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AN ESTIMATE FOR AN ELLIPTIC EQUATION IN DIMENSION 4.

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ABSTRACT. We give a uniform estimate for solutions of prescribed scalar curvature type equation in dimension 4.

1. INTRODUCTION AND MAIN RESULT

We are on a Riemannian manifold (M, g) of dimension $n = 4$. In this paper we denote $\Delta_g = -\nabla^j(\nabla_j)$ the Laplace-Beltrami operator and $N = \frac{2n}{n-2} = 4$ the critical Sobolev exponent in dimension 4.

We consider the following equation (of type prescribed scalar curvature)

$$\Delta_g u + hu = Vu^3, \ u > 0. \tag{1}$$

Where V is a function and h is a smooth bounded function such that $h \leq \frac{S_g}{6}$ with S_g the scalar curvature of (M, g) and h is bounded in C^1 norm by $h_0 \in \mathbb{R}^+$, a number and $h \not\equiv \frac{S_g}{6}$.

With the previous conditions, this equation arise in Physics and in Astronomy. Equations of this type were studied by many authors, see, [1-23]. For $a, b, A > 0$, we consider $u > 0$ solution of the previous equation relative to V(a Lipschitz function) with the following conditions:

$$0 < a \leq V \leq b < +\infty, \text{ and } ||\nabla V||_\infty \leq A.$$

Our main result is:

Theorem 1.1. *For all $a, b, m > 0$, $A \to 0$ and all compact K of M, there is a positive constant $c = c(a, b, A, h_0, m, K, M, g)$ such that:*

$$\sup_K u \leq c \text{ if } \inf_M u \geq m > 0.$$

A consequence of this theorem is: if we take two sequences (u_i, V_i) with u_i solution of (1) relative to V_i with the previous conditions for $a, b > 0$ and $A = A_i \to 0$, we have the following implicit Harnack inequality:

$$\sup_K u_i \leq c(a, b, h_0, (A_i)_i, \inf_M u_i, K, M, g),$$

and, the function c change if we consider other sequences (v_i, W_i) with the same hypothesis for W_i (Lipschitz and the new Lipschitz constant $B_i \to 0$).

2. PROOF OF THE THEOREM.

Let us consider $x_0 \in M$, by a conformal change of the metric $\tilde{g} = \varphi^{4/(n-2)} g = \varphi^2 g$ with $\varphi > 0$ we can consider the equation:

$$\Delta_{\tilde{g}} u + R_{\tilde{g}} u = Vu^3 + (R_g - h)\varphi^{-2} u, \ u > 0. \tag{2}$$

with,

$$Ricci_{\tilde{g}}(x_0) = 0.$$

Here; $R_g = \dfrac{1}{6} S_g$ and $R_{\tilde{g}} = \dfrac{1}{6} S_{\tilde{g}}$, here we have $R_g - h \geq 0$, this function is smooth and uniformly bounded in C^1 norm.

See the computations in previous papers [3,8], Also, we use the notations of the papers [5,7,8].

1) *Blow-up analysis:* we argue by contradiction and we suppose that $\sup$ is not bounded. Let $x_0 \in M$.

We assume that for $a, b, m > 0$ and $A \to 0$:

$\forall\ c, R > 0\ \exists\ u_{c,R}$ solution of (2) such that:

$$R^{n-2} \sup_{B(x_0,R)} u_{c,R} \geq c \text{ and } u_{c,R} \geq m > 0, \qquad (H)$$

Proposition 2.1. *There exist a sequence of points $(y_i)_i$, $y_i \to x_0$ and two sequences of positive real number $(l_i)_i, (L_i)_i$, $l_i \to 0$, $L_i \to +\infty$, such that if we consider $v_i(y) = \dfrac{u_i[\exp_{y_i}(y)]}{u_i(y_i)}$, we have:*

$$i) \qquad 0 < v_i(y) \leq \beta_i \leq 2,\ \beta_i \to 1.$$

$$ii) \qquad v_i(y) \to \frac{1}{1+|y|^2},\ \text{uniformly on every compact set of } \mathbb{R}^n.$$

$$iii) \qquad l_i[u_i(y_i)] \to +\infty$$

Proof: see [3,8].

2) *Polar coordinates and "moving-plane" method*

Let,

$$w_i(t,\theta) = e^t \bar{u}_i(e^t, \theta) = e^t u_i o \exp_{y_i}(e^t\theta),\ \text{and } a(y_i, t, \theta) = \log J(y_i, e^t, \theta).$$

Lemma 2.2. *The function w_i is solution of:*

$$(3) \qquad -\partial_{tt} w_i - \partial_t a \partial_t w_i + \Delta_\theta w_i + c w_i = V_i w_i^3 + (R_g - h)\varphi^{-2} e^{2t} w_i,$$

with,

$$c = c(y_i, t, \theta) = 1 + \partial_t a + R_{\tilde{g}} e^{2t}.$$

Proof of the lemma, see [5,7,8].

