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AN ESTIMATE FOR AN ELLIPTIC EQUATION IN DIMENSION 4.

SAMY SKANDER BAHOURA

ABSTRACT. We give a uniform estimate for solutions of prescribed scalar curvature type equation in dimension 4.

1. INTRODUCTION AND MAIN RESULT

We are on a Riemannian manifold (M, g) of dimension $n = 4$. In this paper we denote $\Delta_g = -\nabla^j(\nabla_j)$ the Laplace-Beltrami operator and $N = \frac{2n}{n-2} = 4$ the critical Sobolev exponent in dimension 4.

We consider the following equation (of type prescribed scalar curvature)

$$(1) \quad \Delta_g u + hu = Vu^3, \quad u > 0.$$

Where V is a function and h is a smooth bounded function such that $h \leq \frac{S_g}{6}$ with S_g the scalar curvature of (M, g) and h is bounded in C^1 norm by $h_0 \in \mathbb{R}^+$, a number and $h \neq \frac{S_g}{6}$.

With the previous conditions, this equation arise in Physics and in Astronomy. Equations of this type were studied by many authors, see, [1-23]. For $a, b, A > 0$, we consider $u > 0$ solution of the previous equation relative to V (a Lipschitz function) with the following conditions:

$$0 < a \leq V \leq b < +\infty, \quad \text{and} \quad \|\nabla V\|_\infty \leq A.$$

Our main result is:

Theorem 1.1. *For all $a, b, m > 0$, $A \rightarrow 0$ and all compact K of M , there is a positive constant $c = c(a, b, A, h_0, m, K, M, g)$ such that:*

$$\sup_K u \leq c \quad \text{if} \quad \inf_M u \geq m > 0.$$

A consequence of this theorem is: if we take two sequences (u_i, V_i) with u_i solution of (1) relative to V_i with the previous conditions for $a, b > 0$ and $A = A_i \rightarrow 0$, we have the following implicit Harnack inequality:

$$\sup_K u_i \leq c(a, b, h_0, (A_i)_i, \inf_M u_i, K, M, g),$$

and, the function c change if we consider other sequences (v_i, W_i) with the same hypothesis for W_i (Lipschitz and the new Lipschitz constant $B_i \rightarrow 0$).

2. PROOF OF THE THEOREM.

Let us consider $x_0 \in M$, by a conformal change of the metric $\tilde{g} = \varphi^{4/(n-2)}g = \varphi^2g$ with $\varphi > 0$ we can consider the equation:

$$(2) \quad \Delta_{\tilde{g}} u + R_{\tilde{g}} u = Vu^3 + (R_g - h)\varphi^{-2}u, \quad u > 0.$$

with,

$$\text{Ricci}_{\tilde{g}}(x_0) = 0.$$

Here; $R_g = \frac{1}{6}S_g$ and $R_{\tilde{g}} = \frac{1}{6}S_{\tilde{g}}$, here we have $R_g - h \geq 0$, this function is smooth and uniformly bounded in C^1 norm.

See the computations in previous papers [3,8], Also, we use the notations of the papers [5,7,8].

1) *Blow-up analysis*: we argue by contradiction and we suppose that sup is not bounded. Let $x_0 \in M$.

We assume that for $a, b, m > 0$ and $A \rightarrow 0$:

$\forall c, R > 0 \exists u_{c,R}$ solution of (2) such that:

$$R^{n-2} \sup_{B(x_0, R)} u_{c,R} \geq c \text{ and } u_{c,R} \geq m > 0, \quad (H)$$

Proposition 2.1. *There exist a sequence of points $(y_i)_i$, $y_i \rightarrow x_0$ and two sequences of positive real number $(l_i)_i, (L_i)_i$, $l_i \rightarrow 0$, $L_i \rightarrow +\infty$, such that if we consider $v_i(y) = \frac{u_i[\exp_{y_i}(y)]}{u_i(y_i)}$, we have:*

$$i) \quad 0 < v_i(y) \leq \beta_i \leq 2, \quad \beta_i \rightarrow 1.$$

$$ii) \quad v_i(y) \rightarrow \frac{1}{1 + |y|^2}, \text{ uniformly on every compact set of } \mathbb{R}^n.$$

$$iii) \quad l_i[u_i(y_i)] \rightarrow +\infty$$

Proof: see [3,8].

