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# Enumeration of walks with small steps avoiding a quadrant

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**Abstract.** We address the enumeration of walks with weighted small steps avoiding a quadrant. In particular we give an exact, integral-expression solution for the generating function  $C(x, y; t)$  counting these walks by length and end-point. Moreover, we determine precisely when this generating function is algebraic, D-finite or D-algebraic with respect to  $x$ , showing that this complexity is the same as for walks in the quarter-plane with the same starting point, as long as the starting point  $(p, q)$  of the walks lies in the quarter plane then. Finally, we give an integral-free expression for the solution in the cases where  $(p, q)$  lies just outside the quarter plane, that is  $p = 0$  or  $q = 0$  with our convention, proving a conjecture of Raschel and Trotignon.

**Keywords:** Lattice path, elliptic function, cone

## 1 Introduction

The systematic study of walks with small steps in the quarter plane was initiated by Bousquet-Mélou and Mishna in 2010 [5], and since then there has been great progress on the model [2, 14, 19, 18, 17, 1, 16, 8, 9]. The model is defined as follows: given a step set  $S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ , determine the generating function

$$Q(x, y; t) = \sum_{n=0}^{\infty} \sum_{i, j \geq 1} q(i, j; n) t^n x^i y^j,$$

where  $q(i, j; n)$  is the number of walks of length  $n$ , starting at  $(1, 1)$ , and ending at  $(i, j)$  using steps in  $S$  and staying in the strictly positive quadrant.<sup>1</sup> A priori, there are 256 distinct step sets  $S$ , but after removing duplicates and cases that are equivalent to half-plane models, Bousquet-Mélou and Mishna indentified 79 non-trivial and combinatorially distinct models. The study of these models is now in some sense complete as it is known for each  $S$  precisely where the generating function fits into the hierarchy

Algebraic  $\subset$  D-finite  $\subset$  D-Algebraic.

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<sup>1</sup>Note that most of the literature has considered the equivalent question of walks starting at  $(0, 0)$  and staying in the non-strictly positive quadrant, for which the resulting generating function is  $\frac{1}{xy}Q(x, y; t)$ .

Recall that a generating function  $G(x)$  is called Algebraic with respect to the variable  $x$  if it satisfies some non-trivial polynomial equation  $P(G(x), x) = 0$ , whose coefficients do not depend on that variable (but may depend on other variables), and it is called D-finite (resp. D-algebraic) if it satisfies a linear (resp. polynomial) differential equation with respect to  $x$  whose coefficients are polynomials in  $x$ . For a multivariate series to be algebraic (resp. D-finite, D-algebraic) it must be algebraic (resp. D-finite, D-algebraic) with respect to each variable.

Of the 79 models proposed by Mishna and Bousquet-Mélou, 4 models admit an algebraic generating function [5, 2], 19 further models admit a D-finite generating function [5, 14], 9 further models admit a D-algebraic generating function [1, 16] and the remaining 47 models admit a generating function which is not D-algebraic [8, 9]. Moreover, in the 74 cases known as *non-singular*, an exact integral expression is known for the generating function [19], while other exact expressions are known in the 5 singular cases [18, 17]. In recent years a number of articles have focused on the equivalent question for walks avoiding a quadrant [4, 20, 6, 11, 3], that is determining the generating function  $C(x, y; t)$  which counts walks starting at  $(1, 1)$  whose intermediate points are required to lie in the three-quadrant cone

$$\mathcal{C} = \{(i, j) : i > 0 \text{ or } j > 0\}.$$

Between them these 5 articles classify 10 models into the complexity hierarchy (algebraic, D-finite, D-algebraic), while excursions have been enumerated for 4 further models [7, 12]. Remarkably the generating function has the same nature in each case as in the quarter plane, a fact that led Dreyfus and Trotignon to conjecture that the nature is the same for any of the 74 non-singular step-sets  $S$  [9]. We give exact integral expression solutions for  $C(x, y; t)$  in each of these 74 cases, analogous to those of Raschel in the quarter-plane [19]. We then prove that the nature of  $C(x, y; t)$  as a function of  $x$  (or  $y$ ) is the same as that of  $Q(x, y; t)$ , and we conjecture that these series also have the same nature as functions of  $t$ . In fact we do this in the more general setting of walks with *weighted* steps and starting at any point in the positive quadrant or on the positive  $x$ -axis.

