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The Condorcet Loser Criterion in Committee Selection

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Abstract

In committee selection setting, we introduce the *Condorcet Loser Committee* (CLC) which when it exists, is a committee such that each of its members is defeated in pairwise comparisons by any outside candidate. It turns out that most popular committee selection rules can elect the CLC when it exists. From the perspective of the Condorcet majority criterion, the election of the CLC is simply not acceptable. We identify the few rules that will never elect the CLC when it exists. We show among others that the k -Borda rule is the only k -scoring committee rule that never select the CLC. This also holds with k -iterative Borda rule, Nanson rule, Kemeny rule, Copeland rule. As a corollary, it appeared that the Chamberlin-Courant rule can elect the CLC when it exists.

Keywords: Committee, Condorcet, Scoring, Borda, Stable rule

JEL Classification: D70 , D71

Statements and Declarations

- The author has no relevant financial or non-financial interests to disclose.
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1 Introduction

Committee selection and the study of the properties of the voting rules for electing committees are increasingly popular topics in social choice theory. In committee selection, the goal of preference aggregation is to elect a fixed number of winners. In the well-supplied family of works on committee selection, one of the approaches is to assume that an elected committee must meet the *Condorcet winner criterion*. In single winner election, the Condorcet winner criterion (Condorcet, 1785) recommends to elect, when he exists, the candidate who defeats each of his competitors in pairwise comparisons; such a candidate is called the *Condorcet winner*. The extension of the Condorcet winner criterion to committee selection led to two main concepts: the *Condorcet Committee à la Gehrlein* (Good, 1971; Gerhlein, 1985) and the *Condorcet Committee à la Fishburn* (Fishburn, 1981).¹ The Condorcet Committee à la Gehrlein (CCG) is a committee such that each of its members defeats any outside candidate in majority duels. The CCG as we have just defined it is still known in the literature as the “Strong Condorcet Committee”; the “Weak Condorcet Committee (à la Gehrlein)” is a committee such that none of its members is defeated in majority duels. When the CCG exists, it is unique, whereas there may be several weak Condorcet Committees. In Condorcet Committee à la Fishburn (CCF) is the committee that is preferred by a majority of individuals to any other committee. In recent work, Kamwa and Merlin (2018); Kaymak and Sanver (2003) have looked at the relationships between these two Condorcet committees. The CCG, like the CCF, does not always exist. Voting rules that always elect the CCG when it exists (or a committee that is close to it when it does not exist) have been proposed in the literature: these are called *stable rules*. We will describe these rules later.

As noted above, Condorcet committees are simply extensions of the Condorcet winner criterion to committee selection. In this paper, we attempt for the first time, an extension of the *Condorcet loser criterion* to the framework of committee selection. In single-winner elections, the Condorcet loser criterion would require that when the *Condorcet loser* exists, he should not be selected. The Condorcet loser is a candidate who is defeated in pairwise comparisons by each of the other candidates. While there is no consensus in the single winner literature that the Condorcet winner should be elected when he exists, there is nonetheless a near consensus that the Condorcet loser should not be elected. According to Lepelley (1993), “[...] the election of the Condorcet loser is disturbing enough to consider the Condorcet loser criterion as a minimal requirement that a voting system should meet: it seems reasonable to rule out the systems which can lead to the election of the Condorcet loser, unless one can demonstrate that the occurrences of this paradoxical result are extremely rare.”

¹Determining the CCF requires voter preferences on all committees. This is not easy in practice. It is for this reason that this concept is less discussed in the literature than the CCG, which only requires the preferences on the candidates.

According to the results by [Diss et al. \(2020\)](#), when electing committees, most of the popular committee selection rule may quite often elect the Condorcet loser as member of an elected committee. By extending the Condorcet loser criterion to the framework of committee selection, we introduce the concept of *Condorcet Loser Committee* (CLC) to describe a committee such that each of its members is defeated in pairwise comparisons by each of the external candidates. It is obvious that when it exists, the CLC is unique. As we will see later, there are situations in which the CLC exists and is elected under a number of popular voting rules; this could even occur in the presence of a CCG. With this in mind, the position we take here, by extension of the point made by [Lepelley \(1993\)](#), is that when the CLC exists, it would be objectionable for it to be elected. From this point on, we strive to identify the committee selection rules that never elect the CLC when it exists. To do so, we focus on four main families of committee selection rules found in the literature: k -scoring rules, k -iterative scoring rules, k -Condorcet consistent rules and stable rules. A k -scoring rule (resp. k -iterative scoring rule) select the k best candidates of the associated scoring rule (resp. iterative scoring rule); we define scoring rules later. A k -Condorcet consistent rule selects the k best candidates of the corresponding Condorcet consistent rule. A Condorcet consistent rule is a voting rule that always picks the Condorcet winner when he exists.