2) *Polar coordinates and "moving-plane" method*

Let,

$$w_i(t, \theta) = e^t \bar{u}_i(e^t, \theta) = e^t u_i \circ \exp_{y_i}(e^t \theta), \text{ and } a(y_i, t, \theta) = \log J(y_i, e^t, \theta).$$

Lemma 2.2. *The function w_i is solution of:*

$$(3) \quad -\partial_{tt} w_i - \partial_t a \partial_t w_i + \Delta_\theta w_i + c w_i = V_i w_i^3 + (R_g - h) \varphi^{-2} e^{2t} w_i,$$

with,

$$c = c(y_i, t, \theta) = 1 + \partial_t a + R_g e^{2t}.$$

Proof of the lemma, see [5,7,8].

$$\text{Now we have, } \partial_t a = \frac{\partial_t b_1}{b_1}, \quad b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0,$$

We can write,

$$-\frac{1}{\sqrt{b_1}} \partial_{tt} (\sqrt{b_1} w_i) + \Delta_\theta w_i + [c(t) + b_1^{-1/2} b_2(t, \theta)] w_i = V_i w_i^3 + (R_g - h) \varphi^{-2} e^{2t} w_i,$$

$$\text{where, } b_2(t, \theta) = \partial_{tt} (\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}} \partial_{tt} b_1 - \frac{1}{4(b_1)^{3/2}} (\partial_t b_1)^2.$$

Let,

$$\tilde{w}_i = \sqrt{b_1} w_i,$$

we have:

Lemma 2.3. *The function \tilde{w}_i is solution of:*

$$\begin{aligned} & -\partial_{tt}\tilde{w}_i + \Delta_\theta(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)\tilde{w}_i = \\ (4) \quad & = V_i \left(\frac{1}{b_1} \right) \tilde{w}_i^3 + (R_g - h)\varphi^{-2}e^{2t}\tilde{w}_i, \end{aligned}$$

where, $c_2 = \left[\frac{1}{\sqrt{b_1}} \Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2 \right]$.

Proof of the lemma, see [5,7,8].

We have,

$$c(y_i, t, \theta) = 1 + \partial_t a + R_g e^{2t}, \quad (\alpha_1)$$

$$b_2(t, \theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}} \partial_{tt} b_1 - \frac{1}{4(b_1)^{3/2}} (\partial_t b_1)^2, \quad (\alpha_2)$$

$$c_2 = \left[\frac{1}{\sqrt{b_1}} \Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2 \right], \quad (\alpha_3)$$

Then,

$$\partial_t c(y_i, t, \theta) = \partial_{tt} a,$$

by proposition 2.1,

$$|\partial_t c_2| + |\partial_t b_1| + |\partial_t b_2| + |\partial_t c| \leq K_1 e^{2t}.$$

We have for $\lambda_i = -\log u_i(y_i)$,

$$w_i(2\xi_i - t, \theta) = w_i[(\xi_i - t + \xi_i - \lambda_i - 2) + (\lambda_i + 2)],$$

Thus,

$$w_i(2\xi_i - t, \theta) = e^{[2(\xi_i - t + \xi_i - \lambda_i - 2)]/2} e^{2\nu_i} [\theta e^2 e^{(\xi_i - t) + (\xi_i - \lambda_i - 2)}] \leq 2e^2 = \bar{c}.$$

3) *The "moving-plane" method:*

Let ξ_i a real number, and suppose $\xi_i \leq t$. We set $t^{\xi_i} = 2\xi_i - t$ and $\tilde{w}_i^{\xi_i}(t, \theta) = \tilde{w}_i(t^{\xi_i}, \theta)$.

We have,

$$\begin{aligned} & -\partial_{tt}\tilde{w}_i^{\xi_i} + \Delta_\theta(\tilde{w}_i^{\xi_i}) + 2\nabla_\theta(\tilde{w}_i^{\xi_i}) \cdot \nabla_\theta \log(\sqrt{b_1}) \tilde{w}_i^{\xi_i} + [c(t^{\xi_i}) + b_1^{-1/2}(t^{\xi_i}, \cdot) b_2(t^{\xi_i}) - c_2^{\xi_i}] \tilde{w}_i^{\xi_i} = \\ & = V_i \left(\frac{1}{b_1^{\xi_i}} \right) (\tilde{w}_i^{\xi_i})^3 + \\ & + (((R_g - h)\varphi^{-2})e^{2t})^{\xi_i} \tilde{w}_i^{\xi_i}, \end{aligned}$$

By using the same arguments than in [3, 5, 7, 8], we have:

Proposition 2.4. *We have for $\lambda_i = -\log u_i(y_i)$:*

$$1) \quad \tilde{w}_i(\lambda_i, \theta) - \tilde{w}_i(\lambda_i + 4, \theta) \geq \tilde{k} > 0, \quad \forall \theta \in \mathbb{S}_3.$$

For all $\beta > 0$, there exist $c_\beta > 0$ such that:

$$2) \quad \frac{1}{c_\beta} e^{(n-2)t/2} \leq \tilde{w}_i(\lambda_i + t, \theta) \leq c_\beta e^{(n-2)t/2}, \quad \forall t \leq \beta, \quad \forall \theta \in \mathbb{S}_3.$$

We set,

$$\bar{Z}_i = -\partial_{tt}(\dots) + \Delta_\theta(\dots) + 2\nabla_\theta(\dots) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)(\dots)$$

Remark: In the operator \bar{Z}_i , by using the proposition 3, the coefficient $c + b_1^{-1/2}b_2 - c_2$ satisfies:

$$c + b_1^{-1/2}b_2 - c_2 \geq k' > 0, \text{ pour } t \ll 0,$$

it is fundamental if we want to apply Hopf maximum principle.

