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# 1 **Acoustic and gravity waves in the ocean: a new** 2 **derivation of a linear model from the compressible** 3 **Euler equation**

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14 In this paper we construct an accurate linear model describing the propagation of both  
15 acoustic and gravity waves in water. This original model is obtained by the linearization of  
16 the compressible Euler equations, written in Lagrangian coordinates. The system is studied  
17 in the isentropic case, with a free surface, an arbitrary bathymetry, and vertical variations of  
18 the background temperature and density. We show that our model is an extension of some  
19 models from the literature to the case of a non-barotropic fluid with a variable sound speed.  
20 Other models from the literature are recovered from our model through two asymptotic  
21 analysis, one for the incompressible regime and one for the acoustic regime. We also propose  
22 a method to write the model in Eulerian coordinates. Our model includes many physical  
23 properties, such as the existence of internal gravity waves or the variation of the sound speed  
24 with depth.

25 **Key words:**

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## 26 **1. Introduction**

27 Several authors have proposed to use the propagation of acoustic waves in the ocean to  
28 detect tsunamis, as the sound travels in water at approximately  $1500 \text{ m s}^{-1}$  and the velocity of  
29 tsunami wave is around  $300 \text{ m s}^{-1}$  (Constantin 2009). The existence of hydro-acoustic signals  
30 generated by tsunami sources such as earthquakes or landslides was shown in Tolstoy (1950).  
31 This motivates the mathematical modelling of the propagation of both surface waves – the

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32 tsunami – and underwater acoustic waves, also called hydroacoustic waves, in a compressible  
33 formulation.

34 The idea of using acoustic-gravity waves for tsunami early-warning systems dates back  
35 from 1950 (Ewing *et al.* 1950). A more recent study (Stiassnie 2010) indicates that the  
36 pressure variations induced by the tsunami are significant enough to be used for the  
37 improvement of the tsunami early-warning systems.

38 For the description of the propagation of sound in water, the most common model is a  
39 linear wave equation for the fluid potential, i.e. for an irrotational fluid (Jensen *et al.* 2011).  
40 When both surface and acoustic waves are considered, different types of models are available.  
41 In his work, Stiassnie (2010) studies the acoustic equation for the fluid potential coupled with  
42 a free-boundary condition. The three-dimensional acoustic equation is analyzed by Nosov  
43 & Kolesov (2007) and a depth-integrated version is proposed by Sammarco *et al.* (2013) in  
44 order to reduce the computational costs. This approach was further developed in a serie of  
45 papers (Cecioni *et al.* 2014; Abdolali *et al.* 2015; Gomez & Kadri 2021).

46 Another approach was proposed by Longuet-Higgins (1950) where the equation is still  
47 on the fluid potential, but includes a gravity term. This equation, including second order  
48 terms, made it possible for the first time to explain the seismic noise generated worldwide by  
49 waves interactions in the ocean (?). This model was also the starting point of an extensive  
50 work to describe the nonlinear interactions between acoustic and gravity waves (Kadri &  
51 Stiassnie 2013). In other works, such as Smith (2015) and Auclair *et al.* (2021), the flow is  
52 not assumed irrotational, so that the equations are written for the fluid velocity. They include  
53 gravity terms and a vertical stratification for the background density, temperature and salinity.  
54 This generalisation allows to study the internal waves caused by the stratification of the fluid,  
55 and dispersion relations for the three types of waves (acoustic, internal, surface) are obtained.

56 The above cited works share one or several of the following assumptions : irrotational  
57 flow, homogeneous background density or barotropic fluid, and a constant speed of sound.  
58 These modelling choices have a strong influence on the structure of the equations, resulting  
59 in a variety of tools for their analysis and their numerical approximation. For example, the  
60 irrotationality assumption allows to reduce the number of unknowns, but the validity of this  
61 assumption in the compressible case is not clear. Furthermore, in the models that do not  
62 assume irrotational flow, the bed is assumed to be flat, even though bed variations are a key  
63 element impacting tsunami and acoustic wave propagation (Caplan-Auerbach *et al.* 2014).  
64 In the ocean, the choice of a constant sound speed may be not appropriate since the variation  
65 of the sound speed creates the SOFAR channel, a horizontal strip in which the acoustic  
66 waves propagate with very little energy loss. Quantifying the impact of these approximations  
67 requires the use of simulations based on a more complete model. Finally, the free-surface  
68 equation induces a strong nonlinearity in the system. Indeed, the domain on which the  
69 equation are written depends on the solution to the equations. The common approach for  
70 the linearization of the system consists in writing the linear equations on the unperturbed  
71 domain. However, by doing so an approximation on the domain is made, in addition to the  
72 approximation made on the solution. It is not clear how to quantify the magnitude of the  
73 error made by the two combined approximations.

74 The aim of this work is to adress these modelling choices by deriving an accurate linear  
75 model as rigorously as possible with only very few assumptions for hydro-acoustic, internal  
76 and surface waves propagating in a fluid over an arbitrary bathymetry. Salinity, thermal  
77 dissipation and viscosity are neglected, and in order to linearize the equations we assume  
78 that the ocean is at equilibrium and at rest before the earthquake or landslide occurs, and  
79 that the tsunami source induces a small displacement of the water. In this model, the speed  
80 of sound results from the imposed background temperature profile, so that the effects of the  
81 SOFAR channel on the propagation of the hydroacoustic waves are naturally present. The

82 obtained model is comparable to the model of Auclair *et al.* (2021), however our model  
 83 includes a bathymetry and a variable sound speed. Moreover our approach differs on several  
 84 aspects:

85 • The problem is formulated as a second-order equation, which allows the use of numerical  
 86 solver dedicated to wave propagation problem such as Specfem (Komatitsch & Tromp 1999).  
 87 Specfem uses spectral finite elements to compute acoustic and/or elastic wave propagations,  
 88 and is widely used in the seismology community, as for example to simulate seismic waves  
 89 generated by landslides (Kuehnert *et al.* 2020). In addition to the acoustic waves already  
 90 modelled in Specfem, the model proposed in the present paper includes the linear water  
 91 waves.

92 • The method used to write the linearization of a free-surface flow is generic, and can  
 93 be applied to extend the model. A possible extension would include second-order terms (a  
 94 similar work was done in Longuet-Higgins (1950) in the barotropic case). Another possibility  
 95 is to take into account the interaction with the Earth. In particular, one can consider the elastic  
 96 deformations of the ocean bottom, that are shown to impact the travel time of tsunami waves  
 97 (Abdolali *et al.* 2019).

98 Another advantage of having a model with few assumptions is that a cascade of simplified  
 99 systems can be obtained from it. We indeed show that with some simplifying assumptions, our  
 100 model reduces to the models proposed in the litterature. The analysis of these simplifications  
 101 helps to understand the mathematical and physical choices made in these models. For  
 102 example, the most common model for the propagation of hydro-acoustic waves (Stiassnie  
 103 2010; Nosov & Kolesov 2007; Sammarco *et al.* 2013) is recovered from the proposed model  
 104 by assuming a barotropic fluid and a constant background density.

105 We also show that our model and the simplified models are energy-preserving. Our model  
 106 is a linear version of the Euler equations, and the equation accounting for the energy  
 107 conservation may be modified by the linearization. To ensure that the obtained model is  
 108 physically relevant, we check that an equation for the energy conservation holds in the  
 109 linear case. Beyond this aspect, the energy preservation allows to write stable numerical  
 110 schemes (Allaire 2015). Indeed, the properties of a numerical scheme are often related to the  
 111 preservation of a discrete energy. For these reasons, the energy preservation is a key feature,  
 112 both in the continuous and in the discrete level.

113 The paper is organized as follows. In Sec. 2 the compressible Euler equations for a free-  
 114 surface flow are written, then the system is transformed in Lagrangian coordinates in order  
 115 to keep an exact description of the free surface. After linearization, a wave-like equation for  
 116 the fluid velocity is obtained and we show that the energy of the system is preserved. In Sec.  
 117 3, we show that with additionnal assumptions, the model reduces to other linear models from  
 118 the litterature. The barotropic case is studied, then the incompressible limit and the acoustic  
 119 limit of the wave equation are written. In Sec. 4, we present a method allowing to write  
 120 the model in Eulerian coordinates. The obtained system can be linearized at the cost of an  
 121 additional approximation, namely that the equations have to be restricted to a fixed domain,  
 122 and we show how to obtain a linear free-surface condition. Finally, in Sec 5 we obtain a  
 123 dispersion relation which includes all of the physical effects mentioned above. In particular,  
 124 it is a generalization of the dispersion relation studied in the work of Auclair *et al.* (2021) to  
 125 the case of a varying sound speed.

## 126 2. Linearization of compressible Euler equations in Lagrangian coordinates

127 We derive here a linear model around a state at rest for the isentropic compressible Euler  
 128 equation with a free surface and an arbitrary bathymetry, valid for a generic equation of state  
 129 and a generic vertical temperature profile. We aim at deriving a model which is physically

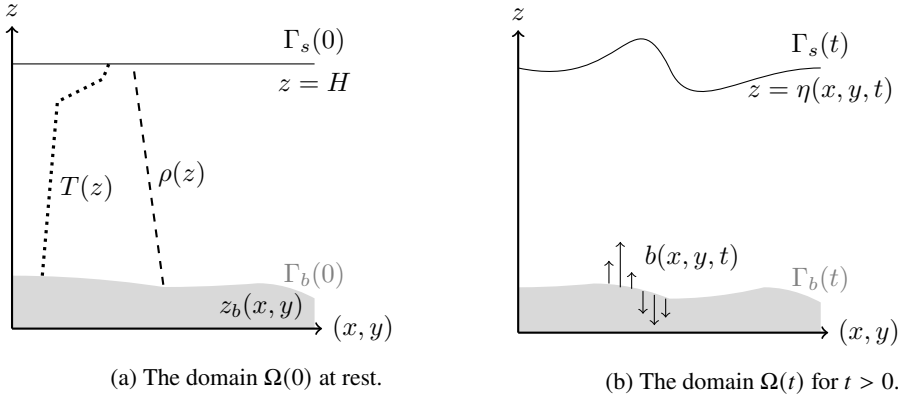


Figure 1: The domain  $\Omega(t)$ , (1a) for time  $t = 0$ , (1b) for time  $t > 0$ . In (1a) typical profiles for the temperature and the density at rest are drawn.

130 relevant in the sense that it preserves or dissipates energy. For this reason, we will analyse  
 131 the energy equation associated to this system and show that preservation or dissipation of  
 132 energy requires a condition on the fluid stratification that is related to the internal waves.

133 We consider a portion of the ocean away from the coast and at equilibrium: there is no  
 134 mean current and the temperature varies only vertically. In this work, we do not take the  
 135 presence of salinity into account, hence the ocean is assimilated to pure water. The bottom  
 136 and the surface of the domain are assumed to be parametrized as graphs, respectively the  
 137 topography  $z_b(x, y) \geq 0$  and the free-surface elevation  $\eta(x, y, t)$ . The reference level  $z = 0$   
 138 is situated inside the earth at an arbitrary level. The ground displacement induced by an  
 139 earthquake or landslide source is assumed to take place away from the coast, so that the  
 140 domain is considered infinite in the  $(x, y)$  plane, see Figure 1. The domain is assumed to  
 141 have the following description, for all time  $t$ ,

$$142 \quad \Omega(t) = \{(x, y, z) \in \mathbb{R}^3 \mid z_b(x, y) < z < \eta(x, y, t)\}. \quad (2.1)$$

143 The boundary of the domain are then defined by

$$144 \quad \Gamma_s(t) = \{(x, y, z) \in \mathbb{R}^3 \mid z = \eta(x, y, t)\}, \quad (2.2)$$

145 and

$$146 \quad \Gamma_b(t) = \{(x, y, z) \in \mathbb{R}^3 \mid z = z_b(x, y) - b(x, y, t)\}. \quad (2.3)$$

147 The function  $b$  is source term, namely the normal displacement of the seabed. It can represent  
 148 for example an earthquake or a landslide. It is assumed that this displacement starts at a time  
 149  $t_0 > 0$ , so that  $b(x, y, 0) = 0$ .

## 150 2.1. Euler equation in Eulerian coordinate

### 151 2.1.1. Equations in the volume

152 The unknowns are the fluid velocity  $\mathbf{U}$ , its density  $\rho$ , its pressure  $p$ , its temperature  $T$ , its  
 153 internal energy  $e$  and its entropy  $s$ . The gradient over the horizontal components will be  
 154 denoted  $\nabla_{x,y}$ .

155 For future reference the equations are written for a viscous fluid with thermal dissipation.  
 156 The stress tensor of a Newtonian fluid  $\mathcal{T}$  has the form

$$157 \quad \mathcal{T} = (-p + \lambda \nabla \cdot \mathbf{U})\mathbf{I} + 2\mu \mathbb{D}(\mathbf{U}), \quad (2.4)$$

158 Where  $\mathbb{D}(\mathbf{U})$  is defined by  $\mathbb{D}(\mathbf{U}) = (\frac{1}{2}(\partial_i \mathbf{U}^j + \partial_j \mathbf{U}^i))_{i,j=x,y,z}$  and  $\mathbb{I}$  is the identity matrix in  
 159  $\mathbb{R}^3$ . The heat flux is denoted by  $\mathbf{q}$  and is a function of  $\rho$  and  $T$ .

The conservation of mass, momentum and energy of a Newtonian fluid read, in the domain  $\Omega(t)$ ,

$$\left\{ \begin{array}{l} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0, \end{array} \right. \quad (2.5)$$

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} (\rho \mathbf{U}) + \nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{g} + \nabla(\lambda \nabla \cdot \mathbf{U}) + \nabla \cdot (2\mu \mathbb{D}(\mathbf{U})), \end{array} \right. \quad (2.6)$$

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \left( \rho \frac{|\mathbf{U}|^2}{2} + \rho e \right) + \nabla \cdot \left( \left( \rho \frac{|\mathbf{U}|^2}{2} + \rho e + p \right) \mathbf{U} \right) \\ = \rho \mathbf{g} \cdot \mathbf{U} + \nabla \cdot (\lambda \mathbf{U} \nabla \cdot \mathbf{U}) + \nabla \cdot (2\mu \mathbb{D}(\mathbf{U}) \cdot \mathbf{U}) - \nabla \cdot \mathbf{q}. \end{array} \right. \quad (2.7)$$

160 The acceleration of gravity is  $\mathbf{g} = -g \mathbf{e}_3$  with  $g > 0$  and  $\mathbf{e}_3$  is the unit vector in the vertical  
 161 direction, oriented upwards.