We show that the k -Borda rule is the k -scoring rule that never elects the CLC when it exists; the same is true of k -iterative Borda rule. We also show that the Kemeny rule, the Nanson rule, and the Copeland rule are Condorcet consistent rule never pick the CLC when it exists. Among the seven stable rule encountered in the literature, only two of them never select the CLC.

The rest of the paper is organized as follows: Section 2 is devoted to notation and definitions; it is in this section that we present each of the families of voting rules we are interested in. Section 3 presents our main results. Section 4 concludes.

2 Basic notation

2.1 Preferences

Let N be the set of n voters ($n \geq 2$) and A the set of m candidates ($m \geq 3$). We denote by R the set of binary relations on A , and by P the subset of complete, transitive, and antisymmetric binary relations on A . The preference profile $\pi = (P_1, P_2, \dots, P_i, \dots, P_n)$ gives all the linear orders of all the n voters on A , where P_i is the strict ranking of a given voter i . For $x, y, z \in A$ and given voter, we write ayz to say that this voter ranks x before y who is ranked before z . We denote by $P(A)^n$ the set of preference profiles. A *voting situation* $\tilde{n} = (n_1, n_2, \dots, n_t, \dots, n_{m!})$ indicates the number of voters for each linear order such that $\sum_{t=1}^{m!} n_t = n$.

We denote by n_{xy} the number of voters who rank x before y . If $n_{xy} > n_{yx}$, we say that x is majority preferred to y ; this is denoted by xMy . In case we get $n_{xy} = n_{yx}$, this means that x and y are tied; we denote this by xTy . In a

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majority graph, the relation xMy will be materialized by an arrow going from x to y ; no arrow between x and y will translate the relation xTy .

Given π and a candidate x , x is said to be the *Condorcet winner* if $xMy \forall y \in A \setminus \{x\}$; he is the *Condorcet loser* if $yMx \forall y \in A \setminus \{x\}$. We have a majority cycle when the majority relation M is not transitive. Thus, we say that $x, y, z \in A$ are in a majority cycle if we have xMy , yMz and zMx .

Assume that we want to elect a committee of size k ($2 \leq k \leq m - 1$). We denote by \mathcal{A}^k the set of all possible committees of size k .

Definition 1 (Condorcet Committee à la Gerhlein (CCG)) A committee $C \in \mathcal{A}^k$ is a CCG if and only if $\forall x \in C, xMy \forall y \in A \setminus C$.

Definition 2 (weak Condorcet Committee à la Gerhlein (wCCG)) A committee $C \in \mathcal{A}^k$ is a wCCG if and only if $\forall x \in C$ and $\forall y \in A \setminus C$ we do not have yMx .

When the CCG exists, it unique while there may exist more than one wCCG. In the remainder of the paper, when we refer simply to the Condorcet Committee (CC), this would mean in the context under consideration that there is no need to distinguish between the CCG and a wCCG. We denote by $\mathcal{C}_k(\pi)$ the set of Condorcet Committees.

Definition 3 (Condorcet Loser Committee (CLC)) A committee $C \in \mathcal{A}^k$ is a CLC if and only if $\forall x \in C$ and $\forall y \in A \setminus C$ we have yMx .

Given π and k the size of the committee to be elected, when the CLC exists, it is unique.

2.2 Voting rules

We will pay attention to four groups of ranking-based committee selection rules: k -scoring rules, k -Iterative scoring rules, k -Condorcet consistent rules and stable rules.

2.2.1 k -scoring rules

Given π and the $m!$ possible complete strict rankings, we denote by $r(t, a)$ ($r(t, a) = 1, 2, \dots, m$) the rank of candidate a in the ranking of voter of type t ($t = 1, 2, \dots, m!$), and by n_t the number of voters of type t ($n_t \geq 0$). A scoring rule gives $w_{r(t, a)}$ points to candidate a each time he is ranked at position $r(t, a)$; a scoring rule is thus defined by a scoring vector $w = (w_1, w_2, \dots, w_m)$ with $w_1 \geq w_2 \geq \dots \geq w_m$ and $w_1 > w_m$. Let us introduce some of the most popular scoring rules. The *Plurality rule* also called *Single non-transferable vote* is characterized by $w = (1, 0, \dots, 0)$; the *Borda rule* is defined by $w = (m - 1, m - 2, \dots, 1, 0)$; the *Antiplurality rule* is defined by $w = (1, 1, \dots, 1, 0)$;

