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Global entropy stability for a class of unlimited second-order schemes for 2D hyperbolic systems of conservation laws on unstructured meshes

Ludovic Martaud*

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Abstract

We introduce a class of second-order accurate finite volume schemes defined on two dimensional unstructured meshes. They satisfy a global discrete entropy stability for an arbitrary strictly convex entropy. These schemes do not require usual limitation techniques. Numerical experiments are performed to illustrate the accuracy and the shock capturing efficiency of the schemes.

Keywords. Systems of conservation laws, Second-order finite Volume schemes, Explicit schemes, Global entropy inequality, entropy stability, 2D Unstructured meshes

Math. classification

1 Introduction

This work is devoted to the numerical approximation of the weak solutions of \(d \geq 1\) conservation laws in two space dimensions given by

\[
\partial_t w + \partial_{x_1} f_1(w) + \partial_{x_2} f_2(w) = 0, \quad (x_1, x_2) \in \mathbb{R}^2, \quad t > 0,
\]

where \(x = (x_1, x_2)^T\) and \(t\) are respectively the space and time variables. The unknown state vector \(w(x, t)\) is defined on \(\Omega\) a non-empty convex open subset of \(\mathbb{R}^d\), \(f_1, f_2 : \Omega \to \mathbb{R}^d\) are two given smooth flux functions. In order to ensure the hyperbolicity of the above system, it is assumed that the \(d \times d\) jacobian matrices \(\nabla f_1(w), \nabla f_2(w)\) are such that the matrix

\[
\nabla f_1(w)n_1 + \nabla f_2(w)n_2,
\]

is diagonalizable for all unit vectors \(n = (n_1, n_2)\) in \(\mathbb{R}^2\) and for all \(w\) in \(\Omega\). We consider \(w_0 : \mathbb{R}^2 \to \Omega\) a given measurable function of \(L^1_{\text{loc}}(\mathbb{R}^2)\) prescribed as an initial value of \(w\) at time \(t = 0\). We recast the system of conservation laws into the following form

\[
\begin{cases}
\partial_t w + \nabla_x \cdot F(w) = 0, & x \in \mathbb{R}^2, \quad t > 0, \\
w(x, t = 0) = w_0(x), & x \in \mathbb{R}^2,
\end{cases}
\]

where \(\nabla_x = (\partial_{x_1}, \partial_{x_2})^T\) is the standard gradient operator and \(F = (f_1, f_2)^T : \Omega \to \mathbb{R}^{2d}\).

As the solutions of (1) may develop discontinuities in a finite time \([50, 51, 66]\) (see also \([72, 37, 36, 32]\)), the weak solutions are in general not unique. To select among weak solutions, we generally require that \(w\) satisfies additionally

\[
\partial_t \eta(w) + \nabla_x \cdot G(w) \leq 0, \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^2 \times (0, +\infty)),
\]

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for all couple entropy-entropy flux \((\eta, G)\), where \(\eta : \Omega \rightarrow \mathbb{R}\) is a convex function and \(G = (g_1, g_2)^T \in C^1(\Omega, \mathbb{R}^2)\) is such that
\[
\nabla \eta(w)^T \nabla f_\mu(w) = \nabla g_\mu(w)^T, \quad \forall (w, \mu) \in \Omega \times \{1, 2\}.
\]

In the case of scalar conservation laws \((i, e d = 1)\) some uniqueness results have been exhibited with the entropy inequalities \([30]\). But, for a general system in two space dimensions, such inequalities are not sufficient to avoid non admissible discontinuities \([9]\). Nevertheless, form a numerical point of view, the entropy inequalities are essential to ensure the computational stability. In particular, a formal integration of (2) over \(\mathbb{R}^2\) gives
\[
\int_{\mathbb{R}^2} \partial_t \eta(w(x, t)) dx \leq 0, \quad t > 0.
\] (3)

Several high-order numerical schemes able to mimic a discrete version of (3) on Cartesian meshes have already been introduced \([32, 16, 31, 48, 12, 46, 65, 55, 34]\). As a matter of fact, high-order schemes on unstructured meshes are sometimes proposed with incomplete stability properties \([23, 5, 60, 46]\) or without stability proof \([35, 30, 45, 56, 54, 3]\). This obstacle has been mainly successfully overcame with the introduction of a dual mesh \([64]\) or within the DG SBP(-SAT) \([1, 18, 31, 19]\) framework. With the DG SBP(-SAT) methods, many high-order schemes working on arbitrary meshes and satisfying a semi-discrete version of (3) have been introduced \([10, 79, 18, 14, 65, 31, 76]\). But, the design of high-order schemes verifying a fully discrete version of (3) is challenging and often require implicit time discretizations \([62, 46]\). From a general point of view, the arbitrary geometry of the mesh is in general an obstacle to conjugate high resolutions and the entropy inequalities \([73, 68, 12, 56]\). As a matter of fact, high-order schemes on unstructured meshes are sometimes proposed with incomplete stability properties \([25, 5, 60, 6, 17]\) or without stability proof \([35, 30, 45, 56, 54, 3]\). This obstacle has been mainly successfully overcame with the introduction of a dual mesh \([64]\) or within the DG SBP(-SAT) \([1, 18, 31, 19]\) framework. With the DG SBP(-SAT) methods, many high-order schemes working on arbitrary meshes and satisfying a semi-discrete version of (3) have been introduced \([10, 79, 18, 14, 65, 31, 76]\). But, the design of high-order schemes verifying a fully discrete version of (3) is challenging and often require implicit time discretizations \([62, 46]\).

In this work, we extend the schemes introduced in \([2]\) to bi-dimensional cases. We propose to study a class of explicit unlimited \((i, e d)\) do not require additional stabilization techniques as one used in MUSCL strategy for instance \([75]\) second-order numerical schemes defined on unstructured meshes and that verify a fully discrete version of (3). In this regard, we approximate the weak solutions of (1) within the standard cell-centered finite volume framework. At first, we consider a set of discrete times \((t^n)_{n \in \mathbb{N}}\) where all of them are separated with a constant step \(\Delta t > 0\). Secondly, we consider a spatial mesh \(\mathcal{T}\) composed of cells \((C_i)_{i \in \mathcal{T}}\) where the surface of each of them is denoted \(|C_i| > 0\). We approximate \(w(\cdot, t^n)\) with piecewise constant functions denoted \(w_{\Delta}(\cdot, t^n)\) and defined on each cells \(C_i\) as follows:
\[
w_{\Delta}(x, t^n) = w^n_i, \quad \text{if} \quad x \in C_i, \quad \text{with} \quad w^n_i = \frac{1}{|C_i|} \int_{C_i} w(x, t^n) dx, \quad \forall i \in \mathcal{T}.
\] (4)

Starting with the sequence \((w^n_i)_{i \in \mathcal{T}}\), the updated sequence \((w^{n+1}_i)_{i \in \mathcal{T}}\) will be computed according to the following explicit cell-centered finite volume scheme
\[
w^{n+1}_i = w^n_i - \frac{\Delta t}{|C_i|} \sum_{j \in \mathcal{N}(i)} |\Gamma_{ij}| \mathcal{F}_{ij}, \quad \forall i \in \mathcal{T},
\] (5)

where \(\mathcal{N}(i)\) denotes a neighborhood of the cell \(i\), \(|\Gamma_{ij}|\) is the length of the edge between the cells \(i\) and \(j\) and \(\mathcal{F}_{ij} \in \mathbb{R}^d\) is a conservative numerical flux function. The rest of the paper is organized as follows. In the section \([2]\) we introduce a methodology to derive explicit second-order schemes \([5]\). In addition, we provide a finite volume approximation of the gradient that does not need standard limitation techniques.

The section \([3]\) is devoted to the proof of the global discrete entropy stability satisfied by our formal second-order schemes. Numerical results are presented in the section \([4]\) to illustrate the accuracy and the stability of the proposed finite volumes schemes.