162 In order to describe the acoustic waves, we derive an equation for the pressure. Among  
 163  $(\rho, e, T, p, s)$  only two variables are independent because of the Gibbs law and of the equation  
 164 of state (Gill 1982). When considering  $\rho$  and  $s$  as independent, it is natural to introduce the  
 165 scalar functions  $f_e, f_p$  and  $f_T$  satisfying

$$166 \quad e = f_e(\rho, s), \quad p = f_p(\rho, s), \quad T = f_T(\rho, s). \quad (2.8)$$

167 With the Gibbs law ( $\partial f_e / \partial \rho = f_p / \rho^2$  and  $\partial f_e / \partial s = f_T$ ), one has

$$168 \quad \frac{\partial e}{\partial t} + \mathbf{U} \cdot \nabla e = \frac{f_p}{\rho^2} \left( \frac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla \rho \right) + f_T \left( \frac{\partial s}{\partial t} + \mathbf{U} \cdot \nabla s \right). \quad (2.9)$$

169 Using (2.7) –  $\mathbf{U} \cdot$  (2.6) and (??), one obtains as an intermediate step the evolution equation  
 170 of the entropy,

$$171 \quad \rho T \left( \frac{\partial s}{\partial t} + \mathbf{U} \cdot \nabla s \right) = \lambda (\nabla \cdot \mathbf{U})^2 + 2\mu \mathbb{D}(\mathbf{U}) : \mathbb{D}(\mathbf{U}) - \nabla \cdot \mathbf{q}. \quad (2.10)$$

172 Now, since  $p = f_p(\rho, s)$  we have

$$173 \quad \frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p = \frac{\partial f_p}{\partial \rho} \left( \frac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla \rho \right) + \frac{1}{T \rho} \frac{\partial f_p}{\partial s} \left( \frac{\partial s}{\partial t} + \mathbf{U} \cdot \nabla s \right), \quad (2.11)$$

174 hence using (2.5) and (??) one obtains

$$175 \quad \frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p = -\frac{\partial f_p}{\partial \rho} (\rho \nabla \cdot \mathbf{U}) + \frac{1}{T \rho} \frac{\partial f_p}{\partial s} \left( \lambda (\nabla \cdot \mathbf{U})^2 + 2\mu \mathbb{D}(\mathbf{U}) : \mathbb{D}(\mathbf{U}) - \nabla \cdot \mathbf{q} \right). \quad (2.12)$$

176 At this point we use the common assumption that the viscous term and the thermal dissipation  
 177 can be neglected compared to the advection term (see Lannes 2013, Chap. 1). With (??) we  
 178 see that this is equivalent to assuming that the flow is isentropic. Moreover, from physical  
 179 considerations the function  $f_p$  must satisfy  $\partial_\rho f_p(\rho, s) \geq 0$ , hence we can introduce the  
 180 speed of sound  $c$  defined by

$$181 \quad c^2 = \frac{\partial f_p}{\partial \rho}(\rho, s). \quad (2.13)$$

182 The equation (??) then reads

$$183 \quad \frac{\partial p}{\partial t} + \mathbf{U} \cdot \nabla p + \rho c^2 \nabla \cdot \mathbf{U} = 0. \quad (2.14)$$

184 Eq. (??) is used for the study of a compressible fluid in the isentropic case, see Gill (1982),  
 185 Chap. 4. Note that the speed of sound  $c$  can also be viewed as a function of  $p$  and  $T$ , in that  
 186 case we have

$$187 \quad c^2(p, T) = \frac{\partial f_p}{\partial \rho} (f_\rho(p, T), f_s(p, T)). \quad (2.15)$$

188 In practice we choose to work directly with the expression  $c = c(p, T)$  tabulated in  
 189 International Association for the Properties of Water and Steam (2009). Note that here  
 190 the temperature intervenes as a side variable, because it is necessary to compute the speed  
 191 of sound. However, we will see later that only the temperature profile of the state at rest is  
 192 needed to close the system.

### 193 2.1.2. Boundary conditions

The following boundary conditions hold,

$$\begin{cases} \mathbf{U} \cdot \mathbf{n}_b = u_b = \partial_t u_b & \text{on } \Gamma_b, \\ p = p^a & \text{on } \Gamma_s. \end{cases} \quad (2.16)$$

$$(2.17)$$

194 The bottom boundary condition (2.9) is a non-penetration condition with a source term. It  
 195 models the tsunami source as a displacement of the ocean bottom with velocity  $u_b$ . We denote  
 196 by  $\mathbf{n}_b$  the unit vector normal to the bottom and oriented outwards. The second condition  
 197 (2.10) is a dynamic condition, where we assume that the surface pressure is at equilibrium  
 198 with a constant atmospheric pressure  $p^a$ . Note that the elevation  $\eta$  is solution of the following  
 199 kinematic equation

$$200 \quad \frac{\partial \eta}{\partial t} + \mathbf{U} \cdot \begin{pmatrix} \partial_x \eta \\ \partial_y \eta \\ -1 \end{pmatrix} = 0 \quad \text{on } \Gamma_s(t). \quad (2.18)$$

### 201 2.1.3. Initial conditions and equilibrium state

202 It is assumed that the initial state corresponds to the rest state, meaning that  $\eta(x, y, 0) = H$   
 203 with the elevation at rest  $H$  being independent of space and  $H > z_b(x, y)$ , therefore

$$204 \quad \Omega(0) = \{(x, y, z) \in \mathbb{R}^3 \mid z_b(x, y) < z < H\}. \quad (2.19)$$

205 We choose the following initial conditions for the velocity, the temperature, the density and  
 206 the pressure,

$$207 \quad \mathbf{U}(x, y, z, 0) = 0, \quad (2.20)$$

$$208 \quad T(x, y, z, 0) = T_0(z), \quad \rho(x, y, z, 0) = \rho_0(z), \quad p(x, y, z, 0) = p_0(z). \quad (2.21)$$

210 Where  $T_0, \rho_0, p_0$  are functions defined on  $(0, H)$  but because of the topography  $z_b(x, y)$ , the  
 211 functions  $T, \rho, p$  need not to be defined from  $z = 0$  for all  $(x, y)$ .

212 When the source term  $u_b$  vanishes, we have an equilibrium state around  $\mathbf{U} \equiv 0$  if the  
 213 functions  $T_0, \rho_0, p_0$  satisfy

$$214 \quad \nabla p_0 = \rho_0 \mathbf{g}, \quad \rho_0 = f_\rho(p_0, T_0), \quad p_0(H) = p^a. \quad (2.22)$$

Hence, if  $T_0(z)$  is given, the system

$$\begin{cases} \frac{dp_0}{dz} = -g f_\rho(p_0, T_0), & z \in (0, H), \\ p_0 = p^a, & z = H. \end{cases} \quad (2.23)$$

$$(2.24)$$

216 can be solved to compute  $p_0$ , and then  $\rho_0$  is computed with  $\rho_0 = f_\rho(p_0, T_0)$ . Note that, in the

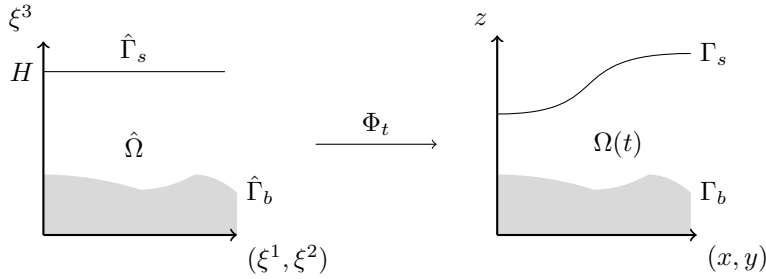


Figure 2: The mapping  $\phi_t$  between the reference domain  $\hat{\Omega} = \Omega(0)$  and the domain  $\Omega(t)$ .

217 forthcoming sections the system (2.5), (2.6), (??) with boundary conditions (2.9), (2.10) and  
 218 initial conditions (2.13), (2.14) will be linearized around the previously defined equilibrium  
 219 state.

220

221

## 2.2. Lagrangian description

222 Although most of the works on free-surface flows are done in Eulerian coordinates, the  
 223 Lagrangian formalism is sometimes preferred, see for example the paper from Nouguiet  
 224 *et al.* (2015) and the references therein, or the work of Godlewski *et al.* (1999) for a precise  
 225 derivation of linear models. Here we choose the Lagrangian description in order to avoid  
 226 any approximation on the shape of the domain when we linearize the equations. The usual  
 227 approximation made on the surface for the linear models in Eulerian coordinates consists  
 228 in evaluating the surface condition on pressure at a fixed height, rather than at the actual,  
 229 time-dependant free surface. The kinematic boundary condition is also replaced by its linear  
 230 approximation. For the derivation and justification of the approximation, see Lighthill (1978,  
 231 Chap. 3).

232 Let  $\hat{\Omega}$  be the domain of the ocean at a reference time, with its surface boundary  $\hat{\Gamma}_s$  and  
 233 bottom boundary  $\hat{\Gamma}_b$ . The reference time is chosen before the tsunami generation, so that the  
 234 surface of the domain is horizontal. In fact the following natural choice is made

$$235 \quad \hat{\Omega} = \Omega(0), \quad \hat{\Gamma}_s = \Gamma_s(0), \quad \hat{\Gamma}_b = \Gamma_b(0). \quad (2.25)$$

236 The position at the reference time of a fluid particle is denoted

$$237 \quad \xi = (\xi^1, \xi^2, \xi^3) \in \hat{\Omega}. \quad (2.26)$$

238 At time  $t$ , the fluid has moved, the domain is  $\Omega(t)$  and the new position of a fluid particle is  
 239  $\mathbf{x} = (x(\xi, t), y(\xi, t), z(\xi, t)) \in \Omega(t)$ . We denote by  $\phi$  the transformation from  $\hat{\Omega}$  to  $\Omega(t)$  that  
 240 maps each particle from its reference position  $\xi$  to its position  $\mathbf{x}$  at time  $t$  (see figure 2).

$$241 \quad \phi : \begin{cases} \hat{\Omega} & \rightarrow \Omega(t) \\ \xi & \mapsto \mathbf{x}(\xi, t) \end{cases} \quad (2.27)$$

242 Hence one has  $\mathbf{x} = \phi(\xi, t)$ . The transformation is assumed invertible, in particular we do not  
 243 consider the case of wave breaking. We also define the displacement of the fluid. For each  
 244 fluid particle with initial position  $\xi$ , its displacement is defined by

$$245 \quad \mathbf{d}(\xi, t) = \phi(\xi, t) - \xi. \quad (2.28)$$

246 The gradient of  $\phi$  with respect to  $\xi$  is denoted  $F$ ,

$$247 \quad F = \nabla_{\xi} \phi, \quad (2.29)$$



248 and its determinant is denoted  $J$ . Both  $F$  and  $J$  can be expressed as functions of the  
249 displacement,

$$250 \quad F = \mathbb{I} + \nabla_{\xi} \mathbf{d}, \quad J = \det F, \quad (2.30)$$

251 where  $\nabla_{\xi}$  is the gradient with respect to the coordinate system  $\xi$ . For a function  $X(\mathbf{x}, t)$   
252 defined on the domain  $\Omega(t)$ , we introduce  $\hat{X}(\xi, t)$  defined on  $\hat{\Omega}$  by

$$253 \quad \hat{X}(\xi, t) = X(\phi(\xi, t), t). \quad (2.31)$$

254 Finally, note that the velocity  $\hat{U}(\xi, t) = U(\phi(\xi, t), t)$  is the time derivative of the displacement  
255  $\mathbf{d}$ .

$$256 \quad \hat{U} = \frac{\partial \mathbf{d}}{\partial t}. \quad (2.32)$$

With this change of coordinates, the system (2.5), (2.6), (??) is now defined in the time-independent reference domain  $\hat{\Omega}$  and it reads

$$\left\{ \begin{array}{l} \frac{\partial \hat{\rho}}{\partial t} + \frac{\hat{\rho}}{|J|} \nabla_{\xi} \cdot (|J| F^{-1} \hat{U}) = 0, \end{array} \right. \quad (2.33)$$

$$\left\{ \begin{array}{l} \hat{\rho} \frac{\partial \hat{U}}{\partial t} + F^{-T} \nabla_{\xi} \hat{p} = \hat{\rho} \mathbf{g}, \end{array} \right. \quad (2.34)$$

$$\left\{ \begin{array}{l} \frac{\partial \hat{p}}{\partial t} + \frac{\hat{\rho} \hat{c}^2}{|J|} \nabla_{\xi} \cdot (|J| F^{-1} \hat{U}) = 0. \end{array} \right. \quad (2.35)$$

The boundary conditions become

$$\left\{ \begin{array}{ll} \hat{U} \cdot \hat{\mathbf{n}}_b = \hat{u}_b & \text{on } \hat{\Gamma}_b, \\ \hat{p} = p^a & \text{on } \hat{\Gamma}_s, \end{array} \right.$$

257 where  $\hat{\mathbf{n}}_b$  is a unit vector normal to  $\hat{\Gamma}_b$  and pointing toward the exterior of the domain. The  
258 variables  $\hat{\rho}, \hat{p}, \hat{T}$  satisfy the same equation of state

$$259 \quad \hat{p} = f_p(\hat{\rho}, \hat{T}), \quad (2.36)$$

260 and the speed of sound is a function of the new variables,  $\hat{c} = c(\hat{\rho}, \hat{s})$ .

### 261 *2.3. Linearization and wave equation*

262 We assume that the source of the tsunami is a displacement of magnitude  $a$  at the seafloor  
263 occurring in an ocean at rest as described in Sec. 2.1.3. In particular, for this rest state  
264 there is no mean current and the temperature, pressure and density vary only vertically. The  
265 magnitude of the displacement is assumed small compared to the water height  $H$ . The ratio  
266 of the bottom displacement amplitude to the water height is denoted  $\varepsilon = a/H \ll 1$ , and the  
267 source term can be expressed as

$$268 \quad \hat{u}_b = \varepsilon \hat{u}_{b,1} + \mathcal{O}(\varepsilon^2). \quad (2.37)$$

269 The linearization of the equations (2.26)-(2.28) around the rest state corresponds to the  
270 following asymptotic expansion,

$$271 \quad \mathbf{d}(\xi, t) = \varepsilon \mathbf{d}_1(\xi, t) + \mathcal{O}(\varepsilon^2), \quad (2.38)$$

$$272 \quad \hat{\rho}(\xi, t) = \hat{\rho}_0(\xi) + \varepsilon \hat{\rho}_1(\xi, t) + \mathcal{O}(\varepsilon^2), \quad (2.39)$$

$$273 \quad \hat{p}(\xi, t) = \hat{p}_0(\xi) + \varepsilon \hat{p}_1(\xi, t) + \mathcal{O}(\varepsilon^2). \quad (2.40)$$

275 Note that the displacement has no zero order term, because the reference configuration used  
276 to define the Lagrangian description is the state given by the initial conditions. It holds then

277  $\mathbf{d}_0 = 0$ ,  $\hat{\mathbf{U}}_0 = 0$  and  $\hat{\Omega} = \Omega(0)$ .

278

279 *Remark:* In comparison with the linearization done in Auclair *et al.* (2021), where the  
280 expansion of the density and the pressure is justified with the decomposition into hydrostatic  
281 and non-hydrostatic components, the asymptotic expansion (2.32) - (2.33) is obtained in a  
282 more straightforward way. Indeed it only requires the assumption of a small perturbation.

283

284 From the expansion one deduces the following Taylor expansions for the other functions,

$$285 \quad \hat{\mathbf{U}} = \varepsilon \hat{\mathbf{U}}_1 + \mathcal{O}(\varepsilon^2), \quad (2.41)$$

$$286 \quad F = \mathbb{I} + \varepsilon \nabla_\xi \mathbf{d}_1 + \mathcal{O}(\varepsilon^2), \quad (2.42)$$

$$287 \quad (F)^{-1} = \mathbb{I} - \varepsilon \nabla_\xi \mathbf{d}_1 + \mathcal{O}(\varepsilon^2), \quad (2.43)$$

$$288 \quad J = 1 + \varepsilon \nabla_\xi \cdot \mathbf{d}_1 + \mathcal{O}(\varepsilon^2). \quad (2.44)$$

Injecting these expressions in Eq.(2.26)-(2.28) yields the system

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} (\hat{\rho}_0 + \varepsilon \hat{\rho}_1) + \varepsilon \hat{\rho}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1 = \mathcal{O}(\varepsilon^2), \quad (2.45) \\ \varepsilon \hat{\rho}_0 \frac{\partial \hat{\mathbf{U}}_1}{\partial t} + (I - \varepsilon \nabla_\xi \mathbf{d}_1)^T \nabla_\xi \hat{p}_0 + \varepsilon \nabla_\xi \hat{p}_1 = (\hat{\rho}_0 + \varepsilon \hat{\rho}_1) \mathbf{g} + \mathcal{O}(\varepsilon^2), \quad (2.46) \\ \frac{\partial}{\partial t} (\hat{p}_0 + \varepsilon \hat{p}_1) + \varepsilon \hat{p}_0 c^2 (\hat{p}_0, \hat{T}_0) \nabla_\xi \cdot \hat{\mathbf{U}}_1 = \mathcal{O}(\varepsilon^2). \quad (2.47) \end{array} \right.$$

290 By separating the powers of  $\varepsilon$  we obtain two systems: a limit system when  $\varepsilon \rightarrow 0$  and  
291 a system for the first order corrections. Since the limit system correspond to the initial  
292 conditions described in Sec. 2.1.3, the model reduces to the first order system.