Note that by our definition, steps directly between  $(1, 0)$  and  $(0, 1)$  are allowed, whereas they are forbidden in [6], for example. We do not expect this to affect the nature of the generating function.

## 2 Functional equations for walks avoiding a quadrant

We start with a step-set  $S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ , a weight  $w_s > 0$  for each  $s \in S$  and a starting point  $(p, q)$  with  $p > 0$ ,  $q \geq 0$ . We will determine the generating function  $C(x, y; t)$  counting walks starting at  $(p, q)$ , taking steps from  $S$  with all intermediate

points lying in the three-quadrant cone  $\mathcal{C}$  and with the weight of the walk being the product of the weights  $w_s$  of the steps. Note that the standard starting point is  $(p, q) = (1, 1)$  and in the unweighted case  $w_s = 1$  for each  $s \in S$ .

The following lemma results from considering the final step of a walk counted by  $C(x, y; t)$ :

**Lemma 1.** *Define the single step generating function  $P(x, y)$  by*

$$P(x, y) = \sum_{(\alpha, \beta) \in S} w_{(\alpha, \beta)} x^\alpha y^\beta$$

*Then there are series  $A_H(\frac{1}{x}; t) \in \frac{t}{x} \mathbb{Z}[\frac{1}{x}][[t]]$ ,  $A_V(\frac{1}{y}; t) \in \frac{t}{y} \mathbb{Z}[\frac{1}{y}][[t]]$  and  $B(t) \in t\mathbb{Z}[[t]]$  which satisfy*

$$C(x, y; t) = x^p y^q + tP(x, y)C(x, y; t) - B(t) - A_H\left(\frac{1}{x}; t\right) - A_V\left(\frac{1}{y}; t\right). \quad (2.1)$$

*Moreover, this equation together with the fact that  $c(i, j; n) = 0$  for  $i, j \leq 0$ , characterises the generating function*

$$C(x, y; t) = \sum_{t \geq 0} \sum_{i, j \in \mathbb{Z}} c(i, j; n) x^i y^j t^n,$$

*as well as the series  $A_H$ ,  $A_V$  and  $B$ .*

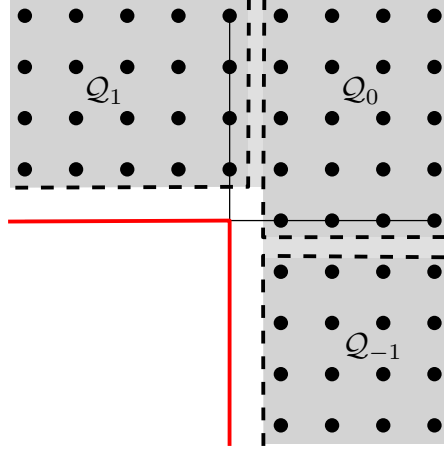
The series  $A_H$ ,  $A_V$  and  $B$  in the lemma above can be understood combinatorially: They count walks starting at  $(p, q)$  and ending just outside  $\mathcal{C}$  whose intermediate points all lie within  $\mathcal{C}$ . More precisely,  $A_H(\frac{1}{x}; t)$  counts those walks ending on the negative  $x$ -axis,  $A_V(\frac{1}{y}; t)$  counts those walks ending on the negative  $y$ -axis, and  $B(t)$  counts those walks ending at  $(0, 0)$ .

The unusual condition that the coefficients  $c(i, j; n)$  of  $C(x, y; t)$  vanish for  $i, j \leq 0$  makes this equation difficult to solve directly, so we partition  $\mathcal{C}$  into three quadrants  $\mathcal{Q}_{-1} = \{(i, j) : i > 0, j < 0\}$ ,  $\mathcal{Q}_0 = \{(i, j) : i > 0, j \geq 0\}$  and  $\mathcal{Q}_1 = \{(i, j) : i \leq 0, j > 0\}$ , as shown in figure 1. A similar decomposition was used in [4, 6], but we have shifted the quadrants  $\mathcal{Q}_{-1}$ ,  $\mathcal{Q}_0$  down one space compared to those articles so that it is impossible to step directly between  $\mathcal{Q}_{-1}$  and  $\mathcal{Q}_1$  and so that our condition on the starting point  $(p, q)$  is now that  $(p, q) \in \mathcal{Q}_0$ .