Now we have, $\partial_t a = \dfrac{\partial_t b_1}{b_1}$, $b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0$,

We can write,

$$-\frac{1}{\sqrt{b_1}} \partial_{tt}(\sqrt{b_1} w_i) + \Delta_\theta w_i + [c(t) + b_1^{-1/2} b_2(t,\theta)] w_i = V_i w_i^3 + (R_g - h)\varphi^{-2} e^{2t} w_i,$$

where, $b_2(t,\theta) = \partial_{tt}(\sqrt{b_1}) = \dfrac{1}{2\sqrt{b_1}} \partial_{tt} b_1 - \dfrac{1}{4(b_1)^{3/2}} (\partial_t b_1)^2$.

Let,

$$\tilde{w}_i = \sqrt{b_1} w_i,$$

we have:

Lemma 2.3. *The function $\tilde{w}_i$ is solution of:*

$$-\partial_{tt}\tilde{w}_i + \Delta_\theta(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i).\nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2} b_2 - c_2)\tilde{w}_i =$$

$$= V_i\left(\frac{1}{b_1}\right)\tilde{w}_i^3 + (R_g - h)\varphi^{-2}e^{2t}\tilde{w}_i, \tag{4}$$

where, $c_2 = [\frac{1}{\sqrt{b_1}}\Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2]$.

Proof of the lemma, see [5,7,8].

We have,

$$c(y_i, t, \theta) = 1 + \partial_t a + R_{\tilde{g}} e^{2t}, \qquad (\alpha_1)$$

$$b_2(t,\theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}}\partial_{tt} b_1 - \frac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2, \qquad (\alpha_2)$$

$$c_2 = [\frac{1}{\sqrt{b_1}}\Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2], \qquad (\alpha_3)$$

Then,

$$\partial_t c(y_i, t, \theta) = \partial_{tt} a,$$

by proposition 2.1,

$$|\partial_t c_2| + |\partial_t b_1| + |\partial_t b_2| + |\partial_t c| \leq K_1 e^{2t}.$$

We have for $\lambda_i = -\log u_i(y_i)$,

$$w_i(2\xi_i - t, \theta) = w_i[(\xi_i - t + \xi_i - \lambda_i - 2) + (\lambda_i + 2)],$$

Thus,

$$w_i(2\xi_i - t, \theta) = e^{[2(\xi_i - t + \xi_i - \lambda_i - 2)]/2} e^2 v_i[\theta e^2 e^{(\xi_i - t) + (\xi_i - \lambda_i - 2)}] \leq 2e^2 = \bar{c}.$$

3)*The "moving-plane" method:*

Let ξ_i a real number, and suppose $\xi_i \leq t$. We set $t^{\xi_i} = 2\xi_i - t$ and $\tilde{w}_i^{\xi_i}(t,\theta) = \tilde{w}_i(t^{\xi_i}, \theta)$.

We have,

$$-\partial_{tt}\tilde{w}_i^{\xi_i} + \Delta_\theta(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i^{\xi_i}).\nabla_\theta \log(\sqrt{b_1})\tilde{w}_i^{\xi_i} + [c(t^{\xi_i}) + b_1^{-1/2}(t^{\xi_i}, .)b_2(t^{\xi_i}) - c_2^{\xi_i}]\tilde{w}_i^{\xi_i} =$$

$$= V_i\left(\frac{1}{b_1^{\xi_i}}\right)(\tilde{w}_i^{\xi_i})^3 +$$

$$+(((R_g - h)\varphi^{-2})e^{2t})^{\xi_i}\tilde{w}_i^{\xi_i},$$

By using the same arguments than in [3, 5, 7, 8], we have:

Proposition 2.4. *We have for $\lambda_i = -\log u_i(y_i)$;*

$$1)\ \tilde{w}_i(\lambda_i, \theta) - \tilde{w}_i(\lambda_i + 4, \theta) \geq \tilde{k} > 0,\ \forall\ \theta \in \mathbb{S}_3.$$

For all $\beta > 0$, there exist $c_\beta > 0$ such that:

$$2)\ \frac{1}{c_\beta}e^{(n-2)t/2} \leq \tilde{w}_i(\lambda_i + t, \theta) \leq c_\beta e^{(n-2)t/2},\ \forall\ t \leq \beta,\ \forall\ \theta \in \mathbb{S}_3.$$

We set,

$$\bar{Z}_i = -\partial_{tt}(...) + \Delta_\theta(...) + 2\nabla_\theta(...).\nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2} b_2 - c_2)(...)$$

Remark: In the operator $\bar{Z}_i$, by using the proposition 3, the coeficient $c + b_1^{-1/2} b_2 - c_2$ satisfies:

$$c + b_1^{-1/2} b_2 - c_2 \geq k' > 0, \text{ pour } t << 0,$$

it is fundamental if we want to apply Hopf maximum principle.

