Quantitative weak propagation of chaos for stable-driven McKean-Vlasov SDEs

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QUANTITATIVE WEAK PROPAGATION OF CHAOS FOR STABLE-DRIVEN MCKEAN-VLASOV SDES

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Abstract. We consider a general McKean-Vlasov stochastic differential equation driven by a rotationally invariant α-stable process on $\mathbb{R}^d$ with $\alpha \in (1,2)$. We assume that the diffusion coefficient is the identity matrix and that the drift is bounded and Hölder continuous in some precise sense with respect to both space and measure variables. The main goal of this work is to prove new propagation of chaos estimates, at the level of semigroup, for the associated mean-field interacting particle system. Our study relies on the regularizing properties and the dynamics of the semigroup associated with the McKean-Vlasov stochastic differential equation, which acts on functions defined on $\mathcal{P}_\beta(\mathbb{R}^d)$, the space of probability measures on $\mathbb{R}^d$ having a finite moment of order $\beta \in (1,\alpha)$. More precisely, the dynamics of the semigroup is described by a backward Kolmogorov partial differential equation defined on the strip $[0,T] \times \mathcal{P}_\beta(\mathbb{R}^d)$.

Contents

1. Introduction

In this work, we are interested in the following McKean-Vlasov Stochastic Differential Equation (SDE) driven by $Z = (Z_t)_t$ a rotationally invariant α-stable process on $\mathbb{R}^d$ with $\alpha \in (1,2)$

$$\begin{cases} dX_t = b(t, X_t, [X_t])\, dt + dZ_t, \quad t \in [0,T], \\ \mu_t := [X_t], \\ X_0 = \xi, \quad [\xi] = \mu \in \mathcal{P}(\mathbb{R}^d), \end{cases} \tag{1.1}$$

where $T > 0$ is fixed, $[\xi]$ denotes the distribution of the random variable ξ which is independent of Z, $\mathcal{P}(\mathbb{R}^d)$ is the space of probability measures on $\mathbb{R}^d$ and with drift coefficient $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$. This kind of SDEs are also called nonlinear SDEs since the associated Fokker-Planck equation solved by the flow of marginal distributions $(\mu_t)_t$ is nonlinear. We also focus on the associated mean-field interacting particle system defined,

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for all $N \geq 1$, by

$$\begin{cases} dX_t^{i,N} = b(t, X_t^{i,N}, \overline{\mu}_t^N)\, dt + dZ_t^i, \quad t \in [0,T], \quad i \in \{1, \dots, N\}, \\ \overline{\mu}_t^N := \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^{j,N}}, \\ X_0^{i,N} = \xi^i, \end{cases} \tag{1.2}$$

where $(Z^i, \xi^i)_{i \geq 1}$ are i.i.d. copies of (Z, ξ). The connection between (1.1) and (1.2) is that the McKean-Vlasov SDE (1.1) describes the dynamics of one particle of the interacting particle system (1.2) when the number of particles N tends to infinity. This property is called the mean-field limit. A stronger property that is also expected to hold true in general is the so-called propagation of chaos. It states that for all $k \geq 1$, the dynamics of k particles, let say $(X^{1,N}, \dots, X^{k,N})$, is described when N tends to infinity by k independent copies of the limiting McKean-Vlasov SDE (1.1). This was originally studied by McKean [24] and then investigated by Sznitman [29] when Z is a Brownian motion. These mean-field systems have many applications for example in physics (kinetic theory), in biology to study the motion of a cell population for example, in neuroscience to model the interactions between neurons, in social sciences to describe self-organization behaviors and also in the Mean-field games theory.

Propagation of chaos can be considered in the weak sense, i.e. in distribution through the convergence of the empirical measure $\overline{\mu}^N$, or in the strong sense, i.e. at the level of paths by coupling. The terminology of weak and strong propagation of chaos that we use is based of the corresponding properties for numerical schemes for SDEs. It can be qualitative or quantitative when a rate of convergence is shown. By quantitative weak propagation of chaos, we precisely mean to find a rate of convergence of $\mathbb{E}|\phi(\overline{\mu}_t^N) - \phi(\mu_t)|$ and $|\mathbb{E}(\phi(\overline{\mu}_t^N) - \phi(\mu_t))|$, for ϕ in a class of functions defined on the space of probability measures. The mean-field limit and propagation of chaos phenomenon have of course been widely studied before, in particular in the Brownian case. We can mention for example Gärtner [14], Méléard [27], Malrieu [22], Mischler et al. [26], Holding [15], Chaudru de Raynal and Frikha [8, 7], Jabin and Wang [16], Lacker [20, 21], Tomašević [30], Jabir [17], Chassagneux et al. [6], Delarue and Tse [9]. For a detailed review on the topics of propagation of chaos, we mention [4, 5].

It is natural to consider other types of noise such as Lévy processes, which are also largely used to model physical systems (Lévy flights and anomalous diffusion), see e.g. [23] for the physical point of view and [19] for the mathematical point of view. The propagation of chaos phenomenon has also been studied for McKean-Vlasov SDEs driven by Lévy processes. In [13], following the approach of Sznitman [29], Graham proves qualitative weak propagation of chaos under Lipschitz assumptions for a mean-field system driven by a Poisson random measure and its compensated measure. He works in the L^1 framework, i.e. the Poisson random measure is associated with a Poisson process having a finite moment of order 1. In this work, the set of jumps is assumed to be discrete, which is not the case for α-stable processes. When the driving noise is a Lévy process having a finite moment of order 2, we refer to Jourdain et al. [18]. The authors prove quantitative rates of convergence for the strong propagation of chaos in L^2 under standard Lipschitz assumptions on the drift and diffusion coefficients. Still in the Lipschitz framework, we mention Neelima et al. [28] where quantitative strong propagation of chaos is proved in L^2 relaxing the assumptions of [18]. We also mention [26] where Mischler, Mouhot and Wennberg exhibit conditions leading to weak propagation of chaos estimates. As an application, they study an inelastic Boltzmann collision jump process. In the one-dimensional case, Frikha and Li [12] study a McKean-Vlasov SDE driven by a compensated Poisson random measure with positive jumps. They prove quantitative rates of convergence for the strong propagation of chaos in L^1 under one-sided Lipschitz assumptions on the coefficients.

The aim of this work is to prove quantitative weak propagation of chaos for (1.1) assuming that the drift b is bounded and Hölder continuous in some precise sense with respect to both space and measure variables (see Assumption (H2) for the precise assumptions). As we are not in a Lipschitz framework, the standard method originally developed by Sznitman [29] in the Brownian case does not apply. We need to benefit from a regularization by noise phenomenon, even only to prove the well-posedness of (1.1). The main difficulty is

that the dependence of the drift b with respect to the measure is general, and we thus need differential calculus on the space of measures. We use the notion of linear (functional) derivative (see Definition 4). The study in the Brownian case has been made by Chaudru de Raynal and Frikha [8, 7] under similar assumptions, and our work is inspired by this. However since Z has an infinite moment of order 2, another difficulty here is that it is not possible to work in L^2 and benefit from all the theory developed in this framework, in particular for Mean-field games.

In the first part of this work, the weak well-posedness of (1.1) is established through the related nonlinear martingale problem by using the Banach fixed point theorem applied on a suitable complete metric space (see Assumption (H1) and Theorem 1). The well-posedness of (1.1) was already shown in [11] by using the same fixed point argument. The fixed point theorem is only shown to hold in small time in [11], which is enough for the well-posedness. However, in the following, we need to apply the Banach fixed point theorem on the whole time interval $[0,T]$ in order to use Picard iterations to approximate the solution to (1.1). That is why we do the proof a bit differently to obtain this result.

We then study the regularity of the transition density associated with (1.1), in particular the differentiability with respect to the initial distribution μ. Let us introduce, for $x \in \mathbb{R}^d$ and $s \in [0,T)$, the following decoupled stochastic flow associated to SDE (1.1)

$$\begin{cases} dX_t^{s,x,\mu} = b(t, X_t^{s,x,\mu}, [X_t^{s,\mu}])\, dt + dZ_t, \quad t \in [s,T], \\ X_s^{s,x,\mu} = x, \end{cases} \tag{1.3}$$

where $[X_t^{s,\mu}]$ denotes the distribution of the solution to (1.1) at time t, starting at time s by any random variable ξ with distribution $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, where $\beta \in (1,\alpha)$ is fixed. The notation $[X_t^{s,\mu}]$ is valid since by weak well-posedness, the distribution of the solution to (1.1) depends on the initial condition only through its distribution μ. The distributions of $X_t^{s,x,\mu}$ and $X_t^{s,\mu}$ admit densities respectively denoted by $p(\mu,s,t,x,\cdot)$ and $p(\mu,s,t,\cdot)$. Moreover, they satisfy the following relation stemming from the well-posedness of the martingale problem

$$p(\mu,s,t,y) = \int_{\mathbb{R}^d} p(\mu,s,t,x,y)\, d\mu(x). \tag{1.4}$$

Let us fix $t \in (0,T]$ and $y \in \mathbb{R}^d$. We study the regularity of the map $(s,x,\mu) \in [0,t) \times \mathbb{R}^d \times \mathcal{P}_\beta(\mathbb{R}^d) \mapsto p(\mu,s,t,x,y)$. Estimates and Hölder controls on the derivatives of p with respect to s, x and μ are proved in Theorem 2 under Assumption (H2) by using Picard iterations and the parametrix method. Moreover, we prove that p is solution to the following backward Kolmogorov PDE

$$\begin{cases} \partial_s p(\mu,s,t,x,y) + \mathscr{L}_s p(\cdot,s,t,\cdot,y)(\mu,x) = 0, \quad \forall (\mu,s,x) \in \mathcal{P}_\beta(\mathbb{R}^d) \times [0,t) \times \mathbb{R}^d, \\ p(\mu,s,t,x,\cdot) \underset{s \to t-}{\longrightarrow} \delta_x, \quad \text{in the weak sense}, \end{cases} \tag{1.5}$$

where $\mathscr{L}_s$ is defined, for smooth enough functions h on $\mathbb{R}^d \times \mathcal{P}_\beta(\mathbb{R}^d)$, by

$$\begin{aligned} \mathscr{L}_s h(\mu,x) := {} & b(s,x,\mu) \cdot \partial_x h(\mu,x) + \int_{\mathbb{R}^d} \left[h(\mu, x+z) - h(\mu,x) - z \cdot \partial_x h(\mu,x) \right] \frac{dz}{|z|^{d+\alpha}} \\ & + \int_{\mathbb{R}^d} b(s,v,\mu) \cdot \partial_v \frac{\delta}{\delta m} h(\mu,x)(v)\, d\mu(v) \\ & + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{\delta}{\delta m} h(\mu,x)(v+z) - \frac{\delta}{\delta m} h(\mu,x)(v) - z \cdot \partial_v \frac{\delta}{\delta m} h(\mu,x)(v) \right] \frac{dz}{|z|^{d+\alpha}}\, d\mu(v), \end{aligned} \tag{1.6}$$

where $\frac{\delta}{\delta m}$ denotes the linear derivative (see Definition 4).

Once that the regularity of the transition density has been studied, we focus on the regularizing properties of the semigroup acting on functions defined on $\mathcal{P}_\beta(\mathbb{R}^d)$ associated with (1.1). For a function $\phi : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$,

the action of the McKean-Vlasov semigroup on ϕ is given by the function U defined by

$$U : (t, \mu) \in [0, T] \times \mathcal{P}_\beta(\mathbb{R}^d) \mapsto \phi([X_T^{t,\mu}]), \tag{1.7}$$

where $[X_T^{t,\mu}]$ denotes the distribution of the solution to (1.1) at time T and starting at time t with a random variable ξ with distribution $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$. More precisely, we prove that if $\phi : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$ has a linear derivative that is uniformly Hölder continuous, then the map U defined by (1.7) is more regular and estimates and Hölder controls on the derivative of U with respect to the measure variable are proved. Inspired by [2] (see (5.128)), [1] and [8], we describe the dynamics of the semigroup in Theorem 3 by showing that U is the unique classical solution to the backward Kolmogorov PDE

$$\begin{cases} \partial_t U(t, \mu) + \mathscr{L}_t U(t, \cdot)(\mu) = 0, \quad \forall (t, \mu) \in [0, T) \times \mathcal{P}_\beta(\mathbb{R}^d), \\ U(T, \mu) = \phi(\mu), \quad \forall \mu \in \mathcal{P}_\beta(\mathbb{R}^d), \end{cases} \tag{1.8}$$

where $\mathscr{L}_t$ was defined in (1.6), and acts here only on functions defined on $\mathcal{P}_\beta(\mathbb{R}^d)$.

Finally, we prove in Theorem 4 new weak propagation of chaos estimates for (1.2). We refer the reader to Remark 5 for a comparison with the existing literature. The method that we use relies on the solution to the backward Kolmogorov PDE (1.8) U defined in (1.7). This strategy was originally described in Chapter 5 of [2] (pages $506 - 508$), inspired by [1] and [26], and was employed for example in [6, 9, 7]. Let us describe the main ideas. We begin by computing the time derivative of the map $t \in [0, T) \mapsto U(t, \overline{\mu}_t^N)$ by applying the standard Itô's formula for the empirical projection $(t, x_1, \dots, x_N) \in [0, T] \times (\mathbb{R}^d)^N \mapsto U\left(T - t, \frac{1}{N}\sum_{k=1}^N \delta_{x_k}\right)$ and for the particle system. Noting that $t \in [0, T] \mapsto U(t, \mu_t)$ is constant, we naturally expect that the time derivative previously computed tends to 0 as N converges to infinity. This convergence has to be shown with an explicit rate of convergence using the PDE satisfied by U and the estimates on U. Finally, we express $\phi(\overline{\mu}_T^N) - \phi(\mu_T) = U(T, \overline{\mu}_T^N) - U(T, \mu_T)$ as the sum of $U(0, \overline{\mu}_0^N) - U(0, \mu_0)$ plus a remainder term related to the time derivative which was previously estimated. Since the initial data are i.i.d., the first term is controlled by standard estimates, for example those in [10].

The paper is organized as follows. In Section 2, we present our results and we comment them. The proofs are given in the next sections. Section 3 is dedicated to prove the weak well-posedness of the McKean-Vlasov SDE (1.1) (Theorem 1). Then, we study the regularity of its transition density in Section 4 (Theorem 2). Section 5 is devoted to the proof of the regularizing properties of the semigroup and backward Kolmogorov PDE that describes its dynamics (Theorem 3). In Section 6, we prove our propagation of chaos estimates (Theorem 4). We prove in Section 7 the technical proposition leading to the regularity of the transition density and the related estimates stated in Theorem 2. In Appendix A, we gather some definitions and propositions related to differential calculus for functions defined on $\mathcal{P}(\mathbb{R}^d)$ or $\mathcal{P}_\beta(\mathbb{R}^d)$ that are used in the article. Finally, Appendix B aims at presenting the parametrix method and quite standard estimates on the density of the solution to a linear stable-driven SDE which are the core of the proof of Theorem 2.

Let us finally introduce some notations used several times in the article.

Notations

- $\mathcal{P}(\mathbb{R}^d)$ denotes the set of probability measures on $\mathbb{R}^d$.
- d_{TV} is the total variation metric on $\mathcal{P}(\mathbb{R}^d)$.
- $\mathcal{P}_\beta(\mathbb{R}^d)$ denotes the set of probability measures μ on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |x|^\beta \, d\mu(x) < +\infty$, for $\beta \geq 1$. It is equipped with the Wasserstein metric of order β denoted by W_β, which makes it complete. Denoting by $\Pi(\mu, \nu)$ the set of couplings between two probability measures $\mu, \nu \in \mathcal{P}_\beta(\mathbb{R}^d)$, the metric W_β is defined by

$$W_\beta(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\beta \, d\pi(x, y) \right)^{\frac{1}{\beta}}.$$

- $[\xi]$ denotes the distribution of the random variable ξ.
- $\overline{\mu}^N_{\boldsymbol{x}} := \frac{1}{N}\sum_{k=1}^N \delta_{x_k}$ denotes the empirical measure, for $\boldsymbol{x} = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$.
- $\tilde{\boldsymbol{z}}_{\boldsymbol{k}} := (0, \dots, z, \dots, 0) \in (\mathbb{R}^d)^N$ for $z \in \mathbb{R}^d$, where z appears in the k-th position.
- B_r is the open ball in $\mathbb{R}^d$ centered at 0 and of radius r for the euclidean norm.
- B_r^c denotes the complementary of B_r.
- $a \wedge b$ denotes the minimum between a and b.
- $a \vee b$ denotes the maximum between a and b.
- C is a generic constant that may depend only on the fixed parameters of the problem and which may change from line to line.

2. Overview on the main results

Let us fix $Z = (Z_t)_t$ a rotationally invariant α-stable process on $\mathbb{R}^d$ with $\alpha \in (1,2)$. Its associated Poisson random measure is denoted by $\mathcal{N}$ and the compensated Poisson random measure by $\widetilde{\mathcal{N}}$. Since $\alpha \in (1,2)$, we can write for all $t \geq 0$

$$Z_t = \int_0^t \int_{\mathbb{R}^d} z \widetilde{\mathcal{N}}(ds, dz).$$

The density of Z_t is denoted by $q(t, \cdot)$ and the Lévy measure ν of Z is given by

$$d\nu(z) := \frac{dz}{|z|^{d+\alpha}}.$$

The related stable operator $\mathcal{L}^\alpha$ is defined for all $f \in \mathcal{C}_b^{1+\gamma}(\mathbb{R}^d; \mathbb{R})$, with $\gamma > \alpha - 1$, i.e. f belongs to $\mathcal{C}_b^1(\mathbb{R}^d; \mathbb{R})$ and ∇f is γ-Hölder and for all $x \in \mathbb{R}^d$ by

$$\mathcal{L}^\alpha f(x) := \int_{\mathbb{R}^d} (f(x+z) - f(x) - \nabla f(x) \cdot z)\, d\nu(z). \tag{2.1}$$

We are interested in the following stable-driven McKean-Vlasov SDE

$$\begin{cases} dX_t^{s,\xi} = b(t, X_t^{s,\xi}, [X_t^{s,\xi}])\, dt + dZ_t, \quad t \in [s,T], \\ X_s^{s,\xi} = \xi, \quad [\xi] = \mu \in \mathcal{P}(\mathbb{R}^d), \end{cases} \tag{2.2}$$

where $[\xi]$ denotes the distribution of the random variable ξ and $s \in [0,T)$.

Let us define, for $k > -\alpha$ the function ρ^k on $(0, +\infty) \times \mathbb{R}^d$ by

$$\forall t > 0,\ x \in \mathbb{R}^d,\ \rho^k(t,x) := t^{-\frac{d}{\alpha}}(1 + t^{-\frac{1}{\alpha}}|x|)^{-d-\alpha-k}. \tag{2.3}$$

These functions are related to gradient estimates on the transition density q of the stable process Z (see Lemma 11). Some useful properties of these functions are given in Lemma 10.

2.1. Well-posedness of the nonlinear martingale problem and Picard iterations.

The first point is to prove the existence and uniqueness, in the weak sense, of the solution to (2.7). Let us first recall the definition of the nonlinear martingale problem associated with (2.7).

Definition 1. Let us fix $(s, \mu) \in [0,T) \times \mathcal{P}(\mathbb{R}^d)$. We say that a probability measure $\mathbb{P}$ on the Skorokhod space $\mathcal{D}([s,T]; \mathbb{R}^d)$, endowed with its canonical filtration $(\mathcal{F}_t)_t$, with time marginal distributions $(\mathbb{P}_t)_{t\in[s,T]} \in \mathcal{C}^0([s,T]; \mathcal{P}(\mathbb{R}^d))$ solves the nonlinear martingale problem associated to the SDE (2.2) with initial distribution μ at time s if the canonical process $(y_t)_{t\in[s,T]}$ satisfies the two following conditions.

(1) We have $\mathbb{P}_s = \mu$.
(2) For any $\phi \in \mathcal{C}_b^{1,2}([s,T] \times \mathbb{R}^d)$, the process defined for $t \in [0,T]$ by

$$\phi(t, y_t) - \phi(s, y_s) - \int_s^t \left(\partial_r + \mathcal{L}_r^{\mathbb{P}}\right) \phi(r, y_r)\, dr$$

is a $(\mathcal{D}([s,T];\mathbb{R}^d),(\mathcal{F}_t)_t,\mathbb{P})$-martingale starting from 0 at time $t=s$ and where

$$\mathcal{L}_r^{\mathbb{P}} f(t,x) := b(r,x,\mathbb{P}_r)\cdot\partial_x f(t,x) + \mathcal{L}^{\alpha} f(t,\cdot)(x). \tag{2.4}$$

We assume that the drift $b:[0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\to\mathbb{R}^d$ satisfies the following properties.

Assumption (H1).

(1) The drift b is measurable and globally bounded on $[0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)$.
(2) For any $(t,\mu)\in[0,T]\times\mathcal{P}(\mathbb{R}^d)$, the map $b(t,\cdot,\mu)$ is η-Hölder continuous on $\mathbb{R}^d$ uniformly with respect to $(t,\mu)\in[0,T]\times\mathcal{P}(\mathbb{R}^d)$, with $\eta\in(0,1]$, i.e. there exists $C>0$ such that for all $t\in[0,T]$, $\mu\in\mathcal{P}(\mathbb{R}^d)$ and $x_1,x_2\in\mathbb{R}^d$
$$|b(t,x_1,\mu)-b(t,x_2,\mu)|\le C|x_1-x_2|^{\eta}.$$
(3) For any $(t,x)\in[0,T]\times\mathbb{R}^d$, the map $b(t,x,\cdot)$ is Lipschitz continuous with respect to the total variation metric d_{TV} uniformly with respect to $(t,x)\in[0,T]\times\mathbb{R}^d$ i.e. there exists $C>0$ such that for all $t\in[0,T]$, $x\in\mathbb{R}^d$ and $\mu_1,\mu_2\in\mathcal{P}(\mathbb{R}^d)$
$$|b(t,x,\mu_1)-b(t,x,\mu_2)|\le C d_{TV}(\mu_1,\mu_2).$$

We can now state the weak well-posedness result.

Theorem 1 (Weak well-posedness)**.** Under Assumption (H1), the martingale problem associated to the McKean-Vlasov SDE (2.2) is well-posed for all initial distribution $\mu\in\mathcal{P}(\mathbb{R}^d)$. In particular, the SDE (2.2) is well-posed in the weak sense. Moreover, for any $P\in\mathcal{C}^0([s,T];\mathcal{P}(\mathbb{R}^d))$ with $P_0=\mu\in\mathcal{P}(\mathbb{R}^d)$, we can define recursively $\overline{X}^{(m)}$ as the unique weak solution to

$$\begin{cases} d\overline{X}_t^{(m)} = b(t,\overline{X}_t^m,[\overline{X}_t^{(m-1)}])\,dt + dZ_t, \quad t\in[s,T],\\ \overline{X}_s^{(m)} = \xi, \end{cases} \tag{2.5}$$

with $[\xi]=\mu$ and $([\overline{X}_t^{(0)}])_{t\in[s,T]}=P$. Then, denoting by P^* the unique solution to the martingale problem associated with (2.2) with initial condition $\mu\in\mathcal{P}(\mathbb{R}^d)$ at time s, we have

$$\sup_{t\in[s,T]} d_{TV}(P_t^*,[\overline{X}_t^{(m)}]) \underset{m\to+\infty}{\longrightarrow} 0, \tag{2.6}$$

where d_{TV} is the total variation metric.

We prove this result in Section 3.

Remark 1. The well-posedness of the nonlinear martingale problem was already proved in [11]. Therein, the fixed point theorem is only shown to hold in small time since it is enough to prove the well-posedness. However, we will need in the following the convergence of the sequence of Picard iterations on the whole interval $[s,T]$. That is why we do the proof a bit differently.

2.2. **Properties of the transition density.** Now that the well-posedness has been established, we focus on the regularity of the transition density associated with (2.7). In particular, our aim is to study its regularity with respect to the initial distribution μ. To do this, we fix $s\in[0,T)$, $\beta\in(1,\alpha)$ and we consider the following stable-driven McKean-Vlasov SDE

$$\begin{cases} dX_t^{s,\xi} = b(t,X_t^{s,\xi},[X_t^{s,\xi}])\,dt + dZ_t, \quad t\in[s,T],\\ X_s^{s,\xi} = \xi, \quad [\xi]=\mu\in\mathcal{P}_\beta(\mathbb{R}^d). \end{cases} \tag{2.7}$$

The associated martingale problem is well-posed using Theorem 1 and there is weak existence and uniqueness for the SDE (2.7). Moreover the distribution of $X_t^{s,\xi}$ depends only on μ and not on the choice of the random variable ξ such that $[\xi]=\mu$ and we denote it by

$$[X_t^{s,\mu}] := [X_t^{s,\xi}]. \tag{2.8}$$

We introduce, for $x\in\mathbb{R}^d$, the following decoupled stochastic flow associated to SDE (2.7)

$$\begin{cases} dX_t^{s,x,\mu} = b(t, X_t^{s,x,\mu}, [X_t^{s,\mu}])\, dt + dZ_t, \quad t \in [s,T], \\ X_s^{s,x,\mu} = x \in \mathbb{R}^d. \end{cases} \tag{2.9}$$

The distribution of $X_t^{s,x,\mu}$ admits a density denoted by $p(\mu, s, t, x, \cdot)$ by using the parametrix expansion given in Theorem 5. The distribution of $X_t^{s,\mu}$ has also a density denoted by $p(\mu, s, t, \cdot)$. Moreover, it satisfies the following relation stemming from the well-posedness of the martingale problem

$$p(\mu, s, t, y) = \int_{\mathbb{R}^d} p(\mu, s, t, x, y)\, d\mu(x). \tag{2.10}$$

We introduce the following assumption, which is stronger than Assumption (H1).

Assumption (H2).

(1) The drift b is jointly continuous and globally bounded on $[0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$.

(2) For any $(t,\mu) \in [0,T] \times \mathcal{P}(\mathbb{R}^d)$, the map $b(t,\cdot,\mu)$ is η-Hölder continuous on $\mathbb{R}^d$, for some fixed $\eta \in (0,1]$, uniformly with respect to $(t,\mu) \in [0,T] \times \mathcal{P}(\mathbb{R}^d)$, i.e. there exists $C > 0$ such that for all $t \in [0,T]$, $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $x_1, x_2 \in \mathbb{R}^d$

$$|b(t, x_1, \mu) - b(t, x_2, \mu)| \le C|x_1 - x_2|^\eta.$$

(3) For any $(t,x) \in [0,T] \times \mathbb{R}^d$, the map $\mu \in \mathcal{P}(\mathbb{R}^d) \mapsto b(t,x,\mu)$ has a linear derivative such that $\frac{\delta}{\delta m} b(t,x,\mu)(\cdot)$ is η-Hölder continuous on $\mathbb{R}^d$ uniformly with respect to $(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ and $\frac{\delta}{\delta m} b$ is bounded on $[0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d$.

(4) For any $(t,x,v) \in [0,T] \times (\mathbb{R}^d)^2$, the map $\mu \in \mathcal{P}(\mathbb{R}^d) \mapsto \frac{\delta}{\delta m} b(t,x,\mu)(v)$ has a linear derivative such that $\frac{\delta^2}{\delta m^2} b(t,x,\mu)(v,\cdot)$ is η-Hölder continuous uniformly with respect to $(t,x,\mu,v) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d$ and $\frac{\delta^2}{\delta m^2} b$ is bounded on $[0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times (\mathbb{R}^d)^2$.

We state in the next theorem the regularity properties satisfied by the transition density p associated with the McKean-Vlasov SDE (2.7). These estimates will be crucial to exhibit the regularizing properties of the semigroup associated with (2.7).

Theorem 2 (Regularity of the transition density)**.** Let us fix $(t,y) \in (0,T] \times \mathbb{R}^d$. Under Assumption (H2), the mapping $(\mu, s, x) \in \mathcal{P}_\beta(\mathbb{R}^d) \times [0,t) \times \mathbb{R}^d \mapsto p(\mu, s, t, x, y)$ belongs to $\mathcal{C}^1(\mathcal{P}_\beta(\mathbb{R}^d) \times [0,t) \times \mathbb{R}^d)$ (see Definition (5)), and is solution to the following backward Kolmogorov PDE

$$\begin{cases} \partial_s p(\mu, s, t, x, y) + \mathscr{L}_s p(\cdot, s, t, \cdot, y)(\mu, x) = 0, \quad \forall (\mu, s, x) \in \mathcal{P}_\beta(\mathbb{R}^d) \times [0,t) \times \mathbb{R}^d, \\ p(\mu, s, t, x, \cdot) \underset{s \to t-}{\longrightarrow} \delta_x, \quad \text{in the weak sense}, \end{cases} \tag{2.11}$$

where $\mathscr{L}_s$ is defined, for smooth enough function h on $\mathbb{R}^d \times \mathcal{P}_\beta(\mathbb{R}^d)$, by

$$\begin{aligned} \mathscr{L}_s h(\mu, x) := b(s, x, \mu) \cdot \partial_x h(\mu, x) &+ \int_{\mathbb{R}^d} [h(\mu, x+z) - h(\mu, x) - z \cdot \partial_x h(\mu, x)] \, \frac{dz}{|z|^{d+\alpha}} \\ &+ \int_{\mathbb{R}^d} b(s, v, \mu) \cdot \partial_v \frac{\delta}{\delta m} h(\mu, x)(v)\, d\mu(v) \\ &+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{\delta}{\delta m} h(\mu, x)(v+z) - \frac{\delta}{\delta m} h(\mu, x)(v) - z \cdot \partial_v \frac{\delta}{\delta m} h(\mu, x)(v) \right] \frac{dz}{|z|^{d+\alpha}}\, d\mu(v). \end{aligned} \tag{2.12}$$

Moreover, it satisfies the following properties.

- There exists $C > 0$ such that for all $j \in \{0,1\}$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \le s < t \le T$ and $x, y \in \mathbb{R}^d$

$$|\partial_x^j p(\mu, s, t, x, y)| \le C(t-s)^{-\frac{j}{\alpha}} \rho^j(t-s, y-x). \tag{2.13}$$

- For all $j \in \{0,1\}$ and $\gamma \in (0,1]$ with $\gamma \in (0, (2\alpha - 2) \wedge (\eta + \alpha - 1))$ if $j = 1$, there exists $C > 0$ such that for all $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \le s < t \le T$ and $x_1, x_2, y \in \mathbb{R}^d$

$$|\partial_x^j p(\mu,s,t,x_1,y)-\partial_x^j p(\mu,s,t,x_2,y)|\le C(t-s)^{-\frac{j+\gamma}{\alpha}}|x_1-x_2|^\gamma\left[\rho^j(t-s,y-x_1)+\rho^j(t-s,y-x_2)\right]. \quad (2.14)$$

- There exists $C>0$ such that for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x,y,v\in\mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v)\right|\le C(t-s)^{1-\frac{1}{\alpha}}\rho^0(t-s,y-x). \quad (2.15)$$

- There exists $C>0$ such that for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x,y,v\in\mathbb{R}^d$

$$\left|\partial_v\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v)\right|\le C(t-s)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}\rho^0(t-s,y-x). \quad (2.16)$$

- There exists a constant $C>0$ such that for all $0\le s<t\le T$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $x,y\in\mathbb{R}^d$

$$|\partial_s p(\mu,s,t,x,y)|\le C(t-s)^{-1}\rho^0(t-s,y-x). \quad (2.17)$$

- For all $\gamma\in(0,1]\cap(0,(2\alpha-2)\wedge(\eta+\alpha-1))$, there exists $C>0$ such that for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x,y,v_1,v_2\in\mathbb{R}^d$

$$\left|\partial_v\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v_1)-\partial_v\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v_2)\right|\le C(t-s)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}}|v_1-v_2|^\gamma\rho^0(t-s,y-x). \quad (2.18)$$

- For all $\gamma\in(0,1]$, there exists $C>0$ such that for all , $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x,y,v_1,v_2\in\mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v_1)-\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v_2)\right|\le C(t-s)^{1-\frac{1+\gamma}{\alpha}}|v_1-v_2|^\gamma\rho^0(t-s,y-x). \quad (2.19)$$

- For all $\gamma\in(0,1]$, there exists $C>0$ such that for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x_1,x_2,y,v\in\mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}p(\mu,s,t,x_1,y)(v)-\frac{\delta}{\delta m}p(\mu,s,t,x_2,y)(v)\right|\le C(t-s)^{1-\frac{1+\gamma}{\alpha}}|x_1-x_2|^\gamma \left[\rho^0(t-s,y-x_1)+\rho^0(t-s,y-x_2)\right]. \quad (2.20)$$

- For all $\gamma\in(0,1]\cap(0,\eta+\alpha-1)$, there exists $C>0$ such that for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x_1,x_2,y,v\in\mathbb{R}^d$

$$\left|\partial_v\frac{\delta}{\delta m}p(\mu,s,t,x_1,y)(v)-\partial_v\frac{\delta}{\delta m}p(\mu,s,t,x_2,y)(v)\right|\le C(t-s)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}}|x_1-x_2|^\gamma \left[\rho^0(t-s,y-x_1)+\rho^0(t-s,y-x_2)\right]. \quad (2.21)$$

- For all $j\in\{0,1\}$ and $\gamma\in(0,1]$ with $\gamma\in(0,\alpha-1+\eta)$ if $j=1$, there exists $C>0$ such that for all $0\le s<t\le T$, $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $x,y\in\mathbb{R}^d$

$$\left|\partial_x^j p(\mu_1,s,t,x,y)-\partial_x^j p(\mu_2,s,t,x,y)\right|\le C(t-s)^{1-\frac{1+\gamma+j}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^j(t-s,y-x). \quad (2.22)$$

- For all $\gamma\in(0,1]$, there exists $C>0$ such that for all $0\le s<t\le T$, $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $x,y,v\in\mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}p(\mu_1,s,t,x,y)(v)-\frac{\delta}{\delta m}p(\mu_2,s,t,x,y)(v)\right|\le C(t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x). \quad (2.23)$$

- For all $\gamma\in(0,1]$, there exists $C>0$ such that for all $0\le s<t\le T$, $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $x,y,v\in\mathbb{R}^d$

$$\left|\partial_v\frac{\delta}{\delta m}p(\mu_1,s,t,x,y)(v)-\partial_v\frac{\delta}{\delta m}p(\mu_2,s,t,x,y)(v)\right|\le C(t-s)^{\frac{\eta-\gamma-1}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x). \quad (2.24)$$

- For all $j \in \{0,1\}$, $\gamma \in \left(0, 1-\frac{j}{\alpha}\right)$, there exists a constant $C>0$ such that for all $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$

$$|\partial_x^j p(\mu, s_1, t, x, y) - \partial_x^j p(\mu, s_2, t, x, y)| \leq C\left[\frac{|s_1-s_2|^\gamma}{(t-s_1)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1, y-x) + \frac{|s_1-s_2|^\gamma}{(t-s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_2, y-x)\right]. \quad (2.25)$$

- For all $\gamma \in (0,1)$, there exists a constant $C>0$ such that for all $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y, v \in \mathbb{R}^d$

$$\left|\frac{\delta}{\delta m} p(\mu, s_1, t, x, y)(v) - \frac{\delta}{\delta m} p(\mu, s_2, t, x, y)(v)\right| \leq C\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1}}\rho^0(t-s_1\vee s_2, y-x) + \frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1}}\rho^0(t-s_1\wedge s_2, y-x)\right]. \quad (2.26)$$

- For all $\gamma \in \left(0, 1+\frac{\eta-1}{\alpha}\right)$, there exists a constant $C>0$ such that for all $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y, v \in \mathbb{R}^d$

$$\left|\partial_v \frac{\delta}{\delta m} p(\mu, s_1, t, x, y)(v) - \partial_v \frac{\delta}{\delta m} p(\mu, s_2, t, x, y)(v)\right| \quad (2.27)$$
$$\leq C\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}\rho^0(t-s_1\vee s_2, y-x) + \frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}\rho^0(t-s_1\wedge s_2, y-x)\right].$$

- For all $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \leq s < t \leq T$, $x, v \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \frac{\delta}{\delta m} p(\mu, s, t, x, y)(v)\, dy = \int_{\mathbb{R}^d} \partial_v \frac{\delta}{\delta m}(\mu, s, t, x, y)(v)\, dy = 0. \quad (2.28)$$

The proof of this theorem is postponed to Section 4.

Remark 2. Contrary to the Brownian case studied in [8, 7], we do not need to prove $\mathcal{C}^2$ regularity with respect to $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$. Such controls would impose stronger assumptions.

2.3. **Backward Kolmogorov PDE on the space of measures.** We can now focus on the study of the semigroup associated with (2.7) acting on functions defined on $\mathcal{P}_\beta(\mathbb{R}^d)$. Let us recall that $\beta \in (1,\alpha)$ is fixed. For a fixed function $\phi : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$, the action of the semigroup on ϕ is given by the map U defined by

$$U(t,\mu) := \phi([X_T^{t,\mu}]), \quad \forall (t,\mu) \in [0,T] \times \mathcal{P}_\beta(\mathbb{R}^d), \quad (2.29)$$

where $[X_T^{t,\mu}]$ is the flow of (2.7) where the terminal distribution is equal to μ and was defined in (2.8). We aim at studying the regularizing properties of the semigroup, i.e. the gain of regularity between ϕ and U with respect to the measure variable.

We define the space of functions on which we make act the semigroup.

Definition 2. Let us fix $\delta \in (0,1]$. The space $\mathcal{C}^\delta(\mathcal{P}_\beta(\mathbb{R}^d))$ is defined as the set of continuous functions $\phi : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$ admitting a linear derivative such that there exists a positive constant C such that for all $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $v_1, v_2 \in \mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}\phi(\mu)(v_1) - \frac{\delta}{\delta m}\phi(\mu)(v_2)\right| \leq C|v_1 - v_2|^\delta.$$

We state in the next theorem the regularizing properties of the semigroup acting on $\mathcal{C}^\delta(\mathcal{P}_\beta(\mathbb{R}^d))$ and we describe its dynamics through the backward Kolmogorov PDE that it satisfies.

Theorem 3 (Backward Kolmogorov PDE)**.** Let us fix $\phi \in \mathcal{C}^\delta(\mathcal{P}_\beta(\mathbb{R}^d))$. Then, under Assumption (H2), the function U defined in (2.29) belongs to $\mathcal{C}^0([0,T]\times\mathcal{P}_\beta(\mathbb{R}^d))\cap\mathcal{C}^1([0,T)\times\mathcal{P}_\beta(\mathbb{R}^d))$ (see Definition 5) and satisfies the following properties.

- There exists a positive constant C such that for all $t\in[0,T)$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $v\in\mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}U(t,\mu)(v)\right| \le C(1+|v|^\delta). \tag{2.30}$$

- There exists a positive constant C such that for all $t\in[0,T)$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $v\in\mathbb{R}^d$

$$\left|\partial_v\frac{\delta}{\delta m}U(t,\mu)(v)\right| \le C(T-t)^{\frac{\delta-1}{\alpha}}. \tag{2.31}$$

- For all $\gamma\in(0,1]\cap(0,(2\alpha-2)\wedge(\eta+\alpha-1))$, there exists a positive constant C such that for all $t\in[0,T)$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $v_1,v_2\in\mathbb{R}^d$

$$\left|\partial_v\frac{\delta}{\delta m}U(t,\mu)(v_1)-\partial_v\frac{\delta}{\delta m}U(t,\mu)(v_2)\right| \le C(T-t)^{\frac{\delta-1-\gamma}{\alpha}}|v_1-v_2|^\gamma. \tag{2.32}$$

Moreover, U is solution to the following backward Kolmogorov PDE

$$\begin{cases}\partial_t U(t,\mu)+\mathscr{L}_t U(t,\cdot)(\mu)=0, \quad \forall(t,\mu)\in[0,T)\times\mathcal{P}_\beta(\mathbb{R}^d),\\ U(T,\mu)=\phi(\mu), \quad \forall\mu\in\mathcal{P}_\beta(\mathbb{R}^d),\end{cases} \tag{2.33}$$

where $\mathscr{L}_t$ was defined in (2.12). It is the unique solution to (2.33) among all functions in $\mathcal{C}^0([0,T]\times\mathcal{P}_\beta(\mathbb{R}^d))\cap\mathcal{C}^1([0,T)\times\mathcal{P}_\beta(\mathbb{R}^d))$ satisfying (2.30), (2.31) and (2.32).

We prove this result in Section 5.

2.4. **Quantitative weak propagation of chaos.** We are now going to use the regularizing properties and the dynamics of the semigroup given in Theorem 3 to prove quantitative propagation of chaos for the mean-field interacting particle system associated with (2.7). Let us introduce $(Z^n)_n$ an i.i.d. sequence of α-stable processes having the same distribution as Z and $(X_0^n)_n$ an i.i.d. sequence of random variables with common distribution $\mu_0\in\mathcal{P}_\beta(\mathbb{R}^d)$, where $\beta\in(1,\alpha)$ is always fixed. For an integer $N\ge1$, the system of N particles associated with (2.7) is defined as the unique weak-solution to the following classical SDE on $(\mathbb{R}^d)^N$

$$\begin{cases} dX_t^{i,N}=b(t,X_t^{i,N},\overline{\mu}_t^N)\,dt+dZ_t^i, \quad t\in[0,T], \quad i\in\{1,\dots,N\},\\ \overline{\mu}_t^N:=\frac{1}{N}\sum_{j=1}^N\delta_{X_t^{j,N}},\\ X_0^{i,N}=X_0^i.\end{cases} \tag{2.34}$$

This linear SDE on $(\mathbb{R}^d)^N$ is well-posed in the weak sense using [25]. The limiting McKean-Vlasov SDE is (2.7) starting at time $s=0$ from any random variable ξ with distribution $\mu_0\in\mathcal{P}_\beta(\mathbb{R}^d)$. We denote by $(\mu_t)_{t\in[0,T]}$ the flow of marginal distributions of its solution. Let us define the space of test functions that we use to quantify the propagation of chaos of the particle system (2.34) towards the McKean-Vlasov SDE (2.7).

Definition 3. For $\delta\in(0,1]$ and $L>0$, we define the space $\mathcal{C}_L^{\delta,\delta}(\mathcal{P}_\beta(\mathbb{R}^d))$ as the set of continuous functions $\phi:\mathcal{P}_\beta(\mathbb{R}^d)\to\mathbb{R}$ admitting two linear derivatives $\frac{\delta}{\delta m}\phi$ and $\frac{\delta^2}{\delta m^2}\phi$ such that for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $v_1,v_2,v_1',v_2'\in\mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}\phi(\mu)(v_1)-\frac{\delta}{\delta m}\phi(\mu)(v_2)\right| \le L|v_1-v_2|^\delta,$$

and

$$\left|\frac{\delta^2}{\delta m^2}\phi(\mu)(v_1,v_1') - \frac{\delta^2}{\delta m^2}\phi(\mu)(v_2,v_2')\right| \le L(|v_1-v_2|^\delta + |v_1'-v_2'|^\delta).$$

Remark 3. Note that $\mathcal{C}_L^{\delta,\delta}(\mathcal{P}_\beta(\mathbb{R}^d))$ is a subspace of $\mathcal{C}^\delta(\mathcal{P}_\beta(\mathbb{R}^d))$ defined in Definition 2.

We now state our quantitative propagation of chaos result.

Theorem 4 (Quantitative weak propagation of chaos)**.**
Let us fix $\delta \in (0,1]$, $L>0$ and $\gamma \in (0,1] \cap (0,(\delta+\alpha-1)\wedge(2\alpha-2)\wedge(\eta+\alpha-1))$. Then, under Assumption (H2), there exists a positive constant $C = C(d,T,\alpha,\beta,(H2),\gamma,\delta,L)$, non-decreasing with respect to T, such that for all $\phi \in \mathcal{C}_L^{\delta,\delta}(\mathcal{P}_\beta(\mathbb{R}^d))$, it holds

$$\mathbb{E}|\phi(\overline{\mu}_T^N) - \phi(\mu_T)| \le C\mathbb{E}W_1(\overline{\mu}_0^N,\mu_0) + \frac{C}{N^{1-\frac{1}{\beta}}}, \tag{2.35}$$

and

$$|\mathbb{E}(\phi(\overline{\mu}_T^N) - \phi(\mu_T))| \le C\mathbb{E}W_1(\overline{\mu}_0^N,\mu_0) + \frac{C}{N^{\gamma}}. \tag{2.36}$$

Moreover, if $\mu_0 \in \mathcal{P}_{2\delta}(\mathbb{R}^d)$, we have for all $\phi \in \mathcal{C}_L^{\delta,\delta}(\mathcal{P}_\beta(\mathbb{R}^d))$

$$|\mathbb{E}(\phi(\overline{\mu}_T^N) - \phi(\mu_T))| \le \frac{C}{N^{\gamma}}. \tag{2.37}$$

The proof is given in Section 6.

Remark 4.

- The initial data terms can be handled using Fournier-Guillin [10], in particular in the case where μ_0 has more moments than β. Indeed, one has if $\mu_0 \in \mathcal{P}_q(\mathbb{R}^d)$ with $q \ge 1$

$$\mathbb{E}W_1(\overline{\mu}_0^N,\mu_0) \le C\begin{cases} N^{-\frac{1}{2}} + N^{-\left(1-\frac{1}{q}\right)}, & \text{if } d=1 \text{ and } q\neq 2,\\ N^{-\frac{1}{2}}\ln(1+N) + N^{-\left(1-\frac{1}{q}\right)}, & \text{if } d=2 \text{ and } q\neq 2,\\ N^{-\frac{1}{d}} + N^{-\left(1-\frac{1}{q}\right)}, & \text{if } d\ge 3 \text{ and } q\neq \frac{d}{d-1}.\end{cases}$$

- Let us denote by $\|\varphi\|_{\mathrm{Lip}} := \sup_{x\neq y}\frac{|\varphi(x)-\varphi(y)|}{|x-y|}$ for $\varphi:\mathbb{R}^d\to\mathbb{R}$. The set

$$\left\{\phi:\mathcal{P}_\beta(\mathbb{R}^d)\to\mathbb{R},\ \exists\varphi:\mathbb{R}^d\to\mathbb{R}, \text{ with } \|\varphi\|_{\mathrm{Lip}}\le 1, \text{ and } \phi(\mu) = \int_{\mathbb{R}^d}\varphi\,d\mu,\ \forall\mu\in\mathcal{P}_\beta(\mathbb{R}^d)\right\}$$

is contained in $\mathcal{C}_1^{1,1}(\mathcal{P}_\beta(\mathbb{R}^d))$. We can thus quantify the mean-field limit with respect to W_1 thanks to Theorem 4. Indeed, we have by the Kantorovich-Rubinstein theorem if $\mu_0\in\mathcal{P}_{2\delta}(\mathbb{R}^d)$

$$\begin{aligned}\sup_{t\in[0,T]} W_1([X_t^{1,N}],\mu_t) &= \sup_{t\in[0,T]}\sup_{\varphi,\,\|\varphi\|_{\mathrm{Lip}}\le 1}\left|\mathbb{E}\varphi(X_t^{1,N}) - \int_{\mathbb{R}^d}\varphi\,d\mu_t\right|\\ &= \sup_{t\in[0,T]}\sup_{\varphi,\,\|\varphi\|_{\mathrm{Lip}}\le 1}\left|\mathbb{E}\left(\frac{1}{N}\sum_{k=1}^N\varphi(X_t^{k,N})\right) - \int_{\mathbb{R}^d}\varphi\,d\mu_t\right|\\ &= \sup_{t\in[0,T]}\sup_{\varphi,\,\|\varphi\|_{\mathrm{Lip}}\le 1}\left|\mathbb{E}\int_{\mathbb{R}^d}\varphi\,d\overline{\mu}_t^N - \mathbb{E}\int_{\mathbb{R}^d}\varphi\,d\mu_t\right|\\ &= \sup_{t\in[0,T]}\sup_{\varphi,\,\|\varphi\|_{\mathrm{Lip}}\le 1}\left|\mathbb{E}\phi(\overline{\mu}_t^N) - \mathbb{E}\phi(\mu_t)\right|\\ &\le \frac{C_T}{N^\gamma},\end{aligned}$$

where $\gamma\in(0,1]\cap(0,(\delta+\alpha-1)\wedge(2\alpha-2)\wedge(\eta+\alpha-1))$ and since the constant C in Theorem 4 is non-decreasing with respect to T.

Let us compare our result with the existing literature.

Remark 5.
- Let us formally take $\alpha = 2$, which corresponds to the Brownian case treated in [7]. Then, we can take $\gamma = 1$ and $\beta = 2$ in Theorem 4. The rates of convergence proved in our theorem are precisely those proved in [7] (see Theorem 3.6), i.e. $N^{-\frac{1}{2}}$ for (2.35) and N^{-1} for (2.36).
- In dimension $d = 1$, we recover with (2.35) the same rate of convergence obtained in [12], for the strong propagation of chaos in L^1, since $\mathbb{E}W_1(\overline{\mu}_0^N, \mu_0) \leq CN^{\frac{1}{\beta}-1}$ by [10].
- In [3], the example of a nonlinear Ornstein-Uhlenbeck is treated using the same method. It corresponds to take
$$b(t, x, \mu) := x + \int_{\mathbb{R}^d} y \, d\mu(y).$$
However, there is preliminary step in the proof which consists in removing the jumps larger than the number of particles N of all the noises. This is due to the unboundedness of b with respect to both space and measures variables in this case. It is proved in [3] that there exists a positive constant C such that for any $\phi \in \mathcal{C}_1^{1,1}(\mathcal{P}_\beta(\mathbb{R}^d))$
$$\mathbb{E}\left|\phi(\overline{\mu}_T^N) - \phi(\mu_T)\right| \leq C\,\mathbb{E}W_1(\overline{\mu}_0^N, \mu_0) + C\frac{\ln(N)^{\frac{1}{\alpha}}}{N^{1-\frac{1}{\alpha}}}, \tag{2.38}$$
which is better than (2.35) in spite of our stronger assumption of boundedness on b. Indeed, by removing the large jumps in a first step, we can take $\beta = \alpha$ in (2.35), up to the logarithmic factor present in (2.38). This factor precisely comes from the fact that $\int_{1\leq|z|\leq N} |z|^\alpha \, d\nu(z) \underset{N\to+\infty}{\sim} \ln(N)$, which is the price to pay to take $\beta = \alpha$ in (2.35). The estimates (2.36) and (2.37) are better in our framework since the rate of the corresponding estimates in [3] is $N^{1-\alpha}$ and we can take $\gamma > \alpha - 1$ in Theorem 4. This is natural since the drift is unbounded in [3].

3. Well-posedness of the nonlinear martingale problem and Picard iterations

The section if dedicated to the proof of Theorem 1. The proof is based on the Banach fixed point theorem on a suitable complete metric space.