the q -approval rule uses the vector $w = (\underbrace{1, \dots, 1}_q, 0, \dots, 0)$, the *Nairu rule* also called the *Harmonic rule* is defined by $w = (1, \frac{1}{2}, \dots, \frac{1}{m-1}, \frac{1}{m})$. Given π and w , the score of candidate a is defines as follows: $\sum_{m!}^{t=1} n_t w_{r(t,a)}$. In committee selection, a k -scoring rule picks the k best candidates of the associate scoring rule. Thus we define Plurality those k -Plurality rule, k -Borda rule, k -Antiplurality rule and so on. q -approval rule is also called *Bloc rule* when $q = k$.

In some situations, the use of a tie-break mechanism may be necessary; we do not need to address this issue here because we will not need to deal with tie scores. This will also be the case for the rules that we will present in the sequel.

2.2.2 k -Iterative scoring rules

Iterative scoring rules can also be used for committee selection and they involve multiple rounds of voting. Here, we consider those k -iterative rules based on One-by-one eliminations. There are two main ways of conceiving iterative k -scoring rules:

- *downward eliminations*: at each round, only the candidate with the lowest score of the considered scoring rule is eliminated until there are only k candidates left. The iterative Borda rule, also known as the Baldwin rule (Baldwin, 1926), belongs to this group.
- *upward eliminations*: at each round, one candidate, the one with the highest score, is declared elected. The process continues until k candidates are elected.

In the family of k -iterative scoring rules, we can also add the particular group formed by *Single Transferable vote* (STV) with its variants.² STV is a multi-round rule where in each round, any candidate whose Plurality score is at least equal to a certain threshold/quota is elected; if there are still seats to be filled, the candidate with the lowest score is eliminated and the process continues until all contested seats are filled. STV tends to promote representative committees. Note that in some situations, STV is equivalent to (iterative) k -Plurality.

2.2.3 k -Condorcet consistent rules

The consistent Condorcet rules have been evoked in the context of committee selection (Coelho, 2004). With these rules, the aim is to select the k best candidates. The Condorcet consistent rules we are interested in are the following:

²According to Tideman and Richardson (2000) "STV is not a specific voting method but rather a family of voting methods. [...] The different STV methods vary primarily in how much of which surplus votes are transferred and in the meanings that are attached to enough votes to be elected and the next available candidate."

- *Kemeny rule*: it operates by computing distances from a given linear order to all the linear orders of a preferences profile; the Kemeny social ranking is the linear order that minimizes this distance, the total distance to the whole profile.
- *Dodgson rule*: this rule elects the candidate who needs the fewest number of adjacency switches to become the Condorcet winner. Based on the ascending number of switches, the Dodgson social ranking is determined.
- *Young rule*: it elects the candidate who needs the fewest number of deletions of voters to become the Condorcet winner. Based on the ascending number of switches, the Young ranking is determined.
- *Maximin/Minimax rule*: This rule determines the support received by each candidate in every pairwise comparison; the candidate with the greatest minimum support received is the winner or appears at top of the collective ranking.
- *Copeland rule*: The Copeland score of a candidate $x \in A$ is defined by $\text{Cop}(x, \pi) = \sum_{y \neq x} \gamma_{x,y}(\pi)$; where for $y \in A \setminus \{x\}$, $\gamma_{x,y}(\pi) = 1$ if xMy , $\gamma_{x,y}(\pi) = \frac{1}{2}$ if xTy and $\gamma_{x,y}(\pi) = 0$ if yMx . For a committee of size k , the Copeland rule will pick the k highest Copeland scores.
- *Nanson rule* (Nanson, 1883): this rule uses the same scoring vector as the Borda rule and at each step, any candidate who obtains (strictly) less than the average of the scores is eliminated. Note that in the context of committee selection, Nanson's rule can pose some problems: there may be situations in which, in the first round, more than $m - k$ candidates obtain scores below the average. To guarantee the election of a committee, it is necessary to describe a procedure to deal with such situations.

To the above list, we can add the iterative Borda rule already defined in Section 2.2.2.