Now, for  $j = -1, 0, 1$ , we define  $Q_j(x, y; t)$  to be the generating function counting walks in  $\mathcal{C}$ , starting at  $(p, q)$  and ending in  $\mathcal{Q}_j$ , so

$$C(x, y; t) = Q_{-1}(x, y; t) + Q_0(x, y; t) + Q_1(x, y; t),$$

and  $Q_{-1} \in \frac{x}{y} \mathbb{Z} \left[ x, \frac{1}{y} \right] [[t]]$ ,  $Q_0 \in x\mathbb{Z} [x, y] [[t]]$  and  $Q_1 \in y\mathbb{Z} \left[ \frac{1}{x}, y \right] [[t]]$ . The following lemma rewrites (2.1) as three equations characterising  $Q_{-1}(x, y; t)$ ,  $Q_0(x, y; t)$  and  $Q_1(x, y; t)$ .



**Figure 1:** The three-quadrant cone  $\mathcal{C}$  partitioned into three quadrants  $\mathcal{Q}_1$ ,  $\mathcal{Q}_0$  and  $\mathcal{Q}_{-1}$ .

**Lemma 2.** Define the kernel  $K(x, y; t) = tP(x, y) - 1$ . There are series  $V_1(y; t), V_2(y; t) \in \mathbb{Z}[y][[t]]$  and  $H_1(x; t), H_2(x; t) \in \mathbb{Z}[x][[t]]$  satisfying the three equations

$$K(x, y; t)Q_{-1}(x, y; t) = A_V \left( \frac{1}{y}; t \right) + H_1(x; t) + \frac{1}{y}H_2(x; t) \quad (2.2)$$

$$K(x, y; t)Q_0(x, y; t) = -xy + B(t) - V_1(y; t) - xV_2(y; t) - H_1(x; t) - \frac{1}{y}H_2(x; t) \quad (2.3)$$

$$K(x, y; t)Q_1(x, y; t) = A_H \left( \frac{1}{x}; t \right) + V_1(y; t) + xV_2(y; t). \quad (2.4)$$

Moreover, these three equations characterise the series  $Q_{-1}(x, y; t)$ ,  $Q_0(x, y; t)$ ,  $Q_1(x, y; t)$ ,  $V_1(y; t)$ ,  $V_2(y; t)$ ,  $H_1(x; t)$ ,  $H_2(x; t)$ ,  $A_H \left( \frac{1}{x}; t \right)$ ,  $A_V \left( \frac{1}{y}; t \right)$  and  $B(t)$ .

Combinatorially, the series  $(V_1(0; t) - V_1(y; t))$  (resp.  $xV_2(y; t)$ ,  $(H_1(x; t) - H_1(0, t))$ ,  $\frac{1}{y}(H_2(0; t) - H_2(x; t))$ ) counts walks whose final step is from  $\mathcal{Q}_0$  (resp.  $\mathcal{Q}_1$ ,  $\mathcal{Q}_{-1}$ ,  $\mathcal{Q}_0$ ) to  $\mathcal{Q}_1$  (resp.  $\mathcal{Q}_0$ ,  $\mathcal{Q}_0$ ,  $\mathcal{Q}_{-1}$ ).

## 2.1 Parameterisation of the kernel curve

Following the method used in the quarter plane pioneered by Fayolle and Raschel [13, 19] we start by fixing  $t \in \left(0, \frac{1}{\sum_{s \in S} w_s}\right)$  and then we consider the curve  $\mathcal{W} = \{(x, y) : K(x, y; t) = 0\}$ . From now on, we will make the following assumption:

**Assumption:**  $S$  is a non-singular step-set. That is, for any line  $\ell$  through the origin, at least one element of  $S$  lies on each side of  $\ell$ .

As explained in [6], if  $S$  did not have this property, the generating function  $C(x, y; t)$  would be algebraic.

Under this assumption, the curve  $\mathcal{W}$  is known to have genus 1, so we will be able to parameterise it using elliptic functions  $X(z)$  and  $Y(z)$ . More precisely the following lemma follows from [10, Proposition 2.1, Lemma 2.6].