293 *First-order system: a wave-like equation for the velocity*

The system for the correction terms reads in  $\hat{\Omega}$ ,

$$\left\{ \begin{array}{l} \hat{\rho}_0 \frac{\partial \hat{\mathbf{U}}_1}{\partial t} + \nabla_\xi \hat{p}_1 - (\nabla_\xi \mathbf{d}_1)^T \nabla_\xi \hat{p}_0 = \hat{\rho}_1 \mathbf{g}, \quad (2.48) \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial \hat{p}_1}{\partial t} + \hat{\rho}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1 = 0, \quad (2.49) \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial \hat{p}_1}{\partial t} + \hat{\rho}_0 \hat{c}_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 = 0, \quad (2.50) \end{array} \right.$$

with the boundary conditions

$$\left\{ \begin{array}{ll} \hat{\mathbf{U}}_1 \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} & \text{on } \hat{\Gamma}_b, \\ \hat{p}_1 = 0 & \text{on } \hat{\Gamma}_s. \end{array} \right. \quad (2.51)$$

$$(2.52)$$

294 In this system the speed of sound is evaluated at the limit – or background – pressure and  
295 temperature,  $\hat{c}_0 = c(\hat{p}_0, \hat{T}_0)$ . In particular,  $\hat{c}_0$  can be written as a function of depth. With an  
296 adapted temperature profile it is then possible to recover the typical speed of sound profile  
297 creating the SOFAR channel.

298 The pressure  $\hat{p}_1$  and density  $\hat{\rho}_1$  can be eliminated in (2.41) thanks to the other equations:  
299 differentiating in time (2.41) and replacing  $\hat{\rho}_1$  and  $\hat{p}_1$  with (2.42), (2.43) we obtain a second  
300 order equation for  $\hat{\mathbf{U}}_1$ ,

$$301 \quad \hat{\rho}_0 \frac{\partial^2 \hat{\mathbf{U}}_1}{\partial t^2} - \nabla_\xi \left( \hat{\rho}_0 \hat{c}_0^2 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \right) - (\nabla_\xi \hat{\mathbf{U}}_1)^T \hat{\rho}_0 \mathbf{g} + \hat{\rho}_0 \nabla_\xi \cdot \hat{\mathbf{U}}_1 \mathbf{g} = 0 \text{ in } \hat{\Omega}. \quad (2.53)$$

Using (2.43), the surface boundary condition (2.45) is formulated for  $\hat{U}_1$ , hence the two boundary conditions for the wave-like equation (2.46) are

$$\begin{cases} \hat{U}_1 \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} & \text{on } \hat{\Gamma}_b, \\ \nabla_{\xi} \cdot \hat{U}_1 = 0 & \text{on } \hat{\Gamma}_s. \end{cases} \quad (2.54)$$

The wave-like equation (2.46) is completed with vanishing initial condition for  $\hat{U}_1(0)$  and  $\partial_t \hat{U}_1(0)$ . The system (2.46) includes both gravity and acoustic terms. This equation, which describes the velocity of a compressible, non-viscous fluid, in Lagrangian description, is called the Galbrun equation. It is used in helioseismology and in aeroacoustics (Legendre 2003; Maeder *et al.* 2020; Hägg & Berggren 2021). However, to our knowledge, this equation has never been used to describe the propagation of hydro-acoustic waves.

We show now that the system (2.46), (2.47), (2.48) is energy-preserving. A model describing a physical system should either preserve or dissipate an energy, and this property makes it also possible to write a stable numerical scheme. Here the energy equation is obtained by taking the scalar product of (2.46) with  $\partial_t \hat{U}_1$  and integrating over the domain. After some computations (see Appendix, Sec. A), we have

$$\frac{d}{dt} \mathcal{E} = \int_{\hat{\Gamma}_b} \hat{\rho}_0 \left( c_0^2 \nabla_{\xi} \cdot \hat{U}_1 - \hat{\rho}_0 g \hat{U}_1 \cdot \mathbf{e}_3 \right) \frac{\partial \hat{u}_{b,1}}{\partial t} d\sigma, \quad (2.56)$$

with the energy being the quadratic functional given by

$$\begin{aligned} \mathcal{E} = \int_{\hat{\Omega}} \rho_0 \frac{1}{2} \left| \frac{\partial \hat{U}_1}{\partial t} \right|^2 d\xi + \frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 \left( \hat{c}_0 \nabla_{\xi} \cdot \hat{U}_1 - \frac{g}{\hat{c}_0} \hat{U}_1 \cdot \mathbf{e}_3 \right)^2 d\xi \\ + \frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 N^2 (\hat{U}_1 \cdot \mathbf{e}_3)^2 d\xi + \frac{1}{2} \int_{\Gamma_s} \hat{\rho}_0 g (\hat{U}_1 \cdot \mathbf{e}_3)^2 d\sigma. \end{aligned} \quad (2.57)$$

The scalar  $N_b$  is the squared Brunt-Väisälä frequency, defined by

$$N_b(\xi^3) = - \left( \frac{g^2}{\hat{c}_0(\xi^3)^2} + g \frac{\hat{\rho}'_0(\xi^3)}{\hat{\rho}_0(\xi^3)} \right). \quad (2.58)$$

The Brunt-Väisälä frequency, or buoyancy frequency, is closely related to the internal waves that appear in a stratified medium (Gill 1982, Chap. 6). In the ocean, the usual values of  $N_b$  are around  $10^{-8} \text{ rad}^2 \text{ s}^{-2}$  (King *et al.* 2012).

The physical interpretation for the different terms in the energy is clearer when one writes the wave-like equation (2.46) in terms of the displacement  $\mathbf{d}_1$  instead of the velocity  $\hat{U}_1$ . Using  $\partial_t \mathbf{d}_1 = \hat{U}_1$  and integrating Eq. (2.46) once in time with the vanishing initial conditions for the displacement, one obtains

$$\hat{\rho}_0 \frac{\partial^2 \mathbf{d}_1}{\partial t^2} - \nabla_{\xi} \left( \hat{\rho}_0 \hat{c}_0^2 \nabla_{\xi} \cdot \mathbf{d}_1 \right) - (\nabla_{\xi} \mathbf{d}_1)^T \hat{\rho}_0 \mathbf{g} + \hat{\rho}_0 \nabla_{\xi} \cdot \mathbf{d}_1 \mathbf{g} = 0 \text{ in } \hat{\Omega}. \quad (2.59)$$

The exact same steps of Appendix A, with  $\mathbf{d}_1$  instead of  $\hat{U}_1$ , yield the energy equation

$$\frac{d}{dt} \mathcal{E}_d = \int_{\hat{\Gamma}_b} \hat{\rho}_0 \left( c_0^2 \nabla_{\xi} \cdot \mathbf{d}_1 - \hat{\rho}_0 g \mathbf{d}_1 \cdot \mathbf{e}_3 \right) \hat{u}_{b,1} d\sigma, \quad (2.60)$$

with the energy

329

$$\begin{aligned} \mathcal{E}_d = \int_{\hat{\Omega}} \hat{\rho}_0 \frac{1}{2} |\hat{U}_1|^2 d\xi + \frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 \left( \hat{c}_0 \nabla_{\xi} \cdot \mathbf{d}_1 - \frac{g}{\hat{c}_0} \mathbf{d}_1 \cdot \mathbf{e}_3 \right)^2 d\xi \\ + \frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 N_b (\mathbf{d}_1 \cdot \mathbf{e}_3)^2 d\xi + \frac{1}{2} \int_{\Gamma_s} \hat{\rho}_0 g (\mathbf{d}_1 \cdot \mathbf{e}_3)^2 d\sigma. \end{aligned} \quad (2.61)$$

331

The first term in Eq. (??) is the kinetic energy. We show that the second term of Eq. (??) corresponds to the acoustic energy. First, using Eq. (2.43) with the vanishing initial conditions yields  $\hat{\rho}_0 \hat{c}_0^2 \cdot \nabla_{\xi} \mathbf{d}_1 = -\hat{p}_1$ . We define then the acoustic pressure

335

$$p_a = \hat{p}_1 - \nabla \hat{p}_0 \cdot \mathbf{d}_1. \quad (2.62)$$

Indeed, in Lagrangian coordinates, the pressure perturbation  $\hat{p}_1$  has two contributions: the small variations in acoustic pressure, and the background pressure being evaluated at a new position. With the definition of  $p_a$  and the equation (2.17), it holds for the second term of (??)

340

$$\frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 \left( \hat{c}_0 \nabla_{\xi} \cdot \mathbf{d}_1 - \frac{g}{\hat{c}_0} \mathbf{d}_1 \cdot \mathbf{e}_3 \right)^2 d\xi = \frac{1}{2} \int_{\hat{\Omega}} \frac{p_a^2}{\hat{\rho}_0 \hat{c}_0^2} d\xi, \quad (2.63)$$

which is the usual expression for the acoustic energy (Lighthill 1978). The last term of (??) is the potential energy associated to the surface waves. Finally, the third term of (??) is the potential energy associated to the internal gravity waves (Lighthill 1978), under the condition

344

$$N_b > 0. \quad (2.64)$$

When  $N_b$  is positive, it is denoted  $N_b = N^2$ , where  $N$  is the buoyancy frequency. The sign of  $N_b$  depends on the choice of the state at equilibrium:  $\hat{\rho}'_0 = d\hat{\rho}_0/dz$  has to be negative and satisfy

348

$$\frac{|\hat{\rho}'_0|}{\hat{\rho}_0} > \frac{g}{\hat{c}_0^2}. \quad (2.65)$$

With the term in  $g^2/\hat{c}_0^2$ , we see that the compressibility tends to take the fluid away from its equilibrium. The stratification of the fluid must be strong enough to counter this effect and keep the system stable (see the discussion in Gill 1982, Chap. 3). As a consequence, if one wants the model to preserve the energy of the system, the background density should not be assumed homogeneous. In the following, we assume that the fluid has a stable stratification, namely that the function  $N_b$  is assumed always positive. We will use the notation  $N^2$  in the rest of this paper.

356

*Remark:* According to the equation of state (when the salinity is neglected)  $\rho = f_{\rho}(p, T)$ , the background density varies because of the variations in temperature and in pressure. The temperature profile can be chosen homogeneous, but the effect of gravity – see Eq. (2.17) – prevents the pressure to be independent of depth. Hence in a model with gravity, the fluid is always stratified with the density increasing with depth.

362

*Remark:* One can notice that the condition (??) is not explicit in Eq. (2.46) We obtain this condition when imposing that the energy  $\mathcal{E}_d$  is positive.

363

364

### 365 3. Derivation of simplified models

366 In order to compare with existing models, we present several simplifications of our model. We  
 367 first show that in the barotropic case the system (2.46) - (2.48) is equivalent to the first-order  
 368 scalar equation of Longuet-Higgins (1950). Our model also reduces to well-known models  
 369 in the acoustic and incompressible asymptotic regimes, as demonstrated below. Further  
 370 numerical implementations of our model will make it possible to quantify the impact of  
 371 assumptions made in more simple models, in particular in the case of acoustic-gravity wave  
 372 generation by earthquakes or landslides in the ocean.

#### 373 3.1. The barotropic case

We consider the barotropic case, which is a very common assumption for the study of hydro-  
 acoustic waves (see for example Longuet-Higgins (1950), Stiassnie (2010)). For a barotropic  
 fluid the pressure is a function of the density only,

$$f_p(\rho, s) = f_p(\rho) = p.$$

374 Then, using Eq. (2.15) and the definition of the speed of sound,

$$375 \hat{\rho}'_0 = \hat{\rho}'_0 \frac{df_p}{d\rho}(\hat{\rho}_0) \Rightarrow -\hat{\rho}_0 g = \hat{\rho}'_0 \hat{c}_0^2, \quad (3.1)$$

376 meaning that the Brunt-Väisälä frequency vanishes,  $N^2 = 0$ . This corresponds to the case  
 377 where the density is stratified because of the variation of pressure only. In order to use this  
 378 equality, we divide Eq. (2.46) by  $\hat{\rho}_0$ ,

$$379 \frac{\partial^2 \hat{U}_1}{\partial t^2} - \nabla_\xi \left( \hat{c}_0^2 \nabla_\xi \cdot \hat{U}_1 \right) - \left( \frac{\hat{\rho}'_0}{\hat{\rho}_0} \hat{c}_0^2 + g \right) \nabla_\xi \cdot \hat{U}_1 \mathbf{e}_3 + \nabla_\xi (\hat{U}_1 \cdot \mathbf{e}_3) g = 0, \quad (3.2)$$

380 and when Eq. (3.1) holds, the equation (3.2) can be simplified and reads

$$381 \frac{\partial^2 \hat{U}_1}{\partial t^2} - \nabla_\xi \left( \hat{c}_0^2 \nabla_\xi \cdot \hat{U}_1 \right) + \nabla_\xi (\hat{U}_1 \cdot \mathbf{e}_3) g = 0. \quad (3.3)$$

382 Taking the curl of Eq. (3.3) yields

$$383 \frac{\partial^2 \nabla_\xi \times \hat{U}_1}{\partial t^2} = 0 \text{ in } \hat{\Omega}. \quad (3.4)$$

384 With the vanishing initial conditions we obtain that the velocity of a barotropic fluid is  
 385 irrotational. This is a well-known result, since the fluid is also inviscid and subject to a  
 386 potential force only (Guyon 2001, Chap. 7). By the Helmholtz decomposition theorem  
 387 (Girault & Raviart 1986) the fluid velocity is written as the gradient of a potential  $\psi$  defined  
 388 up to a constant. The expression  $\hat{U}_1 = \nabla_\xi \psi$  is used in Eq. (3.3), to obtain

$$389 \nabla_\xi \left( \frac{\partial^2 \psi}{\partial t^2} - \hat{c}_0^2 \Delta_\xi \psi + g \frac{\partial \psi}{\partial \xi^3} \right) = 0. \quad (3.5)$$

390 The potential  $\psi$  being defined up to a constant, it can always be sought as the solution of

$$391 \frac{\partial^2 \psi}{\partial t^2} - \hat{c}_0^2 \Delta_\xi \psi + g \frac{\partial \psi}{\partial \xi^3} = 0. \quad (3.6)$$

392 The equation (3.6) is multiplied by  $\hat{\rho}_0 / \hat{c}_0^2$ , and we use  $g / \hat{c}_0^2 = -\hat{\rho}'_0 / \hat{\rho}_0$ ,

$$393 \frac{\hat{\rho}_0}{\hat{c}_0^2} \frac{\partial^2 \psi}{\partial t^2} - \hat{\rho}_0 \Delta_\xi \psi - \hat{\rho}'_0 \frac{\partial \psi}{\partial \xi^3} = 0. \quad (3.7)$$

394 And since  $\hat{\rho}_0$  depends only on  $\xi^3$ , the two last terms can be rewritten,

$$395 \quad \frac{\hat{\rho}_0}{\hat{c}_0^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla_{\xi} \cdot (\hat{\rho}_0 \nabla_{\xi} \psi) = 0. \quad (3.8)$$

Hence  $\psi$  satisfies a wave equation. The boundary conditions are then deduced from Eq. (2.47) and Eq. (2.48),

$$\begin{cases} \nabla_{\xi} \psi \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} \text{ on } \hat{\Gamma}_b, \\ \hat{c}_0^2 \Delta_{\xi} \psi = \frac{\partial^2 \psi}{\partial t^2} + g \frac{\partial \psi}{\partial \xi^3} = 0 \text{ on } \hat{\Gamma}_s. \end{cases} \quad (3.9)$$

396 The system (3.6),(3.7),(3.8) is the first-order system obtained in Longuet-Higgins (1950). In  
397 Longuet-Higgins (1950), the derivation is quite different since the irrotationality assumption  
398 is made independantly from the fact that the fluid is barotropic, and the boundary conditions  
399 are obtained from a linearized surface condition. The linearization made in Longuet-Higgins  
400 (1950) gives exactly the same result as the linearization strategy we have presented.