## 2 Second-order entropy stable finite volume schemes

### 2.1 Second-order unlimited numerical flux

Let consider a mesh \(\mathcal{T}\) of \(\mathbb{R}^2\) composed of cells \((C_i)_{i \in \mathcal{T}}\) for which the mass center is \(x_i\), the surface is \(|C_i| > 0\) and the perimeter is \(|P_i| > 0\). The edge between the cells \(i\) and \(j\) is denoted \(ij\) (see Figure \([1]\)
and $|\Gamma_{ij}|$, $\pi_{ij}$ denote its length and its middle respectively. The normal vector $n_{ij} = (n_{ij1}, n_{ij2})^T \in \mathbb{R}^2$ of the edge $ij$ is oriented from $i$ to $j$. The set of edges is $E$ and we define

$$x_{ij} = ij \cap [x_i, x_j], \quad d_{ij} = \|x_i - x_j\|_2, \quad d_{j,ij} = \|x_{ij} - x_j\|_2, \quad d_{i,ij} = \|x_{ij} - x_i\|_2, \quad \forall ij \in E,$$  

(6)

where $\|\cdot\|_2$ is the standard euclidean norm of $\mathbb{R}^2$.

![Diagram of interface between two cells](image)

**Figure 1:** Interface $ij$ between two cells $i$ and $j$ on a mesh $T$.

We design a second-order of space accuracy scheme according to the following characterization (for more details see for instance [8], Proposition 2.26).

**Proposition 2.1** (Second-order accuracy scheme [8]). Let an integer $l \geq 2$ and $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a continuous function. Consider a numerical scheme under the form

$$w_n^{i+1} = w_n^i - \frac{\Delta t}{|C_i|} \sum_{j \in N(i)} |\Gamma_{ij}| F_{ij}(..., u^n_i, w_j^n, ...), \quad \forall i \in T.$$  

For a given edge $ij$, a numerical flux $F_{ij}$ is a second-order of space accuracy if

$$|\Gamma_{ij}| F_{ij}(..., u_i, u_j, ...) = h \rightarrow 0 n_{ij}^T \int_{ij} F(u(x(\xi))) d\xi + O(h^3),$$

$$= h \rightarrow 0 \sum_{\mu=1}^2 \int_{ij} f_\mu(u(x(\xi))) n_{ij\mu} d\xi + O(h^3),$$

(7)

where $u$ is a given smooth function, $u_i = \frac{1}{|C_i|} \int_{C_i} u(x) dx$ and $h$ is a characteristic length of the mesh cells.

In order to derive unlimited second-order numerical schemes at the interface $ij$, we consider numerical fluxes under the form

$$|\Gamma_{ij}| F_{ij} = n_{ij}^T \int_{ij} F(u(x(\xi))) d\xi - |\Gamma_{ij}| \frac{\lambda}{2} (u_j - u_i) + |\Gamma_{ij}| d_{i,ij} A_{ij}, \quad \forall ij \in E,$$  

(8)

where $\int_{ij} F(u(x(\xi))) d\xi$ is a finite volume discretization of the integral flux $\int_{ij} F(u(x(\xi))) d\xi$. The term $A_{ij} \in \mathbb{R}^d$ verifies $A_{ij} = -A_{ji}$ and $\lambda > 0$ represents the numerical viscosity coefficient. The main idea of this work is to select $A_{ij}$ to compensate at a continuous level the first-order stabilization term $-\frac{\lambda}{2} (u_j - u_i)$. After that, in order to keep the entropy stability proprieties, $A_{ij}$ will be discretized on a different stencil that one used for $-\frac{\lambda}{2} (u_j - u_i)$. As a consequence, the term $-\frac{\lambda}{2} (u_j - u_i) + d_{i,ij} A_{ij}$ will not be null and will drive the numerical diffusion of the scheme. First of all, we proceed to the integral flux discretization with enough accuracy.
\textbf{Lemma 2.1} (Integral flux discretization). Consider an edge $ij$ of an arbitrary mesh $\mathcal{T}$ and $u$ a smooth function for which we set $u_i = \frac{1}{|\mathcal{T}_i|} \int_{\mathcal{C}_i} u(x) dx$. Consider

$$
\int_{ij} F(u(x(\xi))) d\xi = |\Gamma_{ij}| \left( \frac{d_{ij}}{d_{i,j}} (F(u_j) + (\pi_{ij} - x_{ij})^T \nabla_x F_j) + \frac{d_{ij}}{d_{i,j}} (F(u_i) + (\pi_{ij} - x_{ij})^T \nabla_x F_i) \right),
$$

where the distances $d_{i,j}$ are defined in \eqref{eq:distance} and $(\nabla_x F_\alpha)_{\alpha \in \{i,j\}}$ is a finite volume approximation of the gradient of $F$ at the cell $\alpha$ that writes

$$
(\pi_{ij} - x_{ij})^T \nabla_x F_\alpha = \sum_{\mu=1}^{2} (\pi_{ij} - x_{ij})_\mu \partial_{x_\mu} F_\alpha,
$$

$$
= \sum_{\mu=1}^{2} (\pi_{ij} - x_{ij})_\mu \left( \partial_{x_\mu} f_1(u_\alpha) \partial_x u_\alpha, \nabla f_2(u_\alpha) \partial_x u_\alpha \right)^T,
$$

$$
= \sum_{\mu=1}^{2} (\pi_{ij} - x_{ij})_\mu \left( \nabla f_1(u_\alpha) \partial_x u_\alpha, \nabla f_2(u_\alpha) \partial_x u_\alpha \right)^T,
$$

with $(\partial_{x_\mu} u_\alpha)_{\mu \in \{1,2\}}$ a first-order finite volume approximation of $(\partial_{x_\mu} u)_{\mu \in \{1,2\}}$ at the cell $\alpha$, then

$$
\int_{ij} F(u(x(\xi))) d\xi \quad \xrightarrow{h \to 0} \quad \int_{ij} F(u(x(\xi))) d\xi + \mathcal{O}(h^3).
$$

\textbf{Proof.} Since $u$ is assumed regular, a standard Taylor expansion around $x_{ij}$ yields

$$
\int_{\Gamma_{ij}} F(u(x(\xi))) d\xi = \int_{h \to 0} \int_{\Gamma_{ij}} \left( F(u(x_{ij})) + (x(\xi) - x_{ij})^T \nabla_x F(u(x_{ij})) \right) d\xi + \mathcal{O}(h^3)
$$

$$
= \int_{h \to 0} |\Gamma_{ij}| \left( F(u(x_{ij})) + (\pi_{ij} - x_{ij})^T \nabla_x F(u(x_{ij})) \right) + \mathcal{O}(h^3),
$$

then we observe that

$$
F(u(x_{ij})) = \int_{h \to 0} \frac{d_{ij}}{d_{i,j}} F(u_j) + \frac{d_{ij}}{d_{i,j}} F(u_i) + \mathcal{O}(h^2),
$$

$$
\nabla_x F(u(x_{ij})) = \int_{h \to 0} \frac{d_{ij}}{d_{i,j}} \nabla_x F_j + \frac{d_{ij}}{d_{i,j}} \nabla_x F_i + \mathcal{O}(h^2).
$$

Plugging these expressions in the above intermediate result, we deduce the expected formula. \hfill \Box

We work now on the selection of the terms $(A_{ij})_{ij \in \mathcal{E}}$. These correction terms have to be designed to preserve the second-order accuracy of the numerical flux. In this regard, we propose the following lemma.

\textbf{Lemma 2.2} ($(A_{ij})_{ij \in \mathcal{E}}$ characterization). Consider a regular function $u$ and denote $u_i = \frac{1}{|\mathcal{T}_i|} \int_{\mathcal{C}_i} u(x) dx$. For all edges $ij$ localized between two cells $i$ and $j$ of an arbitrary mesh $\mathcal{T}$, consider $(A_{ij})_{ij \in \mathcal{E}}$ under the form

$$
d_{i,j} A_{ij} = d_{i,j} A_i + d_{i,j} A_j, \quad \forall ij \in \mathcal{E},
$$

with $(A_i, A_j) \in (\mathbb{R}^d)^2$. If,

$$
d_{i,j} A_\alpha = \frac{\lambda}{2} (x_j - x_i)^T * \nabla_x u_\alpha = \frac{\lambda}{2} \sum_{\mu=1}^{2} (x_j - x_i)_\mu * \partial_{x_\mu} u_\alpha, \quad \forall \alpha \in \{i, j\},
$$

where $* \nabla_x u_\alpha$ is a first-order discretization of the gradient of $u$ at the cell $\alpha$, then

$$
-\frac{\lambda}{2} (u_j - u_i) + A_{ij} d_{i,j} \xrightarrow{h \to 0} \mathcal{O}(h^2).
$$
Proof. A Taylor expansion around \( x_{ij} \) yields

\[
-\frac{\lambda}{2} (u_j - u_i) + A_{ij} d_{ij} = -\frac{\lambda}{2} (x_j - x_i)^T \nabla_x u(x_{ij}) + A(x_{ij}) d_{ij} + \mathcal{O}(h^2),
\]

which re-writes in a finite volume sense

\[
-\frac{\lambda}{2} (u_j - u_i) + A_{ij} d_{ij} \underset{h \to 0}{=} \frac{d_{ij}}{d_{ij}} \left( -\frac{\lambda}{2} (x_j - x_i)^T \nabla_x u_i + d_{ij} A_i \right) + \frac{d_{ij}}{d_{ij}} \left( -\frac{\lambda}{2} (x_j - x_i)^T \nabla_x u_j + d_{ij} A_j \right) + \mathcal{O}(h^2). \tag{12}
\]

that achieve the proof.