Goal:

Set $\bar{w}_i = \tilde{w}_i - \frac{m}{2}e^t$. Like in [8] we have the some properties for \bar{w}_i , we have:

Lemma 2.5. *There is $\nu < 0$ such that for $\lambda \leq \nu$:*

$$\bar{w}_i^\lambda(t, \theta) - \bar{w}_i(t, \theta) \leq 0, \forall (t, \theta) \in [\lambda, t_i] \times \mathbb{S}_3.$$

Let ξ_i be the following real number,

$$\xi_i = \sup\{\lambda \leq \lambda_i + 2, \bar{w}_i^{\xi_i}(t, \theta) - \bar{w}_i(t, \theta) \leq 0, \forall (t, \theta) \in [\xi_i, t_i] \times \mathbb{S}_3\}.$$

Like in [3,5,7,8], we have elliptic second order operator. Here it is \bar{Z}_i , the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 0, \text{ if } \bar{w}_i^{\xi_i} - \bar{w}_i \leq 0.$$

Clearly, we have:

Lemma 2.6.

$$b_1(y_i, t, \theta) = 1 - \frac{1}{3}\tilde{Ricci}_{y_i}(\theta, \theta)e^{2t} + \dots,$$

$$R_{\bar{g}}(e^t\theta) = R_{\bar{g}}(y_i) + \langle \nabla R_{\bar{g}}(y_i) | \theta \rangle e^t + \dots$$

We have:

$$\partial_t[(R_g - h)\varphi^{-2}e^{2t}] \geq -\epsilon_i e^{2t}, \epsilon_i \rightarrow 0.$$

Thus,

$$[(R_g - h)\varphi^{-2}e^{2t}] - [(R_g - h)\varphi^{-2}e^{2t}]^{\xi_i} \geq -\epsilon_i(e^{2t} - e^{2t^{\xi_i}})$$

Now, we write:

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq \tilde{A}_i(e^t - e^{t^{\xi_i}})(\tilde{w}_i^{\xi_i})^3 + V_i(b_1^{\xi_i})^{-1}(\tilde{w}_i^3 - (\tilde{w}_i^{\xi_i})^3) + o(1)e^{2t}(e^t - e^{t^{\xi_i}}) + o(1)\tilde{w}_i^{\xi_i}(e^{2t} - e^{2t^{\xi_i}})$$

We can write,

$$e^{2t} - e^{2t^{\xi_i}} = (e^t - e^{t^{\xi_i}})(e^t + e^{t^{\xi_i}}) \leq 2e^t(e^t - e^{t^{\xi_i}}).$$

Thus,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 4e\tilde{A}_i(e^t - e^{t^{\xi_i}})(\tilde{w}_i^{\xi_i})^2 + V_i(b_1^{\xi_i})^{-1/2}[(\tilde{w}_i^{\xi_i})^3 - \tilde{w}_i^3] + o(1)e^{2t}(e^t - e^{t^{\xi_i}}) + o(1)e^t\tilde{w}_i^{\xi_i}(e^t - e^{t^{\xi_i}}).$$

But,

$$0 < \tilde{w}_i^{\xi_i} \leq 2e, \quad \tilde{w}_i \geq \frac{m}{2}e^t \quad \text{and} \quad \tilde{w}_i^{\xi_i} - \tilde{w}_i \leq \frac{m}{2}(e^{t^{\xi_i}} - e^t),$$

and,

$$(\tilde{w}_i^{\xi_i})^3 - \tilde{w}_i^3 = (\tilde{w}_i^{\xi_i} - \tilde{w}_i)[(\tilde{w}_i^{\xi_i})^2 + \tilde{w}_i^{\xi_i}\tilde{w}_i + \tilde{w}_i^2] \leq (\tilde{w}_i^{\xi_i} - \tilde{w}_i)(\tilde{w}_i^{\xi_i})^2 + (\tilde{w}_i^{\xi_i} - \tilde{w}_i)\frac{m^2e^{2t}}{4} + (\tilde{w}_i^{\xi_i} - \tilde{w}_i)\frac{m}{2}e^t\tilde{w}_i^{\xi_i},$$

then,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq \left[(\bar{w}_i^{\xi_i})^2 \left[\frac{am}{4} - 4eA_i \right] + \left[\frac{am^3}{16} - o(1) \right] e^{2t} + \left[\frac{am^2}{8} - o(1) \right] e^t \bar{w}_i^{\xi_i} \right] (e^{t\xi_i} - e^t) \leq 0.$$

