + \gamma - z$ here are $z \rightarrow \gamma - z$ and $z \rightarrow -\gamma - z$ in the quarter plane. Raschel used this equation to derive an integral-expression solution determining $Q(x, y; t)$, and the equation has since been used to determine precisely when $Q(x, y; t)$ is differentially algebraic [1, 8, 15] and to determine when it is algebraic or D-finite with respect to x or y [14, 16].

Due to this striking similarity we were able to use these methods to prove the same results for $C(x, y; t)$, in particular showing that it is D-algebraic in exactly the same cases as $Q(x, y; t)$ and that it is algebraic or D-finite with respect to x (or y) in the same cases as $Q(x, y; t)$. We note that Fayolle and Raschel also showed that the unweighted models that are algebraic or D-finite with respect to t have the same nature with respect to t , however these results relied on the precise ratios $\frac{\pi\tau}{\gamma}$ that could occur in these cases, so they do not apply so readily to our equation. Nonetheless, we expect that the same result holds for $C(x, y; t)$.

3.1 Integral expression

In this section we give integral expressions analogous to those of Raschel [19] which determine $P_V(z)$ and $P_H(z)$ exactly. In order to write these expressions, we define an auxiliary elliptic function $\omega(z)$ which satisfies

$$\omega(z) = \omega(\pi\tau - \gamma - z) = \omega(-\pi\tau + \gamma - z) = \omega(z + \pi)$$

and shares the poles of $X(z)$ in $\Omega_1 \cup \Omega_2$, while $1/\omega(z)$ has the same poles as $Y(z)$ in $\Omega_{-1} \cup \Omega_{-2}$. Finally, $\omega(z)/X(z)$ converges to 1 at the poles of $X(z)$ in $\Omega_1 \cup \Omega_2$.

Theorem 2. *Let $z_0 \in \Omega_0$ and let \mathcal{L} be a path from z_0 to $z_0 + \pi$ contained in the closure $\overline{\Omega_0}$ of Ω_0 . Then $P_V(z)$, $P_H(z)$ and $B(t)$ are given by the integrals*

$$P_H(u) = \frac{1}{2\pi it} \int_{\mathcal{L}} X(z)^p Y(z)^q \frac{\omega'(z)}{\omega(z) - \omega(u)} dz \quad \text{for } u \in \Omega_1 \cup \Omega_2 \quad (3.1)$$

$$P_V(u) = -\frac{1}{2\pi it} \int_{\mathcal{L}} X(z)^p Y(z)^q \frac{\omega(u)}{\omega(z)} \frac{\omega'(z)}{\omega(z) - \omega(u)} dz \quad \text{for } u \in \Omega_{-1} \cup \Omega_{-2} \quad (3.2)$$

$$B(t) = -\frac{1}{2\pi it} \int_{\mathcal{L}} X(z)^p Y(z)^q \frac{\omega'(z)}{\omega(z)} dz \quad (3.3)$$

The proof of this theorem involves checking that these expressions satisfy the conditions in Theorem 1.

3.2 Classification of $C(x, y; t)$ into complexity hierarchy

In this section we very briefly describe the properties of the step set S , or equivalently $X(z)$ and $Y(z)$, which determine the nature of the generating function $C(x, y; t)$.

In certain cases, called *finite group cases*, $\gamma = \frac{M}{N}\pi\tau$ for some positive $M, N \in \mathbb{Z}$ independent of t . Then applying (2.15) N times yields

$$P_H(2\pi\tau(N - M) + z) - P_H(z) = P_H((2\pi\tau - 2\gamma)N + z) - P_H(z) = E(z),$$

where $E(z)$ is called the orbit sum of the model. We can use this equation to solve for $P_H(z)$, yielding an expression for $C(x, y; t)$ which is D-finite in x . In the cases where $E(z) = 0$, we can even prove that $C(x, y; t)$ is algebraic in x .

In all other cases we have $\frac{\gamma}{\pi\tau} \notin \mathbb{Q}$ for generic t . The model is then said to decouple if there are rational functions R_1, R_2 satisfying $X(z)^p Y(z)^q = R_1(X(z)) + R_2(Y(z))$. These cases can be solved (with integral-free expressions) as, using (2.14), the function

$$f(z) := R_1(X(z)) - P_H(z) = -R_2(Y(z)) + B(t) + P_v(z)$$

is an elliptic function which can be determined exactly. In these cases every function used is D-algebraic in all of its variables.