At this point, it is clear that \((A_{ij})_{ij \in \mathcal{E}}\) given by \([10]\) combined to the integral flux discretization \([9]\) ensure that the numerical flux \([8]\) verifies the proposition \([2.1]\). As a consequence, we formally have a second-order scheme. Now, for a given edge \( ij \), we have to precise the gradient \((\nabla_x u_\alpha)_{\alpha \in \{i,j\}}\). We use matrices \((\Theta_{am})_{m \in \mathcal{N}(\alpha)}\) of size \(d \times d\) to weight the values at the interfaces of the cell \( \alpha \). Then, we define

\[
\nabla_x u_\alpha = \left( \frac{\partial}{\partial x_1} u_\alpha \right) \left( \frac{\partial}{\partial x_2} u_\alpha \right)^T = \sum_{m \in \mathcal{N}(\alpha)} \left( \Theta_{am} \frac{\partial}{\partial x_1} u_{am} \right) \left( \Theta_{am} \frac{\partial}{\partial x_2} u_{am} \right)^T,
\]

\[
\frac{\partial}{\partial x_\mu} u_{am} = \beta_{am} \frac{\partial}{\partial x_\mu} u_\alpha + (1 - \beta_{am}) \frac{\partial}{\partial x_\mu} u_m,
\]

\[
\sum_{m \in \mathcal{N}(\alpha)} \Theta_{am} = I_d,
\]

for a free parameter \( \beta_{am} \in [0, 1] \) and where \( I_d \) is the identity matrix of size \(d \times d\). In the next section, we will proceed to the stability analysis that will highlight the role and the selection way of the \( \Theta \)-matrices. For the sake of simplicity, we consider the same \( \Theta \)-matrices in both space directions, but more general matrices may be considered. Our scheme design now require a finite volume discretization of the gradient \((\nabla_x u_\alpha)_{i \in \mathcal{T}}\). In the next section we give a method yields the desired approximations.

### 2.2 Unlimited gradient finite volume approximations

A discrete gradient defined on unstructured grids can be derived with several methods \([67, 49, 58, 24]\). The main techniques need the solving of a linear system coupling all the unknown of the mesh \([21, 59]\) or a fix point procedure \([21, 70, 23]\) or a dual mesh \([22, 53, 28]\). The popular methods having an admissible computational time consist in strong assumptions on the mesh \([29, 42]\), or in solving a least square problem \([14, 77]\). The least square methods are very powerful but their accuracy are genuinely dependent to the weights \([11, 74]\) and to the stencil used \([57, 27]\). Almost of these methods require limitations techniques that may be difficult to define on unstructured grids \([21, 0]\). In this work, we assume that for all cells \( i \) of the mesh \( \mathcal{T} \) the following conditions are satisfied

\[
M_i = \left( I_2 - \sum_{j \in \mathcal{N}(i)} \frac{|\Gamma_{ij}|}{|C_i|} n_{ij} (x_{ij} - x_i)^T \right) \in \text{GL}_2(\mathbb{R}), \tag{14a}
\]

\[
M_i^{-1} = h \to 0 \mathcal{O}(1), \tag{14b}
\]

where \( I_2 \) is the identity matrix of size two. The condition \([14a]\) is very few restrictive on the choice of the mesh. Indeed, it is clear that all Cartesian meshes satisfy \([14a]\) in this case \( M_i \) becomes identity matrix. Moreover, the meshes obtained from a local deformation of a Cartesian mesh work as long as \( M_i \) is diagonally dominant at least. Empirically, the meshes containing triangular cells and convex quadrangle cells seem to work. For each edge \( ij \) of given cell \( i \) verifying \([14a]\), we can define the set of \( \mathbb{R}^2 \) vectors \((b_{ij})_{j \in \mathcal{N}(i)}\) such that

\[
b_{ij} = \frac{|\Gamma_{ij}|}{|C_i|} M_i^{-1} n_{ij}, \quad \forall (i, j) \in \mathcal{T} \times \mathcal{N}(i). \tag{15}
\]
Thanks to the well known property \( \sum_{j \in \mathcal{N}(i)} |\Gamma_{ij}| n_{ij} = 0 \) (see for instance [70] for details), we have \( \sum_{j \in \mathcal{N}(i)} b_{ij} = 0 \). Using the conditions (14) and following an approach close to one described in [7, 70], we propose in the sequel a method to approximate the gradient on unstructured meshes. Our method is explicit, linear and local and so easily parallelizable.

**Proposition 2.2** (Discrete gradients). Consider a mesh \( \mathcal{T} \) made of cells verifying (14a), \( u \) a given smooth function and \( u_i = \frac{1}{|\mathcal{C}_i|} \int_{\mathcal{C}_i} u(x)dx \), then a first-order gradient approximation of \( u \) at the cell \( i \) is given by

\[
\nabla_x u_i = (\frac{\partial x}{\partial u} \frac{\partial x}{\partial u})^T = \sum_{j \in \mathcal{N}(i)} \frac{d_{ij}}{d_{ij}} (b_{ij1} b_{ij2})^T, \quad \forall i \in \mathcal{T},
\]

where the distance \( d_{ij} \) and the vectors \( b \) are respectively defined in [6], [15].

**Proof.** From the Green formula we have

\[
\frac{1}{|\mathcal{C}_i|} \int_{\mathcal{C}_i} \nabla_x u dx = \frac{1}{|\mathcal{C}_i|} \int_{\mathcal{C}_i} (\partial_{x_1} u \partial_{x_2} u)^T dx,
\]

\[
= \left( \frac{1}{|\mathcal{C}_i|} \int_{\mathcal{C}_i} \partial_{x_1} u dx \right) \left( \frac{1}{|\mathcal{C}_i|} \int_{\mathcal{C}_i} \partial_{x_2} u dx \right),
\]

\[
= \left( \frac{1}{|\mathcal{C}_i|} \sum_{j \in \mathcal{N}(i)} \int_{ij} u(x(\xi)) \eta_{ij1} d\xi \right) \left( \frac{1}{|\mathcal{C}_i|} \sum_{j \in \mathcal{N}(i)} \int_{ij} u(x(\xi)) \eta_{ij2} d\xi \right),
\]

\[
= \frac{1}{|\mathcal{C}_i|} \sum_{j \in \mathcal{N}(i)} \left( \int_{ij} u(x(\xi)) d\xi \right) n_{ij}.
\]

Using a Taylor expansion around \( x_{ij} \) in the above equation, we obtain

\[
\int_{ij} u(x(\xi)) d\xi = \int_{ij} (u(x_{ij}) + (x(\xi) - x_{ij})^T \nabla_x u(x_{ij})) d\xi + O(h^3),
\]

\[
= \Gamma_{ij} (u(x_{ij}) + (\bar{x}_{ij} - x_{ij})^T \nabla_x u(x_{ij})) + O(h^3).
\]

Now, we have to approximate each terms of the above equation in a finite volumes sense with enough accuracy to get a consistent approximation of the gradient. We use

\[
u(x_{ij}) = \frac{d_{ij} u_{ij} + d_{ij} u_j}{d_{ij}} + O(h^2),
\]

\[
\nabla_x u(x_{ij}) = \frac{1}{|\mathcal{C}_i|} \int_{\mathcal{C}_i} \nabla_x u dx + O(h).
\]

Injecting the previous equations in the Taylor expansion above and using the definition of \( M_i \) given by (14a), we deduce

\[
M_i \left( \frac{1}{|\mathcal{C}_i|} \int_{\mathcal{C}_i} \nabla_x u dx \right) = \sum_{j \in \mathcal{N}(i)} \frac{d_{ij} u_{ij} + d_{ij} u_j}{d_{ij}} M_i b_{ij} + O(h).
\]