401 We show that the system (??),(3.7),(3.8) is energy-preserving. The equation (??) is  
multiplied by  $\partial_t \psi$  and integrated by parts,

$$\begin{aligned} 402 \quad & \int_{\hat{\Omega}} \frac{\hat{\rho}_0}{\hat{c}_0^2} \frac{\partial \psi}{\partial t} \frac{\partial^2 \psi}{\partial t^2} d\xi + \int_{\hat{\Omega}} \hat{\rho}_0 \nabla \left( \frac{\partial \psi}{\partial t} \right) \cdot \nabla \psi d\xi \\ 403 \quad & - \int_{\hat{\Gamma}_s} \hat{\rho}_0 \frac{\partial \psi}{\partial t} \nabla \psi \cdot \mathbf{e}_3 d\sigma + \int_{\hat{\Gamma}_b} \hat{\rho}_0 \frac{\partial \psi}{\partial t} \nabla \psi \cdot \mathbf{n}_b d\sigma = 0. \end{aligned} \quad (3.11)$$

405 With the boundary conditions (3.7) - (3.8) and after simplifications it holds

$$406 \quad \frac{d}{dt} \mathcal{E}_{bar} = - \int_{\Gamma_b} \hat{\rho}_0 \frac{\partial \psi}{\partial t} \hat{u}_{b,1} d\sigma, \quad (3.12)$$

407 where the energy  $\mathcal{E}_{bar}$  is defined by

$$408 \quad \mathcal{E}_{bar} = \frac{1}{2} \int_{\hat{\Omega}} \frac{\hat{\rho}_0}{\hat{c}_0^2} \left( \frac{\partial \psi}{\partial t} \right)^2 d\xi + \frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 |\nabla \psi|^2 d\xi + \frac{1}{2} \int_{\Gamma_s} \frac{\hat{\rho}_0}{g} \left( \frac{\partial \psi}{\partial t} \right)^2 d\xi. \quad (3.13)$$

409 The first term of Eq. (3.10) is the acoustic energy. Indeed, with (??) and (3.1) one can show  
410 that the acoustic pressure  $p_a$  and the potential  $\psi$  satisfy the usual relation  $p_a = -\hat{\rho}_0 \partial_t \psi$   
411 (Lighthill 1978, Chap.3). The second term of (3.10) is the kinetic energy. Finally, with  
412 (3.8), one sees that the third term of (3.10) is the potential energy of the surface waves. In  
413 order to obtain the energy equation for the barotropic system (??), it is necessary to use the  
414 background density  $\hat{\rho}_0$  even if it does not appear in Eq. (??). The correct manipulation for  
415 writing the energy equation was found by comparison with the general case described by Eq.  
416 (2.46).

Finally, note that when assuming a homogeneous density in the equation (??), the system (??)-(3.8) reduce to

$$\begin{cases} \frac{\partial^2 \psi}{\partial t^2} - \hat{c}_0^2 \Delta \psi = 0 \text{ in } \hat{\Omega} \end{cases} \quad (3.14)$$

$$\begin{cases} \nabla_{\xi} \psi \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} \text{ on } \hat{\Gamma}_b, \end{cases} \quad (3.15)$$

$$\begin{cases} \frac{\partial^2 \psi}{\partial t^2} + g \frac{\partial \psi}{\partial \xi^3} = 0 \text{ on } \hat{\Gamma}_s. \end{cases} \quad (3.16)$$

417 and the energy equation (3.10) is not modified by this assumption. However, assuming a

418 homogeneous density is not compatible with the derivation of the system (??)-(3.8), which  
 419 relies on the equality  $g/\hat{c}_0^2 = -\hat{\rho}'_0/\hat{\rho}_0$ . The model (??)-(??) can be understood as a barotropic  
 420 model with the additional assumption that both  $-\hat{\rho}'_0/\hat{\rho}_0$  and  $g/\hat{c}_0^2$  are neglected inside the  
 421 domain.

### 422 3.2. Two asymptotic regimes of the system

423 In this section we write the limit models for two asymptotic regimes of the system (2.46)-  
 424 (2.48). We consider the incompressible regime, where the acoustic waves are neglected, and  
 425 the acoustic regime, where the effect of gravity is neglected. The wave equation (2.46) is  
 426 written in non-dimensional form, and we show that it depends on a small non-dimensional  
 427 parameter. A simplified model is then obtained by passing formally to the limit when the  
 428 small parameter vanishes. By making the appropriate choice for the time scale we obtain first  
 429 an incompressible approximation, then an acoustic approximation.

#### 430 3.2.1. Non-dimensional equation

431 We introduce the following characteristic scales for the system: a time  $\tau$ , a horizontal scale  $L$ ,  
 432 a vertical scale  $H$ , a density  $\bar{\rho}$ , and a fluid velocity  $U$ . Since the speed of sound is not assumed  
 433 constant, we denote by  $C$  its characteristic magnitude. Finally, the surface waves velocity is  
 434 of the order of  $\sqrt{gH}$  (Constantin 2009). We focus on a non-shallow water formulation, hence  
 435 we take  $L = H$ . For a shallow water version of the equation one would choose  $H \ll L$ .

436 Two dimensionless numbers are introduced: the Froude number and the Mach number,  
 437 respectively defined by

$$438 \quad \text{Fr} = \frac{U}{\sqrt{gH}}, \quad \text{Ma} = \frac{U}{C}. \quad (3.17)$$

439 To fix the idea, we choose the following numerical values respectively for the speed of  
 440 sound, the fluid velocity and the surface waves velocity:  $C \sim 1480 \text{ m s}^{-1}$ ,  $U \sim 1 \text{ m s}^{-1}$  and  
 441  $\sqrt{gH} \sim 100 \text{ m s}^{-1}$ . The dimensionless numbers are then

$$442 \quad \text{Fr} = 0.01, \quad \text{Ma} = 6.10^{-4}. \quad (3.18)$$

443 The characteristic scale for time will be fixed later, as it will depend on the regime we want  
 444 to study. The variables are put in non-dimensional form and the dimensionless variables are  
 445 denoted with a  $\tilde{\cdot}$ , except for the space and time variable for the sake of conciseness. The  
 446 adimensionned domain is denoted by  $\tilde{\Omega}$  and its surface and bottom boundary are respectively  
 447  $\tilde{\Gamma}_s$  and  $\tilde{\Gamma}_b$ . The non-dimensional system reads, after simplification by the factor  $\bar{\rho}U$ ,

$$448 \quad \frac{\tilde{\rho}_0}{\tau^2} \frac{\partial^2 \tilde{\mathbf{U}}_1}{\partial t^2} - \frac{C^2}{L^2} \nabla_{\xi} \left( \tilde{\rho}_0 \tilde{c}_0^2 \nabla_{\xi} \cdot \tilde{\mathbf{U}}_1 \right) + \frac{g}{L} \tilde{\rho}_0 \left( \nabla_{\xi} (\tilde{\mathbf{U}}_1 \cdot \mathbf{e}_3) - \nabla_{\xi} \cdot \tilde{\mathbf{U}}_1 \mathbf{e}_3 \right) = 0, \quad (3.19)$$

with the boundary conditions

$$\begin{cases} \tilde{\mathbf{U}}_1 \cdot \tilde{\mathbf{n}}_b = \tilde{u}_{b,1} & \text{on } \tilde{\Gamma}_b, \\ \nabla_{\xi} \cdot \tilde{\mathbf{U}}_1 = 0 & \text{on } \tilde{\Gamma}_s, \end{cases} \quad (3.20)$$

$$(3.21)$$

449 where  $\tilde{u}_{b,1}$  is a dimensionless source term.

#### 450 3.2.2. Incompressible limit

451 We show that in the incompressible regime, our model is an extension of the classical  
 452 free-surface Poisson equation to the case of a variable background density.

453 In order to study the incompressible limit, the characteristic time  $\tau$  is chosen to follow the

454 surface waves, which are much slower than the acoustic waves. We take  $L/\tau = \sqrt{gH}$ . The  
455 equation (3.13) becomes

$$456 \quad \tilde{\rho}_0 \frac{\partial^2 \tilde{U}_1}{\partial t^2} - \frac{\text{Fr}}{\text{Ma}} \nabla_\xi \left( \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{U}_1 \right) + \tilde{\rho}_0 \left( \nabla_\xi (\tilde{U}_1 \cdot \mathbf{e}_3) - \nabla_\xi \cdot \tilde{U}_1 \mathbf{e}_3 \right) = 0. \quad (3.22)$$

457 The small parameter  $\delta = \text{Ma}/\text{Fr} \sim 6 \cdot 10^{-2}$  is introduced in the equation,

$$458 \quad \tilde{\rho}_0 \frac{\partial^2 \tilde{U}_1}{\partial t^2} - \frac{1}{\delta^2} \nabla_\xi \left( \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{U}_1 \right) + \tilde{\rho}_0 \left( \nabla_\xi (\tilde{U}_1 \cdot \mathbf{e}_3) - \nabla_\xi \cdot \tilde{U}_1 \mathbf{e}_3 \right) = 0, \quad (3.23)$$

459 and the goal is now to calculate the limit of equation (3.17) when  $\delta$  goes to zero. We make  
460 the following ansatz for  $\tilde{U}_1$ ,

$$461 \quad \tilde{U}_1 = \tilde{U}_{1,0} + \delta^2 \tilde{U}_{1,2} + \mathcal{O}(\delta^3), \quad (3.24)$$

462 where  $\tilde{U}_{1,0}$ ,  $\tilde{U}_{1,1}$  and  $\tilde{U}_{1,2}$  are independant of  $\delta$ . Since Eq. (3.17) has only even powers of  $\delta$ ,  
463 the term  $\tilde{U}_{1,1}$  is equal to zero. Replacing  $\tilde{U}_1$  by its ansatz in the wave equation (3.17) and  
464 separating the powers of  $\delta$  yields an equation for each term of the asymptotic developement  
465 of  $\tilde{U}_1$ . The equation obtained with the terms in  $\delta^{-2}$  reads

$$466 \quad \nabla_\xi \left( \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{U}_{1,0} \right) = 0, \quad (3.25)$$

467 and the equation obtained with the terms  $\delta^0$  reads

$$468 \quad \tilde{\rho}_0 \frac{\partial^2 \tilde{U}_{1,0}}{\partial t^2} - \nabla_\xi \left( \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{U}_{1,2} \right) + \tilde{\rho}_0 \left( \nabla_\xi (\tilde{U}_{1,0} \cdot \mathbf{e}_3) - \nabla_\xi \cdot \tilde{U}_{1,0} \mathbf{e}_3 \right) = 0. \quad (3.26)$$

With the terms in  $\delta^0$  of the boundary conditions we have

$$\begin{cases} \nabla_\xi \cdot \tilde{U}_{1,0} = 0 & \text{on } \tilde{\Gamma}_s, \\ \tilde{U}_{1,0} \cdot \tilde{\mathbf{n}}_b = \tilde{u}_{b,1} & \text{on } \tilde{\Gamma}_b. \end{cases} \quad (3.27)$$

$$(3.28)$$

And the terms in  $\delta^2$  of the boundary conditions read

$$\begin{cases} \nabla_\xi \cdot \tilde{U}_{1,2} = 0 & \text{on } \tilde{\Gamma}_s, \\ \tilde{U}_{1,2} \cdot \tilde{\mathbf{n}}_b = 0 & \text{on } \tilde{\Gamma}_b. \end{cases} \quad (3.29)$$

$$(3.30)$$

469 We show now that the limit model represents an incompressible flow. The Helmholtz  
470 decomposition of  $\tilde{U}_{1,0}$  reads

$$471 \quad \tilde{U}_{1,0} = \nabla_\xi \varphi_{1,0} + \nabla_\xi \times \boldsymbol{\psi}_{1,0}, \quad (3.31)$$

472 where  $\varphi_{1,0}$  vanishes on  $\tilde{\Gamma}_s$  and  $\tilde{\Gamma}_b$ . Injecting the decomposition of  $\tilde{U}_{1,0}$  in Eq. (3.19) yields

$$473 \quad \nabla_\xi \left( \tilde{\rho}_0 \tilde{c}_0^2 \Delta_\xi \varphi_{1,0} \right) = 0, \quad (3.32)$$

474 hence the term inside the gradient is constant in space. Since the velocity  $\tilde{U}_{1,0}$  is equal to  
475 zero at infinity, we obtain that  $\Delta_\xi \varphi_{1,0} = 0$  in  $\tilde{\Omega}$  (the quantity  $\tilde{\rho}_0 \tilde{c}_0$  being always strictly  
476 positive). With the vanishing boundary conditions for  $\varphi_{1,0}$ , we obtain that  $\varphi_{1,0}$  is equal to  
477 zero everywhere in  $\tilde{\Omega}$ . Then, taking the divergence of  $\tilde{U}_{1,0}$  yields

$$478 \quad \nabla_\xi \cdot \tilde{U}_{1,0} = \nabla_\xi \cdot (\nabla_\xi \times \boldsymbol{\psi}_{1,0}) = 0, \quad (3.33)$$

479 hence  $\tilde{U}_{1,0}$  is divergence-free.



480 Now, using the property  $\nabla \cdot \tilde{\mathbf{U}}_{1,0}$  in the equation (3.20) and rearranging some terms, we  
481 obtain

$$482 \quad \tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_{1,0}}{\partial t^2} - \nabla_\xi \left( \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,2} \right) + \nabla_\xi \left( \tilde{\rho}_0 \tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 \right) - \tilde{\rho}'_0 (\tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3) \mathbf{e}_3 = 0. \quad (3.34)$$

483 Taking the curl of this equation yields

$$484 \quad \nabla_\xi \times \left( \tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_{1,0}}{\partial t^2} - \tilde{\rho}'_0 (\tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3) \mathbf{e}_3 \right) = 0, \quad (3.35)$$

485 This means that these terms can be expressed as the gradient of a potential function defined  
486 up to a constant and denoted  $-\tilde{\varphi}_0$ ,

$$487 \quad \tilde{\rho}_0 \frac{\partial^2 \tilde{\mathbf{U}}_{1,0}}{\partial t^2} - \tilde{\rho}'_0 (\tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3) \mathbf{e}_3 = -\nabla_\xi \tilde{\varphi}_0. \quad (3.36)$$

488 The new function  $\tilde{\varphi}_0$  can be understood as the Lagrange multiplier for the incompressibility  
489 constraint. However, one must be cautious that  $\tilde{\varphi}_0$  is not similar to a pressure in this case, and  
490 rather plays the role of a velocity potential, as we will see later in the case of homogeneous  
491 density. The function  $\tilde{\varphi}_0$  can be expressed differently. By using the definition (3.30) in the  
492 equation (3.28) we have

$$493 \quad \nabla_\xi \left( -\tilde{\varphi}_0 - \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,2} + \tilde{\rho}_0 \tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 \right) = 0, \quad (3.37)$$

494 and since the potential  $\tilde{\varphi}_0$  is defined up to a constant, it can be chosen such that, in  $\hat{\Omega}$ , we  
495 have

$$496 \quad \tilde{\varphi}_0 = -\tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{\mathbf{U}}_{1,2} + \tilde{\rho}_0 \tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3. \quad (3.38)$$

497 We deduce from this equality and (3.23) the boundary condition

$$498 \quad \tilde{\varphi}_0 = \tilde{\rho}_0 \tilde{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 \quad \text{on } \tilde{\Gamma}_s. \quad (3.39)$$

To recover a dimensional system, the terms are multiplied by their corresponding character-  
istic scales, and  $\hat{\varphi}_0 = \bar{\rho} U \tilde{\varphi}_0$  is defined. The limit solution  $\hat{\mathbf{U}}_{1,0} = U \tilde{\mathbf{U}}_{1,0}$  satisfies

$$\begin{cases} \hat{\rho}_0 \frac{\partial^2 \hat{\mathbf{U}}_{1,0}}{\partial t^2} - g \hat{\rho}'_0 (\hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3) \mathbf{e}_3 + g \nabla_\xi \hat{\varphi}_0 = 0 & \text{in } \hat{\Omega}, \\ \nabla_\xi \cdot \hat{\mathbf{U}}_{1,0} = 0 & \text{in } \hat{\Omega}, \end{cases} \quad (3.40)$$

with the boundary conditions

$$\begin{cases} \hat{\mathbf{U}}_{1,0} \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} & \text{on } \hat{\Gamma}_b, \end{cases} \quad (3.42)$$

$$\begin{cases} \nabla_\xi \cdot \hat{\mathbf{U}}_{1,0} = 0 & \text{on } \hat{\Gamma}_s, \end{cases} \quad (3.43)$$

$$\begin{cases} \hat{\varphi}_0 = \hat{\rho}_0 \hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 & \text{on } \hat{\Gamma}_s, \end{cases} \quad (3.44)$$

499 We show that the model (3.34)-(3.38) preserves an energy. Taking the scalar product of  
500 (3.34) with  $\partial_t \tilde{\mathbf{U}}_{1,0}$  and integrating over  $\hat{\Omega}$  yields

$$501 \quad \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0 \left| \frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t} \right|^2 d\xi - \int_{\hat{\Omega}} g \hat{\rho}'_0 (\hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3) \mathbf{e}_3 \cdot \frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t} d\xi + \int_{\hat{\Omega}} g \frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t} \cdot \nabla_\xi \hat{\varphi}_0 d\xi = 0. \quad (3.45)$$

