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# Absolute continuity for semi-extremal holomorphic mappings

Virgile Tapiero

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## Abstract

We study the equilibrium measure  $\mu = T \wedge T$  of endomorphisms  $f$  of  $\mathbb{P}^2$  of degree  $d \geq 2$ , where  $T$  is the Green current of  $f$ . Dujardin proved that if  $\mu$  is absolutely continuous with respect to  $T$  then  $f$  has a minimal Lyapunov exponent [12]. We show the reverse implication under a local uniform assumption on unstable manifolds of the dynamical system.

**Keywords**— Equilibrium measure, Green current, Lyapunov exponents, normal forms, measurable partition, entropy. *MSC* : 37C45, 37F10, 32H50

## 1 Introduction

This article concerns the ergodic theory of holomorphic dynamical systems, it deals with Pesin's formula and its generalizations. We refer to the books [11, 24] for accounts on the dynamical properties of holomorphic mappings of  $\mathbb{P}^k$ . Let  $f$  be a rational map on  $\mathbb{P}^1$  of degree  $d \geq 2$  and let  $\omega_{\mathbb{P}^1}$  be the spherical  $(1, 1)$ -form on  $\mathbb{P}^1$ . The probability measure  $\mu := \lim_n \frac{1}{d^n} f^{n*} \omega_{\mathbb{P}^1}$  is the unique measure of maximal entropy, equal to  $\text{Log } d$ . It satisfies  $f^* \mu = d\mu$  and is mixing. The Lyapunov exponent of  $\mu$  satisfies  $\lambda_f \geq \frac{1}{2} \text{Log } d$  by Margulis-Ruelle inequality, and moreover

$$\lambda_f = \frac{1}{2} \text{Log } d \iff \mu \ll \text{Leb}_{\mathbb{P}^1} = \omega_{\mathbb{P}^1}. \quad (1)$$

The direct implication was proved by Ledrappier [20], the reverse one by Ledrappier-Strelcyn [21], the arguments rely on measurable partitions. In our context the reverse implication can be obtained by inserting the density of  $\mu$  in the relation  $f^* \mu = d\mu$ .

For a holomorphic map  $f$  of algebraic degree  $d \geq 2$  on  $\mathbb{P}^2$ , the probability measure  $\mu$  defined as  $\lim_n \frac{1}{d^{2n}} f^{n*} (\omega_{\mathbb{P}^2} \wedge \omega_{\mathbb{P}^2})$  is the unique measure of maximal entropy, equal to  $\text{Log } d^2$ . Here  $\omega_{\mathbb{P}^2}$  stands for the normalized Fubini-Study  $(1, 1)$ -form. Briend-Duval [7] proved that the Lyapunov exponents  $\lambda_1 \geq \lambda_2$  of  $\mu$  are larger than or equal to  $\frac{1}{2} \text{Log } d$ . This is not a consequence of Margulis-Ruelle inequality, the proof relies on more difficult arguments involving pluripotential theory. Another proof, involving local unstable manifolds and entropy, was given by De Thélin [10]. The counterpart of (1) takes the form

$$\lambda_1 = \lambda_2 = \frac{1}{2} \text{Log } d \iff \mu \ll \text{Leb}_{\mathbb{P}^2} = \omega_{\mathbb{P}^2} \wedge \omega_{\mathbb{P}^2}. \quad (2)$$

The direct implication was obtained by Dupont in [13] following the classical arguments of [20] and in [14, Theorem D] using a central limit theorem for the observable  $\text{Log det } df$ . One can prove the reverse implication by inserting the density of  $\mu$  in  $f^*\mu = d^2\mu$  and using the lower bound on the Lyapunov exponents. The (adapted) equivalence (2) is valid on every  $\mathbb{P}^k, k \geq 1$ .

The measure  $\mu$  is actually equal to  $T \wedge T$ , where  $T$  is the Green current  $\lim_n \frac{1}{d^n} f^{n*} \omega_{\mathbb{P}^2}$ . It satisfies  $f^*T = dT$  and can be interpreted as a singular invariant metric on  $\mathbb{P}^2$ . Berteloot-Loeb [5] proved that if  $T$  is smooth and positive on a non empty open set of  $\mathbb{P}^2$ , then  $f$  is a Lattès map :  $f$  can be lifted to an affine map on a complex torus  $\mathbb{C}^2/\Lambda$ , via a finite ramified covering  $\sigma : \mathbb{C}^2/\Lambda \rightarrow \mathbb{P}^2$ . Moreover  $\sigma^*T$  is equal to the standard hermitian form  $\frac{i}{2}dz \wedge d\bar{z} + \frac{i}{2}dw \wedge d\bar{w}$ . In particular one gets  $\omega_{\mathbb{P}^2} \ll T$  since  $T$  does not charge analytic sets, and one gets  $\lambda_1 = \lambda_2 = \frac{1}{2} \text{Log } d$  by  $f^*T = dT$ . Berteloot-Dupont [2] established later that if  $\mu = T \wedge T \ll \omega_{\mathbb{P}^2} \wedge \omega_{\mathbb{P}^2}$ , then  $T$  satisfies Berteloot-Loeb smoothness condition, the arguments use normal forms, the invariance of  $T$  and pluripotential theory. Those results extend to  $\mathbb{P}^k$ .

More recently, Dujardin studied the Fatou directions associated to endomorphisms of  $\mathbb{P}^k$  and proved the following relation between  $\mu, T$  and the smallest Lyapunov exponent of the measure  $\mu$  [12, Theorem 3.6] :

$$\mu \ll T \wedge \omega_{\mathbb{P}^2} \implies \lambda_2 = \frac{1}{2} \text{Log } d.$$

He also asked the question of the reverse implication :

**Question (Dujardin) :** does  $\lambda_2 = \frac{1}{2} \text{Log } d$  implies  $\mu \ll T \wedge \omega_{\mathbb{P}^2}$  ?

Theorem A below gives a positive answer to Dujardin's question assuming hypothesis  $\mathcal{H}_2$ , defined in Definition 2.1. Roughly speaking,  $\mathcal{H}_2$  asserts that there exists a collection  $\mathcal{A}$  of disjoint unstable manifolds centered at some  $c \in \mathbb{P}^2$  on which the iterated inverse branches of  $f$  and the Oseledec stable cocycle are uniformly controlled.

**Theorem A.** *Let  $f$  be a holomorphic map of degree  $d \geq 2$  on  $\mathbb{P}^2$ . Assume that the smallest Lyapunov exponent  $\lambda_2$  of the maximal entropy measure  $\mu$  is minimal equal to  $\frac{1}{2} \text{Log } d$ . If hypothesis  $\mathcal{H}_2$  holds, then  $\mu \ll T \wedge \omega_{\mathbb{P}^2}$ .*

Let us note that the equivalence (2) gives a positive answer to Dujardin's question when  $\lambda_1 = \lambda_2 = \frac{1}{2} \text{Log } d$ . Indeed, in that case,  $f$  is a Lattès map and  $\omega_{\mathbb{P}^2} \ll T$  as explained above. So we have to focus on the case  $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$ , those mappings are called *semi-extremal*, see [16]. For the present the only examples of semi-extremal mappings that we know preserve a pencil of lines  $\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ . Such mappings have been studied in general by Dupont-Taffin [17, Corollary 1.3]. They obtained that if  $\theta$  stands for the rational map induced by  $f$  on  $\mathbb{P}^1$ , then  $\lambda_\theta \in \{\lambda_1, \lambda_2\}$ ,  $\lambda_1 \geq \text{Log } d$  and  $\mu = T \wedge \pi^* \mu_\theta$ . In particular if  $\theta$  is a Lattès map on  $\mathbb{P}^1$  (hence  $\lambda_\theta = \frac{1}{2} \text{Log } d$  and  $\mu_\theta = \varphi \text{Leb}_{\mathbb{P}^1}$ ), then one gets the following precise description of  $\mu$  :

$$\mu = (\varphi \circ \pi)T \wedge \pi^* \omega_{\mathbb{P}^1} \ll T \wedge \omega_{\mathbb{P}^2}.$$

Our result does not give such a product structure for  $\mu$  with respect to  $T$ . Nonetheless, our hypothesis  $\mathcal{H}_2$  does not rely on the existence of an invariant pencil of lines but on less rigid dynamical aspects, that we hope should be less restrictive. Another open question is to find, if it exists, a geometric characterization of semi-extremal mappings.

The proof of Theorem A relies partly on the classical partition method employed in [20] and [13]. The novelty, in order to deal with semi-extremality, is to introduce normal forms for

the iterated inverse branches of  $f$ . We outline our arguments in Section 2.

**Remark 1.1.** *Our present results on  $\mathbb{P}^2$  extend to higher dimensions with the same strategy, the details are given in dimension 2 in this article for sake of simplicity. The statements on  $\mathbb{C}\mathbb{P}^k$ ,  $k \geq 2$ , are given in details at the end of the paper in Section 9.*

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## 2 Outline of proofs

### 2.1 The classical proof for two minimal exponents

We recall briefly the classical proof of

$$\lambda_1 = \lambda_2 = \frac{1}{2} \text{Log } d \implies \mu \ll \text{Leb}_{\mathbb{P}^2}. \quad (3)$$

We refer the reader to [19, 20] for the one dimensional case (on  $[0, 1]$  and on  $\mathbb{P}^1$ ) and to [13] for its extension to  $\mathbb{P}^k$ . The proof relies on the construction of a decreasing measurable partition  $\eta$ , generator of entropy, such that :

$$\forall n \geq 0, \quad \text{Log } d^{2n} = \int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\widehat{x}} \left( \widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}). \quad (4)$$

We use here the natural extension  $(\widehat{\mathbb{P}^2}, \widehat{f}, \widehat{\mu}) \xrightarrow{\pi_0} (\mathbb{P}^2, f, \mu)$  in order to work with an invertible dynamical system, see Section 8.2 for details. We denote  $\mathcal{B}(\widehat{\mathbb{P}^2})$  the  $\sigma$ -algebra of borel sets of  $\widehat{\mathbb{P}^2}$ . Given a measurable partition  $\zeta$  of  $\widehat{\mathbb{P}^2}$ ,  $\zeta_{\widehat{x}}$  is the atom of  $\zeta$  containing  $\widehat{x} \in \widehat{\mathbb{P}^2}$ . Let  $\mu_{\widehat{x}}$  denote the conditional measure of  $\mu$  on the atom  $\eta_{\widehat{x}}$ . The idea to prove (3) is to introduce for  $\widehat{\mu}$ -almost every  $\widehat{x} \in \widehat{\mathbb{P}^2}$  a probability measure  $p_{\widehat{x}}$  on  $\eta_{\widehat{x}}$  which is absolutely continuous with respect to  $\text{Leb}_{\mathbb{P}^2}$  and such that (4) is satisfied when  $\mu_{\widehat{x}}$  is replaced by  $p_{\widehat{x}}$ . Indeed in this case, for any  $n \geq 0$ , we have the equality :

$$\int_{\widehat{\mathbb{P}^2}} -\text{Log } p_{\widehat{x}} \left( \widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) = \text{Log } d^{2n} = \int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\widehat{x}} \left( \widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}). \quad (5)$$

It implies by Jensen inequality :

$$0 = \int_{\widehat{\mathbb{P}^2}} \text{Log } \frac{p_{\widehat{x}}}{\mu_{\widehat{x}}} \left( \widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) \leq \text{Log } \int_{\widehat{\mathbb{P}^2}} \frac{p_{\widehat{x}}}{\mu_{\widehat{x}}} \left( \widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}). \quad (6)$$