2.2.4 The stable rules

Here we formally define the stable rule encountered in the literature. Given π and a committee $C \in \mathcal{A}^k$:

- The *Kemeny-Ratliff rule* (Ratliff, 2003). Kemeny-Ratliff rule (KR) selects the committee(s) with the smallest total margin of loss in pairwise comparisons versus the outside candidates. KR picks the committee $C \in \mathcal{A}^k$ such that $KR(\pi, C) \leq KR(\pi, C') \quad \forall C' \in \mathcal{A}^g \setminus \{C\}$ with $KR(\pi, C) = \sum_{x \in C, y \in A \setminus C} \max\{0, n_{yx} - n_{xy}\}$. $KR(\pi, C)$ is the total margin of loss in pairwise comparisons by the candidate in C versus the other $m - k$ candidates.
- The *Dodgson-Ratliff rule* (Ratliff, 2003). Dodgson-Ratliff rule (DR) picks the committee(s) with the fewest number of adjacency switches needed to make it becomes a CCG. DR selects the committee $C \in \mathcal{A}^k$ such that $DR(\pi, C) \leq DR(\pi, C') \quad \forall C' \in \mathcal{A}^k \setminus \{C\}$ with $DR(\pi, C)$ the number of adjacency switches needed to make C subset become a Condorcet committee.

- The *Minimal Size of External Opposition rule* (Coelho, 2004). Minimal Size of External Opposition rule (SEO) selects the committee(s) with the smallest margin of loss of its members versus the outside candidates. Let $\text{SEO}(\pi, C) = \max_{x \in C, y \in A \setminus C} n_{yx}$ defines largest margin of loss of the members of $C \in \mathcal{A}^k$ versus the non members. Committee C is selected under SEO if $\text{SEO}(\pi, C) \leq \text{SEO}(\pi, C') \quad \forall C' \in \mathcal{A}^k \setminus \{C\}$.
- The *Minimal Number of External Defeats Rule* (Coelho, 2004). Minimal Number of External Defeats Rule (NED) chooses the committee(s) for which the number of pairwise comparisons lost by its members is minimal. NED selects the committee $C \in \mathcal{A}^k$ such that $\text{NED}(\pi, C) \leq \text{NED}(\pi, C') \quad \forall C' \in \mathcal{A}^k \setminus \{C\}$ with $\text{NED}(\pi, C) = \sum_{x \in C} |\{y \in A \setminus C : n_{xy} < n_{yx}\}|$ the number of pairwise comparisons lost by the members of C .
- The *Young-Condorcet rule* (Kamwa, 2017). Young-Condorcet rule (YC) selects the committee(s) that needs the fewest number of deletions of voters to become a Condorcet committee. Let us denote by π^S be the profile obtained after the deletion of a subset S of voters ($S \subseteq N$). YC picks the committee $C \in \mathcal{A}^k$ such that $YC(\pi, C) \leq YC(\pi, C') \quad \forall C' \in \mathcal{A}^k \setminus \{C\}$ with $YC(\pi, C) = \min_{S \in \Delta(\pi, C)} |S|$ and $\Delta(\pi, C) = \{S \subseteq N : C \in \mathcal{C}_k(\pi^S)\}$.
- The *Minimal number of Deletion of Candidates* (Kamwa, 2017). (MDC) picks the committees that need the fewest number of deletions of candidates to become a Condorcet committee. Let π_B be a voting profile obtained after the deletion of a subset B of A . MDC selects the committee $C \in \mathcal{A}^k$ such that $\min_{B \in \Omega(\pi, C)} |B| \leq \min_{B \in \Omega(\pi, C')} |B| \quad \forall C' \in \mathcal{A}^k \setminus \{C\}$ with $\Omega(\pi, C) = \{B \subseteq A : C \in \mathcal{C}_k(\pi_B)\}$.
- The *Copeland*⁰ (Aziz et al, 2017): This rule differs from the classical Copeland rule only by the fact that when the majority duel between two candidates results in a tie, they both receive 0 point. This rule picks the candidates with the k highest Copeland⁰ scores.

3 Results

Some may wonder if the existence of the CLC is a frequent or rare occurrence. So, before stating our result, Let us take a look at this issue. Note that for a given profile, if there is a CLC of size k this implies that there is a CCG of size $m - k$. This remark is very important if one were to wonder whether the existence of the CLC is a frequent fact or not: the probability of existence of the CLC of size k is equal to the probability of existence of the CCG of size $m - k$. Thus, from the existence probabilities of the CCG we can easily deduce those of the CLC. That what we do from the probabilities of existence of CCG provided by Diss and Doghmi (2016). So, for electorate tending to infinity and a number of candidates between 3 and 6, we record the limiting existence probabilities of the CLC in Table 1 for the *impartial anonymous culture* hypothesis (IAC). IAC is one of the hypothesis often used when computing the likelihood of

voting events. Under IAC, it is assumed that all voting situations are equally likely to be observed. For more on this hypothesis, we refer to the book by [Gehrlein and Lepelley \(2011\)](#).