Introduction of the complete space and parametrix expansion. Let us consider the space $\mathcal{C}^0([s,T];\mathcal{P}(\mathbb{R}^d))$ which is complete under the uniform metric $d_{s,T}$ associated to the total variation metric d_{TV} defined, for $P, Q \in \mathcal{C}^0([s,T];\mathcal{P}(\mathbb{R}^d))$ by
$$d_{s,T}(P,Q) := \sup_{r\in[s,T]} d_{TV}(P_r, Q_r).$$
We introduce the space
$$\mathcal{A}_{s,T,\mu} := \left\{P \in \mathcal{C}^0([s,T];\mathcal{P}(\mathbb{R}^d)),\ P_s = \mu\right\}.$$
Note that it is a closed subspace of $(\mathcal{C}^0([s,T];\mathcal{P}(\mathbb{R}^d)), d_{s,T})$ and thus $(\mathcal{A}_{s,T,\mu}, d_{s,T})$ is complete. For any $P \in \mathcal{A}_{s,T,\mu}$, we consdier the following linear time-inhomogeneous SDE
$$\begin{cases} d\overline{X}_t^{s,\xi,P} = b(t, \overline{X}_t^{s,\xi,P}, P_t)\,dt + dZ_t, \quad t \in [s,T], \\ \overline{X}_s^{s,\xi,P} = \xi. \end{cases}$$
Notice that this SDE is well-posed in the weak sense since it is the case for the related linear martingale problem by [25]. Its flow of marginal distributions $([X_t^{s,\xi,P}])_{t\in[s,T]}$ belongs to $\mathcal{A}_{s,T,\mu}$. We can thus define a map $\mathcal{I} : \mathcal{A}_{s,T,\mu} \to \mathcal{A}_{s,T,\mu}$ such that for any $P \in \mathcal{A}_{s,T,\mu}$, $\mathcal{I}(P)_t = [X_t^{s,\xi,P}]$. We remark that a probability measure $\mathbb{P}$ on the Skorokhod space $\mathcal{D}([s,T];\mathbb{R}^d)$ solves the martingale problem related to the McKean-Vlasov SDE (2.2) if and only if its flow of marginal distributions $(\mathbb{P}_t)_{t\in[s,T]}$ is a fixed point of $\mathcal{I}$. Our goal is thus to prove that

for some $m \geq 1$, the m-th iterate $\mathcal{I}^{(m)}$ is a contraction on $(\mathcal{A}_{s,T,\mu}, d_{s,T})$. We fix $P^1, P^2 \in \mathcal{A}_{s,T,\mu}$ and we define recursively for all $m \geq 1$, $\overline{X}^{1,(m)}$ and $\overline{X}^{2,(m)}$ as the unique weak solutions to

$$\begin{cases} d\overline{X}_t^{i,(m)} = b(t, \overline{X}_t^{i,m}, [\overline{X}_t^{i,(m-1)}])) \, dt + dZ_t, \quad t \in [s,T], \quad i \in \{1,2\}, \\ \overline{X}_s^{i,(m)} = \xi, \end{cases} \tag{3.1}$$

with $([\overline{X}_t^{i,(0)}])_{t\in[s,T]} = P_i$. We also introduce the associated decoupling fields. Namely they are the weak solutions to

$$\begin{cases} d\overline{X}_t^{x,i,(m)} = b(t, \overline{X}_t^{x,i,m}, [\overline{X}_t^{i,(m-1)}])) \, dt + dZ_t, \quad t \in [s,T], \quad i \in \{1,2\}, \\ \overline{X}_s^{x,i,(m)} = x. \end{cases} \tag{3.2}$$

Thanks to Theorem 5, the distribution of $\overline{X}_t^{x,i,(m)}$ has a density with respect to the Lebesgue measure denoted by $p_{i,m}(\mu, s, t, x, \cdot)$. Remark that the notation makes sense since by weak well-posedness, the distribution of $\overline{X}_t^{x,i,(m)}$ depends on the initial condition ξ only through its distribution μ. Moreover, by weak well-posedness of SDE (3.1), $\overline{X}_t^{i,(m)}$ has a density $p_{i,m}(\mu, s, t, \cdot)$ which satisfies the following key relation

$$p_{i,m}(\mu, s, t, y) = \int_{\mathbb{R}^d} p_{i,m}(\mu, s, t, x, y) \, d\mu(x).$$

Let us give the implicit parametrix representation of $p_{i,m}(\mu, s, t, x, y)$. We define for all $0 \leq s \leq r < t \leq T$ and $x, y \in \mathbb{R}^d$

$$\begin{aligned} \widehat{p}(r, t, x, y) &:= q(t - r, y - x), \\ \mathcal{H}_{i,m}(\mu, r, t, x, y) &:= b(r, x, [\overline{X}_r^{i,(m-1)}]) \cdot \partial_x \widehat{p}(r, t, x, y). \end{aligned} \tag{3.3}$$

The space-time convolution between to functions f and g is given by

$$f \otimes g(\mu, r, t, x, y) := \int_r^t \int_{\mathbb{R}^d} f(\mu, r, r', x, z) g(\mu, r', t, z, y) \, dz \, dr', \tag{3.4}$$

when it is well-defined. The convolution iterates of $\mathcal{H}_{i,m}$ are defined recursively by $\mathcal{H}_{i,m}^{k+1} = \mathcal{H}_{i,m} \otimes \mathcal{H}_{i,m}^k$. By convention $\mathcal{H}_{i,m}^0 = \mathrm{Id}$. By Assumption (H1), we can apply Theorem 5 which ensures that

$$p_{i,m}(\mu, s, t, x, y) = \widehat{p}(s, t, x, y) + p_{i,m} \otimes \mathcal{H}_{i,m}(\mu, s, t, x, y). \tag{3.5}$$

Using Theorem 5 and Proposition 8, we deduce that there exists a positive constant C such that for all $i \in \{1,2\}$, $m \geq 1$, $k \geq 1$, $0 \leq s \leq r < t \leq T$ and $x, y \in \mathbb{R}^d$

$$\begin{cases} |p_{i,m}(\mu, s, t, x, y)| \leq C \rho^0(t - s, y - x), \\ \left| \mathcal{H}_{i,m}^k(\mu, r, t, x, y) \right| \leq C^k (t - r)^{-\frac{1}{\alpha} + (k-1)(1 - \frac{1}{\alpha})} \displaystyle\prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1 - \frac{1}{\alpha}\right), 1 - \frac{1}{\alpha} \right) \rho^1(t - r, y - x). \end{cases} \tag{3.6}$$

For any $0 \leq s \leq r < t \leq T$, $x, y \in \mathbb{R}^d$, we denote by

$$\begin{aligned} \Delta p_m(\mu, s, t, x, y) &:= p_{1,m}(\mu, s, t, x, y) - p_{2,m}(\mu, s, t, x, y), \\ \Delta \mathcal{H}_m(\mu, r, t, x, y) &:= \mathcal{H}_{1,m}(\mu, r, t, x, y) - \mathcal{H}_{2,m}(\mu, r, t, x, y). \end{aligned}$$

It follows from the definition of $\mathcal{H}_{i,m}$, the Lipschitz continuity of $b(s, x, \cdot)$ with respect to the total variation metric (and its expression as the L^1 norm of the difference of the densities) and (B.17) that for some constant $C > 0$, one has for all $m \geq 1$

$$\begin{aligned} |\Delta \mathcal{H}_{m+1}(\mu, r, t, x, y)| &\leq C d_{TV}([\overline{X}_r^{1,(m)}], [\overline{X}_r^{2,(m)}]) (t - r)^{-\frac{1}{\alpha}} \rho^1(t - r, y - x) \\ &\leq C \int_{\mathbb{R}^{2d}} \left| \Delta p_m(\mu, s, r, x', y') \right| d\mu(x') \, dy' \, (t - r)^{-\frac{1}{\alpha}} \rho^1(t - r, y - x). \end{aligned} \tag{3.7}$$

We similarly obtain for $m = 0$ that

$$|\Delta\mathcal{H}_1(\mu, r, t, x, y)| \leq C d_{s,r}(P^1, P^2)(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r, y-x). \tag{3.8}$$

Let us prove that for all $m \geq 1$, the following representation formula holds true

$$\Delta p_m(\mu, s, t, x, y) = \sum_{k=0}^{\infty} p_{2,m} \otimes \Delta\mathcal{H}_m \otimes \mathcal{H}_{1,m}^k(\mu, s, t, x, y), \tag{3.9}$$

the series being absolutely convergent. Starting from the implicit parametrix representation formula (3.5), we have

$$\Delta p_m(\mu, s, t, x, y) = \Delta p_m \otimes \mathcal{H}_{1,m}(\mu, s, t, x, y) + p_{2,m} \otimes \Delta\mathcal{H}_m(\mu, s, t, x, y).$$

Iterating this procedure, we easily prove by induction that for all $N \geq 1$

$$\Delta p_m(\mu, s, t, x, y) = \sum_{k=0}^{N} p_{2,m} \otimes \Delta\mathcal{H}_m \otimes \mathcal{H}_{1,m}^k(\mu, s, t, x, y) + \Delta p_m \otimes \mathcal{H}_{1,m}^{N+1}(\mu, s, t, x, y). \tag{3.10}$$

We want to take the limit $N \to +\infty$ in (3.10). By using (3.6) and the convolution inequality (B.16), we obtain that for some positive constant C

$$\begin{aligned}
&\left|\Delta p_m \otimes \mathcal{H}_{1,m}^{N+1}(\mu, s, t, x, y)\right| \\
&\leq \int_s^t \int_{\mathbb{R}^d} C\rho^0(r-s, z-x) C^{N+1}(t-r)^{-\frac{1}{\alpha}+N\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{N} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \rho^1(t-r, y-z)\, dz\, dr \\
&\leq C^{N+2}(t-s)^{(N+1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{N} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \rho^0(t-s, y-x).
\end{aligned}$$

The upper-bounded converges to 0 as N tends to infinity thanks to the asymptotic behavior of the Beta function as it is the general term of a convergent series. Following the same lines, we prove using again (3.6) that the series appearing in (3.10) is absolutely convergent. Letting N tends to infinity in (3.10) yields the representation formula (3.9).

We are now going to prove by induction that there exists a positive constant C such that for all $m \geq 1$, $t \in (s, T]$, $x, y \in \mathbb{R}^d$

$$|\Delta p_m(\mu, s, t, x, y)| \leq C^m (t-s)^{m\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{m-1} \mathcal{B}\left(1 + j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) d_{s,t}(P^1, P^2)\rho^0(t-s, y-x). \tag{3.11}$$

Base case $m = 1$. It follows from (3.6), (3.8) and the convolution inequality (B.16) that

$$\begin{aligned}
|p_{2,1} \otimes \Delta\mathcal{H}_1(\mu, s, t, x, y)| &\leq C \int_s^t \int_{\mathbb{R}^d} \rho^0(r-s, z-x)(t-r)^{-\frac{1}{\alpha}} d_{s,r}(P^1, P^2)\rho^1(t-r, y-z)\, dz\, dr \\
&\leq C(t-s)^{1-\frac{1}{\alpha}} d_{s,t}(P^1, P^2)\rho^0(t-s, y-x),
\end{aligned} \tag{3.12}$$

since $d_{s,r}(P^1, P^2) \leq d_{s,t}(P^1, P^2)$ for $r \in [s, t]$. Following the same lines and using the bound (3.6) on $\mathcal{H}_{1,1}^k$, we show that for some constant $C > 0$, one has for all $k \geq 1$

$$\begin{aligned}
&\left|p_{2,1} \otimes \Delta\mathcal{H}_1 \otimes \mathcal{H}_{1,1}^k(\mu, s, t, x, y)\right| \\
&\leq KC(t-s)^{1-\frac{1}{\alpha}} d_{s,t}(P^1, P^2)\rho^0(t-s, y-x) C^k (t-s)^{k\left(1-\frac{1}{\alpha}\right)} \prod_{l=1}^{k-1} \mathcal{B}\left(l\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right).
\end{aligned}$$

By summing over $k \geq 1$, we deduce using the asymptotic behavior of the Beta function that for some constant $C > 0$, we have

$$\sum_{k=1}^{\infty} \left| p_{2,1} \otimes \Delta\mathcal{H}_1 \otimes \mathcal{H}_{1,1}^k(\mu, s, t, x, y) \right| \leq C(t-s)^{1-\frac{1}{\alpha}} d_{s,t}(P^1, P^2) \rho^0(t-s, y-x). \tag{3.13}$$

Plugging (3.12) and (3.13) into the representation formula (3.9) for $m = 1$ concludes the proof of the base case.

Induction step. We assume that (3.11) holds true at step m for a certain constant C that will be chosen at the end of the induction step to ensure that (3.11) is verified at step $m+1$. We denote by K any constant independent of m and C appearing in the induction step. By using (3.7), one has

$$|\Delta\mathcal{H}_{m+1}(\mu, r, t, x, y)| \leq K(t-r)^{-\frac{1}{\alpha}} \rho^1(t-r, y-x) \\ C^m (r-s)^{m\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{m-1} \mathcal{B}\left(1 + j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) d_{s,r}(P^1, P^2).$$

This inequality together with (3.6) yield

$$\begin{aligned} &|p_{2,m+1} \otimes \Delta\mathcal{H}_{m+1}(\mu, s, t, x, y)| \\ &\leq K \int_s^t \int_{\mathbb{R}^d} \rho^0(r-s, z-x)(t-r)^{-\frac{1}{\alpha}} \rho^1(t-r, y-z) C^m (r-s)^{m\left(1-\frac{1}{\alpha}\right)} \\ &\qquad \prod_{j=1}^{m-1} \mathcal{B}\left(1 + j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) d_{s,r}(P^1, P^2)\, dz\, dr \\ &\leq K C^m (t-s)^{(m+1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{m} \mathcal{B}\left(1 + j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) d_{s,t}(P^1, P^2) \rho^0(t-s, y-x). \end{aligned} \tag{3.14}$$

Following the same lines and using the bound (3.6) on $\mathcal{H}_{1,m+1}^k$, we show that for some constant $K > 0$, one has for all $k \geq 1$

$$\begin{aligned} &\left| p_{2,m+1} \otimes \Delta\mathcal{H}_{m+1} \otimes \mathcal{H}_{1,m+1}^k(\mu, s, t, x, y) \right| \\ &\leq K C^m (t-s)^{(m+1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{m} \mathcal{B}\left(1 + j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) d_{s,t}(P^1, P^2) \rho^0(t-s, y-x) \\ &\qquad K^k (t-s)^{k\left(1-\frac{1}{\alpha}\right)} \prod_{l=1}^{k-1} \mathcal{B}\left(l\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right). \end{aligned}$$

By summing over $k \geq 1$, we deduce thanks to the asymptotic behavior of the Beta function that for some constant $K > 0$, we have

$$\sum_{k=1}^{\infty} \left| p_{2,m+1} \otimes \Delta\mathcal{H}_{m+1} \otimes \mathcal{H}_{1,m+1}^k(\mu, s, t, x, y) \right| \\ \leq K C^m (t-s)^{(m+1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{m} \mathcal{B}\left(1 + j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) d_{s,t}(P^1, P^2) \rho^0(t-s, y-x). \tag{3.15}$$

Finally, plugging (3.14) and (3.15) into the representation formula (3.9), we obtain

$$|\Delta p_{m+1}(\mu,s,t,x,y)| \leq KC^m(t-s)^{(m+1)(1-\frac{1}{\alpha})}\prod_{j=1}^{m}\mathcal{B}\left(1+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)d_{s,t}(P^1,P^2)\rho^0(t-s,y-x).$$

This ends the proof of the induction step provided that we choose $C \geq K$ in (3.11), which is possible since K does not depend on m.

Conclusion of the proof of Theorem 1. Using the asymptotic behavior of the Beta function, we get that for m large enough, we have for all $t \in [s,T]$, $x,y \in \mathbb{R}^d$

$$|\Delta p_m(\mu,s,t,x,y)| \leq \varepsilon d_{s,t}(P^1,P^2)\rho^0(t-s,y-x),$$

where $\varepsilon > 0$ is such that

$$\varepsilon\int_{\mathbb{R}^d}\rho^0(t-s,y)\,dy = \varepsilon\int_{\mathbb{R}^d}\rho^0(1,y)\,dy = \frac{1}{2}.$$

Finally, we have

$$\begin{aligned} d_{s,T}(\mathcal{I}^{(m)}(P^1),\mathcal{I}^{(m)}(P^2)) &= \sup_{t\in(s,T]}\sup_{h,\,\|h\|_\infty\leq 1}\left|\int_{\mathbb{R}^d}h(y)(p_{1,m}(\mu,s,t,y)-p_{2,m}(\mu,s,t,y))\,dy\right| \\ &\leq \sup_{t\in(s,T]}\int_{\mathbb{R}^{2d}}|\Delta p_m(\mu,s,t,x,y)|\,d\mu(x)\,dy \\ &\leq \frac{1}{2}d_{s,T}(P^1,P^2). \end{aligned}$$

The Banach fixed point theorem ensures that $\mathcal{I}$ has a unique fixed point in $\mathcal{A}_{s,T,\mu}$. Thus, the martingale problem associated to the McKean-Vlasov SDE (2.2) is well-posed. Moreover, we know that for any initial data $P \in \mathcal{A}_{s,T,\mu}$, the sequence $(\mathcal{I}^{(m)}(P))_{m\geq 1}$ converges towards the solution to the martingale problem with respect to the metric $d_{s,T}$. This proves (2.6).

4. Properties of the transition density

This section is devoted to prove Theorem 2. Let us introduce the parametrix expansions of the densities that is at the core of the method. We denote by $q(t,\cdot)$ the density of Z_t. We define for all $0 \leq s \leq r < t \leq T$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and $x,y \in \mathbb{R}^d$

$$\begin{aligned} &\widehat{p}(s,r,t,x,y) := q(t-r,y-x), \\ &\mathcal{H}(\mu,s,r,t,x,y) := b(r,x,[X_r^{s,\mu}])\cdot\partial_x\widehat{p}(s,r,t,x,y). \end{aligned} \tag{4.1}$$

Note that the proxy $\widehat{p}(s,r,t,x,\cdot)$ does not depend on μ and s and is the density at time $t > r$ of the solution to

$$\begin{cases} d\widehat{X}_t^{r,x} = dZ_t, \\ \widehat{X}_r^{r,x} = x \in \mathbb{R}^d, \end{cases} \tag{4.2}$$

and $\mathcal{H}$ is the associated parametrix kernel. We also define the space-time convolution operator between to functions f and g by

$$f\otimes g(\mu,s,r,t,x,y) := \int_r^t\int_{\mathbb{R}^d} f(\mu,s,r,r',x,z)g(\mu,s,r',t,z,y)\,dz\,dr', \tag{4.3}$$

when it is well-defined. The space-time convolution iterates $\mathcal{H}^k$ of $\mathcal{H}$ are defined recursively by $\mathcal{H}^1 := \mathcal{H}$ and $\mathcal{H}^{k+1} := \mathcal{H}\otimes\mathcal{H}^k$. By convention $f\otimes\mathcal{H}^0$ is equal to f. In order to simplify a bit the notations, we will write $f\otimes g(\mu,s,t,x,y) := f\otimes g(\mu,s,s,t,x,y)$, $\mathcal{H}(\mu,s,t,x,y) := \mathcal{H}(\mu,s,s,t,x,y)$ and the same for other maps. Finally, we denote by Φ the solution to the following Volterra integral equation

$$\Phi(\mu,s,r,t,x,y) = \mathcal{H}(\mu,s,r,t,x,y) + \mathcal{H}\otimes\Phi(\mu,s,r,t,x,y),$$

which is given by the uniform convergent series

$$\Phi(\mu,s,r,t,x,y)=\sum_{k=1}^{\infty}\mathcal{H}^k(\mu,s,r,t,x,y). \tag{4.4}$$

Step 1: Properties of the Picard approximation of the transition density associated to (2.9). In order to study the regularity with respect to μ of $p(\mu,s,t,x,y)$, we consider an approximation sequence based on Picard iteration. We fix a measure $\nu\in\mathcal{P}_\beta(\mathbb{R}^d)$ and $s\in[0,T)$ and we start by considering the following stable-driven McKean-Vlasov SDE

$$\begin{cases} dX_t^{s,\xi,(1)}=b(t,X_t^{s,\xi,(1)},\nu)\,dt+dZ_t, \quad t\in[s,T],\\ X_s^{s,\xi,(1)}=\xi, \quad [\xi]=\mu\in\mathcal{P}_\beta(\mathbb{R}^d). \end{cases} \tag{4.5}$$

The associated martingale problem is well-posed and there is weak existence and uniqueness for SDE (4.5). As previously, the distribution of $X_t^{s,\xi,(1)}$ is denoted by $[X_t^{s,\mu,(1)}]$. We also introduce, for $x\in\mathbb{R}^d$, the following decoupled stochastic flow associated to SDE (4.5)

$$\begin{cases} dX_t^{s,x,\mu,(1)}=b(t,X_t^{s,x,\mu,(1)},[X_t^{s,\mu,(1)}])\,dt+dZ_t, \quad t\in[s,T],\\ X_s^{s,x,\mu,(1)}=x\in\mathbb{R}^d. \end{cases} \tag{4.6}$$

Then, for all $m\geq 1$, we define recursively

$$\begin{cases} dX_t^{s,\xi,(m+1)}=b(t,X_t^{s,\xi,(m+1)},[X_t^{s,\mu,(m)}])\,dt+dZ_t, \quad t\in[s,T],\\ X_s^{s,\xi,(m+1)}=\xi, \quad [\xi]=\mu\in\mathcal{P}_\beta(\mathbb{R}^d). \end{cases} \tag{4.7}$$

and

$$\begin{cases} dX_t^{s,x,\mu,(m+1)}=b(t,X_t^{s,x,\mu,(m+1)},[X_t^{s,\mu,(m)}])\,dt+dZ_t, \quad t\in[s,T],\\ X_s^{s,x,\mu,(m+1)}=x\in\mathbb{R}^d. \end{cases} \tag{4.8}$$

Note that these are note McKean-Vlasov SDEs. The densities of $[X_t^{s,\mu,(m)}]$ and $[X_t^{s,x,\mu,(m)}]$ are denoted by $p_m(\mu,s,t,\cdot)$ and $p_m(\mu,s,t,x,\cdot)$ and satisfy

$$p_m(\mu,s,t,y)=\int_{\mathbb{R}^d}p_m(\mu,s,t,x,y)\,d\mu(x). \tag{4.9}$$

We define for all $0\leq s<t\leq T$, $r\in[s,t)$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$ and $x,y\in\mathbb{R}^d$

$$\begin{aligned} &\widehat{p}_m(s,r,t,x,y):=\widehat{p}(r,t,x,y)=q(t-r,y-x),\\ &\mathcal{H}_m(\mu,s,r,t,x,y):=b(r,x,[X_r^{s,\mu,(m-1)}])\cdot\partial_x\widehat{p}(s,r,t,x,y). \end{aligned} \tag{4.10}$$

Note that the proxy $\widehat{p}_m(s,r,t,x,\cdot)$ does not depend on m, μ and s. We denote by Φ_m the solution to the following Volterra integral equation

$$\Phi_m(\mu,s,r,t,x,y)=\mathcal{H}_m(\mu,s,r,t,x,y)+\mathcal{H}_m\otimes\Phi_m(\mu,s,r,t,x,y), \tag{4.11}$$

which is given by the uniform convergent series

$$\Phi_m(\mu,s,r,t,x,y)=\sum_{k=1}^{\infty}\mathcal{H}_m^k(\mu,s,r,t,x,y). \tag{4.12}$$

Let us recall that the Beta function $\mathcal{B}$ is defined, for all $x,y>0$ by

$$\mathcal{B}(x,y):=\int_0^1(1-t)^{-1+x}t^{-1+y}\,dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where Γ is the Gamma function.

Applying Proposition 8 and Theorem 5, since all the controls are uniform with respect to the measure argument and thus on $m \geq 1$ too, we deduce the following two propositions.

Proposition 1.

- There exists $C > 0$ such that for all $k \geq 1$, $m \geq 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \leq s \leq r < t \leq T$ and $x, y \in \mathbb{R}^d$

$$|\mathcal{H}_m^k(\mu, s, r, t, x, y)| \leq C^k (t-r)^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \rho^1(t-r, y-x). \tag{4.13}$$

- For $\gamma \in (0, \eta]$ such that $\gamma < \alpha - 1$, there exists $C > 0$ depending on γ such that for all $k \geq 1$, $m \geq 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \leq s \leq r < t \leq T$ and $x_1, x_2, y \in \mathbb{R}^d$

$$|\mathcal{H}_m^k(\mu, s, r, t, x_1, y) - \mathcal{H}_m^k(\mu, s, r, t, x_2, y)| \leq C^k (t-r)^{-\frac{\gamma+1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)} |x_1 - x_2|^\gamma \prod_{j=1}^{k-1} \mathcal{B}\left(-\frac{\gamma}{\alpha} + j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \left[\rho^1(t-r, y-x_1) + \rho^1(t-r, y-x_2)\right]. \tag{4.14}$$

- The series (4.12) defining Φ_m is absolutely convergent and there exists $C > 0$ such that for all $m \geq 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \leq s \leq r < t \leq T$ and $x, y \in \mathbb{R}^d$

$$|\Phi_m(\mu, s, r, t, x, y)| \leq C(t-r)^{-\frac{1}{\alpha}} \rho^1(t-r, y-x). \tag{4.15}$$

- For $\gamma \in (0, \eta]$ such that $\gamma < \alpha - 1$, there exists $C > 0$ depending on γ such that for all $m \geq 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \leq s \leq r < t \leq T$ and $x_1, x_2, y \in \mathbb{R}^d$

$$|\Phi_m(\mu, s, r, t, x_1, y) - \Phi_m(\mu, s, r, t, x_2, y)| \leq C(t-r)^{-\frac{\gamma+1}{\alpha}} |x_1 - x_2|^\gamma \left[\rho^1(t-r, y-x_1) + \rho^1(t-r, y-x_2)\right]. \tag{4.16}$$

Proposition 2. For any $m \geq 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \leq s < t \leq T$ and $x \in \mathbb{R}^d$, the distribution of $X_t^{s,x,\mu,(m)}$ has a density with respect to the Lebesgue measure denoted by $p_m(\mu, s, t, x, \cdot)$ and given by the absolutely convergent parametrix series

$$\begin{aligned} p_m(\mu, s, t, x, y) &= \widehat{p}(s, t, x, y) + \sum_{k=1}^{\infty} \widehat{p} \otimes \mathcal{H}_m^k(\mu, s, t, x, y) \\ &= \widehat{p}(s, t, x, y) + \widehat{p} \otimes \Phi_m(s, t, x, y). \end{aligned} \tag{4.17}$$

For any $m \geq 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \leq s < t \leq T$ and $y \in \mathbb{R}^d$, $p_m(\mu, s, t, \cdot, y)$ is of class $\mathcal{C}^1$ on $\mathbb{R}^d$ and $p_m(\mu, s, t, \cdot, y)$ and $\partial_x p_m(\mu, s, t, \cdot, y)$ are continuous on $\mathbb{R}^d$. Moreover, the following properties hold true.

- There exists $C > 0$ such that for all $j \in \{0, 1\}$, $m \geq 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \leq s < t \leq T$ and $x, y \in \mathbb{R}^d$

$$|\partial_x^j p_m(\mu, s, t, x, y)| \leq C(t-s)^{-\frac{j}{\alpha}} \rho^j(t-s, y-x). \tag{4.18}$$

- For all $j \in \{0, 1\}$ and $\gamma \in (0, 1]$ with $\gamma \in (0, (2\alpha - 2) \wedge (\eta + \alpha - 1))$ if $j = 1$, there exists $C > 0$ such that for all $m \geq 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \leq s < t \leq T$ and $x_1, x_2, y \in \mathbb{R}^d$

$$|\partial_x^j p_m(\mu, s, t, x_1, y) - \partial_x^j p_m(\mu, s, t, x_2, y)| \leq C(t-s)^{-\frac{j+\gamma}{\alpha}} |x_1 - x_2|^\gamma \left[\rho^j(t-s, y-x_1) + \rho^j(t-s, y-x_2)\right]. \tag{4.19}$$

We state in the following proposition all the properties satisfied by the transition densities p_m which are used to prove Theorem 2. As the proof is rather long and technical, it is postponed to Section 7.

Proposition 3. For any $m \geq 1$, $t \in (0, T]$, $y \in \mathbb{R}^d$, the map $(\mu, s, x) \in \mathcal{P}_\beta(\mathbb{R}^d) \times [0, t) \times \mathbb{R}^d \mapsto p_m(\cdot, \cdot, t, \cdot, y)$ belongs to $\mathcal{C}^1(\mathcal{P}_\beta(\mathbb{R}^d) \times [0, t) \times \mathbb{R}^d)$ and satisfies the following properties.

- There exists $C>0$ such that for all $m\geq 1$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\leq s<t\leq T$, $x,y,v\in\mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v)\right|\leq (t-s)^{1-\frac{1}{\alpha}}\rho^0(t-s,y-x)\left(\sum_{k=1}^m C^k(t-s)^{(k-1)\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right). \quad (4.20)$$

- There exists $C>0$ such that for all $m\geq 1$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\leq s<t\leq T$, $x,y,v\in\mathbb{R}^d$

$$\left|\partial_v\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v)\right|\leq (t-s)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}\rho^0(t-s,y-x)\left(\sum_{k=1}^m C^k(t-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+\frac{\eta-1}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right). \quad (4.21)$$

- For all $\tilde{\eta}\in(0,\eta\wedge(\alpha-1))$ there exists a constant $C>0$ such that for all $m\geq 1$, $0\leq s<t\leq T$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $x,y\in\mathbb{R}^d$

$$|\partial_s p_m(\mu,s,t,x,y)|\leq (t-s)^{-1}\rho^{-\tilde{\eta}}(t-s,y-x)\sum_{k=1}^m C^k(t-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(\frac{\eta}{\alpha}+(j-1)\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right). \quad (4.22)$$

- For all $\gamma\in(0,1]\cap(0,(2\alpha-2)\wedge(\eta+\alpha-1))$, there exists $C>0$ such that for all $m\geq 1$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\leq s<t\leq T$, $x,y,v_1,v_2\in\mathbb{R}^d$

$$\left|\partial_v\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v_1)-\partial_v\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v_2)\right|\leq (t-s)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}}|v_1-v_2|^\gamma\rho^0(t-s,y-x)\left(\sum_{k=1}^m C^k(t-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+\frac{\eta-1-\gamma}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right). \quad (4.23)$$

- There exists $C>0$ such that for all $\gamma\in(0,1]$, $m\geq 1$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\leq s<t\leq T$, $x,y,v_1,v_2\in\mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v_1)-\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v_2)\right|\leq C(t-s)^{1-\frac{1+\gamma}{\alpha}}|v_1-v_2|^\gamma\rho^0(t-s,y-x). \quad (4.24)$$

- There exists $C>0$ such that for all $\gamma\in(0,1]$, $m\geq 1$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\leq s<t\leq T$, $x_1,x_2,y,v\in\mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}p_m(\mu,s,t,x_1,y)(v)-\frac{\delta}{\delta m}p_m(\mu,s,t,x_2,y)(v)\right|\leq C(t-s)^{1-\frac{1+\gamma}{\alpha}}|x_1-x_2|^\gamma\left[\rho^0(t-s,y-x_1)+\rho^0(t-s,y-x_2)\right]. \quad (4.25)$$

- For all $\gamma\in(0,1]\cap(0,\eta+\alpha-1)$, there exists $C>0$ such that for all $m\geq 1$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\leq s<t\leq T$, $x_1,x_2,y,v\in\mathbb{R}^d$

$$\left|\partial_v\frac{\delta}{\delta m}p_m(\mu,s,t,x_1,y)(v)-\partial_v\frac{\delta}{\delta m}p_m(\mu,s,t,x_2,y)(v)\right|\leq C(t-s)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}}|x_1-x_2|^\gamma\left[\rho^0(t-s,y-x_1)+\rho^0(t-s,y-x_2)\right]. \quad (4.26)$$

- There exists $C>0$ such that for all $\gamma\in(0,1]$, $m\geq 1$, $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\leq s<t\leq T$, $x,y\in\mathbb{R}^d$

$$|p_m(\mu_1,s,t,x,y)-p_m(\mu_2,s,t,x,y)|\leq C(t-s)^{1-\frac{1+\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x). \quad (4.27)$$

- For all $j \in \{0,1\}$ and $\gamma \in (0,1]$ with $\gamma \in (0, \alpha - 1 + \eta)$ if $j = 1$, there exists $C > 0$ such that for all $m \geq 1$, $0 \leq s < t \leq T$, $\mu_1, \mu_2 \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$

$$\left|\partial_x^j p_m(\mu_1, s, t, x, y) - \partial_x^j p_m(\mu_2, s, t, x, y)\right| \leq C(t-s)^{1-\frac{1+\gamma+j}{\alpha}} W_1^\gamma(\mu_1, \mu_2)\rho^j(t-s, y-x). \tag{4.28}$$

- For all $\gamma \in (0,1]$, there exists $C > 0$ such that for all $m \geq 1$, $0 \leq s < t \leq T$, $\mu_1, \mu_2 \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y, v \in \mathbb{R}^d$

$$\left|\frac{\delta}{\delta m} p_m(\mu_1, s, t, x, y)(v) - \frac{\delta}{\delta m} p_m(\mu_2, s, t, x, y)(v)\right| \leq (t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}} W_1^\gamma(\mu_1, \mu_2)\rho^0(t-s, y-x)$$
$$\sum_{k=1}^{m} C^k (t-s)^{(k-1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(1 - \frac{\gamma}{\alpha} + j\left(1 - \frac{1}{\alpha}\right), 1 - \frac{1}{\alpha}\right). \tag{4.29}$$

- For all $\gamma \in (0,1]$, there exists $C > 0$ such that for all $m \geq 1$, $0 \leq s < t \leq T$, $\mu_1, \mu_2 \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y, v \in \mathbb{R}^d$

$$\left|\partial_v \frac{\delta}{\delta m} p_m(\mu_1, s, t, x, y)(v) - \partial_v \frac{\delta}{\delta m} p_m(\mu_2, s, t, x, y)(v)\right| \leq (t-s)^{\frac{\eta-\gamma-1}{\alpha}+1-\frac{1}{\alpha}} W_1^\gamma(\mu_1, \mu_2)\rho^0(t-s, y-x)$$
$$\sum_{k=1}^{m} C^k (t-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(1 + \frac{\eta-\gamma}{\alpha} + j\left(1 + \frac{\eta-1}{\alpha}\right), 1 - \frac{1}{\alpha}\right). \tag{4.30}$$

- For all $j \in \{0,1\}$, $\gamma \in \left(0, 1 - \frac{j}{\alpha}\right)$, there exists a constant $C > 0$ such that for all $m \geq 1$, $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$

$$|\partial_x^j p_m(\mu, s_1, t, x, y) - \partial_x^j p_m(\mu, s_2, t, x, y)|$$
$$\leq C\left[\frac{|s_1 - s_2|^\gamma}{(t-s_1)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1, y-x) + \frac{|s_1-s_2|^\gamma}{(t-s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_2, y-x)\right]. \tag{4.31}$$

- For all $\gamma \in (0,1)$, there exists a constant $C > 0$ such that for all $m \geq 1$, $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y, v \in \mathbb{R}^d$

$$\left|\frac{\delta}{\delta m} p_m(\mu, s_1, t, x, y)(v) - \frac{\delta}{\delta m} p_m(\mu, s_2, t, x, y)(v)\right|$$
$$\leq \left[\frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma+\frac{1}{\alpha}-1}}\rho^0(t - s_1 \vee s_2, y-x) + \frac{|s_1-s_2|^\gamma}{(t-s_1 \wedge s_2)^{\gamma+\frac{1}{\alpha}-1}}\rho^0(t - s_1 \wedge s_2, y-x)\right] \tag{4.32}$$
$$\sum_{k=1}^{m} C^k (t - s_1 \vee s_2)^{(k-1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(\left[2 - \gamma - \frac{1}{\alpha}\right] \wedge 1 + (j-1)\left(1 - \frac{1}{\alpha}\right), 1 - \frac{1}{\alpha}\right).$$

- For all $\gamma \in \left(0, 1 + \frac{\eta-1}{\alpha}\right)$, there exists a constant $C > 0$ such that for all $m \geq 1$, $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y, v \in \mathbb{R}^d$

$$\left|\partial_v \frac{\delta}{\delta m} p_m(\mu, s_1, t, x, y)(v) - \partial_v \frac{\delta}{\delta m} p_m(\mu, s_2, t, x, y)(v)\right|$$
$$\leq \left[\frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1}{\alpha} - 1 + \frac{1-\eta}{\alpha}}} \rho^0(t - s_1 \vee s_2, y - x) + \frac{|s_1 - s_2|^\gamma}{(t - s_1 \wedge s_2)^{\gamma + \frac{1}{\alpha} - 1 + \frac{1-\eta}{\alpha}}} \rho^0(t - s_1 \wedge s_2, y - x)\right] \tag{4.33}$$
$$\sum_{k=1}^{m} C^k (t - s_1 \vee s_2)^{(k-1)\left(1 + \frac{\eta - 1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(\left[2\left(1 + \frac{\eta - 1}{\alpha}\right) - \gamma\right] \wedge 1 + (j-1)\left(1 + \frac{\eta - 1}{\alpha}\right), 1 - \frac{1}{\alpha}\right).$$

Remark 6. Note that by the asymptotic behavior of the Beta function, we get that all the series appearing in the right-hand side members of the preceding inequalities are convergent.

Step 2: Passage to the limit in the previous estimates. We are going to take the limit $m \to +\infty$ in all the estimates proved in Proposition 3 to deduce that the transition density $p(\mu, s, t, x, y)$ of the McKean-Vlasov SDE (2.7) satisfies the regularity properties and estimates of Theorem 2. This will be done along a converging subsequence given by Arzelà-Ascoli theorem. Notice that all the partial sums of the series appearing in the all the upper-bounds of Proposition 3 have a limit when $m \to +\infty$ using the asymptotic behavior of the Beta function.

First of all, note that Theorem 1 yields

$$\sup_{r \in [s,t]} d_{TV}([X_r^{s,\mu,(m)}], [X_r^{s,\mu}]) \underset{m \to +\infty}{\longrightarrow} 0.$$

It follows that for all $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \leq s < t \leq T$, $x, y \in \mathbb{R}^d$

$$\mathcal{H}_m(\mu, s, t, x, y) \underset{m \to +\infty}{\longrightarrow} \mathcal{H}(\mu, s, t, x, y),$$

where $\mathcal{H}(\mu, s, t, x, y)$ was defined in (4.1). We can thus let m tend to infinity in the parametrix series (4.17) which yields the following pointwise convergence

$$p_m(\mu, s, t, x, y) \underset{m \to +\infty}{\longrightarrow} p(\mu, s, t, x, y) \tag{4.34}$$

thanks to the parametrix expansion (B.8) of p. Let us fix $(t, y) \in (0, T] \times \mathbb{R}^d$ and $\mathcal{K}$ a compact subset of $\mathcal{P}_\beta(\mathbb{R}^d) \times [0, t) \times \mathbb{R}^d$. Using (4.19), (4.28) and (4.31), we deduce that the sequence of function $(p_m(\cdot, \cdot, t, \cdot, y))_m \in \mathcal{C}^0(\mathcal{K})^{\mathbb{N}}$ is uniformly equi-continuous on $\mathcal{K}$. It is also uniformly bounded by (4.18). The Arzelà-Ascoli theorem ensures that we can extract a subsequence of $(p_m(\cdot, \cdot, t, \cdot, y))_m$ which converges uniformly on $\mathcal{K}$ necessarily towards $p(\cdot, \cdot, t, \cdot, y)$ by (4.34). This yields the continuity of $p(\cdot, \cdot, t, \cdot, y)$ on $\mathcal{K}$ and thus, since $\mathcal{K}$ is arbitrary, on $\mathcal{P}_\beta(\mathbb{R}^d) \times [0, t) \times \mathbb{R}^d$. Moreover, passing to the limit in (4.18), (4.19), (4.28) and (4.31) for $j = 1$, along the converging subsequence previously obtained of $(p_m(\cdot, \cdot, t, \cdot, y))_m$, we obtain that (2.13), (2.14), (2.22) and (2.25) hold true for $j = 0$. We now prove that $p(\mu, s, t, \cdot, y)$ is of class $\mathcal{C}^1$ on $\mathbb{R}^d$ for any $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and $s \in [0, t)$. To do this, we fix $R > 0$. By (4.18) and (4.19), we can apply the Arzelà-Ascoli theorem to the sequence $(\partial_x p_m(\mu, s, t, \cdot, y))_m \in \mathcal{C}^0(B_R)^{\mathbb{N}}$, where B_R denotes the open ball of $\mathbb{R}^d$ with radius R. Since R is arbitrary, we can construct, using a diagonal extraction procedure, a continuous function on $\mathbb{R}^d$ which is the limit, uniformly on each compact subset of $\mathbb{R}^d$, of a subsequence of $(\partial_x p_m(\mu, s, t, \cdot, y))_m$. This proves that $p_m(\mu, s, t, \cdot, y)$ is of class $\mathcal{C}^1$. By (4.18), (4.19), (4.28) and (4.31), the continuity of $\partial_x p(\cdot, \cdot, t, \cdot, y)$ on $\mathcal{P}_\beta(\mathbb{R}^d) \times [0, t) \times \mathbb{R}^d$ is again a consequence of the Arzelà-Ascoli theorem applied to $(\partial_x p_m(\cdot, \cdot, t, \cdot, y))_m \in (\mathcal{C}^0(\mathcal{K}))^{\mathbb{N}}$ where $\mathcal{K}$ is an arbitrary compact subset of $\mathcal{P}_\beta(\mathbb{R}^d) \times [0, t) \times \mathbb{R}^d$. Taking the limit $m \to +\infty$ along a converging subsequence in (4.18), (4.19), (4.28) and (4.31), we get that (2.13), (2.14), (2.22) and (2.25) are satisfied.

Let us now focus on the existence of the linear derivative of p. We fix $\tau \leq t$ and $\mathcal{K}$ a closed and bounded subset of $\mathcal{P}_\beta(\mathbb{R}^d) \times [0, \tau] \times (\mathbb{R}^d)^2$. Note that since $\beta > 1$, $\mathcal{K}$ is relatively compact in $\mathcal{P}_1(\mathbb{R}^d) \times [0, t) \times (\mathbb{R}^d)^2$ for

the metric d defined for all $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $s_1,s_2\in[0,\tau]$, $x_1,x_2,v_1,v_2\in\mathbb{R}^d$ by

$$d((\mu_1,s_1,x_1,v_1),(\mu_2,s_2,x_2,v_2)):=W_1(\mu_1,\mu_2)+|s_1-s_2|+|x_1-x_2|+|v_1-v_2|.$$

Using (4.24), (4.25), (4.29), (4.32), we deduce that the sequence of functions $\left(\frac{\delta}{\delta m}p_m(\cdot,\cdot,t,\cdot,y)(\cdot)\right)_m$ is uniformly equi-continuous on $\mathcal{K}$ with respect to the metric d. Moreover (4.20) ensures that

$$\sup_{m\geq 1}\sup_{(\mu,s,x,v)\in\mathcal{P}_\beta(\mathbb{R}^d)\times[0,\tau]\times(\mathbb{R}^d)^2}\left|\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v)\right|<+\infty. \tag{4.35}$$

Then, we apply the Arzelà-Ascoli theorem which gives the existence of a subsequence of $\left(\frac{\delta}{\delta m}p_m(\cdot,\cdot,t,\cdot,y)(\cdot)\right)_m$ which converges uniformly on $\mathcal{K}$ with respect to d. Since this is true for all $\tau<t$ and for every $\mathcal{K}$ bounded subset of $\mathcal{P}_\beta(\mathbb{R}^d)\times[0,\tau]\times(\mathbb{R}^d)^2$, we can use a diagonal extraction procedure. This yields the existence of function g continuous with respect to d on $\mathcal{P}_\beta(\mathbb{R}^d)\times[0,t)\times(\mathbb{R}^d)^2$ such that, up to an extraction, $\left(\frac{\delta}{\delta m}p_m(\cdot,\cdot,t,\cdot,y)(\cdot)\right)_m$ converges towards g uniformly on each compact subset of $\mathcal{P}_\beta(\mathbb{R}^d)\times[0,t)\times(\mathbb{R}^d)^2$. Note that g is also continuous with respect to the usual metric on $\mathcal{P}_\beta(\mathbb{R}^d)\times[0,t)\times(\mathbb{R}^d)^2$ and that (4.35) implies that for each $\tau\in[0,t)$, we have

$$\sup_{(\mu,s,x,v)\in\mathcal{P}_\beta(\mathbb{R}^d)\times[0,\tau]\times(\mathbb{R}^d)^2}|g(\mu,s,x,v)|<+\infty.$$

It is therefore a good candidate to be a linear derivative. We now prove that $p(\cdot,s,t,x,y)$ admits a linear derivative given, for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $s\in[0,t)$, $x,v\in\mathbb{R}^d$ by

$$\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v)=g(\mu,s,x,v), \tag{4.36}$$

which is continuous on $\mathcal{P}_\beta(\mathbb{R}^d)\times[0,t)\times(\mathbb{R}^d)^2$. For all $\mu,\nu\in\mathcal{P}_\beta(\mathbb{R}^d)$, one has

$$p_m(\mu,s,t,x,y)-p_m(\nu,s,t,x,y)=\int_0^1\int_{\mathbb{R}^d}\frac{\delta}{\delta m}p_m(\lambda\mu+(1-\lambda)\nu,s,t,x,y)(v)\,d(\mu-\nu)(v)\,d\lambda.$$

We take the limit $m\to+\infty$ along the subsequence of $\left(\frac{\delta}{\delta m}p_m(\cdot,\cdot,t,\cdot,y)(\cdot)\right)_m$ converging towards g that we have obtained above. By the dominated convergence theorem justified by (4.35) and since $(p_m)_m$ converges pointwise towards p, we obtain that

$$p(\mu,s,t,x,y)-p(\nu,s,t,x,y)=\int_0^1\int_{\mathbb{R}^d}g(\lambda\mu+(1-\lambda)\nu,s,x,v)\,d(\mu-\nu)(v)\,d\lambda.$$

This proves (4.36). Moreover, taking the limit $m\to+\infty$ in (4.20), (4.24), (4.25), (4.29) and (4.32) along the converging subsequence yield (2.15), (2.19), (2.20), (2.23) and (2.26). Using again the Arzelà-Ascoli theorem, we prove that for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $s\in[0,t)$, $x,y\in\mathbb{R}^d$, the map $\frac{\delta}{\delta m}p(\mu,s,t,x,y)(\cdot)$ is of class $\mathcal{C}^1$ on $\mathbb{R}^d$, that $\partial_v\frac{\delta}{\delta m}p(\cdot,\cdot,t,\cdot,y)(\cdot)$ is continuous on $\mathcal{P}_\beta(\mathbb{R}^d)\times[0,t)\times(\mathbb{R}^d)^2$ and that it satisfies (2.16), (2.18), (2.21), (2.24) and (2.27).

Remark 7. We have made the proof of the extraction of converging subsequences for a fixed $y\in\mathbb{R}^d$. However, following exactly the same lines as for the estimates of Hölder-continuity with respect to x (4.19), (4.25) and (4.26), we can prove similar estimates with respect to y. This ensures that the converging subsequences can also be assumed to converge uniformly on each compact subset of $\mathbb{R}^d$ with respect to y.

Let us prove (2.28). We fix $\varepsilon>0$. Thanks to (4.20) and (4.21), we can find a compact subset $\mathcal{K}$ of $\mathbb{R}^d$ such that

$$\int_{\mathcal{K}^c}\sup_{m\geq1}\left\{\left|\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v)\right|+\left|\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v)\right|\right\}dy\leq\varepsilon$$

By Remark (7), up to extracting a subsequence, we can assume that the functions $\left(\frac{\delta}{\delta m}p_m(\mu,s,t,x,\cdot)(v)\right)_m$ and $\left(\partial_v\frac{\delta}{\delta m}p_m(\mu,s,t,x,\cdot)(v)\right)_m$ converge uniformly on $\mathcal{K}$ towards $\frac{\delta}{\delta m}p(\mu,s,t,x,\cdot)(v)$ and $\partial_v\frac{\delta}{\delta m}p(\mu,s,t,x,\cdot)(v)$. Noticing that (2.28) is true for p_m by (7.17), (7.7), (7.18) and (7.8). We can thus write

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v)\,dy\right|\\&=\left|\int_{\mathbb{R}^d}\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v)-\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v)\,dy\right|\\&\leq\int_{\mathcal{K}}\left|\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v)-\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v)\right|dy+\varepsilon.\end{aligned}$$

We conclude by letting m tend to infinity and with a similar reasoning for $\partial_v\frac{\delta}{\delta m}p$.

Step 3: Regularity with respect to time and backward Kolmogorov PDE (2.11). It remains to prove that for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $t\in(0,T]$, $x,y\in\mathbb{R}^d$, the map $s\in[0,t)\mapsto p(\mu,\cdot,t,x,y)$ is of class $\mathcal{C}^1$ on $[0,t)$ and that $\partial_s p(\cdot,\cdot,t,\cdot,y)$ is continuous on $\mathcal{P}_\beta(\mathbb{R}^d)\times[0,t)\times\mathbb{R}^d$ and satisfies (2.11). Let us fix $h\in[0,s]$. By the well-posedness of the nonlinear martingale problem proved in Theorem 1, we deduce that the transition density satisfies the following Markov property

$$p(\mu,s-h,t,x,y)=\mathbb{E}p([X_s^{s-h,\mu}],s,t,X_s^{s-h,x,\mu},y).\tag{4.37}$$

By (2.13), (2.15), (2.16) and (2.14), (2.18) with $\gamma>\alpha-1$, we can apply Itô's formula of Proposition 7 for the function $(\mu,x)\in\mathcal{P}_\beta(\mathbb{R}^d)\times\mathbb{R}^d\mapsto p(\mu,s,t,x,y)$. Taking the expectation in Itô's formula, we obtain that

$$\mathbb{E}p([X_s^{s-h,\mu}],s,t,X_s^{s-h,x,\mu},y)=p(\mu,s,t,x,y)+\int_{s-h}^s\mathscr{L}_r p(\cdot,s,t,\cdot,y)([X_r^{s-h,\mu}],X_r^{s-h,x,\mu})\,dr,$$

where

$$\begin{aligned}\mathscr{L}_r h(\mu,x):=\,&b(r,x,\mu)\cdot\partial_x h(\mu,x)+\int_{\mathbb{R}^d}\left[h(\mu,x+z)-h(\mu,x)-z\cdot\partial_x h(\mu,x)\right]\frac{dz}{|z|^{d+\alpha}}\\&+\int_{\mathbb{R}^d}b(r,v,\mu)\cdot\partial_v\frac{\delta}{\delta m}h(\mu,x)(v)\,d\mu(v)\\&+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[\frac{\delta}{\delta m}h(\mu,x)(v+z)-\frac{\delta}{\delta m}h(\mu,x)(v)-z\cdot\partial_v\frac{\delta}{\delta m}h(\mu,x)(v)\right]\frac{dz}{|z|^{d+\alpha}}\,d\mu(v).\end{aligned}$$

We have thus

$$\frac{1}{h}(p(\mu,s-h,t,x,y)-p(\mu,s,t,x,y))=\frac{1}{h}\int_{s-h}^s\mathbb{E}\mathscr{L}_r p(\cdot,s,t,\cdot,y)([X_r^{s-h,\mu}],X_r^{s-h,x,\mu})\,dr.$$

Using the continuity and the boundedness of b as well as the Hölder continuity and bounds on $p(\cdot,\cdot,t,\cdot,y)$, $\partial_x p(\cdot,\cdot,t,\cdot,y)$, $\frac{\delta}{\delta m}p(\cdot,\cdot,t,\cdot,y)(\cdot)$ and $\partial_v\frac{\delta}{\delta m}p(\cdot,\cdot,t,\cdot,y)(\cdot)$ proved above, we find that

$$\frac{1}{h}(p(\mu,s-h,t,x,y)-p(\mu,s,t,x,y))\underset{h\to0^+}{\longrightarrow}\mathscr{L}_s p(\cdot,s,t,\cdot,y)(\mu,x).$$

The map $s\in[0,t)\mapsto p(\mu,s,t,x,y)$ is thus left-differentiable on $[0,t)$. It also follows that the map $(\mu,s,x)\in\mathcal{P}_\beta(\mathbb{R}^d)\times[0,t)\times\mathbb{R}^d\mapsto\mathscr{L}_s p(\cdot,s,t,\cdot,y)(\mu,x)$ is continuous. This proves that $p(\mu,\cdot,t,x,y)$ is $\mathcal{C}^1$ on $[0,t)$ and that it satisfies for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $s\in[0,t)$, $x,y\in\mathbb{R}^d$

$$\partial_s p(\mu,s,t,x,y)=-\mathscr{L}_s p(\cdot,s,t,\cdot,y)(\mu,x).$$

Let us now fix $f:\mathbb{R}^d\to\mathbb{R}$ a bounded and uniformly continuous function. We fix $\varepsilon>0$. There exists $\delta>0$ such that for all $x,y\in\mathbb{R}^d$ with $|x-y|\le\delta$, we have $|f(x)-f(y)|\le\varepsilon$. Using (2.13), we obtain that

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}f(y)p(\mu,s,t,x,y)\,dy-f(x)\right| &= \sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(f(y)-f(x))p(\mu,s,t,x,y)\,dy\right|\\ &\le \varepsilon+C\|f\|_\infty\int_{|y|>\delta}\rho^0(t-s,y)\,dy\\ &\le \varepsilon+C\|f\|_\infty\int_{|y|>\delta}(t-s)^{-\frac{d}{\alpha}}(1+(t-s)^{-\frac{1}{\alpha}}|y|)^{-d-\alpha}\,dy\\ &= \varepsilon+C\|f\|_\infty\int_{|z|>(t-s)^{-\frac{1}{\alpha}}\delta}(1+|z|)^{-d-\alpha}\,dz.\end{aligned}$$

We conclude taking the lim sup when $s\to t$ in the preceding inequality that $p(\mu,s,t,x,\cdot)\underset{s\to t^-}{\longrightarrow}\delta_x$.

It remains to prove (2.17). Using (2.13), (2.14), (2.15), (2.16), (2.18), the boundedness of b, and the same reasoning as used in the proof of (B.11), we find that there exists a positive constant C such that for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x,y,v\in\mathbb{R}^d$

$$\begin{aligned}&|b(s,x,\mu)\cdot\partial_x p(\mu,s,t,x,y)|\le C(t-s)^{-\frac{1}{\alpha}}\rho^1(t-s,y-x),\\ &\left|\int_{\mathbb{R}^d}[p(\mu,s,t,x+z,y)-p(\mu,s,t,x,y)-z\cdot\partial_x p(\mu,s,t,x,y)]\,\frac{dz}{|z|^{d+\alpha}}\right|\le C(t-s)^{-1}\rho^0(t-s,y-x),\\ &\left|b(s,v,\mu)\cdot\partial_v\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v)\right|\le C(t-s)^{\frac{n}{\alpha}+1-\frac{1}{\alpha}}\rho^0(t-s,y-x),\\ &\left|\int_{\mathbb{R}^d}\left[\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v+z)-\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v)-z\cdot\partial_v\frac{\delta}{\delta m}p(\mu,s,t,x,y)(v)\right]\frac{dz}{|z|^{d+\alpha}}\right|\\ &\qquad\le C(t-s)^{-1+1-\frac{1}{\alpha}}\rho^0(t-s,y-x).\end{aligned}$$

This concludes the proof of (2.17) and thus ends the proof of Theorem 2.

5. Backward Kolmogorov PDE on the space of measures

We prove in this section Theorem 3.