As, according to (14), \( M_i \) is invertible and \( M_i^{-1} = O(1) \), we deduce the expected formula. \( \square \)

In the next section we give our main result about the global entropy stability property.
3 Global entropy stability

Using the section 2 the second-order accuracy scheme writes

\[
    w^{n+1}_i = \sum_{j \in \mathcal{N}(i)} \frac{[\Gamma_{ij}]}{[P_i]} \left( w^n_i - \frac{\Delta t |P_i|}{|C_i|} F_{ij} \right),
\]

\[
    F_{ij} = \frac{d_{i,j}}{d_{i,j}} \left( \sum_{\mu=1}^{2} \left( f_\mu(w^n_j) n_{ij,\mu} + \sum_{\nu=1}^{2} \left( x_{ij} - x_{ij,\nu} \right) \nabla f_\nu(w^n_j) \partial_{x_\mu} w^n_i n_{ij,\mu} \right) \right) \quad \forall (i, j) \in \mathcal{T} \times \mathcal{N}(i), \quad (17)
\]

\[
    + \frac{d_{j,i}}{d_{i,j}} \left( \sum_{\mu=1}^{2} \left( f_\mu(w^n_i) n_{ij,\mu} + \sum_{\nu=1}^{2} \left( x_{ij} - x_{ij,\nu} \right) \nabla f_\nu(w^n_i) \partial_{x_\mu} w^n_i n_{ij,\mu} \right) \right)
\]

\[
    - \frac{\lambda}{2} (w^n_j - w^n_i) + d_{i,j} A_{ij}
\]

with

\[
    d_{i,j} A_{ij} = d_{j,i} A_i + d_{i,j} A_j,
\]

\[
    A_\alpha = \frac{\lambda}{2} (x_j - x_i)^T \nabla_x w^n_i, \quad \forall \alpha \in \{i, j\}, \quad (18)
\]

where the discrete gradient \( \nabla_x \) can be computed as described in [1]. Let state now our result about the global entropy stability satisfied by the unlimited second-order scheme (17). For the convenient, we define the following discrete Sobolev space

\[
    l^2(\mathcal{T}) = \left\{ (w_i)_{i \in \mathcal{T}} \in \mathbb{R}^{|\mathcal{T}|} \mid \sum_{i \in \mathcal{T}} \| w_i \|^2_{2} |C_i| < +\infty \right\},
\]

\[
    l^\infty(\mathcal{T}) = \left\{ (w_i)_{i \in \mathcal{T}} \in \mathbb{R}^{|\mathcal{T}|} \mid \max_{i \in \mathcal{T}} \| w_i \|_{\infty} < +\infty \right\},
\]

where \( \| \cdot \|_2, \| \cdot \|_\infty \) denote the usual euclidean and uniform norms of \( \mathbb{R}^d \). We have:

**Theorem 3.1** (Global entropy stability). For a given mesh \( \mathcal{T} \), consider a strictly convex entropy \( \eta \in \mathcal{C}^{2}(\Omega, \mathbb{R}) \) and a non constant sequence \( (w^n_i)_{i \in \mathcal{T}} \) in \( l^2(\mathcal{T}) \) compactly supported and such that \( \sum_{i \in \mathcal{T}} \eta(w^n_i)|C_i| \) is finite. Assume

i) the existence of a compact set \( K \subset \Omega \) such that \( \text{dist}(K, \partial \Omega) > 0 \), \( (w^n_i)_{i \in \mathcal{T}} \subset K \), \( (w^{n+1}_i)_{i \in \mathcal{T}} \subset K \),

ii) the vectors \( (A_{ij})_{i,j,\in \mathcal{E}} \) verify \( A_{ji} = -A_{ij} \) for all edges \( ij \) and are such that

\[
    \int_0^1 \sum_{i,j} |\Gamma_{ij}| \nabla^2 \eta(w^n_i + s(w^n_j - w^n_i)) (w^n_j - w^n_i) \cdot A_{ij} d_{ij} ds < 0,
\]

then there exists \( \lambda > 0 \) large enough, \( \Delta t > 0 \) small enough such that the updated sequence \( (w^{n+1}_i)_{i \in \mathcal{T}} \) given by the second-order scheme (17) verifies the following global entropy stability

\[
    \sum_{i \in \mathcal{T}} \frac{\eta(w^{n+1}_i) - \eta(w^n_i)}{\Delta t} |C_i| \leq 0.
\]

Before proceeding to the proof of the theorem 3.1 let us precise our assumptions. From a sequence \( (w^n_i)_{i \in \mathcal{T}} \) bounded and included in a compact subset \( K \) of \( \Omega \) such that \( \text{dist}(K, \partial \Omega) > 0 \), we assume that the sequence \( (w^{n+1}_i)_{i \in \mathcal{T}} \) given by the scheme (17) stays in the same compact \( K \) and also belongs in \( l^\infty(\mathcal{T}) \). For instance for the Euler equations with a perfect gas, the admissible set is \( \Omega = \{ w = (\rho, \rho u, \rho v, \rho E)^T \in \mathbb{R}^4 \mid \rho > 0, \ E - u^2/2 - v^2/2 > 0 \} \). The assumption i) ensures that the density \( \rho \) and the pressure \( p = (\gamma - 1)(\rho E - \rho u^2/2 - \rho v^2/2) \), where \( \gamma > 1 \) is the adiabatic constant, are strictly away from the vacuum and bounded from above. Since it is extremely difficult to prove such robustness
without requirement to limitation techniques \[1\] \[61\], this type of assumption is commonly accepted (see for instance \[66\] \[43\]). The inequality \[19\] stands for a dissipation inequality relatively to a given strictly convex entropy \(\eta\). The main originality of this work is to select the \(\Theta\)-matrices required for the discrete gradient \(\nabla_x^*\) (see the equation \[13\]) to satisfy the inequality \[19\]. We give now a lemma that justifies the existence of admissible parameters \((\Theta_{ij})_{ij}\epsilon\) for a particular configuration using our discrete gradients \[16\].

**Lemma 3.0.1** (Dissipation inequality). For a given mesh \(\mathcal{T}\) made of cells verifying the assumptions \[14\], consider a strictly convex entropy \(\eta\epsilon C^2(\Omega, \mathbb{R})\) and a non constant compactly supported sequence \((w^n_i)_{i\in\mathcal{T}}\) in \(l^2(\mathcal{T})\). Assume

- \(i\) the existence of a compact set \(K\subset\Omega\) such that \(\text{dist}(K, \partial\Omega) > 0\), \((w^n_i)_{i\in\mathcal{T}}\subset K\), \((w^{n+1}_i)_{i\in\mathcal{T}}\subset K\),
- \(ii\) for a given edge \(ij\) of \(\mathcal{E}\), the discrete gradients \((\nabla_x w^n_{ij})_{ij\in\mathcal{E}}\) write

\[
\begin{align*}
\nabla_x^* w^n_i &= \Theta_{ij} \nabla_x w^n_j + (1 - \Theta_{ij}) \nabla_x w^n_i, \\
\nabla_x^* w^n_j &= \Theta_{ji} \nabla_x w^n_j + (1 - \Theta_{ji}) \nabla_x w^n_i, \\
\n\forall (ij, \alpha) \in \mathcal{E} \times \{i, j\}. 
\end{align*}
\]

(21)

where \((\Theta_{ij})_{ij\in\mathcal{E}}, (\Theta_{ji})_{ji\in\mathcal{E}}\) are reals sequences.

If,

\[
\begin{align*}
\Theta_{ij} &= -\Theta_{ji} = \frac{-\kappa \text{sign}(\zeta_{ij} - \zeta_{ji})}{2 \max \left(0, \sum_{i\in\mathcal{T}} \sum_{j\in\mathcal{N}(i)} \frac{|\Gamma_{ij}|}{d_{ij}} (x_j - x_i)^T \nabla_x (d_{ij} r_{ij}^n + d_{ij} r_{ij}^n)\right)}, \\
\kappa &> \frac{\sum_{i\in\mathcal{T}} \sum_{j\in\mathcal{N}(i)} |\Gamma_{ij}| \zeta_{ij} - \zeta_{ji}|}{\sum_{i\in\mathcal{T}} \sum_{j\in\mathcal{N}(i)} |\Gamma_{ij}||\zeta_{ij} - \zeta_{ji}|}, \\
\text{where}
\end{align*}
\]

\[
\begin{align*}
\zeta_{ij} &= (x_j - x_i)^T \nabla_x r_{ij}^n - (x_j - x_i)^T \nabla_x r_{ij}^n, \\
(X_j - x_i)^T \nabla_x r_{ij}^n &= \sum_{m\in\mathcal{N}(\alpha)} \frac{d_{m, \alpha} r_{ij}^n + d_{\alpha, m} r_{ij}^n}{d_{\alpha, m}} ((x_j - x_i)^T \eta_{b_{\alpha m}}), \\
r_{ij}^n &= \int_0^1 (1 - s) \nabla_x \eta(w^n_i + s(w^n_m - w^n_i))(w^n_m - w^n_i) \cdot (w^n_m - w^n_i) ds, \\
\forall (ij, \alpha) \in \mathcal{E} \times \{i, j\},
\end{align*}
\]

then the inequality \[19\] is satisfied.