Table 1 Limiting probabilities of existence of the CLC under IAC

	m			
k	3	4	5	6
1	0.937	0.838	0.747	0.677
2	0.937	0.759	0.612	0.483
3	—	0.838	0.602	0.426
4	—	—	0.753	0.494
5	—	—	—	0.683

Recall that our point of view is to say that electing the CLC when it exists is an objectionable situation for committee selection. So, our aim is to search out the committee selection rules that never lead to the CLC when it exists. Theorem 1 to 4 present our results for each of the four groups of voting rules/methods on which we focused.

Theorem 1 *Consider a voting situation with $m \geq 3$ candidates and where one want to elect a committee of size $1 < k < m$; k -Borda rule is the only k -scoring rule that never elects the CLC as the winning committee.*

Proof First of all, let us show that k -Borda rule never elects the CLC as the unique winning committee. Given A , let us consider a voting situation where a committee $C \in \mathcal{A}^k$ is the CLC and is selected by k -Borda; consider the set $H = A \setminus C$ with $|H| = m - k$. It comes that the Borda winner, let us say a , belongs to C and the Borda bottom-ranked candidate belong to H . We know from [Young and Levenglick \(1978\)](#) that if $n_{ab} > n_{ba}$, then b cannot be ranked just above a in a Kemeny social order; since $\forall x \in C$ and $\forall y \in H$, $n_{yx} > n_{xy}$, it follows that the Kemeny winner belongs to H while the Kemeny bottom-ranked candidate belongs to C . As C is assumed to be the Borda winning committee, this means that the Kemeny rule winner is not among the k highest Borda scores: this lead to a contradiction since we know from [Saari and Merlin \(2000\)](#) that for $m \geq 3$ candidates, the Borda rule always ranks the Kemeny rule top-ranked candidate strictly above the Kemeny bottom-ranked candidate and that, the Kemeny rule ranks the Borda rule top-ranked candidate strictly above the Borda rule bottom-ranked candidate. So, k -Borda rule never elects the CLC as the unique winning committee.

A simple way to show that all other scoring rules could elect the CLC would be to assume an election with $m = 3$ and $k = 2$; in this election if there is a CLC, it implies that there is a Condorcet winner. For the CLC to be elected for a given scoring rule in this case, the Condorcet winner would have to be ranked last by this rule. According to [Fishburn and Gehrlein \(1976\)](#), if a scoring rule is not the Borda rule, there is a voting profile where the Condorcet winner scores the worst. Such a situation is known in the literature as the strict Borda paradox. It is established that apart from the Borda rule, all other scoring rules are vulnerable to the strict Borda

paradox; see for instance [Gehrlein and Lepelley \(2011\)](#). Thus, such a voting situation with three candidates is therefore sufficient to say that all other scoring rules can elect the CLC when it exists. \square

Another selection rule much studied in the literature and which we do not dwell on is the *Chamberlin-Courant rule*³. This rule is equivalent to the Borda rule for $k = 1$; it is also equivalent to k -Plurality rule for $k = m - 1$ ([Kamwa, 2014](#)). As we know from Theorem 1 that k -Plurality may pick the CLC when it exists, this may also be the case for the Chamberlin-Courant rule when $k = m - 1$. Thus, we derive Corollary 1.

Corollary 1 *Chamberlin-Courant rule can elect the CLC when it exists.*

Theorem 2 tells us what we get with k -iterative scoring rules.

Theorem 2 *Consider a voting situation with $m \geq 3$ candidates where we want to elect a committee of size k using a k -iterative scoring rule with one-by-one (downward or upward) eliminations. Assume that a CLC of size k ($1 < k \leq m - 1$) exists. Except the k -iterative Borda rule, all the iterative scoring rules may select the CLC as the unique winning committee.*

Proof Given a voting situation with $m \geq 3$ candidates, assume that the CLC of size k denoted $C \in \mathcal{A}^k$ exists and it is elected by a k -iterative scoring rule.