Step 1: Continuity of U on $[0,T]\times\mathcal{P}_\beta(\mathbb{R}^d)$. Reasoning exactly as in the proof of Proposition 6.1 in [8], the continuity of U on $[0,T)\times\mathcal{P}_\beta(\mathbb{R}^d)$ follows from the continuity of the map $(\mu,s,x)\in\mathcal{P}_\beta(\mathbb{R}^d)\times[0,T)\times\mathbb{R}^d\mapsto p(\mu,s,t,x,y)$ and (2.13). Let us prove it on $[0,T]\times\mathcal{P}_\beta(\mathbb{R}^d)$. Let $(t_n)_n\in[0,T)^{\mathbb{N}}$, $(\mu_n)_n\in\mathcal{P}_\beta(\mathbb{R}^d)^{\mathbb{N}}$ and $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$ such that $|t_n-T|\underset{n\to+\infty}{\longrightarrow}0$ and $W_\beta(\mu_n,\mu)\underset{n\to+\infty}{\longrightarrow}0$. Since ϕ is continuous on $\mathcal{P}_\beta(\mathbb{R}^d)$, it is enough to prove that $W_\beta([X_T^{t_n,\mu_n}],\mu)\underset{n\to+\infty}{\longrightarrow}0$. We start with the weak convergence. Let us fix $f:\mathbb{R}^d\to\mathbb{R}$ a bounded and uniformly continuous function. We write

$$\begin{aligned}\int_{\mathbb{R}^{2d}}f(y)p(\mu_n,t_n,T,x,y)\,dy\,d\mu_n(x)-\int_{\mathbb{R}^d}f(x)d\mu(x) &= \int_{\mathbb{R}^{2d}}(f(y)-f(x))p(\mu_n,t_n,T,x,y)\,dy\,d\mu_n(x)\\ &\quad+\int_{\mathbb{R}^d}f(x)d(\mu_n-\mu)(x)\\ &=: I_1+I_2.\end{aligned}$$

Since $W_\beta(\mu_n,\mu)\underset{n\to+\infty}{\longrightarrow}0$, it is clear by definition that $I_2\underset{n\to+\infty}{\longrightarrow}0$. Let us now deal with I_1. We fix $\varepsilon>0$. There exists $\delta>0$ such that for all $x,y\in\mathbb{R}^d$ with $|x-y|\le\delta$, we have $|f(x)-f(y)|\le\varepsilon$. Using (2.13), we

obtain that

$$\begin{aligned}|I_1| &\le \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (f(y)-f(x)) p(\mu_n, t_n, T, x, y)\, dy \right| d\mu_n(x) \\ &\le \varepsilon + C\|f\|_\infty \int_{|y|>\delta} \rho^0(T-t_n, y)\, dy \\ &\le \varepsilon + C\|f\|_\infty \int_{|y|>\delta} (T-t_n)^{-\frac{d}{\alpha}} (1+(T-t_n)^{-\frac{1}{\alpha}}|y|)^{-d-\alpha}\, dy \\ &= \varepsilon + C\|f\|_\infty \int_{|z|>(T-t_n)^{-\frac{1}{\alpha}}\delta} (1+|z|)^{-d-\alpha}\, dz.\end{aligned}$$

We conclude that $I_1 \underset{n\to+\infty}{\longrightarrow} 0$. It remains to show that

$$\int_{\mathbb{R}^{2d}} |y|^\beta p(\mu_n, t_n, T, x, y)\, d\mu_n(x)\, dy \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^d} |x|^\beta d\mu(x).$$

To see this, we write

$$\begin{aligned}\int_{\mathbb{R}^{2d}} |y|^\beta p(\mu_n, t_n, T, x, y)\, dy\, d\mu_n(x) - \int_{\mathbb{R}^d} |x|^\beta d\mu(x) &= \int_{\mathbb{R}^{2d}} (|y|^\beta - |x|^\beta) p(\mu_n, t_n, T, x, y)\, dy\, d\mu_n(x) \\ &\quad + \int_{\mathbb{R}^d} |x|^\beta d(\mu_n - \mu)(x) \\ &=: J_1 + J_2.\end{aligned}$$

Since $W_\beta(\mu_n, \mu) \underset{n\to+\infty}{\longrightarrow} 0$, we deduce that $J_2 \underset{n\to+\infty}{\longrightarrow} 0$. For J_1, by the mean-value theorem, there exists a positive constant C such that for all $x, y \in \mathbb{R}^d$

$$||y|^\beta - |x|^\beta| \le C|y-x|(|y|^{\beta-1} + |x|^{\beta-1}).$$

We obtain by (2.13) and the space-time inequality (B.13) that

$$\begin{aligned}|J_1| &\le C(T-t_n)^{\frac{1}{\alpha}} \int_{\mathbb{R}^{2d}} (|y|^{\beta-1} + |x|^{\beta-1}) \rho^{-1}(T-t_n, y-x)\, dy\, d\mu_n(x) \\ &\le C(T-t_n)^{\frac{1}{\alpha}} \int_{\mathbb{R}^{2d}} \rho^{-\beta}(T-t_n, y-x) + |x|^{\beta-1}\rho^{-1}(T-t_n, y-x)\, dy\, d\mu_n(x) \\ &\le C(T-t_n)^{\frac{1}{\alpha}} \left(1 + \int_{\mathbb{R}^d} |x|^{\beta-1}\, d\mu_n(x)\right) \\ &\le C(T-t_n)^{\frac{1}{\alpha}},\end{aligned}$$

since $\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |x|^{\beta-1}\, d\mu_n(x) < +\infty$. We have thus proved that $W_\beta([X_T^{t_n,\mu_n}], \mu) \underset{n\to+\infty}{\longrightarrow} 0$, which concludes the first step.

Step 2: Estimates (2.30), (2.31) and (2.32). By Proposition 6 and Proposition 3, we know that the map U belongs to $\mathcal{C}^1([0,T) \times \mathcal{P}_\beta(\mathbb{R}^d))$. Moreover, for any $t \in [0,T)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$, we have

$$\begin{aligned}\frac{\delta}{\delta m} U(t,\mu)(v) &= \int_{\mathbb{R}^d} \frac{\delta}{\delta m}\phi([X_T^{t,\mu}])(y) p(\mu, t, T, v, y)\, dy \\ &\quad + \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu}])(x)\right) \frac{\delta}{\delta m} p(\mu, t, T, x, y)(v)\, dy\, d\mu(x),\end{aligned} \tag{5.1}$$

and

$$\partial_v \frac{\delta}{\delta m} U(t,\mu)(v) = \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu}])(v) \right) \partial_x p(\mu,t,T,v,y)\, dy \tag{5.2}$$
$$+ \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu}])(x) \right) \partial_v \frac{\delta}{\delta m} p(\mu,t,T,x,y)(v)\, dy\, d\mu(x).$$

It follows from (2.13), (2.15), the δ-Hölder continuity of $\frac{\delta}{\delta m}\phi(\mu)(\cdot)$ uniformly with respect to $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and the space-time inequality (B.13) that there exists a positive constant C such that for all $t \in [0,T)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$, one has

$$\begin{aligned}\left| \frac{\delta}{\delta m} U(t,\mu)(v) \right| &\leq C((T-t)^{\frac{\delta}{\alpha}} + |v|^\delta) + C(T-t)^{\frac{\delta}{\alpha}+1-\frac{1}{\alpha}} \\ &\leq C(1+|v|^\delta).\end{aligned}$$

By (2.13), (2.16), we similarly get that there exists a positive constant C such that for all $t \in [0,T)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$

$$\begin{aligned}\left| \partial_v \frac{\delta}{\delta m} U(t,\mu)(v) \right| &\leq C(T-t)^{\frac{\delta-1}{\alpha}} + C(T-t)^{\frac{\delta-1}{\alpha}+1+\frac{\eta-1}{\alpha}} \\ &\leq C(T-t)^{\frac{\delta-1}{\alpha}}.\end{aligned}$$

We now prove (2.32). Let us first assume that $|v_1 - v_2| > (T-t)^{\frac{1}{\alpha}}$. In this case, it follows from (2.31) that

$$\begin{aligned}\left| \partial_v \frac{\delta}{\delta m} U(t,\mu)(v_1) - \partial_v \frac{\delta}{\delta m} U(t,\mu)(v_2) \right| &\leq \left| \partial_v \frac{\delta}{\delta m} U(t,\mu)(v_1) \right| + \left| \partial_v \frac{\delta}{\delta m} U(t,\mu)(v_2) \right| \\ &\leq C(T-t)^{\frac{\delta-1}{\alpha}} \\ &\leq C(T-t)^{\frac{\delta-1-\gamma}{\alpha}} |v_1 - v_2|^\gamma.\end{aligned}$$

Assume now that $|v_1 - v_2| < (T-t)^{\frac{1}{\alpha}}$. One has

$$\begin{aligned}&\partial_v \frac{\delta}{\delta m} U(t,\mu)(v_1) - \partial_v \frac{\delta}{\delta m} U(t,\mu)(v_2) \\ &= \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu}])(v_2) \right) (\partial_x p(\mu,t,T,v_1,y) - \partial_x p(\mu,t,T,v_2,y))\, dy \\ &\quad + \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu}])(x) \right) (\partial_v \frac{\delta}{\delta m} p(\mu,t,T,x,y)(v_1) - \partial_v \frac{\delta}{\delta m} p(\mu,t,T,x,y)(v_2))\, dy\, d\mu(x).\end{aligned}$$

Thanks to the uniform δ-Hölder continuity of $\frac{\delta}{\delta m}\phi(\mu)(\cdot)$, (2.14), (2.18) since $\gamma \in (0, (2\alpha-2) \wedge (\eta+\alpha-1))$ and (B.14) since $|v_1 - v_2| < (T-t)^{\frac{1}{\alpha}}$, we obtain that

$$\begin{aligned}&\left| \partial_v \frac{\delta}{\delta m} U(t,\mu)(v_1) - \partial_v \frac{\delta}{\delta m} U(t,\mu)(v_2) \right| \\ &\leq C \int_{\mathbb{R}^d} |y-v_2|^\delta (T-t)^{-\frac{\gamma+1}{\alpha}} |v_1 - v_2|^\gamma \rho^1(T-t, y-v_2)\, dy \\ &\quad + C \int_{\mathbb{R}^{2d}} |y-x|^\delta (T-t)^{-\frac{\gamma+1}{\alpha}+1+\frac{\eta-1}{\alpha}} |v_1 - v_2|^\gamma \rho^0(T-t, y-x)\, dy\, d\mu(x).\end{aligned}$$

Using the space-time inequality (B.13), we deduce that

$$\left|\partial_v \frac{\delta}{\delta m} U(t,\mu)(v_1) - \partial_v \frac{\delta}{\delta m} U(t,\mu)(v_2)\right| \leq C\ (T-t)^{\frac{\delta-\gamma-1}{\alpha}} |v_1 - v_2|^\gamma.$$

Step 3: Time-derivative and backward Kolmogorov PDE (2.33). Let us fix $t \in [0,T)$ and $h \in [0,t]$. From the Markov property stemming from the well-posedness of the associated martingale problem related to (2.7), we obtain that

$$U(t-h,\mu) = \phi([X_T^{t-h,\mu}]) = \phi([X_T^{t,[X_t^{t-h,\mu}]}]) = U(t,[X_t^{t-h,\mu}]).$$

Since $U \in \mathcal{C}^1([0,T)\times\mathcal{P}_\beta(\mathbb{R}^d))$ and thanks to (2.30), (2.31) and (2.32), we can apply Itô's formula of Proposition 7 for the function $U(t,\cdot)$. It yields

$$\begin{aligned} U(t-h,\mu) &= U(t,\mu) + \int_{t-h}^t \mathbb{E}\left(\partial_v \frac{\delta}{\delta m} U(t,[X_s^{t-h,\mu}])(X_s^{t-h,\mu}) \cdot b(s, X_s^{t-h,\mu}, [X_s^{t-h,\mu}])\right) ds \\ &\quad + \int_{t-h}^t \mathbb{E} \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m} U(t,[X_s^{t-h,\mu}])(X_s^{t-h,\mu}+z) - \frac{\delta}{\delta m} U(t,[X_s^{t-h,\mu}])(X_s^{t-h,\mu}) \right. \\ &\qquad \left. - \partial_v \frac{\delta}{\delta m} U(t,[X_s^{t-h,\mu}])(X_s^{t-h,\mu}) \cdot z \right) \frac{dz}{|z|^{d+\alpha}}\, ds \\ &= U(t,\mu) + \int_{t-h}^t \mathscr{L}_s U(t,\cdot)([X_s^{t-h,\mu}])\, ds, \end{aligned}$$

where $\mathscr{L}_s$ was defined in (2.12). Using the continuity and the boundedness of b as well as the regularity properties of the map $\mu \mapsto U(t,\mu)$ obtained above, we find that

$$\frac{1}{h}(U(t-h,\mu) - U(t,\mu)) \underset{h\to 0^+}{\longrightarrow} \mathscr{L}_t U(t,\cdot)(\mu).$$

This proves the left-differentiability of the map $t \in [0,T) \mapsto U(t,\mu)$. It also follows from the regularity of the drift b and of U on $[0,T)\times\mathcal{P}_\beta(\mathbb{R}^d)$ that the map $(t,\mu) \in [0,T)\times\mathcal{P}_\beta(\mathbb{R}^d) \mapsto \mathscr{L}_t U(t,\cdot)(\mu)$ is continuous. Thus, the map $t \in [0,T) \mapsto U(t,\mu)$ is $\mathcal{C}^1$ and satisfies for all $t \in [0,T)$ and $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$

$$\partial_t U(t,\mu) + \mathscr{L}_t U(t,\cdot)(\mu) = 0.$$

This shows that U is solution to the backward Kolmogorov PDE (2.33).

Step 4: Uniqueness of the solution to (2.33). Let us consider $V \in \mathcal{C}^0([0,T]\times\mathcal{P}_\beta(\mathbb{R}^d)) \cap \mathcal{C}^1([0,T)\times\mathcal{P}_\beta(\mathbb{R}^d))$ is another solution to (2.33) satisfying (2.30), (2.31) and (2.32). Let us fix $(t,\mu) \in [0,T)\times\mathcal{P}_\beta(\mathbb{R}^d)$. For any $\tau \in [t,T)$, we can apply Itô's formula of Proposition 7 for the map $(s,\mu) \in [t,\tau]\times\mathcal{P}_\beta(\mathbb{R}^d) \mapsto V(s,\mu)$ which yields

$$V(\tau,[X_\tau^{t,\mu}]) = V(t,\mu) + \int_t^\tau \partial_s V(s,[X_s^{t,\mu}])\, ds + \int_t^\tau \mathscr{L}_s V(s,\cdot)([X_s^{t,\mu}])\, ds.$$

Since V solves (2.33), we obtain that $V(\tau,[X_\tau^{t,\mu}]) = V(t,\mu)$. We then use the continuity of the maps $(t,\mu) \in [0,T]\times\mathcal{P}_\beta(\mathbb{R}^d) \mapsto V(t,\mu)$ and $\tau \in [t,T] \mapsto [X_\tau^{t,\mu}] \in \mathcal{P}_\beta(\mathbb{R}^d)$ to let τ tends to T. This yields

$$\phi([X_T^{t,\mu}]) = U(t,\mu) = V(t,\mu).$$

6. Quantitative weak propagation of chaos

This section is dedicated to prove Theorem 4. We first need to establish additional regularity properties on the solution to the PDE (2.33) to prove our propagation of chaos result.

Proposition 4. Let us fix $\delta \in (0,1]$, $L>0$ and $\gamma \in (0,1] \cap (0, (2\alpha-2) \wedge (\eta+\alpha-1))$. Then, under Assumption (H2), there exists positive constant $C = C(d,T,\alpha,\beta,(H2),\delta,L,\gamma)$ such that for all $\phi \in \mathcal{C}_L^{\delta,\delta}(\mathcal{P}_\beta(\mathbb{R}^d))$ (defined in Definition 3), the solution U of the backward Kolmogorov PDE (2.33) with terminal condition ϕ at time T satisfies the following properties.

- For all $t \in [0,T)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $v \in \mathbb{R}^d$

$$\left| \frac{\delta}{\delta m} U(t,\mu)(v) \right| \leq C(1+|v|^\delta). \tag{6.1}$$

- For all $t \in [0,T)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $v \in \mathbb{R}^d$

$$\left| \partial_v \frac{\delta}{\delta m} U(t,\mu)(v) \right| \leq C(T-t)^{\frac{\delta-1}{\alpha}}. \tag{6.2}$$

- For all $t \in [0,T)$, $\mu_1, \mu_2 \in \mathcal{P}_\beta(\mathbb{R}^d)$, $v_1, v_2 \in \mathbb{R}^d$

$$\left| \partial_v \frac{\delta}{\delta m} U(t,\mu_1)(v_1) - \partial_v \frac{\delta}{\delta m} U(t,\mu_2)(v_2) \right| \leq C(T-t)^{\frac{\delta-1-\gamma}{\alpha}} \left(|v_1-v_2|^\gamma + W_1^\gamma(\mu_1,\mu_2) \right). \tag{6.3}$$

- For all $t \in [0,T)$, $\mu_1, \mu_2 \in \mathcal{P}_\beta(\mathbb{R}^d)$, $v \in \mathbb{R}^d$

$$\left| \frac{\delta}{\delta m} U(t,\mu_1)(v) - \frac{\delta}{\delta m} U(t,\mu_2)(v) \right| \leq C(T-t)^{\frac{\delta-1}{\alpha}} W_1(\mu_1,\mu_2). \tag{6.4}$$

Proof of Proposition 4. First, note that (6.1) and (6.2) have been proved in Theorem 3.

Proof of (6.3). We write

$$\begin{aligned} \partial_v \frac{\delta}{\delta m} U(t,\mu_1)(v_1) - \partial_v \frac{\delta}{\delta m} U(t,\mu_2)(v_2) &= \partial_v \frac{\delta}{\delta m} U(t,\mu_1)(v_1) - \partial_v \frac{\delta}{\delta m} U(t,\mu_1)(v_2) \\ &\quad + \partial_v \frac{\delta}{\delta m} U(t,\mu_1)(v_2) - \partial_v \frac{\delta}{\delta m} U(t,\mu_2)(v_2) \\ &=: I_1 + I_2. \end{aligned} \tag{6.5}$$

Then, using (2.32), we obtain that

$$|I_1| \leq C(T-t)^{\frac{\delta-1-\gamma}{\alpha}} |v_1-v_2|^\gamma. \tag{6.6}$$

We now focus on I_2. Using (5.2), we get

$$
\begin{aligned}
&\partial_v \frac{\delta}{\delta m} U(t,\mu_1)(v_2) - \partial_v \frac{\delta}{\delta m} U(t,\mu_2)(v_2) \\
&= \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(v_2) \right) \partial_x p(\mu_1,t,T,v_2,y)\,dy \\
&\quad + \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(x) \right) \partial_v \frac{\delta}{\delta m} p(\mu_1,t,T,x,y)(v_2)\,dy\,d\mu_1(x) \\
&\quad - \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(v_2) \right) \partial_x p(\mu_2,t,T,v_2,y)\,dy \\
&\quad - \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x) \right) \partial_v \frac{\delta}{\delta m} p(\mu_2,t,T,x,y)(v_2)\,dy\,d\mu_2(x).
\end{aligned}
$$

We decompose it in the following way

$$
\begin{aligned}
&\partial_v \frac{\delta}{\delta m} U(t,\mu_1)(v_2) - \partial_v \frac{\delta}{\delta m} U(t,\mu_2)(v_2) \\
&= \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(v_2) - \left[\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(v_2) \right] \right) \partial_x p(\mu_1,t,T,v_2,y)\,dy \\
&\quad + \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(v_2) \right) \left(\partial_x p(\mu_1,t,T,v_2,y) - \partial_x p(\mu_2,t,T,v_2,y) \right)\,dy \\
&\quad + \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(x) - \left[\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x) \right] \right) \\
&\qquad\qquad \partial_v \frac{\delta}{\delta m} p(\mu_1,t,T,x,y)(v_2)\,dy\,d\mu_1(x) \\
&\quad + \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x) \right) \\
&\qquad\qquad \left(\partial_v \frac{\delta}{\delta m} p(\mu_1,t,T,x,y)(v_2) - \partial_v \frac{\delta}{\delta m} p(\mu_2,t,T,x,y)(v_2) \right)\,dy\,d\mu_1(x) \\
&\quad + \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x) \right) \partial_v \frac{\delta}{\delta m} p(\mu_2,t,T,x,y)(v_2)\,dy\,d(\mu_1-\mu_2)(x) \\
&=: J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned} \tag{6.7}
$$

Before estimating each term of the preceding decomposition, let us prove that for all $\gamma \in (0,1]$, there exists a positive constant C such that for any $\phi \in \mathcal{C}_L^{\delta,\delta}(\mathcal{P}_\beta(\mathbb{R}^d))$, $t \in [0,T)$, $\mu_1,\mu_2, \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x,y \in \mathbb{R}^d$

$$
\begin{aligned}
&\left| \frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(x) - \left[\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x) \right] \right| \\
&\qquad\qquad \le C(T-t)^{-\frac{\gamma}{\alpha}} |x-y|^\delta W_1^\gamma(\mu_1,\mu_2).
\end{aligned} \tag{6.8}
$$

To prove this, we write

$$\begin{aligned}
&\frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(x) - \left[\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x)\right] \\
&= \int_0^1 \int_{\mathbb{R}^d} \left(\frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,v) - \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(x,v)\right)(p(\mu_1,t,T,v) - p(\mu_2,s,T,v))\,dv\,d\lambda \\
&= \int_0^1 \int_{\mathbb{R}^{2d}} \left(\frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,v) - \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(x,v)\right) p(\mu_1,t,T,x',v)\,d(\mu_1-\mu_2)(x')\,dv\,d\lambda \\
&\quad + \int_0^1 \int_{\mathbb{R}^{2d}} \left(\frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,v) - \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(x,v)\right)\big(p(\mu_1,t,T,x',v) - p(\mu_2,t,T,x',v)\big)\,d\mu_2(x')\,dv\,d\lambda \\
&=: K_1 + K_2,
\end{aligned}$$

where $m_\lambda := \lambda[X_T^{t,\mu_1}] + (1-\lambda)[X_T^{t,\mu_2}]$. It follows from (2.22) and the δ-Hölder continuity of $\frac{\delta^2}{\delta m^2}\phi(\mu)(\cdot,v)$ uniformly with respect to $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$ that

$$|K_2| \le C(T-t)^{1-\frac{\gamma+1}{\alpha}}|y-x|^\delta W_1^\gamma(\mu_1,\mu_2). \tag{6.9}$$

Concerning K_1, we set for $x' \in \mathbb{R}^d$

$$F(x') := \int_{\mathbb{R}^d} \left(\frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,v) - \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(x,v)\right) p(\mu_1,t,T,x',v)\,dv.$$

Let us fix $x', x'' \in \mathbb{R}^d$. Thanks to (2.14) and the δ-Hölder continuity of $\frac{\delta^2}{\delta m^2}\phi(\mu)(\cdot,v)$ uniformly with respect to $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$, one gets that

$$|F(x') - F(x'')| \le C|x-y|^\delta (T-t)^{-\frac{\gamma}{\alpha}}|x'-x''|^\gamma.$$

Taking an optimal coupling between μ_1 and μ_1 for W_1 directly yields with Jensen's inequality

$$|K_1| \le C|x-y|^\delta (T-t)^{-\frac{\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2). \tag{6.10}$$

Combining (6.9) and (6.10) concludes the proof of (6.8) since $\alpha \in (1,2)$. We can now turn to estimate each term of (6.7). Using (6.8), (2.13) and the space-time inequality (B.13), one obtains that

$$\begin{aligned}
|J_1| &\le C\int_{\mathbb{R}^d} (T-t)^{-\frac{\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2)|y-v_2|^\delta (T-t)^{-\frac{1}{\alpha}}\rho^1(T-t,y-v_2)\,dy \\
&\le C(T-t)^{\frac{\delta-\gamma-1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\int_{\mathbb{R}^d} \rho^{1-\delta}(T-t,y-v_2)\,dy \\
&\le C(T-t)^{\frac{\delta-\gamma-1}{\alpha}} W_1^\gamma(\mu_1,\mu_2).
\end{aligned} \tag{6.11}$$

For J_2, it follows from the δ-Hölder continuity of $\frac{\delta}{\delta m}\phi(\mu)(\cdot)$ uniformly with respect to $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, (2.22) since $\gamma < \eta + \alpha - 1$ and the space-time inequality (B.13) that

$$\begin{aligned}
|J_2| &\le C\int_{\mathbb{R}^d} |y-v_2|^\delta (T-t)^{1-\frac{2+\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\rho^1(T-t,y-v_2)\,dy \\
&\le C(T-t)^{1+\frac{\delta-2-\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \\
&\le C(T-t)^{\frac{\delta-1-\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2),
\end{aligned} \tag{6.12}$$

since $\alpha \in (1,2)$. For J_3, (6.8), (2.16) and the space-time inequality (B.13) yield

$$
\begin{aligned}
|J_3| &\le C\int_{\mathbb{R}^{2d}} (T-t)^{-\frac{\gamma}{\alpha}}|y-x|^{\delta} W_1^{\gamma}(\mu_1,\mu_2)(T-t)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}\rho^0(T-t,y-x)\,dy\,d\mu_1(x)\\
&\le C(T-t)^{\frac{\delta-1-\gamma}{\alpha}+1+\frac{\eta-1}{\alpha}} W_1^{\gamma}(\mu_1,\mu_2) \qquad (6.13)\\
&\le C(T-t)^{\frac{\delta-1-\gamma}{\alpha}} W_1^{\gamma}(\mu_1,\mu_2).
\end{aligned}
$$

It follows from the δ-Hölder continuity of $\frac{\delta}{\delta m}\phi(\mu)(\cdot)$ uniformly with respect to $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, (2.24) and the space-time inequality (B.13) that

$$
\begin{aligned}
|J_4| &\le C\int_{\mathbb{R}^{2d}} |y-x|^{\delta}(T-t)^{\frac{\eta-\gamma-1}{\alpha}+1-\frac{1}{\alpha}} W_1^{\gamma}(\mu_1,\mu_2)\rho^0(T-t,y-x)\,dy\,d\mu_1(x)\\
&\le C(T-t)^{\frac{\delta-1-\gamma}{\alpha}+1+\frac{\eta-1}{\alpha}} W_1^{\gamma}(\mu_1,\mu_2) \qquad (6.14)\\
&\le C(T-t)^{\frac{\delta-1-\gamma}{\alpha}} W_1^{\gamma}(\mu_1,\mu_2).
\end{aligned}
$$

Let us prove that

$$
|H(x)-H(x')| \le C(T-t)^{\frac{\delta-1-\gamma}{\alpha}+1+\frac{\eta-1}{\alpha}}|x-x'|^{\gamma}. \qquad (6.15)
$$

We finally deal with J_5. We set for $x\in\mathbb{R}^d$

$$
H(x) := \int_{\mathbb{R}^d}\left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y)-\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x)\right)\partial_v\frac{\delta}{\delta m}p(\mu_2,t,T,x,y)(v_2)\,dy.
$$

Let us fix $x,x'\in\mathbb{R}^d$. Assume first that $|x-x'|>(T-t)^{\frac{1}{\alpha}}$. The δ-Hölder continuity of $\frac{\delta}{\delta m}\phi(\mu)(\cdot)$ uniformly with respect to $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, (2.16) and the space-time inequality (B.13) ensure that

$$
\begin{aligned}
|H(x)-H(x')| &\le |H(x)|+|H(x')|\\
&\le C\int_{\mathbb{R}^d}|y-x|^{\delta}(T-t)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}\rho^0(T-t,y-x)\,dy\\
&\le C(T-t)^{\frac{\delta-1}{\alpha}+1+\frac{\eta-1}{\alpha}}\\
&\le C(T-t)^{\frac{\delta-1-\gamma}{\alpha}+1+\frac{\eta-1}{\alpha}}|x-x'|^{\gamma}.
\end{aligned}
$$

In the case where $|x-x'|\le(T-t)^{\frac{1}{\alpha}}$, thanks to (2.28) we write

$$
\begin{aligned}
&H(x)-H(x')\\
&=\int_{\mathbb{R}^d}\left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x')-\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x)\right)\partial_v\frac{\delta}{\delta m}p(\mu_2,t,T,x,y)(v_2)\,dy\\
&\quad+\int_{\mathbb{R}^d}\left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y)-\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x')\right)\left(\partial_v\frac{\delta}{\delta m}p(\mu_2,t,T,x,y)(v_2)-\partial_v\frac{\delta}{\delta m}p(\mu_2,t,T,x',y)(v_2)\right)dy\\
&=\int_{\mathbb{R}^d}\left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y)-\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x')\right)\left(\partial_v\frac{\delta}{\delta m}p(\mu_2,t,T,x,y)(v_2)-\partial_v\frac{\delta}{\delta m}p(\mu_2,t,T,x',y)(v_2)\right)dy.
\end{aligned}
$$

Thanks to the δ-Hölder continuity of $\frac{\delta}{\delta m}\phi(\mu)(\cdot)$ uniformly with respect to $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$ and (2.21) since $\gamma<\eta+\alpha-1$, we directly get that

$$
|H(x)-H(x')|\le C\int_{\mathbb{R}^d}|y-x'|^{\delta}(T-t)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}}|x-x'|^{\gamma}\left[\rho^0(T-t,y-x')+\rho^0(T-t,y-x)\right]dy.
$$

Since $|x-x'|\le(T-t)^{\frac{1}{\alpha}}$, it follows from (B.14) and the space-time inequality (B.13) that

$$
\begin{aligned}
|H(x)-H(x')| &\le C\int_{\mathbb{R}^d}|y-x'|^{\delta}(T-t)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}}|x-x'|^{\gamma}\rho^0(T-t,y-x')\,dy\\
&\le C(T-t)^{\frac{\delta-1-\gamma}{\alpha}+1+\frac{\eta-1}{\alpha}}|x-x'|^{\gamma}.
\end{aligned}
$$

Then, it follows from (6.15) that

$$\begin{aligned}|J_5| &\leq C(T-t)^{\frac{\delta-1-\gamma}{\alpha}+1+\frac{\eta-1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\\ &\leq C(T-t)^{\frac{\delta-1-\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2).\end{aligned} \tag{6.16}$$

Coming back to (6.5) and using (6.7), (6.11), (6.12), (6.13), (6.14), (6.16), we have shown that

$$|I_2| \leq C(T-t)^{\frac{\delta-1-\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2).$$

This estimate together with (6.6) ends the proof of (6.3).

Proof of (6.4)**.** It follows from (5.2) that

$$\begin{aligned}&\frac{\delta}{\delta m}U(t,\mu_1)(v) - \frac{\delta}{\delta m}U(t,\mu_2)(v)\\ &= \int_{\mathbb{R}^d} \frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) p(\mu_1,t,T,v,y)\,dy\\ &\quad + \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(x)\right) \frac{\delta}{\delta m} p(\mu_1,t,T,x,y)(v)\,dy\,d\mu_1(x)\\ &\quad - \int_{\mathbb{R}^d} \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) p(\mu_2,t,T,v,y)\,dy\\ &\quad - \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x)\right) \frac{\delta}{\delta m} p(\mu_2,t,T,x,y)(v)\,dy\,d\mu_2(x).\end{aligned}$$

We decompose it in the following way

$$\begin{aligned}&\frac{\delta}{\delta m}U(t,\mu_1)(v) - \frac{\delta}{\delta m}U(t,\mu_2)(v)\\ &= \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y)\right) p(\mu_1,t,T,v,y)\,dy\\ &\quad + \int_{\mathbb{R}^d} \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y)\left(p(\mu_1,t,T,v,y) - p(\mu_2,t,T,v,y)\right)\,dy\\ &\quad + \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(x) - \left[\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x)\right]\right)\\ &\qquad\qquad \frac{\delta}{\delta m} p(\mu_1,t,T,x,y)(v_2)\,dy\,d\mu_1(x)\\ &\quad + \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x)\right)\\ &\qquad\qquad \left(\frac{\delta}{\delta m} p(\mu_1,t,T,x,y)(v_2) - \frac{\delta}{\delta m} p(\mu_2,t,T,x,y)(v_2)\right)\,dy\,d\mu_1(x)\\ &\quad + \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(x)\right) \frac{\delta}{\delta m} p(\mu_2,t,T,x,y)(v_2)\,dy\,d(\mu_1-\mu_2)(x)\\ &=: J_1+J_2+J_3+J_4+J_5.\end{aligned} \tag{6.17}$$

We first focus of J_1. Let us prove that there exists a positive constant C such that for all $t \in [0,T)$, $\mu_1,\mu_2 \in \mathcal{P}_\beta(\mathbb{R}^d)$, $y \in \mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y)\right| C(T-t)^{\frac{\delta-1}{\alpha}} W_1(\mu_1,\mu_2). \tag{6.18}$$

To prove this, we write

$$\begin{aligned}
&\frac{\delta}{\delta m}\phi([X_T^{t,\mu_1}])(y) - \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y)\\
&= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,z)(p(\mu_1,t,T,z) - p(\mu_2,t,T,z))\,dz\,d\lambda\\
&= \int_0^1 \int_{\mathbb{R}^{2d}} \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,z)(p(\mu_1,t,T,x,z) - p(\mu_2,t,T,x,z))\,d\mu_1(x)\,dz\,d\lambda\\
&\quad + \int_0^1 \int_{\mathbb{R}^{2d}} \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,z)p(\mu_2,t,T,x,z)\,d(\mu_1-\mu_2)(x)\,dz\,d\lambda\\
&=: K_1 + K_2,
\end{aligned}$$

where $m_\lambda := \lambda[X_T^{t,\mu_1}] + (1-\lambda)[X_T^{t,\mu_2}]$. We rewrite K_1 as

$$K_1 = \int_0^1 \int_{\mathbb{R}^{2d}} \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,z) \int_0^1 \int_{\mathbb{R}^d} \frac{\delta}{\delta m}p(M_r,t,T,x,z)(w)\,d(\mu_1-\mu_2)(w)\,dr\,d\mu_1(x)\,dz\,d\lambda,$$

where $M_r = r\mu_1 + (1-r)\mu_2$. We set for $w \in \mathbb{R}^d$

$$F(w) := \int_{\mathbb{R}^d} \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,z)\frac{\delta}{\delta m}p(M_r,t,T,x,z)(w)\,dz.$$

For $w_1, w_2 \in \mathbb{R}^d$, one has

$$\begin{aligned}
&F(w_1) - F(w_2)\\
&= \int_{\mathbb{R}^d} \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,z) \int_0^1 \partial_v \frac{\delta}{\delta m}p(M_r,t,T,x,z)(sw_1+(1-s)w_2)\cdot(w_1-w_2)\,ds\,dz\\
&= \int_0^1 \int_{\mathbb{R}^d} \left(\frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,z) - \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,x)\right) \partial_v \frac{\delta}{\delta m}p(M_r,t,T,x,z)(sw_1+(1-s)w_2)\cdot(w_1-w_2)\,dz\,ds.
\end{aligned}$$

It follows from the δ-Hölder continuity of $\frac{\delta^2}{\delta m^2}\phi(\mu)(y,\cdot)$ uniformly with respect to $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, (2.16) and the space-time inequality (B.13) that

$$|F(w_1) - F(w_2)| \le C(T-t)^{\frac{\delta-1}{\alpha}+1+\frac{\eta-1}{\alpha}}|w_1-w_2|.$$

This yields

$$\begin{aligned}
|K_1| &\le C(T-t)^{\frac{\delta-1}{\alpha}+1+\frac{\eta-1}{\alpha}} W_1(\mu_1,\mu_2)\\
&\le C(T-t)^{\frac{\delta-1}{\alpha}} W_1(\mu_1,\mu_2).
\end{aligned} \tag{6.19}$$

We control K_2 as for K_1 by studying the regularity with respect to x of the function G given by

$$G(x) := \int_{\mathbb{R}^d} \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,z)p(\mu_2,t,T,x,z)\,dz.$$

For $x_1, x_2 \in \mathbb{R}^d$, one has

$$\begin{aligned}
G(x_1) - G(x_2) &= \int_{\mathbb{R}^d} \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,z) \int_0^1 \partial_x p(\mu_2,t,T,rx_1+(1-r)x_2,z)\cdot(x_1-x_2)\,dr\,dz\\
&= \int_0^1 \int_{\mathbb{R}^d} \left(\frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,z) - \frac{\delta^2}{\delta m^2}\phi(m_\lambda)(y,rx_1+(1-r)x_2)\right)\\
&\qquad\qquad \partial_x p(\mu_2,t,T,rx_1+(1-r)x_2,z)\cdot(x_1-x_2)\,dz\,dr.
\end{aligned}$$

Using δ-Hölder continuity of $\frac{\delta^2}{\delta m^2}\phi(\mu)(y,\cdot)$ uniformly with respect to $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, (2.13) and the space-time inequality (B.13), we obtain that

$$|G(x_1) - G(x_2)| \leq C(T-t)^{\frac{\delta-1}{\alpha}}|x_1 - x_2|.$$

It proves that

$$|K_2| \leq C(T-t)^{\frac{\delta-1}{\alpha}} W_1(\mu_1, \mu_2). \tag{6.20}$$

Combining (6.19) and (6.20) concludes the proof of (6.18). We can now turn to estimate J_1 in (6.17). Using (6.18), one gets

$$\begin{aligned} |J_1| &\leq C \int_{\mathbb{R}^d} (T-t)^{\frac{\delta-1}{\alpha}} W_1(\mu_1, \mu_2) p(\mu_2, t, T, v, y)\, dy \\ &\leq C(T-t)^{\frac{\delta-1}{\alpha}} W_1(\mu_1, \mu_2). \end{aligned} \tag{6.21}$$

For J_2, we rewrite it as

$$J_2 = \int_{\mathbb{R}^d} \frac{\delta}{\delta m}\phi([X_T^{t,\mu_2}])(y) \int_0^1 \int_{\mathbb{R}^d} \frac{\delta}{\delta m} p(M_r, t, T, v, y)(z)\, d(\mu_1 - \mu_2)(z)\, dr\, dy,$$

where $M_r := r\mu_1 + (1-r)\mu_2$. Following the same lines as for the proof of the above estimates on K_1 (6.19), we obtain that

$$|J_2| \leq C(T-t)^{\frac{\delta-1}{\alpha}} W_1(\mu_1, \mu_2). \tag{6.22}$$

Following same lines as in the proof of (6.13), (6.14) and (6.16) by using (2.15), (2.23) and (2.20), we can prove the following estimates

$$|J_3| \leq C(T-t)^{\frac{\delta-1}{\alpha}+1-\frac{1}{\alpha}} W_1(\mu_1, \mu_2), \tag{6.23}$$

$$|J_4| \leq C(T-t)^{\frac{\delta-1}{\alpha}+1-\frac{1}{\alpha}} W_1(\mu_1, \mu_2), \tag{6.24}$$

$$|J_5| \leq C(T-t)^{\frac{\delta-1}{\alpha}+1-\frac{1}{\alpha}} W_1(\mu_1, \mu_2). \tag{6.25}$$

Gathering (6.17) (6.21), (6.22), (6.23), (6.24), (6.25), we have proved (6.4).

□

Proof of Theorem 4. Let us first introduce some notations. We can write for all $i \leq N$ and for all $t \in [0,T]$

$$Z_t^i = \int_0^t \int_{\mathbb{R}^d} z\, \widetilde{\mathcal{N}}^i(ds, dz),$$

where $\mathcal{N}^i$ is the Poisson random measure associated with Z^i and $\widetilde{\mathcal{N}}^i$ is the associated compensated Poisson random measure. Then, we set for all $t \in [0,T]$

$$\mathbf{Z}_t^N = \begin{pmatrix} Z_t^1 \\ \vdots \\ Z_t^N \end{pmatrix} \in (\mathbb{R}^d)^N.$$

As the Lévy processes $(Z^n)_n$ are independent, the process $(\mathbf{Z}_t^N)_t$ is a Lévy process in $(\mathbb{R}^d)^N$. Its Poisson random measure $\boldsymbol{\mathcal{N}}^N$ and its Lévy measure $\boldsymbol{\nu}^N$ are defined as follows. For all $\phi : [0,T] \times (\mathbb{R}^d)^N \to \mathbb{R}^+$, one has

$$\int_0^T \int_{\mathbb{R}^{Nd}} \phi(s, \boldsymbol{x})\, \boldsymbol{\mathcal{N}}^N(ds, d\boldsymbol{x}) = \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d} \phi(s, 0, \dots, 0, x_i, 0, \dots, 0)\, \mathcal{N}^i(ds, dx_i). \tag{6.26}$$

For all $\phi : \mathbb{R}^{Nd} \to \mathbb{R}^+$, one has

$$\int_{\mathbb{R}^{Nd}} \phi(\boldsymbol{x})\, d\boldsymbol{\nu}^N(\boldsymbol{x}) = \sum_{i=1}^N \int_{\mathbb{R}^d} \phi(0,\dots,0,x_i,0,\dots,0)\, d\nu(x_i). \tag{6.27}$$

Note that since $(Z^n)_n$ are independent processes, for all $t \in [0,T]$, the support of the random measure $\boldsymbol{\mathcal{N}}^N(t, d\boldsymbol{x})$ is contained in

$$\bigcup_{i=0}^{N-1} \{0\}^i_{\mathbb{R}^d} \times \mathbb{R}^d \times \{0\}^{N-1-i}_{\mathbb{R}^d} \subset (\mathbb{R}^d)^N.$$

Let us define for all $t \in [0,T]$, $\boldsymbol{x} = (x_1,\dots,x_N) \in (\mathbb{R}^d)^N$

$$\boldsymbol{b}^N(t,\boldsymbol{x}) := \begin{pmatrix} b(t,x_1,\overline{\mu}^N_{\boldsymbol{x}}) \\ \vdots \\ b(t,x_N,\overline{\mu}^N_{\boldsymbol{x}}) \end{pmatrix} \in (\mathbb{R}^d)^N,$$

where $\overline{\mu}^N_{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$. Thus, writing

$$\boldsymbol{X}^N_t = \begin{pmatrix} X^{1,N}_t \\ \vdots \\ X^{N,N}_t \end{pmatrix},$$

the SDE (2.34) defining the particle system can be rewritten as

$$\begin{cases} d\boldsymbol{X}^N_t = \boldsymbol{b}^N(t,\boldsymbol{X}^N_t)\, dt + d\mathbf{Z}^N_t, \quad t \in [0,T], \\ \boldsymbol{X}^N_0 = \begin{pmatrix} X^1_0 \\ \vdots \\ X^N_0 \end{pmatrix}. \end{cases} \tag{6.28}$$

Proof of (2.35). Let us consider U the solution to the backward Kolmogorov PDE (2.33) with terminal condition ϕ at time T given in Theorem 3. Using Lemma 5 and the fact that $U \in \mathcal{C}^1([0,T) \times \mathcal{P}_\beta(\mathbb{R}^d))$, we obtain that the function $(t,\boldsymbol{x}) \in [0,T) \times (\mathbb{R}^d)^N \mapsto U(t,\overline{\mu}^N_{\boldsymbol{x}})$ belongs to $\mathcal{C}^1([0,T) \times (\mathbb{R}^d)^N)$. Moreover, for all $t \in [0,T)$ and $\boldsymbol{x} = (x_1,\dots,x_N) \in (\mathbb{R}^d)^N$, we have

$$\partial_{\boldsymbol{x}} U(t,\overline{\mu}^N_{\boldsymbol{x}}) = \frac{1}{N} \begin{pmatrix} \partial_v \frac{\delta}{\delta m} U(t,\overline{\mu}^N_{\boldsymbol{x}})(x_1) \\ \vdots \\ \partial_v \frac{\delta}{\delta m} U(t,\overline{\mu}^N_{\boldsymbol{x}})(x_N) \end{pmatrix}.$$

Let us fix $\gamma \in (0,1] \cap (\alpha-1, (\delta+\alpha-1) \wedge (2\alpha-2) \wedge (\eta+\alpha-1))$. We easily see using (6.3) that the map $\boldsymbol{x} \mapsto \partial_{\boldsymbol{x}} U(t,\overline{\mu}^N_{\boldsymbol{x}})$ is γ-Hölder continuous locally uniformly with respect to $t \in [0,T)$. This comes from the fact that for all $\boldsymbol{x} = (x_1,\dots,x_N), \boldsymbol{y} = (y_1,\cdot,y_N) \in (\mathbb{R}^d)^N$,

$$W_1^\gamma(\overline{\mu}^N_{\boldsymbol{x}}, \overline{\mu}^N_{\boldsymbol{y}}) \le \frac{1}{N^\gamma} \sum_{k=1}^N |x_k - y_k|^\gamma.$$

We denote by $\boldsymbol{\mathcal{N}}^N$ the Poisson random measure associated with $\boldsymbol{Z}^N = (Z^1,\dots,Z^N)$ defined in (6.26) and by $\boldsymbol{\nu}^N$ its associated Lévy measure defined in (6.27). Since $\gamma > \alpha - 1$, we can apply the standard Itô formula for this function and the $(\mathbb{R}^d)^N$-valued process $(\boldsymbol{X}^N_t)_t$. Noticing that $t \in [0,T] \mapsto U(t,\mu_t)$ is constant, we obtain

that for all $t \in [0,T)$

$$
\begin{aligned}
&U(t,\overline{\mu}_t^N) - U(t,\mu_t) - \big(U(0,\overline{\mu}_0^N) - U(0,\mu_0)\big) \\
&= \int_0^t \partial_t U(s,\overline{\mu}_s^N)\, ds \\
&\quad + \frac{1}{N}\sum_{i=1}^N \int_0^t \partial_v \frac{\delta}{\delta m} U(s,\overline{\mu}_s^N)(X_s^{i,N}) \cdot b(s, X_s^{i,N}, \overline{\mu}_s^N)\, ds \\
&\quad + \int_0^t \int_{(\mathbb{R}^d)^N} \left[U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\boldsymbol{z}}) - U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) - \partial_{\boldsymbol{x}} U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \cdot \boldsymbol{z} \right] d\boldsymbol{\nu}^N(\boldsymbol{z})\, ds \qquad (6.29) \\
&\quad + \int_0^t \int_{(B_1^c)^N} \left[U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\boldsymbol{z}}) - U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right] \widetilde{\boldsymbol{\mathcal{N}}}^N(ds, d\boldsymbol{z}) \\
&\quad + \int_0^t \int_{(B_1)^N} \left[U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\boldsymbol{z}}) - U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right] \widetilde{\boldsymbol{\mathcal{N}}}^N(ds, d\boldsymbol{z}) \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

Note that

$$
\int_0^t \int_{(B_1^c)^N} \left[U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\boldsymbol{z}}) - U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right] d\boldsymbol{\nu}^N(\boldsymbol{z})\, ds
$$

is well-defined. Indeed, if we set, for $h \in \mathbb{R}^d$, $\tilde{\boldsymbol{h_j}} := (0,\dots,0,h,0,\dots,0) \in (\mathbb{R}^d)^N$, where h appears in the j-th coordinate, one has using (6.2)

$$
\begin{aligned}
&\int_0^t \int_{(B_1^c)^N} \left| U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\boldsymbol{z}}) - U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right| d\boldsymbol{\nu}^N(\boldsymbol{z})\, ds \\
&= \sum_{i=1}^N \int_0^t \int_{B_1^c} \left| U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\tilde{\boldsymbol{z_i}}}) - U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right| d\nu(z)\, ds \\
&\le \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{B_1^c} \int_0^1 \left| \partial_v \frac{\delta}{\delta m} U(s, m^i_{s,z,w})(X_{s^-}^{i,N} + hz) \right| |z|\, dw\, d\nu(z)\, ds \\
&\le C \int_0^t (T-s)^{\frac{\delta-1}{\alpha}}\, ds \int_{B_1^c} |z|\, d\nu(z),
\end{aligned}
$$

where $m^i_{s,z,w} := w\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\tilde{\boldsymbol{z_i}}} + (1-w)\overline{\mu}^N_{\boldsymbol{X}^N_{N,s^-}}$. We conclude since $\alpha \in (1,2)$.

By (6.27), we write

$$
\begin{aligned}
I_3 &= \sum_{i=1}^N \int_0^t \int_{(\mathbb{R}^d)^N} \Big[U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\tilde{\boldsymbol{z_i}}}) - U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \\
&\qquad\qquad - \frac{1}{N} \partial_v \frac{\delta}{\delta m} U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X_{s^-}^{i,N}) \cdot z \Big]\, d\nu(z)\, ds \\
&= \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d} \int_0^1 \Big[\frac{\delta}{\delta m} U(s, m^i_{s,z,w})(X_{s^-}^{i,N} + z) \\
&\qquad\qquad - \frac{\delta}{\delta m} U(s, m^i_{s,z,w})(X_{s^-}^{i,N}) - \partial_v \frac{\delta}{\delta m} U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X_{s^-}^{i,N}) \cdot z \Big]\, dw\, d\nu(z)\, ds,
\end{aligned}
$$

where $m^i_{s,z,w} = w\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\tilde{\boldsymbol{z_i}}} + (1-w)\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}$. In order to make appear the backward Kolmogorov PDE (2.33), we decompose I_3 in the following way

$$
\begin{aligned}
I_3 = &\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{\delta}{\delta m} U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(x+z) - \frac{\delta}{\delta m} U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(x) - \partial_v \frac{\delta}{\delta m} U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(x) \cdot z \right] d\nu(z)\, d\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}(x)\, ds \\
&+ \frac{1}{N} \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d} \int_0^1 \left[\frac{\delta}{\delta m} U(s, m^i_{s,z,w})(X^{i,N}_{s^-} + z) - \frac{\delta}{\delta m} U(s, m^i_{s,z,w})(X^{i,N}_{s^-}) \right. \\
&\qquad \left. + \frac{\delta}{\delta m} U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X^{i,N}_{s^-}) - \frac{\delta}{\delta m} U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X^{i,N}_{s^-} + z) \right] dw\, d\nu(z)\, ds \\
=: &\, I_{3,A} + I_{3,B}.
\end{aligned}
$$

Since $(\boldsymbol{X}^N_s)_{s\in[0,T]}$ is càd-làg, we deduce that almost surely for almost all $s \in [0,t]$ we have $\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}} = \overline{\mu}^N_{\boldsymbol{X}^N_s} = \overline{\mu}^N_s$. Owing to the backward Kolmogorov PDE (2.33) in Theorem 3, one has

$$
\begin{aligned}
I_1 + I_2 + I_{3,A} &= \int_0^t \partial_s U(s, \overline{\mu}^N_{N,s}) + \mathscr{L}_s U(s, \cdot)(\overline{\mu}^N_s)\, ds \\
&= 0.
\end{aligned}
$$

Thus, we obtain the following decomposition

$$
\begin{aligned}
&U(t, \overline{\mu}^N_t) - U(t, \mu_t) - \big(U(0, \overline{\mu}^N_0) - U(0, \mu_0) \big) \\
&= \frac{1}{N} \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d} \int_0^1 \left[\frac{\delta}{\delta m} U(s, m^i_{s,z,w})(X^{i,N}_{s^-} + z) - \frac{\delta}{\delta m} U(s, m^i_{s,z,w})(X^{i,N}_{s^-}) \right. \\
&\qquad \left. + \frac{\delta}{\delta m} U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X^{i,N}_{s^-}) - \frac{\delta}{\delta m} U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X^{i,N}_{s^-} + z) \right] dw\, d\nu(z)\, ds \\
&\quad + \int_0^t \int_{B_1^c} \left[U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-} + \boldsymbol{z}}) - U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right] \widetilde{\boldsymbol{\mathcal{N}}}^N(ds, d\boldsymbol{z}) \\
&\quad + \int_0^t \int_{B_1} \left[U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-} + \boldsymbol{z}}) - U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right] \widetilde{\boldsymbol{\mathcal{N}}}^N(ds, d\boldsymbol{z}) \\
&= I_{3,B} + I_4 + I_5.
\end{aligned} \tag{6.30}
$$

It follows that for all $t \in [0,T)$

$$
\mathbb{E}|U(t, \overline{\mu}^N_t) - U(t, \mu_t)| \le \mathbb{E}|U(0, \overline{\mu}^N_0) - U(0, \mu_0)| + \mathbb{E}(|I_{3,B}| + |I_4| + |I_5|). \tag{6.31}
$$

We now treat each term separately. For $I_{3,B}$, one has

$$\begin{aligned}
\mathbb{E}|I_{3,B}| \leq \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{B_1}\int_0^1 \Big|\frac{\delta}{\delta m}U(s,m^i_{s,z,w})(X^{i,N}_{s^-}+z) - \frac{\delta}{\delta m}U(s,m^i_{s,z,w})(X^{i,N}_{s^-}) \\
+ \frac{\delta}{\delta m}U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X^{i,N}_{s^-}) - \frac{\delta}{\delta m}U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X^{i,N}_{s^-}+z)\Big| \, dw\, d\nu(z)\, ds \\
+ \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{B_1^c}\int_0^1 \Big|\frac{\delta}{\delta m}U(s,m^i_{s,z,w})(X^{i,N}_{s^-}+z) - \frac{\delta}{\delta m}U(s,m^i_{s,z,w})(X^{i,N}_{s^-}) \\
+ \frac{\delta}{\delta m}U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X^{i,N}_{s^-}) - \frac{\delta}{\delta m}U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X^{i,N}_{s^-}+z)\Big| \, dw\, d\nu(z)\, ds \\
\leq \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{B_1}\int_0^1\int_0^1 \Big|\partial_v\frac{\delta}{\delta m}U(s,m^i_{s,z,w})(X^{i,N}_{s^-}+\lambda z) \\
-\partial_v\frac{\delta}{\delta m}U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X^{i,N}_{s^-}+\lambda z)\Big| \, |z|\, d\lambda\, dw\, d\nu(z)\, ds \\
+ \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{B_1^c}\int_0^1 \Big|\frac{\delta}{\delta m}U(s,m^i_{s,z,w})(X^{i,N}_{s^-}+z) - \frac{\delta}{\delta m}U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X^{i,N}_{s^-}+z)\Big| \\
+ \Big|\frac{\delta}{\delta m}U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}})(X^{i,N}_{s^-}) - \frac{\delta}{\delta m}U(s,m^i_{s,z,w})(X^{i,N}_{s^-})\Big| \, dw\, d\nu(z)\, ds.
\end{aligned}$$

Recall that $m^i_{s,z,w} = w\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\tilde{\boldsymbol{z}}_{\boldsymbol{i}}} + (1-w)\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}$. Moreover, one has

$$\begin{aligned}
W_1(m^i_{s,z,w},\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) &\leq W_1(\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\tilde{\boldsymbol{z}}_{\boldsymbol{i}}},\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \\
&\leq \frac{|z|}{N}.
\end{aligned}$$

Then, by (6.3) with $\gamma \in (0,1] \cap (\alpha-1, (\delta+\alpha-1)\wedge(2\alpha-2)\wedge(\eta+\alpha-1))$ and (6.4), we deduce that

$$\begin{aligned}
\mathbb{E}|I_{3,B}| \leq \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{B_1} (T-s)^{\frac{\delta-1-\gamma}{\alpha}} \frac{|z|^{1+\gamma}}{N^\gamma}\, d\nu(z)\, ds \\
+ \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{B_1^c} (T-s)^{\frac{\delta-1}{\alpha}} \frac{|z|}{N} d\nu(z)\, ds.
\end{aligned}$$

Since $\gamma < \delta+\alpha-1$, we have $\frac{\delta-1-\gamma}{\alpha} > -1$ and therefore the map $s \in [0,T) \mapsto (T-s)^{\frac{\delta-1-\gamma}{\alpha}}$ is integrable and $z \in B_1 \mapsto |z|^{1+\gamma} \in L^1(B_1,\nu)$. It yields

$$\mathbb{E}|I_{3,B}| \leq \frac{C}{N^\gamma}. \tag{6.32}$$

Let us now focus on I_4. Recall that $m^i_{s,z,w} = w\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\tilde{\boldsymbol{z}}_{\boldsymbol{i}}} + (1-w)\overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}$. Thanks to Cauchy-Schwarz inequality and the L^2-isometry of compensated Poisson random integrals, there exists a constant $C>0$ independent of

N such that we have for all $t \in (0,T]$

$$\begin{aligned}
\mathbb{E}|I_4| &\le \left(\mathbb{E}\left(\int_0^t \int_{(B_1)^N} \left| U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\boldsymbol{z}}) - U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right] \widetilde{\boldsymbol{\mathcal{N}}}^N(ds, d\boldsymbol{z}) \right)^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^N \mathbb{E} \int_0^t \int_{B_1} \left| U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\tilde{\boldsymbol{z}}_{\boldsymbol{i}}}) - U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right|^2 d\nu(z)\, ds \right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^N \mathbb{E} \int_0^t \int_{B_1} \left| \frac{1}{N} \int_{[0,1]^2} \partial_v \frac{\delta}{\delta m} U(s, m^i_{s,z,w})(X^{i,N}_{s^-} + hz) \cdot z \, dh \, dw \right|^2 d\nu(z)\, ds \right)^{\frac{1}{2}} \\
&\le C \left(\sum_{i=1}^N \mathbb{E} \int_0^t \int_{B_1} \frac{1}{N^2} (T-s)^{\frac{2\delta-2}{\alpha}} |z|^2 \, d\nu(z)\, ds \right)^{\frac{1}{2}} \\
&\le \frac{C}{\sqrt{N}}.
\end{aligned} \tag{6.33}$$

Finally for I_5, BDG's inequalities and the fact that $1 < \beta < 2$ yields for all $t \in [0,T)$

$$\begin{aligned}
\mathbb{E}|I_5| &\le \left(\mathbb{E}\left(\int_0^t \int_{(B_1^c)^N} \left| U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\boldsymbol{z}}) - U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right] \widetilde{\boldsymbol{\mathcal{N}}}^N(ds, d\boldsymbol{z}) \right)^\beta \right)^{\frac{1}{\beta}} \\
&\le C \left(\mathbb{E}\left[\int_0^t \int_{(B_1^c)^N} \left| U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\boldsymbol{z}}) - U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right|^2 \boldsymbol{\mathcal{N}}^N(ds, d\boldsymbol{z}) \right]^{\frac{\beta}{2}} \right)^{\frac{1}{\beta}} \\
&\le C \left(\mathbb{E} \int_0^t \int_{(B_1^c)^N} \left| U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}+\boldsymbol{z}}) - U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s^-}}) \right|^\beta \boldsymbol{\mathcal{N}}^N(ds, d\boldsymbol{z}) \right)^{\frac{1}{\beta}} \\
&= C \left(\sum_{i=1}^N \mathbb{E} \int_0^t \int_{B_1^c} \left| \frac{1}{N} \int_{[0,1]^2} \partial_v \frac{\delta}{\delta m} U(s, m^i_{s,z,w})(X^{i,N}_{s^-} + hz) \cdot z \, dh \, dw \right|^\beta d\nu(z)\, ds \right)^{\frac{1}{\beta}} \\
&\le C \left(\sum_{i=1}^N \mathbb{E} \int_0^t \int_{B_1^c} \left| \frac{1}{N} (T-s)^{\frac{\delta-1}{\alpha}} \right|^\beta |z|^\beta \, d\nu(z)\, ds \right)^{\frac{1}{\beta}} \\
&\le \frac{C}{N^{1-\frac{1}{\beta}}}.
\end{aligned} \tag{6.34}$$

As a consequence of (6.30), (6.32), (6.33), (6.34), and the fact that $\gamma \wedge \frac{1}{2} > 1 - \frac{1}{\beta}$ since $1 < \beta < \alpha < 1 + \gamma$, we have for a constant $C > 0$ that for all $N \ge 1$ and $t \in [0,T)$

$$\mathbb{E}|U(t, \overline{\mu}^N_t) - U(t, \mu_t)| \le \mathbb{E}|U(0, \overline{\mu}^N_0) - U(0, \mu_0)| + \frac{C}{N^{1-\frac{1}{\beta}}}. \tag{6.35}$$

It follows from (6.2) that the map $\frac{\delta}{\delta m} U(0,\mu)(\cdot)$ is Lipschitz uniformly with respect to $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$. The Kantorovich-Rubinstein theorem ensures that

$$\mathbb{E}|U(0, \overline{\mu}^N_0) - U(0, \mu_0)| \le C W_1(\overline{\mu}^N_0, \mu_0).$$

By Fatou's lemma, the continuity of U on $[0,T] \times \mathcal{P}_\beta(\mathbb{R}^d)$ and since $\overline{\mu}^N_{T^-} = \overline{\mu}^N_T$ almost surely, we can let t tends to T in (6.35) which ends the proof of (2.35).