**Proof.** Thanks to the explicit definitions \[21\] and to the assumption \(i\), the proof of the lemma is based on a direct computation of the inequality \[19\]. At first, substituting the definition of \(\nabla_x w^n_{ij}\) given by \[21\] in the definition of \(A_{ij}\) given by \[18\], it is clear that \(A_{ji} = -A_{ij}\). As a consequence, a discrete
integration by parts of (19) coupled to the definitions (18)-(21) involves

\[
\int_0^1 \sum_{ij \in \mathcal{E}} \nabla^2 \eta(w_i^n + s(w_j^n - w_i^n)(w_j^n - w_i^n) \cdot A_{ij}d_{ij}ds = - \sum_{i \in T} \sum_{j \in \mathcal{N}(i)} |\Gamma_{ij}| \nabla \eta(w_i^n) \cdot A_{ij}d_{ij},
\]

\[
= - \sum_{i \in T, j \in \mathcal{N}(i)} |\Gamma_{ij}| \nabla \eta(w_i^n) \cdot (A_{ij}d_{ij} + A_{j,ij}),
\]

\[
= - \frac{\lambda}{2} \sum_{i \in T, j \in \mathcal{N}(i)} |\Gamma_{ij}| \nabla \eta(w_i^n) \cdot ((x_j - x_i)^T \nabla \eta(w_i^n) - (x_i - x_j)^T \nabla \eta(w_j^n)) + (1 - \Theta_{ij}) \nabla \eta(w_i^n) \cdot ((x_j - x_i)^T \nabla \eta(w_j^n)),
\]

\[
= - \frac{\lambda}{2} \sum_{i \in T, j \in \mathcal{N}(i)} \frac{d_{ij}}{d_{ij}} \left( \Theta_{ij} \nabla \eta(w_i^n) \cdot ((x_j - x_i)^T \nabla \eta(w_j^n)) + (1 - \Theta_{ij}) \nabla \eta(w_i^n) \cdot ((x_j - x_i)^T \nabla \eta(w_j^n)) \right),
\]

But, according to the gradient approximations (16), we have

\[
\nabla \eta(w_i^n) \cdot ((x_j - x_i)^T \nabla \eta(w_i^n)) = \sum_{m \in \mathcal{N}(i)} \left( \frac{d_{m,am}}{d_{a,m}} \nabla \eta(w_i^n) \cdot w_m^n + \frac{d_{a,m}}{d_{a,m}} \nabla \eta(w_i^n) \cdot w_m^n \right) ((x_j - x_i)^T b_{am}),
\]

\[
= \sum_{m \in \mathcal{N}(i)} \left( \frac{d_{m,am}}{d_{a,m}} \nabla \eta(w_i^n) \cdot (w_m^n - w_i^n) + \frac{d_{a,m}}{d_{a,m}} \nabla \eta(w_i^n) \cdot (w_m^n - w_i^n) \right) ((x_j - x_i)^T b_{am}),
\]

\[
= \sum_{m \in \mathcal{N}(i)} \left( \frac{d_{m,am}}{d_{a,m}} \eta(w_i^n) - \eta(w_m^n) - r_{i,a} + \frac{d_{a,m}}{d_{a,m}} \eta(w_i^n) - \eta(w_m^n) - r_{i,a} \right) ((x_j - x_i)^T b_{am}),
\]

\[
= (x_j - x_i)^T \nabla \eta(w_i^n) - (x_j - x_i)^T \nabla r_{i,a},
\]

where we have used the properties of the vectors $b$, $\sum_{j \in \mathcal{N}(i)} b_{ij} = 0$ and a Taylor expansion with the integral form of the reminder that writes $\eta(w_i^n) \cdot (w_m^n - w_i^n) = \eta(w_m^n) - \eta(w_i^n) - r_{i,m}$. So, injecting the previous considerations in the first computation, we deduce

\[
\int_0^1 \sum_{ij \in \mathcal{E}} \nabla^2 \eta(w_i^n + s(w_j^n - w_i^n)(w_j^n - w_i^n) \cdot A_{ij}d_{ij}ds = - \sum_{i \in T} \sum_{j \in \mathcal{N}(i)} |\Gamma_{ij}| \nabla \eta(w_i^n) \cdot A_{ij}d_{ij}
\]

\[
= - \frac{\lambda}{2} \sum_{i \in T} \sum_{j \in \mathcal{N}(i)} |\Gamma_{ij}|(x_j - x_i)^T \left( \Theta_{ij} \nabla \eta(w_i^n) + (1 - \Theta_{ij}) \nabla \eta(w_j^n) \right) \frac{d_{ij}}{d_{ij}}
\]

\[
+ \frac{\lambda}{2} \sum_{i \in T} \sum_{j \in \mathcal{N}(i)} |\Gamma_{ij}|(x_j - x_i)^T \left( \Theta_{ij} \nabla \eta(w_i^n) + (1 - \Theta_{ij}) \nabla \eta(w_j^n) \right) \frac{d_{ij}}{d_{ij}}
\]

\[
+ \frac{\lambda}{2} \sum_{i \in T} \sum_{j \in \mathcal{N}(i)} |\Gamma_{ij}|(x_j - x_i)^T \left( \Theta_{ij} \nabla \eta(w_i^n) + (1 - \Theta_{ij}) \nabla \eta(w_j^n) \right) \frac{d_{ij}}{d_{ij}}.
\]

where we have defined

\[
* \nabla \eta(w_i^n) = \Theta_{ij} \nabla \eta(w_j^n) + (1 - \Theta_{ij}) \nabla \eta(w_j^n),
\]

\[
* \nabla \eta(w_j^n) = \Theta_{ij} \nabla \eta(w_j^n) + (1 - \Theta_{ij}) \nabla \eta(w_j^n),
\]

\[
\forall ij \in \mathcal{E}.
\]
Now, introducing \( \zeta_{i,j} \), and \( \zeta_{j,i} \) in the previous computation and rearranging the terms, we have

\[
\frac{2}{\lambda} \int_0^1 \sum_{ij \in E} \nabla^2 \eta(w_{i}^{n} + s(w_{j}^{n} - w_{i}^{n})) (w_{j}^{n} - w_{i}^{n}) \cdot A_{ij}d_{ij}ds
\]

\[
= \sum_{i \in T, j \in N(i)} \left| \Gamma_{ij} \right| (d_{j,ij} \Theta_{ij} - d_{i,ij} \Theta_{ji}) \zeta_{i,j} + \sum_{i \in T, j \in N(i)} \left| \Gamma_{ij} \right| (x_j - x_i)^T \nabla x (d_{j,ij} r_{i,j}^{n} + d_{i,ij} r_{j,i}^{n}),
\]

\[
= \sum_{i \in T, j \in N(i)} \left( \frac{\left| \Gamma_{ij} \right|}{2d_{ij}} \right) (d_{j,ij} \Theta_{ij} - d_{i,ij} \Theta_{ji}) (\zeta_{i,j} - \zeta_{j,i}) + \sum_{i \in T, j \in N(i)} \left| \Gamma_{ij} \right| (x_j - x_i)^T \nabla x (d_{j,ij} r_{i,j}^{n} + d_{i,ij} r_{j,i}^{n}),
\]

which is negative according to (22).

From a general point of view, many choices of discrete gradients \( \nabla_{x} \) and \( \Theta \)-matrices are possible.