Other properties of  $\eta$  (see for instance Theorem 3.9) allow to prove  $\int \frac{p_{\widehat{x}}}{\mu_{\widehat{x}}} \left( \widehat{f}^{-n} \eta \right)_{\widehat{x}} d\widehat{\mu}(\widehat{x}) = 1$ , hence (6) is an equality. The strict concavity of  $\text{Log}$  then implies  $p_{\widehat{x}} \left( \widehat{f}^{-n} \eta \right)_{\widehat{x}} = \mu_{\widehat{x}} \left( \widehat{f}^{-n} \eta \right)_{\widehat{x}}$ . Finally, the generating property of  $\eta$  yields  $p_{\widehat{x}} = \mu_{\widehat{x}}$ , hence  $\widehat{\mu} = \int p_{\widehat{x}} d\widehat{\mu}(\widehat{x}) =: \widehat{p}$  on  $\widehat{\mathbb{P}^2}$ . Using the fact that  $p_{\widehat{x}}$  is absolutely continuous (which has not be used so far), we get as desired

$$\mu = (\pi_0)_* \widehat{p} \ll \text{Leb}_{\mathbb{P}^2}.$$

Practically, the measures  $p_{\hat{x}}$  are defined by :

$$\forall A \in \mathcal{B}(\widehat{\mathbb{P}^2}), p_{\hat{x}}(A) = \frac{1}{L(\hat{x})} \int_{\pi_0(A \cap \eta_{\hat{x}})} \prod_{i=1}^{+\infty} \frac{\det d_{x_{-i}} f}{\det d_{y_{-i}} f} d\text{Leb}(y_0), \quad (7)$$

where  $\hat{y} = (y_{-i})_i$  is given by the injectivity of  $\pi_0$  on the atoms of  $\eta$ , and  $L(\hat{x}) > 0$  ensures  $p_{\hat{x}}(\eta_{\hat{x}}) = 1$ . Using the change of variable formula for the Lebesgue measure, the decreasing property of  $\eta$  and the injectivity of  $f^n$  on  $\pi_0((\hat{f}^{-n}\eta)_{\hat{x}})$  (see Theorem 3.9) one obtains the fundamental formula

$$\int_{\widehat{\mathbb{P}^2}} -\text{Log } p_{\hat{x}}(\hat{f}^{-n}\eta)_{\hat{x}} d\hat{\mu}(\hat{x}) = \int_{\widehat{\mathbb{P}^2}} \text{Log } (\det d_{x_0} f^n) d\hat{\mu}(\hat{x}). \quad (8)$$

The right hand side is equal to  $2n\lambda_1 + 2n\lambda_2$  by classical ergodic theory. It is equal to  $\text{Log } d^{2n}$  when  $\lambda_1$  and  $\lambda_2$  are minimal, in this case (5) is satisfied and absolute continuity follows.

An important step is to establish the convergence of the integral (7). The problem is settled in [13, 20] by using a Pesin box  $\mathcal{A}$ . Roughly speaking  $\mathcal{A}$  is a disjoint union  $\bigsqcup_{\hat{z} \in \mathcal{T}} W^u(\hat{z}, \mathcal{R})$  of unstable manifolds, where  $\mathcal{T}$  is a subset of a fiber  $\pi_0^{-1}(c)$ . The unstable manifolds are defined using inverse branches  $f_{\hat{x}}^{-n}$ . A crucial property of  $\mathcal{A}$  is that  $\eta_{\hat{x}} \subset W^u(\hat{z}, \mathcal{R})$  for any  $\hat{x} \in W^u(\hat{z}, \mathcal{R})$ . Indeed, this inclusion implies the convergence of the infinite product  $y_0 \in \pi_0(\eta_{\hat{x}}) \mapsto \prod_{i=1}^{+\infty} \frac{\det d_{x_{-i}} f}{\det d_{y_{-i}} f}$ , see for instance [13, Lemme 3.3]. When  $\hat{x}$  does not belong to  $\mathcal{A}$ , Birkhoff ergodic theorem allows us to get back to  $\mathcal{A}$ . In our work we shall use Pesin boxes, see Section 3, but we will not have to prove the convergence of an infinite product, that step will indeed be contained in a normal form theorem for inverse branches.

## 2.2 Outline of the proof of Theorem A

We resume our proof of  $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d \implies \mu \ll T \wedge \omega_{\mathbb{P}^2}$  under  $\mathcal{H}_2$ . We start by proving the following theorem, whose first formula replaces Equation (8). The hypothesis  $\mathcal{H}_1$  is given in Definition 2.1, it is less restrictive than  $\mathcal{H}_2$ .

**Theorem B.** *Let  $f$  be a holomorphic map of degree  $d \geq 2$  on  $\mathbb{P}^2$ . Assume that the Lyapunov exponents of the maximal entropy measure  $\mu$  satisfy  $\lambda_1 > \lambda_2$  and that  $\mathcal{H}_1$  holds. There exist a measurable partition  $\eta$  of  $\mathbb{P}^2$  and a measurable family  $\hat{x} \mapsto q_{\hat{x}}$  of probability measures on  $\widehat{\mathbb{P}^2}$  supported on  $\eta_{\hat{x}}$  such that*

$$(\pi_0)_* q_{\hat{x}} \ll T \wedge \omega_{\mathbb{P}^2},$$

$$\forall n \geq 0, \int_{\widehat{\mathbb{P}^2}} -\text{Log } q_{\hat{x}}(\hat{f}^{-n}\eta)_{\hat{x}} d\hat{\mu}(x) = \text{Log } d^n + 2n\lambda_2.$$

The measurable partition  $\eta$  also satisfies the formula :

$$\forall n \geq 0, \int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\hat{x}}(\hat{f}^{-n}\eta)_{\hat{x}} d\hat{\mu}(x) = \text{Log } d^n + \text{Log } d^n.$$

Theorem B implies Theorem A when  $\lambda_2 = \frac{1}{2} \text{Log } d$ . Indeed, in this case, Formula (5) is satisfied replacing  $p_{\hat{x}}$  by  $q_{\hat{x}}$ . The same arguments than before (involving Jensen inequality and the properties of  $\eta$ ) then imply  $q_{\hat{x}} = \mu_{\hat{x}}$ . We deduce  $\mu \ll T \wedge \omega_{\mathbb{P}^2}$  in Section 6.2 from the fact that  $\mu$  and  $T \wedge \omega_{\mathbb{P}^2}$  do not charge the critical set.

Compared with the classical method, our novelty is to use normal forms for inverse branches  $f_{\widehat{x}}^{-n}$  to construct  $q_{\widehat{x}}$ . More precisely, by Theorem 8.2, there exist controlled holomorphic change of coordinates  $(Z_{\widehat{x}}, W_{\widehat{x}})$  such that the following diagram commutes

$$\begin{array}{ccc} B(x_{-n}, 2\eta_\varepsilon(\widehat{x}_{-n})) & \xleftarrow{f_{\widehat{x}}^{-n}} & B(x_0, 2\eta_\varepsilon(\widehat{x})) \\ \xi_{\widehat{x}_{-n}}=(Z_{\widehat{x}_{-n}}, W_{\widehat{x}_{-n}}) \downarrow & & \downarrow (Z_{\widehat{x}}, W_{\widehat{x}})=\xi_{\widehat{x}} \\ \mathbb{D}^2(\rho_\varepsilon(\widehat{x}_{-n})) & \xleftarrow{R_{n,\widehat{x}}} & \mathbb{D}^2(\rho_\varepsilon(\widehat{x})) \end{array} \quad (9)$$

where the second coordinate of  $R_{n,\widehat{x}}$  is linear and satisfies

$$R_{n,\widehat{x}}^{(2)}(z, w) = \beta_{n,\widehat{x}} w \quad , \quad e^{-n(\lambda_2+\varepsilon)} \leq |\beta_{n,\widehat{x}}| \leq e^{-n(\lambda_2-\varepsilon)}.$$

Using  $f^*T = dT$  and the diagram above, we obtain the change of variable formula :

$$(f_{\widehat{x}}^{-n})^* (T \wedge dd^c |W_{\widehat{x}_{-n}}|^2) = d^{-n} |\beta_{n,\widehat{x}}|^2 \times (T \wedge dd^c |W_{\widehat{x}}|^2).$$

We define the probability measures  $q_{\widehat{x}}$  on  $\eta_{\widehat{x}}$  by

$$q_{\widehat{x}} = \frac{(T \wedge dd^c |W_{\widehat{x}}|^2) \circ \pi_0}{(T \wedge dd^c |W_{\widehat{x}}|^2)(\pi_0(\eta_{\widehat{x}}))}. \quad (10)$$

The crucial part of our work is the proof of the first formula of Theorem B. It is important that  $\lambda_2$  comes in this formula without any  $e^{\pm n\varepsilon}$  error term (such terms appearing in the definition of  $\beta_{n,\widehat{x}}$ ), this is settled by Jensen inequality. Actually  $q_{\widehat{x}}$  is defined by (10) only when  $\widehat{x} \in \mathcal{A}$ . If  $\widehat{x} \notin \mathcal{A}$  we have to introduce an adapted formula, see Section 4.

## 2.3 Hypothesis $\mathcal{H}_1$ and $\mathcal{H}_2$

We fix once for all  $\varepsilon$  small with respect to the Lyapunov exponents  $\lambda_1 > \lambda_2$ . By Theorem 8.1, there exists of borel set  $A_{os}$  of full  $\mu$ -measure on which the unitary stable vector  $\vec{v}_s : A_{os} \rightarrow T\mathbb{P}^2$  is defined. We denote

$$\Delta(x, n) := \|d_x f^n \cdot \vec{v}_s(x)\|.$$

Let  $\mathcal{FN}_\varepsilon$  be the set of full  $\widehat{\mu}$ -measure provided by the normal form Theorem 8.2. In particular, the inverse branches

$$f_{\widehat{x}}^{-n} : B(x_0, 2\eta_\varepsilon(\widehat{x})) \rightarrow B(x_{-n}, 2\eta_\varepsilon(\widehat{x}_{-n}))$$

are defined for  $\widehat{x} \in \mathcal{FN}_\varepsilon$ . Let  $\beta_\varepsilon(\widehat{x})$  be the distortion of the change of coordinates  $\xi_{\widehat{x}}$  specified in the diagram (9). As in Briend's article [6], for every  $0 < r \leq 2\eta_\varepsilon(\widehat{x})$ , we define the unstable manifold by

$$W^u(\widehat{x}, r) := \left\{ \widehat{z} \in \widehat{\mathbb{P}}^2, \exists t \in B(x_0, r) : z_{-n} = f_{\widehat{x}}^{-n}(t), \forall n \in \mathbb{N} \right\}.$$

For  $\mu$ -almost every  $c \in \mathbb{P}^2$ , Theorem 3.5 (due to Briend) asserts that for  $r > 0$  small enough and  $\rho > 0$  large enough,  $\widehat{\mu} = \mu|_{B(c,r)} \otimes \mu_{\pi_0,c}$  on  $\bigsqcup_{\widehat{z} \in \mathcal{T}} W^u(\widehat{z}, r)$ , where  $\mathcal{T} \subset \pi_0^{-1}(c) \cap \{\eta_\varepsilon \geq 1/\rho, \beta_\varepsilon \leq \rho\}$ . We shall say that  $\mathcal{T}$  is a regular tree if  $\mu_{\pi_0,c}(\mathcal{T}) > 0$ .

**Definition 2.1.**

$\mathcal{H}_1$  : there exist a regular tree  $\mathcal{T}$  and  $\rho_0 > 0$  such that for every  $\hat{z} \in \mathcal{T}$

$$\mathcal{R}_{\hat{z}} := \inf\{ \eta_\varepsilon(\hat{x}), \hat{x} \in W^u(\hat{z}, \eta_\varepsilon(\hat{z})) \cap \mathcal{FN}_\varepsilon \} \geq \frac{1}{\rho_0}.$$

$\mathcal{H}_2$  :  $\mathcal{H}_1$  is satisfied and for every  $\hat{z} \in \mathcal{T}$ , there exists  $\Delta_{\hat{z}} > 1$  such that :

$$\forall n \geq 0, \forall \hat{x} \in W^u(\hat{z}, \eta_\varepsilon(\hat{z})) \cap A_{os}, \frac{1}{\Delta_{\hat{z}}} \leq \frac{\Delta(z_{-n}, n)}{\Delta(x_{-n}, n)} \leq \Delta_{\hat{z}}.$$

Let us explain why we introduce  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . The measure  $q_{\hat{x}}$  is built using the submersion  $W_{\hat{x}}$  defined on  $B(x_0, \eta_\varepsilon(\hat{x}))$ , but this ball does not contain  $\pi_0(\eta_{\hat{x}})$  in general. This induces difficulties to implement the classical partition method. This did not occur for the construction of  $p_{\hat{x}}$ , which used the function  $x \mapsto \det(d_x f)$  defined on  $\mathbb{P}^2$ . The hypothesis  $\mathcal{H}_1$  thus sets an uniform control for the function  $\hat{x} \mapsto \eta_\varepsilon(\hat{x})$  on the Pesin box  $\mathcal{A}$ .