Proof of (2.36) and (2.37). Coming back to (6.30) and using (6.33) and (6.34), we see that I_4 an I_5 are centered random variables because of the martingale property of compensated Poisson random integrals. We thus obtain that

$$|\mathbb{E}(U(t,\overline{\mu}_t^N)-U(t,\mu_t))| \le |\mathbb{E}(U(0,\overline{\mu}_0^N)-U(0,\mu_0))| + \mathbb{E}|I_{3,B}|.$$

The initial data term is handled in the same way as in the proof of (2.36) and the control of $\mathbb{E}|I_{3,B}|$ has already been done in (6.32). We thus get that for all $t \in [0,T)$

$$|\mathbb{E}(U(t,\overline{\mu}_t^N)-U(t,\mu_t))| \le C\,\mathbb{E}W^1(\overline{\mu}_0^N,\mu_0) + \frac{C}{N^\gamma},$$

and we conclude by letting t tend to T that (2.36) holds true. Finally, we prove (2.37) using the same argument as in [7]. Indeed, reasoning as for $\frac{\delta}{\delta m}U$ and the estimate (6.1), we can prove that $U(0,\cdot)$ admits a second order linear derivative $\frac{\delta^2}{\delta m^2}U(0,\cdot)(\cdot,\cdot)$ and that it satisfies that for all $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $v,v' \in \mathbb{R}^d$

$$\left|\frac{\delta^2}{\delta m^2}U(0,\mu)(v,v')\right| \le C(1+|v|^\delta)(1+|v'|^\delta),$$

for some positive constant C. Reasoning as in [7] (after (5.24)), we obtain that

$$\mathbb{E}(U(0,\overline{\mu}_0^N)-U(0,\mu_0)) = \frac{1}{N}\int_0^1\int_0^1 \lambda_1 \mathbb{E}\left(\frac{\delta^2}{\delta m^2}U(0,\tilde{\mu}_0^{N,\lambda_1,\lambda_2})(\tilde{\xi},\tilde{\xi}) - \frac{\delta^2}{\delta m^2}U(0,\tilde{\mu}_0^{N,\lambda_1,\lambda_2})(\tilde{\xi},\xi^1)\right) d\lambda_1\, d\lambda_2,$$

where $\mu_0^{N,\lambda_1} := \lambda_1\overline{\mu}_0^N + (1-\lambda_1)\mu_0$, $\tilde{\mu}_0^{N,\lambda_1} := \lambda_1\tilde{\mu}_0^N + (1-\lambda_1)\mu_0$, $\tilde{\mu}_0^N := \overline{\mu}_0^N + \frac{1}{N}(\delta_{\tilde{\xi}} - \delta_{\xi^1})$, $\tilde{\mu}_0^{N,\lambda_1,\lambda_2} := \lambda_2\tilde{\mu}_0^{N,\lambda_1} + (1-\lambda_2)\mu_0^{N,\lambda_1}$, $(\xi^i)_i$ being an i.i.d. sequence of random variable with common distribution μ_0 and $\tilde{\xi}$ a random variable independent of $(\xi^i)_i$ with distribution μ_0. This ensures that there exists a positive constant C such that for all $\phi \in \mathcal{C}^{\delta,\delta}(\mathcal{P}_\beta(\mathbb{R}^d))$ and $N \ge 1$

$$|U(0,\overline{\mu}_0^N)-U(0,\mu_0)| \le \frac{C}{N},$$

because $\mu_0 \in \mathcal{P}_{2\delta}(\mathbb{R}^d)$. This concludes the proof of (2.37).

□

7. Proof of Proposition 3

This section is dedicated to prove Proposition 3, which is rather long and technical. We proceed by induction on $m \ge 1$. The base case $m=1$ is immediate. It is enough to apply Theorem 5 since $p_1(\mu,s,t,x,y)$ does not depend on μ. For the induction step, we assume that all the estimates of Proposition 3 are satisfied for p_m. For the sake of clarity, we denote by K a positive constant depending only on (d,α,b,T) appearing in the induction step and independent of the induction assumption, which may change from line to line and which will determine the choice of the constant C appearing in Proposition 3. Let us introduce a notation used in this section. If $f:(\mu,s,x,v) \in \mathcal{P}_\beta(\mathbb{R}^d)\times[0,t)\times(\mathbb{R}^d)^2$ is a function, we define $\Delta_{\mu_1,\mu_2}f(\cdot,s,x,v) := f(\mu_1,s,x,v) - f(\mu_2,s,x,v)$ for $\mu_1,\mu_2 \in \mathcal{P}_\beta(\mathbb{R}^d)$. The same notations holds with respect to the variables x,v. For the time variable, if $s_1,s_2 \in [0,t)$, we define $\Delta_{s_1,s_2}f(\mu,\cdot,x,v) := f(s_1\vee s_2,x,v) - f(s_1\wedge s_2,x,v)$.

7.1. Preparatory technical results.

Lemma 1. • There exists $K>0$ such that for all $0 \le s \le r \le T$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x,v \in \mathbb{R}^d$, one has

$$\left|\frac{\delta}{\delta m}\left[b(r,x,[X_r^{s,\mu,(m)}])\right](v)\right| \le K\left[1+\sum_{k=1}^m C^k(r-s)^{k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right]. \tag{7.1}$$

- There exists $K>0$ such that for all $0\le s\le r\le T$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $x,v\in\mathbb{R}^d$, one has

$$\left|\partial_v\frac{\delta}{\delta m}\left[b(r,x,[X_r^{s,\mu,(m)}])\right](v)\right| \le K(r-s)^{\frac{\eta-1}{\alpha}}\left[1+\sum_{k=1}^m C^k(r-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+\frac{\eta-1}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right]. \quad (7.2)$$

- There exists $K>0$ such that for all $0\le s\le r<t\le T$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $x,y,v\in\mathbb{R}^d$, one has

$$\left|\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,s,r,t,x,y)(v)\right| \le K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x) \left[1+\sum_{k=1}^m C^k(r-s)^{k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right]. \quad (7.3)$$

- There exists $K>0$ such that for all $0\le s\le r<t\le T$, $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $x,y,v\in\mathbb{R}^d$, one has

$$\left|\partial_v\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,s,r,t,x,y)(v)\right| \le K(r-s)^{\frac{\eta-1}{\alpha}}(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x) \left[1+\sum_{k=1}^m C^k(r-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+\frac{\eta-1}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right]. \quad (7.4)$$

Proof of Lemma 1. Using the induction assumption, we deduce that for all $y\in\mathbb{R}^d$, the map $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)\mapsto p_m(\mu,s,r,y)$ has a linear derivative given, for all $v\in\mathbb{R}^d$, by

$$\frac{\delta}{\delta m}p_m(\mu,s,t,y)(v)=p_m(\mu,s,r,v,y)+\int_{\mathbb{R}^d}\frac{\delta}{\delta m}p_m(\mu,s,r,x',y)(v)\,d\mu(x').$$

Thus, the map $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)\mapsto b(r,x,[X_r^{s,\mu,(m)}])$ has a linear derivative given by

$$\begin{aligned}\frac{\delta}{\delta m}\left[b(r,x,[X_r^{s,\mu,(m)}])\right](v)&=\int_{\mathbb{R}^d}\frac{\delta}{\delta m}b(r,x,[X_r^{s,\mu,(m)}])(z)p_m(\mu,s,r,v,z)\,dz \\ &\quad+\int_{\mathbb{R}^{2d}}\frac{\delta}{\delta m}b(r,x,[X_r^{s,\mu,(m)}])(z)\frac{\delta}{\delta m}p_m(\mu,s,r,x',z)(v)\,dz\,d\mu(x').\end{aligned} \quad (7.5)$$

Using the boundedness of $\frac{\delta}{\delta m}b$ on $[0,T]\times\mathbb{R}^d\times\mathcal{P}_\beta(\mathbb{R}^d)\times\mathbb{R}^d$, (4.18), and the induction assumption, we deduce that

$$\begin{aligned}&\left|\frac{\delta}{\delta m}\left[b(r,x,[X_r^{s,\mu,(m)}])\right](v)\right| \\ &\le K+K\int_{\mathbb{R}^{2d}}\left|\frac{\delta}{\delta m}p_m(\mu,s,r,x',z)(v)\right|dz\,d\mu(x') \\ &\le K+K\int_{\mathbb{R}^{2d}}\left(\sum_{k=1}^m C^k(r-s)^{(k-1)\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right) \\ &\qquad\qquad(r-s)^{1-\frac{1}{\alpha}}\rho^0(r-s,z-x')\,dz\,d\mu(x') \\ &\le K+K\sum_{k=1}^m C^k(r-s)^{k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right).\end{aligned}$$

This proves (7.1). Let us now focus on (7.2). Note that for all $v, x' \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \partial_x p_m(\mu, s, t, v, z)\, dz = \int_{\mathbb{R}^d} \partial_v \frac{\delta}{\delta m} p_m(\mu, s, t, x', z)(v)\, dz = 0.$$

Thus, using the induction assumption and differentiating (7.5) with respect to v, one has for all $v \in \mathbb{R}^d$

$$\begin{aligned} &\partial_v \frac{\delta}{\delta m}\left[b(r, x, [X_r^{s,\mu,(m)}])\right](v) \\ &\quad = \int_{\mathbb{R}^d} \left[\frac{\delta}{\delta m} b(r, x, [X_r^{s,\mu,(m)}])(z) - \frac{\delta}{\delta m} b(r, x, [X_r^{s,\mu,(m)}])(v)\right] \partial_x p_m(\mu, s, r, v, z)\, dz \\ &\qquad + \int_{\mathbb{R}^{2d}} \left[\frac{\delta}{\delta m} b(r, x, [X_r^{s,\mu,(m)}])(z) - \frac{\delta}{\delta m} b(r, x, [X_r^{s,\mu,(m)}])(x')\right] \partial_v \frac{\delta}{\delta m} p_m(\mu, s, r, x', z)(v)\, dz\, d\mu(x'). \end{aligned} \tag{7.6}$$

Using the η-Hölder continuity of $\frac{\delta}{\delta m} b$ with respect to v and uniformly with respect to the other variables, (4.18), and the induction assumption, we deduce that

$$\begin{aligned} &\left|\partial_v \frac{\delta}{\delta m}\left[b(r, x, [X_r^{s,\mu,(m)}])\right](v)\right| \\ &\le K(r-s)^{\frac{\eta-1}{\alpha}} + K \int_{\mathbb{R}^{2d}} |z - x'|^\eta \left|\partial_v \frac{\delta}{\delta m} p_m(\mu, s, r, x', z)(v)\right| dz\, d\mu(x') \\ &\le K(r-s)^{\frac{\eta-1}{\alpha}} + K \int_{\mathbb{R}^{2d}} \left(\sum_{k=1}^m C^k (r-s)^{(k-1)(1+\frac{\eta-1}{\alpha})} \prod_{j=1}^{k-1} \mathcal{B}\left(1 + \frac{\eta-1}{\alpha} + j\left(1 + \frac{\eta-1}{\alpha}\right), 1 - \frac{1}{\alpha}\right)\right) \\ &\qquad\qquad (r-s)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}} |z - x'|^\eta \rho^0(r-s, z-x')\, dz\, d\mu(x') \\ &\le K(r-s)^{\frac{\eta-1}{\alpha}} + K(r-s)^{\frac{\eta-1}{\alpha}} \sum_{k=1}^m C^k (r-s)^{k(1+\frac{\eta-1}{\alpha})} \prod_{j=1}^{k-1} \mathcal{B}\left(1 + \frac{\eta-1}{\alpha} + j\left(1 + \frac{\eta-1}{\alpha}\right), 1 - \frac{1}{\alpha}\right). \end{aligned}$$

It ends the proof of (7.2). Using (7.1), it is clear that $\mathcal{H}_{m+1}$ has a linear derivative given for all $v \in \mathbb{R}^d$ by

$$\frac{\delta}{\delta m} \mathcal{H}_{m+1}(\mu, s, r, t, x, y)(v) = \left(\frac{\delta}{\delta m}\left[b(r, x, [X_r^{s,\mu,(m)}])\right](v)\right) \cdot \partial_x \widehat{p}(r, t, x, y), \tag{7.7}$$

and that

$$\partial_v \frac{\delta}{\delta m} \mathcal{H}_{m+1}(\mu, s, r, t, x, y)(v) = \left(\partial_v \frac{\delta}{\delta m}\left[b(r, x, [X_r^{s,\mu,(m)}])\right](v)\right) \cdot \partial_x \widehat{p}(r, t, x, y). \tag{7.8}$$

Thus, (7.3) and (7.4) follow directly from (7.1), (7.2), and (B.17). □

Lemma 2. For any $k \ge 1$, $0 \le s \le r < t \le T$, $x, y \in \mathbb{R}^d$, $\mathcal{H}_{m+1}^k(\cdot, s, r, t, x, y)$ has a linear derivative which is $\mathcal{C}^1$ and satisfies the following properties.

- There exists a positive constant K_m depending on m and a positive constant C independent of k and m such that for all $k \ge 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \le s \le r < t \le T$, $x, y, v \in \mathbb{R}^d$, one has

$$\begin{aligned} \left|\frac{\delta}{\delta m} \mathcal{H}_{m+1}^k(\mu, s, r, t, x, y)(v)\right| &\le K_m k C^{k-1} (t-r)^{-\frac{1}{\alpha} + (k-1)(1-\frac{1}{\alpha})} \\ &\qquad \prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1 - \frac{1}{\alpha}\right), 1 - \frac{1}{\alpha}\right) \rho^1(t-r, y-x), \end{aligned} \tag{7.9}$$

and

$$\left|\partial_v \frac{\delta}{\delta m}\mathcal{H}^k_{m+1}(\mu,s,r,t,x,y)(v)\right| \le K_m k C^{k-1}(r-s)^{\frac{\eta-1}{\alpha}}(t-r)^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^1(t-r,y-x). \quad (7.10)$$

- There exists a positive constant K_m depending on m and a positive constant C independent of k and m such that for all $k \ge 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \le s < t \le T$, $x, y, v \in \mathbb{R}^d$, one has

$$\left|\widehat{p} \otimes \frac{\delta}{\delta m}\mathcal{H}^k_{m+1}(\mu,s,t,x,y)(v)\right| \le K_m k C^{k-1}(t-s)^{1-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)}\mathcal{B}\left(k\left(1-\frac{1}{\alpha}\right),1\right) \prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^0(t-s,y-x), \quad (7.11)$$

and

$$\left|\widehat{p} \otimes \partial_v \frac{\delta}{\delta m}\mathcal{H}^k_{m+1}(\mu,s,t,x,y)(v)\right| \le K_m k C^{k-1}(t-s)^{\frac{\eta-1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)}\mathcal{B}\left(k\left(1-\frac{1}{\alpha}\right),1+\frac{\eta-1}{\alpha}\right) \prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^0(t-s,y-x). \quad (7.12)$$

Proof of Lemma 2. We proceed by induction on k to prove (7.9) for all $m \ge 1$. The base case $k = 1$ is a direct consequence of (7.3). Assume now that (7.9) holds for $\mathcal{H}^k_{m+1}$ and let us prove it for $\mathcal{H}^{k+1}_{m+1}$. By definition, we have

$$\mathcal{H}^{k+1}_{m+1}(\mu,s,r,t,x,y) = \int_r^t \int_{\mathbb{R}^d} \mathcal{H}_{m+1}(\mu,s,r,r',x,z)\mathcal{H}^k_{m+1}(\mu,s,r',t,z,y)\,dz\,dr'.$$

Using Lemma 1 and the induction assumption, we deduce that $\mathcal{H}^{k+1}_{m+1}(\cdot,s,r,t,x,y)$ has a linear derivative given by

$$\frac{\delta}{\delta m}\mathcal{H}^{k+1}_{m+1} = \frac{\delta}{\delta m}\mathcal{H}_{m+1} \otimes \mathcal{H}^k_{m+1} + \mathcal{H}_{m+1} \otimes \frac{\delta}{\delta m}\mathcal{H}^k_{m+1}.$$

Using (4.13), the induction assumption (7.9) and the convolution inequality (B.16), we deduce that

$$
\begin{aligned}
&\left|\frac{\delta}{\delta m}\mathcal{H}_{m+1}^{k+1}(\mu,s,r,t,x,y)(v)\right| \\
&\leq K_m \int_r^t \int_{\mathbb{R}^d} (r'-r)^{-\frac{1}{\alpha}} \rho^1(r'-r,z-x) C^k (t-r')^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)} \\
&\qquad\qquad \prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \rho^1(t-r',y-z)\,dz\,dr' \\
&\quad + \int_r^t \int_{\mathbb{R}^d} K(r'-r)^{-\frac{1}{\alpha}} \rho^1(r'-r,z-x) K_m k C^{k-1} (t-r')^{(k-1)\left(1-\frac{1}{\alpha}\right)} \\
&\qquad\qquad \prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) (t-r')^{-\frac{1}{\alpha}} \rho^1(t-r',y-z)\,dz\,dr \\
&\leq K_m C^k (t-r)^{-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \rho^1(t-r,y-x) \\
&\quad + K K_m k C^{k-1} (t-r)^{-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \rho^1(t-r,y-x) \\
&\leq K_m (k+1) C^k (t-r)^{-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \rho^1(t-r,y-x),
\end{aligned}
$$

if we chose $C \geq K$. Following the same lines, we prove that for any $k \geq 1$, $\frac{\delta}{\delta m}\mathcal{H}_{m+1}^k$ is $\mathcal{C}^1$ with respect to $v \in \mathbb{R}^d$ and that it satisfies

$$
\partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1}^{k+1} = \partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1} \otimes \mathcal{H}_{m+1}^k + \mathcal{H}_{m+1} \otimes \partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1}^k.
$$

Using (4.13), the induction assumption (7.10) and the convolution inequality (B.16), we deduce that

$$\begin{aligned}
&\left|\partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1}^{k+1}(\mu,s,r,t,x,y)(v)\right| \\
&\le K_m \int_r^t \int_{\mathbb{R}^d} (r-s)^{\frac{\eta-1}{\alpha}}(r'-r)^{-\frac{1}{\alpha}}\rho^1(r'-r,z-x)C^k(t-r')^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)} \\
&\qquad\qquad \prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^1(t-r',y-z)\,dz\,dr' \\
&\quad + \int_r^t \int_{\mathbb{R}^d} K(r'-r)^{-\frac{1}{\alpha}}\rho^1(r'-r,z-x)K_m k C^{k-1}(r'-s)^{\frac{\eta-1}{\alpha}}(t-r')^{(k-1)\left(1-\frac{1}{\alpha}\right)} \\
&\qquad\qquad \prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)(t-r')^{-\frac{1}{\alpha}}\rho^1(t-r',y-z)\,dz\,dr \\
&\le K_m C^k (r-s)^{\frac{\eta-1}{\alpha}}(t-r)^{-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^1(t-r,y-x) \\
&\quad + KK_m k C^{k-1}(r-s)^{\frac{\eta-1}{\alpha}}(t-r)^{-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^1(t-r,y-x) \\
&\le K_m(k+1)C^k(r-s)^{\frac{\eta-1}{\alpha}}(t-r)^{-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^1(t-r,y-x),
\end{aligned}$$

if we chose $C \ge K$. It proves (7.10). The estimates (7.11) and (7.12) follows immediately from (7.9), (7.10), Lemma 11 and the convolution inequality (B.16).

□

Lemma 3.

- For any $0 \le s < t \le T$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, we have

$$\left|\partial_s\left[b(t,x,[X_t^{s,\mu,(m)}])\right]\right| \le K\int_{\mathbb{R}^{2d}}(1\wedge|x'-y|^\eta)|\partial_s p_m(\mu,s,t,x,y)|\,dy\,d\mu(x'). \tag{7.13}$$

- For any $0 \le s < r < t \le T$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x,y \in \mathbb{R}^d$, we have

$$|\partial_s\mathcal{H}_{m+1}(\mu,s,r,t,x,y)| \le K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x)\int_{\mathbb{R}^{2d}}(1\wedge|x'-y|^\eta)|\partial_s p_m(\mu,s,r,x,y)|\,dy\,d\mu(x'). \tag{7.14}$$

Proof of Lemma 3. **Proof of (7.13).**

By the induction assumption at step m, we see by the dominated convergence theorem that the map $s \in [0,t) \mapsto b(t,x,[X_t^{s,\mu,(m)}])$ is differentiable and that

$$\partial_s\left[b(t,x,[X_t^{s,\mu,(m)}])\right] = \int_{\mathbb{R}^{2d}}\left(\frac{\delta}{\delta m}b(t,x,[X_t^{s,\mu,(m)}])(y) - \frac{\delta}{\delta m}b(t,x,[X_t^{s,\mu,(m)}])(x')\right)\partial_s p_m(\mu,s,t,x,y)\,d\mu(x')\,dy.$$

The boundedness of $\frac{\delta}{\delta m}b$ and the η-Hölder continuity of $\frac{\delta}{\delta m}b(t,x,\mu)(\cdot)$ allows to conclude.

Proof of (7.14).

Because of the expression (4.10) of $\mathcal{H}_{m+1}$ and since $\widehat{p}(\mu,s,r,t,x,y)$ does not depend on s, (7.14) follows from (7.13) and (B.17). □

Lemma 4.

- There exists a positive constant K_m depending on m and a positive constant C such that for all $k \ge 1$, $m \ge 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \le s < r < t \le T$, $x,y \in \mathbb{R}^d$

$$\left|\partial_s \mathcal{H}^k_{m+1}(\mu,s,r,t,x,y)\right| \le K_m k C^{k-1}(r-s)^{\frac{\eta}{\alpha}-1}(t-r)^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)}\rho^1(t-r,y-x) \prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right). \quad (7.15)$$

- There exists a positive constant K_m depending on m such that for all $k \ge 1$, $m \ge 1$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \le s < r < t \le T$, $x, y \in \mathbb{R}^d$

$$|\partial_s \Phi_{m+1}(\mu,s,r,t,x,y)| \le K_m (r-s)^{\frac{\eta}{\alpha}-1}(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x). \quad (7.16)$$

Proof of Lemma 4. **Proof of (7.15).** We proceed by induction on $k \ge 1$. The base case $k = 1$ comes from (7.14) and the induction assumption (4.22) for $\partial_s p_m$ Indeed, the induction assumption and the space-time inequality (B.13) ensure that

$$\int_{\mathbb{R}^{2d}} (1 \wedge |x'-y|^\eta)|\partial_s p_m(\mu,s,r,x,y)|\,dy\,d\mu(x') \le K_m (r-s)^{\frac{\eta}{\alpha}-1}\int_{\mathbb{R}^{2d}} \rho^{-\tilde\eta-\eta}(r-s,y-x)\,dy\;d\mu(x).$$

We conclude since $\eta+\tilde\eta < \alpha$, the map $\rho^{-\tilde\eta-\eta}(r-s,\cdot)$ belongs to $L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d}\rho^{-\tilde\eta-\eta}(t-s,y)\,dy$ is equal to a constant independent of s and t. For the induction step, we assume that (7.15) holds at step k. By definition, we have

$$\mathcal{H}^{k+1}_{m+1}(\mu,s,r,t,x,y) = \int_r^t\int_{\mathbb{R}^d} \mathcal{H}_{m+1}(\mu,s,r,r',x,z)\mathcal{H}^k_{m+1}(\mu,s,r',t,z,y)\,dz\,dr'.$$

It follows from the induction assumption and the dominated convergence theorem that the map $s \in [0,t) \mapsto \mathcal{H}^{k+1}_{m+1}(\mu,s,r,t,x,y)$ is differentiable and that

$$\begin{aligned}\partial_s \mathcal{H}^{k+1}_{m+1}(\mu,s,r,t,x,y) &= \int_r^t\int_{\mathbb{R}^d} \partial_s\mathcal{H}_{m+1}(\mu,s,r,r',x,z)\mathcal{H}^k_{m+1}(\mu,s,r',t,z,y)\,dz\,dr' \\ &\quad + \int_r^t\int_{\mathbb{R}^d} \mathcal{H}_{m+1}(\mu,s,r,r',x,z)\partial_s\mathcal{H}^k_{m+1}(\mu,s,r',t,z,y)\,dz\,dr' \\ &=: I_1 + I_2.\end{aligned}$$

Using the base case $k = 1$ and (4.13), we deduce that

$$\begin{aligned}|I_1| &\le \int_r^t\int_{\mathbb{R}^d} K_m(r-s)^{\frac{\eta}{\alpha}-1}(r'-r)^{-\frac{1}{\alpha}}\rho^1(r'-r,z-x)C^k(t-r')^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)} \\ &\qquad \prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^1(t-r',y-z)\,dr'\,dz \\ &\le K_m C^k (r-s)^{\frac{\eta}{\alpha}-1}(t-r)^{-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^1(t-r,y-x).\end{aligned}$$

For I_2, from the induction assumption (7.15), (4.13), the convolution inequality (B.16) and since $\frac{\eta}{\alpha} < 1$, we obtain that

$$
\begin{aligned}
|I_2| &\leq \int_r^t \int_{\mathbb{R}^d} K(r'-r)^{-\frac{1}{\alpha}} \rho^1(r'-r, z-x) K_m k C^{k-1} (r'-s)^{\frac{\eta}{\alpha}-1} (t-r')^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)} \\
&\qquad \prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \rho^1(t-r', y-z)\, dz\, dr' \\
&\leq (r-s)^{\frac{\eta}{\alpha}-1} \int_r^t \int_{\mathbb{R}^d} K(r'-r)^{-\frac{1}{\alpha}} \rho^1(r'-r, z-x) K_m k C^{k-1} (t-r')^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)} \\
&\qquad \prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \rho^1(t-r', y-z)\, dz\, dr' \\
&\leq K_m k C^k (r-s)^{\frac{\eta}{\alpha}-1} (t-r)^{-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)} \rho^1(t-r, y-x) \prod_{j=1}^{k} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right),
\end{aligned}
$$

provided that we choose $C \geq K$ in (7.15). This concludes the induction step for (7.15).

Proof of (7.16). Using the definition of Φ_{m+1} (4.12) and (7.15), we obtain by the dominated convergence theorem that $s \in [0, r) \mapsto \phi_{m+1}(\mu, s, r, t, x, y)$ is continuously differentiable with

$$
\partial_s \Phi_{m+1}(\mu, s, r, t, x, y) = \sum_{k=1}^{\infty} \partial_s \mathcal{H}_{m+1}^k(\mu, s, r, t, x, y).
$$

Then, (7.16) follows immediately from (7.15).

□

7.2. **First part of the proof of the induction step.** We split the proof of the induction step into different parts for the sake of clarity. We start by proving the estimates (4.20), (4.21) and (4.22).

Proof of (4.20) and (4.21). We start by showing that for all $0 \leq s < t \leq T$, $x, y \in \mathbb{R}^d$, the map $p_{m+1}(\cdot, s, t, x, y)$ admits a linear derivative given which is $\mathcal{C}^1$ with respect to $v \in \mathbb{R}^d$ and such that for all $v \in \mathbb{R}^d$

$$
\frac{\delta}{\delta m} p_{m+1}(\mu, s, t, x, y)(v) = \sum_{k=1}^{\infty} \widehat{p} \otimes \frac{\delta}{\delta m} \mathcal{H}_{m+1}^k(\mu, s, t, x, y)(v), \tag{7.17}
$$

and

$$
\partial_v \frac{\delta}{\delta m} p_{m+1}(\mu, s, t, x, y)(v) = \sum_{k=1}^{\infty} \widehat{p} \otimes \partial_v \frac{\delta}{\delta m} \mathcal{H}_{m+1}^k(\mu, s, t, x, y)(v). \tag{7.18}
$$

where the series are absolutely convergent. Moreover, we also have the following representation formulas

$$
\frac{\delta}{\delta m} p_{m+1}(\mu, s, t, x, y)(v) = \sum_{k=0}^{\infty} p_{m+1} \otimes \frac{\delta}{\delta m} \mathcal{H}_{m+1} \otimes \mathcal{H}_{m+1}^k(\mu, s, t, x, y)(v), \tag{7.19}
$$

and

$$
\partial_v \frac{\delta}{\delta m} p_{m+1}(\mu, s, t, x, y)(v) = \sum_{k=0}^{\infty} p_{m+1} \otimes \partial_v \frac{\delta}{\delta m} \mathcal{H}_{m+1} \otimes \mathcal{H}_{m+1}^k(\mu, s, t, x, y)(v). \tag{7.20}
$$

To prove (7.17), we fix $\mu_1, \mu_2 \in \mathcal{P}_\beta(\mathbb{R}^d)$ and we write thanks to Lemma 2 and Fubini's theorem

$$
\begin{aligned}
&p_{m+1}(\mu_1,s,t,x,y)-p_{m+1}(\mu_2,s,t,x,y)\\
&=\sum_{k=1}^{\infty}\widehat{p}\otimes\mathcal{H}^k_{m+1}(\mu_1,s,t,x,y)-\widehat{p}\otimes\mathcal{H}^k_{m+1}(\mu_2,s,t,x,y)\\
&=\sum_{k=1}^{\infty}\int_s^t\int_{\mathbb{R}^d}\widehat{p}(s,r,x,z)(\mathcal{H}^k_{m+1}(\mu_1,s,r,t,z,y)-\mathcal{H}^k_{m+1}(\mu_2,s,r,t,z,y))\,dz\,dr\\
&=\sum_{k=1}^{\infty}\int_s^t\int_{\mathbb{R}^d}\widehat{p}(s,r,x,z)\int_0^1\int_{\mathbb{R}^d}\frac{\delta}{\delta m}\mathcal{H}^k_{m+1}(\lambda\mu_1+(1-\lambda)\mu_2,s,r,t,z,y)(v)\,d(\mu_1-\mu_2)(v)\,d\lambda\,dz\,dr\\
&=\int_0^1\int_{\mathbb{R}^d}\sum_{k=1}^{\infty}\int_s^t\int_{\mathbb{R}^d}\widehat{p}(s,r,x,z)\frac{\delta}{\delta m}\mathcal{H}^k_{m+1}(\lambda\mu_1+(1-\lambda)\mu_2,s,r,t,z,y)(v)\\
&\qquad\qquad d(\mu_1-\mu_2)(v)\,d\lambda\,dz\,dr\,d(\mu_1-\mu_2)(v)\,d\lambda\\
&=\int_0^1\int_{\mathbb{R}^d}\sum_{k=1}^{\infty}\widehat{p}\otimes\frac{\delta}{\delta m}\mathcal{H}^k_{m+1}(\lambda\mu_1+(1-\lambda)\mu_2,s,t,x,y)(v)\,d(\mu_1-\mu_2)(v)\,d\lambda.
\end{aligned}
$$

Note that owing to (7.11) the series (7.17) is absolutely convergent, locally uniformly with respect to $(\mu,s,x,v)\in\mathcal{P}_\beta(\mathbb{R}^d)\times[0,t)\times\mathbb{R}^d\times\mathbb{R}^d$. This concludes the proof of (7.17). Moreover, we have proved that the map $(\mu,s,x,v)\mapsto\frac{\delta}{\delta m}p_{m+1}(\mu,s,t,x,y)(v)$ is continuous. By differentiation under the integral using (7.10), we obtain (7.18). The dominated convergence theorem yields the continuity of the map $(\mu,s,x,v)\mapsto\partial_v\frac{\delta}{\delta m}p_{m+1}(\mu,s,t,x,y)(v)$ is continuous. Let us now focus on the representation formula (7.19). Using the parametrix expansion (4.17) of p_{m+1}, one has for all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x,y\in\mathbb{R}^d$

$$
p_{m+1}(\mu,s,t,x,y)=\widehat{p}(s,t,x,y)+p_{m+1}\otimes\mathcal{H}_{m+1}(\mu,s,t,x,y).
$$

We can easily see by induction thanks to (7.9) that for any $k\ge1$, one has

$$
\frac{\delta}{\delta m}\mathcal{H}^k_{m+1}(\mu,s,r,t,x,y)(v)=\sum_{j=1}^{k}\mathcal{H}^{k-j}_{m+1}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}\otimes\mathcal{H}^{j-1}_{m+1}(\mu,s,r,t,x,y)(v).
$$

We plug this expression into (7.17), and since the series is absolutely convergent, we obtain setting $l=k-j$ and $i=j-1$ and by Fubini's theorem that

$$
\begin{aligned}
\frac{\delta}{\delta m}p_{m+1}(\mu,s,t,x,y)(v)&=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\widehat{p}\otimes\mathcal{H}^{k-j}_{m+1}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}\otimes\mathcal{H}^{j-1}_{m+1}(\mu,s,t,x,y)(v)\\
&=\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\widehat{p}\otimes\mathcal{H}^{l}_{m+1}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}\otimes\mathcal{H}^{i}_{m+1}(\mu,s,t,x,y)(v)\\
&=\sum_{i=0}^{\infty}p_{m+1}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}\otimes\mathcal{H}^{i}_{m+1}(\mu,s,t,x,y)(v).
\end{aligned}
$$

This is exactly (7.19). Let us now prove that the estimate (4.20) is still true at step $m+1$. It follows from (7.3), (4.18) and the convolution inequality (B.16) that for some positive constant K independent of m one has

$$\begin{aligned}
&\left|\widehat{p}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,s,t,x,y)(v)\right| \\
&\leq K\int_s^t\int_{\mathbb{R}^d}\left[1+\sum_{k=1}^m C^k(r-s)^{k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right] \\
&\qquad\qquad \rho^0(r-s,z-x)(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr \\
&\leq K(t-s)^{1-\frac{1}{\alpha}}+K\sum_{k=1}^m C^k(t-s)^{(k+1)\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(1+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^0(t-s,y-x).
\end{aligned}$$

Using this inequality, the bound (4.13) for $\mathcal{H}^k_{m+1}$, the convolution inequality (B.16) and summing over $k\geq 0$, we find that there exists a constant K independent of m such that

$$\begin{aligned}
&\left|\frac{\delta}{\delta m}p_{m+1}(\mu,s,t,x,y)(v)\right| \\
&\leq\sum_{k=0}^\infty\left|p_{m+1}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}\otimes\mathcal{H}^k_{m+1}(\mu,s,t,x,y)(v)\right| \\
&\leq K(t-s)^{1-\frac{1}{\alpha}}+K\sum_{k=1}^m C^k(t-s)^{(k+1)\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(1+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^0(t-s,y-x) \\
&\leq (t-s)^{1-\frac{1}{\alpha}}\rho^0(t-s,y-x)\left(\sum_{k=1}^{m+1}C^{k+1}(t-s)^{k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right),
\end{aligned}$$

provided that we choose $C\geq K$ in (4.20). It ends the proof of the induction step for (4.20).

Notice that by differentiating under the integral (7.19) using (4.18), (7.4) and (4.13), we obtain the representation formula (7.20) for $\partial_v\frac{\delta}{\delta m}p_{m+1}$. Let us now prove that the estimate (4.21) is verified at step $m+1$. It follows from (7.4), (4.18) and the convolution inequality (B.16) that for some positive constant K independent of m one has

$$\begin{aligned}
&\left|\widehat{p}\otimes\partial_v\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,s,t,x,y)(v)\right| \\
&\leq K\int_s^t\int_{\mathbb{R}^d}(r-s)^{\frac{\eta-1}{\alpha}}\left[1+\sum_{k=1}^m C^k(r-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+\frac{\eta-1}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right] \\
&\qquad\qquad \rho^0(r-s,z-x)(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr \\
&\leq K(t-s)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}\left[1+\sum_{k=1}^m C^k(t-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(1+\frac{\eta-1}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right]\rho^0(t-s,y-x).
\end{aligned}$$

Using this inequality, the bound (4.13) for $\mathcal{H}^k_{m+1}$, the convolution inequality (B.16) and summing over $k\geq 0$, we find that there exists a constant K independent of m such that

$$
\begin{aligned}
&\left|\partial_v \frac{\delta}{\delta m} p_{m+1}(\mu,s,t,x,y)(v)\right| \\
&\leq \sum_{k=0}^{\infty} \left|p_{m+1} \otimes \partial_v \frac{\delta}{\delta m} \mathcal{H}_{m+1} \otimes \mathcal{H}_{m+1}^k(\mu,s,t,x,y)(v)\right| \\
&\leq K(t-s)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}} \left[1+\sum_{k=1}^{m} C^k (t-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k} \mathcal{B}\left(1+\frac{\eta-1}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right), 1-\frac{1}{\alpha}\right)\right] \\
&\qquad\qquad (t-s)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}} \rho^0(t-s,y-x) \\
&\leq (t-s)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}} \rho^0(t-s,y-x) \left(\sum_{k=1}^{m+1} C^k (t-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(1+\frac{\eta-1}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right), 1-\frac{1}{\alpha}\right)\right),
\end{aligned}
$$

provided that we choose $C \geq K$ in (4.21). It ends the proof of the induction step for (4.21).

Proof of (4.22)**.** Let us first prove the following representation formula for $\partial_s \Phi_{m+1}$

$$
\begin{aligned}
\partial_s \Phi_{m+1}(\mu,s,r,t,x,y) &= \left[\partial_s \mathcal{H}_{m+1} + \partial_s \mathcal{H}_{m+1} \otimes \Phi_{m+1}\right](\mu,s,r,t,x,y) \\
&\quad + \Phi_{m+1} \otimes \left[\partial_s \mathcal{H}_{m+1} + \partial_s \mathcal{H}_{m+1} \otimes \Phi_{m+1}\right](\mu,s,r,t,x,y). \quad (7.21)
\end{aligned}
$$

By differentiating the relation $\Phi_{m+1} = \mathcal{H}_{m+1} + \mathcal{H}_{m+1} \otimes \Phi_{m+1}$, we obtain that

$$
\begin{aligned}
\partial_s \Phi_{m+1}(\mu,s,r,t,x,y) &= \partial_s \mathcal{H}_{m+1}(\mu,s,r,t,x,y) + \partial_s \mathcal{H}_{m+1} \otimes \Phi_{m+1}(\mu,s,r,t,x,y) \\
&\quad + \mathcal{H}_{m+1} \otimes \partial_s \Phi_{m+1}(\mu,s,r,t,x,y).
\end{aligned}
$$

Notice that by (7.15) for $k=1$ and (4.15), we get that

$$
\left|\left[\partial_s \mathcal{H}_{m+1} + \partial_s \mathcal{H}_{m+1} \otimes \Phi_{m+1}\right](\mu,s,r,t,x,y)\right| \leq K_m (r-s)^{\frac{\eta}{\alpha}-1} (t-r)^{-\frac{1}{\alpha}} \rho^1(t-r,y-x).
$$

The kernel $\left[\partial_s \mathcal{H}_{m+1} + \partial_s \mathcal{H}_{m+1} \otimes \Phi_{m+1}\right](\mu,s,r,t,x,y)$ yields a time-integrable singularity in $(r-t)^{-\frac{1}{\alpha}}$, we can iterate the previous relation to obtain that

$$
\begin{aligned}
&\partial_s \Phi_{m+1}(\mu,s,r,t,x,y) \\
&= \sum_{k=0}^{\infty} \mathcal{H}_{m+1}^k \otimes \left[\partial_s \mathcal{H}_{m+1} + \partial_s \mathcal{H}_{m+1} \otimes \Phi_{m+1}\right](\mu,s,r,t,x,y) \\
&= \left[\partial_s \mathcal{H}_{m+1} + \partial_s \mathcal{H}_{m+1} \otimes \Phi_{m+1}\right](\mu,s,r,t,x,y) + \Phi_{m+1} \otimes \left[\partial_s \mathcal{H}_{m+1} + \partial_s \mathcal{H}_{m+1} \otimes \Phi_{m+1}\right](\mu,s,r,t,x,y).
\end{aligned}
$$

In order to deal with the differentiability of the map $s \in [0,t) \mapsto p_{m+1}(\mu,s,t,x,y)$, keeping in mind the parametrix expansion (4.17), we first study the differentiability of the map

$$
s \in [0,r) \mapsto \int_{\mathbb{R}^d} \widehat{p}(s,r,x,z) \Phi_{m+1}(\mu,s,r,t,z,y)\, dz.
$$

Since $\int_{\mathbb{R}^d} \partial_s \widehat{p}(s,r,x,z)\, dz = 0$, we deduce by the dominated convergence theorem that

$$
\begin{aligned}
\partial_s \left(\int_{\mathbb{R}^d} \widehat{p}(s,r,x,z) \Phi_{m+1}(\mu,s,r,t,z,y)\, dz\right) &= \int_{\mathbb{R}^d} \partial_s \widehat{p}(s,r,x,z) \left(\Phi_{m+1}(\mu,s,r,t,z,y) - \Phi_{m+1}(\mu,s,r,t,x,y)\right) dz \\
&\quad + \int_{\mathbb{R}^d} \widehat{p}(s,r,x,z) \partial_s \Phi_{m+1}(\mu,s,r,t,z,y)\, dz \quad (7.22) \\
&=: A_1 + A_2,
\end{aligned}
$$

which is continuous with respect to $(\mu,s,x)\in\mathcal{P}_\beta(\mathbb{R}^d)\times[0,r)\times\mathbb{R}^d$. We now control A_1 and A_2. For A_1, it follows from (B.11) and (4.16) with $\gamma=\tilde\eta$, the space-time inequality (B.13) that

$$\begin{aligned}|A_1|&\le K\int_{\mathbb{R}^d}(r-s)^{-1}\rho^0(r-s,z-x)(t-r)^{-\frac{\tilde\eta+1}{\alpha}}|z-x|^{\tilde\eta}\left[\rho^1(t-r,y-z)+\rho^1(t-r,y-x)\right]\,dz\\&\le K\int_{\mathbb{R}^d}(r-s)^{\frac{\tilde\eta}{\alpha}-1}\rho^{-\tilde\eta}(r-s,z-x)(t-r)^{-\frac{\tilde\eta+1}{\alpha}}\left[\rho^{-\tilde\eta}(t-r,y-z)+\rho^{-\tilde\eta}(t-r,y-x)\right]\,dz.\end{aligned}$$

Note that $\int_{\mathbb{R}^d}\rho^{-\tilde\eta}(r-s,z-x)\,dz$ is a constant independent of s and r . By the definition of $\rho^{-\tilde\eta}$ (B.7), the fact that $r>s$ and the convolution inequality (B.16), one has

$$|A_1|\le K(r-s)^{\frac{\tilde\eta}{\alpha}-1}(t-r)^{-\frac{\tilde\eta+1}{\alpha}}\left[\rho^{-\tilde\eta}(t-s,y-x)+(t-r)^{-\frac{d}{\alpha}}(1+(t-s)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha+\tilde\eta}\right]. \tag{7.23}$$

Concerning A_2, it follows from (7.16) and (B.17) that

$$\begin{aligned}|A_2|&\le\int_{\mathbb{R}^d}K\rho^0(r-s,z-x)K_m(r-s)^{\frac{\eta}{\alpha}-1}(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\\&\le K_m(r-s)^{\frac{\eta}{\alpha}-1}(t-r)^{-\frac{1}{\alpha}}\rho^0(t-s,y-x).\end{aligned}$$

By the dominated convergence theorem justified by the controls previously obtained on A_1 and A_2, we obtain that the map $s\in[0,t)\mapsto\widehat{p}\otimes\Phi_{m+1}(\mu,s,t,x,y)$ is differentiable with

$$\begin{aligned}\partial_s\left(\widehat{p}\otimes\Phi_{m+1}\right)(\mu,s,t,x,y)&=-\Phi_{m+1}(\mu,s,t,x,y)+\partial_s\widehat{p}\otimes\Phi_{m+1}(\mu,s,t,x,y)+\widehat{p}\otimes\partial_s\Phi_{m+1}(\mu,s,t,x,y)\\&=-\Phi_{m+1}(\mu,s,t,x,y)\\&\quad+\int_s^t\int_{\mathbb{R}^d}\partial_s\widehat{p}(s,r,x,z)\left(\Phi_{m+1}(\mu,s,r,t,z,y)-\Phi_{m+1}(\mu,s,r,t,x,y)\right)\,dz\,dr\\&\quad+\int_s^t\int_{\mathbb{R}^d}\widehat{p}(s,r,x,z)\partial_s\Phi_{m+1}(\mu,s,r,t,z,y)\,dz\,dr.\end{aligned}$$

Thus, the map $s\in[0,t)\mapsto p_{m+1}(\mu,s,t,x,y)$ is differentiable with

$$\begin{aligned}\partial_s p_{m+1}(\mu,s,t,x,y)&=\partial_s\widehat{p}(s,t,x,y)-\Phi_{m+1}(\mu,s,t,x,y)\\&\quad+\partial_s\widehat{p}\otimes\Phi_{m+1}(\mu,s,t,x,y)+\widehat{p}\otimes\partial_s\Phi_{m+1}(\mu,s,t,x,y).\end{aligned}\tag{7.24}$$

Then, plugging (7.21) into (7.24), we obtain that

$$\begin{aligned}\partial_s p_{m+1}(\mu,s,t,x,y)&=\partial_s\widehat{p}(s,t,x,y)-\Phi_{m+1}(\mu,s,t,x,y)+\partial_s\widehat{p}\otimes\Phi_{m+1}(\mu,s,t,x,y)\\&\quad+\widehat{p}\otimes\left(\partial_s\mathcal{H}_{m+1}+\partial_s\mathcal{H}_{m+1}\otimes\Phi_{m+1}\right)(\mu,s,t,x,y)\\&\quad+\widehat{p}\otimes\Phi_{m+1}\otimes\left(\partial_s\mathcal{H}_{m+1}+\partial_s\mathcal{H}_{m+1}\otimes\Phi_{m+1}\right)(\mu,s,t,x,y).\end{aligned}$$

By the parametrix expansion (4.17) of p_{m+1}, we obtain the following representation formula

$$\begin{aligned}\partial_s p_{m+1}(\mu,s,t,x,y)&=\partial_s\widehat{p}(s,t,x,y)-\Phi_{m+1}(\mu,s,t,x,y)\\&\quad+\partial_s\widehat{p}\otimes\Phi_{m+1}(\mu,s,t,x,y)\\&\quad+p_{m+1}\otimes\left(\partial_s\mathcal{H}_{m+1}+\partial_s\mathcal{H}_{m+1}\otimes\Phi_{m+1}\right)(\mu,s,t,x,y)\\&=:I_1+I_2+I_3.\end{aligned}$$

We can now prove that the estimate (4.22) is still true at step $m+1$ for some choice of the constant C which appears in (4.22).

For I_1, we note that (B.18) and (4.15) yield

$$|I_1|\le K(t-s)^{-1}\rho^0(t-s,y-x)\le K(t-s)^{-1}\rho^{-\tilde\eta}(t-s,y-x),$$

since $\alpha\in(1,2)$ and $\rho^0(t-s,y-x)\le\rho^{-\tilde\eta}(t-s,y-x)$.