Let us first consider the k -iterative scoring rules with downward eliminations. As C is elected, this means that at the last round, the set of candidates is $C \cup \{x\}$ with $x \in A \setminus C$. As C is the CLC, candidate x is by definition, the Condorcet winner on $C \cup \{x\}$. In such a situation, for C to be elected, this needs x to scores the worst on $C \cup \{x\}$. It is well known that on $C \cup \{x\}$, x the Condorcet winner can never obtains the lowest score only under the Borda rule (see [Fishburn, 1974](#); [Smith, 1973](#); [Daunou, 1803](#)) while this can occur under the other scoring rules ([Fishburn and Gehrlein, 1976](#)). So, k -iterative Borda is the only k -iterative scoring rule with downward eliminations that never picks the CLC when it exists.

Let us now consider the k -iterative scoring rules with upward eliminations. Consider $y \in C$ the candidate elected at round k th; it follows that at this round, y is the Condorcet loser on $\{y\} \cup A \setminus C$ and gets the highest score. It is well known that it is only under the Borda rule that the Condorcet loser never scores the best while this may occur under the other scoring rules. Thus, k -iterative Borda is the only k -iterative scoring rule with upward eliminations that never picks the CLC when it exists. \square

³Recall that *Chamberlin-Courant rule* ([Chamberlin and Courant, 1983](#)) is a voting rule that combines the Borda rule and proportional representation. Given a profile π , CCR selects the committee C which maximizes the representativeness value $\alpha(C, \pi) = \sum_{x \in C} \sum_{i \in N_x(C, \pi)} w(r_{ix})$ where r_{ix} is the rank of candidate x in voter i 's ranking, $w(r_{ix}) = m - r_{ix}$ and $N_x(C, \pi)$ the set of voters for which the representative in committee C is candidate x for profile π . This is the utilitarian version of the CCR; to more on the other versions/approximations of the CCR, the reader may refer to [Elkind et al. \(2017\)](#).

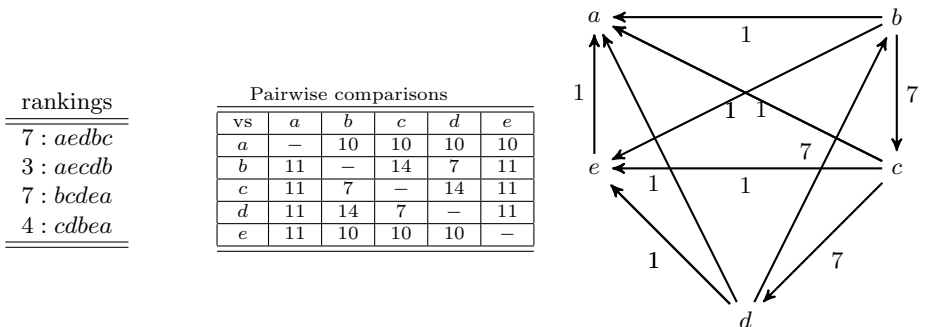
Since STV can be similar to (iterative) Plurality in some situations, Corollary 2 follows from Theorems 1 et 2

Corollary 2 *STV can elect the CLC when it exists.*

Theorem 3 tells us what we get with the Condorcet consistent rule on which we focus.

Theorem 3 *Consider a voting situation with $m > 4$ candidates such that there is no Condorcet winner but there is a CLC of size $1 < k < m - 1$. Dodgson rule, Maximin rule and Young rule may lead to the CLC. This never occurs with the Kemeny rule, Nanson rule and the Copeland rule.*

Proof The cases $m = 3$ or $m = 4$ with $k = m - 1$ are automatically excluded because the existence of the CLC implies that of a Condorcet winner who would be elected in all cases. Similarly, in the case $m = 4$ with $k = 2$ the existence of the CLC implies that of a Condorcet Committee of which at least one of the members would in all cases be elected by our Condorcet consistent rules. To show that Dodgson rule, Young rule and Maximin rule may lead to the CLC when it exists, let us consider the following voting profile with 21 voters and 5 candidates.⁴



In this profile candidate a is the Condorcet loser. Assume that we want a committee of size $k = 2$; in our profile there is no (Weak) Condorcet Committee while (a, e) is the CLC. After computations, it comes for the Dodgson ranking that e is ranked at the top while all the other candidates tie for the second place; thus for $k = 2$, committee (a, e) the CLC belongs to the set of elected committee for the Dodgson rule. Candidates a and e share the first place in the collective ranking of the Young and Maximin rules while the others share the second place; thus, (a, e) the CLC is elected by the Young rule and the Maximin rule. So, it comes from the above profile that Dodgson rule, Young rule and Maximin rule may pick the Condorcet loser as a member of an elected committee.