Another difficulty, which did not appear for the family  $(p_{\hat{x}})_{\hat{x}}$ , is to check that  $q_{\hat{x}} = q_{\hat{y}}$  when  $\hat{x}$  and  $\hat{y}$  belong to the same atom of  $\eta$ . That property is fundamental to show Theorem 6.1. To get it we show that it suffices to control  $(n, \hat{x}) \mapsto \Delta(z_{-n}, n)/\Delta(x_{-n}, n)$  when  $\hat{x}$  runs over the unstable manifolds  $W^u(\hat{z}, \mathcal{R})$  of the Pesin box  $\mathcal{A}$ , see Proposition 4.5 (proved in Section 7). It leads us to the hypothesis  $\mathcal{H}_2$ .

## 3 Construction of Pesin boxes and partitions

### 3.1 Unstable manifolds and Buzzi's partition

The following proposition leads to the definition of  $\mathcal{P}$ -address.

**Proposition 3.1** (Buzzi [8, Section 4], see also Dupont [13, Section 2.4]). *There exists a partition  $\mathcal{P} = (\mathcal{P}_j)_{j \in \{1, \dots, N\}}$  of a full  $\mu$ -measure set of  $\mathbb{P}^2$ , whose atoms are open sets of  $\mathbb{P}^2$  and such that :*

1. *The map  $f$  is injective on each atom  $\mathcal{P}_j$ .*
2. *If  $\mathcal{P}_x$  denotes the atom of  $\mathcal{P}$  which contains  $x$ , then  $f_{\hat{x}}^{-n}(B(x_0, 2\eta_\varepsilon(\hat{x}))) \subset \mathcal{P}_{x_{-n}}$  for every  $n \geq 0$  and for  $\hat{\mu}$ -almost every  $\hat{x}$ .*

**Definition 3.2.**

1. *The  $\hat{f}$ -invariant set  $\hat{\mathcal{P}} := \bigcap_{n \in \mathbb{Z}} \hat{f}^n \left( \pi_0^{-1} \left( \bigcup_{j=1}^N \mathcal{P}_j \right) \right)$  has full  $\hat{\mu}$ -measure. We say that  $\hat{x}, \hat{y} \in \hat{\mathcal{P}}$  have the same  $\mathcal{P}$ -address if  $\mathcal{P}_{x_{-n}} = \mathcal{P}_{y_{-n}}$  for every  $n \geq 0$ . By Proposition 3.1, the  $\mathcal{P}$ -address stay constant on the unstable manifolds.*
2. *Let*

$$\Lambda_\varepsilon := \pi_0^{-1}(A_{os}) \cap \mathcal{FN}_\varepsilon \cap \hat{\mathcal{P}},$$

*it is a totally invariant set of full  $\mu$ -measure on which  $\vec{v}_s$ , the  $\mathcal{P}$ -address and the inverse branches  $f_{\hat{x}}^{-n}$  are defined.*

**Lemma 3.3.** *For any  $\hat{z} \in \Lambda_\varepsilon$  we have :*

1. *Let  $\hat{x} \in W^u(\hat{z}, 2\eta_\varepsilon(\hat{z})) \cap \Lambda_\varepsilon$  and  $n \geq 0$ . Then the maps  $f_{\hat{x}}^{-n}$  and  $f_{\hat{z}}^{-n}$  coincide on  $B(x_0, 2\eta_\varepsilon(\hat{x})) \cap B(z_0, 2\eta_\varepsilon(\hat{z}))$ .*

2. Let  $0 < r \leq 2\eta_\varepsilon(\widehat{z})$  and  $\widehat{x} \in W^u(\widehat{z}, r)$ . If  $\widehat{w} \in \widehat{\mathcal{P}}$  satisfies
- (a)  $\widehat{x}$  and  $\widehat{w}$  have the same  $\mathcal{P}$ -address,
  - (b)  $w_0 \in B(z_0, r)$ ,
- then  $\widehat{w} \in W^u(\widehat{z}, r)$ . If moreover  $w_0 = z_0$  then  $\widehat{w} = \widehat{z}$ .

**Proof :** By Proposition 3.1 and Definition 3.2 we have for every  $k \geq 0$  :

$$\mathcal{P}_{f_{\widehat{x}}^{-k}(t_0)} = \mathcal{P}_{x_{-k}} = \mathcal{P}_{z_{-k}} = \mathcal{P}_{f_{\widehat{z}}^{-k}(t_0)}. \quad (11)$$

Let  $t_0 \in B(x_0, 2\eta_\varepsilon(\widehat{x})) \cap B(z_0, 2\eta_\varepsilon(\widehat{z}))$ . If  $f_{\widehat{x}}^{-j}(t_0) = f_{\widehat{z}}^{-j}(t_0)$  then  $f(f_{\widehat{x}}^{-(j+1)}(t_0)) = f(f_{\widehat{z}}^{-(j+1)}(t_0))$ . By using (11) and the fact that  $f$  is injective on the atoms of  $\mathcal{P}$  (see Proposition 3.1), we get  $f_{\widehat{x}}^{-(j+1)}(t_0) = f_{\widehat{z}}^{-(j+1)}(t_0)$ . An induction thus implies  $f_{\widehat{x}}^{-j} = f_{\widehat{z}}^{-j}$  for every  $j \geq 0$ . The second item can be proved similarly.  $\square$

## 3.2 Pesin boxes and Briend's theorem

**Definition 3.4.** A Pesin box is a quintuplet  $(P, r, \rho, \mathcal{T}, c)$  where  $c \in \mathbb{P}^2$  and

1.  $\mathcal{T}$  is a borel subset of  $\Lambda_\varepsilon \cap \{\pi_0 = c\}$  such that  $2\eta_\varepsilon(\widehat{z}) \geq r > 0$  for every  $\widehat{z} \in \mathcal{T}$ ,
2.  $\beta_\varepsilon(\widehat{z}) \leq \rho$  for every  $\widehat{z} \in \mathcal{T}$ ,
3.  $P = \bigsqcup_{\widehat{z} \in \mathcal{T}} W^u(\widehat{z}, r)$ .

In the third item the unstable manifolds are pairwise disjoint since the  $\mathcal{P}$ -address is constant on unstable manifolds and  $f$  is injective on the atoms of  $\mathcal{P}$ , see the proof of Lemma 3.3. For any  $\widehat{z} \in \mathcal{T}$ , let us define

$$\varphi_{\widehat{z}} : y_0 \in B(c, r) \mapsto (f_{\widehat{z}}^{-n}(y_0))_{n \in \mathbb{Z}} \in \widehat{\mathbb{P}}^2,$$

where  $f_{\widehat{z}}^n = f^n$  if  $n \geq 0$ . For any  $\widehat{y} \in P$ , let  $\tilde{\pi}(\widehat{y})$  be the unique  $\widehat{z} \in \mathcal{T}$  such that  $\varphi_{\widehat{z}}(y_0) = \widehat{y}$ . Then the Pesin box  $P$  is homeomorphic to  $B(c, r) \times \mathcal{T}$  via the following continuous bijective mappings, inverse map for one to the other :

$$\Psi : \begin{cases} B(c, r) \times \mathcal{T} & \longrightarrow P \\ (y_0, \widehat{z}) & \longmapsto \varphi_{\widehat{z}}(y_0) \end{cases}, \quad \Theta : \begin{cases} P & \longrightarrow B(c, r) \times \mathcal{T} \\ \widehat{y} & \longmapsto (y_0, \tilde{\pi}(\widehat{y})) \end{cases}$$

In particular the Pesin box  $P$  is a borel set of  $\widehat{\mathbb{P}}^2$ . The continuity of  $\Psi$  and  $\Theta$  can be verified by hands using the product topology of  $\widehat{\mathbb{P}}^2$ , they are implicit in [6].

By Rokhlin's theorem [22, §3.1],  $\widehat{\mu}$  admits a decomposition on the fibers of  $\pi_0$  :

$$\forall A \in \mathcal{B}(\widehat{\mathbb{P}}^2), \quad \widehat{\mu}(A) = \int_{\mathbb{P}^2} \mu_{\pi_0, c}(A \cap T_c) \, d\mu(c), \quad (12)$$

where  $\mu_{\pi_0, c}$  is the conditional measure of  $\widehat{\mu}$  on the fiber  $T_c := \{\pi_0 = c\}$ . The measure  $\widehat{\mu}$  is a product on Pesin boxes :

**Theorem 3.5** (Briend [6, Theorem 4.1]). *For  $\mu$ -almost every  $c \in \mathbb{P}^2$ , there exists  $r_c > 0$  and  $\rho_c > 0$  such that :*

$$\Theta_*(\widehat{\mu}|_P) = \mu|_{B(c, r)} \otimes \mu_{\pi_0, c}|_{\mathcal{T}},$$

for every Pesin box  $(P, r, \rho, \mathcal{T}, c)$  satisfying  $r \leq r_c$  and  $\rho \geq \rho_c$ .



### 3.3 A special Pesin box $\mathcal{A}$

In this section we construct a special Pesin box  $\mathcal{A}$  of  $\widehat{\mu}$ -positive measure on which the dynamic has good properties. We use it to construct the decreasing partition  $\eta$  in Section 3.4 and the family of conditional measures  $q_{\widehat{x}}$  under the hypothesis  $\mathcal{H}_1$  and  $\mathcal{H}_2$  in Section 5. We proceed in several steps. First let  $Br_\varepsilon$  be the set of full  $\mu$ -measure of points  $c \in \text{Supp}(\mu)$  that satisfy Theorem 3.5. Then for  $\rho > 0$  we define :

$$V := \{\widehat{z} \in \Lambda_\varepsilon : \eta_\varepsilon(\widehat{z}) \geq 1/\rho, \beta_\varepsilon(\widehat{z}) \leq \rho\}$$

which is of positive  $\widehat{\mu}$ -measure for  $\rho > 0$  large enough. Thus applying (12) with  $A = V \cap Br_\varepsilon$ , there exists a set of positive  $\mu$ -measure  $E \subset Br_\varepsilon$  such that for every  $c \in E$ , the conditional measure  $\mu_{\pi_0, c}$  charges the set  $V \cap T_c$ . Then for every  $c \in E$  and  $r \in ]0, \frac{1}{\rho}]$  we define

$$P_{c,r} := \bigsqcup_{\widehat{z} \in V \cap T_c} W^u(\widehat{z}, r).$$

We deduce that  $(P_{c,r}, r, \rho, V \cap T_c, c)$  is a Pesin box with respect to the Definition 3.4. Moreover by Theorem 3.5, we get for every  $\rho \geq \rho_c$  and  $0 < r \leq r_c$  :

$$\widehat{\mu}(P_{c,r}) = \mu(B(c, r)) \times \mu_{\pi_0, c}(V \cap T_c) > 0. \quad (13)$$

**Lemma 3.6.** [13, Lemma 2.2] *There exists  $S \subset [1/4\rho, 1/2\rho]$  a borel set of full Lebesgue measure such that for every  $s \in S$ , there exists  $\gamma : \widehat{\mathbb{P}}^2 \rightarrow ]0, 1]$  a measurable function satisfying*

$$\forall \widehat{x} - \widehat{\mu} - a.e., \forall n \in \mathbb{N}, |d(x_{-n}, c) - s| \geq \gamma(\widehat{x})e^{-n\varepsilon}.$$

Let us fix  $\mathcal{R} \in S$ , set  $\mathcal{T} := V \cap T_c$  and define

$$\mathcal{A} := P_{c, \mathcal{R}} = \bigsqcup_{\widehat{z} \in \mathcal{T}} W^u(\widehat{z}, \mathcal{R}) \subset \pi_0^{-1}(B(c, \mathcal{R})). \quad (14)$$

We still have  $\widehat{\mu}(\mathcal{A}) > 0$  by (13). The hypothesis  $\mathcal{H}_1$  and  $\mathcal{H}_2$  allow to get much properties that will be crucial to define the measures  $q_{\widehat{x}}$ .