Concerning I_2, it can be rewritten as

$$I_2 = \int_s^t \int_{\mathbb{R}^d} \partial_s \widehat{p}(s,r,x,z) \left(\Phi_{m+1}(\mu,s,r,t,z,y) - \Phi_{m+1}(\mu,s,r,t,x,y)\right) dz\, dr.$$

Owing to the bound (7.23) obtained for A_1, which was defined in (7.22), we deduce that

$$\begin{aligned}
|I_2| &\leq K \int_s^t (r-s)^{\frac{\tilde{\eta}}{\alpha}-1}(t-r)^{-\frac{\tilde{\eta}+1}{\alpha}} \left[\rho^{-\tilde{\eta}}(t-s,y-x) + (t-r)^{-\frac{d}{\alpha}}(1+(t-s)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha+\tilde{\eta}}\right] dr \\
&\leq K(t-s)^{-\frac{1}{\alpha}}\rho^{-\tilde{\eta}}(t-s,y-x) + K(t-s)^{-\frac{1}{\alpha}-\frac{d}{\alpha}}(1+(t-s)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha+\tilde{\eta}} \\
&\leq K(t-s)^{-\frac{1}{\alpha}}\rho^{-\tilde{\eta}}(t-s,y-x) \\
&\leq K(t-s)^{-1}\rho^{-\tilde{\eta}}(t-s,y-x).
\end{aligned}$$

We now focus on I_3. Using the induction assumption (4.22), (7.14), the fact that $\eta+\tilde{\eta} < \alpha$ and the space-time inequality (B.13), we get that

$$\begin{aligned}
&|\partial_s \mathcal{H}_{m+1}(\mu,s,r,t,x,y)| \\
&\leq K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x) \int_{\mathbb{R}^{2d}} |x'-y|^{\eta}(r-s)^{-1}\rho^{-\tilde{\eta}}(r-s,y-x') \\
&\qquad \sum_{k=1}^m C^k (r-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(\frac{\eta}{\alpha} + (j-1)\left(1+\frac{\eta-1}{\alpha}\right), 1-\frac{1}{\alpha}\right) dy\, d\mu(x') \\
&\leq K(r-s)^{\frac{\eta}{\alpha}-1}(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x) \\
&\qquad \sum_{k=1}^m C^k (r-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(\frac{\eta}{\alpha} + (j-1)\left(1+\frac{\eta-1}{\alpha}\right), 1-\frac{1}{\alpha}\right).
\end{aligned}$$

Since the kernel Φ_{m+1} yields a time-integrable singularity by (4.15), we deduce with the preceding inequality that

$$\begin{aligned}
&|[\partial_s \mathcal{H}_{m+1} + \partial_s \mathcal{H}_{m+1} \otimes \Phi_{m+1}](\mu,s,r,t,x,y)| \\
&\leq K(r-s)^{\frac{\eta}{\alpha}-1}(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x) \\
&\qquad \sum_{k=1}^m C^k (r-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(\frac{\eta}{\alpha} + (j-1)\left(1+\frac{\eta-1}{\alpha}\right), 1-\frac{1}{\alpha}\right).
\end{aligned}$$

It follows from this inequality, (4.18) and the convolution inequality (B.16) that

$$\begin{aligned}
|I_3| &\leq \int_s^t \int_{\mathbb{R}^d} \rho^0(r-s,z-x) K(r-s)^{\frac{\eta}{\alpha}-1}(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z) \\
&\qquad \sum_{k=1}^m C^k (r-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(\frac{\eta}{\alpha} + (j-1)\left(1+\frac{\eta-1}{\alpha}\right), 1-\frac{1}{\alpha}\right) dz\, dr \\
&\leq K(t-s)^{-1}\rho^0(t-s,y-x) \sum_{k=1}^m C^k (r-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k} \mathcal{B}\left(\frac{\eta}{\alpha} + (j-1)\left(1+\frac{\eta-1}{\alpha}\right), 1-\frac{1}{\alpha}\right).
\end{aligned}$$

Gathering the previous estimates on I_1, I_2 and I_3, we have

$$
\begin{aligned}
&|\partial_s p_{m+1}(\mu,s,t,x,y)| \\
&\le (t-s)^{-1}\rho^{-\tilde{\eta}}(t-s,y-x)K\left[1+\sum_{k=1}^{m}C^k(r-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(\frac{\eta}{\alpha}+(j-1)\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right] \\
&\le (t-s)^{-1}\rho^{-\tilde{\eta}}(t-s,y-x)\sum_{k=1}^{m+1}C^k(t-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(\frac{\eta}{\alpha}+(j-1)\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)
\end{aligned}
$$

provided that we choose $C \ge K$ in (4.22). This concludes the induction step.

7.3. **Preparatory technical results.**

Lemma 5.

- For all $\gamma \in (0,1]\cap(0,(2\alpha-2)\wedge(\eta+\alpha-1))$, there exists a positive constant K independent of m and such that for all $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \le s \le r \le T$, $x, v_1, v_2 \in \mathbb{R}^d$, one has

$$
\begin{aligned}
&\left|\partial_v \frac{\delta}{\delta m}\left[b(r,x,[X_r^{s,\mu,(m)}])\right](v_1) - \partial_v \frac{\delta}{\delta m}\left[b(r,x,[X_r^{s,\mu,(m)}])\right](v_2)\right| \qquad (7.25)\\
&\le K(r-s)^{\frac{\eta-1-\gamma}{\alpha}}|v_1-v_2|^\gamma\left[1+\sum_{k=1}^{m}C^k(r-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+\frac{\eta-1-\gamma}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right].
\end{aligned}
$$

- For all $\gamma \in (0,1]\cap(0,(2\alpha-2)\wedge(\eta+\alpha-1))$, there exists a positive constant K independent of m and such that for all $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \le s \le r < t \le T$, $x, y, v_1, v_2 \in \mathbb{R}^d$, one has

$$
\begin{aligned}
&\left|\partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,s,r,t,x,y)(v_1) - \partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,s,r,t,x,y)(v_2)\right| \qquad (7.26)\\
&\le K(r-s)^{\frac{\eta-1-\gamma}{\alpha}}(t-r)^{-\frac{1}{\alpha}}|v_1-v_2|^\gamma\rho^1(t-r,y-x) \\
&\qquad\qquad\left[1+\sum_{k=1}^{m}C^k(r-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+\frac{\eta-1-\gamma}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right].
\end{aligned}
$$

Proof of Lemma 5. First, note that we only need to show (7.25) since it implies (7.26) because of (7.7). We can write

$$
\begin{aligned}
&\Delta_{v_1,v_2}\partial_v \frac{\delta}{\delta m}\left[b(r,x,[X_r^{s,\mu,(m)}])\right](\cdot) \\
&= \int_{\mathbb{R}^d}\frac{\delta}{\delta m}b(r,x,[X_r^{s,\mu,(m)}])(z)\Delta_{v_1,v_2}\partial_x p_m(\mu,s,r,\cdot,z)\,dz \\
&\quad + \int_{\mathbb{R}^{2d}}\frac{\delta}{\delta m}b(r,x,[X_r^{s,\mu,(m)}])(z)\Delta_{v_1,v_2}\partial_v\frac{\delta}{\delta m}p_m(\mu,s,r,x',z)(\cdot)\,dz\,d\mu(x') \\
&= \int_{\mathbb{R}^d}\left(\frac{\delta}{\delta m}b(r,x,[X_r^{s,\mu,(m)}])(z) - \frac{\delta}{\delta m}b(r,x,[X_r^{s,\mu,(m)}])(v_2)\right)\Delta_{v_1,v_2}\partial_x p_m(\mu,s,r,\cdot,z)\,dz \\
&\quad + \int_{\mathbb{R}^{2d}}\left(\frac{\delta}{\delta m}b(r,x,[X_r^{s,\mu,(m)}])(z) - \frac{\delta}{\delta m}b(r,x,[X_r^{s,\mu,(m)}])(x')\right) \\
&\qquad\qquad \Delta_{v_1,v_2}\partial_v\frac{\delta}{\delta m}p_m(\mu,s,r,x',z)(\cdot)\,dz\,d\mu(x') \\
&=: I_1 + I_2.
\end{aligned}
$$

Let us first focus on I_1. We start by assuming that $|v_1 - v_2| \leq (r-s)^{\frac{1}{\alpha}}$. Using the η-Hölder continuity of $\frac{\delta}{\delta m} b(r,x,\mu)(\cdot)$, (4.19), (B.14) since $|v_1 - v_2| \leq (r-s)^{\frac{1}{\alpha}}$ and (B.13), we get that

$$\begin{aligned} |I_1| &\leq K \int_{\mathbb{R}^d} |z - v_2|^\eta (r-s)^{-\frac{\gamma+1}{\alpha}} |v_1 - v_2|^\gamma \rho^1(r-s, z-v_2)\, dz \\ &\leq K(r-s)^{\frac{\eta-\gamma-1}{\alpha}} |v_1 - v_2|^\gamma. \end{aligned}$$

Now assume that $|v_1 - v_2| > (r-s)^{\frac{1}{\alpha}}$. The gradient estimate (4.18) yields

$$\begin{aligned} |I_1| &= \left| \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m} b(r,x,[X_r^{s,\mu,(m)}])(z) - \frac{\delta}{\delta m} b(r,x,[X_r^{s,\mu,(m)}])(v_1) \right) \partial_x p_m(\mu,s,r,v_1,z)\, dz \right. \\ &\quad \left. + \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m} b(r,x,[X_r^{s,\mu,(m)}])(z) - \frac{\delta}{\delta m} b(r,x,[X_r^{s,\mu,(m)}])(v_2) \right) \partial_x p_m(\mu,s,r,v_2,z)\, dz \right| \\ &\leq K(r-s)^{\frac{\eta-1}{\alpha}} \\ &\leq K(r-s)^{\frac{\eta-1-\gamma}{\alpha}} |v_1 - v_2|^\gamma. \end{aligned}$$

We have thus shown that

$$|I_1| \leq K(r-s)^{\frac{\eta-1-\gamma}{\alpha}} |v_1 - v_2|^\gamma.$$

Then, we have

$$\begin{aligned} |I_2| &\leq K \int_{\mathbb{R}^{2d}} |z - x'|^\eta (r-s)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}} |v_1 - v_2|^\gamma \rho^0(r-s, z-x') \\ &\qquad \left(\sum_{k=1}^m C^k (r-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(1 + \frac{\eta-1-\gamma}{\alpha} + j\left(1+\frac{\eta-1}{\alpha}\right), 1 - \frac{1}{\alpha} \right) \right) dz\, d\mu(x') \\ &\leq K(r-s)^{\frac{\eta-1-\gamma}{\alpha}+1+\frac{\eta-1}{\alpha}} |v_1 - v_2|^\gamma \\ &\qquad \left(\sum_{k=1}^m C^k (r-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(1 + \frac{\eta-1-\gamma}{\alpha} + j\left(1+\frac{\eta-1}{\alpha}\right), 1 - \frac{1}{\alpha} \right) \right). \end{aligned}$$

□

7.4. **Second part of the induction step.** We prove here that the estimates (4.23), (4.24), (4.25), (4.26) and (4.27) hold true.

Proof of (4.23). It follows from (7.26), (4.18) and the convolution inequality (B.16) that for some positive constant K independent of m one has

$$\begin{aligned} &\left| \Delta_{v_1,v_2} \left(\widehat{p} \otimes \partial_v \frac{\delta}{\delta m} \mathcal{H}_{m+1} \right)(\mu,s,t,x,y)(\cdot) \right| \\ &\leq K \int_s^t \int_{\mathbb{R}^d} (r-s)^{\frac{\eta-1-\gamma}{\alpha}} (t-r)^{-\frac{1}{\alpha}} |v_1 - v_2|^\gamma \rho^1(t-r, y-x) \rho^0(r-s, z-x) \\ &\qquad \left[1 + \sum_{k=1}^m C^k (r-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(1 + \frac{\eta-1-\gamma}{\alpha} + j\left(1+\frac{\eta-1}{\alpha}\right), 1 - \frac{1}{\alpha} \right) \right] dz\, dr \\ &\leq K(t-s)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}} |v_1 - v_2|^\gamma \rho^0(t-s, y-x) \left[1 + \sum_{k=1}^m C^k (r-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)} \right. \\ &\qquad \left. \prod_{j=1}^{k} \mathcal{B}\left(1 + \frac{\eta-1-\gamma}{\alpha} + j\left(1+\frac{\eta-1}{\alpha}\right), 1 - \frac{1}{\alpha} \right) \right]. \end{aligned}$$

Starting from the representation formula (7.20), using this inequality, the bound (4.13) for $\mathcal{H}^k_{m+1}$, the convolution inequality (B.16) and finally summing over $k \geq 0$, we find that there exists a constant K independent of m such that

$$\begin{aligned}
&\left|\Delta_{v_1,v_2}\partial_v \frac{\delta}{\delta m} p_{m+1}(\mu,s,t,x,y)(\cdot)\right| \\
&\leq \sum_{k=0}^{\infty}\left|\Delta_{v_1,v_2}\left(p_{m+1}\otimes\partial_v\frac{\delta}{\delta m}\mathcal{H}_{m+1}\otimes\mathcal{H}^k_{m+1}\right)(\mu,s,t,x,y)(\cdot)\right| \\
&\leq K(t-s)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}}|v_1-v_2|^\gamma\rho^0(t-s,y-x)\left[1+\sum_{k=1}^{m}C^k(r-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\right. \\
&\qquad\qquad\left.\prod_{j=1}^{k}\mathcal{B}\left(1+\frac{\eta-1-\gamma}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right] \\
&\leq (t-s)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}}|v_1-v_2|^\gamma\rho^0(t-s,y-x) \\
&\qquad\left(\sum_{k=1}^{m+1}C^k(t-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+\frac{\eta-1-\gamma}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right).
\end{aligned}$$

provided that we choose $C \geq K$ in (4.23). It ends the proof of the induction step for (4.23).

Proof of (4.24).

Let us first assume that $|v_1-v_2| \geq (t-s)^{\frac{1}{\alpha}}$. Using (4.20) and the fact the series appearing in this bound has a limit when m tends to infinity, one has in this case

$$\begin{aligned}
\left|\Delta_{v_1,v_2}\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(\cdot)\right| &\leq C(t-s)^{1-\frac{1}{\alpha}}\rho^0(t-s,y-x) \\
&\leq C(t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}}|v_1-v_2|^\gamma\rho^0(t-s,y-x).
\end{aligned}$$

In the case where $|v_1-v_2| < (t-s)^{\frac{1}{\alpha}}$, the mean value theorem and (4.21) yield

$$\begin{aligned}
\left|\Delta_{v_1,v_2}\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(\cdot)\right| &\leq C(t-s)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}|v_1-v_2|\rho^0(t-s,y-x) \\
&\leq C(t-s)^{\frac{\eta}{\alpha}-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}}|v_1-v_2|^\gamma\rho^0(t-s,y-x) \\
&\leq C(t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}}|v_1-v_2|^\gamma\rho^0(t-s,y-x).
\end{aligned}$$

Proof of (4.25).

We first note that since (4.20) has been proved before, (7.3) ensures that there exists a positive constant C such that for all $m \geq 1$

$$\left|\frac{\delta}{\delta m}\mathcal{H}_m(\mu,s,r,t,x,y)(v)\right| \leq C(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x).$$

Using this inequality, (4.19) and the convolution inequality (B.16), we obtain that

$$\begin{aligned}
&\left|\Delta_{x_1,x_2}\left[p_m\otimes\frac{\delta}{\delta m}\mathcal{H}_m\right](\mu,s,r,t,\cdot,y)(v)\right|\\
&=\left|\int_r^t\int_{\mathbb{R}^d}\Delta_{x_1,x_2}p_m(\mu,r,r',\cdot,z)\frac{\delta}{\delta m}\mathcal{H}_m(\mu,s,r',t,z,y)\,dz\,dr'\right|\\
&\le C\int_r^t\int_{\mathbb{R}^d}(r'-r)^{-\frac{\gamma}{\alpha}}|x_1-x_2|^\gamma\left[\rho^0(r'-r,z-x_1)+\rho^0(r'-r,z-x_2)\right](t-r')^{-\frac{1}{\alpha}}\rho^1(t-r',y-z)\,dz\,dr'\\
&\le C(t-r)^{1-\frac{1+\gamma}{\alpha}}|x_1-x_2|^\gamma\left[\rho^0(t-r,y-x_1)+\rho^0(t-r,y-x_2)\right].
\end{aligned}$$

We conclude by the representation formula (7.19) and (4.13) (since the series appearing is convergent) that we have

$$\begin{aligned}
\left|\Delta_{x_1,x_2}\frac{\delta}{\delta m}p_m(\mu,s,t,\cdot,y)(v)\right|&\le\sum_{k=0}^\infty\left|\Delta_{x_1,x_2}\left[p_m\otimes\frac{\delta}{\delta m}\mathcal{H}_m\otimes\mathcal{H}_m^k\right](\mu,s,t,\cdot,y)(v)\right|\\
&\le C(t-s)^{1-\frac{1+\gamma}{\alpha}}|x_1-x_2|^\gamma\left[\rho^0(t-s,y-x_1)+\rho^0(t-s,y-x_2)\right].
\end{aligned}$$

Proof of (4.26). We first note that since (4.21) has been proved before, (7.4) ensures that there exists a positive constant C such that for all $m\ge 1$

$$\left|\partial_v\frac{\delta}{\delta m}\mathcal{H}_m(\mu,s,r,t,x,y)(v)\right|\le C(r-s)^{\frac{\eta-1}{\alpha}}(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x).$$

Using this inequality, (4.19), the convolution inequality (B.16) and the fact that $\gamma<\alpha-1-\eta$, we obtain that

$$\begin{aligned}
&\left|\Delta_{x_1,x_2}\left[p_m\otimes\partial_v\frac{\delta}{\delta m}\mathcal{H}_m\right](\mu,s,s,t,\cdot,y)(v)\right|\\
&=\left|\int_s^t\int_{\mathbb{R}^d}\Delta_{x_1,x_2}p_m(\mu,s,r,\cdot,z)\partial_v\frac{\delta}{\delta m}\mathcal{H}_m(\mu,s,r,t,z,y)\,dz\,dr\right|\\
&\le C\int_s^t\int_{\mathbb{R}^d}(r-s)^{-\frac{\gamma}{\alpha}}|x_1-x_2|^\gamma\left[\rho^0(r-s,z-x_1)+\rho^0(r-s,z-x_2)\right](r-s)^{\frac{\eta-1}{\alpha}}(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr\\
&\le C(t-s)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}}|x_1-x_2|^\gamma\left[\rho^0(t-r,y-x_1)+\rho^0(t-r,y-x_2)\right].
\end{aligned}$$

We conclude by the representation formula (7.20) and (4.13) that we have

$$\begin{aligned}
\left|\Delta_{x_1,x_2}\partial_v\frac{\delta}{\delta m}p_m(\mu,s,t,\cdot,y)(v)\right|&\le\sum_{k=0}^\infty\left|\Delta_{x_1,x_2}\left[p_m\otimes\partial_v\frac{\delta}{\delta m}\mathcal{H}_m\otimes\mathcal{H}_m^k\right](\mu,s,t,\cdot,y)(v)\right|\\
&\le C(t-s)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}}|x_1-x_2|^\gamma\left[\rho^0(t-r,y-x_1)+\rho^0(t-r,y-x_2)\right].
\end{aligned}$$

Proof of (4.27). Let us first assume that $W_1(\mu_1,\mu_2)\le(t-s)^{\frac{1}{\alpha}}$. In this case, by definition of the linear derivative, the Kantorovich-Rubinstein theorem, and (4.21), we get

$$\begin{aligned}
|\Delta_{\mu_1,\mu_2}p_m(\cdot,s,t,x,y)|&=\left|\int_0^1\int_{\mathbb{R}^d}\frac{\delta}{\delta m}p_m(\lambda\mu_1+(1-\lambda)\mu_2,s,t,x,y)(v)\,d(\mu_1-\mu_2)(v)\,d\lambda\right|\\
&\le\sup_{\lambda\in[0,1],\,v\in\mathbb{R}^d}\left|\partial_v\frac{\delta}{\delta m}p_m(\lambda\mu_1+(1-\lambda)\mu_2,s,t,x,y)(v)\right|W_1(\mu_1,\mu_2)\\
&\le C(t-s)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}W_1(\mu_1,\mu_2)\rho^0(t-s,y-x)\\
&\le C(t-s)^{1-\frac{1+\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x).
\end{aligned}$$

In the case where $W_1(\mu_1,\mu_2)>(t-s)^{\frac{1}{\alpha}}$, the parametrix expansion (4.17), (B.17) and (4.15) yield

$$\begin{aligned}|\Delta_{\mu_1,\mu_2} p_m(\cdot,s,t,x,y)| &\le |\Delta_{\mu_1,\mu_2}\left[\widehat{p}\otimes\Phi_m\right](\cdot,s,t,x,y)|\\ &\le |\widehat{p}\otimes\Phi_m(\mu_1,s,t,x,y)| + |\widehat{p}\otimes\Phi_m(\mu_2,s,t,x,y)|\\ &\le C(t-s)^{1-\frac{1}{\alpha}}\rho^0(t-s,y-x)\\ &\le C(t-s)^{1-\frac{1+\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x).\end{aligned}$$

7.5. **Preparatory technical results.**

Lemma 6. • For any $\gamma\in[\eta,1]$, there exists a positive constant K such that for all $m\ge 1$, $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x\in\mathbb{R}^d$

$$|b(t,x,[X_t^{s,\mu_1,(m)}]) - b(t,x,[X_t^{s,\mu_2,(m)}])| \le K(t-s)^{\frac{\eta-\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2). \tag{7.27}$$

- For any $\gamma\in[\eta,1]$, there exists a positive constant K such that for all $m\ge 1$, $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s\le r<t\le T$, $x,y\in\mathbb{R}^d$

$$|\mathcal{H}_m(\mu_1,s,r,t,x,y) - \mathcal{H}_m(\mu_2,s,r,t,x,y)| \le K(r-s)^{\frac{\eta-\gamma}{\alpha}}(t-r)^{-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r,y-x). \tag{7.28}$$

- For any $\gamma\in[\eta,1]$, there exists a positive constant K such that for all $m\ge 1$, $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s\le r<t\le T$, $x,y\in\mathbb{R}^d$

$$|\Phi_m(\mu_1,s,r,t,x,y) - \Phi_m(\mu_2,s,r,t,x,y)| \le K(r-s)^{\frac{\eta-\gamma}{\alpha}}(t-r)^{-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r,y-x). \tag{7.29}$$

Proof of Lemma 6. **Proof of (7.27) and (7.28).**

By definition of the linear derivative and denoting by $m_l := l[X_t^{s,\mu_1,(m)}] + (1-l)[X_t^{s,\mu_2,(m)}]$, we have

$$\begin{aligned}&b(t,x,[X_t^{s,\mu_1,(m)}]) - b(t,x,[X_t^{s,\mu_2,(m)}])\\ &= \int_0^1\int_{\mathbb{R}^d}\frac{\delta}{\delta m} b(t,x,m_l)(y')[p_m(\mu_1,s,t,y') - p_m(\mu_2,s,t,y')]\,dy'\,dl\\ &= \int_0^1\int_{\mathbb{R}^{2d}}\frac{\delta}{\delta m} b(t,x,m_l)(y') p_m(\mu_1,s,t,x',y')\,dy'\,d(\mu_1-\mu_2)(x')dl\\ &\quad + \int_0^1\int_{\mathbb{R}^{2d}}\frac{\delta}{\delta m} b(t,x,m_l)(y')[p_m(\mu_1,s,t,x',y') - p_m(\mu_2,s,t,x',y')]\,dy'\,d\mu_2(x')dl\\ &=: I_1 + I_2.\end{aligned}$$

For I_1, we need to control, for $x',x''\in\mathbb{R}^d$

$$\int_{\mathbb{R}^d}\frac{\delta}{\delta m} b(t,x,m_l)(y')[p_m(\mu_1,s,t,x',y') - p_m(\mu_1,s,t,x'',y')]\,dy'.$$

In the case where $|x'-x''|\le (t-s)^{\frac{1}{\alpha}}$, we write

$$\begin{aligned}&\int_{\mathbb{R}^d}\frac{\delta}{\delta m} b(t,x,m_l)(y')[p_m(\mu_1,s,t,x',y') - p_m(\mu_1,s,t,x'',y')]\,dy'\\ &= \int_0^1\int_{\mathbb{R}^d}\left[\frac{\delta}{\delta m} b(t,x,m_l)(y') - \frac{\delta}{\delta m} b(t,x,m_l)(l'x' + (1-l')x'')\right]\\ &\qquad\qquad \partial_x p_m(\mu_1,s,t,l'x'+(1-l')x'',y')\cdot(x'-x'')\,dy'\,dl'.\end{aligned}$$

Using the η-Hölder continuity of $\frac{\delta}{\delta m}b$ and (4.18), we obtain that

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\frac{\delta}{\delta m}b(t,x,m_l)(y')[p_m(\mu_1,s,t,x',y')-p_m(\mu_1,s,t,x'',y')]\,dy'\right|\\&\leq K(t-s)^{\frac{\eta-1}{\alpha}}|x'-x''|\\&\leq K(t-s)^{\frac{\eta-\gamma}{\alpha}}|x'-x''|^\gamma.\end{aligned}$$

Assume now that $|x'-x''|>(t-s)^{\frac{1}{\alpha}}$. One can write

$$\begin{aligned}&\int_{\mathbb{R}^d}\frac{\delta}{\delta m}b(t,x,m_l)(y')[p_m(\mu_1,s,t,x',y')-p_m(\mu_1,s,t,x'',y')]\,dy'\\&=\frac{\delta}{\delta m}b(t,x,m_l)(x')-\frac{\delta}{\delta m}b(t,x,m_l)(x'')\\&\quad+\int_{\mathbb{R}^d}\left[\frac{\delta}{\delta m}b(t,x,m_l)(y')-\frac{\delta}{\delta m}b(t,x,m_l)(x')\right]p_m(\mu_1,s,t,x',y')\,dy'\\&\quad+\int_{\mathbb{R}^d}\left[\frac{\delta}{\delta m}b(t,x,m_l)(y')-\frac{\delta}{\delta m}b(t,x,m_l)(x'')\right]p_m(\mu_1,s,t,x'',y')\,dy'\end{aligned}$$

It follows from the η-Hölder continuity of $\frac{\delta}{\delta m}b$, (4.18) and the space-time inequality (B.13) that

$$\begin{aligned}&\int_{\mathbb{R}^d}\frac{\delta}{\delta m}b(t,x,m_l)(y')[p_m(\mu_1,s,t,x',y')-p_m(\mu_1,s,t,x'',y')]\,dy'\\&\leq K(|x'-x''|^\eta+(t-s)^{\frac{\eta}{\alpha}})\\&\leq K(|x'-x''|^\eta+(t-s)^{\frac{\eta}{\alpha}})\\&\leq K(t-s)^{\frac{\eta-\gamma}{\alpha}}|x'-x''|^\gamma,\end{aligned}$$

since $\gamma\geq\eta$. Jensen's inequality yields

$$|I_1|\leq K(t-s)^{\frac{\eta-\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2).$$

It remains to study I_2 which can be rewritten as

$$I_2=\int_0^1\int_{\mathbb{R}^{2d}}\left[\frac{\delta}{\delta m}b(t,x,m_l)(y')-\frac{\delta}{\delta m}b(t,x,m_l)(x')\right][p_m(\mu_1,s,t,x',y')-p_m(\mu_2,s,t,x',y')]\,dy'\,d\mu_2(x')dl. \tag{7.30}$$

Thanks to (4.27), (4.18) and the space-time inequality (B.13), one has since $\alpha\in(1,2)$

$$\begin{aligned}|I_2|&\leq K(t-s)^{1-\frac{1+\gamma}{\alpha}+\frac{\eta}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\\&\leq K(t-s)^{\frac{\eta-\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2).\end{aligned}$$

This concludes the proof of (7.27). The proof of (7.28) immediately follows from the definition (4.10) of $\mathcal{H}_m$ and (B.17).

Proof of (7.29).

We prove by induction of $k\geq 2$ that there exists a constant K independent of k such that for any $k\geq 2$, $m\geq 1$, $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\leq s\leq r<t\leq T$, $x,y\in\mathbb{R}^d$

$$\begin{aligned}|\Delta_{\mu_1,\mu_2}\mathcal{H}_m^k(\cdot,s,r,t,x,y)|&\leq(r-s)^{\frac{\eta-\gamma}{\alpha}}(t-r)^{-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r,y-x)\\&\quad\sum_{i=1}^k C^i(t-r)^{i\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^i\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right).\end{aligned} \tag{7.31}$$

We do not prove the base case $k = 2$ since it relies on the same computations as the induction step using (7.28). For the induction step, assume that (7.31) holds for $\mathcal{H}_m^k$ and let us prove it for $\mathcal{H}_m^{k+1}$. One has

$$\begin{aligned}|\Delta_{\mu_1,\mu_2}\mathcal{H}_m^{k+1}(\cdot,s,r,t,x,y)| &\le \left|\int_r^t\int_{\mathbb{R}^d}\mathcal{H}_m(\mu_1,s,r,r',x,z)\Delta_{\mu_1,\mu_2}\mathcal{H}_m^k(\cdot,s,r',t,z,y)\,dz\,dr'\right| \\ &\quad+\left|\int_r^t\int_{\mathbb{R}^d}\Delta_{\mu_1,\mu_2}\mathcal{H}_m(\cdot,s,r,r',x,z)\mathcal{H}_m^k(\mu_2,s,r',t,z,y)\,dz\,dr'\right| \\ &=: I_1+I_2.\end{aligned}$$

By (7.28) and the bound of $\mathcal{H}_m^k$ (4.13) (note that the series appearing in this bound in convergent and thus can be bounded independently of k and m), we have

$$\begin{aligned}I_2 &\le K\int_r^t\int_{\mathbb{R}^d}(r-s)^{\frac{\eta-\gamma}{\alpha}}(r'-r)^{-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^1(r'-r,z-x)(t-r')^{-\frac{1}{\alpha}}\rho^1(t-r',y-z)\,dz\,dr' \\ &\le K(r-s)^{\frac{\eta-\gamma}{\alpha}}(t-r)^{-\frac{1}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r,y-x)\mathcal{B}\left(1-\frac{1}{\alpha},1-\frac{1}{\alpha}\right).\end{aligned}$$

By induction and using (4.13), one has

$$\begin{aligned}I_1 &\le K\int_r^t\int_{\mathbb{R}^d}(r'-r)^{-\frac{1}{\alpha}}\rho^1(r'-r,z-x)(r'-s)^{\frac{\eta-\gamma}{\alpha}}(t-r')^{-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r',y-z) \\ &\qquad\qquad\sum_{i=1}^k C^i(t-r')^{i\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^i\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)dz\,dr' \\ &\le K(r-s)^{\frac{\eta-\gamma}{\alpha}}\int_r^t\int_{\mathbb{R}^d}(r'-r)^{-\frac{1}{\alpha}}\rho^1(r'-r,z-x)(t-r')^{-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r',y-z) \\ &\qquad\qquad\sum_{i=1}^k C^i(t-r')^{i\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^i\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)dz\,dr' \\ &\le K(r-s)^{\frac{\eta-\gamma}{\alpha}}(t-r)^{-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r,y-x)\sum_{i=1}^k C^i(t-r)^{(i+1)\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{i+1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\end{aligned}$$

Summing the bounds obtained for I_1 and I_2, we have proved that

$$\begin{aligned}|\Delta_{\mu_1,\mu_2}\mathcal{H}_m^{k+1}(\cdot,s,r,t,x,y)| &\le (r-s)^{\frac{\eta-\gamma}{\alpha}}(t-r)^{-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r,y-x) \\ &\qquad\sum_{i=1}^{k+1}C^i(t-r)^{i\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^i\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right),\end{aligned}$$

if we choose $C \ge K$ since the beginning. Finally, summing (7.31) over $k \ge 1$ yields (7.29).

□

Lemma 7.
- For all $\gamma \in [\eta, 1]$, there exists $K > 0$ such that for all $m \ge 1$, $\mu_1, \mu_2 \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \le s < t \le T$, $x, v \in \mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}b(t,x,[X_t^{s,\mu_1,(m)}])(v)-\frac{\delta}{\delta m}b(t,x,[X_t^{s,\mu_2,(m)}])(v)\right| \le K(t-s)^{\frac{\eta-\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2). \tag{7.32}$$

- For all $\gamma \in (0, 1]$, there exists $K > 0$ such that for all $m \ge 1$, $\mu_1, \mu_2 \in \mathcal{P}_\beta(\mathbb{R}^d)$, $0 \le s < t \le T$, $x, v \in \mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}\left[b(t,x,[X_t^{s,\mu_1,(m)}])\right](v)-\frac{\delta}{\delta m}\left[b(t,x,[X_t^{s,\mu_2,(m)}])\right](v)\right|\le K(t-s)^{-\frac{\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2)$$
$$+K\int_{\mathbb{R}^{2d}}\left|\Delta_{\mu_1,\mu_2}\frac{\delta}{\delta m}p_m(\cdot,s,t,x',y')\right|dy'\,d\mu_2(x'). \quad (7.33)$$

- For all $\gamma\in(0,1]\cap(0,\eta+\alpha-1)$, there exists $K>0$ such that for all $m\ge1$, $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x,v\in\mathbb{R}^d$

$$\left|\partial_v\frac{\delta}{\delta m}\left[b(t,x,[X_t^{s,\mu_1,(m)}])\right](v)-\partial_v\frac{\delta}{\delta m}\left[b(t,x,[X_t^{s,\mu_2,(m)}])\right](v)\right|\le K(t-s)^{\frac{\eta-\gamma-1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)$$
$$+K\int_{\mathbb{R}^{2d}}(1\wedge|x'-y'|^\eta)\left|\Delta_{\mu_1,\mu_2}\partial_v\frac{\delta}{\delta m}p_m(\cdot,s,t,x',y')\right|dy'\,d\mu_2(x'). \quad (7.34)$$

- For all $\gamma\in(0,1]$, there exists $K>0$ such that for all $m\ge1$, $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x,v\in\mathbb{R}^d$

$$\left|\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu_1,s,r,t,x,y)(v)-\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu_2,s,r,t,x,y)(v)\right|\le K(r-s)^{-\frac{\gamma}{\alpha}}(t-r)^{-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r,y-x)$$
$$+K(t-r)^{-\frac{1}{\alpha}}\int_{\mathbb{R}^{2d}}\left|\Delta_{\mu_1,\mu_2}\frac{\delta}{\delta m}p_m(\cdot,s,r,x',y')\right|dy'\,d\mu_2(x')\rho^1(t-r,y-x). \quad (7.35)$$

- For all $\gamma\in(0,1]\cap(0,\eta+\alpha-1)$, there exists $K>0$ such that for all $m\ge1$, $\mu_1,\mu_2\in\mathcal{P}_\beta(\mathbb{R}^d)$, $0\le s<t\le T$, $x,v\in\mathbb{R}^d$

$$\begin{aligned}&\left|\partial_v\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu_1,s,r,t,x,y)(v)-\partial_v\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu_1,s,r,t,x,y)(v)\right|\\&\le K(r-s)^{\frac{\eta-\gamma-1}{\alpha}}(t-r)^{-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r,y-x)\\&\quad+K(t-r)^{-\frac{1}{\alpha}}\int_{\mathbb{R}^{2d}}(1\wedge|x'-y'|^\eta)\left|\Delta_{\mu_1,\mu_2}\partial_v\frac{\delta}{\delta m}p_m(\cdot,s,r,x',y')\right|dy'\,d\mu_2(x')\rho^1(t-r,y-x).\end{aligned} \quad (7.36)$$

Proof of Lemma 7. **Proof of (7.32).** By definition of the linear derivative and denoting by $m_l:=l[X_t^{s,\mu_1,(m)}]+(1-l)[X_t^{s,\mu_2,(m)}]$, we have

$$\begin{aligned}&\frac{\delta}{\delta m}b(t,x,[X_t^{s,\mu_1,(m)}])(v)-\frac{\delta}{\delta m}b(t,x,[X_t^{s,\mu_2,(m)}])(v)\\&=\int_0^1\int_{\mathbb{R}^d}\frac{\delta^2}{\delta m^2}b(t,x,m_l)(v,y')[p_m(\mu_1,s,t,y')-p_m(\mu_2,s,t,y')]\,dy'\,dl\\&=\int_0^1\int_{\mathbb{R}^{2d}}\frac{\delta^2}{\delta m^2}b(t,x,m_l)(v,y')p_m(\mu_1,s,t,x',y')\,dy'\,d(\mu_1-\mu_2)(x')dl\\&\quad+\int_0^1\int_{\mathbb{R}^{2d}}\frac{\delta^2}{\delta m^2}b(t,x,m_l)(v,y')[p_m(\mu_1,s,t,x',y')-p_m(\mu_2,s,t,x',y')]\,dy'\,d\mu_2(x')dl\\&=:I_1+I_2.\end{aligned}$$

For I_1, we need to control, for $x',x''\in\mathbb{R}^d$

$$\int_{\mathbb{R}^d}\frac{\delta^2}{\delta m^2}b(t,x,m_l)(v,y')[p_m(\mu_1,s,t,x',y')-p_m(\mu_1,s,t,x'',y')]\,dy'.$$

In the case where $|x'-x''|\le(t-s)^{\frac{1}{\alpha}}$, we write

$$\int_{\mathbb{R}^d} \frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,y')[p_m(\mu_1,s,t,x',y') - p_m(\mu_1,s,t,x'',y')]\,dy'$$
$$= \int_0^1 \int_{\mathbb{R}^d} \left[\frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,y') - \frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,l'x' + (1-l')x'')\right]$$
$$\partial_x p_m(\mu_1,s,t,l'x' + (1-l')x'',y') \cdot (x' - x'')\,dy'\,dl'.$$

Using the η-Hölder continuity of $\frac{\delta^2}{\delta m^2} b(t,x,\mu)(v,\cdot)$ and (4.18), we obtain that

$$\left|\int_{\mathbb{R}^d} \frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,y')[p_m(\mu_1,s,t,x',y') - p_m(\mu_1,s,t,x'',y')]\,dy'\right|$$
$$\leq K(t-s)^{\frac{\eta-1}{\alpha}}|x' - x''|$$
$$\leq K(t-s)^{\frac{\eta-\gamma}{\alpha}}|x' - x''|^\gamma.$$

Assume now that $|x' - x''| > (t-s)^{\frac{1}{\alpha}}$. One can write

$$\int_{\mathbb{R}^d} \frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,y')[p_m(\mu_1,s,t,x',y') - p_m(\mu_1,s,t,x'',y')]\,dy'$$
$$= \frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,x') - \frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,x'')$$
$$+ \int_{\mathbb{R}^d} \left[\frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,y') - \frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,x')\right] p_m(\mu_1,s,t,x',y')\,dy'$$
$$+ \int_{\mathbb{R}^d} \left[\frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,y') - \frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,x'')\right] p_m(\mu_1,s,t,x'',y')\,dy'$$

It follows from the η-Hölder continuity of $\frac{\delta^2}{\delta m^2} b(t,x,\mu)(v,\cdot)$, (4.18) and the space-time inequality (B.13) that

$$\int_{\mathbb{R}^d} \frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,y')[p_m(\mu_1,s,t,x',y') - p_m(\mu_1,s,t,x'',y')]\,dy'$$
$$\leq K(|x' - x''|^\eta + (t-s)^{\frac{\eta}{\alpha}})$$
$$\leq K(|x' - x''|^\eta + (t-s)^{\frac{\eta}{\alpha}})$$
$$\leq K(t-s)^{\frac{\eta-\gamma}{\alpha}}|x' - x''|^\gamma,$$

since $\gamma \geq \eta$. Jensen's inequality yields

$$|I_1| \leq K(t-s)^{\frac{\eta-\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2).$$

It remains to study I_2 which can be rewritten as

$$I_2 = \int_0^1 \int_{\mathbb{R}^{2d}} \left[\frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,y') - \frac{\delta^2}{\delta m^2} b(t,x,m_l)(v,x')\right] [p_m(\mu_1,s,t,x',y') - p_m(\mu_2,s,t,x',y')]\,dy'\,d\mu_2(x')dl. \tag{7.37}$$

Thanks to (4.27), (4.18) and the space-time inequality (B.13), one has since $\alpha \in (1,2)$

$$|I_2| \leq K(t-s)^{1-\frac{1+\gamma}{\alpha}+\frac{\eta}{\alpha}} W_1^\gamma(\mu_1,\mu_2)$$
$$\leq K(t-s)^{\frac{\eta-\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2).$$

This concludes the proof of (7.32).

Proof of (7.33)**.** We begin to treat the case where $W_1(\mu_1,\mu_2) \geq (t-s)^{\frac{1}{\alpha}}$. Using (7.1) and noting that the series appearing in the bound is convergent, one has

$$\begin{aligned}\left|\Delta_{\mu_1,\mu_2}\frac{\delta}{\delta m}\left[b(t,x,[X_t^{s,\cdot,(m)}])\right](v)\right| &\le \left|\frac{\delta}{\delta m}\left[b(t,x,[X_t^{s,\mu_1,(m)}])\right](v)\right| + \left|\frac{\delta}{\delta m}\left[b(t,x,[X_t^{s,\mu_2,(m)}])\right](v)\right| \\ &\le K \\ &\le K(t-s)^{-\frac{\gamma}{\alpha}}W_1(\mu_1,\mu_2)^\gamma.\end{aligned}$$

We now focus on the case where $W_1(\mu_1,\mu_2) < (t-s)^{\frac{1}{\alpha}}$. Using (7.5), one has

$$\begin{aligned}&\Delta_{\mu_1,\mu_2}\frac{\delta}{\delta m}\left[b(t,x,[X_t^{s,\cdot,(m)}])\right](v) \\ &= \int_{\mathbb{R}^d}\Delta_{\mu_1,\mu_2}\frac{\delta}{\delta m}b(t,x,[X_t^{s,\cdot,(m)}])(y)p_m(\mu_1,s,t,v,y)\,dy \\ &\quad + \int_{\mathbb{R}^d}\frac{\delta}{\delta m}b(t,x,[X_t^{s,\mu_2,(m)}])(y)\Delta_{\mu_1,\mu_2}p_m(\cdot,s,t,v,y)\,dy \\ &\quad + \int_{\mathbb{R}^{2d}}\Delta_{\mu_1,\mu_2}\frac{\delta}{\delta m}b(t,x,[X_t^{s,\cdot,(m)}])(y)\frac{\delta}{\delta m}p_m(\mu_1,s,t,x',y)(v)\,dy\,d\mu_1(x') \\ &\quad + \int_{\mathbb{R}^{2d}}\frac{\delta}{\delta m}b(t,x,[X_t^{s,\mu_2,(m)}])(y)\frac{\delta}{\delta m}p_m(\mu_1,s,t,x',y)(v)\,dy\,d(\mu_1-\mu_2)(x') \\ &\quad + \int_{\mathbb{R}^{2d}}\frac{\delta}{\delta m}b(t,x,[X_t^{s,\mu_2,(m)}])(y)\Delta_{\mu_1,\mu_2}\frac{\delta}{\delta m}p_m(\cdot,s,t,x',y)(v)\,dy\,d\mu_2(x') \\ &=: I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

Thanks to (4.18) and (7.32) which is true for all $\gamma\in(0,1]$ since $W_1(\mu_1,\mu_2) < (t-s)^{\frac{1}{\alpha}}$, we obtain that

$$\begin{aligned}|I_1| &\le K(t-s)^{\frac{\eta-\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2) \\ &\le K(t-s)^{-\frac{\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2).\end{aligned}$$

It follows from the boundedness of $\frac{\delta}{\delta m}b$ and (4.27) that

$$\begin{aligned}|I_2| &\le K(t-s)^{1-\frac{1+\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2) \\ &\le K(t-s)^{-\frac{\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2).\end{aligned}$$

Owing to (7.32) which is true for all $\gamma\in(0,1]$ since $W_1(\mu_1,\mu_2) < (t-s)^{\frac{1}{\alpha}}$ and (4.20) (the series appearing in this estimates is convergent), one has

$$\begin{aligned}|I_3| &\le K(t-s)^{\frac{\eta-\gamma}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2) \\ &\le K(t-s)^{-\frac{\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2).\end{aligned}$$

Let us now deal with I_4. We need to control

$$J := \int_{\mathbb{R}^d}\frac{\delta}{\delta m}b(t,x,[X_t^{s,\mu_2,(m)}])(y)\left[\frac{\delta}{\delta m}p_m(\mu_1,s,t,x',y)(v) - \frac{\delta}{\delta m}p_m(\mu_1,s,t,x'',y)(v)\right]dy.$$

Thanks to (4.25) and the boundedness $\frac{\delta}{\delta m}b$, we obtain that

$$\begin{aligned}|J| &\le K\int_{\mathbb{R}^d}(t-s)^{1-\frac{1+\gamma}{\alpha}}|x'-x''|^\gamma\left[\rho^0(t-s,y-x')+\rho^0(t-s,y-x'')\right]dy \\ &\le K(t-s)^{1-\frac{1+\gamma}{\alpha}}|x'-x''|^\gamma.\end{aligned}$$

Jensen's inequality implies that

$$\begin{aligned}|I_4| &\le K(t-s)^{1-\frac{1+\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\\ &\le K(t-s)^{-\frac{\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2).\end{aligned}$$

Finally, for I_5, one has

$$\begin{aligned}|I_5| &= \left| \int_{\mathbb{R}^{2d}} \frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(y) \Delta_{\mu_1,\mu_2} \frac{\delta}{\delta m} p_m(\cdot,s,t,x',y)(v)\, dy\, d\mu_2(x') \right| \\ &\le K \int_{\mathbb{R}^{2d}} \left| \Delta_{\mu_1,\mu_2} \frac{\delta}{\delta m} p_m(\cdot,s,t,x',y) \right| dy\, d\mu_2(x').\end{aligned}$$

Gathering all the previous estimates, we have proved that

$$\begin{aligned}\left| \frac{\delta}{\delta m} \left[b(t,x,[X_t^{s,\mu_1,(m)}]) \right](v) - \frac{\delta}{\delta m} \left[b(t,x,[X_t^{s,\mu_2,(m)}]) \right](v) \right| &\le K(t-s)^{-\frac{\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \\ &\quad + K \int_{\mathbb{R}^d} \left| \Delta_{\mu_1,\mu_2} \frac{\delta}{\delta m} p_m(\cdot,s,t,x',y') \right| dy\, d\mu_2(x').\end{aligned}$$

Proof of (7.34)**.** We begin to treat the case where $W_1(\mu_1,\mu_2) \ge (t-s)^{\frac{1}{\alpha}}$. Using (7.2) and noting that the series appearing in the bound is convergent, one has

$$\begin{aligned}\left| \Delta_{\mu_1,\mu_2} \partial_v \frac{\delta}{\delta m} \left[b(t,x,[X_t^{s,\cdot,(m)}]) \right](v) \right| &\le \left| \partial_v \frac{\delta}{\delta m} \left[b(t,x,[X_t^{s,\mu_1,(m)}]) \right](v) \right| + \left| \partial_v \frac{\delta}{\delta m} \left[b(t,x,[X_t^{s,\mu_2,(m)}]) \right](v) \right| \\ &\le K(t-s)^{\frac{\eta-1}{\alpha}} \\ &\le K(t-s)^{\frac{\eta-1-\gamma}{\alpha}} W_1(\mu_1,\mu_2)^\gamma.\end{aligned}$$

We now focus on the case where $W_1(\mu_1,\mu_2) < (t-s)^{\frac{1}{\alpha}}$. In this case, by (7.6), we have

$$\begin{aligned}&\Delta_{\mu_1,\mu_2} \partial_v \frac{\delta}{\delta m} \left[b(t,x,[X_t^{s,\cdot,(m)}]) \right](v) \\ &= \int_{\mathbb{R}^d} \Delta_{\mu_1,\mu_2} \frac{\delta}{\delta m} b(t,x,[X_t^{s,\cdot,(m)}])(y) \partial_x p_m(\mu_1,s,t,v,y)\, dy \\ &\quad + \int_{\mathbb{R}^d} \frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(y) \Delta_{\mu_1,\mu_2} \partial_x p_m(\cdot,s,t,v,y)\, dy \\ &\quad + \int_{\mathbb{R}^{2d}} \Delta_{\mu_1,\mu_2} \frac{\delta}{\delta m} b(t,x,[X_t^{s,\cdot,(m)}])(y) \partial_v \frac{\delta}{\delta m} p_m(\mu_1,s,t,x',y)(v)\, dy\, d\mu_1(x') \\ &\quad + \int_{\mathbb{R}^{2d}} \frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(y) \partial_v \frac{\delta}{\delta m} p_m(\mu_1,s,t,x',y)(v)\, dy\, d(\mu_1-\mu_2)(x') \\ &\quad + \int_{\mathbb{R}^{2d}} \frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(y) \Delta_{\mu_1,\mu_2} \partial_v \frac{\delta}{\delta m} p_m(\cdot,s,t,x',y)(v)\, dy\, d\mu_2(x') \\ &=: I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

Thanks to (4.18) and (7.32) which is true for all $\gamma \in (0,1]$ since $W_1(\mu_1,\mu_2) < (t-s)^{\frac{1}{\alpha}}$, we obtain that

$$|I_1| \le K(t-s)^{\frac{\eta-\gamma-1}{\alpha}} W_1^\gamma(\mu_1,\mu_2).$$

I_2 can be rewritten as

$$I_2 = \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(y) - \frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(v) \right) \Delta_{\mu_1,\mu_2} \partial_x p_m(\cdot,s,t,v,y)\, dy.$$

It follows from the η-Hölder continuity of $\frac{\delta}{\delta m}b(t,x,\mu)(\cdot)$, (4.28) since $\gamma \in (0, \eta+\alpha-1)$ and the space-time inequality (B.13) that

$$\begin{aligned}|I_2| &\leq K(t-s)^{\frac{\eta}{\alpha}+1-\frac{1+\gamma+1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\\ &\leq K(t-s)^{\frac{\eta-\gamma-1}{\alpha}} W_1^\gamma(\mu_1,\mu_2).\end{aligned}$$

Owing to (7.32) which is true for all $\gamma \in (0,1]$ since $W_1(\mu_1,\mu_2) < (t-s)^{\frac{1}{\alpha}}$ and (4.21) (the series appearing in this estimates is convergent), one has

$$\begin{aligned}|I_3| &\leq K(t-s)^{\frac{\eta-\gamma}{\alpha}+\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\\ &\leq K(t-s)^{\frac{\eta-\gamma-1}{\alpha}} W_1^\gamma(\mu_1,\mu_2).\end{aligned}$$

Let us now deal with I_4. We need to control

$$J := \int_{\mathbb{R}^d} \frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(y) \left[\partial_v \frac{\delta}{\delta m} p_m(\mu_1,s,t,x',y)(v) - \partial_v \frac{\delta}{\delta m} p_m(\mu_1,s,t,x'',y)(v)\right] dy.$$

Let us prove that

$$|J| \leq K(t-s)^{\frac{\eta-\gamma-1}{\alpha}+1+\frac{\eta-1}{\alpha}} |x'-x''|^\gamma.$$

Assume first that $|x'-x''| \leq (t-s)^{\frac{1}{\alpha}}$. In this case using (7.20) and (7.7), we have $\int_{\mathbb{R}^d} \partial_v \frac{\delta}{\delta m} p_m(\mu_1,s,t,x',y)(v)\, dy = 0$. We can thus rewrite

$$\begin{aligned}J = \int_{\mathbb{R}^d} &\left[\frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(y) - \frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(x'')\right]\\ &\left[\partial_v \frac{\delta}{\delta m} p_m(\mu_1,s,t,x',y)(v) - \partial_v \frac{\delta}{\delta m} p_m(\mu_1,s,t,x'',y)(v)\right] dy\end{aligned}$$

Thanks to (4.26), (B.14) since $|x'-x''| \leq (t-s)^{\frac{1}{\alpha}}$ and the η-Hölder continuity of $\frac{\delta}{\delta m}b(t,x,\mu)(\cdot)$, we obtain that

$$|J| \leq K \int_{\mathbb{R}^d} |y-x''|^\eta (t-s)^{\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}} |x'-x''|^\gamma \rho^1(t-s, y-x'')\, dy.$$

The space-time inequality (B.13) yields

$$|J| \leq K(t-s)^{\frac{\eta}{\alpha}+\frac{\eta-1-\gamma}{\alpha}+1-\frac{1}{\alpha}} |x'-x''|^\gamma.$$

Assume now that $|x'-x''| > (t-s)^{\frac{1}{\alpha}}$. In this case, we rewrite

$$\begin{aligned}J = &\int_{\mathbb{R}^d} \left[\frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(y) - \frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(x')\right] \partial_v \frac{\delta}{\delta m} p_m(\mu_1,s,t,x',y)(v)\, dy\\ &+ \int_{\mathbb{R}^d} \left[\frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(y) - \frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(x'')\right] \partial_v \frac{\delta}{\delta m} p_m(\mu_1,s,t,x'',y)(v)\, dy.\end{aligned}$$

Then, we use (4.21) (the series appearing in the bound being convergent) and the η-Hölder continuity of $\frac{\delta}{\delta m}b(t,x,\mu)(\cdot)$ which yield

$$\begin{aligned}|J| &\leq K(t-s)^{\frac{\eta}{\alpha}+\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}\\ &\leq K(t-s)^{\frac{\eta}{\alpha}+\frac{\eta-\gamma-1}{\alpha}+1-\frac{1}{\alpha}} |x'-x''|^\gamma.\end{aligned}$$

Jensen's inequality implies that

$$\begin{aligned}|I_4| &\leq K(t-s)^{\frac{\eta-\gamma-1}{\alpha}+1+\frac{\eta-1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\\ &\leq K(t-s)^{\frac{\eta-\gamma-1}{\alpha}} W_1^\gamma(\mu_1,\mu_2).\end{aligned}$$

Finally, for I_5, one has

$$|I_5| = \left| \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(y) - \frac{\delta}{\delta m} b(t,x,[X_t^{s,\mu_2,(m)}])(x') \right) \Delta_{\mu_1,\mu_2} \partial_v \frac{\delta}{\delta m} p_m(\cdot,s,t,x',y)(v)\, dy\, d\mu_2(x') \right|$$
$$\leq K \int_{\mathbb{R}^{2d}} (1 \wedge |x'-y|^\eta) \left| \Delta_{\mu_1,\mu_2} \partial_v \frac{\delta}{\delta m} p_m(\cdot,s,t,x',y) \right| dy\, d\mu_2(x').$$

Gathering all the previous estimates, we have proved that

$$\left| \partial_v \frac{\delta}{\delta m} \left[b(t,x,[X_t^{s,\mu_1,(m)}]) \right](v) - \partial_v \frac{\delta}{\delta m} \left[b(t,x,[X_t^{s,\mu_2,(m)}]) \right](v) \right| \leq K(t-s)^{\frac{\eta-\gamma-1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)$$
$$+ K \int_{\mathbb{R}^d} (1 \wedge |x'-y|^\eta) \left| \Delta_{\mu_1,\mu_2} \partial_v \frac{\delta}{\delta m} p_m(\cdot,s,t,x',y') \right| dy\, d\mu_2(x').$$

Proof of (7.35) and (7.36). Both estimates are immediate consequences of (7.33) and (7.34) using the expression of $\frac{\delta}{\delta m}\mathcal{H}_{m+1}$ given by (7.7) and (4.18).