**Lemma 3.** *There are meromorphic functions  $X(z), Y(z) : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$  and numbers  $\gamma, \tau \in i\mathbb{R}$  with  $\Im(\pi\tau) > \Im(2\gamma) > 0$  satisfying the following conditions*

- $K(X(z), Y(z)) = 0$
- $X(z) = X(z + \pi) = X(z + \pi\tau) = X(-\gamma - z)$
- $Y(z) = Y(z + \pi) = X(z + \pi\tau) = Y(\gamma - z)$
- $|X(-\frac{\gamma}{2})|, |Y(\frac{\gamma}{2})| < 1$
- *Counting with multiplicity, the functions  $X(z)$  and  $Y(z)$  each contain two poles and two roots in each fundamental domain  $\{z_c + r_1\pi + r_2\pi\tau : r_1, r_2 \in [0, 1)\}$ .*

Moreover,  $X(z)$  and  $Y(z)$  are differentially algebraic with respect to  $z$ .

We intend to substitute  $x \rightarrow X(z)$  and  $y \rightarrow Y(z)$  into (2.2), (2.3) and (2.4), however we can only do this as long as the series in these equations converge, which occurs as long as  $|x| \leq 1 \leq |y|$  for (2.2),  $|x|, |y| \leq 1$  for (2.3) and  $|y| \leq 1 \leq |x|$  for (2.4). So to substitute  $x \rightarrow X(z)$  and  $y \rightarrow Y(z)$ , we need to understand how  $|X(z)|$  and  $|Y(z)|$  compare to 1, for which we use the following lemma, which follows from [10, Lemma 2.9].

**Lemma 4.** *The complex plane can be partitioned into simply connected regions  $\{\Omega_s\}_{s \in \mathbb{Z}}$  (see Figure 2) satisfying*

$$\bigcup_{s \in \mathbb{Z}} \Omega_{4s} \cup \Omega_{4s+1} = \{z \in \mathbb{C} : |Y(z)| < 1\},$$

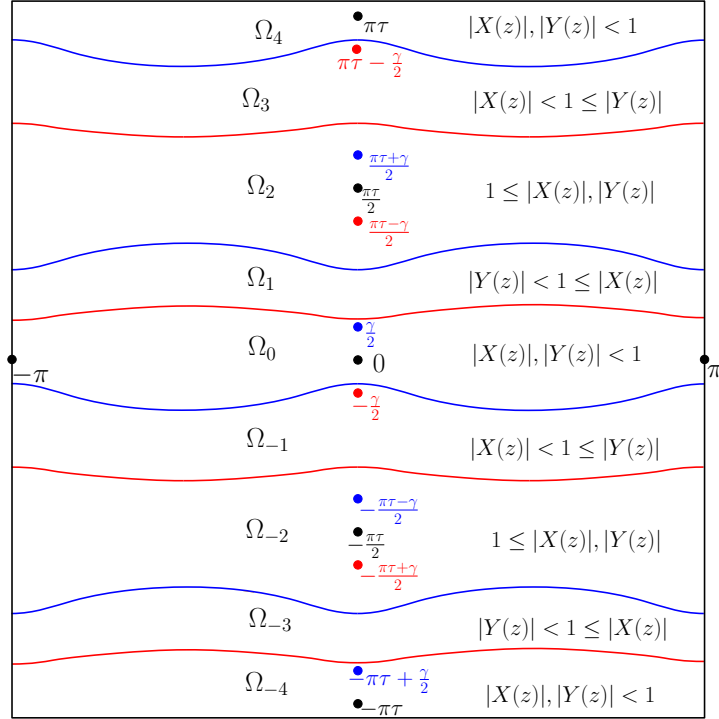
$$\bigcup_{s \in \mathbb{Z}} \Omega_{4s} \cup \Omega_{4s-1} = \{z \in \mathbb{C} : |X(z)| < 1\}$$

and for  $s \in \mathbb{Z}$ ,

$$\begin{aligned} \pi + \Omega_s &= \Omega_s, \\ s\pi\tau + \gamma - \Omega_{2s} \cup \Omega_{2s+1} &= \Omega_{2s} \cup \Omega_{2s+1}, \\ s\pi\tau - \gamma - \Omega_{2s} \cup \Omega_{2s-1} &= \Omega_{2s} \cup \Omega_{2s-1}. \end{aligned}$$

## 2.2 Analytic reformulation of functional equations

Using the results in the previous section, we can substitute  $x = X(z)$  and  $y = Y(z)$  into (2.2), (2.3) and (2.4) for  $z$  in the regions  $\Omega_{-1}$ ,  $\Omega_0$  and  $\Omega_1$ , respectively, yielding (2.9), (2.10) and (2.11) in the following proposition:



**Figure 2:** The complex plane partitioned into regions  $\Omega_j$ . For  $z$  on the blue lines,  $|Y(z)| = 1$ , while on the red lines  $|X(z)| = 1$ .