**Proposition 3.7** (Use of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ ).

1. *Under hypothesis  $\mathcal{H}_1$ , we can assume that for every  $\widehat{x} \in \mathcal{A} \cap \Lambda_\varepsilon$  :*

$$2\mathcal{R} \leq \eta_\varepsilon(\widehat{x}) \text{ and } \beta_\varepsilon(\widehat{x}) \leq \frac{1}{2\mathcal{R}}.$$

2. *Under hypothesis  $\mathcal{H}_2$ , we can assume moreover that for every  $\widehat{z} \in \mathcal{T}$  there exists  $\Delta_{\widehat{z}} > 1$  such that :*

$$\forall n \geq 0, \forall \widehat{x} \in W^u(\widehat{z}, \mathcal{R}) \cap \Lambda_\varepsilon, \frac{1}{\Delta_{\widehat{z}}} \leq \frac{\Delta(z_{-n}, n)}{\Delta(x_{-n}, n)} \leq \Delta_{\widehat{z}}.$$

**Proof :** We focus on the first item, the second one is get similarly. One can assume  $\rho \geq \rho_0$  in the preceding discussion, where  $\rho_0$  is provided by  $\mathcal{H}_1$  in Definition 2.1. Then by definition of  $\mathcal{H}_1$  and  $\mathcal{R} \in S \subset [1/4\rho, 1/2\rho]$ , one has for every  $\widehat{x} \in W^u(\widehat{z}, \mathcal{R}) \cap \mathcal{FN}_\varepsilon$  :

$$\eta_\varepsilon(\widehat{x}) \geq \mathcal{R}_{\widehat{z}} \geq \frac{1}{\rho_0} \geq \frac{1}{\rho} \geq 2\mathcal{R}.$$

From Remark 8.3 we get  $\beta_\varepsilon(\widehat{x}) \leq \frac{1}{\eta_\varepsilon(\widehat{x})} \leq \frac{1}{2\mathcal{R}}$ . □

### 3.4 A decreasing partition $\eta$ of the natural extension

We construct the measurable partition  $\eta$ . We refer to [1, 9, 13, 22, 23] for accounts on the theory on measurable partitions and entropy. We mention that in [1] (resp. in [9]) the theory of measurable partitions is used to prove the uniqueness of the measure of maximal entropy for polynomial automorphisms of  $\mathbb{C}^2$  (resp. for automorphisms of K3 surfaces).

For sake of simplicity, we denote by  $g$  the left shift  $\widehat{f}$ . The arguments are borrowed from [13]. We use the partition  $\mathcal{P}$  and the Pesin box  $\mathcal{A}$  respectively defined in Proposition 3.1 and in Section 3.3. We define the measurable partitions

$$\xi := \pi_0^{-1}(\mathcal{P}) \bigvee \{\mathcal{A}, \mathcal{A}^c\} \quad \text{and} \quad \eta := \bigvee_{p \in \mathbb{N}} g^p(\xi).$$

The partition  $\eta$  satisfies the following properties.

**Lemma 3.8.**

1. For  $\widehat{\mu}$ -almost every  $\widehat{x}$ , the elements of  $\eta_{\widehat{x}}$  have the same  $\mathcal{P}$ -address of  $\widehat{x}$ .
2.  $\forall \widehat{z} \in \mathcal{T}, \forall \widehat{x} \in W^u(\widehat{z}, \mathcal{R}), \eta_{\widehat{x}} \subset W^u(\widehat{z}, \mathcal{R}) \subset \mathcal{A}$ .
3. There exists a measurable function  $\eta'_\varepsilon : \widehat{\mathbb{P}}^2 \rightarrow ]0, 1]$  such that for  $\widehat{\mu}$ -almost every  $\widehat{x}$  :
  - (a)  $0 < \eta'_\varepsilon(\widehat{x}) \leq \eta_\varepsilon(\widehat{x}) \leq 1$ .
  - (b)  $\forall n \geq 0, f_{\widehat{x}}^{-n}(B(x_0, \eta'_\varepsilon(\widehat{x}))) \subset B(c, s)$  or  $f_{\widehat{x}}^{-n}(B(x_0, \eta'_\varepsilon(\widehat{x}))) \subset B(c, s)^c$ .
  - (c)  $W^u(\widehat{x}, \eta'_\varepsilon(\widehat{x})) \subset \eta_{\widehat{x}}$ .

**Proof :**

1. The  $\mathcal{P}$ -address is constant on the atoms of  $\widehat{\mathcal{P}}$ , hence on the atoms of  $\eta$ .
2. Let  $\widehat{w} \in \eta_{\widehat{x}}$ , by the previous point  $\widehat{w}$  and  $\widehat{x}$  have the same  $\mathcal{P}$ -address that is the one of  $\widehat{z}$  since the  $\mathcal{P}$ -address is constant on unstable manifolds. One has  $\widehat{x} \in \mathcal{A}$  by definition of  $\mathcal{A}$ , hence  $\eta_{\widehat{x}} \subset \mathcal{A}$  because  $\eta$  is thinner than  $\{\mathcal{A}, \mathcal{A}^c\}$ . Thus  $\widehat{w}$  belongs to  $\mathcal{A}$  and there is  $\widehat{z}' \in \mathcal{T}$  such that  $\widehat{w} \in W^u(\widehat{z}', \mathcal{R})$ . Since the  $\mathcal{P}$ -address is constant on unstable manifolds, we deduce that the  $\mathcal{P}$ -address of  $\widehat{z}'$  is the one of  $\widehat{w}$ , that is the one of  $\widehat{z}$ . Then, since  $z_0 = c = z'_0$ , we can apply the point 2. of Lemma 3.3 to conclude that  $\widehat{z} = \widehat{z}'$  and thus  $\widehat{w} \in W^u(\widehat{z}, \mathcal{R})$ .
3. Let  $\eta'_\varepsilon(\widehat{x}) := \min \left\{ \mathcal{R}, \eta_\varepsilon(\widehat{x}), \frac{\gamma(\widehat{x})}{2\beta_\varepsilon(\widehat{x})} \right\}$ , where  $\gamma(\widehat{x})$  is the function given by Lemma 3.6. Then one can follow the arguments of [13, Lemma 4.2] to complete the proof.  $\square$

The other important properties of the partition  $\eta$  are gathered in the following theorem. Given  $\zeta$  a measurable partition, we denote  $\mathcal{M}(\zeta)$  the completion with respect to  $\widehat{\mu}$  of the  $\sigma$ -algebra generated by the atoms of  $\zeta$ . Let  $\mathcal{M}$  be the completion with respect to  $\widehat{\mu}$  of the  $\sigma$ -algebra of borel sets of  $\widehat{\mathbb{P}}^2$ .

**Theorem 3.9.** *The following properties holds for the partition  $\eta$  :*

1.  $\eta$  is decreasing :  $\forall n \geq 0$ , for  $\widehat{\mu}$ -almost every  $\widehat{x}$ ,  $(g^{-n}\eta)_{\widehat{x}} \subset \eta_{\widehat{x}}$ .
2.  $\pi_0$  is injective on  $\eta_{\widehat{x}}$  for  $\widehat{\mu}$ -almost every  $\widehat{x}$ .
3. For  $\widehat{\mu}$ -almost every  $\widehat{x}$ ,  $f^n$  is injective on  $\pi_0((g^{-n}\eta)_{\widehat{x}})$  for every  $n \geq 0$ .
4. for every  $n \geq 0$  and for  $\widehat{\mu}$ -almost every  $\widehat{x}$ ,  $\eta_{\widehat{x}}$  is a countable union of atoms of  $g^{-n}\eta$ .
5.  $\bigvee_{n \geq 0} \mathcal{M}(g^{-n}\eta)$  coincide with  $\mathcal{M}$ .

6. for every  $n \geq 0$ ,  $\int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\widehat{x}}(g^{-n}\eta)_{\widehat{x}} d\widehat{\mu}(x) = \text{Log } d^{2n}$ .

The proof follows classical arguments, developed for instance in [13]. Let us explain the last item. By definition  $\int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\widehat{x}}((g^{-n}\eta)_{\widehat{x}}) d\widehat{\mu}(\widehat{x})$  is equal to the conditional entropy  $H(g^{-n}\eta|\eta)$ . Moreover, following [13, Section 4.3] one can prove that  $H(g^{-n}\eta|\eta)$  is equal to the relative entropy  $h_{\widehat{\mu}}(g^{-n}, \xi_n)$ , where  $\xi_n := g^{-1}\xi \vee \dots \vee g^{-n}\xi$ . But  $\xi_n$  is a generator of finite entropy for  $g^{-n}$  [13, Proposition 4.1], hence  $h_{\widehat{\mu}}(g^{-n}, \xi_n) = h_{\widehat{\mu}}(g^{-n}) = nh_{\widehat{\mu}}(g)$  by Kolmogorov-Sinai theorem [23, §9]. One finally obtains the last item from  $h_{\widehat{\mu}}(g) = h_{\mu}(f) = \text{Log } d^2$ .

From the point 3. we deduce the following elementary lemma. The purpose of the lemma is to prove that for any  $\widehat{x} \in \mathcal{A}$ , the identity  $f_{\widehat{x}}^{-n} \circ f^n = \text{Id}$  on  $\pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}]$  is true. It is used in the proof of Proposition 4.7 to compute the equality  $(T \wedge dd^c |W_{\widehat{x}_{-p}}|^2) = d^{q-p} \times \left( T \wedge dd^c |W_{\widehat{x}_{-p}} \circ f_{\widehat{x}_{-q}}^{-(p-q)}|^2 \right) \circ f^{p-q}$  on  $\pi_0[(g^{-p}\eta)_{\widehat{x}_{-p}}]$ , where  $p \geq q$  satisfy  $\widehat{x}_{-p} \in \mathcal{A}$  and  $\widehat{x}_{-q} \in \mathcal{A}$ .

**Lemma 3.10.** *Assume  $\mathcal{H}_1$ . For any  $\widehat{x} \in \mathcal{A}$  and for every  $n \in \mathbb{N}$  one has :*

1.  $f_{\widehat{x}}^{-n}[\pi_0(\eta_{\widehat{x}})] = \pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}]$ .
2.  $f_{\widehat{x}}^{-n} \circ f^n|_{\pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}]} = \text{Id}|_{\pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}]}$ .

**Proof :** Let  $\widehat{y} \in \eta_{\widehat{x}}$ . Since  $\widehat{x} \in \mathcal{A}$  there exists  $\widehat{z} \in \mathcal{T}$  such that  $\eta_{\widehat{x}} \subset W^u(\widehat{z}, \mathcal{R})$  (by Lemma 3.8). Then  $\widehat{x}, \widehat{y} \in W^u(\widehat{z}, \mathcal{R})$  and so by Lemma 3.3 we have  $f_{\widehat{x}}^{-n}(y_0) = f_{\widehat{z}}^{-n}(y_0) = y_{-n} \in \pi_0((g^{-n}\eta)_{\widehat{x}_{-n}})$  since  $g^{-n}\widehat{y} \in g^{-n}(\eta_{\widehat{x}}) = (g^{-n}\eta)_{\widehat{x}_{-n}}$ . Thus we have the inclusion "  $\subset$  ". The reverse inclusion "  $\supset$  " is obtained with the same arguments.

For the second item, observe that by Proposition 4.1 (given below for convenience) we have using  $\mathcal{H}_1$

$$\pi_0(\eta_{\widehat{x}}) \subset B(x_0, \eta_{\varepsilon}(\widehat{x})) \subset \text{Dom}(f_{\widehat{x}}^{-n})$$

and thus  $f^n(\pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}]) \subset \pi_0(\eta_{\widehat{x}})$  belongs to the domain of definition of the holomorphic map  $f_{\widehat{x}}^{-n}$ . At this stage the map  $h := f_{\widehat{x}}^{-n} \circ f^n|_{\pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}]}$  is well defined. To see that  $h = \text{Id}$  on  $\pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}]$  we use the point 3. of Theorem 3.9, indeed by definition one has :

$$\forall p \in \pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}], f^n(h(p)) = f^n(p),$$

and using the point 1.

$$h(\pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}]) = f_{\widehat{x}}^{-n} \circ \pi_0[g^n g^{-n}(\eta_{\widehat{x}})] = f_{\widehat{x}}^{-n}(\pi_0[\eta_{\widehat{x}}]) = \pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}],$$

so by injectivity of  $f^n$  on  $\pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}]$  we have  $h(p) = p$  on  $\pi_0[(g^{-n}\eta)_{\widehat{x}_{-n}}]$ .  $\square$

## 4 The measures $q_{\widehat{x}}$

### 4.1 Introduction of normal forms

For every  $\widehat{x} \in \Lambda_{\varepsilon}$ , Theorem 8.2 provides holomorphic coordinates

$$\xi_{\widehat{x}} : B(x_0, 2\eta_{\varepsilon}(\widehat{x})) \longrightarrow \mathbb{D}^2(\rho_{\varepsilon}(\widehat{x})) \quad , \quad \xi_{\widehat{x}} = (Z_{\widehat{x}}, W_{\widehat{x}}). \quad (15)$$

We will define  $q_{\widehat{x}}$  on  $\eta_{\widehat{x}}$ . The following lemma ensures that  $\pi_0(\eta_{\widehat{x}})$  is contained in the domain of definition  $W_{\widehat{x}}$  for every  $\widehat{x} \in \mathcal{A}$ . The definition of  $\mathcal{A}$  is given in Equation (14).