We will give some examples for the configuration (21) in the next section that will be devoted to the numerical experiments. Now we proceed to the proof of the theorem 3.1

**Proof.** Let consider a strictly convex entropy \( \eta \in C^2(\Omega, \mathbb{R}) \). Thanks to the assumption i) and to the convexity proprieties of \( \eta \), a Taylor expansion with the integral form of the remainder leads to

\[
\eta(w_{i}^{n+1}) \leq \sum_{j \in N(i)} \frac{\left| \Gamma_{ij} \right|}{|P_i|} \eta(w_{i}^{n} - \frac{\Delta t|P_i|}{|C_i|} F_{ij}),
\]

\[
\leq \eta(w_{i}^{n}) - \frac{\Delta t|P_i|}{|C_i|} \sum_{j \in N(i)} \left| \Gamma_{ij} \right| \nabla \eta(w_{i}^{n}) \cdot F_{ij} \quad \forall i \in T.
\]

\[
+ \left( \frac{\Delta t|P_i|}{|C_i|} \right)^2 \int_0^1 (1-s) \sum_{j \in N(i)} \left| \Gamma_{ij} \right| \nabla^2 \eta \left( w_{i}^{n} - s \frac{\Delta t|P_i|}{|C_i|} F_{ij} \right) F_{ij} \cdot F_{ij} ds,
\]

But, the above equation integrated over the domain reads

\[
\sum_{i \in T} \frac{\eta(w_{i}^{n+1}) - \eta(w_{i}^{n})}{\Delta t} \leq - \sum_{i \in T} \sum_{j \in N(i)} \left| \Gamma_{ij} \right| \nabla \eta(w_{i}^{n}) \cdot F_{ij} + \sum_{i \in T} \frac{\Delta t|P_i|}{|C_i|} \int_0^1 (1-s) \sum_{j \in N(i)} \left| \Gamma_{ij} \right| \nabla^2 \eta \left( w_{i}^{n} - s \frac{\Delta t|P_i|}{|C_i|} F_{ij} \right) F_{ij} \cdot F_{ij} ds.
\]

Now, we have to prove that the right hand side of the above inequality can be negative. So, we give a more convenient form of the term \(- \sum_{i \in T} \sum_{j \in N(i)} \left| \Gamma_{ij} \right| \nabla \eta(w_{i}^{n}) \cdot F_{ij} \). Using the definition of the numerical flux given by (17), we have

\[
- \sum_{i \in T} \sum_{j \in N(i)} \left| \Gamma_{ij} \right| \nabla \eta(w_{i}^{n}) \cdot F_{ij} = \sum_{i \in T, j \in N(i)} \left| \Gamma_{ij} \right| R_{ij} - \sum_{i \in T} \sum_{j \in N(i)} \left| \Gamma_{ij} \right| \nabla \eta(w_{i}^{n}) \cdot \left( - \frac{\lambda}{2} (w_{j}^{n} - w_{i}^{n}) + d_{i,ij} A_{ij} \right),
\]

with

\[
R_{ij} = \frac{d_{i,ij}}{d_{ij}} \nabla \eta(w_{i}^{n}) \cdot \left( \sum_{\mu=1}^{2} \left( f_{\mu}(w_{j}^{n}) n_{ij,\mu} + d_{i,ij} A_{ij} \right) \left( \sum_{\nu=1}^{2} (\pi_{ij} - x_{ij}) \mu \nabla f_{\nu}(w_{i}^{n}) \partial w_{i}^{n} n_{ij,\nu} \right) \right) + \frac{d_{i,ij}}{d_{ij}} \nabla \eta(w_{i}^{n}) \cdot \left( \sum_{\mu=1}^{2} \left( f_{\mu}(w_{i}^{n}) n_{ij,\mu} + d_{i,ij} A_{ij} \right) \left( \sum_{\nu=1}^{2} (\pi_{ij} - x_{ij}) \mu \nabla f_{\nu}(w_{i}^{n}) \partial w_{i}^{n} n_{ij,\nu} \right) \right).
\]
Since the vectors \((A_{ij})_{ij \in E}\) verifies \(A_{ij} = -A_{ij}\) for all edges \(ij\), we can use a discrete integration by parts and write
\[
-\sum_{i \in T} \sum_{j \in N(i)} |\Gamma_{ij}| \left( \nabla \eta(w^n_i) \cdot \left( -\frac{\lambda}{2} (w^n_j - w^n_i) + d_{ij} A_{ij} \right) \right)
\]
\[
= \sum_{ij \in E} |\Gamma_{ij}| \left( \nabla \eta(w^n_j) - \nabla \eta(w^n_i) \right) \cdot \left( -\frac{\lambda}{2} (w^n_j - w^n_i) + d_{ij} A_{ij} \right),
\]
\[
= -\frac{\lambda}{2} \int_0^1 \sum_{ij \in E} |\Gamma_{ij}| \left( \nabla \eta \cdot \left( w^n_j + s(w^n_j - w^n_i) \right) \right) \left( w^n_j - w^n_i \right) \cdot \left( w^n_j - w^n_i \right) \, ds
\]
\[
+ \int_0^1 \sum_{ij \in E} |\Gamma_{ij}| \left( \nabla \eta \cdot \left( w^n_j + s(w^n_j - w^n_i) \right) \right) \left( w^n_j - w^n_i \right) \cdot \left( w^n_j - w^n_i \right) \, ds
\]
\[
+ \frac{\lambda}{2} \int_0^1 \sum_{ij \in E} |\Gamma_{ij}| \left( \nabla \eta \cdot \left( w^n_j + s(w^n_j - w^n_i) \right) \right) \left( w^n_j - w^n_i \right) \cdot \left( \frac{d_{ij}}{d_{ij}}(x_j - x_i)^T \nabla w^n_i \right) \, ds
\]
\[
+ \frac{\lambda}{2} \int_0^1 \sum_{ij \in E} |\Gamma_{ij}| \left( \nabla \eta \cdot \left( w^n_j + s(w^n_j - w^n_i) \right) \right) \left( w^n_j - w^n_i \right) \cdot \left( \frac{d_{ij}}{d_{ij}}(x_j - x_i)^T \nabla w^n_j \right) \, ds < 0,
\]
which is negative according to the inequality (19). Thanks to the assumption \(i\), the term \(\sum_{i \in T} \sum_{j \in N(i)} |\Gamma_{ij}| |R_{ij}|\) is necessary finite. As a consequence, we deduce the existence of a finite numerical diffusion \(\lambda > 0\) such that
\[
-\sum_{i \in T} \sum_{j \in N(i)} |\Gamma_{ij}| \mathcal{F}_{ij} \cdot \nabla \eta(w^n_i) < 0.
\]
Finally, if \(\Delta t\) verifies
\[
0 < \Delta t \leq \frac{\sum_{i \in T} \sum_{j \in N(i)} |\Gamma_{ij}| \mathcal{F}_{ij} \cdot \nabla \eta(w^n_i)}{\int_0^1 \left( 1 - s \right) \sum_{i \in T} \sum_{j \in N(i)} \frac{|\Gamma_{ij}| |P_i|}{|C_i|} \nabla \eta \left( w^n_i - s \frac{\Delta t |P_i|}{|C_i|} \mathcal{F}_{ij} \right) \mathcal{F}_{ij} \cdot \mathcal{F}_{ij} \, ds},
\]
then we deduce the expected result. \(\square\)

The next section deals with the numerical experiments.

4 Numerical results

4.1 Methods

The scheme (17) was implemented in C and parallelized with some OpenMP instructions. In order to reduce the time computations, the all constant coefficients needed for the discrete gradient (16) are computed before the time loop. As usual, the updated sequence \((w^n_{i+1})_{i \in T}\) is computed with a loop on the edges of the mesh. We illustrate the accuracy and the stability performances of our schemes (17) with the Euler equations for a diatomic perfect gas in which
\[
w = (\rho, \rho u, \rho v, \rho E)^T, \quad F(w) = \left( \begin{array}{c} (\rho u, \rho u^2 + p, \rho u v, (\rho E + p) u) \\ (\rho v, \rho u v, \rho v^2 + p, (\rho E + p) v) \end{array} \right),
\]
where the pressure \(p\) is given by \(p = (\gamma - 1) \left( \frac{\rho E - \rho u^2}{2} - \rho v^2/2 \right)\), with \(\gamma = 1.4\) and the convex set \(\Omega\) is \(\{w = (\rho, \rho u, \rho v, \rho E)^T \in \mathbb{R}^4 \mid \rho > 0, p > 0\}\). In order to preserve this convex set, we add a MOOD
procedure \[20\] to our scheme. Within this paradigm, if the second-order scheme \[17\] does not preserve $\Omega$, then a correction is done with the standard first-order numerical scheme. We endow the Euler equations with the pair entropy, entropy-flux

$$
\eta(w) = -\rho \ln(p/\rho^\gamma), \quad G(w) = -\rho \ln \left(\frac{p}{\rho^\gamma}\right)(u,v)^T.
$$

We use standard Cartesian meshes and irregular triangle meshes generated with FreeFem++ \[41\]. The Cartesian meshes verify the assumptions \[14\] and we numerically verify that the unstructured triangle meshes satisfy the assumption \[14a\]. The Figure 2 shows an example of unstructured meshes used.