□

7.6. **Third part of the induction step.** We prove here that the estimates (4.28), (4.29), and (4.30) hold true.

Proof of (4.28). Assume first that $W_1(\mu_1,\mu_2) > (t-s)^{\frac{1}{\alpha}}$. In this case, the parametrix expansion (4.17) (which can be differentiated with respect to x), (B.17) and (4.15) yield

$$\begin{aligned}
\left| \Delta_{\mu_1,\mu_2} \partial_x^j p_m(\cdot,s,t,x,y) \right| &\leq \left| \Delta_{\mu_1,\mu_2} \left[\partial_x^j \widehat{p} \otimes \Phi_m \right](\cdot,s,t,x,y) \right| \\
&\leq \left| \partial_x^j \widehat{p} \otimes \Phi_m(\mu_1,s,t,x,y) \right| + \left| \partial_x^j \widehat{p} \otimes \Phi_m(\mu_2,s,t,x,y) \right| \\
&\leq C(t-s)^{1-\frac{1+j}{\alpha}} \rho^j(t-s,y-x) \\
&\leq C(t-s)^{1-\frac{1+j+\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \rho^j(t-s,y-x).
\end{aligned}$$

We now focus on the case where $W_1(\mu_1,\mu_2) \leq (t-s)^{\frac{1}{\alpha}}$. Let us prove that the following representation formula holds true

$$\Delta_{\mu_1,\mu_2} \partial_x^j p_{m+1}(\cdot,s,t,x,y) = \sum_{k=0}^{\infty} \left(\partial_x^j p_{m+1} \otimes \Delta_{\mu_1,\mu_2} \mathcal{H}_{m+1} \right) \otimes \mathcal{H}_{m+1}^k(\mu_2,s,t,x,y). \tag{7.38}$$

Indeed, using the representation formula (4.17) which can be differentiated with respect to $x \in \mathbb{R}^d$, we get that

$$\Delta_{\mu_1,\mu_2} \partial_x^j p_{m+1}(\cdot,s,t,x,y) = \partial_x^j p_{m+1} \otimes \Delta_{\mu_1,\mu_2} \mathcal{H}_{m+1}(\mu_2,s,t,x,y) + \Delta_{\mu_1,\mu_2} \partial_x^j p_{m+1} \otimes \mathcal{H}_{m+1}(\mu_1,s,t,x,y).$$

Hence, we easily prove by induction that for all $n \geq 1$, one has

$$\Delta_{\mu_1,\mu_2} \partial_x^j p_{m+1}(\cdot,s,t,x,y) = \sum_{k=0}^{n} \left(\partial_x^j p_{m+1} \otimes \Delta_{\mu_1,\mu_2} \mathcal{H}_{m+1} \right) \otimes \mathcal{H}_{m+1}^k(\mu_2,s,t,x,y)$$
$$+ \Delta_{\mu_1,\mu_2} \partial_x^j p_{m+1} \otimes \mathcal{H}_{m+1}^{n+1}(\mu_1,s,t,x,y). \tag{7.39}$$

Thanks to (4.18) and (4.13), we deduce that $\Delta_{\mu_1,\mu_2} \partial_x^j p_{m+1} \otimes \mathcal{H}_{m+1} \otimes \mathcal{H}_{m+1}^{n+1}(\mu_1,s,t,x,y)$ converges to 0 as n tends to infinity. Then, it follows from (4.18), (4.13) and the convolution inequality (B.16) that

$$\begin{aligned}
&\left| \partial_x^j p_{m+1} \otimes \Delta_{\mu_1,\mu_2} \mathcal{H}_{m+1}(\mu_2,s,t,x,y) \right| \\
&\leq \int_s^t \int_{\mathbb{R}^d} \left| \partial_x^j p_{m+1}(\mu_2,s,r,x,z) \Delta_{\mu_1,\mu_2} \mathcal{H}_{m+1}(\cdot,s,r,t,z,y) \right| dz\, dr \\
&\leq K \int_s^t \int_{\mathbb{R}^d} (r-s)^{-\frac{j}{\alpha}} \rho^j(r-s,z-x)(t-r)^{-\frac{1}{\alpha}} \rho^1(t-r,y-z)\, dz\, dr \\
&\leq K(t-s)^{-\frac{j}{\alpha}+1-\frac{1}{\alpha}} \rho^j(t-s,y-x).
\end{aligned}$$

From this estimate and (4.13), we deduce that for any $k \geq 1$

$$\left|\left(\partial_x^j p_{m+1} \otimes \Delta_{\mu_1,\mu_2}\mathcal{H}_{m+1}\right) \otimes \mathcal{H}_{m+1}^k(\mu_2,s,t,x,y)\right| \leq K(t-s)^{-\frac{j}{\alpha}+1-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)}\rho^j(t-s,y-x)$$
$$\mathcal{B}\left(k\left(1-\frac{1}{\alpha}\right),1-\frac{j}{\alpha}+1-\frac{1}{\alpha}\right)\prod_{i=1}^{k-1}\mathcal{B}\left(i\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right).$$

Thanks to the asymptotic behavior of the Beta function, we deduce that the series appearing in (7.39) is absolutely convergent which yields the representation formula (7.38). Now, since $\gamma \in (0,1]$ if $j=0$ and $\gamma \in (0,\eta+\alpha-1)$ if $j=1$ we can chose $\tilde{\gamma} \geq \gamma$ such that $\tilde{\gamma} \in [\eta,1]$ if $j=0$ and $\tilde{\gamma} \in [\eta,\eta+\alpha-1)$ if $j=1$. It follows from (4.18), (7.28), the convolution inequality (B.16) and since $\tilde{\gamma} \in [\eta,\eta+\alpha-1)$ when $j=1$ that

$$\begin{aligned}
&\left|\partial_x^j p_{m+1} \otimes \Delta_{\mu_1,\mu_2}\mathcal{H}_{m+1}(\mu_2,s,t,x,y)\right| \\
&\leq \int_s^t \int_{\mathbb{R}^d} \left|\partial_x^j p_{m+1}(\mu_2,s,r,x,z)\Delta_{\mu_1,\mu_2}\mathcal{H}_{m+1}(\cdot,s,r,t,z,y)\right| \, dz\, dr \\
&\leq K\int_s^t \int_{\mathbb{R}^d} (r-s)^{-\frac{j}{\alpha}}\rho^j(r-s,z-x)(r-s)^{\frac{\eta-\tilde{\gamma}}{\alpha}}(t-r)^{-\frac{1}{\alpha}}W_1^{\tilde{\gamma}}(\mu_1,\mu_2)\rho^1(t-r,y-z)\, dz\, dr \\
&\leq K(t-s)^{\frac{\eta-\tilde{\gamma}-j}{\alpha}+1-\frac{1}{\alpha}}W_1^{\tilde{\gamma}}(\mu_1,\mu_2)\rho^j(t-s,y-x).
\end{aligned}$$

From this estimate and (4.13), we deduce that for any $k \geq 1$

$$\left|\left(\partial_x^j p_{m+1} \otimes \Delta_{\mu_1,\mu_2}\mathcal{H}_{m+1}\right) \otimes \mathcal{H}_{m+1}^k(\mu_2,s,t,x,y)\right| \leq K(t-s)^{\frac{\eta-\tilde{\gamma}-j}{\alpha}+1-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)}W_1^{\tilde{\gamma}}(\mu_1,\mu_2)\rho^j(t-s,y-x)$$
$$\mathcal{B}\left(k\left(1-\frac{1}{\alpha}\right),1+\frac{\eta-\tilde{\gamma}-j}{\alpha}+1-\frac{1}{\alpha}\right)\prod_{i=1}^{k-1}\mathcal{B}\left(i\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right).$$

Summing over $k \geq 0$, we find that since $\gamma \leq \tilde{\gamma}$ and $W_1(\mu_1,\mu_2) < (t-s)^{\frac{1}{\alpha}}$, we have

$$\begin{aligned}
\left|\Delta_{\mu_1,\mu_2}\partial_x^j p_{m+1}(\cdot,s,t,x,y)\right| &\leq K(t-s)^{\frac{\eta-\tilde{\gamma}-j}{\alpha}+1-\frac{1}{\alpha}}W_1^{\tilde{\gamma}}(\mu_1,\mu_2)\rho^j(t-s,y-x) \\
&\leq K(t-s)^{\frac{\eta-\gamma-j}{\alpha}+1-\frac{1}{\alpha}}W_1^{\gamma}(\mu_1,\mu_2)\rho^j(t-s,y-x) \\
&\leq K(t-s)^{1-\frac{1+\gamma+j}{\alpha}}W_1^{\gamma}(\mu_1,\mu_2)\rho^j(t-s,y-x).
\end{aligned}$$

This concludes the proof of (4.28).

Proof of (4.29)**.** We first assume that $W_1(\mu_1,\mu_2) > (t-s)^{\frac{1}{\alpha}}$. In this case, using (4.20), one has

$$\begin{aligned}
&\left|\Delta_{\mu_1,\mu_2}\frac{\delta}{\delta m}p_{m+1}(\cdot,s,t,x,y)(v)\right| \\
&\leq K(t-s)^{1-\frac{1}{\alpha}}\rho^0(t-s,y-x) \\
&\leq K(t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}}W_1^{\gamma}(\mu_1,\mu_2)\rho^0(t-s,y-x) \\
&\leq (t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}}W_1^{\gamma}(\mu_1,\mu_2)\rho^0(t-s,y-x) \\
&\qquad \sum_{k=1}^{m+1} C^k(t-s)^{(k-1)\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1-\frac{\gamma}{\alpha}+(j-1)\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right),
\end{aligned}$$

provided that we choose $C \geq K$ in (4.29). We now treat the case $W_1(\mu_1,\mu_2) \leq (t-s)^{\frac{1}{\alpha}}$. By the representation formula (7.19), we have the following decomposition

$$\begin{aligned}\Delta_{\mu_1,\mu_2}\frac{\delta}{\delta m}p_{m+1}(\cdot,s,t,x,y)(v) &= p_{m+1}\otimes\Delta_{\mu_1,\mu_2}\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu_2,s,t,x,y)(v)\\ &\quad+\Delta_{\mu_1,\mu_2}p_{m+1}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu_1,s,t,x,y)(v)\\ &\quad+\left(\Delta_{\mu_1,\mu_2}\left[p_{m+1}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}\right]\right)\otimes\Phi_{m+1}(\mu_1,s,t,x,y)(v)\\ &\quad+\left[p_{m+1}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}\right]\otimes\Delta_{\mu_1,\mu_2}\Phi_{m+1}(\mu_2,s,t,x,y)(v)\\ &=: I_1+I_2+I_3+I_4.\end{aligned}$$

For I_1, using (4.18), (7.35), the induction assumption (4.29) and the convolution inequality (B.16), one has

$$\begin{aligned}|I_1| &\leq K\int_s^t\int_{\mathbb{R}^d}\rho^0(r-s,z-x)\left[(r-s)^{-\frac{\gamma}{\alpha}}(t-r)^{-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r,y-z)\right.\\ &\qquad\left.+(t-r)^{-\frac{1}{\alpha}}\int_{\mathbb{R}^{2d}}\left|\Delta_{\mu_1,\mu_2}\frac{\delta}{\delta m}p_m(\cdot,s,r,x',y')\right|\,dy'\,d\mu_2(x')\rho^1(t-r,y-z)\right]\,dz\,dr\\ &\leq K(t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x)\\ &\quad+K(t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x)\sum_{k=1}^m C^k(t-s)^{k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^k\mathcal{B}\left(1-\frac{\gamma}{\alpha}+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right).\end{aligned}$$

For I_2, it follows from (4.27), (7.3) (the series appearing in the bound being convergent) and the convolution inequality (B.16) that

$$\begin{aligned}|I_2| &\leq K\int_s^t\int_{\mathbb{R}^d}(r-s)^{1-\frac{1+\gamma}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(r-s,z-x)(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr\\ &\leq K(t-s)^{1-\frac{1+\gamma}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x)\\ &\leq K(t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x).\end{aligned}$$

Concerning I_3, we note that it writes

$$I_3=(I_1+I_2)\otimes\Phi_{m+1}(\mu_1,s,t,x,y)(v).$$

Thus, using the preceding bounds obtained for I_1 and I_2, (4.15) and the convolution inequality (B.16), we get that

$$\begin{aligned}|I_3| &\leq K(t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x)\\ &\qquad\left[1+\sum_{k=1}^m C^k(t-s)^{k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^k\mathcal{B}\left(1-\frac{\gamma}{\alpha}+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right]\\ &\leq K(t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x)\\ &\qquad\left[1+\sum_{k=1}^m C^k(t-s)^{k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^k\mathcal{B}\left(1-\frac{\gamma}{\alpha}+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right].\end{aligned}$$

We finally deal with I_4. The convolution inequality (B.16), (4.18) and (7.3) (the series appearing in this bound being convergent) yield

$$\left|p_{m+1}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu_2,s,t,x,y)(v)\right|\leq K(t-s)^{1-\frac{1}{\alpha}}\rho^0(t-s,y-x).$$

Then, it follows from (7.29), which is valid for all $\gamma \in (0,1]$ since $W_1(\mu_1,\mu_2) < (t-s)^{\frac{1}{\alpha}}$ that

$$\begin{aligned}
|I_4| &\leq K \int_s^t \int_{\mathbb{R}^d} K(r-s)^{1-\frac{1}{\alpha}} \rho^0(r-s, z-x)(r-s)^{\frac{\eta-\gamma}{\alpha}} (t-r)^{-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\rho^1(t-r, y-z)\, dz\, dr \\
&\leq K(t-s)^{\frac{\eta-\gamma}{\alpha}+1-\frac{1}{\alpha}+1-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s, y-x) \\
&\leq K(t-s)^{\frac{\eta-\gamma}{\alpha}+1-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s, y-x).
\end{aligned}$$

Gathering all the previous estimates, we have proved that

$$\begin{aligned}
&\left|\Delta_{\mu_1,\mu_2} \frac{\delta}{\delta m} p_{m+1}(\cdot, s, t, x, y)(v)\right| \\
&\leq K(t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s, y-x) \\
&\qquad \left[1 + \sum_{k=1}^m C^k (t-s)^{k\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^k \mathcal{B}\left(1 - \frac{\gamma}{\alpha} + j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right)\right] \\
&\leq (t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s, y-x) \sum_{k=1}^{m+1} C^k (t-s)^{(k-1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(1 - \frac{\gamma}{\alpha} + j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right),
\end{aligned}$$

provided that we choose $C \geq K$ in (4.29). This ends the proof of the induction step for (4.29).

Proof of (4.30)**.** Assume first that $W_1(\mu_1,\mu_2) > (t-s)^{\frac{1}{\alpha}}$. In this case, (4.20) yields

$$\begin{aligned}
&\left|\Delta_{\mu_1,\mu_2} \partial_v \frac{\delta}{\delta m} p_{m+1}(\cdot, s, t, x, y)(v)\right| \\
&\leq K(t-s)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}} \rho^0(t-s, y-x) \\
&\leq K(t-s)^{\frac{\eta-\gamma-1}{\alpha}+1-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s, y-x) \\
&\leq (t-s)^{\frac{\eta-\gamma-1}{\alpha}+1-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s, y-x) \\
&\qquad \sum_{k=1}^{m+1} C^k (t-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(1 + \frac{\eta-\gamma}{\alpha} + (j-1)\left(1+\frac{\eta-1}{\alpha}\right), 1-\frac{1}{\alpha}\right),
\end{aligned}$$

provided that we choose $C \geq K$ in (4.29). We now treat the case $W_1(\mu_1,\mu_2) \leq (t-s)^{\frac{1}{\alpha}}$. By the representation formula (7.20), we have the following decomposition

$$\begin{aligned}
\Delta_{\mu_1,\mu_2} \partial_v \frac{\delta}{\delta m} p_{m+1}(\cdot, s, t, x, y)(v) &= p_{m+1} \otimes \Delta_{\mu_1,\mu_2} \partial_v \frac{\delta}{\delta m} \mathcal{H}_{m+1}(\mu_2, s, t, x, y)(v) \\
&\quad + \Delta_{\mu_1,\mu_2} p_{m+1} \otimes \partial_v \frac{\delta}{\delta m} \mathcal{H}_{m+1}(\mu_1, s, t, x, y)(v) \\
&\quad + \left(\Delta_{\mu_1,\mu_2}\left[p_{m+1} \otimes \partial_v \frac{\delta}{\delta m} \mathcal{H}_{m+1}\right]\right) \otimes \Phi_{m+1}(\mu_1, s, t, x, y)(v) \\
&\quad + \left[p_{m+1} \otimes \partial_v \frac{\delta}{\delta m} \mathcal{H}_{m+1}\right] \otimes \Delta_{\mu_1,\mu_2} \Phi_{m+1}(\mu_2, s, t, x, y)(v) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For I_1, using (4.18), (7.36), the induction assumption (4.30), the space-time inequality (B.13) and the convolution inequality (B.16), one has since $\gamma \in (0, \eta + \alpha - 1)$

$$\begin{aligned}
|I_1| &\le K \int_s^t \int_{\mathbb{R}^d} \rho^0(r-s, z-x) \Big[(r-s)^{\frac{\eta-\gamma-1}{\alpha}} (t-r)^{-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \rho^1(t-r, y-z) \\
&\qquad + (t-r)^{-\frac{1}{\alpha}} \int_{\mathbb{R}^{2d}} (1 \wedge |x'-y'|^\eta) \left| \Delta_{\mu_1,\mu_2} \frac{\delta}{\delta m} p_m(\cdot, s, r, x', y') \right| dy'\, d\mu_2(x') \rho^1(t-r, y-z) \Big] \, dz\, dr \\
&\le K (t-s)^{\frac{\eta-\gamma-1}{\alpha} + 1 - \frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \rho^0(t-s, y-x) \\
&\qquad \left[1 + \sum_{k=1}^m C^k (t-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^k \mathcal{B}\left(1 + \frac{\eta-\gamma}{\alpha} + j\left(1 + \frac{\eta-1}{\alpha}\right), 1 - \frac{1}{\alpha} \right) \right].
\end{aligned}$$

For I_2, it follows from (4.27), (7.4) and the convolution inequality (B.16) that

$$\begin{aligned}
|I_2| &\le K \int_s^t \int_{\mathbb{R}^d} (r-s)^{1 - \frac{1+\gamma}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \rho^0(r-s, z-x) (r-s)^{\frac{\eta-1}{\alpha}} (t-r)^{-\frac{1}{\alpha}} \rho^1(t-r, y-z)\, dz\, dr \\
&\le K (t-s)^{\frac{\eta-1}{\alpha} + 1 - \frac{1+\gamma}{\alpha} + 1 - \frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \rho^0(t-s, y-x) \\
&\le K (t-s)^{\frac{\eta-\gamma-1}{\alpha} + 1 - \frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \rho^0(t-s, y-x).
\end{aligned}$$

Concerning I_3, we note that it writes

$$I_3 = (I_1 + I_2) \otimes \Phi_{m+1}(\mu_1, s, t, x, y)(v).$$

Thus, using the preceding bounds obtained for I_1 and I_2, (4.15) and the convolution inequality (B.16), we get that

$$\begin{aligned}
|I_3| &\le K (t-s)^{\frac{\eta-\gamma-1}{\alpha} + 1 - \frac{1}{\alpha} + 1 - \frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \rho^0(t-s, y-x) \\
&\qquad \left[1 + \sum_{k=1}^m C^k (t-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^k \mathcal{B}\left(1 + \frac{\eta-\gamma}{\alpha} + j\left(1 + \frac{\eta-1}{\alpha}\right), 1 - \frac{1}{\alpha} \right) \right] \\
&\le K (t-s)^{\frac{\eta-\gamma-1}{\alpha} + 1 - \frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \rho^0(t-s, y-x) \\
&\qquad \left[1 + \sum_{k=1}^m C^k (t-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)} \prod_{j=1}^k \mathcal{B}\left(1 + \frac{\eta-\gamma}{\alpha} + j\left(1 + \frac{\eta-1}{\alpha}\right), 1 - \frac{1}{\alpha} \right) \right].
\end{aligned}$$

We finally deal with I_4. The convolution inequality (B.16), (4.18) and (7.4) (the series appearing in this bound being convergent) yield

$$\left| p_{m+1} \otimes \partial_v \frac{\delta}{\delta m} \mathcal{H}_{m+1}(\mu_2, s, t, x, y)(v) \right| \le K (t-s)^{\frac{\eta-1}{\alpha} + 1 - \frac{1}{\alpha}} \rho^0(t-s, y-x).$$

Then, it follows from (7.29), which is valid for all $\gamma \in (0, 1]$ since $W_1(\mu_1, \mu_2) < (t-s)^{\frac{1}{\alpha}}$ that

$$\begin{aligned}
|I_4| &\le K \int_s^t \int_{\mathbb{R}^d} K (r-s)^{\frac{\eta-1}{\alpha} + 1 - \frac{1}{\alpha}} \rho^0(r-s, z-x) (r-s)^{\frac{\eta-\gamma}{\alpha}} (t-r)^{-\frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \rho^1(t-r, y-z)\, dz\, dr \\
&\le K (t-s)^{\frac{\eta-1}{\alpha} + \frac{\eta-\gamma}{\alpha} + 1 - \frac{1}{\alpha} + 1 - \frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \rho^0(t-s, y-x) \\
&\le K (t-s)^{\frac{\eta-\gamma-1}{\alpha} + 1 - \frac{1}{\alpha}} W_1^\gamma(\mu_1,\mu_2) \rho^0(t-s, y-x).
\end{aligned}$$

Gathering all the previous estimates, we have proved that

$$\begin{aligned}
&\left|\Delta_{\mu_1,\mu_2}\partial_v\frac{\delta}{\delta m}p_{m+1}(\cdot,s,t,x,y)(v)\right| \\
&\leq K(t-s)^{\frac{\eta-\gamma-1}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x) \\
&\qquad\left[1+\sum_{k=1}^m C^k(t-s)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^k\mathcal{B}\left(1+\frac{\eta-\gamma}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right] \\
&\leq (t-s)^{\frac{\eta-\gamma-1}{\alpha}+1-\frac{1}{\alpha}}W_1^\gamma(\mu_1,\mu_2)\rho^0(t-s,y-x) \\
&\qquad\sum_{k=1}^{m+1} C^k(t-s)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(1+\frac{\eta-\gamma}{\alpha}+j\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right),
\end{aligned}$$

provided that we choose $C \geq K$ in (4.30). This ends the proof of the induction step for (4.30).

7.7. **Preparatory technical results.**

Lemma 8. • For any $\gamma \in (0,1]$, $\tilde{\eta} \in (0, \eta \wedge (\alpha-1))$, there exists a positive constant K such that for all $m \geq 1$, $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y, \in \mathbb{R}^d$

$$|\Delta_{s_1,s_2}p_m(\mu,\cdot,t,x,y)| \leq K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1)^\gamma}\rho^{-\tilde{\eta}}(t-s_1,y-x)+\frac{|s_1-s_2|^\gamma}{(t-s_2)^\gamma}\rho^{-\tilde{\eta}}(t-s_2,y-x)\right]. \tag{7.40}$$

- For any $\gamma \in (0,1]$, there exists a positive constant K such that for all $m \geq 1$, $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x \in \mathbb{R}^d$

$$\left|\Delta_{s_1,s_2}b(t,x,[X_t^{\cdot,\mu,(m)}])\right| \leq K|s_1-s_2|^\gamma\left[(t-s_1)^{\frac{\eta}{\alpha}-\gamma}+(t-s_2)^{\frac{\eta}{\alpha}-\gamma}\right]. \tag{7.41}$$

- For any $\gamma \in (0,1]$, there exists a positive constant K such that for all $m \geq 1$, $0 \leq r < t \leq T$, $s_1, s_2 \in [0,r)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$

$$|\Delta_{s_1,s_2}\mathcal{H}_{m+1}(\mu,\cdot,r,t,x,y)| \leq K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x)|s_1-s_2|^\gamma\left[(r-s_1)^{\frac{\eta}{\alpha}-\gamma}+(r-s_2)^{\frac{\eta}{\alpha}-\gamma}\right], \tag{7.42}$$

and

$$|\Delta_{s_1,s_2}\Phi_{m+1}(\mu,\cdot,r,t,x,y)| \leq K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x)|s_1-s_2|^\gamma\left[(r-s_1)^{\frac{\eta}{\alpha}-\gamma}+(r-s_2)^{\frac{\eta}{\alpha}-\gamma}\right]. \tag{7.43}$$

Proof of Lemma (8). **Proof of** (7.40)**.** Assume first that $|s_1-s_2| > t-s_1\vee s_2$. Then, using (4.18), we get that

$$\begin{aligned}
&|\Delta_{s_1,s_2}p_m(\mu,\cdot,t,x,y)| \\
&\leq K\left[\rho^0(t-s_1\vee s_2,y-x)+\rho^0(t-s_1\wedge s_2,y-x)\right] \\
&\leq K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^\gamma}\rho^0(t-s_1\vee s_2,y-x)+\frac{(t-s_1\wedge s_2)^\gamma+|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^\gamma}\rho^0(t-s_1\wedge s_2,y-x)\right] \\
&\leq K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^\gamma}\rho^0(t-s_1\vee s_2,y-x)+\frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^\gamma}\rho^0(t-s_1\wedge s_2,y-x)\right].
\end{aligned}$$

We now focus on the case $|s_1 - s_2| \le t - s_1 \vee s_2$. For $\lambda \in [0,1]$, we set $s_\lambda := \lambda s_1 \vee s_2 + (1-\lambda) s_1 \wedge s_2$. We have thus by (4.22) (the series appearing being convergent)

$$\begin{aligned}
&|\Delta_{s_1,s_2} p_m(\mu,\cdot,t,x,y)| \\
&\le \int_0^1 |\partial_s p_m(\mu, s_\lambda, t, x, y)|\, |s_1 - s_2|\, d\lambda \\
&\le K|s_1 - s_2| \int_0^1 (t - s_\lambda)^{-1} \rho^{-\tilde\eta}(t - s_\lambda, y - x)\, d\lambda \\
&\le C|s_1 - s_2| \int_0^1 (t - s_\lambda)^{-1-\frac{d}{\alpha}} (1 + (t - s_\lambda)^{-\frac{1}{\alpha}} |y - x|)^{-d-\alpha+\tilde\eta}\, d\lambda \\
&\le K|s_1 - s_2|^\gamma (t - s_1 \vee s_2)^{1-\gamma} (t - s_1 \vee s_2)^{-1-\frac{d}{\alpha}} \Big[(1 + (t - s_1 \vee s_2)^{-\frac{1}{\alpha}} |y - x|)^{-d-\alpha+\tilde\eta} \\
&\qquad\qquad + (1 + (t - s_1 \wedge s_2)^{-\frac{1}{\alpha}} |y - x|)^{-d-\alpha+\tilde\eta} \Big],
\end{aligned}$$

for some $\tilde\eta \in (0, \eta \wedge (\alpha - 1))$. Since $|s_1 - s_2| \le t - s_1 \vee s_2$, we easily check that $(t - s_1 \vee s_2)^{-1} \le 2(t - s_1 \wedge s_2)^{-1}$. It follows that

$$\begin{aligned}
&|\Delta_{s_1,s_2} p_m(\mu,\cdot,t,x,y)| \\
&\le K|s_1 - s_2|^\gamma \Big[(t - s_1 \vee s_2)^{-\gamma-\frac{d}{\alpha}} (1 + (t - s_1 \vee s_2)^{-\frac{1}{\alpha}} |y - x|)^{-d-\alpha+\tilde\eta} \\
&\qquad\qquad + (t - s_1 \wedge s_2)^{-\gamma-\frac{d}{\alpha}} (1 + (t - s_1 \wedge s_2)^{-\frac{1}{\alpha}} |y - x|)^{-d-\alpha+\tilde\eta} \Big] \\
&\le K \left[\frac{|s_1 - s_2|^\gamma}{(t - s_1 \wedge s_2)^\gamma} \rho^{-\tilde\eta}(t - s_1 \wedge s_2, y - x) + \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^\gamma} \rho^{-\tilde\eta}(t - s_1 \vee s_2, y - x) \right].
\end{aligned}$$

This concludes the proof of (7.40).

Proof of (7.41)**.** By definition of the linear derivative and an centering argument, one can write setting $m_\lambda := \lambda [X_t^{s_1 \vee s_2, \mu, (m)}] + (1-\lambda)[X_t^{s_1 \wedge s_2, \mu, (m)}]$ for $\lambda \in [0,1]$

$$\Delta_{s_1,s_2} b(t, x, [X_t^{\cdot,\mu,(m)}]) = \int_{\mathbb{R}^{2d}} \left(\frac{\delta}{\delta m} b(t, x, m_\lambda)(y) - \frac{\delta}{\delta m} b(t, x, m_\lambda)(x') \right) \Delta_{s_1,s_2} p_m(\mu, \cdot, t, x', y)\, d\mu(x')\, dy\, d\lambda.$$

We deduce using (7.40) for some $\tilde\eta \in (0, \eta \wedge (\alpha - 1))$, the boundedness of $\frac{\delta}{\delta m} b$ and the η-Hölder continuity of $\frac{\delta}{\delta m} b(t, x, \mu)(\cdot)$ that

$$\begin{aligned}
&\left| \Delta_{s_1,s_2} b(t, x, [X_t^{\cdot,\mu,(m)}]) \right| \\
&\le K \int_{\mathbb{R}^{2d}} |y - x'|^\eta \left[\frac{|s_1 - s_2|^\gamma}{(t - s_1 \wedge s_2)^\gamma} \rho^{-\tilde\eta}(t - s_1 \wedge s_2, y - x') \right. \\
&\qquad\qquad \left. + \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^\gamma} \rho^{-\tilde\eta}(t - s_1 \vee s_2, y - x') \right] dy\, d\mu(x').
\end{aligned}$$

Notice that because $\eta + \tilde\eta < \alpha$, $\rho^{-\tilde\eta-\eta}(t - s, \cdot)$ is integrable. By the space-time inequality (B.13) and since $\int_{\mathbb{R}^d} \rho^{-\tilde\eta-\eta}(t - s, y)\, dy$ is a finite constant independent of s and t, we obtain (7.41).

Proof of (7.42) **and** (7.43)**.** By the definition (4.10) of $\mathcal{H}_{m+1}$ and since $\widehat{p}(\mu, s, r, t, x, y)$ does not depend on μ and s, we immediately deduce (7.42) from (7.41) and (B.17).

Concerning the proof of (7.43), we start from the Volterra integral equation (4.11) which yields

$$\Delta_{s_1,s_2}\Phi_{m+1}(\mu,\cdot,r,t,x,y) = \Delta_{s_1,s_2}\mathcal{H}_{m+1}(\mu,\cdot,r,t,x,y) + \Delta_{s_1,s_2}\mathcal{H}_{m+1}\otimes\Phi_{m+1}(\mu,s_1\vee s_2,r,t,x,y) + \mathcal{H}_{m+1}\otimes\Delta_{s_1,s_2}\Phi_{m+1}(\mu,s_1\wedge s_2,r,t,x,y). \tag{7.44}$$

Using (7.42) and (4.15), we deduce that

$$\begin{aligned}
&|\Delta_{s_1,s_2}\mathcal{H}_{m+1}\otimes\Phi_{m+1}(\mu,s_1\vee s_2,r,t,x,y)|\\
&\le K\int_r^t\int_{\mathbb{R}^d}(r'-r)^{-\frac{1}{\alpha}}\rho^1(r'-r,y-x)|s_1-s_2|^\gamma\left[(r'-s_1)^{\frac{\eta}{\alpha}-\gamma}+(r'-s_2)^{\frac{\eta}{\alpha}-\gamma}\right](t-r')^{-\frac{1}{\alpha}}\rho^1(t-r',y-z)\,dz\,dr'\\
&\le K(t-r)^{-\frac{1}{\alpha}+1-\frac{1}{\alpha}}\left[(r-s_1)^{\frac{\eta}{\alpha}-\gamma}+(r-s_2)^{\frac{\eta}{\alpha}-\gamma}\right]\rho^1(t-r,y-x).
\end{aligned}$$

This inequality and (7.42) ensure that

$$|[\Delta_{s_1,s_2}\mathcal{H}_{m+1}+\Delta_{s_1,s_2}\mathcal{H}_{m+1}\otimes\Phi_{m+1}](\mu,s_1\vee s_2,r,t,x,y)| \le K(t-r)^{-\frac{1}{\alpha}}\left[(r-s_1)^{\frac{\eta}{\alpha}-\gamma}+(r-s_2)^{\frac{\eta}{\alpha}-\gamma}\right]\rho^1(t-r,y-x). \tag{7.45}$$

As Φ_{m+1} yields a time-integrable singularity, we can iterate the implicit representation formula (7.44). We thus obtain

$$\Delta_{s_1,s_2}\Phi_{m+1}(\mu,\cdot,t,x,y) = [\Delta_{s_1,s_2}\mathcal{H}_{m+1}+\Delta_{s_1,s_2}\mathcal{H}_{m+1}\otimes\Phi_{m+1}](\mu,s_1\vee s_2,t,x,y) + \sum_{k=1}^\infty\mathcal{H}_{m+1}^k\otimes[\Delta_{s_1,s_2}\mathcal{H}_{m+1}+\Delta_{s_1,s_2}\mathcal{H}_{m+1}\otimes\Phi_{m+1}](\mu,s_1\wedge s_2,t,x,y).$$

After standard computations using (7.45) and (4.13) that we omit, we conclude that (7.43) holds true.

□

7.8. **Fourth part of the induction step.** We prove here that the estimate (4.31) holds true.

Proof of (4.31). We start by assuming that $|s_1-s_2|>t-s_1\vee s_2$. In this case, (4.18) directly yields

$$\begin{aligned}
&\left|\Delta_{s_1,s_2}\partial_x^j p_m(\mu,\cdot,t,x,y)\right|\\
&\le\left|\partial_x^j p_m(\mu,s_1,t,x,y)\right|+\left|\partial_x^j p_m(\mu,s_2,t,x,y)\right|\\
&\le K\left[(t-s_1\vee s_2)^{-\frac{j}{\alpha}}\rho^j(t-s_1\vee s_2,y-x)+(t-s_1\wedge s_2)^{-\frac{j}{\alpha}}\rho^j(t-s_1\wedge s_2,y-x)\right]\\
&\le K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\vee s_2,y-x)+\frac{|s_1-s_2|^\gamma+(t-s_1\vee s_2)^\gamma}{(t-s_1\wedge s_2)^\gamma(t-s_1\wedge s_2)^{\frac{j}{\alpha}}}\rho^j(t-s_1\wedge s_2,y-x)\right]\\
&\le K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\vee s_2,y-x)+\frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\wedge s_2,y-x)\right],
\end{aligned}$$

which proves (4.31). We now turn to the case $|s_1-s_2|\le t-s_1\vee s_2$.
Differentiating with respect to x the parametrix expansion (4.17), we get

$$\partial_x^j p_m(\mu,s,t,x,y) = \partial_x^j\widehat{p}(s,t,x,y)+\partial_x^j\widehat{p}\otimes\Phi_m(\mu,s,t,x,y).$$

We are going to use the following decomposition

$$\begin{aligned}\Delta_{s_1,s_2}\partial_x^j p_m(\mu,\cdot,t,x,y) &= \Delta_{s_1,s_2}\partial_x^j \widehat{p}(\cdot,t,x,y)\\ &\quad+\int_{s_1\vee s_2}^t\int_{\mathbb{R}^d}\Delta_{s_1,s_2}\partial_x^j\widehat{p}(\cdot,r,x,z)\Phi_m(\mu,s_1\vee s_2,r,t,z,y)\,dz\,dr\\ &\quad+\int_{s_1\vee s_2}^t\int_{\mathbb{R}^d}\partial_x^j\widehat{p}(s_1\wedge s_2,r,x,z)\Delta_{s_1,s_2}\Phi_m(\mu,\cdot,r,t,z,y)\,dz\,dr\\ &\quad-\int_{s_1\wedge s_2}^{s_1\vee s_2}\int_{\mathbb{R}^d}\partial_x^j\widehat{p}(s_1\wedge s_2,r,x,z)\Phi_m(\mu,s_1\wedge s_2,r,t,z,y)\,dz\,dr\\ &=: I_1+I_2+I_3+I_4.\end{aligned}$$

By (B.20), we obtain that

$$|I_1|\le K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\wedge s_2,y-x)+\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\vee s_2,y-x)\right].$$

We now focus on I_2. Thanks to (B.20), (4.15) and the convolution inequality (B.16), one has

$$\begin{aligned}|I_2|&\le K\int_{s_1\vee s_2}^t\int_{\mathbb{R}^d}\left[\frac{|s_1-s_2|^\gamma}{(r-s_1\wedge s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(r-s_1\wedge s_2,z-x)+\frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(r-s_1\vee s_2,z-x)\right]\\ &\qquad\qquad(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr\\ &\le K\int_{s_1\wedge s_2}^t\int_{\mathbb{R}^d}\frac{|s_1-s_2|^\gamma}{(r-s_1\wedge s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(r-s_1\wedge s_2,z-x)(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr\\ &\quad+K\int_{s_1\vee s_2}^t\int_{\mathbb{R}^d}\frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(r-s_1\vee s_2,z-x)(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr\\ &\le K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^{\gamma+\frac{j}{\alpha}-1+\frac{1}{\alpha}}}\rho^j(t-s_1\wedge s_2,y-x)+\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{j}{\alpha}-1+\frac{1}{\alpha}}}\rho^j(t-s_1\vee s_2,y-x)\right]\\ &\le K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\wedge s_2,y-x)+\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\vee s_2,y-x)\right].\end{aligned}$$

For I_3, it follows from (B.17), (7.43) and the convolution inequality (B.16) that

$$\begin{aligned}|I_3|&\le K\int_{s_1\vee s_2}^t\int_{\mathbb{R}^d}(r-s_1\wedge s_2)^{-\frac{j}{\alpha}}\rho^j(r-s_1\wedge s_2,z-x)(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\\ &\qquad\qquad|s_1-s_2|^\gamma\left[\frac{1}{(r-s_1\vee s_2)^\gamma}+\frac{1}{(r-s_1\wedge s_2)^\gamma}\right]dz\,dr\\ &\le K|s_1-s_2|^\gamma\int_{s_1\vee s_2}^t(r-s_1\vee s_2)^{-\frac{j}{\alpha}-\gamma}(t-r)^{-\frac{1}{\alpha}}\,dr\left[\rho^j(t-s_1\vee s_2,y-x)+\rho^j(t-s_1\wedge s_2,y-x)\right]\\ &\le K|s_1-s_2|^\gamma(t-s_1\vee s_2)^{-\frac{j}{\alpha}-\gamma+1-\frac{1}{\alpha}}\left[\rho^j(t-s_1\vee s_2,y-x)+\rho^j(t-s_1\wedge s_2,y-x)\right]\\ &\le K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\frac{j}{\alpha}+\gamma}}\rho^j(t-s_1\vee s_2,y-x)+\frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^{\frac{j}{\alpha}+\gamma}}\rho^j(t-s_1\wedge s_2,y-x)\right],\end{aligned}$$

where the last inequality comes from the fact that $t-s_1\wedge s_1\le 2(t-s_1\vee s_2)$. We finally deal with I_4. Owing to (B.17) and (4.15), we have

$$
\begin{aligned}
|I_4| &\leq K \int_{s_1\wedge s_2}^{s_1\vee s_2} \int_{\mathbb{R}^d} (r-s_1\wedge s_2)^{-\frac{j}{\alpha}} \rho^j(r-s_1\wedge s_2, z-x)(t-r)^{-\frac{1}{\alpha}} \rho^1(t-r, y-z)\, dz\, dr \\
&\leq K \int_{s_1\wedge s_2}^{s_1\vee s_2} (r-s_1\wedge s_2)^{-\frac{j}{\alpha}} (t-r)^{-\frac{1}{\alpha}}\, dr \rho^j(t-s_1\wedge s_2, y-x) \\
&\leq K(t-s_1\vee s_2)^{-\frac{1}{\alpha}} \int_{s_1\wedge s_2}^{s_1\vee s_2} (r-s_1\wedge s_2)^{-\frac{j}{\alpha}}\, dr \rho^j(t-s_1\wedge s_2, y-x) \\
&\leq K(t-s_1\vee s_2)^{-\frac{1}{\alpha}} |s_1-s_2|^{1-\frac{j}{\alpha}} \left[\rho^j(t-s_1\vee s_2, y-x) + \rho^j(t-s_1\wedge s_2, y-x)\right].
\end{aligned}
$$

Since by assumption $\gamma \leq 1-\frac{j}{\alpha}$, we obtain

$$
\begin{aligned}
|I_4| &\leq K(t-s_1\vee s_2)^{-\frac{1}{\alpha}-\gamma+1-\frac{j}{\alpha}} |s_1-s_2|^{\gamma} \left[\rho^j(t-s_1\vee s_2, y-x) + \rho^j(t-s_1\wedge s_2, y-x)\right] \\
&\leq K\left[\frac{|s_1-s_2|^{\gamma}}{(t-s_1\vee s_2)^{\frac{j}{\alpha}+\gamma}} \rho^j(t-s_1\vee s_2, y-x) + \frac{|s_1-s_2|^{\gamma}}{(t-s_1\wedge s_2)^{\frac{j}{\alpha}+\gamma}} \rho^j(t-s_1\wedge s_2, y-x)\right],
\end{aligned}
$$

where the last inequality comes from the fact that $t-s_1\wedge s_1 \leq 2(t-s_1\vee s_2)$ and $1-\frac{1}{\alpha} > 0$. This concludes the proof of (4.31).

7.9. **Preparatory technical results.**

Lemma 9.

- For any $\gamma \in (0,1]$, there exists a positive constant K such that for all $m \geq 1$, $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, v \in \mathbb{R}^d$

$$
\left|\Delta_{s_1,s_2} \frac{\delta}{\delta m} b(t,x,[X_t^{\cdot,\mu,(m)}])(v)\right| \leq K|s_1-s_2|^{\gamma} \left[(t-s_1)^{\frac{\eta}{\alpha}-\gamma} + (t-s_2)^{\frac{\eta}{\alpha}-\gamma}\right]. \tag{7.46}
$$

- For any $\gamma \in (0,1]$, there exists a positive constant K such that for all $m \geq 1$, $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, v \in \mathbb{R}^d$

$$
\left|\Delta_{s_1,s_2} \frac{\delta}{\delta m}\left[b(t,x,[X_t^{\cdot,\mu,(m)}])\right](v)\right| \leq K\frac{|s_1-s_2|^{\gamma}}{(t-s_1\vee s_2)^{\gamma}} + K\int_{\mathbb{R}^{2d}} \left|\Delta_{s_1,s_2} \frac{\delta}{\delta m} p_m(\mu,\cdot,t,x',y)\right| dy\, d\mu(x'). \tag{7.47}
$$

- For any $\gamma \in (0,1]$, there exists a positive constant K such that for all $m \geq 1$, $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $r \in [s_1\vee s_2, t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y, v \in \mathbb{R}^d$

$$
\begin{aligned}
\left|\Delta_{s_1,s_2} \frac{\delta}{\delta m} \mathcal{H}_{m+1}(\mu,\cdot,r,t,x,y)(v)\right| &\leq K(t-r)^{-\frac{1}{\alpha}} \rho^1(t-r, y-x) \\
&\quad \left[\frac{|s_1-s_2|^{\gamma}}{(r-s_1\vee s_2)^{\gamma}} + \int_{\mathbb{R}^{2d}} \left|\Delta_{s_1,s_2} \frac{\delta}{\delta m} p_m(\mu,\cdot,r,x',y)\right| dy\, d\mu(x')\right].
\end{aligned} \tag{7.48}
$$

- For any $\gamma \in (0,1]$, there exists a positive constant K such that for all $m \geq 1$, $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, v \in \mathbb{R}^d$

$$
\begin{aligned}
\left|\Delta_{s_1,s_2} \partial_v \frac{\delta}{\delta m}\left[b(t,x,[X_t^{\cdot,\mu,(m)}])\right](v)\right| &\leq K\frac{|s_1-s_2|^{\gamma}}{(t-s_1\vee s_2)^{\gamma+\frac{1-\eta}{\alpha}}} \\
&\quad + K\int_{\mathbb{R}^{2d}} (1\wedge|y-x'|^{\eta}) \left|\Delta_{s_1,s_2} \partial_v \frac{\delta}{\delta m} p_m(\mu,\cdot,t,x',y)\right| dy\, d\mu(x').
\end{aligned} \tag{7.49}
$$

- For any $\gamma \in (0,1]$, there exists a positive constant K such that for all $m \geq 1$, $t \in (0,T]$, $s_1, s_2 \in [0,t)$, $r \in [s_1 \vee s_2, t)$, $\mu \in \mathcal{P}_\beta(\mathbb{R}^d)$, $x, y, v \in \mathbb{R}^d$

$$\left|\Delta_{s_1,s_2}\partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,\cdot,r,t,x,y)(v)\right| \leq K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x) \left[\frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^{\gamma+\frac{1-\eta}{\alpha}}} + \int_{\mathbb{R}^{2d}}(1\wedge|y-x'|^\eta)\left|\Delta_{s_1,s_2}\partial_v\frac{\delta}{\delta m}p_m(\mu,\cdot,r,x',y)\right| dy\, d\mu(x')\right]. \quad (7.50)$$

Proof of Lemma 9. **Proof of (7.46).** By definition of the linear derivative and a centering argument, one can write setting $m_\lambda := \lambda[X_t^{s_1\vee s_2,\mu,(m)}] + (1-\lambda)[X_t^{s_1\wedge s_2,\mu,(m)}]$ for $\lambda \in [0,1]$

$$\Delta_{s_1,s_2}\frac{\delta}{\delta m}b(t,x,[X_t^{\cdot,\mu,(m)}])(v) = \int_{\mathbb{R}^{2d}}\left(\frac{\delta^2}{\delta m^2}b(t,x,m_\lambda)(v,y) - \frac{\delta^2}{\delta m^2}b(t,x,m_\lambda)(v,x')\right) \Delta_{s_1,s_2}p_m(\mu,\cdot,t,x',y)\, d\mu(x')\, dy\, d\lambda.$$

We deduce using (7.40) and the η-Hölder continuity of $\frac{\delta^2}{\delta m^2}b(t,x,\mu)(v,\cdot)$ that

$$\begin{aligned}&\left|\Delta_{s_1,s_2}\frac{\delta}{\delta m}b(t,x,[X_t^{\cdot,\mu,(m)}])(v)\right| \\ &\leq K\int_{\mathbb{R}^{2d}}|y-x'|^\eta\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^\gamma}\rho^{-\tilde\eta}(t-s_1\wedge s_2,y-x') \right. \\ &\qquad \left. + \frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^\gamma}\rho^{-\tilde\eta}(t-s_1\vee s_2,y-x')\right] dy\, d\mu(x').\end{aligned}$$

Notice that because $\eta+\tilde\eta<\alpha$, $\rho^{-\tilde\eta-\eta}(t-s,\cdot)$ is integrable. We conclude by the space-time inequality (B.13) and since $\int_{\mathbb{R}^d}\rho^{-\tilde\eta-\eta}(t-s,y)\,dy$ is a finite constant independent of s and t.

Proof of (7.47) and (7.48). Coming back to (7.5), we have the following decomposition

$$\begin{aligned}&\Delta_{s_1,s_2}\frac{\delta}{\delta m}\left[b(t,x,[X_t^{\cdot,\mu,(m)}])\right](v) \\ &= \int_{\mathbb{R}^d}\Delta_{s_1,s_2}\frac{\delta}{\delta m}b(t,x,[X_t^{\cdot,\mu,(m)}])(y)p_m(\mu,s_1\vee s_2,t,v,y)\,dy \\ &\quad + \int_{\mathbb{R}^d}\frac{\delta}{\delta m}b(t,x,[X_t^{s_1\wedge s_2,\mu,(m)}])(y)\Delta_{s_1,s_2}p_m(\mu,\cdot,t,v,y)\,dy \\ &\quad + \int_{\mathbb{R}^{2d}}\Delta_{s_1,s_2}\frac{\delta}{\delta m}b(t,x,[X_t^{\cdot,\mu,(m)}])(y)\frac{\delta}{\delta m}p_m(\mu,s_1\vee s_2,t,x',y)(v)\,d\mu(x')\,dy \\ &\quad + \int_{\mathbb{R}^{2d}}\frac{\delta}{\delta m}b(t,x,[X_t^{s_1\wedge s_2,\mu,(m)}])(y)\Delta_{s_1,s_2}\frac{\delta}{\delta m}p_m(\mu,\cdot,t,x',y)(v)\,d\mu(x')\,dy \\ &=: I_1+I_2+I_3+I_4.\end{aligned}$$

It immediately follows from (7.46) and (4.18) that

$$\begin{aligned}|I_1| &\leq K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1)^{\gamma-\frac{\eta}{\alpha}}} + \frac{|s_1-s_2|^\gamma}{(t-s_2)^{\gamma-\frac{\eta}{\alpha}}}\right] \\ &\leq K\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^\gamma}.\end{aligned}$$

Then, the boundedness of $\frac{\delta}{\delta m}b$ and (4.31) yield

$$|I_2| \leq K \int_{\mathbb{R}^d} \frac{|s_1 - s_2|^\gamma}{(t-s_1)^\gamma} \rho^{-\tilde{\eta}}(t - s_1, y - v) + \frac{|s_1 - s_2|^\gamma}{(t-s_2)^\gamma} \rho^{-\tilde{\eta}}(t - s_2, y - v)\, dy$$
$$\leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^\gamma}.$$

Similarly to I_1 and using (4.20), we deduce that

$$|I_3| \leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^\gamma}.$$

Gathering the previous estimates on I_1, I_2, I_3 and noting that $\frac{\delta}{\delta m} b$ is bounded, we conclude that (7.47) holds true. Remark that (7.48) follows directly from the expression of $\frac{\delta}{\delta m} \mathcal{H}_{m+1}$ given in (7.7) and from (7.47).

Proof of (7.49)**.** We use (7.6), which yields the following decomposition

$$\begin{aligned}
&\Delta_{s_1,s_2} \partial_v \frac{\delta}{\delta m} \left[b(t, x, [X_t^{\cdot,\mu,(m)}]) \right](v) \\
&= \int_{\mathbb{R}^d} \Delta_{s_1,s_2} \frac{\delta}{\delta m} b(t, x, [X_t^{\cdot,\mu,(m)}])(y) \partial_x p_m(\mu, s_1 \vee s_2, t, v, y)\, dy \\
&\quad + \int_{\mathbb{R}^d} \frac{\delta}{\delta m} b(t, x, [X_t^{s_1 \wedge s_2,\mu,(m)}])(y) \Delta_{s_1,s_2} \partial_x p_m(\mu, \cdot, t, v, y)\, dy \\
&\quad + \int_{\mathbb{R}^{2d}} \Delta_{s_1,s_2} \frac{\delta}{\delta m} b(t, x, [X_t^{\cdot,\mu,(m)}])(y) \partial_v \frac{\delta}{\delta m} p_m(\mu, s_1 \vee s_2, t, x', y)(v)\, d\mu(x')\, dy \\
&\quad + \int_{\mathbb{R}^{2d}} \frac{\delta}{\delta m} b(t, x, [X_t^{s_1 \wedge s_2,\mu,(m)}])(y) \Delta_{s_1,s_2} \partial_v \frac{\delta}{\delta m} p_m(\mu, \cdot, t, x', y)(v)\, d\mu(x')\, dy \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

It immediately follows from (7.46) and (4.18) that

$$|I_1| \leq K \left[\frac{|s_1 - s_2|^\gamma}{(t-s_1)^{\gamma - \frac{\eta}{\alpha}}} + \frac{|s_1 - s_2|^\gamma}{(t-s_2)^{\gamma - \frac{\eta}{\alpha}}} \right] (t - s_1 \vee s_2)^{-\frac{1}{\alpha}}$$
$$\leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1-\eta}{\alpha}}}.$$

We rewrite I_2 in the following form

$$I_2 = \int_{\mathbb{R}^d} \left(\frac{\delta}{\delta m} b(t, x, [X_t^{s_1 \wedge s_2,\mu,(m)}])(y) - \frac{\delta}{\delta m} b(t, x, [X_t^{s_1 \wedge s_2,\mu,(m)}])(v) \right) \Delta_{s_1,s_2} \partial_x p_m(\mu, \cdot, t, v, y)\, dy.$$

Then, the η-Hölder continuity of $\frac{\delta}{\delta m} b(t, x, \mu)(\cdot)$ and (4.31) yield

$$|I_2| \leq K \int_{\mathbb{R}^d} \frac{|s_1 - s_2|^\gamma}{(t-s_1)^{\gamma + \frac{1}{\alpha}}} |y - v|^\eta \rho^j(t - s_1, y - v) + \frac{|s_1 - s_2|^\gamma}{(t-s_2)^{\gamma + \frac{1}{\alpha}}} |y - v|^\eta \rho^j(t - s_2, y - v)\, dy.$$

Since $\rho^{j-\eta}(t - s, \cdot)$ is integrable and $\int_{\mathbb{R}^d} \rho^{j-\eta}(t - s, z)\, dz$ is a constant independent of s and t. Thus, the space-time inequality (B.13) ensures that

$$|I_2| \leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1-\eta}{\alpha}}}.$$

By using (7.46) and (4.21), we deduce that

$$|I_3| \le K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1)^{\gamma-\frac{\eta}{\alpha}}}+\frac{|s_1-s_2|^\gamma}{(t-s_2)^{\gamma-\frac{\eta}{\alpha}}}\right](t-s_1\vee s_2)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}$$
$$\le K\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1-\eta}{\alpha}}}.$$

Note that since $\int_{\mathbb{R}^d}\partial_v\frac{\delta}{\delta m}p_m(\mu,s,t,x,y)(v)\,dy=0$, one can rewrite I_4 as

$$I_4=\int_{\mathbb{R}^{2d}}\left(\frac{\delta}{\delta m}b(t,x,[X_t^{s_1\wedge s_2,\mu,(m)}])(y)-\frac{\delta}{\delta m}b(t,x,[X_t^{s_1\wedge s_2,\mu,(m)}])(x')\right)\Delta_{s_1,s_2}\partial_v\frac{\delta}{\delta m}p_m(\mu,\cdot,t,x',y)(v)\,d\mu(x')\,dy.$$

The boundedness of $\frac{\delta}{\delta m}b$ and the η-Hölder continuity of $\frac{\delta}{\delta m}b(t,x,\mu)(\cdot)$ yield

$$|I_4|\le\int_{\mathbb{R}^{2d}}(1\wedge|y-x'|^\eta)\left|\Delta_{s_1,s_2}\partial_v\frac{\delta}{\delta m}p_m(\mu,\cdot,r,x',y)\right|\,dy\,d\mu(x').$$

Gathering the previous estimates on I_1, I_2, I_3 and I_4, we conclude that (7.49) holds true. Finally, notice that (7.50) follows directly from the expression of $\partial_v\frac{\delta}{\delta m}\mathcal{H}_{m+1}$ obtained by differentiating with respect to v (7.7) and from (7.49).