**Proposition 4.1.** *Assume  $\mathcal{H}_1$  and let  $\hat{x} \in \mathcal{A} \cap \Lambda_\varepsilon$ . Let  $\hat{z}$  be the unique element of  $\mathcal{T}$  such that  $\hat{x} \in W^u(\hat{z}, \mathcal{R})$ . Then*

$$\pi_0(\eta_{\hat{x}}) \subset B(z_0, \mathcal{R}) \subset B(x_0, \eta_\varepsilon(\hat{x})) \subset \text{Dom}(W_{\hat{x}}).$$

*In particular,  $\pi_0(\eta_{\hat{x}}) \subset B(z_0, \mathcal{R}) \subset B(x_0, \eta_\varepsilon(\hat{x})) \cap B(y_0, \eta_\varepsilon(\hat{y}))$  for every  $\hat{y} \in \eta_{\hat{x}}$ .*

**Proof :** By Lemma 3.8 one has  $\eta_{\hat{x}} \subset W^u(\hat{z}, \mathcal{R})$ , which proves the first inclusion. Since  $x_0 \in B(z_0, \mathcal{R})$ , we have  $B(z_0, \mathcal{R}) \subset B(x_0, 2\mathcal{R})$ . Proposition 3.7 finally gives  $2\mathcal{R} \leq \eta_\varepsilon(\hat{x})$ .  $\square$

**Proposition 4.2.** *Let  $\hat{x} \in \Lambda_\varepsilon$ .*

1. *The measure  $T \wedge dd^c |W_{\hat{x}}|^2$  is not null on every ball  $B(x_0, r)$  with  $0 < r \leq \eta_\varepsilon(\hat{x})$ .*
2. *For every  $n \geq 0$ ,  $(T \wedge dd^c |W_{\hat{x}_{-n}}|^2) \circ \pi_0 \circ g^{-n}(\eta_{\hat{x}}) > 0$ .*

**Proof :** The first assertion is a consequence of [15, § 3.3], which asserts that if  $h$  is a holomorphic submersion defined on a neighborhood of a point  $x \in \text{Supp}(\mu)$  then  $(T \wedge dd^c |h|^2)(B(x, r)) > 0$  for every small  $r > 0$ . For the second assertion, we use the inclusion

$$\eta_{\hat{x}} \supset W^u(\hat{x}, \eta'_\varepsilon(\hat{x}))$$

given by the point 3.(c) of Lemma 3.8. We deduce that :

$$\pi_0[g^{-n}(\eta_{\hat{x}})] \supset \pi_0(\{\hat{y}_{-n}, \hat{y} \in W^u(\hat{x}, \eta'_\varepsilon(\hat{x}))\}) = f_{\hat{x}}^{-n}(B(x_0, \eta'_\varepsilon(\hat{x}))) \supset B(x_{-n}, r'),$$

where  $r' > 0$  is small enough to get the last inclusion. Since  $x_{-n} \in \text{Supp}(\mu)$ , the first item allows to conclude.  $\square$

Let us now define the measures  $q_{\hat{x}}$ . The set  $\mathcal{B}(\widehat{\mathbb{P}^2})$  below stands for the sigma-algebra of borel sets of  $\widehat{\mathbb{P}^2}$ . We recall that  $\xi_{\hat{x}}$  is defined in (15). Let

$$M(\hat{x}) := \|d_0 \xi_{\hat{x}}^{-1} \cdot (0, 1)\| \quad , \quad \Delta(x_{-p}, p) = \|d_{x_{-p}} f^p \cdot \vec{v}_s(x_{-p})\|.$$

**Definition 4.3.** *Assume  $\mathcal{H}_1$ . For every  $\hat{x} \in \Lambda_\varepsilon$ , we define*

$$\tilde{q}_{\hat{x}} := d^p (\Delta(x_{-p}, p))^2 M(\hat{x}_{-p})^2 \times (T \wedge dd^c |W_{\hat{x}_{-p}}|^2) \circ \pi_0 \circ g^{-p} \text{ on } \eta_{\hat{x}},$$

*where  $p \geq 0$  is any integer such that  $\hat{x}_{-p}$  belongs to  $\mathcal{A}$ . The probability measure  $q_{\hat{x}}$  is then*

$$\forall A \in \mathcal{B}(\widehat{\mathbb{P}^2}), \quad q_{\hat{x}}(A) := \frac{\tilde{q}_{\hat{x}}(A \cap \eta_{\hat{x}})}{L(\hat{x})},$$

*the normalization  $L(\hat{x}) := \tilde{q}_{\hat{x}}(\eta_{\hat{x}})$  being  $> 0$  by Proposition 4.2.*

**Remark 4.4.**

1. *By Birkhoff ergodic theorem, the set  $\text{Rec}_{\hat{x}}(\mathcal{A}) := \{p \geq 0, \hat{x}_{-p} \in \mathcal{A}\}$  is infinite for every  $\hat{x} \in \Lambda_\varepsilon$ .*
2. *The fact that  $\tilde{q}_{\hat{x}}$  does not depend on  $p$  is established in Proposition 4.7. We need to introduce  $p$  so that  $T \wedge dd^c |W_{\hat{x}_{-p}}|^2$  is well defined on  $\pi_0(\eta_{\hat{x}_{-p}})$  (see Proposition 4.1), which contains  $\pi_0(g^{-p}(\eta_{\hat{x}}))$  by the decreasing property of  $\eta$ .*
3. *By Theorem 3.9,  $\pi_0$  is injective on the atoms of  $g^{-p}\eta$ , thus  $\tilde{q}_{\hat{x}}$  is a measure.*
4. *If  $\hat{x} \in \mathcal{A}$ , then  $\tilde{q}_{\hat{x}} = M(\hat{x})^2 (T \wedge dd^c |W_{\hat{x}}|^2) \circ \pi_0$  on  $\eta_{\hat{x}}$ .*

**Proposition 4.5.** Assume  $\mathcal{H}_2$ , then for  $\widehat{\mu}$ -almost every  $\widehat{x} \in \widehat{\mathbb{P}^2}$  :

$$\widehat{y} \in \eta_{\widehat{x}} \cap \Lambda_\varepsilon \implies q_{\widehat{x}} = q_{\widehat{y}}.$$

In particular we can define a measure  $\widehat{q}$  on  $\mathcal{B}(\widehat{\mathbb{P}^2})$  whose conditionals on  $\eta$  are given by the measures  $q_{\widehat{x}}$  :

$$\forall A \in \mathcal{B}(\widehat{\mathbb{P}^2}), \widehat{q}(A) = \int_{\widehat{\mathbb{P}^2}} q_{\widehat{x}}(A \cap \eta_{\widehat{x}}) d\widehat{\mu}(\widehat{x}).$$

**Remark 4.6.** From the proof of Theorem 8.2, one can construct  $\widehat{x} \mapsto \xi_{\widehat{x}}$  as limits of holomorphic maps hence the dependence on  $\widehat{x}$  is measurable. Thus for every borel set  $A \subset \widehat{\mathbb{P}^2}$ , the function  $\widehat{x} \mapsto (T \wedge dd^c |W_{\widehat{x}}|^2)(\pi_0(A \cap \eta_{\widehat{x}}))$  is measurable and the integrals above  $A \mapsto \int q_{\widehat{x}}(A \cap \eta_{\widehat{x}}) d\widehat{\mu}(\widehat{x})$  make sense.

The proof of Proposition 4.5 is provided in Section 7.

## 4.2 The definition of $q_{\widehat{x}}$ does not depend on $p \in \text{Rec}_{\widehat{x}}(\mathcal{A})$

The set of integers  $\text{Rec}_{\widehat{x}}(\mathcal{A})$  is defined in Remark 4.4.

**Proposition 4.7.** Assume  $\mathcal{H}_1$ . Let  $\widehat{x} \in \Lambda_\varepsilon$ ,  $(p, q) \in \text{Rec}_{\widehat{x}}(\mathcal{A})^2$  and  $A \subset \eta_{\widehat{x}}$  be a borel set. Let

$$E^p := d^p \Delta(x_{-p}, p)^2 M(\widehat{x}_{-p})^2 (T \wedge dd^c |W_{\widehat{x}_{-p}}|^2) \circ \pi_0 \circ g^{-p}(A),$$

and let  $E^q$  be defined similarly replacing  $p$  by  $q$ . Then  $E^p = E^q$ .

We need two lemmas.

**Lemma 4.8.** For every  $\widehat{x} \in \Lambda_\varepsilon$  the following formula holds on  $B(x_0, 2\eta_\varepsilon(\widehat{x}))$  :

$$W_{\widehat{x}_{-n}} \circ f_{\widehat{x}}^{-n} = \beta_{n, \widehat{x}} \times W_{\widehat{x}}, \quad \forall n \geq N(\widehat{x}).$$

In particular, for every  $n \geq N(\widehat{x})$ , we get on  $B(x_0, 2\eta_\varepsilon(\widehat{x}))$  :

$$(f_{\widehat{x}}^{-n})^*(T \wedge dd^c |W_{\widehat{x}_{-n}}|^2) = d^{-n} |\beta_{n, \widehat{x}}|^2 \times (T \wedge dd^c |W_{\widehat{x}}|^2).$$

**Proof :** The commutative diagram (9) given by Theorem 8.2 provides for  $n \geq N(\widehat{x})$  :

$$W_{\widehat{x}_{-n}} \circ f_{\widehat{x}}^{-n} = W \circ (\xi_{\widehat{x}_{-n}} \circ f_{\widehat{x}}^{-n}) = (W \circ R_{n, \widehat{x}}) \circ \xi_{\widehat{x}} = \beta_{n, \widehat{x}} \times W_{\widehat{x}},$$

where  $W : \mathbb{C}^2 \rightarrow \mathbb{C}$  is the projection on the second coordinate. □

**Lemma 4.9.**

1. For every  $\widehat{x} \in \Lambda_\varepsilon$  and  $n, k \geq 0$ ,  $\Delta(x, n)\Delta(x_n, k) = \Delta(x, n+k)$ .
2. For every  $\widehat{x} \in \Lambda_\varepsilon$  and  $n \geq N(\widehat{x})$ ,  $\Delta(x_{-n}, n) |\beta_{n, \widehat{x}}| = \frac{M(\widehat{x})}{M(\widehat{x}_{-n})}$ .