![Figure 2](image)

Figure 2: On the left, the square $[0,10]^2$ meshed with FreeFem++ \[41\] ($\approx 5000$ cells), on the right, zoom on the triangular cells generated by FreeFem++ \[41\].

We proceed to high-order time discretization using SSP Runge-Kutta methods \[39, 40, 38\]. Since these methods only depend on convex combination of first-order times sub-steps, the main entropy stability result \[3.1\] is preserved. Our main stability result \[3.1\] is established without consideration for the boundaries conditions. But, in finite domains as ones used for the following test cases, the boundaries conditions may play a central role in the entropy dissipation. As a consequence, we consider the criterion

$$
\sum_{i \in T} \eta(w_{i}^{n+1}) - \eta(w_{i}^{n}) |C_{i}| + \sum_{ij \in \partial E} |\Gamma_{ij}|n_{ij}^{T}G(w_{j}^{n}), \quad (DEI)
$$

where $\partial E$ and $j$ denote respectively the edges on the boundaries of the mesh $T$ and the ghost cells. The criterion (DEI) includes the boundaries conditions and it is a modified version of \[20\]. If the boundaries conditions are compatible to the entropy stability, then according to the theorem \[3.1\], (DEI) will be negative. For a given edge $ij$ of $E$, we select the discrete gradients \(\left(\nabla_{x}w_{\alpha}^{n}\right)_{\alpha \in \{i,j\}}\) as follows:

\[
\nabla_{x}w_{\alpha}^{n} = \left(\frac{\partial w_{\alpha}^{n}}{\partial x_{1}}, \frac{\partial w_{\alpha}^{n}}{\partial x_{2}}\right)^{T},
\]

\[
\frac{\partial w_{\alpha}^{n}}{\partial x_{\mu}} = \Theta_{ij}\frac{\partial w_{\alpha}^{n}}{\partial x_{\mu}} + (I_{d} - \Theta_{ij})\frac{\partial w_{\mu}^{n}}{\partial x_{\mu}},
\]

\[
\frac{\partial w_{\alpha}^{n}}{\partial x_{\mu}} = \Theta_{ji}\frac{\partial w_{\mu}^{n}}{\partial x_{\mu}} + (I_{d} - \Theta_{ji})\frac{\partial w_{\alpha}^{n}}{\partial x_{\alpha}}, \quad \forall (ij, \alpha, \mu) \in E \times \{i,j\} \times \{1,2\}.
\]

For the matrix $(\Theta_{ij})_{ij \in E}$, we set two distinct configurations

\[
\Theta_{a,ij} = \frac{d_{j,ij}}{d_{i,j}} I_{d}, \quad \text{and} \quad \Theta_{b,ij} = \text{diag tanh}(\delta_{ij,l} - \delta_{i,j,l}),
\]

(23)
with
\[ \delta_{ij,l} = \left( \frac{(x_j - x_i)^T \nabla_x w^n_{i,l}}{d_{ij}} \right)^2 = \left( \sum_{m \in \mathcal{N}(i)} \frac{d_{m,im}(w^n_{i,l}) + d_{i,im}(w^n_{m,l})}{d_{i,m}d_{i,j}}((x_j - x_i)^T b_{im}) \right)^2. \]

The theorem 3.1 ensures the existence of a couple \((\lambda, \Delta t)\) such that the discrete entropy stability is verified. But their explicit formulations arising from the proof of the theorem 3.1 may be very difficultly used in practical test cases. We take the usual parameters
\[ \lambda = \max_{ij \in E} \left( |r_{ij}(w^n_j)|, |r_{ij}(w^n_i)| \right), \quad \frac{\lambda \Delta t |P_i|}{|C_i|} \leq \frac{1}{2}, \quad \forall i \in T, \]
where \(|r_{ij}(w)|\) is the spectral radius of the matrix \(\nabla f(w)_{n_{ij1}} + \nabla f_2(w)n_{ij2}\).

### 4.2 Accuracy performances

The measure of the accuracy performance of the second-order scheme (17) is led with the isentropic vertex problem as described in [18, 69]. We work on the square \([0, 10]^2\), and we assume that the gas verifies at the initial time
\[ \rho = \left( 1 - \frac{\gamma - 1}{2\gamma} \phi^2(x) \right)^{\frac{1}{\gamma - 1}}, \quad (1 - u, 1 - v)^T = \phi(x) (x - x_0)^T, \quad p = \left( 1 - \frac{\gamma - 1}{2\gamma} \phi^2(x) \right)^{\frac{1}{\gamma - 1}}, \]
with \(\phi(x) = \epsilon e^{-\alpha(1 - \|x - x_0\|^2)}\), \(\epsilon = \frac{5}{3\pi}, \alpha = 0.5, x_0 = (5, 5)^T\). Under these conditions, the exact solution is a regular vertex advected in the diagonal direction. The final time is 0.1, We prescribe periodic conditions on all the boundaries of the domain. We select \(\lambda\) and \(\Delta t\) as described in 4.1. The Figure 3 shows the results on Cartesian grid and on unstructured mesh. The tables 1, 2 report the rate of convergence of the variable \(\rho\) for both types of meshes and for the parameters \((\Theta_{a,ij})_{ij \in E} - (\Theta_{b,ij})_{ij \in E}\).
Figure 3: Density profile at time \( t = 0.1 \) for the isentropic vertex problem. On the top left and right second-order approximation given by the scheme $\Theta_{ij}$ endowed with the $(\Theta_{a,ij})_{ij\in\mathcal{E}}$ matrices on Cartesian grid (6400 cells) and on unstructured mesh ($\approx$ 5000 cells) respectively. On the bottom left and right second-order approximation given by the scheme $\Theta_{b,ij}$ endowed with the $(\Theta_{b,ij})_{ij\in\mathcal{E}}$ matrices on Cartesian grid (6400 cells) and on unstructured mesh ($\approx$ 5000 cells) respectively.

<table>
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<th>$L^2$ order</th>
<th>$L^\infty$ order</th>
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Table 1: Errors and order evaluations for the second-order accurate scheme $\Theta_{ij}$ for the continuous Euler solution, with $h = \max_{ij\in\mathcal{E}} d_{ij}$ and for the $(\Theta_{a,ij})_{ij\in\mathcal{E}}$ configuration given by (23).
Second-order scheme errors on irregular triangle grids

<table>
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<td>2.3</td>
<td>1.2E-03</td>
</tr>
</tbody>
</table>

Table 2: Errors and order evaluations for the second-order accurate scheme (17) for the continuous Euler solution, with \( h = \max_{ij \in E} d_{ij} \) and for the \( (\Theta_{b,ij})_{ij \in E} \) configuration given by (23).

Thanks to the free limitations strategy, the second-order is observed on Cartesian meshes. On unstructured grids, some cells do not seem satisfy the assumption (14b) which slightly spoils the results, but the order of convergence remains accurate.

### 4.3 Riemann problems

We study three Riemann problems from [47]. The unit square is divided into four quadrants formed with the line \( x = 1/2 \) and \( y = 1/2 \) and for each problems \( (\rho, u, v, p)^T \) is initialized by constant states in each quadrant as described in the Table 3.