<u>*Goal:*</u>

Set $\bar{w}_i = \tilde{w}_i - \frac{m}{2} e^t$. Like in [8] we have the some properties for $\bar{w}_i$, we have:

Lemma 2.5. *There is $\nu < 0$ such that for $\lambda \leq \nu$:*

$$\bar{w}_i^\lambda(t,\theta) - \bar{w}_i(t,\theta) \leq 0, \ \forall \ (t,\theta) \in [\lambda, t_i] \times \mathbb{S}_3.$$

Let ξ_i be the following real number,

$$\xi_i = \sup\{\lambda \leq \lambda_i + 2, \bar{w}_i^{\xi_i}(t,\theta) - \bar{w}_i(t,\theta) \leq 0, \ \forall \ (t,\theta) \in [\xi_i, t_i] \times \mathbb{S}_3\}.$$

Like in [3,5,7,8], we have elliptic second order operator. Here it is $\bar{Z}_i$, the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 0, \text{ if } \bar{w}_i^{\xi_i} - \bar{w}_i \leq 0.$$

Clearly, we have:

Lemma 2.6.

$$b_1(y_i, t, \theta) = 1 - \frac{1}{3} \tilde{R}icci_{y_i}(\theta,\theta) e^{2t} + \ldots,$$

$$R_{\tilde{g}}(e^t\theta) = R_{\tilde{g}}(y_i) + < \nabla R_{\tilde{g}}(y_i) | \theta > e^t + \ldots.$$

We have:

$$\partial_t[(R_g - h)\varphi^{-2} e^{2t}] \geq -\epsilon_i e^{2t}, \epsilon_i \to 0.$$

Thus,

$$[(R_g - h)\varphi^{-2} e^{2t}] - [(R_g - h)\varphi^{-2} e^{2t}]^{\xi_i} \geq -\epsilon_i (e^{2t} - e^{2t^{\xi_i}})$$

Now, we write:

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq \tilde{A}_i(e^t - e^{t^\xi})(\tilde{w}_i^{\xi_i})^3 + V_i(b_1^{\xi_i})^{-1}(\tilde{w}_i^3 - (\tilde{w}_i^{\xi_i})^3) + o(1)e^{2t}(e^t - e^{t^{\xi_i}}) + o(1)\tilde{w}_i^{\xi_i}(e^{2t} - e^{2t^{\xi_i}})$$

We can write,

$$e^{2t} - e^{2t^{\xi_i}} = (e^t - e^{t^{\xi_i}})(e^t + e^{t^{\xi_i}}) \leq 2e^t(e^t - e^{t^{\xi_i}}).$$

Thus,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 4e\tilde{A}_i(e^t - e^{t^{\xi_i}})(\tilde{w}_i^{\xi_i})^2 + V_i(b_1^{\xi_i})^{-1/2}[(\tilde{w}_i^{\xi_i})^3 - \tilde{w}_i^3] + o(1)e^{2t}(e^t - e^{t^{\xi_i}}) + o(1)e^t\tilde{w}_i^{\xi_i}(e^t - e^{t^{\xi_i}}).$$

But,

$$0 < \tilde{w}_i^{\xi_i} \leq 2e, \ \ \tilde{w}_i \geq \frac{m}{2} e^t \ \text{ and } \ \tilde{w}_i^{\xi_i} - \tilde{w}_i \leq \frac{m}{2}(e^{t^{\xi_i}} - e^t),$$

and,

$$(\tilde{w}_i^{\xi_i})^3 - \tilde{w}_i^3 = (\tilde{w}_i^{\xi_i} - \tilde{w}_i)[(\tilde{w}_i^{\xi_i})^2 + \tilde{w}_i^{\xi_i}\tilde{w}_i + \tilde{w}_i^2] \leq (\tilde{w}_i^{\xi_i} - \tilde{w}_i)(\tilde{w}_i^{\xi_i})^2 + (\tilde{w}_i^{\xi_i} - \tilde{w}_i)\frac{m^2 e^{2t}}{4} + (\tilde{w}_i^{\xi_i} - \tilde{w}_i)\frac{m}{2} e^t \tilde{w}_i^{\xi_i},$$

then,

$$\bar{Z}_i(\bar{w}_i^{\xi_i}-\bar{w}_i) \leq \left[(\tilde{w}_i^{\xi_i})^2[\frac{am}{4}-4eA_i]+[\frac{am^3}{16}-o(1)]e^{2t}+[\frac{am^2}{8}-o(1)]e^t\tilde{w}_i^{\xi_i} \right](e^{t^{\xi_i}}-e^t) \leq 0.$$

I fwe use the Hopf maximum principle, we obtain (like in [3,5,7,8]):

$$\min_{\theta\in\mathbb{S}_3} w_i(t_i,\theta) \leq \max_{\theta\in\mathbb{S}_3} w_i(2\xi_i-t_i,\theta),$$

we can write (by using the proposition 2.4):

$$l_i u_i(y_i) \leq c,$$

Contradiction.

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