**Proof :** The first item comes from the chain rule formula. For the second item, the commutative diagram (9) of Theorem 8.2 gives for  $n \geq N(\widehat{x})$  :

$$|\beta_{n, \widehat{x}}| = \left| \left| d_{x_{-n}} \xi_{\widehat{x}_{-n}} \circ d_{x_0} f_{\widehat{x}}^{-n} \circ d_0 (\xi_{\widehat{x}})^{-1} \cdot (0, 1) \right| \right|.$$

One concludes by Theorems 8.1 and 8.2, which yield  $d_0(\xi_{\widehat{x}})^{-1} \cdot (0, 1) = M(\widehat{x})\vec{v}_s(x_0)$ ,

$$d_{x_0} f_{\widehat{x}}^{-n} \cdot \vec{v}_s(x_0) = \Delta(x_{-n}, n)^{-1} \vec{v}_s(x_{-n})$$

and  $\|d_{x_{-n}} \xi_{\widehat{x}_{-n}} \cdot \vec{v}_s(x_{-n})\| = M(\widehat{x}_{-n})^{-1}$ .  $\square$

**Proof of Proposition 4.7 :** Let  $\widehat{x} \in \Lambda_\varepsilon$  and let  $A \subset \eta_{\widehat{x}}$  be a borel set. We have to prove  $E^p = E^q$  for any  $p, q \in \text{Rec}_{\widehat{x}}(\mathcal{A})$ . It suffices to show the following property :

$$\forall p, q \in \text{Rec}_{\widehat{x}}(\mathcal{A}), (p - q \geq N(\widehat{x}) + N(\widehat{x}_{-q})) \implies E^p = E^q : (P)$$

Indeed since  $\text{Rec}_{\widehat{x}}(\mathcal{A})$  is infinite, for any  $p, q \in \text{Rec}_{\widehat{x}}(\mathcal{A})$  we can find  $r \in \text{Rec}_{\widehat{x}}(\mathcal{A})$  such that  $r - p \geq N(\widehat{x}) + N(\widehat{x}_{-p})$  and  $r - q \geq N(\widehat{x}) + N(\widehat{x}_{-q})$ . Thus if (P) is true,  $E^r = E^p$  and  $E^r = E^q$ , which implies  $E^p = E^q$ . Let us prove now the property (P). Let  $p, q \in \text{Rec}_{\widehat{x}}(\mathcal{A})$  such that  $p - q \geq N(\widehat{x}) + N(\widehat{x}_{-q})$ . We define :

$$F^p := (T \wedge dd^c |W_{\widehat{x}_{-p}}|^2) \circ \pi_0 \circ g^{-p}(A).$$

and similarly  $F^q$  by replacing  $p$  by  $q$ . Proving  $E^p = E^q$  amounts to show the equality :

$$d^p \Delta(x_{-p}, p)^2 M(\widehat{x}_{-p})^2 F^p = d^q \Delta(x_{-q}, q)^2 M(\widehat{x}_{-q})^2 F^q. \quad (16)$$

Since  $\widehat{x}_{-q} \in \mathcal{A}$  and  $g^{-p}(\eta_{\widehat{x}}) \subset (g^{-(p-q)}\eta)_{\widehat{x}_{-q-(p-q)}}$ , one gets by Lemma 3.10 (applied to  $\widehat{x}_{-q}$  with  $n = p - q$ ) :

$$f_{\widehat{x}_{-q}}^{-(p-q)} \circ f_U^{p-q}|_{\pi_0(g^{-p}(\eta_{\widehat{x}}))} = \text{Id}|_{\pi_0(g^{-p}(\eta_{\widehat{x}}))}. \quad (17)$$

Let us denote  $U := \pi_0(g^{-p}(\eta_{\widehat{x}}))$  and  $f_U^{p-q} := f_U^{p-q}|_U$ . Using hypothesis  $\mathcal{H}_1$ , Proposition 4.1 tells us that  $\pi_0(\eta_{\widehat{x}_{-p}}) \subset B(x_{-p}, \eta_\varepsilon(\widehat{x}_{-p})) \subset \text{Dom}(W_{\widehat{x}_{-p}})$ , and thus  $\pi_0(g^{-p}(A)) \subset \pi_0(g^{-p}(\eta_{\widehat{x}})) \subset \pi_0(\eta_{\widehat{x}_{-p}}) \subset \text{Dom}(W_{\widehat{x}_{-p}})$ . By (17) and by  $(f_U^{p-q})^* T = d^{p-q} T$ , we obtain

$$\begin{aligned} F^p &= \left[ T \wedge (f_{\widehat{x}_{-q}}^{-(p-q)} \circ f_U^{p-q})^* dd^c |W_{\widehat{x}_{-p}}|^2 \right] \circ \pi_0 \circ g^{-p}(A) \\ &= d^{q-p} \left[ (f_U^{p-q})^* T \wedge (f_U^{p-q})^* dd^c \left| W_{\widehat{x}_{-p}} \circ f_{\widehat{x}_{-q}}^{-(p-q)} \right|^2 \right] \circ \pi_0 \circ g^{-p}(A). \end{aligned}$$

Then by injectivity of  $f_U^{p-q}$  we have :

$$F^p = d^{q-p} \left[ T \wedge dd^c \left| W_{\widehat{x}_{-p}} \circ f_{\widehat{x}_{-q}}^{-(p-q)} \right|^2 \right] \circ (f_U^{p-q}) \circ \pi_0 \circ g^{-p}(A).$$

Using the relation  $f \circ \pi_0 = \pi_0 \circ g$ , we have  $(f_U^{p-q}) \circ \pi_0 \circ g^{-p}(A) = \pi_0 \circ g^{-q}(A)$ . Moreover, by Lemma 4.8 with  $\widehat{x}$  replaced by  $\widehat{x}_{-q}$  and  $n = p - q \geq N(\widehat{x}_{-q})$ , we get :

$$W_{\widehat{x}_{-p}} \circ f_{\widehat{x}_{-q}}^{-(p-q)} = \beta_{p-q, \widehat{x}_{-q}} \times W_{\widehat{x}_{-q}} \text{ on } B(x_{-q}, \eta_\varepsilon(\widehat{x}_{-q})) \subset \text{Dom}(W_{\widehat{x}_{-q}}).$$

We deduce :

$$F^p = d^{q-p} |\beta_{p-q, \widehat{x}_{-q}}|^2 \times F^q.$$

Now we replace  $|\beta_{p-q, \widehat{x}_{-q}}|^2$  by applying the second item of Lemma 4.9 to  $\widehat{x}_{-q}$  with  $n = p - q \geq N(\widehat{x}_{-q})$ . We get

$$d^p M(\widehat{x}_{-p})^2 F^p = d^q M(\widehat{x}_{-q})^2 F^q \times \frac{1}{\Delta(x_{-p}, p - q)^2}.$$

To conclude, we use  $\Delta(x_{-p}, p - q) \Delta(x_{-q}, q) = \Delta(x_{-p}, p)$  which comes from Lemma 4.9 applied to  $\widehat{x}_{-p}$ ,  $n = p - q$  and  $k = q$ .  $\square$

## 5 Proof of Theorem B

The property  $(\pi_0)_* q_{\widehat{x}} \ll T \wedge \omega_{\mathbb{P}^2}$  directly follows from the definition of  $q_{\widehat{x}}$ . The second formula of Theorem B is contained in Theorem 3.9. It remains to show for every  $n \geq 0$  :

$$\text{Log } d^n + 2n\lambda_2 = \int_{\widehat{\mathbb{P}^2}} -\text{Log } q_{\widehat{x}}(g^{-n}\eta)_{\widehat{x}} \, d\widehat{\mu}(\widehat{x}).$$

**Proposition 5.1.** *For  $\widehat{\mu}$ -almost every  $\widehat{x}$  and for every  $n \geq 0$  :*

$$L(\widehat{x})q_{\widehat{x}}((g^{-n}\eta)_{\widehat{x}}) = \frac{1}{d^n} \frac{L(\widehat{x}_n)}{\Delta(x_0, n)^2}.$$

**Proof :** Let  $\widehat{x} \in \Lambda_\varepsilon$  and  $p \in \mathbb{N}$  such that  $\widehat{x}_{-p} \in \mathcal{A}$ . Let us fix  $n \in \mathbb{N}$  and let  $q := n + p$  so that  $\widehat{x}_{n-q} = \widehat{x}_{-p} \in \mathcal{A}$ . By definition of  $q_{\widehat{x}}$  and the inclusion  $(g^{-n}\eta)_{\widehat{x}} \subset \eta_{\widehat{x}}$  :

$$L(\widehat{x})q_{\widehat{x}}((g^{-n}\eta)_{\widehat{x}}) = d^p \Delta(x_{-p}, p)^2 M(\widehat{x}_{-p})^2 \times [T \wedge dd^c |W_{\widehat{x}_{-p}}|^2] \circ \pi_0 \circ g^{-p}((g^{-n}\eta)_{\widehat{x}}).$$

Using  $p = q - n$  and  $g^{-p}((g^{-n}\eta)_{\widehat{x}}) = g^{-q}(\eta_{\widehat{x}_n})$  we have :

$$L(\widehat{x})q_{\widehat{x}}((g^{-n}\eta)_{\widehat{x}}) = \frac{1}{d^n} \Delta(x_{n-q}, p)^2 d^q M(\widehat{x}_{n-q})^2 [T \wedge dd^c |W_{\widehat{x}_{n-q}}|^2] \circ \pi_0 \circ g^{-q}(\eta_{\widehat{x}_n}).$$

By definition of  $L(\widehat{x}_n)$  (recall that  $(\widehat{x}_{n-q}) \in \mathcal{A}$ ) and since  $n - q + p = 0$ , we have

$$L(\widehat{x})q_{\widehat{x}}((g^{-n}\eta)_{\widehat{x}}) = \frac{\Delta(x_{n-q}, p)^2}{d^n \Delta(x_{n-q}, q)^2} \times L(\widehat{x}_n).$$

The first item of Lemma 4.9 applied to  $\widehat{x}_{n-q}$ ,  $p$  and  $q - p$  completes the proof.  $\square$

By taking Log in Proposition 5.1, we obtain :

$$\text{Log } d^n + 2 \text{Log } \Delta(x_0, n) = -\text{Log } q_{\widehat{x}}(g^{-n}\eta)_{\widehat{x}} + \text{Log } \frac{L(\widehat{x}_n)}{L(\widehat{x})}.$$

The third item of Theorem 8.1 asserts that  $\int_{\widehat{\mathbb{P}^2}} \text{Log } \Delta(x_0, n) \, d\widehat{\mu}(\widehat{x}) = n\lambda_2$ . To finish the proof of Theorem B, it remains to show that  $h_n := \text{Log} \left( \frac{L \circ g^n}{L} \right) \in L^1(\widehat{\mu})$  and satisfies  $\int h_n \, d\widehat{\mu} = 0$ . We use the following classical lemma, the original statement is stated with  $\text{Log}^-$  instead of  $\text{Log}^+$ , but the proof also works with  $\text{Log}^+$ .

**Lemma 5.2** (Ledrappier-Strelcyn, [21, Proposition 2.2]). *Let  $n \geq 0$  and let  $\varphi$  be a positive measurable function on  $\widehat{\mathbb{P}^2}$ . If  $\text{Log}^+ \left( \frac{\varphi \circ g^n}{\varphi} \right) \in L^1(\widehat{\mu})$  then :*

$$\text{Log} \left( \frac{\varphi \circ g^n}{\varphi} \right) \in L^1(\widehat{\mu}) \text{ and } \int \text{Log} \left( \frac{\varphi \circ g^n}{\varphi} \right) \, d\widehat{\mu} = 0.$$

The assumptions of that Lemma are satisfied. Indeed, Proposition 5.1 implies for  $\widehat{\mu}$ -almost every  $\widehat{x}$  in  $\{h_n \geq 0\}$  :

$$0 \leq h_n(\widehat{x}) = \text{Log}(d^n \Delta(x_0, n)^2) + \text{Log } q_{\widehat{x}}((g^{-n}\eta)_{\widehat{x}}) \leq \text{Log}(d^n \Delta(x_0, n)^2).$$

Moreover,  $\widehat{x} \mapsto \text{Log } \Delta(x_0, n)$  belongs to  $L^1(\widehat{\mu})$  by Theorem 8.1. Therefore we can apply Lemma 5.2 to get  $h_n \in L^1(\widehat{\mu})$  and  $\int h_n \, d\widehat{\mu} = 0$  as desired.  $\square$

## 6 Absolute continuity of $\mu$ with respect to $T$

### 6.1 Proof of $\hat{q} = \hat{\mu}$

The probability measure  $\hat{q}$  is defined in Proposition 4.5.