If we use the Hopf maximum principle, we obtain (like in [3,5,7,8]):

$$\min_{\theta \in \mathbb{S}_3} w_i(t_i, \theta) \leq \max_{\theta \in \mathbb{S}_3} w_i(2\xi_i - t_i, \theta),$$

we can write (by using the proposition 2.4):

$$l_i u_i(y_i) \leq c,$$

Contradiction.

REFERENCES

- [1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag 1998
- [2] S.S.Bahoura. Différentes estimations du $\sup u \times \inf u$ pour l'équation de la courbure scalaire prescrite en dimension $n \geq 3$. J. Math. Pures Appl. (9), 82 (1) (2003), pp. 43-66
- [3] S.S.Bahoura. Majorations du type $\sup u \times \inf u \leq c$ pour l'équation de la courbure scalaire sur un ouvert de \mathbb{R}^n , $n \geq 3$. J. Math. Pures. Appl.(9) 83 2004 no, 9, 1109-1150.
- [4] S.S.Bahoura. Estimations du type $\sup \times \inf$ sur une variété compacte. Bull.Sci.math. 130 (7), 2006, pp 624-636.
- [5] S.S.Bahoura. Estimations uniformes pour l'équation de Yamabe en dimensions 5 et 6. Journal.Func.Anal. 242 (2), 2007, pp 550-562.
- [6] S.S.Bahoura. Lower bounds for $\sup + \inf$ and $\sup * \inf$ and an extension of Chen-Lin result in dimension 3. Acta. Math. Scientia. 28 (4), 2008, pp 749-758.
- [7] S.S.Bahoura. Harnack inequalities for Yamabe type equations. Bull.Sci.Math. 133 (8), 2009, pp 875-892.
- [8] S.S.Bahoura. A uniform estimate for scalar curvature equation on manifolds of dimension 4. J.Math.Anal.Appl. Volume 388, 1, (2012) pp 386-392.
- [9] H. Brezis, YY. Li, I. Shafir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.
- [10] H.Brezis and F.Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V e^u$ in two dimensions, Commun Partial Differential Equations 16 (1991), 1223-1253.
- [11] L. Caffarelli, B. Gidas, J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. 42 (3) (1989) 271-297.
- [12] C.C.Chen.C.S.Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in \mathbb{R}^2 . Commun. Anal. Geom. 6, No.1, 1-19 (1998).
- [13] C-C.Chen, C-S. Lin. Estimates of the conformal scalar curvature equation via the method of moving planes. Comm. Pure Appl. Math. L(1997) 0971-1017.
- [14] B. Gidas, W-M. Ni, L. Nirenberg. Symmetry and Related Properties via the Maximum Principle. Commun. Math. Phys. 68, 209-243 (1979).
- [15] D. Gilbarg, N.S. Trudinger. Elliptic Partial Differential Equations of Second order, Berlin Springer-Verlag, Second edition, Grundlehern Math. Wiss.,224, 1983.
- [16] E. Hebey, Analyse non lineaire sur les Variétés, Editions Diderot.
- [17] E. Hebey, M. Vaugon. The best constant problem in the Sobolev embedding theorem for complete Riemannian manifolds. Duke Math. J. 79 (1995), no. 1, 235-279.
- [18] J.M. Lee, T.H. Parker. The Yamabe problem. Bull.Amer.Math.Soc (N.S) 17 (1987), no.1, 37 -91.
- [19] YY. Li. Harnack Type Inequality: the Method of Moving Planes. Commun. Math. Phys. 200,421-444 (1999).
- [20] YY. Li. Prescribing scalar curvature on \mathbb{S}_n and related Problems. C.R. Acad. Sci. Paris 317 (1993) 159-164. Part I: J. Differ. Equations 120 (1995) 319-410. Part II: Existence and compactness. Comm. Pure Appl.Math.49 (1996) 541-597.
- [21] YY. Li, L. Zhang. A Harnack type inequality for the Yamabe equation in low dimensions. Calc. Var. Partial Differential Equations 20 (2004), no. 2, 133-151.
- [22] F.C. Marques. A Priori Estimates for the Yamabe Problem in the non-locally conformally flat case. J. Diff. Geom. 71 (2005) 315-346.
- [23] I. Shafir. A sup+inf inequality for the equation $-\Delta u = V e^u$. C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.

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