502 The last term of Eq. (3.39) is integrated by parts. With the vanishing divergence of  $\hat{\mathbf{U}}_{1,0}$  and

503 the bottom condition (3.36) it holds

$$504 \quad \int_{\hat{\Omega}} g \frac{\partial \hat{U}_{1,0}}{\partial t} \cdot \nabla_{\xi} \hat{\varphi}_0 \, d\xi = \int_{\hat{\Gamma}_s} g \hat{\varphi}_0 \frac{\partial \hat{U}_{1,0}}{\partial t} \cdot \mathbf{e}_3 \, d\sigma - \int_{\hat{\Gamma}_b} g \hat{\varphi}_0 \frac{\partial \hat{u}_{b,1}}{\partial t} \, d\sigma, \quad (3.46)$$

then  $\hat{\varphi}_0$  is replaced in the surface integral using Eq. (3.38),

$$505 \quad \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0 \left| \frac{\partial \hat{U}_{1,0}}{\partial t} \right|^2 \, d\xi - \int_{\hat{\Omega}} g \hat{\rho}'_0 (\hat{U}_{1,0} \cdot \mathbf{e}_3) \mathbf{e}_3 \cdot \frac{\partial \hat{U}_{1,0}}{\partial t} \, d\xi$$

$$506 \quad + \int_{\hat{\Gamma}_s} g \hat{\rho}_0 \hat{U}_{1,0} \cdot \mathbf{e}_3 \frac{\partial \hat{U}_{1,0}}{\partial t} \cdot \mathbf{e}_3 \, d\sigma = \int_{\hat{\Gamma}_b} g \hat{\varphi}_0 \frac{\partial \hat{u}_{b,1}}{\partial t} \, d\sigma. \quad (3.47)$$

508 By defining the energy

$$509 \quad \mathcal{E}_{incomp} = \frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 \left| \frac{\partial \hat{U}_{1,0}}{\partial t} \right|^2 \, d\xi - \frac{1}{2} \int_{\hat{\Omega}} g \hat{\rho}'_0 |\hat{U}_{1,0} \cdot \mathbf{e}_3|^2 \, d\xi + \frac{1}{2} \int_{\hat{\Gamma}_s} g \hat{\rho}_0 |\hat{U}_{1,0} \cdot \mathbf{e}_3|^2 \, d\sigma, \quad (3.48)$$

510 the equation (3.39) can be formulated in the following way,

$$511 \quad \frac{d}{dt} \mathcal{E}_{incomp} = \int_{\Gamma_b} g \varphi_0 \frac{\partial \hat{u}_{b,1}}{\partial t}. \quad (3.49)$$

512 Each term of  $\mathcal{E}_{incomp}$  has the same interpretation as in  $\mathcal{E}$ . Note that the acoustic term of  $\mathcal{E}$  is  
513 not present in  $\mathcal{E}_{incomp}$ . The potential energy associated to the internal waves is also written  
514 differently, as in the formal limit  $\hat{c}_0 \rightarrow \infty$  the buoyancy frequency reads  $N^2 = -g\hat{\rho}'_0/\hat{\rho}_0$ .

515

516 *Remark:* The condition  $|\hat{\rho}'_0|/\hat{\rho}_0 > g/\hat{c}_0^2$  is no longer required because the destabilizing  
517 effects in the energy equation (2.49) come from the compressibility, and here it is neglected.  
518 This can be seen by formally assuming that the sound velocity is infinite, then the squared  
519 buoyancy frequency reads  $N^2 = -g\hat{\rho}'_0/\hat{\rho}_0$ . Density must still decrease with depth, but can  
520 be homogeneous.

521

522 The system (3.35)-(3.34) represents an incompressible fluid. However, this system is  
523 different from the classical Poisson equation found in the litterature (Lighthill 1978) because  
524 of the assumption of a nonhomogeneous background density. For the sake of comparison  
525 with other models, assume now that the ocean at rest has a homogeneous density,  $\hat{\rho}'_0 = 0$ .  
526 Taking the divergence of Eq. (3.34) yields

$$527 \quad \Delta_{\xi} \hat{\varphi}_0 = 0. \quad (3.50)$$

528 The boundary condition are written differently to ease the comparison. The bottom boundary  
529 condition is obtained by taking the scalar product of Eq. (3.34) with  $\hat{\mathbf{n}}_b$ , and replacing the  
530 first term with Eq. (3.36) differentiated twice in time,

$$531 \quad -\hat{\rho}_0 \frac{\partial^2 \hat{u}_{b,1}}{\partial t^2} - g \hat{\rho}'_0 (\hat{U}_{1,0} \cdot \mathbf{e}_3) \mathbf{e}_3 \cdot \hat{\mathbf{n}}_b + g \nabla_{\xi} \hat{\varphi}_0 \cdot \hat{\mathbf{n}}_b = 0. \quad (3.51)$$

532 For the surface condition, the equation (3.38) is differentiated twice in time and the term in  
533  $\partial_{tt}^2 \hat{U}_{1,0}$  is replaced with (3.34),

$$534 \quad \frac{\partial^2 \hat{\varphi}_0}{\partial t^2} - g \hat{\rho}'_0 (\hat{U}_{1,0} \cdot \mathbf{e}_3) + g \frac{\partial \hat{\varphi}_0}{\partial \xi^3} = 0 \quad \text{on } \tilde{\Gamma}_s. \quad (3.52)$$

With the assumption of a homogeneous density, the boundary conditions (??), (??) read then

$$\begin{cases} \nabla_{\xi} \hat{\varphi}_0 \cdot \hat{\mathbf{n}}_b = -\hat{\rho}_0 g \hat{u}_{b,1} & \text{on } \hat{\Gamma}_b, \\ \frac{\partial^2 \hat{\varphi}_0}{\partial t^2} + g \frac{\partial \hat{\varphi}_0}{\partial \xi^3} = 0 & \text{on } \hat{\Gamma}_s. \end{cases} \quad (3.53)$$

535 The Poisson equation (3.44) with boundary conditions (3.45) - (3.46) is the system satisfied  
 536 by the velocity flow of an incompressible homogeneous free-surface fluid (Lighthill 1978,  
 537 Chap. 3.1). Note that it was required that  $\hat{\rho}'_0 \neq 0$  in the system (2.46) to obtain an a priori  
 538 positive energy. Here this assumption is dropped, however a rather simple expression for the  
 539 preserved energy can be derived: multiplying Eq. (3.44) by  $\partial_t \hat{\varphi}_0$ , integrating by parts and  
 using Eq. (3.45)- (3.46), we obtain

$$\begin{aligned} 540 \quad \int_{\hat{\Omega}} \Delta_{\xi} \hat{\varphi}_0 \frac{\partial \hat{\varphi}_0}{\partial t} d\xi &= - \int_{\hat{\Omega}} \nabla_{\xi} \varphi \cdot \nabla_{\xi} \left( \frac{\partial \hat{\varphi}_0}{\partial t} \right) d\xi \\ 541 \quad &- \int_{\Gamma_s} \frac{1}{g} \frac{\partial \hat{\varphi}_0}{\partial t} \frac{\partial^2 \hat{\varphi}_0}{\partial t^2} d\sigma + \int_{\Gamma_b} \hat{\rho}_0 g \frac{\partial \hat{\varphi}_0}{\partial t} \hat{u}_{b,1} d\sigma. \end{aligned} \quad (3.55)$$

542 We define the energy

$$543 \quad \mathcal{E}_{Poisson} = \frac{1}{2} \left( \int_{\hat{\Omega}} |\nabla_{\xi} \hat{\varphi}|^2 d\xi + \int_{\Gamma_s} \frac{1}{g} \left( \frac{\partial \hat{\varphi}_0}{\partial t} \right)^2 d\sigma \right), \quad (3.56)$$

544 Then it holds

$$545 \quad \frac{d}{dt} \mathcal{E}_{Poisson} = - \int_{\Gamma_b} \hat{\rho}_0 g \frac{\partial \hat{\varphi}_0}{\partial t} \hat{u}_{b,1} d\sigma. \quad (3.57)$$

546 By comparison with the energy of the barotropic system (3.10), we see that the first term of  
 547 (3.48) is the kinetic energy and the second term of (3.48) is the potential energy associated  
 548 to the surface waves.

### 550 3.2.3. Acoustic limit

551 Another possible simplification of the system (2.46) - (2.48) is to keep only the acoustic  
 552 terms. This choice is justified for short time scale, because the propagation speed of the  
 553 acoustic waves and the gravity waves have different orders of magnitude (Longuet-Higgins  
 554 1950). Here we show that in the acoustic limit, the model reduces to a classical acoustic  
 555 equation.

556 With the timescale  $L/\tau = C$ , corresponding to the acoustic wave, and with the same small  
 557 parameter  $\delta = \text{Ma}/\text{Fr}$  as before, the system (3.13) becomes

$$558 \quad \tilde{\rho}_0 \frac{\partial^2 \tilde{U}_1}{\partial t^2} - \nabla_{\xi} \left( \tilde{\rho}_0 \tilde{c}_0^2 \nabla_{\xi} \cdot \tilde{U}_1 \right) + \delta^2 \tilde{\rho}_0 \left( \nabla_{\xi} (\tilde{U}_1 \cdot \mathbf{e}_3) - \nabla_{\xi} \cdot \tilde{U}_1 \mathbf{e}_3 \right) = 0 \quad \text{in } \tilde{\Omega}, \quad (3.58)$$

With the boundary conditions

$$\begin{cases} \tilde{U}_1 \cdot \tilde{\mathbf{n}}_b = \tilde{u}_{b,1} & \text{on } \tilde{\Gamma}_b, \\ \nabla_{\xi} \cdot \tilde{U}_1 = 0 & \text{on } \tilde{\Gamma}_s. \end{cases} \quad (3.59)$$

559 As before, we make the following ansatz for  $\tilde{U}_1$ ,

$$560 \quad \tilde{U}_1 = \tilde{U}_{1,0} + \delta^2 \tilde{U}_{1,2} + \mathcal{O}(\delta^3). \quad (3.61)$$

562 One can see that the limit term  $\delta \rightarrow 0$  for the volumic equation (3.50) is

$$563 \quad \tilde{\rho}_0 \frac{\partial^2 \tilde{U}_{1,0}}{\partial t^2} - \nabla_\xi \left( \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{U}_{1,0} \right) = 0. \quad (3.62)$$

564 Taking the curl of this equation yields

$$565 \quad \frac{\partial^2}{\partial t^2} \left( \nabla_\xi \times (\tilde{\rho}_0 \tilde{U}_{1,0}) \right) = 0, \quad (3.63)$$

566 hence the curl of  $\tilde{\rho}_0 \tilde{U}_{1,0}$  is affine in time. Moreover, it is equal to zero due to the vanishing  
567 initial conditions. By the Helmholtz decomposition theorem, the term  $\tilde{\rho}_0 \tilde{U}_{1,0}$  can be expressed  
568 as the gradient of some function  $\tilde{\psi}_0$  defined up to a constant,

$$569 \quad \tilde{\rho}_0 \tilde{U}_{1,0} = \nabla_\xi \tilde{\psi}_0. \quad (3.64)$$

570 By subsituting in equation (3.54) we have

$$571 \quad \nabla_\xi \left( \frac{\partial^2 \tilde{\psi}_0}{\partial t^2} - \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \left( \tilde{\rho}_0^{-1} \nabla_\xi \tilde{\psi}_0 \right) \right) = 0, \quad (3.65)$$

572 then it holds

$$573 \quad \frac{\partial^2 \tilde{\psi}_0}{\partial t^2} - \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \left( \tilde{\rho}_0^{-1} \nabla_\xi \tilde{\psi}_0 \right) = 0, \quad (3.66)$$

574 since  $\tilde{\psi}_0$  is defined up to a constant. We need the boundary conditions to conclude. Evaluating  
575 (3.54) at the surface yields

$$576 \quad \tilde{\rho}_0 \frac{\partial^2 \tilde{U}_{1,0}}{\partial t^2} - \frac{\partial}{\partial \xi^3} \left( \tilde{\rho}_0 \tilde{c}_0^2 \nabla_\xi \cdot \tilde{U}_{1,0} \right) \mathbf{e}_3 = 0 \quad \text{on } \tilde{\Gamma}_s. \quad (3.67)$$

577 Using the surface condition (3.52) in the equation (3.59) yields

$$578 \quad \tilde{\rho}_0 \frac{\partial^2 \tilde{U}_{1,0}}{\partial t^2} = 0 \quad \text{on } \tilde{\Gamma}_s. \quad (3.68)$$

579 With the definition of the potential  $\tilde{\psi}_0$  it holds

$$580 \quad \frac{\partial^2 \nabla \tilde{\psi}_0}{\partial t^2} = 0 \quad \text{on } \tilde{\Gamma}_s, \quad (3.69)$$

581 hence one has

$$582 \quad \frac{\partial^2 \tilde{\psi}_0}{\partial t^2} = C(t) \quad \text{on } \tilde{\Gamma}_s, \quad (3.70)$$

583 where  $C$  does not depend on space. Moreover, since  $\tilde{\psi}_0$  vanishes at infinity the constant  $C$   
584 is equal to zero, hence  $\partial_{tt}^2 \tilde{\psi}_0 = 0$  on  $\tilde{\Gamma}_s$ . With the vanishing initial conditions this implies  
585 that  $\tilde{\psi}_0 = 0$  on  $\tilde{\Gamma}_s$ . To recover a dimensional system, the terms are multiplied by their  
586 corresponding characteristic scales, and  $\hat{\psi}_0 = \bar{\rho} U L \tilde{\psi}_0$  is defined. The system reads then

$$587 \quad \frac{\partial^2 \hat{\psi}_0}{\partial t^2} - \hat{\rho}_0 \hat{c}_0^2 \nabla_\xi \cdot \left( \hat{\rho}_0^{-1} \nabla_\xi \hat{\psi}_0 \right) = 0 \quad \text{in } \hat{\Omega}, \quad (3.71)$$

with the boundary conditions

$$\begin{cases} \nabla_\xi \hat{\psi}_0 \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1} & \text{on } \hat{\Gamma}_b, \\ \hat{\psi}_0 = 0 & \text{on } \hat{\Gamma}_s. \end{cases} \quad (3.72)$$

$$(3.73)$$

588 The system (3.60)-(3.62) is the classical wave equation for the potential  $\hat{\psi}_0$ , with a propagation  
589 speed  $\hat{c}_0^2$  and a non-homogeneous density.

590 An energy equation can be obtained by multiplying Eq. (3.60) by  $\partial_t \psi / (\rho_0 \hat{c}_0^2)$  and  
591 integrating over the domain. The result reads after an integration by parts

$$592 \quad \frac{d}{dt} \mathcal{E}_{acoustic} = - \int_{\hat{\Gamma}_b} \frac{1}{\hat{\rho}_0} \frac{\partial \hat{\psi}_0}{\partial t} \hat{u}_{b,1} d\sigma, \quad (3.74)$$

593 where the acoustic energy is

$$594 \quad \mathcal{E}_{acoustic} = \frac{1}{2} \int_{\hat{\Omega}} \frac{1}{\hat{\rho}_0 \hat{c}_0^2} \left( \frac{\partial \hat{\psi}_0}{\partial t} \right)^2 d\xi + \frac{1}{2} \int_{\hat{\Omega}} \frac{1}{\hat{\rho}_0} |\nabla \hat{\psi}_0|^2 d\xi. \quad (3.75)$$

595 With the same analysis as in the previous cases, one can show that the first term of (3.64) is  
596 the acoustic energy, and the second term is the kinetic energy.

597

598 *Remark:* In the sections 3.2.2 and 3.2.3, the equations (3.44) - (3.46) and (3.60)-(3.62)  
599 use the Lagrangian description whereas the equations from the litterature use the Eulerian  
600 description. In the general case, the use of different coordinate systems would cause two  
601 problems. First, when doing the change of coordinates, new terms should appear from the  
602 space or time differentiation. Second, the description of the domain is different, and this  
603 implies that the boundary conditions are not evaluated at the same location. In the next  
604 section we will show that the first problem does not exist in our case, due to the lack of a  
605 background velocity. As for the second problem, the linear eulerian models are obtained by  
606 evaluating the boundary conditions at a fixed water height. To this regard they use the same  
607 boundary as if they were in a Lagrangian description of the domain, so that the comparison  
608 remains valid.

609

610 The equations with their boundary conditions and the associated energy, for the general  
611 model and its different simplifications, are summarized in Table 1.