□

7.10. **Fifth part of the induction step.** We prove here that the estimates (4.32) and (4.33) hold true.

Proof of (4.32). We separate the proof of the induction step into two disjoint cases. First, we assume that $|s_1-s_2|>t-s_1\vee s_2$. By (4.20) (the series appearing in the bound being convergent), we can write

$$\begin{aligned}
&\left|\Delta_{s_1,s_2}\frac{\delta}{\delta m}p_m(\mu,\cdot,t,x,y)(v)\right|\\
&\le\left|\frac{\delta}{\delta m}p_m(\mu,s_1,t,x,y)(v)\right|+\left|\frac{\delta}{\delta m}p_m(\mu,s_2,t,x,y)(v)\right|\\
&\le K\left[(t-s_1\vee s_2)^{1-\frac{1}{\alpha}}\rho^0(t-s_1\vee s_2,y-x)+(t-s_1\wedge s_2)^{1-\frac{1}{\alpha}}\rho^0(t-s_1\wedge s_2,y-x)\right]\\
&\le K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1}}\rho^0(t-s_1\vee s_2,y-x)+\frac{|s_1-s_2|^\gamma+(t-s_1\vee s_2)^\gamma}{(t-s_1\wedge s_2)^\gamma(t-s_1\wedge s_2)^{\frac{1}{\alpha}-1}}\rho^0(t-s_1\wedge s_2,y-x)\right]\\
&\le K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1}}\rho^0(t-s_1\vee s_2,y-x)+\frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1}}\rho^0(t-s_1\wedge s_2,y-x)\right],
\end{aligned}$$

This shows (4.32) holds true at step $m+1$ provided that we choose $C\ge K$ in (4.32). We now turn to the case $|s_1-s_2|\le t-s_1\vee s_2$. Using the representation formula (7.19), we get that

$$\begin{aligned}
\Delta_{s_1,s_2}\frac{\delta}{\delta m}p_{m+1}(\mu,\cdot,t,x,y)(v)&=\Delta_{s_1,s_2}\left(p_{m+1}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}\right)(\mu,\cdot,t,x,y)(v)\\
&\quad+\Delta_{s_1,s_2}\left(p_{m+1}\otimes\frac{\delta}{\delta m}\mathcal{H}_{m+1}\otimes\Phi_{m+1}\right)(\mu,\cdot,t,x,y)(v)\\
&=:I_1+I_2.
\end{aligned}$$

Then, we decompose I_1 in the following way

$$
\begin{aligned}
I_1 &= \int_{s_1\vee s_2}^t \int_{\mathbb{R}^d} \Delta_{s_1,s_2} p_{m+1}(\mu,\cdot,r,x,z)\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,s_1\vee s_2,r,t,z,y)\,dz\,dr \\
&\quad + \int_{s_1\vee s_2}^t \int_{\mathbb{R}^d} p_{m+1}(\mu,s_1\wedge s_2,r,x,z)\Delta_{s_1,s_2}\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,\cdot,r,t,z,y)\,dz\,dr \\
&\quad - \int_{s_1\wedge s_2}^{s_1\vee s_2} \int_{\mathbb{R}^d} p_{m+1}(\mu,s_1\wedge s_2,r,x,z)\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,s_1\wedge s_2,r,t,z,y)\,dz\,dr \\
&=: I_{1,1}+I_{1,2}+I_{1,3}.
\end{aligned}
$$

It follows from (4.31), (7.3) (the series appearing in the bound being convergent) and the convolution inequality (B.16) that

$$
\begin{aligned}
|I_{1,1}| &\le K\int_{s_1\vee s_2}^t \int_{\mathbb{R}^d} \frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^\gamma}\left[\rho^0(r-s_1\vee s_2,y-x)+\rho^0(r-s_1\wedge s_2,y-x)\right] \\
&\qquad\qquad (t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr \\
&\le K\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1}}\left[\rho^0(t-s_1\vee s_2,y-x)\rho^0(t-s_1\wedge s_2,y-x)\right].
\end{aligned}
$$

We now deal with $I_{1,2}$. Using the induction assumption and (7.48), we deduce that

$$
\begin{aligned}
&\left|\Delta_{s_1,s_2}\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,\cdot,r,t,x,y)(v)\right| \\
&\le K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x)\left[\frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^\gamma}+\int_{\mathbb{R}^{2d}}\left|\Delta_{s_1,s_2}\frac{\delta}{\delta m}p_m(\mu,\cdot,r,x',y)\right|\,dy\,d\mu(x')\right] \\
&\le K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x)\frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^\gamma} \qquad (7.51)\\
&\quad + K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x)\left[\frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1}}+\frac{|s_1-s_2|^\gamma}{(r-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1}}\right] \\
&\qquad \sum_{k=1}^m C^k(r-s_1\vee s_2)^{(k-1)(1-\frac{1}{\alpha})}\prod_{j=1}^{k-1}\mathcal{B}\left(\left[2-\gamma-\frac{1}{\alpha}\right]\wedge 1+(j-1)\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right).
\end{aligned}
$$

Notice that

$$
\begin{cases}
(r-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1}\ge (r-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1} & \text{if } \gamma+\frac{1}{\alpha}-1\ge 0,\\
(r-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1}\ge (t-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1}\ge 2^{\gamma+\frac{1}{\alpha}-1}(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1} & \text{if } \gamma+\frac{1}{\alpha}-1<0
\end{cases}
$$

since $t-s_1\wedge s_2\le 2(t-s_1\vee s_2)$. Using these inequalities, one has

$$
\begin{aligned}
&\int_{s_1\vee s_2}^t (t-r)^{-\frac{1}{\alpha}}\left[\frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1}}+\frac{|s_1-s_2|^\gamma}{(r-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1}}\right](r-s_1\vee s_2)^{(k-1)(1-\frac{1}{\alpha})}\,dr \\
&\le K\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1}}(t-s_1\vee s_2)^{k(1-\frac{1}{\alpha})}\mathcal{B}\left(1-\gamma+k\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right) \\
&\quad + K\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1}}(t-s_1\vee s_2)^{k(1-\frac{1}{\alpha})}\mathcal{B}\left(1+(k-1)\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right) \\
&\le K\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1}}(t-s_1\vee s_2)^{k(1-\frac{1}{\alpha})}\mathcal{B}\left(\left[2-\gamma-\frac{1}{\alpha}\right]\wedge 1+(k-1)\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right),
\end{aligned}
$$

since the Beta function is decreasing with respect to its first argument. From this inequality, (7.51) and (4.18), we obtain

$$|I_{1,2}| \leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1}{\alpha} - 1}} \rho^0(t - s_1 \wedge s_2, y - x) \tag{7.52}$$
$$\left[1 + \sum_{k=1}^{m} C^k (t - s_1 \vee s_2)^{k\left(1 - \frac{1}{\alpha}\right)} \prod_{j=1}^{k} \mathcal{B}\left(\left[2 - \gamma - \frac{1}{\alpha}\right] \wedge 1 + (j-1)\left(1 - \frac{1}{\alpha}\right), 1 - \frac{1}{\alpha}\right)\right].$$

We now turn to the estimate on $I_{1,3}$. Thanks to (4.18) and (7.3), one has

$$\begin{aligned}
|I_{1,3}| &\leq \int_{s_1 \wedge s_2}^{s_1 \vee s_2} \rho^0(r - s_1 \wedge s_2, z - x)(t - r)^{-\frac{1}{\alpha}} \rho^1(t - r, y - z)\, dz\, dr \\
&\leq K \frac{|s_1 - s_2|}{(t - s_1 \vee s_2)^{\frac{1}{\alpha}}} \rho^0(t - s_1 \wedge s_2, y - x) \\
&\leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1}{\alpha} - 1}} \rho^0(t - s_1 \wedge s_2, y - x).
\end{aligned}$$

Gathering the preceding estimates, we have proved that

$$|I_1| \leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1}{\alpha} - 1}} \left[\rho^0(t - s_1 \vee s_2, y - x) + \rho^0(t - s_1 \wedge s_2, y - x)\right] \tag{7.53}$$
$$\left[1 + \sum_{k=1}^{m} C^k (t - s_1 \vee s_2)^{k\left(1 - \frac{1}{\alpha}\right)} \prod_{j=1}^{k} \mathcal{B}\left(\left[2 - \gamma - \frac{1}{\alpha}\right] \wedge 1 + (j-1)\left(1 - \frac{1}{\alpha}\right), 1 - \frac{1}{\alpha}\right)\right].$$

As done before for I_1, we decompose I_2 in the following way

$$\begin{aligned}
I_2 &= \int_{s_1 \vee s_2}^{t} \int_{\mathbb{R}^d} \Delta_{s_1, s_2} \left(p_{m+1} \otimes \frac{\delta}{\delta m} \mathcal{H}_{m+1}\right)(\mu, \cdot, r, x, z) \Phi_{m+1}(\mu, s_1 \vee s_2, r, t, z, y)\, dz\, dr \\
&\quad + \int_{s_1 \vee s_2}^{t} \int_{\mathbb{R}^d} \left(p_{m+1} \otimes \frac{\delta}{\delta m} \mathcal{H}_{m+1}\right)(\mu, s_1 \wedge s_2, r, x, z) \Delta_{s_1, s_2} \Phi_{m+1}(\mu, \cdot, r, t, z, y)\, dz\, dr \\
&\quad - \int_{s_1 \wedge s_2}^{s_1 \vee s_2} \int_{\mathbb{R}^d} \left(p_{m+1} \otimes \frac{\delta}{\delta m} \mathcal{H}_{m+1}\right)(\mu, s_1 \wedge s_2, r, x, z) \Phi_{m+1}(\mu, s_1 \wedge s_2, r, t, z, y)\, dz\, dr \\
&=: I_{2,1} + I_{2,2} + I_{2,3}.
\end{aligned}$$

We follow the same line of reasoning as for I_1. Using the bound (7.53) previously obtained for I_1 and (4.15), we have that

$$\begin{aligned}
|I_{2,1}| \leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1}{\alpha} - 1}} &(t - s_1 \vee s_2)^{1 - \frac{1}{\alpha}} \left[\rho^0(t - s_1 \vee s_2, y - x) + \rho^0(t - s_1 \wedge s_2, y - x)\right] \\
&\left[1 + \sum_{k=1}^{m} C^k (t - s_1 \vee s_2)^{k\left(1 - \frac{1}{\alpha}\right)} \prod_{j=1}^{k} \mathcal{B}\left(\left[2 - \gamma - \frac{1}{\alpha}\right] \wedge 1 + (j-1)\left(1 - \frac{1}{\alpha}\right), 1 - \frac{1}{\alpha}\right)\right].
\end{aligned}$$

Notice that (4.18) (7.3) and the convolution inequality (B.16) yield

$$\left|p_{m+1} \otimes \frac{\delta}{\delta m} \mathcal{H}_{m+1}(\mu, s, t, x, y)(v)\right| \leq K (t - s)^{1 - \frac{1}{\alpha}} \rho^0(t - s, y - x). \tag{7.54}$$

Using this inequality and (7.43), we deduce that

$$|I_{2,2}| \leq K \int_{s_1 \vee s_2}^{t} \int_{\mathbb{R}^d} (t - s_1 \wedge s_2)^{1-\frac{1}{\alpha}} \rho^0(r - s_1 \wedge s_2, z - x) \frac{|s_1 - s_2|^\gamma}{(r - s_1 \vee s_2)^\gamma} (t-r)^{-\frac{1}{\alpha}} \rho^1(t-r, y-z)\, dz\, dr.$$

Since $t - s_1 \wedge s_2 \leq 2(t - s_1 \vee s_2)$, we get

$$|I_{2,2}| \leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1}{\alpha} - 1}} (t - s_1 \vee s_2)^{1-\frac{1}{\alpha}} \rho^0(t - s_1 \wedge s_2, y - x).$$

For $I_{2,3}$, (7.54) and (4.15) yield

$$\begin{aligned}
|I_{2,3}| &\leq \int_{s_1 \wedge s_2}^{s_1 \vee s_2} (r - s_1 \wedge s_2)^{1-\frac{1}{\alpha}} \rho^0(r - s_1 \wedge s_2, z - x)(t-r)^{-\frac{1}{\alpha}} \rho^1(t-r, y-z)\, dz\, dr \\
&\leq K \frac{|s_1 - s_2|^{1+1-\frac{1}{\alpha}}}{(t - s_1 \vee s_2)^{\frac{1}{\alpha}}} \rho^0(t - s_1 \wedge s_2, y - x) \\
&\leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1}{\alpha} - 1}} (t - s_1 \vee s_2)^{1-\frac{1}{\alpha}} \rho^0(t - s_1 \wedge s_2, y - x),
\end{aligned}$$

since $|s_1 - s_2| \leq t - s_1 \vee s_2$. Gathering the estimates obtained, we have proved that

$$\begin{aligned}
|I_2| \leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1}{\alpha} - 1}} (t - s_1 \vee s_2)^{1-\frac{1}{\alpha}} \left[\rho^0(t - s_1 \vee s_2, y - x) + \rho^0(t - s_1 \wedge s_2, y - x)\right] \\
\left[1 + \sum_{k=1}^{m} C^k (t - s_1 \vee s_2)^{k\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k} \mathcal{B}\left(\left[2 - \gamma - \frac{1}{\alpha}\right] \wedge 1 + (j-1)\left(1 - \frac{1}{\alpha}\right), 1 - \frac{1}{\alpha}\right)\right]. \qquad (7.55)
\end{aligned}$$

We conclude by (7.53), (7.55) and the fact that $t - s_1 \vee s_2 \leq t - s_1 \wedge s_2 \leq 2(t - s_1 \vee s_2)$ that

$$\begin{aligned}
&\left|\Delta_{s_1,s_2} \frac{\delta}{\delta m} p_{m+1}(\mu, \cdot, t, x, y)(v)\right| \\
&\leq K \frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1}{\alpha} - 1}} \left[\rho^0(t - s_1 \vee s_2, y - x) + \rho^0(t - s_1 \wedge s_2, y - x)\right] \\
&\qquad \left[1 + \sum_{k=1}^{m} C^k (t - s_1 \vee s_2)^{k\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k} \mathcal{B}\left(\left[2 - \gamma - \frac{1}{\alpha}\right] \wedge 1 + (j-1)\left(1 - \frac{1}{\alpha}\right), 1 - \frac{1}{\alpha}\right)\right] \\
&\leq \left[\frac{|s_1 - s_2|^\gamma}{(t - s_1 \vee s_2)^{\gamma + \frac{1}{\alpha} - 1}} \rho^0(t - s_1 \vee s_2, y - x) + \frac{|s_1 - s_2|^\gamma}{(t - s_1 \wedge s_2)^{\gamma + \frac{1}{\alpha} - 1}} \rho^0(t - s_1 \wedge s_2, y - x)\right] \\
&\qquad \sum_{k=1}^{m+1} C^k (t - s_1 \vee s_2)^{(k-1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(\left[2 - \gamma - \frac{1}{\alpha}\right] \wedge 1 + (j-1)\left(1 - \frac{1}{\alpha}\right), 1 - \frac{1}{\alpha}\right),
\end{aligned}$$

provided that we choose $C \geq K$ in (4.32). This ends the proof of the induction step for (4.32).

Proof of (4.33)**.** We start by treating the case $|s_1 - s_2| > t - s_1 \vee s_2$. Reasoning as before by using (4.21) (the series appearing in the bound being convergent), we get that

$$\begin{aligned}
&\left|\Delta_{s_1,s_2}\partial_v \frac{\delta}{\delta m} p_m(\mu,\cdot,t,x,y)(v)\right| \\
&\leq K\left[(t-s_1\vee s_2)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}\rho^0(t-s_1\vee s_2,y-x)+(t-s_1\wedge s_2)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}\rho^0(t-s_1\wedge s_2,y-x)\right] \\
&\leq K\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}\rho^0(t-s_1\vee s_2,y-x)+\frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}\rho^0(t-s_1\wedge s_2,y-x)\right],
\end{aligned}$$

This shows that (4.33) holds true at step $m+1$ provided that we choose $C \geq K$ in (4.33). We now turn to the case $|s_1 - s_2| \leq t - s_1 \vee s_2$. By the representation formula (7.20), we write

$$\begin{aligned}
\Delta_{s_1,s_2}\partial_v \frac{\delta}{\delta m} p_{m+1}(\mu,\cdot,t,x,y)(v) &= \Delta_{s_1,s_2}\left(p_{m+1}\otimes \partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1}\right)(\mu,\cdot,t,x,y)(v) \\
&\quad + \Delta_{s_1,s_2}\left(p_{m+1}\otimes \partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1}\otimes \Phi_{m+1}\right)(\mu,\cdot,t,x,y)(v) \\
&=: I_1 + I_2.
\end{aligned}$$

Then, we decompose I_1 in the following way

$$\begin{aligned}
I_1 &= \int_{s_1\vee s_2}^t \int_{\mathbb{R}^d} \Delta_{s_1,s_2} p_{m+1}(\mu,\cdot,r,x,z)\partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,s_1\vee s_2,r,t,z,y)\,dz\,dr \\
&\quad + \int_{s_1\vee s_2}^t \int_{\mathbb{R}^d} p_{m+1}(\mu,s_1\wedge s_2,r,x,z)\Delta_{s_1,s_2}\partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,\cdot,r,t,z,y)\,dz\,dr \\
&\quad - \int_{s_1\wedge s_2}^{s_1\vee s_2} \int_{\mathbb{R}^d} p_{m+1}(\mu,s_1\wedge s_2,r,x,z)\partial_v \frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,s_1\wedge s_2,r,t,z,y)\,dz\,dr \\
&=: I_{1,1} + I_{1,2} + I_{1,3}.
\end{aligned}$$

It follows from (4.31), (7.4) (the series appearing in the bound being convergent), the convolution inequality (B.16) and since $\gamma < 1 + \frac{\eta-1}{\alpha}$ that

$$\begin{aligned}
|I_{1,1}| &\leq K\int_{s_1\vee s_2}^t \int_{\mathbb{R}^d} \frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^\gamma}\left[\rho^0(r-s_1\vee s_2,y-x)+\rho^0(r-s_1\wedge s_2,y-x)\right] \\
&\qquad\qquad (r-s_1\vee s_2)^{\frac{\eta-1}{\alpha}}(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr \\
&\leq K\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}\left[\rho^0(t-s_1\vee s_2,y-x)+\rho^0(t-s_1\wedge s_2,y-x)\right].
\end{aligned}$$

We now deal with $I_{1,2}$. Using the induction assumption and (7.50), we deduce that

$$
\begin{aligned}
&\left|\Delta_{s_1,s_2}\partial_v\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,\cdot,r,t,x,y)(v)\right| \\
&\leq K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x)\left[\frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^{\gamma+\frac{1-\eta}{\alpha}}}+\int_{\mathbb{R}^{2d}}(1\wedge|y-x'|^\eta)\left|\Delta_{s_1,s_2}\partial_v\frac{\delta}{\delta m}p_m(\mu,\cdot,r,x',y)\right|\,dy\,d\mu(x')\right] \\
&\leq K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x)\frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^{\gamma+\frac{1-\eta}{\alpha}}} \\
&\quad+K(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-x)\left[\frac{|s_1-s_2|^\gamma(r-s_1\vee s_2)^{\frac{\eta}{\alpha}}}{(r-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}+\frac{|s_1-s_2|^\gamma(r-s_1\wedge s_2)^{\frac{\eta}{\alpha}}}{(r-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}\right] \\
&\qquad\sum_{k=1}^m C^k(r-s_1\vee s_2)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(\left[2\left(1+\frac{\eta-1}{\alpha}\right)-\gamma\right]\wedge 1+(j-1)\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right).
\end{aligned}
$$

Note that

$$
\begin{cases}
(r-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-2\eta}{\alpha}}\geq(r-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-2\eta}{\alpha}} & \text{if } \gamma+\frac{1}{\alpha}-1+\frac{1-2\eta}{\alpha}\geq 0,\\
(r-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-2\eta}{\alpha}}\geq 2^{\gamma+\frac{1}{\alpha}-1+\frac{1-2\eta}{\alpha}}(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-2\eta}{\alpha}} & \text{if } \gamma+\frac{1}{\alpha}-1+\frac{1-2\eta}{\alpha}<0
\end{cases}
\tag{7.56}
$$

since $t-s_1\wedge s_2\leq 2(t-s_1\vee s_2)$. It yields

$$
\begin{aligned}
&\int_{s_1\vee s_2}^t(t-r)^{-\frac{1}{\alpha}}\left[\frac{|s_1-s_2|^\gamma}{(r-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-2\eta}{\alpha}}}+\frac{|s_1-s_2|^\gamma}{(r-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-2\eta}{\alpha}}}\right](r-s_1\vee s_2)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)}\,dr \\
&\leq K\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}(t-s_1\vee s_2)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\mathcal{B}\left(2\left(1+\frac{\eta-1}{\alpha}\right)-\gamma+(k-1)\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right) \\
&\quad+K\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}(t-s_1\vee s_2)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\mathcal{B}\left(1+(k-1)\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right) \\
&\leq K\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}(t-s_1\vee s_2)^{k\left(1+\frac{\eta-1}{\alpha}\right)} \\
&\qquad\qquad\mathcal{B}\left(\left[2\left(1+\frac{\eta-1}{\alpha}\right)-\gamma\right]\wedge 1+(j-1)\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right),
\end{aligned}
$$

since the Beta function is decreasing with respect to its first argument. From this inequality, (7.51) and (4.18), we deduce that

$$
\begin{aligned}
|I_{1,2}|\leq K&\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}\rho^0(t-s_1\wedge s_2,y-x) \\
&\left[1+\sum_{k=1}^m C^k(t-s_1\vee s_2)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^k\mathcal{B}\left(\left[2\left(1+\frac{\eta-1}{\alpha}\right)-\gamma\right]\wedge 1+(j-1)\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right].
\end{aligned}
$$

We now turn to the estimate on $I_{1,3}$. Thanks to (4.18) and (7.4), one has

$$
\begin{aligned}
|I_{1,3}| &\le \int_{s_1\wedge s_2}^{s_1\vee s_2} \rho^0(r-s_1\wedge s_2, z-x)(r-s_1\wedge s_2)^{\frac{\eta-1}{\alpha}}(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr \\
&\le K\frac{|s_1-s_2|^{1+\frac{\eta-1}{\alpha}}}{(t-s_1\vee s_2)^{\frac{1}{\alpha}}}\rho^0(t-s_1\wedge s_2, y-x) \\
&\le K\frac{|s_1-s_2|^{\gamma}}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}\rho^0(t-s_1\wedge s_2, y-x),
\end{aligned}
$$

since $\gamma < 1+\frac{\eta-1}{\alpha}$. Gathering the estimates obtained, we have proved that

$$
\begin{aligned}
|I_1| \le K\frac{|s_1-s_2|^{\gamma}}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}\left[\rho^0(t-s_1\vee s_2, y-x)+\rho^0(t-s_1\wedge s_2, y-x)\right] \qquad (7.57)\\
\left[1+\sum_{k=1}^{m} C^k (t-s_1\vee s_2)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(\left[2\left(1+\frac{\eta-1}{\alpha}\right)-\gamma\right]\wedge 1+(j-1)\left(1+\frac{\eta-1}{\alpha}\right), 1-\frac{1}{\alpha}\right)\right].
\end{aligned}
$$

Then, we decompose I_2 in the following way

$$
\begin{aligned}
I_2 &= \int_{s_1\vee s_2}^{t}\int_{\mathbb{R}^d} \Delta_{s_1,s_2}\left(p_{m+1}\otimes\partial_v\frac{\delta}{\delta m}\mathcal{H}_{m+1}\right)(\mu,\cdot,r,x,z)\Phi_{m+1}(\mu,s_1\vee s_2,r,t,z,y)\,dz\,dr \\
&\quad+\int_{s_1\vee s_2}^{t}\int_{\mathbb{R}^d}\left(p_{m+1}\otimes\partial_v\frac{\delta}{\delta m}\mathcal{H}_{m+1}\right)(\mu,s_1\wedge s_2,r,x,z)\Delta_{s_1,s_2}\Phi_{m+1}(\mu,\cdot,r,t,z,y)\,dz\,dr \\
&\quad-\int_{s_1\wedge s_2}^{s_1\vee s_2}\int_{\mathbb{R}^d}\left(p_{m+1}\otimes\partial_v\frac{\delta}{\delta m}\mathcal{H}_{m+1}\right)(\mu,s_1\wedge s_2,r,x,z)\Phi_{m+1}(\mu,s_1\wedge s_2,r,t,z,y)\,dz\,dr \\
&=: I_{2,1}+I_{2,2}+I_{2,3}.
\end{aligned}
$$

We follow the same line of reasoning as for I_1. Using the bound (7.57) previously obtained for I_1 and (4.15), we show that

$$
\begin{aligned}
|I_{2,1}| \le K\frac{|s_1-s_2|^{\gamma}}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}(t-s_1\vee s_2)^{1-\frac{1}{\alpha}}\left[\rho^0(t-s_1\vee s_2, y-x)+\rho^0(t-s_1\wedge s_2, y-x)\right] \\
\left[1+\sum_{k=1}^{m} C^k (t-s_1\vee s_2)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(\left[2\left(1+\frac{\eta-1}{\alpha}\right)-\gamma\right]\wedge 1+(j-1)\left(1+\frac{\eta-1}{\alpha}\right), 1-\frac{1}{\alpha}\right)\right].
\end{aligned}
$$

Notice that it follows from (4.18) (7.4) and the convolution inequality (B.16) that

$$
\left|p_{m+1}\otimes\partial_v\frac{\delta}{\delta m}\mathcal{H}_{m+1}(\mu,s,t,x,y)(v)\right| \le K(t-s)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}\rho^0(t-s,y-x). \qquad (7.58)
$$

Using this inequality and (7.43), we deduce that

$$
|I_{2,2}| \le K\int_{s_1\vee s_2}^{t}\int_{\mathbb{R}^d}(r-s_1\wedge s_2)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}\rho^0(r-s_1\wedge s_2,z-x)\frac{|s_1-s_2|^{\gamma}}{(r-s_1\vee s_2)^{\gamma}}(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr.
$$

Reasoning as in (7.56) to bound $(r-s_1\wedge s_2)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}$ since $t-s_1\vee s_2 \le t-s_1\wedge s_2 \le 2(t-s_1\vee s_2)$, we get that

$$
|I_{2,2}| \le K\frac{|s_1-s_2|^{\gamma}}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}(t-s_1\vee s_2)^{1-\frac{1}{\alpha}}\rho^0(t-s_1\wedge s_2,y-x).
$$

For $I_{2,3}$, (7.58), (4.15) and the convolution inequality (B.16) yield

$$
\begin{aligned}
|I_{2,3}| &\leq \int_{s_1\wedge s_2}^{s_1\vee s_2} (r-s_1\vee s_2)^{\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}\rho^0(r-s_1\wedge s_2, z-x)(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr \\
&\leq K\frac{|s_1-s_2|^{1+\frac{\eta-1}{\alpha}+1-\frac{1}{\alpha}}}{(t-s_1\vee s_2)^{\frac{1}{\alpha}}}\rho^0(t-s_1\wedge s_2, y-x) \\
&\leq K\frac{|s_1-s_2|^{\gamma}}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}(t-s_1\vee s_2)^{1-\frac{1}{\alpha}}\rho^0(t-s_1\wedge s_2, y-x),
\end{aligned}
$$

since $|s_1-s_2|\leq t-s_1\vee s_2$. Gathering the estimates obtained, we have proved that

$$
\begin{aligned}
|I_2| \leq K\frac{|s_1-s_2|^{\gamma}}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}(t-s_1\vee s_2)^{1-\frac{1}{\alpha}}\left[\rho^0(t-s_1\vee s_2,y-x)+\rho^0(t-s_1\wedge s_2,y-x)\right] \qquad (7.59)\\
\left[1+\sum_{k=1}^{m} C^k(t-s_1\vee s_2)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(\left[2\left(1+\frac{\eta-1}{\alpha}\right)-\gamma\right]\wedge 1+(j-1)\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right].
\end{aligned}
$$

We conclude from (7.57), (7.59) and the fact that $t-s_1\wedge s_2\leq 2(t-s_1\vee s_2)$ that

$$
\begin{aligned}
&\left|\Delta_{s_1,s_2}\partial_v\frac{\delta}{\delta m}p_{m+1}(\mu,\cdot,t,x,y)(v)\right| \\
&\leq K\frac{|s_1-s_2|^{\gamma}}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}\left[\rho^0(t-s_1\vee s_2,y-x)+\rho^0(t-s_1\wedge s_2,y-x)\right] \\
&\quad\left[1+\sum_{k=1}^{m} C^k(t-s_1\vee s_2)^{k\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(\left[2\left(1+\frac{\eta-1}{\alpha}\right)-\gamma\right]\wedge 1+(j-1)\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right)\right] \\
&\leq \left[\frac{|s_1-s_2|^{\gamma}}{(t-s_1\vee s_2)^{\gamma+\frac{1}{\alpha}-1+\frac{1-\eta}{\alpha}}}\rho^0(t-s_1\vee s_2,y-x)+\frac{|s_1-s_2|^{\gamma}}{(t-s_1\wedge s_2)^{\gamma+\frac{1}{\alpha}-1}}\rho^0(t-s_1\wedge s_2,y-x)\right] \\
&\quad\sum_{k=1}^{m+1} C^k(t-s_1\vee s_2)^{(k-1)\left(1+\frac{\eta-1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(\left[2\left(1+\frac{\eta-1}{\alpha}\right)-\gamma\right]\wedge 1+(j-1)\left(1+\frac{\eta-1}{\alpha}\right),1-\frac{1}{\alpha}\right),
\end{aligned}
$$

provided that we choose $C\geq K$ in (4.33). This ends the proof of the induction step for (4.33).

Appendix A. Differential calculus for functions of a measure variable

Let us fix $\beta\in[0,2]$. We use the following convention $\mathcal{P}_0(\mathbb{R}^d):=\mathcal{P}(\mathbb{R}^d)$, endowed with the weak topology.

Definition 4 (Linear derivative)**.** A function $u:\mathcal{P}_\beta(\mathbb{R}^d)\to\mathbb{R}$ is said to have a linear derivative if there exists a function $\frac{\delta}{\delta m}u\in\mathcal{C}^0(\mathcal{P}_\beta(\mathbb{R}^d)\times\mathbb{R}^d;\mathbb{R})$ satisfying the following properties.

(1) For all compact subset $\mathcal{K}\subset\mathcal{P}_\beta(\mathbb{R}^d)$, there exists a constant $C_{\mathcal{K}}>0$ such that

$$
\forall\mu\in\mathcal{K},\ \forall v\in\mathbb{R}^d,\ \left|\frac{\delta}{\delta m}u(\mu)(v)\right|\leq C_{\mathcal{K}}(1+|v|^{\beta}).
$$

(2) For all $\mu,\nu\in\mathcal{P}_\beta(\mathbb{R}^d)$, we have

$$
u(\mu)-u(\nu)=\int_0^1\int_{\mathbb{R}^d}\frac{\delta}{\delta m}u(t\mu+(1-t)\nu)(v)\,d(\mu-\nu)(v)\,dt.
$$

The function u is said to have a linear derivative of order two if for all $v \in \mathbb{R}^d$, the map $\frac{\delta}{\delta m}u(\cdot)(v)$ admits a linear derivative $\frac{\delta^2}{\delta m^2}(\cdot)(v,\cdot)$ such that $\frac{\delta^2}{\delta m^2}u$ is continuous on $\mathcal{P}_\beta(\mathbb{R}^d)\times\mathbb{R}^d\times\mathbb{R}^d$ and for all compact subset $\mathcal{K}\subset\mathcal{P}_\beta(\mathbb{R}^d)$, there exists a constant $C_\mathcal{K}>0$ such that

$$\forall \mu\in\mathcal{K},\ \forall v,v'\in\mathbb{R}^d,\ \left|\frac{\delta^2}{\delta m^2}u(\mu)(v,v')\right|\leq C_\mathcal{K}(1+|v|^\beta+|v'|^\beta).$$

Definition 5. We define the space $\mathcal{C}^1([0,T]\times\mathbb{R}^d\times\mathcal{P}_\beta(\mathbb{R}^d))$ as the set of continuous functions $u:[0,T]\times\mathbb{R}^d\times\mathcal{P}_\beta(\mathbb{R}^d)\to\mathbb{R}$ satisfying the following properties.

(1) For any $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, the map $u(\cdot,\cdot,\mu)$ belongs to $\mathcal{C}^1([0,T]\times\mathbb{R}^d)$ with $\partial_t u$ and $\partial_x u$ continuous on $[0,T]\times\mathbb{R}^d\times\mathcal{P}_\beta(\mathbb{R}^d)$.
(2) For any $(t,x)\in[0,T]\times\mathbb{R}^d$, the map $u(t,x,\cdot)$ admits a linear derivative $(\mu,v)\in\mathcal{P}_\beta(\mathbb{R}^d)\times\mathbb{R}^d\mapsto\frac{\delta}{\delta m}u(t,x,\mu)(v)$ such that $\frac{\delta}{\delta m}u$ is continuous on $[0,T]\times\mathbb{R}^d\times\mathcal{P}_\beta(\mathbb{R}^d)$.
(3) For any $(t,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_\beta(\mathbb{R}^d)$, the map $\frac{\delta}{\delta m}u(t,x,\mu)$ is of class $\mathcal{C}^1$ on $\mathbb{R}^d$ and $\partial_v\frac{\delta}{\delta m}u$ is continuous on $[0,T]\times\mathbb{R}^d\times\mathcal{P}_\beta(\mathbb{R}^d)\times\mathbb{R}^d$.

We now introduce the notion of empirical projection.

Definition 6 (Empirical projection)**.** Fix $u:\mathcal{P}_\beta(\mathbb{R}^d)\to\mathbb{R}$. For all $N\geq 1$, the empirical projection u^N of u is defined for all $\boldsymbol{x}=(x_1,\dots,x_N)\in(\mathbb{R}^d)^N$ by

$$u^N(\boldsymbol{x})=u(\overline{\mu}^N_{\boldsymbol{x}}),$$

where $\overline{\mu}^N_{\boldsymbol{x}}=\frac{1}{N}\sum_{j=1}^N\delta_{x_j}$.

The following proposition is the analogue of Proposition 5.91 of [2] where $\beta=2$, so we don't give the proof.

Proposition 5. Let $u:\mathcal{P}_\beta(\mathbb{R}^d)\to\mathbb{R}$ be a function admitting a linear derivative $\frac{\delta}{\delta m}u$ such that all $\mu\in\mathcal{P}_\beta(\mathbb{R}^d)$, $\frac{\delta}{\delta m}u(\mu)(\cdot)\in\mathcal{C}^1(\mathbb{R}^d)$ and $\partial_v\frac{\delta}{\delta m}u$ is continuous on $\mathcal{P}_\beta(\mathbb{R}^d)\times\mathbb{R}^d$. Then, for all $N\geq 1$, the empirical projection u^N of u defined for all $\boldsymbol{x}=(x_1,\dots,x_N)\in(\mathbb{R}^d)^N$ by

$$u^N(\boldsymbol{x}):=u(\overline{\mu}^N_{\boldsymbol{x}})$$

is of class $\mathcal{C}^1$. Moreover for all $\boldsymbol{x}=(x_1,\dots,x_N)\in(\mathbb{R}^d)^N$

$$\partial_{x_i}u^N(x_1,\dots,x_N)=\frac{1}{N}\partial_v\frac{\delta}{\delta m}u(\overline{\mu}^N_{\boldsymbol{x}})(x_i).$$

The next proposition illustrates how a smooth flow of measures admitting a transition density can regularize a function defined on $\mathcal{P}_\beta(\mathbb{R}^d)$. It is clearly reminiscent of Proposition 2.3 in [7]. We don't prove it since it can be done in a completely analogous manner.

Proposition 6 (Regularization by a smooth flow of density functions)**.** Let us fix $\phi:\mathcal{P}_\beta(\mathbb{R}^d)\to\mathbb{R}$ a function admitting a linear derivative and consider a map $(\mu,s,x)\in\mathcal{P}_\beta(\mathbb{R}^d)\times[0,T)\times\mathbb{R}^d\mapsto p(\mu,s,T,x,y)$, where $T>0$ is fixed and such that $p(\mu,s,T,x,\cdot)$ is a density function. We define the measure-valued map $\Theta:(s,\mu)\in[0,T)\times\mathcal{P}_\beta(\mathbb{R}^d)\mapsto\Theta(s,\mu)(dy):=\left(\int_{\mathbb{R}^d}p(\mu,s,T,x,y)\,d\mu(x)\right)\,dy\in\mathcal{P}(\mathbb{R}^d)$. We assume that the following properties hold true.

(1) For any compact subset $\mathcal{K}$ of $[0,T)\times\mathcal{P}_\beta(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\sup_{(s,\mu)\in\mathcal{K}}|y|^\beta\Theta(s,\mu)(dy)<+\infty.$$

(2) For all $y\in\mathbb{R}^d$, the map $(\mu,s,x)\in\mathcal{P}_\beta(\mathbb{R}^d)\times[0,T)\times\mathbb{R}^d\mapsto p(\mu,s,T,x,y)$ belongs to $\mathcal{C}^1(\mathcal{P}_\beta(\mathbb{R}^d)\times[0,T)\times\mathbb{R}^d)$.

(3) For any compact subset $\mathcal{K}$ of $\mathcal{P}_\beta(\mathbb{R}^d)\times[0,T)\times\mathbb{R}^d\times\mathbb{R}^d$ and for any $j\in\{0,1\}$, there exists a positive constant C such that
$$\int_{\mathbb{R}^d}\sup_{(\mu,s,x,v)\in\mathcal{K}}\left\{\left|\partial_v^j\frac{\delta}{\delta m}p(\mu,s,T,x,y)(v)\right|+\left|\partial_x^j p(\mu,s,T,x,y)\right|+\left|\partial_s p(\mu,s,T,x,y)\right|\right\}dy<+\infty.$$

(4) For any compact subset $\mathcal{K}$ of $\mathcal{P}_\beta(\mathbb{R}^d)\times[0,T)$, there exists a positive constant C such that
$$\int_{\mathbb{R}^d}(1+|y|^\beta)\sup_{(\mu,s)\in\mathcal{K}}\left|\frac{\delta}{\delta m}p(\mu,s,T,x,y)(v)\right|dy\le C(1+|x|^\beta)(1+|v|^\beta),$$
$$\int_{\mathbb{R}^d}(1+|y|^\beta)\sup_{(\mu,s)\in\mathcal{K}}|p(\mu,s,T,v,y)|\,dy\le C(1+|v|^\beta),$$
and
$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\sup_{(\mu,s)\in\mathcal{K}}\left|\frac{\delta}{\delta m}p(\mu,s,T,x,y)(v)\right|dy\le C(1+|v|^\beta).$$

Then, the function $(s,\mu)\in[0,T)\times\mathcal{P}_\beta(\mathbb{R}^d)\mapsto\phi(\Theta(s,\mu))$ belongs to $\mathcal{C}^1([0,T)\times\mathcal{P}_\beta(\mathbb{R}^d))$. Moreover, we have
$$\partial_s\left[\phi(\Theta(s,\mu))\right]=\int_{\mathbb{R}^{2d}}\left(\frac{\delta}{\delta m}\phi(\Theta(s,\mu))(y)-\frac{\delta}{\delta m}\phi(\Theta(s,\mu))(x)\right)\partial_s p(\mu,s,T,x,y)\,dy\,d\mu(x),$$
$$\begin{aligned}\frac{\delta}{\delta m}\left[\phi(\Theta(s,\mu))\right](v)&=\int_{\mathbb{R}^d}\frac{\delta}{\delta m}\phi(\Theta(s,\mu))(y)p(\mu,s,T,v,y)\,dy\\&\quad+\int_{\mathbb{R}^{2d}}\left(\frac{\delta}{\delta m}\phi(\Theta(s,\mu))(y)-\frac{\delta}{\delta m}\phi(\Theta(s,\mu))(x)\right)\frac{\delta}{\delta m}p(\mu,s,T,x,y)(v)\,dy\,d\mu(x),\end{aligned}$$
and
$$\begin{aligned}\partial_v\frac{\delta}{\delta m}\left[\phi(\Theta(s,\mu))\right](v)&=\int_{\mathbb{R}^d}\left(\frac{\delta}{\delta m}\phi(\Theta(s,\mu))(y)-\frac{\delta}{\delta m}\phi(\Theta(s,\mu))(v)\right)\partial_x p(\mu,s,T,v,y)\,dy\\&\quad+\int_{\mathbb{R}^{2d}}\left(\frac{\delta}{\delta m}\phi(\Theta(s,\mu))(y)-\frac{\delta}{\delta m}\phi(\Theta(s,\mu))(x)\right)\partial_v\frac{\delta}{\delta m}p(\mu,s,T,x,y)(v)\,dy\,d\mu(x).\end{aligned}$$

We now focus on Itô's formula along the flow of probability measures associated with a jump process. Let us fix $Z^1=(Z^1_t)_t$ and $Z^2=(Z^2_t)_t$ two α-stable processes on $\mathbb{R}^d$ with $\alpha\in(1,2)$. Their associated Poisson random measures are respectively denoted by $\mathcal{N}^1$ and $\mathcal{N}^2$, their compensated Poisson random measures by $\widetilde{\mathcal{N}}^1$ and $\widetilde{\mathcal{N}}^2$ and their Lévy measures by ν^1 and ν^2. Since $\alpha\in(1,2)$, we can write for all $t\ge0$
$$Z^1_t=\int_0^t\int_{\mathbb{R}^d}z\,\widetilde{\mathcal{N}}^1(ds,dz)\quad\text{and}\quad Z^2_t=\int_0^t\int_{\mathbb{R}^d}z\,\widetilde{\mathcal{N}}^2(ds,dz).$$

We fix $\beta\in(1,\alpha)$ and $\gamma\in(0,1]$ such that $\gamma>\alpha-1$. We consider two jump processes $X=(X_t)_{t\in[0,T]}$ and $Y=(Y_t)_{t\in[0,T]}$ defined for all $t\in[0,T]$ by
$$X_t:=X_0+\int_0^t b_s\,ds+Z^1_t,\quad\text{and}\quad Y_t=Y_0+\int_0^t\eta_s\,ds+Z^2_t,\tag{A.1}$$
where $X_0,Y_0\in L^\beta(\Omega,\mathcal{F}_0)$, $b,\eta:[0,T]\times\Omega\to\mathbb{R}^d$ are bounded predictable processes. The distribution of X_t is denoted by μ_t.

We state in the next proposition Itô's formula, which is deduced from Theorem 2 in [3] for the specific type of processes that are considered in the present work.

Proposition 7 (Itô's formula)**.** Let $u:[0,T]\times\mathbb{R}^d\times\mathcal{P}_\beta(\mathbb{R}^d)\to\mathbb{R}$ be a continuous function satisfying the following properties.

(1) The function u belongs to $\mathcal{C}^1([0,T]\times\mathbb{R}^d\times\mathcal{P}_\beta(\mathbb{R}^d))$ and $\partial_x u(t,\cdot,\mu)$ is γ-Hölder continuous and bounded uniformly with respect to t and μ.

(2) For all compact $\mathcal{K} \subset \mathbb{R}^d \times \mathcal{P}_\beta(\mathbb{R}^d)$, there exists $C_\mathcal{K} > 0$ such that

$$\forall t \in [0,T],\ \forall (x,\mu) \in \mathcal{K},\ \forall v \in \mathbb{R}^d,\ \left| \frac{\delta}{\delta m} u(t,x,\mu)(v) \right| \leq C_\mathcal{K}(1 + |v|^\beta).$$

(3) If $\gamma > 0$, for any compact $\mathcal{K} \subset \mathbb{R}^d \times \mathcal{P}_\beta(\mathbb{R}^d)$, there exists $C_\mathcal{K} > 0$ such that

$$\forall t \in [0,T],\ \forall (x,\mu) \in \mathcal{K},\ \forall v, v' \in \mathbb{R}^d,\ \left| \partial_v \frac{\delta}{\delta m} u(t,x,\mu)(v) - \partial_v \frac{\delta}{\delta m} u(t,x,\mu)(v') \right| \leq C_\mathcal{K} |v - v'|^\gamma.$$

(4) For any compact $\mathcal{K} \subset \mathbb{R}^d \times \mathcal{P}_\beta(\mathbb{R}^d)$, we have

$$\sup_{t \in [0,T]} \sup_{(x,\mu) \in \mathcal{K}} \int_{\mathbb{R}^d} \left| \partial_v \frac{\delta}{\delta m} u(t,x,\mu)(v) \right|^{\beta'} d\mu(v) < +\infty.$$

Then, the function $(t,x) \in [0,T] \times \mathbb{R}^d \mapsto u(t,x,\mu_t)$ is of class $\mathcal{C}^1$, with $\partial_x u(t,\cdot,\mu_t)$ γ-Hölder continuous uniformly with respect to t. Moreover, we have almost surely for all $t \in [0,T]$

$$\begin{aligned}
&u(t,Y_t,\mu_t) - u(0,Y_0,\mu_0) \\
&= \int_0^t \partial_t u(s,Y_s,\mu_s)\, ds + \int_0^t \overline{\mathbb{E}} \left(\partial_v \frac{\delta}{\delta m} u(s,Y_s,\mu_s)(\overline{X}_s) \cdot \overline{b}_s \right) ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \overline{\mathbb{E}} \left[\frac{\delta}{\delta m} u(s,Y_s,\mu_s)(\overline{X}_{s^-} + z) - \frac{\delta}{\delta m} u(s,Y_s,\mu_s)(\overline{X}_{s^-}) \right. \\
&\qquad\qquad \left. - \partial_v \frac{\delta}{\delta m} u(s,Y_s,\mu_s)(\overline{X}_{s^-}) \cdot z \right] d\nu^1(z)\, ds \qquad \text{(A.2)} \\
&\quad + \int_0^t \partial_x u(s,Y_s,\mu_s) \cdot \eta_s\, ds + \int_0^t \int_{\mathbb{R}^d} u(s,Y_{s^-} + z,\mu_s) - u(s,Y_{s^-},\mu_s)\, \widetilde{\mathcal{N}}^2(ds,dz) \\
&\quad + \int_0^t \int_{\mathbb{R}^d} [u(s,Y_{s^-} + z,\mu_s) - u(s,Y_{s^-},\mu_s) - \partial_x u(s,Y_{s^-},\mu_s) \cdot z]\, d\nu^2(z)\, ds
\end{aligned}$$

where $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ is an independent copy of $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\overline{b}, \overline{X})$ is a copy of (b, X).

Appendix B. Parametrix expansion for stable-driven SDEs

Let us fix $Z = (Z_t)_t$ a rotationally invariant α-stable process on $\mathbb{R}^d$ with $\alpha \in (1,2)$. Its associated Poisson random measure is denoted by $\mathcal{N}$, the compensated Poisson random measure by $\widetilde{\mathcal{N}}$. Since $\alpha \in (1,2)$, we can write for all $t \geq 0$

$$Z_t = \int_0^t \int_{\mathbb{R}^d} z \widetilde{\mathcal{N}}(ds,dz).$$

The Lévy measure ν of Z is given by

$$d\nu(z) := \frac{dz}{|z|^{d+\alpha}}.$$

We consider a function $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the following properties.

(1) b is jointly continuous and globally bounded on $[0,T] \times \mathbb{R}^d$.
(2) b is η-Hölder continuous on $\mathbb{R}^d$ uniformly in time, with $\eta \in (0,1]$, i.e. there exists $C > 0$ such that for all $t \in [0,T]$ and $x_1, x_2 \in \mathbb{R}^d$

$$|b(t,x_1) - b(t,x_2)| \leq C|x_1 - x_2|^\eta.$$

We fix $s \in [0,T)$ and we consider the following stable-driven SDE

$$\begin{cases} dX_t^{s,x} = b(t, X_t^{s,x})\, dt + dZ_t, \quad t \in [s,T], \\ X_s^{s,x} = x \in \mathbb{R}^d. \end{cases} \qquad \text{(B.1)}$$

The density of Z_t is denoted by $q(t,\cdot)$. We denote by $\mathcal{L}^\alpha$ the stable operator associated with Z defined for all $f \in \mathcal{C}_b^{1+\gamma}(\mathbb{R}^d;\mathbb{R})$, with $\gamma > \alpha - 1$, i.e. f belongs to $\mathcal{C}_b^1(\mathbb{R}^d;\mathbb{R})$ and ∇f is γ-Hölder and for all $x \in \mathbb{R}^d$ by

$$\mathcal{L}^\alpha f(x) := \int_{\mathbb{R}^d} (f(x+z) - f(x) - \nabla f(x)\cdot z)\, d\nu(z). \tag{B.2}$$

We define for all $s \in [0,T)$, $t \in (s,T]$ and $x, y \in \mathbb{R}^d$

$$\begin{aligned} \widehat{p}(s,t,x,y) &:= q(t-s, y-x), \\ \mathcal{H}(s,t,x,y) &:= b(s,x)\cdot \partial_x \widehat{p}(s,t,x,y). \end{aligned} \tag{B.3}$$

Note that the proxy $\widehat{p}(s,t,x,\cdot)$ is the density at time $t > s$ of the solution to

$$\begin{cases} d\widehat{X}_t^{s,x} = dZ_t, \quad t \in [s,T], \\ \widehat{X}_s^{s,x} = x \in \mathbb{R}^d, \end{cases} \tag{B.4}$$

and $\mathcal{H}$ is the associated parametrix kernel. We also defined the space-time convolution operator between to functions f and g by

$$f \otimes g(s,t,x,y) := \int_s^t \int_{\mathbb{R}^d} f(s,r,x,z) g(r,t,z,y)\, dz\, dr, \tag{B.5}$$

when it is well-defined. The space-time convolution iterates $\mathcal{H}^k$ of $\mathcal{H}$ are defined recursively by $\mathcal{H}^1 := \mathcal{H}$ and $\mathcal{H}^{k+1} := \mathcal{H} \otimes \mathcal{H}^k$. By convention $f \otimes \mathcal{H}^0$ is equal to f. Finally, we denote by Φ the solution to the following Volterra integral equation

$$\Phi(s,t,x,y) = \mathcal{H}(s,t,x,y) + \mathcal{H} \otimes \Phi(s,t,x,y),$$

which is given by the uniform convergent series

$$\Phi(s,t,x,y) = \sum_{k=1}^{\infty} \mathcal{H}^k(s,t,x,y). \tag{B.6}$$

Let us also define, for $k > -\alpha$ the function ρ^k by

$$\forall t > 0,\ x \in \mathbb{R}^d,\ \rho^k(t,x) := t^{-\frac{d}{\alpha}} (1 + t^{-\frac{1}{\alpha}}|x|)^{-d-\alpha-k}. \tag{B.7}$$

Theorem 5. For any $s \in [0,T)$, $s < t \le T$ and $x \in \mathbb{R}^d$, the distribution of $X_t^{s,x}$ has a density with respect to the Lebesgue measure denoted by $p(s,t,x,\cdot)$ and given by the absolutely convergent parametrix series

$$\begin{aligned} p(s,t,x,y) &= \widehat{p}(s,t,x,y) + \sum_{k=1}^{\infty} \widehat{p} \otimes \mathcal{H}^k(s,t,x,y) \\ &= \widehat{p}(s,t,x,y) + \widehat{p} \otimes \Phi(s,t,x,y). \end{aligned} \tag{B.8}$$

For any $t \in (0,T]$ and $y \in \mathbb{R}^d$, $p(\cdot,t,\cdot,y)$ is of class $\mathcal{C}^1$ on $[0,t)\times\mathbb{R}^d$ and $p(\cdot,t,\cdot,y)$, $\partial_s p(\cdot,t,\cdot,y)$ and $\partial_x p(\cdot,t,\cdot,y)$ are continuous on $[0,t)\times\mathbb{R}^d$. The function $p(\cdot,t,\cdot,y)$ is solution to the following backward Kolmogorov PDE

$$\begin{cases} \partial_s p(s,t,x,y) + b(s,x)\cdot\partial_x p(s,t,x,y) + \mathcal{L}^\alpha p(s,t,\cdot,y)(x) = 0, \quad \forall (s,x) \in [0,t)\times\mathbb{R}^d, \\ p(s,t,x,\cdot) \underset{s\to t^-}{\longrightarrow} \delta_x, \end{cases} \tag{B.9}$$

where $p(s,t,x,\cdot) \underset{s\to t^-}{\longrightarrow} \delta_x$ means that for all function $f : \mathbb{R}^d \to \mathbb{R}$ bounded and uniformly continuous, one has

$$\sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y) p(s,t,x,y)\, dy - f(x) \right| \underset{s\to t^-}{\to} 0.$$

Moreover, p satisfies the following estimates.