**Theorem 6.1.** *Assume  $\mathcal{H}_2$  and  $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$ .*

1. *For  $\hat{\mu}$ -almost every  $\hat{x}$  and for every  $n \geq 0$ ,  $q_{\hat{x}}(g^{-n}\eta)_{\hat{x}} = \mu_{\hat{x}}(g^{-n}\eta)_{\hat{x}}$ .*
2. *For  $\hat{\mu}$ -almost every  $\hat{x}$ ,  $q_{\hat{x}} = \mu_{\hat{x}}$  and so  $\hat{q} = \hat{\mu}$ .*

**Proof :** The second item is a classical consequence of the first one by using the generating property of  $\eta$ , see for instance [13, Lemme 5.6]. Let us prove the first item. By Theorem 3.9 we know that

$$\forall n \geq 0, \quad \text{Log } d^{2n} = \int_{\widehat{\mathbb{P}^2}} -\text{Log } \mu_{\hat{x}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}).$$

But we have by Theorem B (using  $\mathcal{H}_1$ ) :

$$\forall n \geq 0, \quad \text{Log } d^n + 2n\lambda_2 = \int_{\widehat{\mathbb{P}^2}} -\text{Log } q_{\hat{x}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}).$$

If  $\lambda_2$  is minimal equal to  $\frac{1}{2} \text{Log } d$ , we get :

$$\int_{\widehat{\mathbb{P}^2}} -\text{Log } q_{\hat{x}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}) = \text{Log } d^{2n} = - \int_{\widehat{\mathbb{P}^2}} \text{Log } \mu_{\hat{x}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}).$$

By Jensen inequality we deduce :

$$0 = \int_{\widehat{\mathbb{P}^2}} \text{Log } \frac{q_{\hat{x}}}{\mu_{\hat{x}}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}) \leq \text{Log } \int_{\widehat{\mathbb{P}^2}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}). \quad (18)$$

By definition of the conditional measures of  $\hat{\mu}$  with respect to  $\eta$ , we get

$$\int_{\widehat{\mathbb{P}^2}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}\left((g^{-n}\eta)_{\hat{x}}\right) \, d\hat{\mu}(\hat{x}) = \int_{\widehat{\mathbb{P}^2}} \int_{\eta_{\hat{x}}} \frac{q_{\hat{y}}}{\mu_{\hat{y}}}\left((g^{-n}\eta)_{\hat{y}}\right) \, d\mu_{\hat{x}}(\hat{y}) \, d\hat{\mu}(\hat{x}).$$

But by Proposition 4.5 we have  $q_{\hat{x}} = q_{\hat{y}}$  for every  $\hat{y} \in \eta_{\hat{x}} \cap \Lambda_\varepsilon$ . Thus we have

$$\int_{\eta_{\hat{x}}} \frac{q_{\hat{y}}}{\mu_{\hat{y}}}\left((g^{-n}\eta)_{\hat{y}}\right) \, d\mu_{\hat{x}}(\hat{y}) = \int_{\eta_{\hat{x}}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}\left((g^{-n}\eta)_{\hat{y}}\right) \, d\mu_{\hat{x}}(\hat{y}).$$

Now by Theorem 3.9  $\eta_{\hat{x}}$  is a countable union  $\bigsqcup_{j \in \mathbb{N}} A_{j,n}^{\hat{x}}$  of atoms of  $g^{-n}\eta$ . We infer

$$\int_{\eta_{\hat{x}}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}\left((g^{-n}\eta)_{\hat{y}}\right) \, d\mu_{\hat{x}}(\hat{y}) = \sum_{j=0}^{+\infty} \int_{A_{j,n}^{\hat{x}}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}\left(A_{j,n}^{\hat{x}}\right) \, d\mu_{\hat{x}}(\hat{y}) = \sum_{j=0}^{+\infty} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}\left(A_{j,n}^{\hat{x}}\right) \times \mu_{\hat{x}}\left(A_{j,n}^{\hat{x}}\right)$$

which is equal to  $q_{\hat{x}}\left(\bigsqcup_j A_{j,n}^{\hat{x}}\right) = 1$ . Finally,  $\int_{\widehat{\mathbb{P}^2}} \frac{q_{\hat{x}}}{\mu_{\hat{x}}}(g^{-n}\eta)_{\hat{x}} \, d\hat{\mu}(\hat{x}) = 1$ , which implies the equality in Equation (18). The strict concavity of  $\text{Log}$  completes the proof.  $\square$



## 6.2 Proof of Theorem A

**Definition 6.2.** For every  $\hat{x} \in \Lambda_\varepsilon$ , the measure  $T \wedge dd^c|W_{\hat{x}}|^2$  is absolutely continuous with respect to the trace measure  $\sigma_T = T \wedge \omega_{\mathbb{P}^2}$ . We can define the Radon-Nikodym derivative :

$$D_{\hat{x}} := \frac{d(T \wedge dd^c|W_{\hat{x}}|^2)}{d\sigma_T} \text{ on } B(x_0, 2\eta_\varepsilon(\hat{x})).$$

Now let us assume  $\mathcal{H}_2$  and  $\lambda_1 > \lambda_2 = \frac{1}{2} \text{Log } d$ . Our aim is to prove

$$\mu \ll \sigma_T \text{ on } \mathbb{P}^2.$$

We will show  $\sigma_T(B) = 0 \Rightarrow \mu(B) = 0$  for every borel set  $B$  of  $\mathbb{P}^2$ . One has

$$\mu(B) = \hat{\mu}(\pi_0^{-1}(B)) = \hat{q}(\pi_0^{-1}(B)) \leq \sum_{N=0}^{+\infty} \int_{g^N(\mathcal{A})} q_{\hat{x}}(\pi_0^{-1}(B)) \, d\hat{\mu}(\hat{x}),$$

where the second equality comes from Theorem 6.1 and the inequality comes from the fact that  $\cup_{N \geq 0} g^N(\mathcal{A})$  has full  $\hat{\mu}$ -measure (a consequence of Birkhoff ergodic theorem and  $\hat{\mu}(\mathcal{A}) > 0$ ). Let us now fix  $N \geq 0$  and  $\hat{x} \in g^N(\mathcal{A}) \cap \Lambda_\varepsilon$ . Let  $C(N, \hat{x}) := d^N \Delta(x_{-N}, N)^2 M(\hat{x}_{-N})^2 / L(\hat{x})$ , so that

$$q_{\hat{x}}(\pi_0^{-1}(B)) = C(N, \hat{x}) \times (T \wedge dd^c|W_{\hat{x}_{-N}}|^2) \circ \pi_0 \circ g^{-N}[\eta_{\hat{x}} \cap \pi_0^{-1}(B)].$$

Now observe that  $g^{-N}[\eta_{\hat{x}} \cap \pi_0^{-1}B] = (g^{-N}\eta)_{\hat{x}_{-N}} \cap (g^{-N}\pi_0^{-1}B) = (g^{-N}\eta)_{\hat{x}_{-N}} \cap \pi_0^{-1}(f^{-N}B)$ . Recall that  $f^N$  is injective on  $\pi_0[(g^{-N}\eta)_{\hat{x}_{-N}}]$  by Theorem 3.9. Let  $h_N$  denote the inverse of the restriction of  $f^N$  to  $\pi_0[(g^{-N}\eta)_{\hat{x}_{-N}}]$ . We obtain

$$q_{\hat{x}}(\pi_0^{-1}(B))/C(N, \hat{x}) \leq (T \wedge dd^c|W_{\hat{x}_{-N}}|^2)(h_N(B')) =: l_{N, \hat{x}}(B'),$$

where  $B' := B \cap \pi_0(\eta_{\hat{x}})$ . Using Definition 6.2, we have

$$l_{N, \hat{x}}(B') = \int_{h_N(B')} D_{\hat{x}_{-N}} \, d\sigma_T.$$

Now it suffices to show that  $\sigma_T(B) = 0 \implies l_{N, \hat{x}}(B') = 0$ . Let  $E_N$  be the smooth positive function defined on  $\mathbb{P}^2 \setminus \text{Crit}(f^N)$  by  $E_N(p) := \|(d_p f^N)^{-1}\|^{-2}$ . It satisfies  $E_N \times \omega_{\mathbb{P}^2} \leq (f^N)^* \omega_{\mathbb{P}^2}$  and therefore

$$\sigma_T \leq E_N^{-1} \times T \wedge (f^N)^* \omega_{\mathbb{P}^2} = E_N^{-1} \times d^{-N} (f^N)^* (T \wedge \omega_{\mathbb{P}^2}) \text{ on } \mathbb{P}^2 \setminus \text{Crit}(f^N),$$

where we used  $(f^N)^* T = d^N T$  for the equality. The preceding estimate holds on  $\mathbb{P}^2$  since  $\sigma_T$  and  $(f^N)^*(T \wedge \omega_{\mathbb{P}^2})$  do not charge  $\text{Crit}(f^N)$ . Hence

$$l_{N, \hat{x}}(B') \leq d^{-N} \int_{h_N(B')} E_N^{-1} D_{\hat{x}_{-N}} \, d(f^N)^* [T \wedge \omega_{\mathbb{P}^2}].$$

Since  $f^N : h_N(B') \rightarrow B'$  is invertible with inverse map  $h_N|_{B'}$ , we get

$$l_{N, \hat{x}}(B') \leq d^{-N} \int_{B'} (E_N^{-1} D_{\hat{x}_{-N}}) \circ h_N \, d[T \wedge \omega_{\mathbb{P}^2}].$$

That implies  $\sigma_T(B) = 0 \implies l_{N, \hat{x}}(B') = 0$ , as desired.  $\square$

## 7 Proof of Proposition 4.5

In this Section we assume  $\mathcal{H}_2$  and prove for  $\widehat{\mu}$ -almost every  $\widehat{x} \in \widehat{\mathbb{P}}^2$  :

$$\widehat{y} \in \eta_{\widehat{x}} \cap \Lambda_\varepsilon \implies q_{\widehat{x}} = q_{\widehat{y}} \text{ on } \eta_{\widehat{x}}. \quad (19)$$

Section 7.1 only needs  $\mathcal{H}_1$  to consider the measures  $q_{\widehat{x}}$ . We will need  $\mathcal{H}_2$  in Section 7.2.

### 7.1 Reduction to the case $\widehat{x} \in \mathcal{A}$

**Proposition 7.1.** *If Equation (19) holds for  $\widehat{\mu}$ -almost every  $\widehat{x}$  in  $\mathcal{A}$ , then it holds for  $\widehat{\mu}$ -almost every  $\widehat{x}$  in  $\widehat{\mathbb{P}}^2$ .*

**Proof :** The assumption implies that there exists  $E \subset \Lambda_\varepsilon \cap \mathcal{A}$  of measure  $\widehat{\mu}(E) = \widehat{\mu}(\mathcal{A})$  such that for every  $\widehat{x} \in E$ , (19) is true. Then by Birkhoff ergodic theorem, there exists  $F \subset \Lambda_\varepsilon$  of measure  $\widehat{\mu}(F) = 1$  such that for every  $\widehat{x} \in F$ , there exists  $n \geq 0$  satisfying  $\widehat{x}_{-n} \in E$ . Now let us fix  $\widehat{x} \in F$  and let  $n \geq 0$  such that  $\widehat{x}_{-n} \in E$ . Let  $\widehat{y} \in \Lambda_\varepsilon \cap \eta_{\widehat{x}}$ . Then because  $\eta$  is decreasing and is thinner than  $\{\mathcal{A}, \mathcal{A}^c\}$ , one has  $\widehat{y}_{-n} \in \eta_{\widehat{x}_{-n}} \subset \{\mathcal{A}, \mathcal{A}^c\}_{\widehat{x}_{-n}} = \mathcal{A}$ . So we have  $q_{\widehat{y}_{-n}} = q_{\widehat{x}_{-n}}$  by definition of  $E$ , which can be written as

$$\frac{1}{L(\widehat{y}_{-n})} \tilde{q}_{\widehat{y}_{-n}} = \frac{1}{L(\widehat{x}_{-n})} \tilde{q}_{\widehat{x}_{-n}}. \quad (20)$$

Now using the definition of  $q_{\widehat{y}}$  we have :

$$L(\widehat{y})q_{\widehat{y}} = d^n \Delta(y_{-n}, n)^2 \times M(\widehat{y}_{-n})^2 (T \wedge dd^c |W_{\widehat{y}_{-n}}|^2) \circ \pi_0 \circ g^{-n} \lfloor_{\eta_{\widehat{y}}}.$$

Since  $g^{-n}(\eta_{\widehat{y}}) \subset \eta_{\widehat{y}_{-n}}$ , one can replace  $\pi_0 \circ g^{-n} \lfloor_{\eta_{\widehat{y}}}$  by  $\pi_0 \lfloor_{\eta_{\widehat{y}_{-n}}} \circ g^{-n} \lfloor_{\eta_{\widehat{y}}}$ . Moreover, because  $\widehat{y}_{-n} \in \mathcal{A}$ , we get by Remark 4.4 :

$$\tilde{q}_{\widehat{y}_{-n}} = M(\widehat{y}_{-n})^2 (T \wedge dd^c |W_{\widehat{y}_{-n}}|^2) \circ \pi_0 \lfloor_{\eta_{\widehat{y}_{-n}}}.$$

We finally deduce

$$L(\widehat{y})q_{\widehat{y}} = d^n \Delta(y_{-n}, n)^2 \times \tilde{q}_{\widehat{y}_{-n}} \circ g^{-n} \lfloor_{\eta_{\widehat{y}}}$$

and similarly  $L(\widehat{x})q_{\widehat{x}} = d^n \Delta(x_{-n}, n)^2 \times \tilde{q}_{\widehat{x}_{-n}} \circ g^{-n} \lfloor_{\eta_{\widehat{x}}}$ . Equation (20) yields

$$q_{\widehat{y}} = \frac{\Delta(y_{-n}, n)^2 L(\widehat{y}_{-n}) L(\widehat{x})}{\Delta(x_{-n}, n)^2 L(\widehat{x}_{-n}) L(\widehat{y})} \times q_{\widehat{x}}.$$

Therefore the ratio is equal to 1 and  $q_{\widehat{y}} = q_{\widehat{x}}$  since these are probability measures on  $\eta_{\widehat{x}}$ .  $\square$

### 7.2 The case $\widehat{x} \in \mathcal{A}$

In this section we assume  $\mathcal{H}_2$  and  $\lambda_1 > \lambda_2$ , and we prove Equation (19) for  $\widehat{x} \in \mathcal{A}$ . Let us begin with some remarks.