## 612 4. The model in Eulerian coordinates

613 The equations we have been working on are defined on the reference domain  $\hat{\Omega}$ . However, the  
614 linear equations for the acoustic-gravity waves are generally written in Eulerian coordinates.  
615 In order to compare our model with those from the litterature, the equations must be  
616 formulated on the moving domain  $\Omega(t)$ . In this section we present a method to write the  
617 system in Eulerian coordinate.

618

### 4.1. General method

619 The aim is to write the equation on a moving domain  $\Omega(t)$ , hence a transformation  $\phi : \hat{\Omega} \rightarrow$   
620  $\Omega(t)$  is needed. We start by using a first order approximation of the real transformation  $\phi$ .  
621 The transformation  $\phi$  is developed for small displacements,

$$622 \quad \phi(\xi, t) = \mathbb{I} + \varepsilon \phi_1(\xi, t) + \mathcal{O}(\varepsilon^2). \quad (4.1)$$

623 Let  $\phi_\varepsilon(\xi, t) = \mathbb{I} + \varepsilon \phi_1(\xi, t)$  be its first order approximation.  $\phi_\varepsilon$  is used to define the equivalent  
624 domain and its boundary,

$$625 \quad \Omega_\varepsilon(t) = \phi_\varepsilon(\hat{\Omega}), \quad \Gamma_{s,eq} = (\phi_\varepsilon(\hat{\Gamma}_s)), \quad \Gamma_{b,eq} = (\phi_\varepsilon(\hat{\Gamma}_b)). \quad (4.2)$$

626 The coordinates on the equivalent domain are written  $\mathbf{x} = (x, y, z)$ . For any generic function  
 627  $\hat{X}(\boldsymbol{\xi}, t)$  defined in  $\hat{\Omega}$ , a function  $X(\mathbf{x}, t)$  is defined in  $\Omega_\varepsilon$  by the following change of variables

$$628 \quad X(\mathbf{x}, t) = \hat{X}(\boldsymbol{\phi}_\varepsilon^{-1}(\mathbf{x}, t), t), \quad (4.3)$$

629 which is equivalent to

$$630 \quad \hat{X}(\boldsymbol{\xi}, t) = X(\boldsymbol{\phi}_\varepsilon(\boldsymbol{\xi}, t), t), \quad (4.4)$$

as long as  $\boldsymbol{\phi}_\varepsilon$  is invertible. Then, if the function  $\hat{X}$  has a first-order approximation  $\hat{X} = \hat{X}_0 + \varepsilon \hat{X}_1 + \mathcal{O}(\varepsilon^2)$ , then the function  $X$  also has a first-order approximation  $X = X_0 + \varepsilon X_1 + \mathcal{O}(\varepsilon^2)$  and it holds (see Appendix, Sec. B)

$$\left\{ \begin{array}{l} \nabla_\xi \hat{X}_0 = \nabla X_0, \end{array} \right. \quad (4.5)$$

$$\left\{ \begin{array}{l} \frac{\partial \hat{X}_0}{\partial t} = \frac{\partial X_0}{\partial t}, \end{array} \right. \quad (4.6)$$

$$\left\{ \begin{array}{l} \nabla_\xi \hat{X}_1 = (\nabla_\xi \mathbf{d}_1)^T \nabla X_0 + \nabla X_1, \end{array} \right. \quad (4.7)$$

$$\left\{ \begin{array}{l} \frac{\partial \hat{X}_1}{\partial t} = \frac{\partial X_1}{\partial t} + \mathbf{U}_1 \cdot \nabla X_0. \end{array} \right. \quad (4.8)$$

631 In the following, when writing the equations satisfied by the free surface of  $\Omega_\varepsilon$ , we will also  
 632 use

$$633 \quad \frac{\partial \boldsymbol{\phi}_\varepsilon}{\partial t} = \varepsilon \hat{\mathbf{U}}_1. \quad (4.9)$$

#### 634 4.2. The model in Eulerian coordinates

Using the change of variable (4.4) in the system (2.41)-(2.43) and with the equalities (4.5)-(4.8) we obtain the following system for  $\mathbf{U}_1, p_1, \rho_1$  defined in  $\Omega_\varepsilon$ ,

$$\left\{ \begin{array}{l} \rho_0 \frac{\partial \mathbf{U}_1}{\partial t} + \nabla p_1 = \rho_1 \mathbf{g}, \end{array} \right. \quad (4.10)$$

$$\left\{ \begin{array}{l} \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{U}_1) = 0, \end{array} \right. \quad (4.11)$$

$$\left\{ \begin{array}{l} \frac{\partial p_1}{\partial t} + \nabla p_0 \cdot \mathbf{U}_1 + \rho_0 c_0^2 \nabla \cdot \mathbf{U}_1 = 0. \end{array} \right. \quad (4.12)$$

And  $p_0, \rho_0$  satisfy the limit equations

$$\left\{ \begin{array}{l} \frac{\partial \rho_0}{\partial t} = 0, \end{array} \right. \quad (4.13)$$

$$\left\{ \begin{array}{l} \nabla p_0 = \rho_0 \mathbf{g}. \end{array} \right. \quad (4.14)$$

635 In order to close the System (4.10)-(4.14), boundary conditions should be prescribed. To get  
 636 a linear problem one wants to prescribe this condition on the fixed domain  $\hat{\Omega}$ . To do so we  
 637 assume in the following that the equations (4.10)-(4.14) are defined in  $\hat{\Omega}$ . It would be true if  
 638  $\hat{\Omega} \subset \Omega_\varepsilon$ , but the inclusion is in general not verified. Because of this approximation, errors of  
 639 order  $\mathcal{O}(\varepsilon)$  may be introduced. For this reason the system in Lagrangian coordinates should  
 640 be preferred, at least for future extension of this work.

#### 641 *Boundary conditions and free surface description*

642 Following the approach of Nougier *et al.* (2015), we show that a description for the free  
 643 surface can be obtained. In the following, the components of the fluid velocity are denoted  
 644  $\mathbf{U}_1 = (U_1^1, U_1^2, U_1^3)^T$ . The surface is defined by  $\Gamma_{s,eq} = \boldsymbol{\phi}_\varepsilon(\hat{\Gamma}_s)$ , and we assume that at each

645 time  $t$  it can be parametrized as the graph  $\eta_\varepsilon$ . The elevation  $\eta_\varepsilon$  is a function of  $x, y$ , and  $t$   
 646 and can be decomposed in the following way,

$$647 \quad \eta_\varepsilon(x, y, t) = H + \varepsilon\eta_1(x, y, t). \quad (4.15)$$

648 From the correspondence between the free surface and the particle displacement, it holds

$$649 \quad \phi_\varepsilon^3(\xi^1, \xi^2, H, t) = \eta_\varepsilon(x(\xi^1, \xi^2, H, t), y(\xi^1, \xi^2, H, t), t). \quad (4.16)$$

650 Differentiating (4.16) in time and using the equation (4.9) yields

$$651 \quad \varepsilon\hat{U}_1^3(\xi^1, \xi^2, H, t) = \frac{\partial\eta_\varepsilon}{\partial t} + \varepsilon\hat{U}_1^1(\xi^1, \xi^2, H, t) \frac{\partial\eta_\varepsilon}{\partial x} + \varepsilon\hat{U}_1^2(\xi^1, \xi^2, H, t) \frac{\partial\eta_\varepsilon}{\partial y}. \quad (4.17)$$

We use the change of variables  $\phi_\varepsilon(\xi, t) = Id + \varepsilon\phi_1(\xi, t)$ ,

$$652 \quad \varepsilon U_1^3(\phi_\varepsilon(\xi^1, \xi^2, H, t), t) = \frac{\partial\eta_\varepsilon}{\partial t} + \varepsilon U_1^1(\phi_\varepsilon(\xi^1, \xi^2, H, t), t) \frac{\partial\eta_\varepsilon}{\partial x}$$

$$653 \quad + \varepsilon U_1^2(\phi_\varepsilon(\xi^1, \xi^2, H, t), t) \frac{\partial\eta_\varepsilon}{\partial y}. \quad (4.18)$$

655 After a Taylor developement and keeping only the terms in  $\varepsilon$ , it holds

$$656 \quad U_1^3(x, y, H, t) = \frac{\partial\eta_1}{\partial t}, \quad (4.19)$$

this is the linearized equation for the free surface. Then the dynamic boundary conditions are linearized. With the change of variables, the boundary conditions (2.17), (2.44) and (2.45) become

$$\begin{cases} U_1 \cdot \mathbf{n}_b = u_{b,1} & \text{on } \Gamma_{b,eq}, & (4.20) \\ p_0 = p^a & \text{on } \Gamma_{s,eq}, & (4.21) \\ p_1 = 0 & \text{on } \Gamma_{s,eq}. & (4.22) \end{cases}$$

658 If we linearize (4.22) only we would miss the first-order term coming from (4.21). From  
 659 (4.21) and (4.22) we deduce the boundary condition for the pressure

$$660 \quad p_0 + \varepsilon p_1 = p^a \quad \text{on } \Gamma_{s,eq}. \quad (4.23)$$

661 A Taylor developement of  $p_0$  and  $p_1$  around  $z = H$  on  $\Gamma_{s,eq}$  yields

$$662 \quad p_0(H) + \varepsilon(p_1(x, y, H, t) + p_0'(H)\eta_1) + \mathcal{O}(\varepsilon^2) = p^a. \quad (4.24)$$

663 After an identification of the powers of  $\varepsilon$  it holds

$$664 \quad p_0(H) = p^a, \quad p_1(x, y, H, t) = \rho_0(x, y, H, t)g\eta_1(x, y, t). \quad (4.25)$$

665 In a similar way, the linearization of Eq. (4.20) reads

$$666 \quad U_1^3(x, y, z_b) - U_1^1(x, y, z_b) \partial_x z_b - U_1^2(x, y, z_b) \partial_y z_b = u_{b,1}(x, y, t). \quad (4.26)$$

667 Hence the equations for  $U_1, \rho_1, p_1$  can be fully defined on the domain  $\hat{\Omega}$ , with an error in  
 668  $\mathcal{O}(\varepsilon^2)$ . Finally, note that the system (4.10)-(4.12) with the boundary conditions (4.25),(4.26)  
 669 and the kinematic condition (4.19), is shown to be energy-preserving, locally as well as over  
 670 a whole water column (Lighthill 1978; Lotto & Dunham 2015).

671 In this section we have derived the linear equation in Eulerian coordinates, even though  
 672 an approximation on the domain in which the equations are defined was necessary. The  
 673 computations of Sec. 4.1 also justify that in the absence of mean flow and with the evaluation  
 674 of the boundary conditions at a fixed height, the linear system in Eulerian coordinates is

675 similar to the one in Lagrangian coordinate, up to terms in  $O(\varepsilon^2)$ . At the same time, the  
 676 linearization in the Lagrangian coordinates is better defined. For this reason the system in  
 677 Lagrangian coordinates is preferred for the rest of this work. We conclude this paper by the  
 678 study of the dispersion relation obtained from Eq. (2.46).

## 679 5. Dispersion relation

680 A key aspect of wave models is the related dispersion relation, that we derive here from Eq.  
 681 (2.46) and solve numerically. First note that if one defines the equivalent pressure  $p_\varepsilon$ , the  
 682 equivalent density  $\rho_\varepsilon$  and the equivalent velocity  $\mathbf{U}_\varepsilon$  by

$$683 \quad p_\varepsilon = p_0 + \varepsilon p_1, \quad \rho_\varepsilon = \rho_0 + \varepsilon \rho_1, \quad \mathbf{U}_\varepsilon = \varepsilon \mathbf{U}_1, \quad (5.1)$$

then a combination of the equations (4.10)-(4.14) yields the following system for  $p_\varepsilon$ ,  $\rho_\varepsilon$  and  $\mathbf{U}_\varepsilon$ ,

$$\begin{cases} \rho_0 \frac{\partial \mathbf{U}_\varepsilon}{\partial t} + \nabla p_\varepsilon = \rho_\varepsilon \mathbf{g} + O(\varepsilon^2), & (5.2) \\ \frac{\partial \rho_\varepsilon}{\partial t} + \nabla \cdot (\rho_0 \mathbf{U}_\varepsilon) = O(\varepsilon^2), & (5.3) \\ \frac{\partial p_\varepsilon}{\partial t} + \nabla p_0 \cdot \mathbf{U}_\varepsilon + \rho_0 c_0^2 \nabla \cdot \mathbf{U}_\varepsilon = O(\varepsilon^2). & (5.4) \end{cases}$$

684 This system is comparable – up to the terms in  $O(\varepsilon^2)$  – to the system studied in the paper from  
 685 Auclair *et al.* (2021). In Auclair *et al.* (2021), the author thoroughly analyze the dispersion  
 686 relation for the model of a stratified compressible fluid with a constant sound speed.

687 In order to make the computations clearer, the problem is restricted to a 2-dimensional  
 688 configuration in  $\xi^1$  and  $\xi^3$ . We also assume that the bottom is flat. Following the approach  
 689 of Auclair *et al.* (2021), the wave angular frequency  $\omega$  and the horizontal wave-number  
 690 frequency  $k_x$  are defined, and we seek a solution of the form

$$691 \quad \hat{\rho}_0 \hat{\mathbf{U}}_1(\xi^1, \xi^3, t) = \begin{pmatrix} \tilde{U}^1(\xi^3) \\ \tilde{U}^3(\xi^3) \end{pmatrix} e^{i(k_x \xi^1 - \omega t)}. \quad (5.5)$$

692 First Eq. (2.46) is written differently to make the unknown  $\hat{\rho}_0 \hat{\mathbf{U}}_1$  appear.

$$693 \quad \frac{\partial^2 \hat{\rho}_0 \hat{\mathbf{U}}_1}{\partial t^2} - \nabla_\xi (\hat{c}_0^2 \nabla_\xi \cdot (\hat{\rho}_0 \hat{\mathbf{U}}_1)) - \nabla_\xi \left( \frac{\hat{c}_0^2 N_0^2}{g} \rho_0 \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right) - g \nabla_\xi \cdot (\hat{\rho}_0 \hat{\mathbf{U}}_1) \mathbf{e}_3 = 0 \quad (5.6)$$

694 Injecting the ansatz (5.5) in Eq. (5.6) yields

$$695 \quad \omega^2 \tilde{U}^1 + ik_x \left( \hat{c}_0^2 (ik_x \tilde{U}^1 + (\tilde{U}^3)') + \frac{\hat{c}_0^2 N_0^2}{g} \tilde{U}^3 \right) = 0, \quad (5.7)$$

$$696 \quad \omega^2 \tilde{U}^3 + \partial_3 \left( \hat{c}_0^2 (ik_x \tilde{U}^1 + (\tilde{U}^3)') \right) + \partial_3 \left( \frac{\hat{c}_0^2 N_0^2}{g} \tilde{U}^3 \right) + g (ik_x \tilde{U}^1 + (\tilde{U}^3)') = 0. \quad (5.8)$$

698 Using the equation (5.7), the horizontal component  $\tilde{U}^1$  is expressed as a function of the  
 699 vertical component,

$$700 \quad \tilde{U}^1 = -ik_x \frac{\hat{c}_0^2 D(\tilde{U}^3)' + (\hat{c}_0^2 - gD)\tilde{U}^3}{D(\omega^2 - \hat{c}_0^2 k_x^2)}, \quad (5.9)$$



701 where  $D$  is a depth scale, defined by

$$702 \quad \frac{1}{D} = \frac{N^2}{g} + \frac{g}{\hat{c}_0^2} = \frac{\hat{\rho}'_0}{\hat{\rho}_0}. \quad (5.10)$$

703 We also define the quantity

$$704 \quad S = 2 \frac{\hat{c}'_0}{\hat{c}_0}. \quad (5.11)$$

705 Replacing  $\tilde{U}^1$  in the equation (5.8) yields, after some computations,

$$706 \quad (\tilde{U}^3)'' + \left( \frac{1}{D} + \omega^2 S^2 \right) (\tilde{U}^3)' + \left( \frac{\omega^2}{\hat{c}_0^2} + k_x^2 \frac{N^2 - \omega^2}{\omega^2} - \frac{D'}{D^2} + S \left( \frac{g}{\hat{c}_0^2} + \frac{N^2}{g} \frac{\omega^2}{\omega^2 - \hat{c}_0^2 k_x^2} \right) \right) \tilde{U}^3 = 0. \quad (5.12)$$

707 In order to write an harmonic equation the following change of variable is made,

$$708 \quad \tilde{U}^3(z) = \tilde{U}^3(H) F(z) \exp \left( \int_z^H \frac{\alpha}{2} dz' \right), \quad \alpha = \frac{1}{D} + \omega^2 S. \quad (5.13)$$

709 Then  $F(0) = 0$ ,  $F(H) = 1$  and  $F$  satisfies the equation

$$710 \quad F'' + k_z^2 F = 0 \quad (5.14)$$

711 where the vertical wave-number  $k_z$  is defined by

$$712 \quad k_z^2 + k_x^2 \frac{N^2 - \omega^2}{\omega^2} + \frac{\omega^2}{\hat{c}_0^2} - \frac{1 + 2D'}{4D^2} - \frac{1}{2} \omega^2 S' \\ 713 \quad + S \left( \frac{g}{\hat{c}_0^2} + \frac{N^2}{g} \frac{\omega^2}{\omega^2 - \hat{c}_0^2 k_x^2} - \frac{\omega^2}{2D} - \frac{1}{4} \omega^4 S \right) = 0, \quad (5.15)$$

714 The equation (5.15) is the dispersion relation for the two wave-numbers  $k_x$ ,  $k_z$  and the  
715 frequency  $\omega$ . It is a generalization of the inner dispersion relation in Auclair *et al.* (2021) to  
716 the case of a non-constant sound speed. Indeed, with a constant sound speed one has  $S = 0$   
717 and (5.15) is exactly the inner dispersion relation in Auclair *et al.* (2021)..