**Proposition 1.** *The functions*

$$L_H(z) := H_1(X(z); t) + \frac{1}{Y(z)} H_2(X(z); t), \quad \text{for } z \in \Omega_0 \cup \Omega_{-1}, \quad (2.5)$$

$$L_V(z) := V_1(Y(z); t) + X(z) V_2(Y(z); t), \quad \text{for } z \in \Omega_0 \cup \Omega_1, \quad (2.6)$$

$$P_V(z) := A_V \left( \frac{1}{Y(z)}; t \right), \quad \text{for } z \in \Omega_{-1} \cup \Omega_{-2}, \quad (2.7)$$

$$P_H(z) := A_H \left( \frac{1}{X(z)}; t \right), \quad \text{for } z \in \Omega_1 \cup \Omega_2. \quad (2.8)$$

are well defined and satisfy the equations

$$0 = P_V(z) + L_H(z) \quad \text{for } z \in \Omega_{-1}, \quad (2.9)$$

$$0 = -X(z)^p Y(z)^q + B(t) - L_V(z) - L_H(z) \quad \text{for } z \in \Omega_0, \quad (2.10)$$

$$0 = P_H(z) + L_V(z) \quad \text{for } z \in \Omega_1 \quad (2.11)$$

$$P_H(z) = P_H(\pi\tau - \gamma - z) = P_H(z + \pi) \quad (2.12)$$

$$P_V(z) = P_V(-\pi\tau + \gamma - z) = P_V(z + \pi) \quad (2.13)$$

While these equations are a priori defined on different sets, they can be used to show that the functions extend meromorphically to all of  $\mathbb{C}$ , and so the equations hold on all of  $\mathbb{C}$ . Simply taking the sum of the three equations (2.9), (2.10) and (2.11) yields (2.14) in the theorem below.

**Theorem 1.** *The functions  $P_H(z)$  and  $P_V(z)$  extend to meromorphic functions on  $\mathbb{C}$  which, along with the constant  $B(t)$ , are uniquely defined by the equation*

$$X(z)^p Y(z)^q = P_V(z) + B(t) + P_H(z), \quad (2.14)$$

along with (2.12), (2.13) and the conditions

- $P_H(z)$  has no poles in  $\Omega_0 \cup \Omega_1 \cup \Omega_2$ ,
- the poles of  $X(z)$  for  $z \in \Omega_1 \cup \Omega_2$  are roots of  $P_H(z)$ ,
- $P_V(z)$  has no poles in  $\Omega_0 \cup \Omega_{-1} \cup \Omega_{-2}$ ,
- the poles of  $Y(z)$  for  $z \in \Omega_{-1} \cup \Omega_{-2}$  are roots of  $P_V(z)$ .

Note that combining (2.14), (2.12), (2.13) yields

$$P_H(2\pi\tau - 2\gamma + z) - P_H(z) = W(z), \quad (2.15)$$

where  $W(z)$  is an elliptic function with periods  $\pi$  and  $\pi\tau$  given by

$$W(z) := (X(z - 2\gamma)^p - X(z)^p) Y(z)^q.$$

### 3 Solving the functional equation

In the previous section we reduced the problem to finding the unique meromorphic functions  $P_V, P_H : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$  and constant  $B(t)$  characterised by Theorem 1 (for each  $t$ ), as these determine  $A_H(\frac{1}{x}, t)$  and  $A_V(\frac{1}{y}, t)$  using 2.8 and 2.7, respectively, after which  $C(x, y; t)$  is determined by (2.1).

an equation analogous to (2.14) was found by Raschel for walks in the quarter plane [19], the only difference being that the transformations  $z \rightarrow \pi\tau - \gamma - z$  and  $z \rightarrow -\pi\tau + \gamma - z$  here are  $z \rightarrow \gamma - z$  and  $z \rightarrow -\gamma - z$  in the quarter plane. Raschel used this equation to derive an integral-expression solution determining  $Q(x, y; t)$ , and the equation has since been used to determine precisely when  $Q(x, y; t)$  is differentially algebraic [1, 8, 15] and to determine when it is algebraic or D-finite with respect to  $x$  or  $y$  [14, 16].