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>KT3 problem</th>
<th>KT5 problem</th>
<th>KT6 problem</th>
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<td>([0, \frac{1}{2}] \times [0, \frac{1}{2}])</td>
<td>((0.138, 1.206, 1.206, 0.029)^T)</td>
<td>((1, 0.75, 0.5, 1)^T)</td>
<td>((1, -0.75, 0.5, 1)^T)</td>
</tr>
<tr>
<td>([0, \frac{1}{2}] \times [\frac{1}{2}, 1])</td>
<td>((0.5323, 1.206, 0, 0.3)^T)</td>
<td>((2, -0.75, 0.5, 1)^T)</td>
<td>((2, 0.75, 0.5, 1)^T)</td>
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<tr>
<td>([\frac{1}{2}, 1] \times [0, \frac{1}{2}])</td>
<td>((0.5323, 0, 1.206, 0.3)^T)</td>
<td>((3, 0.75, -0.5, 1)^T)</td>
<td>((3, -0.75, -0.5, 1)^T)</td>
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<tr>
<td>([\frac{1}{2}, 1] \times [\frac{1}{2}, 1])</td>
<td>((1.5, 0, 0, 1.5)^T)</td>
<td>((1, -0.75, -0.5, 1)^T)</td>
<td>((1, 0.75, -0.5, 1)^T)</td>
</tr>
</tbody>
</table>

Table 3: Initial data for Riemann problems.

For all the problems, the final time is 0.3 and we impose homogeneous Neumann boundary conditions on all the boundaries of the domain. The domain is dicretized with a Cartesian mesh having 1000 cells per direction and with an unstructured triangle mesh having approximately \( 10^6 \) cells. The time step \( \Delta t \) and the numerical viscosity \( \lambda \) are selected as described in the section 4.1. The Figure 4-5-6 show the numerical results for the problem KT3, KT5, KT6 respectively. For the comparative, we also plot the result of the standard first-order scheme.
Figure 4: Density profile at time $t = 0.3$ for the Riemann problem KT3. On the top left, standard first-order approximation, on the top right, time evolution of the discrete global entropy (DEI), on the bottom left and right, second-order approximation given by the scheme $[17]$ endowed with the $(\Theta_{b,ij})_{ij \in E}$ matrices given by $[23]$ on Cartesian grid and on unstructured mesh respectively.

For the problem KT3, the matrices $(\Theta_{a,ij})_{ij \in E}$ do not work. We observe a good accuracy of the solution computed with the matrices $(\Theta_{b,ij})_{ij \in E}$ on the Cartesian grid. A slight deformation of the waves localized close to the boundaries of the unstructured mesh is observed. The MOOD procedure was only necessary on the unstructured mesh. The number of cells modified with the MOOD procedure does not exceed $10^{-4}\%$ of the total cell numbers. The criterion (DEI) is negative. Due to the MOOD procedure used on the unstructured mesh, we observe some spurious oscillations in the criterion (DEI), but it remains negative.
Figure 5: Density profile at time $t = 0.3$ for the Riemann problem KT5. On the top left, standard first-order approximation, on the top right, time evolution of (DEI), on the center left and right, second-order approximation with the $(\Theta_{a,ij})_{ij \in E}$ matrices on Cartesian grid and on unstructured mesh respectively, on the bottom left and right, second-order approximation with the $(\Theta_{b,ij})_{ij \in E}$ matrices on Cartesian grid and on unstructured mesh respectively.

For the KT5 problem, the solutions are acceptable for both $\Theta$-matrices and for both type of meshes. But, the solution is more accurate with the matrices $(\Theta_{a,ij})_{ij \in E}$. For this test case, the MOOD procedure is useless for both type of meshes and for both matrices $(\Theta_{a,ij})_{ij \in E}$, $(\Theta_{b,ij})_{ij \in E}$. The discrete entropy stability criterion (DEI) is negative.
Figure 6: Density profile at time $t = 0.3$ for the Riemann problem KT6. On the top left, standard first-order approximation, on the top right, time evolution of $[DEI]$, on the center left and right, second-order approximation with the $(\Theta_{a,ij})_{ij \in E}$ matrices on Cartesian grid and on unstructured mesh respectively, on the bottom left and right, second-order approximation with the $(\Theta_{b,ij})_{ij \in E}$ matrices on Cartesian grid and on unstructured mesh respectively.

For the KT6 problem the solution is sharper and more accurate with the matrices $(\Theta_{a,ij})_{ij \in E}$. The
MOOD procedure is useless for both type of meshes and for both matrices \((\Theta_{a,ij})_{ij \in E}\), \((\Theta_{b,ij})_{ij \in E}\). As for the KT5 problem, the discrete entropy stability criterion \([DEI]\) is negative.

4.4 Double mach reflection of a strong shock

In this section we investigate a gas flow in a wind along a reflecting wall [78]. The domain is included in the box \([0, 1.9] \times [0, 1.3]\). At \(x = 1/6\), the reflecting wall makes a 30° angle with the horizontal axis and extends to the right boundary [9]. At the initial time the gas is inert and verify \(\rho = 1.4\), \((u, v) = (0, 0)\), \(p = 1\). In order to create a shock wave, we impose on the left boundary a constant in flow verifying \(\rho = 8\), \((u, v) = (8.25, 0)\) and \(p = 116.5\). On the top, on the bottom and on the reflecting wall, we impose reflecting boundary conditions that write \(un_1 + vn_2 = 0\), with \((n_1, n_2) \in \mathbb{R}^2\) the unit normal vector to the boundary. As the gradients are assumed to be null on the right, we lay down a homogeneous Neumann condition. The final time is 0.15, the time step and the numerical viscosity are chosen as mentioned in the section 4.1. We work with the matrix \((\Theta_{b,ij})_{ij \in E}\) given by (23). The figure 7 shows the density results on an unstructured triangle mesh having approximately \(10^6\) cells. For the comparative, we also plot the result the standard first-order scheme.

![Figure 7: Density profile at time \(t = 0.15\) on unstructured triangle mesh (\(\approx 10^6\) cells) for the double mach reflection of a strong shock problem. On the left, standard first-order approximation, on the right second-order approximation given by the scheme (17) endowed with the \((\Theta_{b,ij})_{ij \in E}\) matrices given by (23).](image)

The matrices \((\Theta_{a,ij})_{ij \in E}\) do not work for this test case. The scheme results are stable and accurate with the matrices \((\Theta_{b,ij})_{ij \in E}\). Due to the boundaries conditions imposed for this test case, the criteria \([DEI]\) is not negative even for the usual first-order scheme. As a consequence, we do not plot it. The MOOD process was applied at most on \(10^{-4}\%\) of the total cell numbers.

4.5 A mach 3 wind tunnel with a step

We study a diatomic Mach 3 perfect gas flow in a wind tunnel containing a step on the bottom as introduced in [78]. At the initial time the flow verify \(\rho = 1.4\), \((u, v) = (3, 0)\), \(p = 1\). On the left we impose a Dirichlet boundary condition with the initial condition values. On the top, on the bottom and on the step, we fix reflection boundaries conditions. On the right, as the gradients are assumed to be equal to zero, we prescribe homogeneous Neumann boundaries conditions. Under these conditions, the corner in the step is singular and the flow develops a rarefaction wave. The final time is 4. We use the parameters \((\lambda, \Delta t)\) described in the section 4.1 and the \((\Theta_{b,ij})_{ij \in E}\) matrices. The figure 8 reports the density results for the second-order scheme (17) on an unstructured triangle mesh having approximately \(10^5\) cells. For the comparative, we also plot the result the standard first-order scheme.

![Figure 8: Density results for the second-order scheme (17) on an unstructured triangle mesh having approximately \(10^5\) cells.](image)
Figure 8: Density profile at time $t = 4$ on unstructured triangle mesh ($\approx 10^5$ cells) for the Mach 3 wind tunnel with a step problem. On the left, standard first-order approximation, on the right second-order approximation given by the scheme (17) endowed with the $(\Theta_{b,ij})_{ij \in E}$ matrices given by (23).

As for the double mach reflection of a strong shock problem, the here boundaries conditions involve a positive values of the criterion $|DEI|$, even for the usual first-order scheme. As a consequence, we do not plot it. The matrices $(\Theta_{a,ij})_{ij \in E}$ do not work for this test case. The scheme results are stable and accurate with the matrices $(\Theta_{b,ij})_{ij \in E}$. The MOOD process was applied at most on $2 \cdot 10^{-3}\%$ of the total cell numbers.

5 Conclusion

In this work, we have introduced a class of explicit unlimited second-order schemes defined on two dimensional unstructured meshes. If a dissipation inequality is satisfied and under an appropriate choice of the artificial numerical coefficient and the time step, the unlimited schemes satisfy a global discrete entropy stability. From a numerical point of view, we have proposed two particular second-order discretizations. Depending on problems, one discretization may yields good results or does not work. The other discretization is more robust and provide accurate results for continuous and discontinuous solutions. The study of this second-order discretization could be the purpose of a future work.

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References