Regarding the Kemeny rule, we can use an argument that we have used earlier. Consider a voting situation where a committee $C \in \mathcal{A}^k$ is the CLC and let us consider the set $H = A \setminus C$ such that $|H| = m - k$. We know from Young and Levenglick (1978)

⁴This profile is drawn from Kamwa (2017).

that if $n_{ab} > n_{ba}$, then b cannot be ranked just above a in a Kemeny social order. So, since for every $x \in C$ and $y \in H$, $n_{yx} > n_{xy}$, it follows that every candidate in H are ranked ahead of those in C in the Kemeny social order; then, the Kemeny winner thus belongs to H while C contains the Kemeny k bottom-ranked candidate: thus, Kemeny rule cannot select the CLC. It follows that if there is no CLC while a Condorcet loser exists, this candidate cannot be a member of an elected committee.

Let us consider the Nanson rule. If there is a Condorcet winner, the proof is direct since we know that this candidate will never have a Borda score below the average. Let us assume that there is no Condorcet winner. Given C and $H = A \setminus C$, it follows from Theorems 1 and 2 that at each round, there is always at least one candidate $x \in H$ who score better than at least one candidate $y \in C$; so, C cannot be elected. Thus Nanson rule never elects the CLC.

Let us show that the Copeland rule never elects the CLC when it exists. Consider the best candidate in the CLC; this candidate will get, in the best case, a Copeland score equal to $k - 1$. Let us take a candidate outside the CLC and suppose that he loses all his pairwise comparisons against his fellows; he will then have a Copeland score equal to k . Thus the best candidate in the CLC has a lower score than the worst external candidate. Thus, the CLC cannot be elected with the Copeland rule. It is also obvious that whenever the CLC does not exist, a Condorcet loser cannot be a member of an elected committee under the Copeland rule. \square

We know from Kamwa (2017) that for voting profiles where there is a Condorcet winner (resp. a Condorcet loser) but no (Weak) Condorcet Committee of size k , KR, DR, YR, and SEO may pick a committee that does not contain (resp. that contains) the Condorcet winner (resp. the Condorcet loser) while this is never the case with MDC and NED. Theorem 4 tells us that among the Condorcet Consistent rules under consideration, only NED and Copeland⁰ never lead to the CLC.

Theorem 4 *Consider a voting situation $m \geq 4$ where for a given k , there is no (Weak) Condorcet Committee while there is a CLC. Among the identified stable rules, NED and Copeland⁰ never elect the CLC while this can be the case for KR, DR, YR, SEO and MDC. More, the CLC cannot be the unique element in the outcome set of MDC.*

Proof The case $m = 3$ is de facto eliminated because if there is no (Weak) Condorcet Committee for a given $k = 1, 2$, it means that we are in the presence of a majority cycle and therefore of no CLC. Kamwa (2017) showed that when there is no (Weak) Condorcet Committee, KR, DR, YC and SEO may pick the Condorcet loser when he exists and that these rule may fail to picks the Condorcet winner while this is not the case with MDC and NED.

Let us prove that NED and Copeland⁰ never select the CLC when it exists. Consider a profile π with $m \geq 4$ such that for a given k ($2 \leq k \leq m - 1$), $X \in \mathcal{A}^k$ is the CLC and $\mathcal{C}_k = \emptyset$.

Notice that $\forall a \in X$, Cop⁰(a) the Copeland⁰ score is as follows: $0 \leq \text{Cop}^0(a) \leq k - 1$. Let us consider b the worst candidate in $A \setminus X$; we get Cop⁰(b) = k then

$\text{Cop}^0(b) > \text{Cop}^0(a)$. So, the worst candidate outside the CLC always scores more than every candidates in this set. Thus, Copeland^0 never selects the CLC when it exists.

Under NED, we get $NED(\pi, X) = k(m - k)$. Assume that $k < m - k$ and let us take a committee Y such that $|Y| = k$ and $X \cap Y = \emptyset$. Since $\mathcal{C}_k = \emptyset$, the NED score of Y is as follows $1 \leq NED(\pi, Y) \leq k(m - 2k)$; as $k(m - 2k) < k(m - k)$, it follows that $NED(\pi, Y) < NED(\pi, X)$. So, NED cannot pick the CLC. Let now assume that $k > m - k$ and consider a committee Z such that $|Z| = k$ and $X \cap Z \neq \emptyset$ with $|X \cap Z| = 2k - m$. Since $\mathcal{C}_k = \emptyset$, the NED score of Z here is as follows $1 \leq NED(\pi, Z) \leq (m - k)(2k - m)$. Since $k(m - k) > (m - k)(2k - m)$, it then follows that $NED(\pi, X) > NED(\pi, Z)$. Thus, NED cannot pick the CLC.