Finally in the infinite group cases which do not decouple, the generating function can be shown to be non-D-algebraic in x using galois theory of q -difference equations, as in [8, 15] for $Q(x, y; t)$.

3.3 Special case: walks starting on x -axis

We will now study the special cases where the walk starts at some point $(p, 0)$ for $p > 0$. Trotignon and Raschel conjectured that with this starting point all finite group models admit algebraic generating functions [20]. Indeed, if $q = 0$ it is easy to see that the orbit sum $E(z)$ defined in section 3.2 is equal to 0, so this follows from our more general results. Moreover, if $q = 0$, the model trivially decouples, so even in the infinite group case the generating function $C(x, y; t)$ is D-algebraic.

The following lemma follows directly from Theorem 1, where $\omega(z)$ is defined as in Section 3.1.

Lemma 5. *If the starting point of the walks is $(p, 0)$ for some $p \geq 1$, then there is a degree p polynomial H satisfying*

$$P_V(z) = H(\omega(z)) - H(0), \quad (3.4)$$

$$B(t) = H(0), \quad (3.5)$$

$$P_H(z) = X(z)^p - H(\omega(z)). \quad (3.6)$$

Moreover, this polynomial is uniquely determined by the fact that the right hand side of (3.6) has a root at $z = \delta$.

In fact, we can convert these directly to formulae for A_H , A_V and B using series $W_1\left(\frac{1}{x}; t\right) \in x\mathbb{Z}\left[\frac{1}{x}\right][[t]]$ and $W_2\left(\frac{1}{y}; t\right) \in \frac{1}{y}\mathbb{Z}\left[\frac{1}{y}\right][[t]]$ satisfying $W_1\left(\frac{1}{X(z)}; t\right) = \omega(z)$ for $z \in \Omega_{-1} \cup \Omega_{-2}$ and $W_2\left(\frac{1}{Y(z)}; t\right) = \omega(z)$ for $z \in \Omega_1 \cup \Omega_2$. Note that these depend on the step-set but not the starting point (p, q) of the walk. For general p , we can rewrite Lemma 5 as the following theorem:

Theorem 3. *If the starting point of the walks is $(p, 0)$ for some $p \geq 1$, then there is a degree p polynomial H satisfying*

$$A_V\left(\frac{1}{y}\right) = H\left(W_2\left(\frac{1}{y}; t\right)\right) - H(0), \quad (3.7)$$

$$B(t) = H(0), \quad (3.8)$$

$$A_H\left(\frac{1}{x}\right) = x^p - H\left(W_1\left(\frac{1}{x}; t\right)\right). \quad (3.9)$$

Moreover, this polynomial is uniquely determined by the fact that the right hand side of (3.9) is a series in $\frac{1}{x}\mathbb{Z}\left[\frac{1}{x}\right][[t]]$.

In the $p = 1$ case, we have $W_2\left(\frac{1}{y}; t\right) = A_V\left(\frac{1}{y}\right)$ and $W_1\left(\frac{1}{x}; t\right) = x - B(t) - A_H\left(\frac{1}{x}\right)$, which can be used as alternative definitions for W_1 and W_2 .

3.4 Special case: simple walks

We now describe the case of simple walks, that is, unweighted walks with step-set $S = \{(0, 1), (1, 0), (0, -1), (-1, 0)\}$. In this case $X(z)$ and $Y(z)$ can be written in terms of the Jacobi theta function

$$\vartheta(z, \tau) := \sum_{n=0}^{\infty} e^{i\pi\tau n(n+1)} \left(e^{(2n+1)iz} - e^{-(2n+1)iz} \right).$$

as

$$X(z) = e^{-i\gamma} \frac{\vartheta(z, \tau)\vartheta(z + \gamma, \tau)}{\vartheta(z - \gamma, \tau)\vartheta(z + 2\gamma, \tau)} \quad \text{and} \quad Y(z) = e^{-i\gamma} \frac{\vartheta(z, \tau)\vartheta(z - \gamma, \tau)}{\vartheta(z + \gamma, \tau)\vartheta(z - 2\gamma, \tau)},$$

where $\gamma = \frac{\pi\tau}{4}$ and is related to t by

$$e^{-i\gamma} \frac{\vartheta\left(\frac{\gamma}{2}, \tau\right)^2}{\vartheta\left(\frac{3\gamma}{2}, \tau\right)^2} = \frac{1 + 2t - \sqrt{1 + 4t}}{2t}.$$