Due to this striking similarity we were able to use these methods to prove the same results for  $C(x, y; t)$ , in particular showing that it is D-algebraic in exactly the same cases as  $Q(x, y; t)$  and that it is algebraic or D-finite with respect to  $x$  (or  $y$ ) in the same cases as  $Q(x, y; t)$ . We note that Fayolle and Raschel also showed that the unweighted models that are algebraic or D-finite with respect to  $t$  have the same nature with respect to  $t$ , however these results relied on the precise ratios  $\frac{\pi\tau}{\gamma}$  that could occur in these cases, so they do not apply so readily to our equation. Nonetheless, we expect that the same result holds for  $C(x, y; t)$ .



### 3.1 Integral expression

In this section we give integral expressions analogous to those of Raschel [19] which determine  $P_V(z)$  and  $P_H(z)$  exactly. In order to write these expressions, we define an auxiliary elliptic function  $\omega(z)$  which satisfies

$$\omega(z) = \omega(\pi\tau - \gamma - z) = \omega(-\pi\tau + \gamma - z) = \omega(z + \pi)$$

and shares the poles of  $X(z)$  in  $\Omega_1 \cup \Omega_2$ , while  $1/\omega(z)$  has the same poles as  $Y(z)$  in  $\Omega_{-1} \cup \Omega_{-2}$ . Finally,  $\omega(z)/X(z)$  converges to 1 at the poles of  $X(z)$  in  $\Omega_1 \cup \Omega_2$ .

**Theorem 2.** *Let  $z_0 \in \Omega_0$  and let  $\mathcal{L}$  be a path from  $z_0$  to  $z_0 + \pi$  contained in the closure  $\overline{\Omega_0}$  of  $\Omega_0$ . Then  $P_V(z)$ ,  $P_H(z)$  and  $B(t)$  are given by the integrals*

$$P_H(u) = \frac{1}{2\pi it} \int_{\mathcal{L}} X(z)^p Y(z)^q \frac{\omega'(z)}{\omega(z) - \omega(u)} dz \quad \text{for } u \in \Omega_1 \cup \Omega_2 \quad (3.1)$$

$$P_V(u) = -\frac{1}{2\pi it} \int_{\mathcal{L}} X(z)^p Y(z)^q \frac{\omega(u)}{\omega(z)} \frac{\omega'(z)}{\omega(z) - \omega(u)} dz \quad \text{for } u \in \Omega_{-1} \cup \Omega_{-2} \quad (3.2)$$

$$B(t) = -\frac{1}{2\pi it} \int_{\mathcal{L}} X(z)^p Y(z)^q \frac{\omega'(z)}{\omega(z)} dz \quad (3.3)$$

The proof of this theorem involves checking that these expressions satisfy the conditions in Theorem 1.

### 3.2 Classification of $C(x, y; t)$ into complexity hierarchy

In this section we very briefly describe the properties of the step set  $S$ , or equivalently  $X(z)$  and  $Y(z)$ , which determine the nature of the generating function  $C(x, y; t)$ .

In certain cases, called *finite group cases*,  $\gamma = \frac{M}{N}\pi\tau$  for some positive  $M, N \in \mathbb{Z}$  independent of  $t$ . Then applying (2.15)  $N$  times yields

$$P_H(2\pi\tau(N - M) + z) - P_H(z) = P_H((2\pi\tau - 2\gamma)N + z) - P_H(z) = E(z),$$

where  $E(z)$  is called the orbit sum of the model. We can use this equation to solve for  $P_H(z)$ , yielding an expression for  $C(x, y; t)$  which is D-finite in  $x$ . In the cases where  $E(z) = 0$ , we can even prove that  $C(x, y; t)$  is algebraic in  $x$ .

In all other cases we have  $\frac{\gamma}{\pi\tau} \notin \mathbb{Q}$  for generic  $t$ . The model is then said to decouple if there are rational functions  $R_1, R_2$  satisfying  $X(z)^p Y(z)^q = R_1(X(z)) + R_2(Y(z))$ . These cases can be solved (with integral-free expressions) as, using (2.14), the function

$$f(z) := R_1(X(z)) - P_H(z) = -R_2(Y(z)) + B(t) + P_v(z)$$

is an elliptic function which can be determined exactly. In these cases every function used is D-algebraic in all of its variables.

Finally in the infinite group cases which do not decouple, the generating function can be shown to be non-D-algebraic in  $x$  using galois theory of  $q$ -difference equations, as in [8, 15] for  $Q(x, y; t)$ .