To show that KR, DR, YR and SEO can select the CLC, let us consider the following profile drawn from Ratliff (2003, p.436) with four candidates and 33 voters.⁵

preferences				Pairwise comparisons				
6 : $abcd$	10 : $cdba$			vs	a	b	c	d
5 : $bcda$	1 : $bacd$			a	—	17	17	17
10 : $adbc$	1 : $dcab$			b	16	—	22	12
				c	16	11	—	22
				d	16	21	11	—

Assume that we want to select a committee of size $k = 3$. It is easy to check that for $k = 3$, there is no (Weak) Condorcet Committee while $\{b, c, d\}$ is the CLC. After all computations, we report the scores for $k = 3$ as it follows:

committees	Scores				
	KR	DR	SEO	YC	MDC
$\{a, b, c\}$	9	5	21	9	1
$\{a, b, d\}$	11	6	22	11	1
$\{a, c, d\}$	11	6	22	11	1
$\{b, c, d\}$	3	3	17	1	1

It comes that KR, DR, SEO and YC selects $\{b, c, d\}$; MDC selects $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, and $\{b, c, d\}$. Thus the CLC in our profile belongs to the outcome set of each of the rules KR, DR, YC and MDC. So they may pick the CLC when it exists. We know from Kamwa (2017) that MDC never selects a committee containing the Condorcet loser picks; so, if the CLC contains a CL, this committee cannot be selected by MDC. Thus, this is the only case where MDC never picks the CLC. \square

4 Concluding comments

In the context of committee selection, we have introduced the notion of Condorcet loser Committee (CLC), a committee such that each of its members is beaten in pairwise comparisons by each of the external candidates. We have highlighted the fact that the most popular committee selection rules can select the selection CLC. Such a situation is proved to be questionable. Focusing on four groups of popular committee selection rules, we showed that:

⁵This profile was used by Kamwa (2017) to show that KR, DR, YR and SEO can pick the Condorcet loser among the member of the winning committee.

- The k -Borda rule (resp. the k -iterative Borda rule) is the only k -scoring rule (resp. k -iterative scoring rule) that never picks the CLC when it exists.
- Among the k -iterative scoring rules (with upward or downward eliminations), only k -iterative Borda rule never picks the CLC when it exists.
- Dodgson rule, Maximin rule and Young rule are k -Condorcet consistent rule that may select the CLC while this is never the case for the Kemeny rule, the Nanson rule and the Copeland rule.
- Among the seven stable rules encountered in the literature, only NED and Copeland⁰ never elect the CLC.

Our results allow us to discriminate between the committee selection rules: those which can pick the CLC from those from those that do not. This study could be pushed forward by considering whether the fact that some rules elect the CLC when it exists is a rare fact or not. Table 2 goes a bit in this direction; it gives us for elections with 4 candidates, the limiting probabilities under IAC that the CLC of size k exists and is elected for some committee selection rules: k -Plurality (k -PR), k -Antiplurality (k -APR), Bloc, k -Nairu and Chamberlin-Courant rule (CCR).

Table 2 Limiting probabilities under IAC that some committee selection rules pick the CLC when it exists

k	Voting rules				
	k -Plurality	k -APR	Bloc	k -Nairu	CCR
1	0.0227	0.0238	0.0227	0.0033	0
2	0.0394	0.0094	0.0015	0.0011	0.0032
3	0.0056	0.0227	0.0227	0.0029	0.0056

The message that can be drawn from Table 2 varies from one rule to another. We see that with CCR, the probabilities tend to increase with k the size of the committee. With k -Plurality, we note that when we go from $k = 1$ to $k = 2$, probability increases then decreases when we go from $k = 2$ to $k = 3$. We have an inverse pattern with k -Antiplurality, Bloc and k -Nairu. Results with more candidates and more voting rules would help us to draw more accurate conclusions. Nonetheless, such a work is not an easy task. One solution would be to rely on Monte-Carlo simulations or to investigate what happens with real-world data. Our Theorems 1 and 2 have allowed us to derive conclusions about committee selection rules that are not k -scoring rules such as Bloc and Chamberlin-Courant rule. It would also be interesting to look at the *committee scoring rules* introduced by Elkind et al. (2017) as a broad class of rules encompassing almost all of the most common committee selection rules. By doing so, this would give a generalization of our Theorem 1.

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