Moreover, the function $\omega(z)$ defined in Section 3.1 is given by

$$\omega(z) = e^{-3i\gamma} \frac{\vartheta(2\gamma, \tau)\vartheta'(0, \frac{3\tau}{2})\vartheta(\gamma, \frac{3\tau}{2})}{\vartheta'(0, \tau)\vartheta(2\gamma, \frac{3\tau}{2})\vartheta(3\gamma, \frac{3\tau}{2})} \cdot \frac{\vartheta(z + \gamma, \frac{3\tau}{2})\vartheta(z + 2\gamma, \frac{3\tau}{2})}{\vartheta(z - \gamma, \frac{3\tau}{2})\vartheta(z + 4\gamma, \frac{3\tau}{2})},$$

which has π and $\frac{3\pi\tau}{2}$ as periods. Since $X(z)$ and $\omega(z)$ share the periods π and $3\pi\tau$, they are related by a polynomial equation. One such equation is

$$\frac{1}{2t} - Y(z) - \frac{1}{Y(z)} = X(z) + \frac{1}{X(z)} - \frac{1}{2t} = \frac{\omega(z) + c_1}{\omega(z) - c_1} \left(\omega(z) + \frac{c_1^2}{\omega(z)} + c_2 \right),$$

where c_1 and c_2 are given by

$$c_1 = -e^{-i\gamma} \frac{\vartheta(2\gamma, \tau)\vartheta'(0, \frac{3\tau}{2})\vartheta(\gamma, \frac{3\tau}{2})}{\vartheta'(0, \tau)\vartheta(2\gamma, \frac{3\tau}{2})\vartheta(3\gamma, \frac{3\tau}{2})},$$

$$c_2 = \frac{1 + 4t}{2t} \cdot \frac{1 + c_3}{1 - c_3} + c_1 c_3 + \frac{c_1}{c_3}, \quad \text{where} \quad c_3 = -e^{i\gamma} \frac{\vartheta\left(\frac{5\gamma}{2}, \frac{3\pi\tau}{2}\right)}{\vartheta\left(\frac{\gamma}{2}, \frac{3\pi\tau}{2}\right)}.$$

So the series $W_1\left(\frac{1}{x}; t\right) \in x\mathbb{Z}\left[\frac{1}{x}\right]$ and $W_2\left(\frac{1}{y}; t\right) \in \frac{1}{y}\mathbb{Z}\left[\frac{1}{y}\right]$ defined in Section 3.3 satisfy

$$-\frac{1}{2t} + x + \frac{1}{x} = \frac{W_1\left(\frac{1}{x}; t\right) + c_1}{W_1\left(\frac{1}{x}; t\right) - c_1} \left(W_1\left(\frac{1}{x}; t\right) + \frac{c_1^2}{W_1\left(\frac{1}{x}; t\right)} + c_2 \right),$$

$$\frac{1}{2t} - y - \frac{1}{y} = \frac{W_2\left(\frac{1}{y}; t\right) + c_1}{W_2\left(\frac{1}{y}; t\right) - c_1} \left(W_2\left(\frac{1}{y}; t\right) + \frac{c_1^2}{W_2\left(\frac{1}{y}; t\right)} + c_2 \right).$$

Then for any starting point $(p, 0)$, the generating functions A_H, A_V, B and hence $C(x, y; t)$ can be determined by Theorem 3. Moreover, c_1, c_2 and t can be shown to be modular functions of τ , so they are all algebraically related. Hence in these cases the generating function $C(x, y; t)$ is algebraic in t as well as the other variables.

4 Nature of $C(x, y; t)$ with respect to t

The main remaining problem is to prove that the generating function $C(x, y; t)$ has the same nature (algebraic, D-finite, D-algebraic) as a function of t as it does as a function of x and y . Even for $Q(x, y; t)$, which counts walks confined to a quadrant, this has not been proven for weighted models, so it is not surprising that we have so far been unable to prove it for $C(x, y; t)$. However, we expect that if this is proven for $Q(x, y; t)$, the result for $C(x, y; t)$ will follow using the same method applied to Theorem 1.

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