### 3.3 Special case: walks starting on $x$ -axis

We will now study the special cases where the walk starts at some point  $(p, 0)$  for  $p > 0$ . Trotignon and Raschel conjectured that with this starting point all finite group models admit algebraic generating functions [20]. Indeed, if  $q = 0$  it is easy to see that the orbit sum  $E(z)$  defined in section 3.2 is equal to 0, so this follows from our more general results. Moreover, if  $q = 0$ , the model trivially decouples, so even in the infinite group case the generating function  $C(x, y; t)$  is D-algebraic.

The following lemma follows directly from Theorem 1, where  $\omega(z)$  is defined as in Section 3.1.

**Lemma 5.** *If the starting point of the walks is  $(p, 0)$  for some  $p \geq 1$ , then there is a degree  $p$  polynomial  $H$  satisfying*

$$P_V(z) = H(\omega(z)) - H(0), \quad (3.4)$$

$$B(t) = H(0), \quad (3.5)$$

$$P_H(z) = X(z)^p - H(\omega(z)). \quad (3.6)$$

Moreover, this polynomial is uniquely determined by the fact that the right hand side of (3.6) has a root at  $z = \delta$ .

In fact, we can convert these directly to formulae for  $A_H$ ,  $A_V$  and  $B$  using series  $W_1\left(\frac{1}{x}; t\right) \in x\mathbb{Z}\left[\frac{1}{x}\right][[t]]$  and  $W_2\left(\frac{1}{y}; t\right) \in \frac{1}{y}\mathbb{Z}\left[\frac{1}{y}\right][[t]]$  satisfying  $W_1\left(\frac{1}{X(z)}; t\right) = \omega(z)$  for  $z \in \Omega_{-1} \cup \Omega_{-2}$  and  $W_2\left(\frac{1}{Y(z)}; t\right) = \omega(z)$  for  $z \in \Omega_1 \cup \Omega_2$ . Note that these depend on the step-set but not the starting point  $(p, q)$  of the walk. For general  $p$ , we can rewrite Lemma 5 as the following theorem:

**Theorem 3.** *If the starting point of the walks is  $(p, 0)$  for some  $p \geq 1$ , then there is a degree  $p$  polynomial  $H$  satisfying*

$$A_V\left(\frac{1}{y}\right) = H\left(W_2\left(\frac{1}{y}; t\right)\right) - H(0), \quad (3.7)$$

$$B(t) = H(0), \quad (3.8)$$

$$A_H\left(\frac{1}{x}\right) = x^p - H\left(W_1\left(\frac{1}{x}; t\right)\right). \quad (3.9)$$

Moreover, this polynomial is uniquely determined by the fact that the right hand side of (3.9) is a series in  $\frac{1}{x}\mathbb{Z}\left[\frac{1}{x}\right][[t]]$ .

In the  $p = 1$  case, we have  $W_2\left(\frac{1}{y}; t\right) = A_V\left(\frac{1}{y}\right)$  and  $W_1\left(\frac{1}{x}; t\right) = x - B(t) - A_H\left(\frac{1}{x}\right)$ , which can be used as alternative definitions for  $W_1$  and  $W_2$ .

### 3.4 Special case: simple walks

We now describe the case of simple walks, that is, unweighted walks with step-set  $S = \{(0, 1), (1, 0), (0, -1), (-1, 0)\}$ . In this case  $X(z)$  and  $Y(z)$  can be written in terms of the Jacobi theta function

$$\vartheta(z, \tau) := \sum_{n=0}^{\infty} e^{i\pi\tau n(n+1)} \left( e^{(2n+1)iz} - e^{-(2n+1)iz} \right).$$

as

$$X(z) = e^{-i\gamma} \frac{\vartheta(z, \tau)\vartheta(z + \gamma, \tau)}{\vartheta(z - \gamma, \tau)\vartheta(z + 2\gamma, \tau)} \quad \text{and} \quad Y(z) = e^{-i\gamma} \frac{\vartheta(z, \tau)\vartheta(z - \gamma, \tau)}{\vartheta(z + \gamma, \tau)\vartheta(z - 2\gamma, \tau)},$$

where  $\gamma = \frac{\pi\tau}{4}$  and is related to  $t$  by

$$e^{-i\gamma} \frac{\vartheta\left(\frac{\gamma}{2}, \tau\right)^2}{\vartheta\left(\frac{3\gamma}{2}, \tau\right)^2} = \frac{1 + 2t - \sqrt{1 + 4t}}{2t}.$$