718

719 *Remark:* In the most general case, the scalars  $N$ ,  $D$  and  $S$  depend on the depth  $z$ , hence  
720  $k_z$  also depends on  $z$ . It is then not clear whether the solution to Eq. (5.14) and the profile  
721  $\tilde{U}^3$ , can be written explicitly. When  $k_z$  does not depend on  $z$ , as in the study from Auclair  
722 *et al.* (2021), the expression of the profile  $\tilde{U}^3$  is used with the boundary conditions to obtain  
723 a boundary dispersion relation. In our case  $k_z$  is not a constant, and the boundary dispersion  
724 relation is not easily deduced.

### 725 Numerical approximation of the dispersion relation

726 An evaluation of the equation (5.15) is possible once the limit state for the pressure and the  
727 density is computed. The differential equation for the pressure (2.15) is numerically solved  
728 for the temperature profile shown in figure 3a.

729 Then the density and the speed of sound are computed from the tabulations given in  
730 IAPWS-SR7 (2009). Figure 3b, 3c show the obtained density and speed of sound. With  
731 these profiles the dispersion relation (5.15) is computed. Figure 4 shows the contours of the  
732 vertical wave-number as a function of the horizontal wave-number and the angular frequency,  
733 at different depths. For the sake of comparison, the plotted variables are the adimensionned

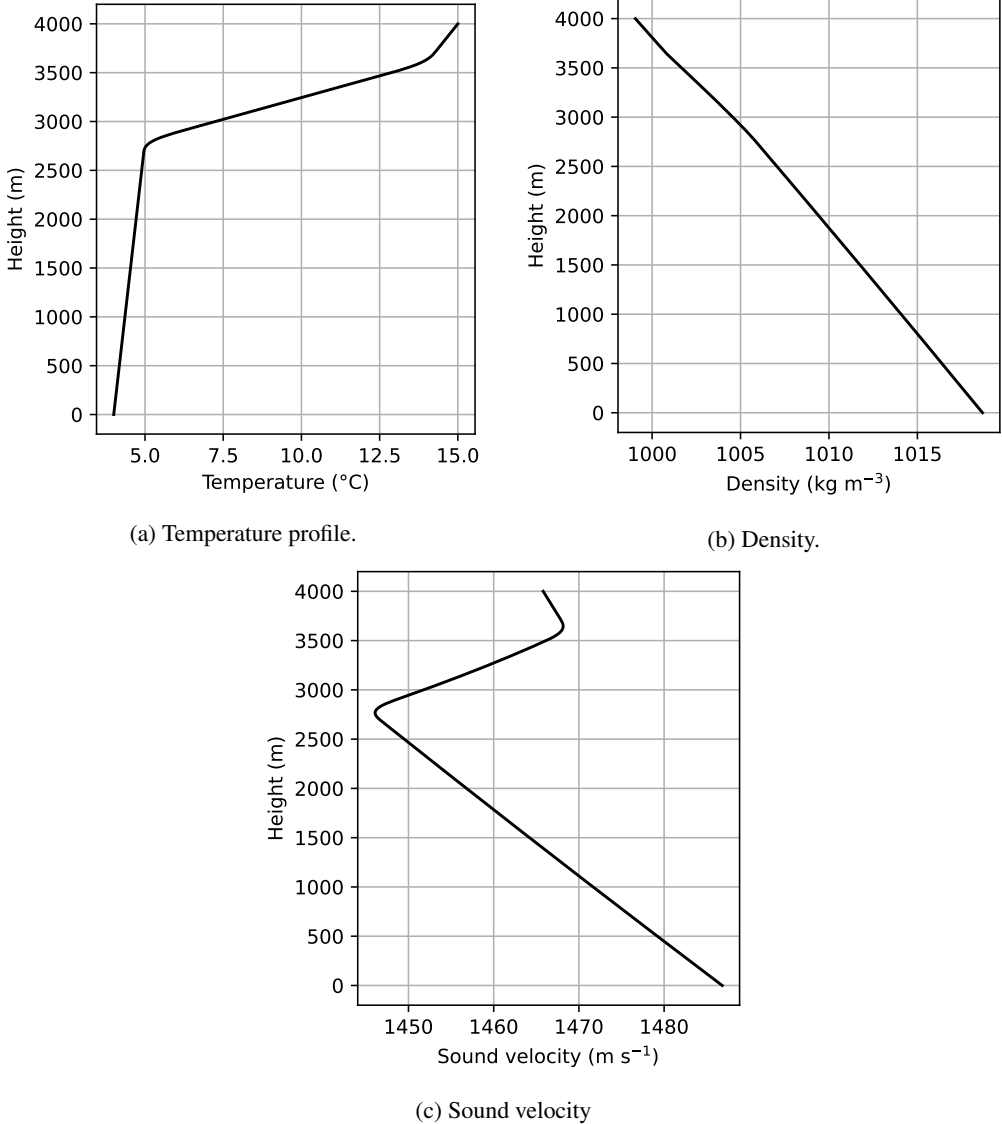


Figure 3: Temperature, density and sound velocity profiles used for the computation of the dispersion relation where  $\xi^3 = 0$  is the seafloor and  $\xi^3 = 4000$  m is the ocean surface.

734 variables  $\delta_x = k_x H$ ,  $\delta_z = k_z H$  and  $\log_{10}(\delta_\omega)$ , where  $\delta_\omega = \omega \sqrt{H/g}$ .

735

736 Although Fig 4 is close to the one in the paper from Auclair *et al.* (2021), one can notice  
 737 the influence of the ocean depth on the contours. This first result suggests that the variation  
 738 of the parameters  $\hat{c}_0$ ,  $N$ ,  $D$  with depth plays a non-negligible role in the waves dispersion.

## 739 6. Conclusion and future work

740 In this work we have presented a system describing the propagation of acoustic-gravity  
 741 waves in a free-surface fluid over an varying bed (bathymetry) and with a variable sound

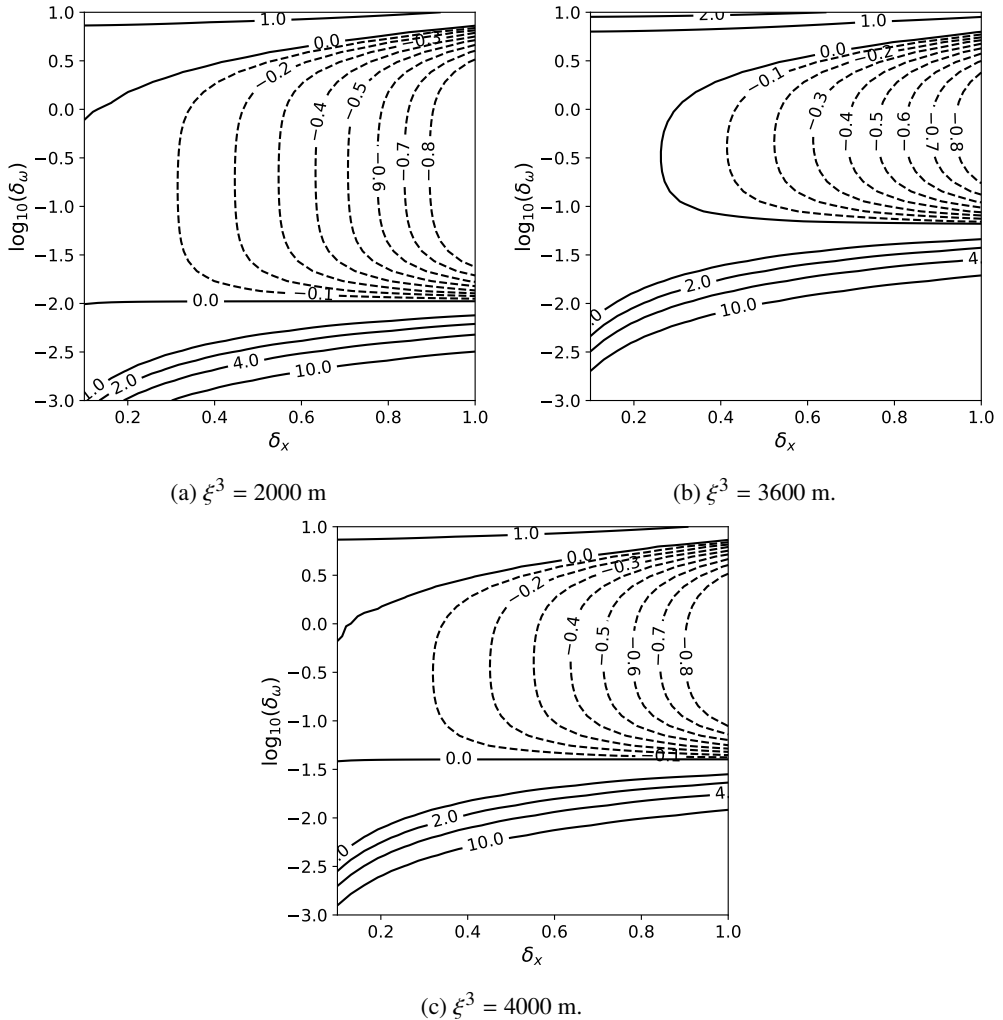


Figure 4: Contour of the vertical wave-number as a function of the horizontal wave-number  $\delta_x$  and the angular frequency  $\delta_\omega$ , at different depths  $\xi^3$ . **4a:**  $\xi^3 = 2000$  m, **4b:**  $\xi^3 = 3600$  m, **4c:**  $\xi^3 = 4000$  m.

742 speed, applicable to describe in particular hydro-acoustic and tsunami waves generated by  
 743 earthquakes or landslides in the ocean. Through a rigorous linearization of the compressible  
 744 Euler equation we have obtained a model able to represent many physical phenomena, such  
 745 as the SOFAR channel or the propagation of internal waves. The variety of these phenomena  
 746 is well represented in the dispersion relation.

747 In the derivation only few assumptions are made and some common simplifying hypotheses  
 748 were avoided. In particular, the fluid is not necessarily assumed barotropic and it is not  
 749 assumed irrotational. Thanks to this approach, many terms representing different physical  
 750 phenomena are kept in the wave-like equation. With a numerical approximation one could  
 751 then compute their respective magnitude, and justify which terms can be neglected. Note  
 752 also that in the present work the source term is a displacement of the seabed, but this is  
 753 not restrictive and other source terms could be used (a change in the surface pressure for  
 754 example).

755 With additional assumptions compatible with the derivation of the system, such as  
 756 considering a barotropic fluid, or restricting the model to the incompressible regime or to the  
 757 acoustic regime, we are able to recover simpler models widely studied in the literature. The  
 758 mathematical study of the more complete model can help gain insight on the other ones. For  
 759 example, we could clearly identify the assumptions made in the hydro-acoustic waves model  
 760 used by Stiassnie (2010); Sammarco *et al.* (2013) and others. Namely, in those models the  
 761 fluid is assumed barotropic, and the effects of stratification and gravity are neglected inside  
 762 the domain. The study of the more complete model also helped to write the conservation  
 763 of energy in each simplified case. The linear model in Lagrangian coordinates can also be  
 764 used to recover the linearized Euler equations in Eulerian coordinates. This brings a clear  
 765 understanding of the usual – nevertheless non satisfactory – assumption that is used to derive  
 766 the aforementioned models in Eulerian coordinates.

767 The wave-like formulation of the model makes it a good candidate for a numerical  
 768 approximation by the finite elements method. The fact that it preserves an energy suggests  
 769 that the problem is well-posed, which motivates a more thorough study of the mathematical  
 770 problem. Numerical implementation of this model will make it possible to simulate acoustic-  
 771 gravity waves generated by earthquakes and landslide sources accounting for the complex  
 772 bathymetry, thus contributing to improve early-warning systems. It will also help quantifying  
 773 the errors made in more simple models, such as the hypothesis of an irrotational flow. These  
 774 two aspects will be investigated in a future work.

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 802 theory and algorithms*, 1st edn. *Springer series in computational mathematics* 5. Berlin Heidelberg:  
 803 Springer.

	Equation in $\hat{\Omega}$	Boundary conditions	Energy
General model (Sec. 2.3)	$\hat{\rho}_0 \frac{\partial^2 \hat{\mathbf{U}}_1}{\partial t^2} - \nabla \left( \hat{\rho}_0 \hat{c}_0^2 \nabla \cdot \hat{\mathbf{U}}_1 \right) - \rho_{0g} \left( (\nabla \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3) + \nabla \cdot \hat{\mathbf{U}}_1 \mathbf{e}_3 \right) = 0$	$\hat{\mathbf{U}}_1 \cdot \mathbf{n}_b = \hat{u}_{b,1}, \quad \hat{\Gamma}_b$ $\nabla \cdot \hat{\mathbf{U}}_1 = 0, \quad \hat{\Gamma}_s$	$\mathcal{E} = \frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 \left( \left  \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \right ^2 + \left( \hat{c}_0 \nabla \cdot \hat{\mathbf{U}}_1 - \frac{g}{\hat{c}_0} \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right)^2 \right) + \frac{1}{2} \int_{\Gamma_s} \hat{\rho}_{0g} (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3)^2 d\sigma$
Barotropic case (Sec. 3.1)	$\frac{\hat{\rho}_0}{\hat{c}_0^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla \cdot (\hat{\rho}_0 \nabla \psi) = 0, = 0,$ $\hat{\mathbf{U}}_1 = \nabla \psi$	$\nabla \psi \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1}, \quad \hat{\Gamma}_b$ $\frac{\partial^2 \psi}{\partial t^2} + g \frac{\partial \psi}{\partial \xi^3} = 0, \quad \hat{\Gamma}_s$	$\mathcal{E}_{bar} = \frac{1}{2} \int_{\hat{\Omega}} \hat{\rho}_0 \left( \frac{1}{\hat{c}_0^2} \left( \frac{\partial \psi}{\partial t} \right)^2 +  \nabla \psi ^2 \right) d\xi + \frac{1}{2} \int_{\Gamma_s} \frac{\hat{\rho}_0}{g} \left( \frac{\partial \psi}{\partial t} \right)^2 d\sigma$
Incompressible limit (Sec. 3.2.2)	$\hat{\rho}_0 \frac{\partial^2 \hat{\mathbf{U}}_{1,0}}{\partial t^2} - g \hat{\rho}'_0 (\hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3) \mathbf{e}_3 + g \nabla \hat{\varphi}_0 = 0,$ $\nabla \cdot \hat{\mathbf{U}}_{1,0} = 0$	$\hat{\mathbf{U}}_{1,0} \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1}, \quad \hat{\Gamma}_b$ $\nabla \cdot \hat{\mathbf{U}}_{1,0} = 0, \quad \hat{\Gamma}_s$ $\hat{\varphi}_0 = g \hat{\rho}_0 \hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3, \quad \hat{\Gamma}_s$	$\mathcal{E}_{incomp} = \frac{1}{2} \int_{\hat{\Omega}} \left( \hat{\rho}_0 \left  \frac{\partial \hat{\mathbf{U}}_{1,0}}{\partial t} \right ^2 - g \hat{\rho}'_0  \hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 ^2 \right) d\xi + \frac{1}{2} \int_{\Gamma_s} g \hat{\rho}_0  \hat{\mathbf{U}}_{1,0} \cdot \mathbf{e}_3 ^2 d\sigma$
Incompressible limit, homogeneous density (Sec. 3.2.2)	$\Delta \hat{\varphi}_0 = 0,$ $\frac{\partial^2 \hat{\mathbf{U}}_{1,0}}{\partial t^2} = -\frac{g}{\hat{\rho}_0} \nabla \hat{\varphi}_0$	$\nabla \hat{\varphi}_0 \cdot \hat{\mathbf{n}}_b = -\hat{\rho}_{0g} \hat{u}_{b,1}, \quad \hat{\Gamma}_b$ $\frac{\partial^2 \hat{\varphi}_0}{\partial t^2} + g \frac{\partial \hat{\varphi}_0}{\partial \xi^3} = 0, \quad \hat{\Gamma}_s$	$\mathcal{E}_{Poisson} = \frac{1}{2} \left( \int_{\hat{\Omega}}  \nabla \hat{\varphi} ^2 d\xi + \int_{\Gamma_s} \frac{1}{g} \left( \frac{\partial \hat{\varphi}_0}{\partial t} \right)^2 d\sigma \right)$
Acoustic limit (Sec. 3.2.3)	$\frac{\partial^2 \hat{\psi}_0}{\partial t^2} - \hat{\rho}_0 \hat{c}_0^2 \nabla \cdot (\hat{\rho}_0^{-1} \nabla \hat{\psi}_0) = 0,$ $\hat{\mathbf{U}}_1 = \nabla \hat{\psi}_0$	$\nabla \hat{\psi}_0 \cdot \hat{\mathbf{n}}_b = \hat{u}_{b,1}, \quad \hat{\Gamma}_b$ $\hat{\psi}_0 = 0, \quad \hat{\Gamma}_s$	$\mathcal{E}_{acoustic} = \frac{1}{2} \int_{\hat{\Omega}} \frac{1}{\hat{\rho}_0} \left( \hat{c}_0^2 \left( \frac{\partial \hat{\psi}_0}{\partial t} \right)^2 +  \nabla \hat{\psi}_0 ^2 \right) d\xi$