- There exists $C>0$ such that for all $j\in\{0,1\}$, $0\le s<t\le T$ and $x,y\in\mathbb{R}^d$

$$|\partial_x^j p(s,t,x,y)|\le C(t-s)^{-\frac{j}{\alpha}}\rho^j(t-s,y-x). \tag{B.10}$$

$$|\partial_s p(s,t,x,y)|\le C(t-s)^{-1}\rho^0(t-s,y-x). \tag{B.11}$$

- For all $j\in\{0,1\}$ and $\gamma\in(0,1]$ with $\gamma\in(0,(2\alpha-2)\wedge(\eta+\alpha-1))$ if $n=1$, there exists $C>0$ such that for all $0\le s<t\le T$ and $x_1,x_2,y\in\mathbb{R}^d$

$$|\partial_x^j p(s,t,x_1,y)-\partial_x^j p(s,t,x_2,y)|\le C(t-s)^{-\frac{j+\gamma}{\alpha}}|x_1-x_2|^\gamma\left[\rho^j(t-s,y-x_1)+\rho^j(t-s,y-x_2)\right]. \tag{B.12}$$

Before proving Theorem 5, we recall some properties satisfied by the functions ρ^k.

Lemma 10.

- For all $k>-\alpha$ and $\gamma\in[0,1]$ with $k-\gamma>-\alpha$, we have for all $t>0$ and $x\in\mathbb{R}^d$

$$|x|^\gamma t^{-\frac{\gamma}{\alpha}}\rho^k(t,x)\le\rho^{k-\gamma}(t,x). \tag{B.13}$$

- Let us fix $-\alpha<k_1\le k_2$. Then, for all function $y:(0,+\infty)\to\mathbb{R}^d$ such that $t\in(0,+\infty)\mapsto t^{-\frac{1}{\alpha}}y(t)$ is bounded, there exists $C>0$ such that for all $t>0$ and $x\in\mathbb{R}^d$

$$\rho^{k_2}(t,x+y(t))\le C\rho^{k_1}(t,x). \tag{B.14}$$

- For all $k>-\alpha$ and $R>0$, there exists C such that for all $t>0$, $y\in\mathbb{R}^d$ and $x\in\mathbb{R}^d$ with $|x|\le R$

$$\rho^k(t,y+x)\le(1+ct^{-\frac{1}{\alpha}}R)^{d+\alpha+k}\rho^k(t,y). \tag{B.15}$$

- For all $k_1,k_2>-\alpha$, there exists $C>0$ such that for all $s\ge0$, $t>s$ and $y\in\mathbb{R}^d$

$$\int_{\mathbb{R}^d}\rho^{k_1}(t-s,y-z)\rho^{k_2}(s,z)\,dz\le C\rho^{k_1\wedge k_2}(t,y). \tag{B.16}$$

The following lemma gathers the properties that we need on the proxy $\widehat{p}$.

Lemma 11. For all $t\in(0,T]$, $y\in\mathbb{R}^d$, $\widehat{p}(\cdot,t,\cdot,y)$ is of class $\mathcal{C}^{1,\infty}$ on $[0,t)\times\mathbb{R}^d$. Moreover, it satisfies the following gradient estimates.

- For all $j\in\mathbb{N}$, there exists $C>0$ such that for all $t\in(0,T]$, $s\in[0,t)$, $x,y\in\mathbb{R}^d$, we have

$$|\partial_x^j\widehat{p}(s,t,x,y)|\le C(t-s)^{-\frac{j}{\alpha}}\rho^j(t-s,y-x). \tag{B.17}$$

- There exists a constant $C>0$ such that for all $j\in\{0,1\}$, $t\in(0,T]$, $s\in[0,t)$, $x,y\in\mathbb{R}^d$

$$|\partial_s\partial_x^j\widehat{p}(s,t,x,y)|\le C(t-s)^{-1-\frac{j}{\alpha}}\rho^j(t-s,y-x). \tag{B.18}$$

- For all $j\in\mathbb{N}$, there exists $C>0$ such that for all $\gamma\in(0,1]$, $t\in(0,T]$, $s\in[0,t)$, $x_1,x_2,y\in\mathbb{R}^d$, we have

$$|\partial_x^j\widehat{p}(s,t,x_1,y)-\partial_x^j\widehat{p}(s,t,x_2,y)|\le C(t-s)^{-\frac{j+\gamma}{\alpha}}|x_1-x_2|^\gamma\left[\rho^j(t-s,y-x_1)+\rho^j(t-s,y-x_2)\right]. \tag{B.19}$$

- For all $j\in\{0,1\}$, there exists $C>0$ such that for all $\gamma\in(0,1]$, $t\in(0,T]$, $s_1,s_2\in[0,t)$, $x,y\in\mathbb{R}^d$, we have

$$\begin{aligned}&|\partial_x^j\widehat{p}(s_1,t,x,y)-\partial_x^j\widehat{p}(s_2,t,x,y)|\\&\quad\le C\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\wedge s_2,y-x)+\frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\vee s_2,y-x)\right].\end{aligned} \tag{B.20}$$

Proof. Notice that (B.17) and (B.19) are quite standard since $\widehat{p}(s,t,x,y) = q(t-s, y-x)$, where $q(t-s,\cdot)$ is the density of the α-stable random variable Z_{t-s} and $(Z_t)_t$ is a rotationally invariant stable process. We prove (B.18). We remark that the auto-similarity of the α-stable process Z implies that

$$\widehat{p}(s,t,x,y) = q(t-s,y-x) = (t-s)^{-\frac{d}{\alpha}} q\left(1, \frac{y-x}{(t-s)^{\frac{1}{\alpha}}}\right) = (t-s)^{-\frac{d}{\alpha}} \widehat{p}(0,1,(t-s)^{-\frac{1}{\alpha}}x, (t-s)^{-\frac{1}{\alpha}}y).$$

This yields

$$\begin{aligned}\partial_s \widehat{p}(s,t,x,y) = \frac{d}{\alpha}(t-s)^{-\frac{d}{\alpha}-1}\widehat{p}(0,1,(t-s)^{-\frac{1}{\alpha}}x,(t-s)^{-\frac{1}{\alpha}}y) \\ + (t-s)^{-\frac{d}{\alpha}}\partial_x \widehat{p}(0,1,(t-s)^{-\frac{1}{\alpha}}x,(t-s)^{-\frac{1}{\alpha}}y)\cdot\left(\frac{1}{\alpha}(t-s)^{-\frac{1}{\alpha}-1}(x-y)\right).\end{aligned} \tag{B.21}$$

Using (B.17), we obtain that

$$|\partial_s \widehat{p}(,s,t,xy)| \le C(t-s)^{-1}\rho^0(t-s,y-x) + C(t-s)^{-1}((t-s)^{-\frac{1}{\alpha}}|x-y|)\rho^1(1,(t-s)^{-\frac{1}{\alpha}}(y-x)).$$

The space-time inequality (B.13) finally yields

$$|\partial_s \widehat{p}(,s,t,xy)| \le C(t-s)^{-1}\rho^0(t-s,y-x).$$

By differentiating (B.21) with respect to x, one has

$$\begin{aligned}\partial_x\partial_s \widehat{p}(s,t,x,y) = \frac{d}{\alpha}(t-s)^{-\frac{1}{\alpha}}(t-s)^{-\frac{d}{\alpha}-1}\partial_x\widehat{p}(0,1,(t-s)^{-\frac{1}{\alpha}}x,(t-s)^{-\frac{1}{\alpha}}y) \\ + (t-s)^{-\frac{1}{\alpha}}(t-s)^{-\frac{d}{\alpha}}\partial_x^2\widehat{p}(0,1,(t-s)^{-\frac{1}{\alpha}}x,(t-s)^{-\frac{1}{\alpha}}y)\left(\frac{1}{\alpha}(t-s)^{-\frac{1}{\alpha}-1}(x-y)\right) \\ + \frac{1}{\alpha}(t-s)^{-\frac{1}{\alpha}-1}(t-s)^{-\frac{d}{\alpha}}\partial_x\widehat{p}(0,1,(t-s)^{-\frac{1}{\alpha}}x,(t-s)^{-\frac{1}{\alpha}}y).\end{aligned} \tag{B.22}$$

As previously, it follows from (B.17) and the space-time inequality (B.13) that

$$|\partial_s\partial_x \widehat{p}(s,t,x,y)| \le C(t-s)^{-1-\frac{j}{\alpha}}\rho^1(t-s,y-x).$$

We now use (B.18) to prove (B.20). We fix $j\in\{0,1\}$, $\gamma\in(0,1]$ and we start with the case $|s_1-s_2| > t-s_1\vee s_2$. In this case, using (B.17), we deduce that for some constant $C>0$, one has

$$\begin{aligned}&|\partial_x^j\widehat{p}(s_1,t,x,y) - \partial_x^j\widehat{p}(s_2,t,x,y)| \\ &\le C\left[(t-s_1\vee s_2)^{-\frac{j}{\alpha}}\rho^j(t-s_1\vee s_2,y-x) + (t-s_1\wedge s_2)^{-\frac{j}{\alpha}}\rho^j(t-s_1\wedge s_2,y-x)\right] \\ &\le C\left[\frac{|s_1-s_2|^\gamma}{|t-s_1\vee s_2|^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\vee s_2,y-x) + \frac{|s_1-s_2|^\gamma + |t-s_1\vee s_2|^\gamma}{|t-s_1\wedge s_2|^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\wedge s_2,y-x)\right] \\ &\le C\left[\frac{|s_1-s_2|^\gamma}{|t-s_1\vee s_2|^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\vee s_2,y-x) + \frac{|s_1-s_2|^\gamma}{|t-s_1\wedge s_2|^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\wedge s_2,y-x)\right].\end{aligned}$$

We now focus on the case $|s_1 - s_2| \le t - s_1 \vee s_2$. For $\lambda \in [0,1]$, we set $s_\lambda := \lambda s_1 + (1-\lambda)s_2$. We can thus write thanks to (B.18)

$$\begin{aligned}
&|\partial_x^j \widehat{p}(s_1,t,x,y) - \partial_x^j \widehat{p}(s_2,t,x,y)| \\
&\le \int_0^1 |\partial_s \partial_x^j \widehat{p}(s_\lambda,t,x,y)|\,|s_1-s_2|\,d\lambda \\
&\le C|s_1-s_2| \int_0^1 (t-s_\lambda)^{-1-\frac{j}{\alpha}} \rho^j(t-s_\lambda, y-x)\,d\lambda \\
&\le C|s_1-s_2| \int_0^1 (t-s_\lambda)^{-1-\frac{j+d}{\alpha}} (1+(t-s_\lambda)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha-j}\,d\lambda \\
&\le C|s_1-s_2|^\gamma (t-s_1\vee s_2)^{1-\gamma}(t-s_1\vee s_2)^{-1-\frac{j+d}{\alpha}} \Big[(1+(t-s_1\vee s_2)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha-j} \\
&\qquad\qquad +(1+(t-s_1\wedge s_2)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha-j}\Big].
\end{aligned}$$

Since $|s_1-s_2| \le t - s_1 \vee s_2$, we easily check that $(t-s_1\vee s_2)^{-1} \le 2(t-s_1\wedge s_2)^{-1}$. It follows that

$$\begin{aligned}
&|\partial_x^j \widehat{p}(s_1,t,x,y) - \partial_x^j \widehat{p}(s_2,t,x,y)| \\
&\le C|s_1-s_2|^\gamma \Big[(t-s_1\vee s_2)^{-\gamma-\frac{j+d}{\alpha}}(1+(t-s_1\vee s_2)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha-j} \\
&\qquad\qquad +(t-s_1\wedge s_2)^{-\gamma-\frac{j+d}{\alpha}}(1+(t-s_1\wedge s_2)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha-j}\Big] \\
&\le C\left[\frac{|s_1-s_2|^\gamma}{(t-s_1\wedge s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\wedge s_2, y-x) + \frac{|s_1-s_2|^\gamma}{(t-s_1\vee s_2)^{\gamma+\frac{j}{\alpha}}}\rho^j(t-s_1\vee s_2, y-x)\right].
\end{aligned}$$

This concludes the proof. □

Recall that the Beta function $\mathcal{B}$ is defined, for all $x, y > 0$ by

$$\mathcal{B}(x,y) := \int_0^1 (1-t)^{-1+x} t^{-1+y}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where Γ is the Gamma function.

The next proposition gathers the controls we need on the parametrix kernel $\mathcal{H}$ and the solution de the Volterra integral equation Φ.

Proposition 8.

- There exists $C > 0$ such that for all $k \ge 1$, $0 \le s < t \le T$ and $x, y \in \mathbb{R}^d$

$$|\mathcal{H}^k(s,t,x,y)| \le C^k (t-s)^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \rho^1(t-s, y-x). \tag{B.23}$$

- For $\gamma \in (0, \eta]$ such that $\gamma < \alpha - 1$, there exists $C > 0$ depending on γ such that for all $k \ge 1$, $0 \le s < t \le T$ and $x_1, x_2, y \in \mathbb{R}^d$

$$\begin{aligned}
|\mathcal{H}^k(s,t,x_1,y) - \mathcal{H}^k(s,t,x_2,y)| &\le C^k (t-s)^{-\frac{\gamma+1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)} |x_1-x_2|^\gamma \prod_{j=1}^{k-1} \mathcal{B}\left(-\frac{\gamma}{\alpha} + j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha}\right) \\
&\qquad \left[\rho^1(t-s, y-x_1) + \rho^1(t-s, y-x_2)\right].
\end{aligned} \tag{B.24}$$

- There exists $C>0$ such that for all $k\geq 1$, $0\leq s<t\leq T$ and $x,y\in\mathbb{R}^d$

$$|\widehat{p}\otimes\mathcal{H}^k(s,t,x,y)|\leq C^{k+1}(t-s)^{k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k}\mathcal{B}\left(\frac{1}{\alpha}+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^0(t-s,y-x). \tag{B.25}$$

- The series (B.6) defining Φ is absolutely convergent and there exists $C>0$ such that for all $0\leq s<t\leq T$ and $x,y\in\mathbb{R}^d$

$$|\Phi(s,t,x,y)|\leq C(t-s)^{-\frac{1}{\alpha}}\rho^1(t-s,y-x). \tag{B.26}$$

- For $\gamma\in(0,\eta]$ such that $\gamma<\alpha-1$, there exists $C>0$ depending on γ such that for all $0\leq s<t\leq T$ and $x_1,x_2,y\in\mathbb{R}^d$

$$|\Phi(s,t,x_1,y)-\Phi(s,t,x_2,y)|\leq C(t-s)^{-\frac{\gamma+1}{\alpha}}|x_1-x_2|^\gamma\left[\rho^1(t-s,y-x_1)+\rho^1(t-s,y-x_2)\right]. \tag{B.27}$$

Proof. **Proof of (B.23).** We reason by induction on k. The base case $k=1$ is clear since b is bounded and by Lemma 11. We assume now that (B.23) holds for $\mathcal{H}^k$ and we want to prove it for $\mathcal{H}^{k+1}$. We have thanks to Lemma 10

$$\begin{aligned}
|\mathcal{H}^{k+1}(s,t,x,y)| &= \left|\int_s^t\int_{\mathbb{R}^d}\mathcal{H}(s,r,x,z)\mathcal{H}^k(r,t,z,y)\,dz\,dr\right| \\
&\leq \int_s^t\int_{\mathbb{R}^d}C(r-s)^{-\frac{1}{\alpha}}\rho^1(r-s,z-x)C^k(t-r)^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)} \\
&\qquad\qquad\prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^1(t-r,y-z)\,dz\,dr \\
&\leq C^{k+1}\int_s^t(r-s)^{-\frac{1}{\alpha}}(t-r)^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)}dr\prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^1(t-s,y-x).
\end{aligned}$$

Changing variables in $r=s+\lambda(t-s)$ yields

$$\begin{aligned}
|\mathcal{H}^{k+1}(s,t,x,y)| &\leq C^{k+1}(t-s)^{-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)}\int_0^1\lambda^{-\frac{1}{\alpha}}(1-\lambda)^{-\frac{1}{\alpha}+(k-1)\left(1-\frac{1}{\alpha}\right)}d\lambda \\
&\qquad\qquad\prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^1(t-s,y-x) \\
&\leq C^{k+1}(t-s)^{-\frac{1}{\alpha}+k\left(1-\frac{1}{\alpha}\right)}\mathcal{B}\left(k\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\rho^1(t-s,y-x).
\end{aligned}$$

Proof of (B.24). We start with the case $k=1$. We write

$$\begin{aligned}
|\mathcal{H}(s,t,x_1,y)-\mathcal{H}(s,t,x_2,y)| &= |b(s,x_1)\cdot\partial_x\widehat{p}(s,t,x_1,y)-b(s,x_2)\cdot\partial_x\widehat{p}(s,t,x_2,y)| \\
&\leq |b(s,x_1)|\,|\partial_x\widehat{p}(s,t,x_1,y)-\partial_x\widehat{p}(s,t,x_2,y)| \\
&\quad+|\partial_x\widehat{p}(s,t,x_2,y)|\,|b(s,x_1)-b(s,x_2)| \\
&=: I_1+I_2.
\end{aligned}$$

Using that b is bounded and Lemma 11, we deduce that

$$I_1\leq C(t-s)^{-\frac{1+\gamma}{\alpha}}|x_1-x_2|^\gamma\left[\rho^1(t-s,y-x_1)+\rho^1(t-s,y-x_2)\right].$$

Since b is uniformly η-Hölder continuous and bounded, and thus uniformly γ-Hölder continuous because $\gamma\leq\eta$, one has by Lemma 11

$$I_2\leq C(t-s)^{-\frac{1}{\alpha}}|x_1-x_2|^\gamma\rho^1(t-s,y-x_2).$$

We now prove that (B.24) holds for $\mathcal{H}^{k+1}$, with $k \geq 1$. Using the case $k = 1$, (B.23) and Lemma 10, we obtain that

$$
\begin{aligned}
&|\mathcal{H}^{k+1}(s,t,x_1,y) - \mathcal{H}^{k+1}(s,t,x_2,y)| \\
&= \left| \int_s^t \int_{\mathbb{R}^d} (\mathcal{H}(s,r,x_1,z) - \mathcal{H}(s,r,x_2,z)) \mathcal{H}^k(r,t,z,y)\, dz\, dr \right| \\
&\leq \int_s^t \int_{\mathbb{R}^d} C(r-s)^{-\frac{1+\gamma}{\alpha}} |x_1 - x_2|^\gamma \left[\rho^1(r-s, z-x_1) + \rho^1(r-s, z-x_2) \right] C^k (t-r)^{-\frac{1}{\alpha} + (k-1)\left(1-\frac{1}{\alpha}\right)} \\
&\qquad \prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha} \right) \rho^1(t-r, y-z)\, dz\, dr \\
&\leq C^{k+1} (t-s)^{-\frac{1+\gamma}{\alpha} + k\left(1-\frac{1}{\alpha}\right)} \mathcal{B}\left(k\left(1-\frac{1}{\alpha}\right), 1-\frac{1+\gamma}{\alpha} \right) \prod_{j=1}^{k} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha} \right) \\
&\qquad \left[\rho^1(t-s, y-x_1) + \rho^1(t-s, y-x_2) \right].
\end{aligned}
$$

We conclude noting that

$$
\begin{aligned}
\mathcal{B}\left(k\left(1-\frac{1}{\alpha}\right), 1-\frac{1+\gamma}{\alpha} \right) \prod_{j=1}^{k} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha} \right) &= \frac{\Gamma\left(k\left(1-\frac{1}{\alpha}\right)\right) \Gamma\left(1-\frac{1+\gamma}{\alpha}\right)}{\Gamma\left(1-\frac{1+\gamma}{\alpha} + k\left(1-\frac{1}{\alpha}\right)\right)} \prod_{j=1}^{k-1} \frac{\Gamma\left(j\left(1-\frac{1}{\alpha}\right)\right) \Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left((j+1)\left(1-\frac{1}{\alpha}\right)\right)} \\
&= \frac{\Gamma\left(k\left(1-\frac{1}{\alpha}\right)\right) \Gamma\left(1-\frac{1+\gamma}{\alpha}\right)}{\Gamma\left(1-\frac{1+\gamma}{\alpha} + k\left(1-\frac{1}{\alpha}\right)\right)} \frac{\left[\Gamma\left(1-\frac{1}{\alpha}\right)\right]^k}{\Gamma\left(k\left(1-\frac{1}{\alpha}\right)\right)} \\
&= \frac{\Gamma\left(1-\frac{1+\gamma}{\alpha}\right) \left[\Gamma\left(1-\frac{1}{\alpha}\right)\right]^k}{\Gamma\left(1-\frac{1+\gamma}{\alpha} + k\left(1-\frac{1}{\alpha}\right)\right)} \\
&= \prod_{j=1}^{k} \frac{\Gamma\left(-\frac{\gamma}{\alpha} + j\left(1-\frac{1}{\alpha}\right)\right) \Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(-\frac{\gamma}{\alpha} + (j+1)\left(1-\frac{1}{\alpha}\right)\right)} \\
&= \prod_{j=1}^{k} \mathcal{B}\left(-\frac{\gamma}{\alpha} + j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha} \right).
\end{aligned}
$$

Proof of (B.25)**.** It follows from the control of $\widehat{p}$ given by Lemma 11 and from (B.23). Indeed, one has thanks to Lemma 10

$$
\begin{aligned}
&|\widehat{p} \otimes \mathcal{H}^k(s,t,x,y)| \\
&\leq C \int_s^t \int_{\mathbb{R}^d} \rho^0(r-s, z-x) C^k (t-r)^{-\frac{1}{\alpha} + (k-1)\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha} \right) \rho^1(t-r, y-z)\, dz\, dr \\
&\leq C^{k+1} \frac{(t-s)^{k\left(1-\frac{1}{\alpha}\right)}}{k\left(1-\frac{1}{\alpha}\right)} \prod_{j=1}^{k-1} \mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right), 1-\frac{1}{\alpha} \right) \rho^0(t-s, y-x)
\end{aligned}
$$

We conclude noting that

$$\begin{aligned}\frac{1}{k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k-1}\mathcal{B}\left(j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right) &= \frac{1}{k\left(1-\frac{1}{\alpha}\right)}\prod_{j=1}^{k-1}\frac{\Gamma\left(j\left(1-\frac{1}{\alpha}\right)\right)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left((j+1)\left(1-\frac{1}{\alpha}\right)\right)}\\ &= \frac{1}{k\left(1-\frac{1}{\alpha}\right)}\frac{\left[\Gamma\left(1-\frac{1}{\alpha}\right)\right]^k}{\Gamma\left(k\left(1-\frac{1}{\alpha}\right)\right)}\\ &= \frac{\left[\Gamma\left(1-\frac{1}{\alpha}\right)\right]^k}{\Gamma\left(1+k\left(1-\frac{1}{\alpha}\right)\right)}\\ &= \prod_{j=1}^{k}\mathcal{B}\left(\frac{1}{\alpha}+j\left(1-\frac{1}{\alpha}\right),1-\frac{1}{\alpha}\right)\end{aligned}$$

Finally, the proof of (B.26) and (B.27) follows directly from (B.23) and (B.24) using the asymptotic expansion of the Beta function.

□

Proof of Theorem 5. The existence of the density and its representation (B.8) is a consequence of (B.17), (B.25) and (B.26). Indeed, using the asymptotic expansion of the Beta function, we obtain that the series (B.8) is absolutely convergent, locally uniformly with respect to $(s,x)\in[0,t)\times\mathbb{R}^d$. The permutation of the series and the convolution in the representation is clearly justified by the dominated convergence theorem (B.8). The regularity of the density p with respect to x and the controls (B.10) and (B.12) follow from Lemma 11 for the proxy, i.e. the first term of the parametrix series (B.8). We now prove that they also hold for the other term $\widehat{p}\otimes\Phi$ of (B.8).

Proof of (B.10) for $j=1$. We use (B.26) and Lemma 11 to deduce that

$$|\partial_x\widehat{p}\otimes\Phi(s,t,x,y)|\le C\int_s^t\int_{\mathbb{R}^d}(r-s)^{-\frac{1}{\alpha}}\rho^1(r-s,z-x)(t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr.$$

Then, Lemma 10 yields

$$\begin{aligned}&|\partial_x\widehat{p}\otimes\Phi(s,t,x,y)|\\ &\le C\int_s^t(r-s)^{-\frac{1}{\alpha}}(t-r)^{-\frac{1}{\alpha}}\,dr\rho^1(t-s,y-x)\\ &\le C(t-s)^{-\frac{1}{\alpha}1-\frac{1}{\alpha}}\rho^1(t-s,y-x).\end{aligned}$$

Proof of (B.12) for $j=0$. We use again the control of $\widehat{p}$ given by Lemma 11 and (B.26). Thanks to Lemma 10, we obtain

$$\begin{aligned}&|\widehat{p}\otimes\Phi(s,t,x_1,y)-\widehat{p}\otimes\Phi(s,t,x_2,y)|\\ &=\left|\int_s^t\int_{\mathbb{R}^d}(\widehat{p}(s,r,x_1,z)-\widehat{p}(s,r,x_2,z))\Phi(r,t,z,y)\,dz\,dr\right|\\ &\le C\int_s^t\int_{\mathbb{R}^d}(r-s)^{-\frac{\gamma}{\alpha}}|x_1-x_2|^\gamma\left[\rho^0(r-s,z-x_1)+\rho^0(r-s,z-x_2)\right](t-r)^{-\frac{1}{\alpha}}\rho^1(t-r,y-z)\,dz\,dr\\ &\le C(t-s)^{-\frac{\gamma}{\alpha}+1-\frac{1}{\alpha}}|x_1-x_2|^\gamma\left[\rho^0(t-s,y-x_1)+\rho^0(t-s,y-x_2)\right].\end{aligned}$$

Proof of (B.12) **for** $j = 1$**.** We use the following decomposition

$$\begin{aligned}\partial_x\widehat{p}\otimes\Phi(s,t,x_1,y)-\partial_x\widehat{p}\otimes\Phi(s,t,x_2,y) &= \int_s^t\int_{\mathbb{R}^d}(\partial_x\widehat{p}(s,r,x_1,z)-\partial_x\widehat{p}(s,r,x_2,z))\Phi(r,t,z,y)\,dz\,dr\\ &= \int_{D_1}\int_{\mathbb{R}^d}(\partial_x\widehat{p}(s,r,x_1,z)-\partial_x\widehat{p}(s,r,x_2,z))\Phi(r,t,z,y)\,dz\,dr\\ &\quad+\int_{D_2}\int_{\mathbb{R}^d}(\partial_x\widehat{p}(s,r,x_1,z)-\partial_x\widehat{p}(s,r,x_2,z))\Phi(r,t,z,y)\,dz\,dr\\ &=: I_1+I_2,\end{aligned}$$

where $D_1 := \{r\in(s,t),\ |x_1-x_2|>(r-s)^{\frac{1}{\alpha}}\}$ and $D_2 := \{r\in(s,t),\ |x_1-x_2|\le(r-s)^{\frac{1}{\alpha}}\}$. For I_1, one can write

$$\begin{aligned}I_1 &= \int_{D_1}\int_{\mathbb{R}^d}\partial_x\widehat{p}(s,r,x_1,z)(\Phi(r,t,z,y)-\Phi(r,t,x_1,y))\,dz\,dr\\ &\quad-\int_{D_1}\int_{\mathbb{R}^d}\partial_x\widehat{p}(s,r,x_2,z)(\Phi(r,t,z,y)-\Phi(r,t,x_2,y))\,dz\,dr,\\ &=: I_{1,1}+I_{1,2},\end{aligned}$$

since $\int_{\mathbb{R}^d}\partial_x\widehat{p}(s,r,x,z)\,dz=0$ for all $x\in\mathbb{R}^d$. Since $\alpha\in(1,2)$ and $\gamma<(2\alpha-2)\wedge(\eta+\alpha-1)$, we can pick $\delta\in(0,(\alpha-1)\wedge\eta)$ such that $\gamma<\delta+\alpha-1$. Then, we can use Lemma 11 and (B.27) with δ which yields

$$|I_{1,1}|\le C\int_{D_1}\int_{\mathbb{R}^d}(r-s)^{-\frac{1}{\alpha}}\rho^1(r-s,z-x_1)(t-r)^{-\frac{\delta+1}{\alpha}}|z-x_1|^\delta\left[\rho^1(t-r,y-z)+\rho^1(t-r,y-x_1)\right]\,dz\,dr.$$

Using the space-time inequality (B.13) in Lemma 10, we deduce that

$$|I_{1,1}|\ \le\ C\int_{D_1}\int_{\mathbb{R}^d}(r-s)^{\frac{\delta-1}{\alpha}}\rho^{1-\delta}(r-s,z-x_1)(t-r)^{-\frac{\delta+1}{\alpha}}\left[\rho^1(t-r,y-z)+\rho^1(t-r,y-x_1)\right]\,dz\,dr.$$

Note that $\int_{\mathbb{R}^d}\rho^{1-\delta}(r-s,z-x_1)\,dz$ is a constant independent of r,s,x_1, that if $r\in(s,t)$

$$\rho^1(t-r,y-x_1)\le(t-r)^{-\frac{d}{\alpha}}(1+(t-s)^{-\frac{1}{\alpha}}|y-x_1|)^{-d-\alpha-1}$$

and that if $r\in D_1$, $(r-s)^{\frac{\gamma}{\alpha}}<|x_1-x_2|^\gamma$. Applying Lemma 10, we get that

$$\begin{aligned}|I_{1,1}| &\le C\left[\int_s^t(r-s)^{\frac{\delta-1-\gamma}{\alpha}}(t-r)^{-\frac{\delta+1}{\alpha}}\,dr|x_1-x_2|^\gamma\rho^{1-\delta}(t-s,y-x_1)\right.\\ &\qquad\left.\int_s^t(r-s)^{\frac{\delta-1-\gamma}{\alpha}}(t-r)^{-\frac{\delta+1+d}{\alpha}}\,dr|x_1-x_2|^\gamma(1+(t-s)^{-\frac{1}{\alpha}}|y-x_1|)^{-d-\alpha-1}\right]\\ &\le C(t-s)^{-\frac{1+\gamma}{\alpha}+1-\frac{1}{\alpha}}|x_1-x_2|^\gamma\rho^{1-\delta}(t-s,y-x_1).\end{aligned}$$

Similarly, we obtain

$$|I_{1,2}|\le C(t-s)^{-\frac{1+\gamma}{\alpha}+1-\frac{1}{\alpha}}|x_1-x_2|^\gamma\rho^{1-\delta}(t-s,y-x_2).$$

This proves (B.12) for I_1. We now focus on I_2. One can write

$$\begin{aligned}I_2 &= \int_{D_2}\int_{\mathbb{R}^d}(\partial_x\widehat{p}(s,r,x_1,z)-\partial_x\widehat{p}(s,r,x_2,z))\Phi(r,t,z,y)\,dz\,dr\\ &= \int_{D_2}\int_{\mathbb{R}^d}(\partial_x\widehat{p}(s,r,x_1,z)-\partial_x\widehat{p}(s,r,x_2,z))(\Phi(r,t,z,y)-\Phi(r,t,x_2,y))\,dz\,dr.\end{aligned}$$

Note that if $r \in D_2$, we have

$$\begin{aligned}|\partial_x \widehat{p}(s,r,x_1,z) - \partial_x \widehat{p}(s,r,x_2,z)| &\leq C(r-s)^{-\frac{2}{\alpha}}|x_1-x_2|\left[\rho^2(r-s,z-x_1)+\rho^2(r-s,z-x_1)\right]\\ &\leq C(r-s)^{-\frac{\gamma+1}{\alpha}}|x_1-x_2|^{\gamma}\rho^2(r-s,z-x_2),\end{aligned}$$

since $|x_1-x_2| \leq (r-s)^{\frac{1}{\alpha}}$ and by (B.14). Using (B.27) with $\delta \in (0,(\alpha-1)\wedge\eta)$ such that $\gamma < \delta+\alpha-1$ and the space-time inequality (B.13) we get

$$\begin{aligned}|I_2| &\leq C\int_{D_2}\int_{\mathbb{R}^d}(r-s)^{-\frac{\gamma+1}{\alpha}}|x_1-x_2|^{\gamma}\rho^2(r-s,z-x_2)(t-r)^{-\frac{\delta+1}{\alpha}}|z-x_2|^{\delta}\\ &\qquad\qquad\left[\rho^1(t-r,y-z)+\rho^1(t-r,y-x_2)\right]\,dz\,dr\\ &\leq C\int_{D_2}\int_{\mathbb{R}^d}(r-s)^{-\frac{\gamma+1}{\alpha}+\frac{\delta}{\alpha}}|x_1-x_2|^{\gamma}\rho^{2-\delta}(r-s,z-x_2)(t-r)^{-\frac{\delta+1}{\alpha}}\\ &\qquad\qquad\left[\rho^1(t-r,y-z)+\rho^1(t-r,y-x_2)\right]\,dz\,dr.\end{aligned}$$

As done previously to deal with $I_{1,1}$, Lemma 10 yields

$$|I_2| \leq C(t-s)^{-\frac{\gamma+1}{\alpha}+1-\frac{1}{\alpha}}\left[\rho^1(t-r,y-x_1)+\rho^1(t-r,y-x_2)\right].$$

It concludes the proof of (B.12) for $j=1$.

Proof of (B.9)**.** Let us now prove that $p(\cdot,t,\cdot,y)$ is a fundamental solution to (B.9). We fix $0 \leq s < t \leq T$ and $x,y \in \mathbb{R}^d$. From the Markov property satisfied by the SDE (B.1), stemming from the well-posedness of the related martingale problem, one has for all $h>0$ such that $s-h \geq 0$

$$p(s-h,t,x,y) = \mathbb{E}(p(s,t,X_s^{s-h,x},y)).$$

Applying Itô's formula to the function $p(s,t,\cdot,y)$ which belongs to $\mathcal{C}_b^{1+\gamma}(\mathbb{R}^d;\mathbb{R})$ for $\gamma > \alpha-1$, we obtain that

$$\begin{aligned}p(s,t,X_s^{s-h,x},y) &= p(s,t,x,y) + \int_{s-h}^{s} b(r,X_r^{s-h,x})\cdot\partial_x p(s,t,X_r^{s-h,x},y)\,dr\\ &\quad+\int_{s-h}^{s}\int_{\mathbb{R}^d} p(s,t,X_{r^-}^{s-h,x}+z,y)-p(s,t,X_{r^-}^{s-h,x},y)\,\widetilde{\mathcal{N}}(dr,dz)\\ &\quad+\int_{s-h}^{s}\int_{\mathbb{R}^d} p(s,t,X_{r^-}^{s-h,x}+z,y)-p(s,t,X_{r^-}^{s-h,x},y)-\partial_x p(s,t,X_{r^-}^{s-h,x},y)\cdot z\,d\nu(z)\,dr\end{aligned}$$

□

We can take the expectation in the preceding formula using (B.10), (B.12). It yields

$$\begin{aligned}p(s-h,t,x,y) &= p(s,t,x,y) + \int_{s-h}^{s}\mathbb{E}(b(r,X_r^{s-h,x})\cdot\partial_x p(s,t,X_r^{s-h,x},y))\,dr\\ &\quad+\int_{s-h}^{s}\int_{\mathbb{R}^d}\mathbb{E}(p(s,t,X_r^{s-h,x}+z,y)-p(s,t,X_r^{s-h,x},y)-\partial_x p(s,t,X_r^{s-h,x},y)\cdot z)\,d\nu(z)\,dr\end{aligned}$$

Let us prove that

$$\frac{1}{h}\int_{s-h}^{s}\mathbb{E}(b(r,X_r^{s-h,x})\cdot\partial_x p(s,t,X_r^{s-h,x},y))\,dr \underset{h\to 0}{\longrightarrow} b(s,x)\cdot\partial_x p(s,t,x,y). \tag{B.28}$$

We can write

$$\begin{aligned}
&\left|\frac{1}{h}\int_{s-h}^{s}\mathbb{E}(b(r,X_r^{s-h,x})\cdot\partial_x p(s,t,X_r^{s-h,x},y))\,dr - b(s,x)\cdot\partial_x p(s,t,x,y)\right| \\
&\leq \left|\frac{1}{h}\int_{s-h}^{s}\mathbb{E}(b(r,X_r^{s-h,x})\cdot\partial_x p(s,t,X_r^{s-h,x},y)) - \mathbb{E}(b(r,X_s^{s-h,x})\cdot\partial_x p(s,t,X_s^{s-h,x},y))\,dr\right| \\
&\quad + \left|\frac{1}{h}\int_{s-h}^{s}\mathbb{E}(b(r,X_s^{s-h,x})\cdot\partial_x p(s,t,X_s^{s-h,x},y)) - (b(r,x)\cdot\partial_x p(s,t,x,y))\,dr\right| \\
&\quad + \left|\frac{1}{h}\int_{s-h}^{s}(b(r,x)\cdot\partial_x p(s,t,x,y)) - b(s,x)\cdot\partial_x p(s,t,x,y)\,dr\right| \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

It is clear that I_3 converges to 0 as h tends to 0 since b is continuous. Let us deal with I_2, which we decompose in the following way

$$\begin{aligned}
I_2 &\leq \frac{1}{h}\int_{s-h}^{s}\mathbb{E}|b(r,X_s^{s-h,x})\cdot\partial_x p(s,t,X_s^{s-h,x},y)) - (b(r,x)\cdot\partial_x p(s,t,x,y)|\,dr \\
&\leq \frac{1}{h}\int_{s-h}^{s}\mathbb{E}|b(r,X_s^{s-h,x})|\,|\partial_x p(s,t,X_s^{s-h,x},y) - \partial_x p(s,t,x,y)|\,dr \\
&\quad + \frac{1}{h}\int_{s-h}^{s}\mathbb{E}|b(r,X_s^{s-h,x}) - b(r,x)|\,|\partial_x p(s,t,x,y)|\,dr \\
&=: I_{2,1} + I_{2,2}.
\end{aligned}$$

Since b is bounded and by the Hölder control (B.12), we obtain that for some constant $\gamma\in(0,1)$

$$\begin{aligned}
I_{2,1} &\leq C_{s,t,\gamma}\mathbb{E}|X_s^{s-h,x} - x|^{\gamma} \\
&\leq C_{s,t,\gamma}h^{\frac{\gamma}{\alpha}}.
\end{aligned}$$

The same reasoning based on the Hölder regularity of b proves that $I_2 \underset{h\to 0}{\longrightarrow} 0$.

Finally, we decompose I_1 in the following way

$$\begin{aligned}
I_1 &\leq \frac{1}{h}\int_{s-h}^{s}\mathbb{E}|b(r,X_r^{s-h,x})|\,|\partial_x p(s,t,X_r^{s-h,x},y) - \partial_x p(s,t,X_s^{s-h,x},y)|\,dr \\
&\quad + \frac{1}{h}\int_{s-h}^{s}\mathbb{E}|b(r,X_r^{s-h,x}) - b(r,X_s^{s-h,x})|\,|\partial_x p(s,t,X_s^{s-h,x},y)|\,dr \\
&=: I_{1,1} + I_{1,2}.
\end{aligned}$$

Note that for all $\gamma\in(0,1]$, there exists $C_{s,t,\gamma}>0$ such that for all $r\in[s-h,s]$

$$\mathbb{E}|X_r^{s-h,x} - X_s^{s-h,x}|^{\gamma} \leq C_{s,t,\gamma}(s-r)^{\frac{\gamma}{\alpha}},$$

The same reasoning as done for I_2 can be applied since b and $\partial_x p(s,t,\cdot,y)$ are globally bounded and b is uniformly Hölder continuous. It yields $I_1 \underset{h\to 0}{\longrightarrow} 0$.

Let us now show that

$$\begin{aligned}
&\frac{1}{h}\int_{s-h}^{s}\int_{\mathbb{R}^d}\mathbb{E}(p(s,t,X_r^{s-h,x}+z,y) - p(s,t,X_r^{s-h,x},y) - \partial_x p(s,t,X_r^{s-h,x},y)\cdot z)\,d\nu(z)\,dr \\
&\qquad\qquad \underset{h\to 0}{\longrightarrow} \mathcal{L}^{\alpha}p(s,t,\cdot,y)(x). \quad \text{(B.29)}
\end{aligned}$$

One can write

$$\begin{aligned}
&\left|\frac{1}{h}\int_{s-h}^{s}\int_{\mathbb{R}^d}\mathbb{E}(p(s,t,X_r^{s-h,x}+z,y)-p(s,t,X_r^{s-h,x},y)-\partial_x p(s,t,X_r^{s-h,x},y)\cdot z)\,d\nu(z)\,dr-\mathcal{L}^\alpha p(s,t,\cdot,y)(x)\right|\\
&\le\left|\int_{\mathbb{R}^d}\frac{1}{h}\int_{s-h}^{s}\mathbb{E}(p(s,t,X_r^{s-h,x}+z,y)-p(s,t,X_r^{s-h,x},y)-\partial_x p(s,t,X_r^{s-h,x},y)\cdot z)\,dr\,d\nu(z)\right.\\
&\qquad\left.-\int_{\mathbb{R}^d}\mathbb{E}(p(s,t,X_s^{s-h,x}+z,y)-p(s,t,X_s^{s-h,x},y)-\partial_x p(s,t,X_s^{s-h,x},y)\cdot z)\,d\nu(z)\right|\\
&\quad+\left|\int_{\mathbb{R}^d}\mathbb{E}(p(s,t,X_s^{s-h,x}+z,y)-p(s,t,X_s^{s-h,x},y)-\partial_x p(s,t,X_s^{s-h,x},y)\cdot z)\,d\nu(z)-\mathcal{L}^\alpha p(s,t,\cdot,y)(x)\right|\\
&=: J_1+J_2.
\end{aligned}$$

For J_1, we obtain that

$$\begin{aligned}
J_1\le\int_{\mathbb{R}^d}\frac{1}{h}\int_{s-h}^{s}\int_0^1\mathbb{E}\Big|&(\partial_x p(s,t,X_r^{s-h,x}+\lambda z,y)-\partial_x p(s,t,X_r^{s-h,x},y))\\
&-(\partial_x p(s,t,X_s^{s-h,x}+\lambda z,y)-\partial_x p(s,t,X_s^{s-h,x},y))\Big|\,|z|\,d\lambda\,dr\,d\nu(z)
\end{aligned}$$

We are going to use the dominated convergence theorem in the integral with respect to ν. By the Hölder control (B.12) on $\partial_x p$, we deduce that for some $\gamma>\alpha-1$, there exists a constant $C_{s,t,\gamma}>0$ such that for all $r\in[s-h,s]$, $z\in\mathbb{R}^d$

$$\begin{aligned}
&\frac{1}{h}\int_{s-h}^{s}\int_0^1\mathbb{E}\Big|(\partial_x p(s,t,X_r^{s-h,x}+\lambda z,y)-\partial_x p(s,t,X_r^{s-h,x},y))\\
&\qquad-(\partial_x p(s,t,X_s^{s-h,x}+\lambda z,y)-\partial_x p(s,t,X_s^{s-h,x},y))\Big|\,|z|\,d\lambda\,dr\\
&\le C_{s,t,\gamma}\frac{1}{h}\int_{s-h}^{s}\mathbb{E}|X_r^{s-h,x}-X_s^{s-h,x}|^\gamma\,dr\,|z|\\
&\le C_{s,t,\gamma}\frac{1}{h}\int_{s-h}^{s}(r-s)^{\frac{\gamma}{\alpha}}\,dr\,|z|\\
&\le C_{s,t,\gamma}h^{\frac{\gamma}{\alpha}}|z|.
\end{aligned}$$

The right-hand side term tends to 0 when $h\to0$. We start by justifying the domination on the ball B_1. In this case, we use again the Hölder continuity of $\partial_x p$ with respect to x, which yields

$$\begin{aligned}
&\frac{1}{h}\int_{s-h}^{s}\int_0^1\mathbb{E}\Big|(\partial_x p(s,t,X_r^{s-h,x}+\lambda z,y)-\partial_x p(s,t,X_r^{s-h,x},y))\\
&\qquad-(\partial_x p(s,t,X_s^{s-h,x}+\lambda z,y)-\partial_x p(s,t,X_s^{s-h,x},y))\Big|\,|z|\,d\lambda\,dr\\
&\le C_{s,t,\gamma}|z|^{1+\gamma}.
\end{aligned}$$

Since $\gamma>\alpha-1$, $z\in B_1\mapsto|z|^{1+\gamma}$ belongs to $L^1(B_1,\nu)$. The domination on B_1^c is clear since $\partial_x\partial_x p(s,t,\cdot,y)$ is globally bounded by (B.10) and $z\in B_1^c\mapsto|z|$ belongs to $L^1(B_1^c,\nu)$.

We now deal with J_2. Thanks to the dominated convergence theorem, J_2 converges to 0 as h tends to 0. The Hölder control (B.12) of $\partial_x p$ ensures that for some constant $\gamma>\alpha-1$, there exists $C_{s,t,\gamma}>0$ such that

$$\mathbb{E}|p(s,t,X_s^{s-h,x}+z,y)-p(s,t,X_s^{s-h,x},y)-\partial_x p(s,t,X_s^{s-h,x},y)\cdot z|\le C_{s,t,\gamma}|z|^{1+\gamma}.$$

This proves the domination on the ball B_1. The domination on B_1^c is a consequence of (B.10), and the fact that $\alpha\in(1,2)$.

We have thus proved that the map $s \in [0,t) \mapsto p(s,t,x,y)$ is left-differentiable by (B.28) and (B.29). Moreover, since the map $(s,x) \in [0,t) \times \mathbb{R}^d \mapsto b(s,x) \cdot \partial_x p(s,t,x,y) + \mathcal{L}^\alpha p(s,t,\cdot,y)(x)$ is continuous, we deduce that the map $s \in [0,t) \mapsto p(s,t,x,y)$ is of class $\mathcal{C}^1$ and that it solves

$$\partial_s p(s,t,x,y) + b(s,x) \cdot \partial_x p(s,t,x,y) + \mathcal{L}^\alpha p(s,t,\cdot,y)(x) = 0, \quad \forall (s,x) \in [0,t) \times \mathbb{R}^d.$$

Let us now fix $f : \mathbb{R}^d \to \mathbb{R}$ a bounded and uniformly continuous function. We fix $\varepsilon > 0$. There exists $\delta > 0$ such that for all $x, y \in \mathbb{R}^d$ with $|x-y| \le \delta$, we have $|f(x) - f(y)| \le \varepsilon$. Using (B.10), we obtain that

$$\begin{aligned}
\sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y) p(s,t,x,y)\, dy - f(x) \right| &= \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} (f(y) - f(x)) p(s,t,x,y)\, dy \right| \\
&\le \varepsilon + C\|f\|_\infty \int_{|y|>\delta} (t-s)^{-\frac{d}{\alpha}} (1 + (t-s)^{-\frac{1}{\alpha}} |y|)^{-d-\alpha}\, dy \\
&= \varepsilon + C\|f\|_\infty \int_{|z|>(t-s)^{-\frac{1}{\alpha}}\delta} (1+|z|)^{-d-\alpha}\, dz.
\end{aligned}$$

We conclude taking the lim sup when $s \to t$ in the preceding inequality that $p(s,t,x,\cdot) \underset{s \to t^-}{\longrightarrow} \delta_x$.

Proof of (B.11)**.** Note that we only need to prove that

$$|\mathcal{L}^\alpha p(s,t,\cdot,y)(x)| \le C(t-s)^{-1} \rho^0(t-s, y-x),$$

using the PDE (B.9) and the fact that b is bounded and (B.10). Note that by symmetry, we have

$$\mathcal{L}^\alpha p(s,t,\cdot,y)(x) = \frac{1}{2} \int_{\mathbb{R}^d} p(s,t,x+z,y) + p(s,t,x-z,y) - 2p(s,t,x,y)\, \frac{dz}{|z|^{d+\alpha}}.$$

We decompose it in the following way

$$\begin{aligned}
|\mathcal{L}^\alpha p(s,t,\cdot,y)(x)| &\le \int_{|z| \le (t-s)^{\frac{1}{\alpha}}} |p(s,t,x+z,y) + p(s,t,x-z,y) - 2p(s,t,x,y)|\, \frac{dz}{|z|^{d+\alpha}} \\
&\quad + \int_{|z|>(t-s)^{\frac{1}{\alpha}}} |p(s,t,x+z,y) + p(s,t,x-z,y) - 2p(s,t,x,y)|\, \frac{dz}{|z|^{d+\alpha}} \\
&= \int_{|z| \le (t-s)^{\frac{1}{\alpha}}} \left| \int_0^1 (\partial_x p(s,t,x+\lambda z,y) - \partial_x p(s,t,x-\lambda z,y)) \cdot z\, d\lambda \right| \frac{dz}{|z|^{d+\alpha}} \\
&\quad + \int_{|z|>(t-s)^{\frac{1}{\alpha}}} |p(s,t,x+z,y) + p(s,t,x-z,y) - 2p(s,t,x,y)|\, \frac{dz}{|z|^{d+\alpha}} \\
&=: I_1 + I_2.
\end{aligned}$$

We start with I_1. Using the Hölder control (B.12), we obtain that for some $\gamma \in (0, (2\alpha-2) \wedge (\eta + \alpha - 1))$ with $\gamma > \alpha - 1$, there exists a constant $C > 0$ such that

$$I_1 \le C(t-s)^{-\frac{1+\gamma}{\alpha}} \int_{|z| \le (t-s)^{\frac{1}{\alpha}}} \int_0^1 (\rho^1(t-s, y-x-\lambda z) + \rho^1(t-s, y-x+\lambda z))\, d\lambda |z|^{1+\gamma}\, \frac{dz}{|z|^{d+\alpha}}.$$

Since $|z| \le (t-s)^{\frac{1}{\alpha}}$, the space-time inequality (B.13) ensures that

$$\begin{aligned}
I_1 &\le C(t-s)^{-\frac{1+\gamma}{\alpha}} \int_{|z| \le (t-s)^{\frac{1}{\alpha}}} \rho^1(t-s, y-x) |z|^{1+\gamma}\, \frac{dz}{|z|^{d+\alpha}} \\
&\le C(t-s)^{-1} \rho^1(t-s, y-x).
\end{aligned} \tag{B.30}$$

For I_2, we have with (B.10)

$$I_2 \le C \int_{|z|>(t-s)^{\frac{1}{\alpha}}} \left[\rho^0(t-s, y-x-z) + \rho^0(t-s, y-x+z) + \rho^0(t-s, y-x) \right] \frac{dz}{|z|^{d+\alpha}}$$

We distinguish two cases. Firstly, assume that $|y-x| \leq 2(t-s)^{\frac{1}{\alpha}}$. Then, using that for all $s<t$ and $y \in \mathbb{R}^d$ $\rho^0(t-s,x) \leq C(t-s)^{-\frac{d}{\alpha}}$, we get that

$$\begin{aligned} I_2 &\leq C(t-s)^{-\frac{d}{\alpha}} \int_{|z|>(t-s)^{\frac{1}{\alpha}}} \frac{dz}{|z|^{d+\alpha}} \\ &\leq C(t-s)^{-1-\frac{d}{\alpha}} \\ &\leq C(t-s)^{-1}\rho^1(t-s,y-x), \end{aligned} \tag{B.31}$$

since $|y-x| \leq 2(t-s)^{\frac{1}{\alpha}}$. Finally, if $|y-x| > 2(t-s)^{\frac{1}{\alpha}}$, we write

$$\begin{aligned} I_2 \leq\ & C \int_{(t-s)^{\frac{1}{\alpha}}<|z|\leq\frac{|x-y|}{2}} \left[\rho^0(t-s,y-x-z)+\rho^0(t-s,y-x+z)+\rho^0(t-s,y-x)\right] \frac{dz}{|z|^{d+\alpha}} \\ &+ C \int_{|z|>\frac{|x-y|}{2}} \left[\rho^0(t-s,y-x-z)+\rho^0(t-s,y-x+z)+\rho^0(t-s,y-x)\right] \frac{dz}{|z|^{d+\alpha}} \\ =:\ & I_{2,1}+I_{2,2}. \end{aligned}$$

Then, the reverse triangle inequality yields

$$\begin{aligned} I_{2,1} &\leq C \int_{(t-s)^{\frac{1}{\alpha}}<|z|\leq\frac{|x-y|}{2}} \left[\rho^0(t-s,y-x)+\rho^0(t-s,\frac{y-x}{2})\right] \frac{dz}{|z|^{d+\alpha}} \\ &\leq C\rho^0(t-s,y-x) \int_{(t-s)^{\frac{1}{\alpha}}<|z|} \frac{dz}{|z|^{d+\alpha}} \\ &\leq C(t-s)^{-1}\rho^0(t-s,y-x). \end{aligned} \tag{B.32}$$

For $I_{2,2}$, we have

$$\begin{aligned} I_{2,2} &\leq C|y-x|^{-d-\alpha} \int_{\mathbb{R}^d} \rho^0(t-s,y-x-z)+\rho^0(t-s,y-x+z)\,dz \\ &\quad + C\rho^0(t-s,y-x) \int_{|z|>\frac{|x-y|}{2}} \frac{dz}{|z|^{d+\alpha}} \\ &\leq C(t-s)^{\frac{-d-\alpha}{\alpha}}((t-s)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha} + C\rho^0(t-s,y-x)|x-y|^{-\alpha} \\ &\leq C(t-s)^{\frac{-d-\alpha}{\alpha}}((t-s)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha} + C(t-s)^{-1}\rho^0(t-s,y-x). \end{aligned}$$

Since $1 < \frac{1}{2}(t-s)^{-\frac{1}{\alpha}}|y-x|$, we deduce that for some constant $C>0$, one has

$$((t-s)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha} \leq C(1+(t-s)^{-\frac{1}{\alpha}}|y-x|)^{-d-\alpha}.$$

It follows that

$$I_{2,2} \leq C(t-s)^{-1}\rho^0(t-s,y-x). \tag{B.33}$$

Combining (B.30), (B.31), (B.32) and (B.33), we have proved that (B.11) holds true.

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