Moreover, the function  $\omega(z)$  defined in Section 3.1 is given by

$$\omega(z) = e^{-3i\gamma} \frac{\vartheta(2\gamma, \tau)\vartheta'(0, \frac{3\tau}{2})\vartheta(\gamma, \frac{3\tau}{2})}{\vartheta'(0, \tau)\vartheta(2\gamma, \frac{3\tau}{2})\vartheta(3\gamma, \frac{3\tau}{2})} \cdot \frac{\vartheta(z + \gamma, \frac{3\tau}{2})\vartheta(z + 2\gamma, \frac{3\tau}{2})}{\vartheta(z - \gamma, \frac{3\tau}{2})\vartheta(z + 4\gamma, \frac{3\tau}{2})},$$

which has  $\pi$  and  $\frac{3\pi\tau}{2}$  as periods. Since  $X(z)$  and  $\omega(z)$  share the periods  $\pi$  and  $3\pi\tau$ , they are related by a polynomial equation. One such equation is

$$\frac{1}{2t} - Y(z) - \frac{1}{Y(z)} = X(z) + \frac{1}{X(z)} - \frac{1}{2t} = \frac{\omega(z) + c_1}{\omega(z) - c_1} \left( \omega(z) + \frac{c_1^2}{\omega(z)} + c_2 \right),$$

where  $c_1$  and  $c_2$  are given by

$$c_1 = -e^{-i\gamma} \frac{\vartheta(2\gamma, \tau)\vartheta'(0, \frac{3\tau}{2})\vartheta(\gamma, \frac{3\tau}{2})}{\vartheta'(0, \tau)\vartheta(2\gamma, \frac{3\tau}{2})\vartheta(3\gamma, \frac{3\tau}{2})},$$

$$c_2 = \frac{1 + 4t}{2t} \cdot \frac{1 + c_3}{1 - c_3} + c_1 c_3 + \frac{c_1}{c_3}, \quad \text{where} \quad c_3 = -e^{i\gamma} \frac{\vartheta\left(\frac{5\gamma}{2}, \frac{3\pi\tau}{2}\right)}{\vartheta\left(\frac{\gamma}{2}, \frac{3\pi\tau}{2}\right)}.$$

So the series  $W_1\left(\frac{1}{x}; t\right) \in x\mathbb{Z}\left[\frac{1}{x}\right]$  and  $W_2\left(\frac{1}{y}; t\right) \in \frac{1}{y}\mathbb{Z}\left[\frac{1}{y}\right]$  defined in Section 3.3 satisfy

$$-\frac{1}{2t} + x + \frac{1}{x} = \frac{W_1\left(\frac{1}{x}; t\right) + c_1}{W_1\left(\frac{1}{x}; t\right) - c_1} \left( W_1\left(\frac{1}{x}; t\right) + \frac{c_1^2}{W_1\left(\frac{1}{x}; t\right)} + c_2 \right),$$

$$\frac{1}{2t} - y - \frac{1}{y} = \frac{W_2\left(\frac{1}{y}; t\right) + c_1}{W_2\left(\frac{1}{y}; t\right) - c_1} \left( W_2\left(\frac{1}{y}; t\right) + \frac{c_1^2}{W_2\left(\frac{1}{y}; t\right)} + c_2 \right).$$

Then for any starting point  $(p, 0)$ , the generating functions  $A_H, A_V, B$  and hence  $C(x, y; t)$  can be determined by Theorem 3. Moreover,  $c_1, c_2$  and  $t$  can be shown to be modular functions of  $\tau$ , so they are all algebraically related. Hence in these cases the generating function  $C(x, y; t)$  is algebraic in  $t$  as well as the other variables.

## 4 Nature of $C(x, y; t)$ with respect to $t$

The main remaining problem is to prove that the generating function  $C(x, y; t)$  has the same nature (algebraic, D-finite, D-algebraic) as a function of  $t$  as it does as a function of  $x$  and  $y$ . Even for  $Q(x, y; t)$ , which counts walks confined to a quadrant, this has not been proven for weighted models, so it is not surprising that we have so far been unable to prove it for  $C(x, y; t)$ . However, we expect that if this is proven for  $Q(x, y; t)$ , the result for  $C(x, y; t)$  will follow using the same method applied to Theorem 1.

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