Table 1: Summary of the equations, boundary conditions and energy for the general model and its simplifications. For conciseness the operator  $\nabla_\xi$  and  $\Delta_\xi$  are denoted respectively  $\nabla$  and  $\Delta$ .

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854 **Appendix A.**

855 In this section an energy equation for the system (2.46) is obtained. Recall that the system  
856 (2.46) reads in  $\hat{\Omega}$ ,

$$857 \quad \hat{\rho}_0 \frac{\partial^2 \hat{U}_1}{\partial t^2} - \nabla_{\xi} \left( \hat{\rho}_0 \hat{c}_0^2 \nabla_{\xi} \cdot \hat{U}_1 \right) - (\nabla_{\xi} \hat{U}_1)^T \hat{\rho}_0 \mathbf{g} + \hat{\rho}_0 \nabla_{\xi} \cdot \hat{U}_1 \mathbf{g} = 0, \quad (\text{A } 1)$$

with the boundary conditions

$$\begin{cases} \hat{U}_1 \cdot \mathbf{n}_b = \hat{u}_{b,1} & \text{on } \hat{\Gamma}_b, \\ \nabla_{\xi} \cdot \hat{U}_1 = 0 & \text{on } \hat{\Gamma}_s. \end{cases} \quad (\text{A } 2)$$

$$(\text{A } 3)$$

By taking the scalar product of (2.46) with  $\partial_t \hat{U}_1$  and integrating over the domain we have

858

$$859 \quad \int_{\hat{\Omega}} \frac{\partial \hat{U}_1}{\partial t} \cdot \left( \hat{\rho}_0 \frac{\partial^2 \hat{U}_1}{\partial t^2} \right) d\xi - \int_{\hat{\Omega}} \frac{\partial \hat{U}_1}{\partial t} \cdot \left( \nabla_{\xi} \left( \hat{\rho}_0 \hat{c}_0^2 \nabla_{\xi} \cdot \hat{U}_1 \right) \right) d\xi$$

$$860 \quad + \int_{\hat{\Omega}} \frac{\partial \hat{U}_1}{\partial t} \cdot \left( \nabla_{\xi} (\hat{U}_1 \cdot \mathbf{e}_3) \hat{\rho}_0 g \right) d\xi - \int_{\hat{\Omega}} \frac{\partial \hat{U}_1}{\partial t} \cdot \left( \hat{\rho}_0 \nabla_{\xi} \cdot \hat{U}_1 g \mathbf{e}_3 \right) d\xi = 0. \quad (\text{A } 4)$$

861 For the first integral of (A 4) it holds

$$862 \quad \int_{\hat{\Omega}} \frac{\partial \hat{U}_1}{\partial t} \cdot \left( \hat{\rho}_0 \frac{\partial^2 \hat{U}_1}{\partial t^2} \right) d\xi = \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0 \frac{1}{2} \left| \frac{\partial \hat{U}_1}{\partial t} \right|^2 d\xi. \quad (\text{A } 5)$$

863 The second term of (A 4) is integrated by parts, using  $\nabla_{\xi} \cdot \hat{U}_1 = 0$  on the surface and  
864  $\hat{U}_1 \cdot \hat{\mathbf{n}}_b = \hat{b}_1$  at the bottom (hence  $\partial_t (\hat{U}_1 \cdot \hat{\mathbf{n}}_b) = \partial_t \hat{b}_1$ ),

$$865 \quad - \int_{\hat{\Omega}} \frac{\partial \hat{U}_1}{\partial t} \cdot \nabla_{\xi} \left( \hat{\rho}_0 \hat{c}_0^2 \nabla_{\xi} \cdot \hat{U}_1 \right) d\xi = \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0 \hat{c}_0^2 |\nabla_{\xi} \cdot \hat{U}_1|^2 d\xi$$

$$866 \quad - \int_{\hat{\Gamma}_b} \hat{\rho}_0 \hat{c}_0^2 \nabla_{\xi} \cdot \hat{U}_1 \frac{\partial \hat{u}_{b,1}}{\partial t} d\sigma. \quad (\text{A } 6)$$

867 For the computation of the two last integral of (A 4), we define

$$868 \quad (I) = \int_{\hat{\Omega}} \frac{\partial \hat{U}_1}{\partial t} \cdot \left( \nabla_{\xi} (\hat{U}_1 \cdot \mathbf{e}_3) \hat{\rho}_0 g \right) d\xi - \int_{\hat{\Omega}} \frac{\partial \hat{U}_1}{\partial t} \cdot \left( \hat{\rho}_0 \nabla_{\xi} \cdot \hat{U}_1 g \mathbf{e}_3 \right) d\xi, \quad (\text{A } 7)$$

869 an we denote by  $\hat{\mathbf{n}}$  be the vector normal to the boundary  $\partial \hat{\Omega}$ . (I) is integrated by parts and  
870 reads

$$871 \quad (I) = \int_{\partial \hat{\Omega}} \hat{\rho}_0 g \hat{U}_1 \cdot \mathbf{e}_3 \frac{\partial \hat{U}_1}{\partial t} \cdot \hat{\mathbf{n}} d\sigma - \int_{\hat{\Omega}} g \hat{U}_1 \cdot \mathbf{e}_3 \nabla_{\xi} \cdot \left( \hat{\rho}_0 \frac{\partial \hat{U}_1}{\partial t} \right) d\xi$$

$$872 \quad - \int_{\hat{\Omega}} \hat{\rho}_0 g \frac{\partial \hat{U}_1}{\partial t} \cdot \mathbf{e}_3 \nabla_{\xi} \cdot \hat{U}_1 d\xi. \quad (\text{A } 8)$$

873 The boundary term is simplified using  $\partial_t (\hat{U}_1 \cdot \hat{\mathbf{n}}_b) = \partial_t \hat{b}_1$  at the bottom. On the boundary  $\hat{\Gamma}_s$ ,  
874 the surface is horizontal hence the normal vector is the unit vector  $\mathbf{e}_3$ , so it holds

$$875 \quad (I) = \int_{\hat{\Gamma}_b} \hat{\rho}_0 g \hat{U}_1 \cdot \mathbf{e}_3 \frac{\partial \hat{b}_1}{\partial t} d\sigma + \int_{\hat{\Gamma}_s} \hat{\rho}_0 g \hat{U}_1 \cdot \mathbf{e}_3 \frac{\partial \hat{U}_1}{\partial t} \cdot \mathbf{e}_3 d\sigma$$

$$876 \quad - \int_{\hat{\Omega}} g \hat{U}_1 \cdot \mathbf{e}_3 \nabla_{\xi} \cdot \left( \hat{\rho}_0 \frac{\partial \hat{U}_1}{\partial t} \right) d\xi - \int_{\hat{\Omega}} \hat{\rho}_0 g \frac{\partial \hat{U}_1}{\partial t} \cdot \mathbf{e}_3 \nabla_{\xi} \cdot \hat{U}_1 d\xi. \quad (\text{A } 9)$$

877 Next we develop the gradient in the third integral of Eq. (A 9). Note that  $\hat{\rho}_0$  depends only on  
878 the vertical coordinate, then we have

$$879 \quad - \int_{\hat{\Omega}} g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \nabla_{\xi} \hat{\rho}_0 = - \int_{\hat{\Omega}} g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \frac{\partial \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3}{\partial t} \frac{d\hat{\rho}_0}{d\xi^3} = - \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}'_0 g |\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3|^2, \quad (\text{A } 10)$$

hence we obtain

$$880 \quad (I) = \int_{\hat{\Gamma}_b} \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \frac{\partial \hat{b}_1}{\partial t} d\sigma + \frac{1}{2} \frac{d}{dt} \int_{\hat{\Gamma}_s} \hat{\rho}_0 g |\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3|^2 d\sigma$$

$$881 \quad - \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} g \hat{\rho}'_0 |\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3|^2 d\xi - \int_{\hat{\Omega}} \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \frac{\partial}{\partial t} (\nabla_{\xi} \cdot \hat{\mathbf{U}}_1) d\xi$$

$$882 \quad - \int_{\hat{\Omega}} \hat{\rho}_0 g \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \cdot \mathbf{e}_3 \nabla_{\xi} \cdot \hat{\mathbf{U}}_1 d\xi. \quad (\text{A } 11)$$

The two last terms of (A 11) are put together,

$$883 \quad (I) = \int_{\hat{\Gamma}_b} \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \frac{\partial \hat{b}_1}{\partial t} d\sigma + \frac{1}{2} \frac{d}{dt} \int_{\hat{\Gamma}_s} \hat{\rho}_0 g |\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3|^2 d\sigma$$

$$884 \quad - \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} g \hat{\rho}'_0 |\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3|^2 d\xi - \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \nabla_{\xi} \cdot \hat{\mathbf{U}}_1 d\xi. \quad (\text{A } 12)$$

Summing the terms (A 5), (A 6) and

$$885 \quad \frac{d}{dt} \int_{\hat{\Omega}} \rho_0 \frac{1}{2} \left| \frac{\partial \hat{\mathbf{U}}_1}{\partial t} \right|^2 d\xi + \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0 \left( \hat{c}_0 \nabla_{\xi} \cdot \hat{\mathbf{U}}_1 - \frac{g}{\hat{c}_0} \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right)^2 d\xi$$

$$886 \quad - \frac{1}{2} \frac{d}{dt} \int_{\hat{\Omega}} \hat{\rho}_0 (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3)^2 \left( \frac{g^2}{\hat{c}_0^2} + \frac{g \hat{\rho}'_0}{\hat{\rho}_0} \right) d\xi + \frac{1}{2} \frac{d}{dt} \int_{\Gamma_s} \hat{\rho}_0 g (\hat{\mathbf{U}}_1 \cdot \mathbf{e}_3)^2 d\sigma$$

$$887 \quad = \int_{\hat{\Gamma}_b} \rho_0 \left( c_0^2 \nabla_{\xi} \cdot \hat{\mathbf{U}}_1 - \hat{\rho}_0 g \hat{\mathbf{U}}_1 \cdot \mathbf{e}_3 \right) \frac{\partial \hat{u}_{b,1}}{\partial t} d\sigma, \quad (\text{A } 13)$$

888 and by defining

$$889 \quad N_b = - \left( \frac{g^2}{\hat{c}_0^2} + \frac{g \hat{\rho}'_0}{\hat{\rho}_0} \right), \quad (\text{A } 14)$$

890 we obtain the energy equation (2.49).

## 891 Appendix B.

892 In this section we derive the relations between the zero- and first-order approximation  
893 in Eulerian and in Lagrangian coordinates, when differentiating with respect to time or  
894 space. First note that  $\phi_0$  and  $\phi_1$  can be expressed in terms of the displacement  $\mathbf{d}$ . From the  
895 assumption of small displacements it holds  $\mathbf{d} = \varepsilon \mathbf{d}_1 + O(\varepsilon^2)$ , then identifying the powers of  
896  $\varepsilon$  and summing yields

$$897 \quad \boldsymbol{\phi}_{\varepsilon}(\boldsymbol{\xi}, t) = \boldsymbol{\xi} + \varepsilon \mathbf{d}_1(\boldsymbol{\xi}, t). \quad (\text{B } 1)$$

898 From the change of coordinate we have

$$899 \quad \nabla_{\xi} \hat{X} = (\nabla_{\xi} \boldsymbol{\phi}_{\varepsilon})^T \nabla X = (Id + \varepsilon \nabla_{\xi} \mathbf{d}_1)^T \nabla X, \quad (\text{B } 2)$$



903 and using this identity for  $\hat{X} = \hat{X}_0 + \varepsilon \hat{X}_1$  yields

$$904 \quad \nabla_{\xi}(\hat{X}_0 + \varepsilon \hat{X}_1) = \nabla X_0 + \varepsilon \left( (\nabla_{\xi} \mathbf{d}_1)^T \nabla X_0 + \nabla X_1 \right) + \mathcal{O}(\varepsilon^2). \quad (\text{B } 3)$$

906 By identifying the powers of  $\varepsilon$  it holds

$$907 \quad \nabla_{\xi} \hat{X}_0 = \nabla X_0, \quad \nabla_{\xi} \hat{X}_1 = (\nabla_{\xi} \mathbf{d}_1)^T \nabla X_0 + \nabla X_1. \quad (\text{B } 4)$$

908 The same method is used for the time derivative. Starting with

$$909 \quad \frac{\partial \hat{X}}{\partial t}(\xi, t) = \frac{\partial X}{\partial t}(\phi_{\varepsilon}(\xi), t) + \frac{\partial \phi_{\varepsilon}}{\partial t}(\xi) \cdot \nabla X(\phi_{\varepsilon}(\xi), t), \quad (\text{B } 5)$$

910 we obtain after replacing  $X$  and  $\hat{X}$  by their first order approximation,

$$911 \quad \frac{\partial \hat{X}_0}{\partial t} + \varepsilon \frac{\partial \hat{X}_1}{\partial t} = \frac{\partial X_0}{\partial t} + \varepsilon \left( \frac{\partial X_1}{\partial t} + \frac{\partial \mathbf{d}_1}{\partial t} \cdot \nabla X_0 \right) + \mathcal{O}(\varepsilon^2). \quad (\text{B } 6)$$

913 With  $\partial_t \mathbf{d}_1(\xi, t) = \hat{U}_1(\xi, t) = \mathbf{U}_1(\mathbf{x}, t)$  it holds

$$914 \quad \frac{\partial X_0}{\partial t} + \varepsilon \left( \frac{\partial X_1}{\partial t} + \mathbf{U}_1 \cdot \nabla X_0 \right) + \mathcal{O}(\varepsilon^2). \quad (\text{B } 7)$$

915 We identify the powers of  $\varepsilon$ ,

$$916 \quad \frac{\partial \hat{X}_0}{\partial t} = \frac{\partial X_0}{\partial t}, \quad \frac{\partial \hat{X}_1}{\partial t} = \frac{\partial X_1}{\partial t} + \mathbf{U}_1 \cdot \nabla X_0. \quad (\text{B } 8)$$