Let  $\widehat{z}$  be the unique element of  $\mathcal{T}$  such that  $\widehat{x} \in W^u(\widehat{z}, \mathcal{R})$ . Lemma 3.8 yields  $\eta_{\widehat{x}} \subset W^u(\widehat{z}, \mathcal{R}) \subset \mathcal{A}$ , and Proposition 4.1 gives

$$\forall \widehat{y} \in \eta_{\widehat{x}}, \pi_0(\eta_{\widehat{x}}) \subset B(z_0, \mathcal{R}) \subset B(x_0, \eta_\varepsilon(\widehat{x})) \cap B(y_0, \eta_\varepsilon(\widehat{y})). \quad (21)$$

By Lemma 3.3, the maps  $f_{\widehat{x}}^{-n}$ ,  $f_{\widehat{y}}^{-n}$  and  $f_{\widehat{z}}^{-n}$  coincide on  $B(z_0, \mathcal{R})$ . By using the normal forms Theorem, we get

$$f_{\widehat{z}}^{-n} \circ \pi_0(\eta_{\widehat{x}}) \subset f_{\widehat{z}}^{-n}(B(z_0, \mathcal{R})) \subset B(x_{-n}, \eta_\varepsilon(\widehat{x}_{-n})) \cap B(y_{-n}, \eta_\varepsilon(\widehat{y}_{-n})). \quad (22)$$

We will denote for each  $n \geq 0$ ,  $B_n := B(x_n, 2\eta_\varepsilon(\hat{x}_{-n})) \cap B(y_n, 2\eta_\varepsilon(\hat{y}_{-n}))$ .

**Proposition 7.2.** *Assume  $\mathcal{H}_2$  and  $\lambda_1 > \lambda_2$ . Let  $\hat{x} \in \mathcal{A} \cap \Lambda_\varepsilon$ ,  $\hat{z} \in \mathcal{T}$  and  $\hat{y} \in \eta_{\hat{x}} \cap \Lambda_\varepsilon$  as before. Assume that  $\hat{x}_{-n} \in \mathcal{A} \cap \{\beta_\varepsilon \leq \tau\}$ , where  $n \geq \max\{N(\hat{x}), N(\hat{y})\}$  and  $\tau > 0$ . Then*

$$q_{\hat{y}} = u_n |C_n \circ f_{\hat{z}}^{-n} \circ \pi_0|^2 \times q_{\hat{x}} + J_n \quad \text{on } \eta_{\hat{x}},$$

where

1.  $0 \leq u_n \leq \frac{L(\hat{x})}{L(\hat{y})} 4\tau^2 \Delta_{\hat{z}}^4$ ,
2.  $C_n : f_{\hat{z}}^{-n}(B_0) \rightarrow \mathbb{C}$  is a holomorphic function bounded by  $1/\mathcal{R}$ ,
3.  $J_n$  is a real valued measure on  $\eta_{\hat{x}}$  satisfying

$$|J_n| \leq c_0 e^{-n(\lambda_1 - \lambda_2 - 4\varepsilon)} \sigma_T \circ \pi_0$$

for some  $c_0 = c_0(\hat{x}, \hat{y}, \mathcal{R})$ .

We give the proof in Section 7.3 and Section 7.4. Let us see how this Proposition implies Equation (19), completing the proof of Proposition 4.5. Taking  $\tau > 0$  large enough, by Poincaré recurrence theorem, for  $\hat{\mu}$ -almost every  $\hat{x} \in \mathcal{A} \cap \{\beta_\varepsilon \leq \tau\} \cap \Lambda_\varepsilon$ , for any  $\hat{y} \in \eta_{\hat{x}} \cap \Lambda_\varepsilon$ , there exists  $(n_k)_k$  satisfying

$$\hat{x}_{-n_k} \in \mathcal{A} \cap \{\beta_\varepsilon \leq \tau\} \quad \text{and} \quad n_k \geq \max\{N(\hat{x}), N(\hat{y})\}.$$

Proposition 7.2 and the fact that  $\text{Lip}(f_{\hat{z}}^{-n}) \rightarrow 0$  (see Theorem 8.2) imply that, up to a subsequence,  $(C_n \circ f_{\hat{z}}^{-n})_n$  converges uniformly on  $\pi_0(\eta_{\hat{x}}) \subset \frac{1}{2}B_0$  to a constant function  $F$ . Proposition 7.2 also implies that  $(u_n)_n$  converges to some  $u \geq 0$  and that  $\lim_n J_n = 0$  (since  $\lambda_1 > \lambda_2$ ). Therefore

$$q_{\hat{y}} = \lim_{n \rightarrow +\infty} u_n |C_n \circ f_{\hat{z}}^{-n} \circ \pi_0|^2 \times q_{\hat{x}} = uF \times q_{\hat{x}} \quad \text{on } \eta_{\hat{x}}.$$

Hence  $uF = 1$  and  $q_{\hat{y}} = q_{\hat{x}}$ , since they are probability measures.

### 7.3 Proof of Proposition 7.2 in the non resonant case

Let  $\hat{x}$  and  $\hat{y}$  as in Proposition 7.2. Recall that  $B_n := B(x_{-n}, 2\eta_\varepsilon(\hat{x}_{-n})) \cap B(y_{-n}, 2\eta_\varepsilon(\hat{y}_{-n}))$ , it is not empty by Equation (22). We define on  $B_n$  the functions

$$C_n^Z := \frac{\partial W_{\hat{y}_{-n}}}{\partial Z_{\hat{x}_{-n}}} := \frac{\partial}{\partial z} \left[ W \circ \xi_{\hat{y}_{-n}} \circ \xi_{\hat{x}_{-n}}^{-1} \right] \circ \xi_{\hat{x}_{-n}}$$

$$C_n^W := \frac{\partial W_{\hat{y}_{-n}}}{\partial W_{\hat{x}_{-n}}} = \frac{\partial}{\partial w} \left[ W \circ \xi_{\hat{y}_{-n}} \circ \xi_{\hat{x}_{-n}}^{-1} \right] \circ \xi_{\hat{x}_{-n}}.$$

We have

$$\frac{i}{2} dW_{\hat{y}_{-n}} \wedge d\overline{W}_{\hat{y}_{-n}} = \sum_{A, B \in \{Z, W\}} C_n^A \overline{C_n^B} \times \frac{i}{2} dA_{\hat{x}_{-n}} \wedge d\overline{B}_{\hat{x}_{-n}} \quad \text{on } B_n. \quad (23)$$

We now pull back Equation (23) by  $f_{\hat{z}}^{-n} : B_0 \rightarrow B_n$ . We assume in this section that the Lyapunov exponents are not resonant, see Section 7.4 for the resonant case. By Theorem 8.2, the map  $(z, w) \mapsto R_{n, \hat{x}}(z, w)$  is linear and for every  $n \geq N(\hat{x})$  :

$$Z_{\hat{x}_{-n}} \circ f_{\hat{x}}^{-n} = \alpha_{n, \hat{x}} \times Z_{\hat{x}} \quad \text{and} \quad W_{\hat{x}_{-n}} \circ f_{\hat{x}}^{-n} = \beta_{n, \hat{x}} \times W_{\hat{x}}.$$

As explain before, by Equation (21)  $\pi_0(\eta_{\hat{x}})$  is included in the intersection of the domains of definition of the maps  $f_{\hat{x}}^{-n}, f_{\hat{y}}^{-n}$  and  $f_{\hat{z}}^{-n}$ , and by Lemma 3.3 these maps coincide on this intersection, thus on  $\pi_0(\eta_{\hat{x}})$ . The pull back of Equation (23) by  $f_{\hat{z}}^{-n} : \pi_0(\eta_{\hat{x}}) \rightarrow f_{\hat{z}}^{-n}(\pi_0(\eta_{\hat{x}}))$  then gives :

$$|\beta_{n,\hat{y}}|^2 \times \frac{i}{2} dW_{\hat{y}} \wedge d\overline{W}_{\hat{y}} = \sum_{A,B \in \{Z,W\}} F_n^{AB} \times \alpha_{n,\hat{x}}^A \overline{\alpha_{n,\hat{x}}^B} \times \frac{i}{2} dA_{\hat{x}} \wedge d\overline{B}_{\hat{x}} \text{ on } \pi_0(\eta_{\hat{x}}), \quad (24)$$

where

$$F_n^{AB} := C_n^A \circ f_{\hat{z}}^{-n} \times \overline{C_n^B \circ f_{\hat{z}}^{-n}}, \quad \alpha_{n,\hat{x}}^Z := \alpha_{n,\hat{x}}, \quad \alpha_{n,\hat{x}}^W := \beta_{n,\hat{x}}.$$

Let us denote  $\lambda_{AB} := T \wedge \frac{i}{2} dA_{\hat{x}} \wedge d\overline{B}_{\hat{x}}$ . Since  $\hat{y} \in \mathcal{A}$  the definition of  $q_{\hat{y}}$  yields

$$(T \wedge \frac{i}{2} dW_{\hat{y}} \wedge d\overline{W}_{\hat{y}}) \circ \pi_0|_{\eta_{\hat{x}}} = \frac{L(\hat{y})}{M(\hat{y})^2} \times q_{\hat{y}},$$

with a similar formula for  $\hat{x} \in \mathcal{A}$ . Wedging Equation (24) by  $T$ , we deduce on  $\eta_{\hat{x}}$  :

$$q_{\hat{y}} = u_n |C_n \circ f_{\hat{z}}^{-n} \circ \pi_0|^2 \times q_{\hat{x}} + J_n, \quad (25)$$

where

$$u_n := \frac{L(\hat{x})M(\hat{y})^2|\beta_{n,\hat{x}}|^2}{L(\hat{y})M(\hat{x})^2|\beta_{n,\hat{y}}|^2}, \quad C_n := C_n^W, \quad (26)$$

$$J_n := \frac{M(\hat{y})^2}{L(\hat{y})|\beta_{n,\hat{y}}|^2} \sum_{(A,B) \neq (W,W)} (F_n^{AB} \circ \pi_0) \times \alpha_{n,\hat{x}}^A \overline{\alpha_{n,\hat{x}}^B} \cdot \lambda_{AB} \circ \pi_0|_{\eta_{\hat{x}}}.$$

— Let us now prove that  $u_n$  is bounded by a constant independent of  $n$ . Lemma 4.9 implies  $|\beta_{n,\hat{x}}|^2/M(\hat{x})^2 = M(\hat{x}_{-n})^{-2} \Delta(x_{-n}, n)^{-2}$  (with a similar formula for  $\hat{y}$ ). We deduce

$$u_n = \frac{L(\hat{x})M(\hat{y}_{-n})^2 \Delta(y_{-n}, n)^2}{L(\hat{y})M(\hat{x}_{-n})^2 \Delta(x_{-n}, n)^2} = \frac{L(\hat{x})}{L(\hat{y})} \times \frac{M(\hat{y}_{-n})^2}{M(\hat{x}_{-n})^2} \times \frac{\Delta(y_{-n}, n)^2}{\Delta(z_{-n}, n)^2} \frac{\Delta(z_{-n}, n)^2}{\Delta(x_{-n}, n)^2}.$$

We then get  $0 \leq u_n \leq \frac{L(\hat{x})}{L(\hat{y})} 4\tau^2 \Delta_{\hat{z}}^4$  by using the inequalities  $\beta_{\varepsilon}^{-1}(\hat{w}) \leq M(\hat{w}) \leq 2$  (see Theorem 8.2),  $\hat{x}_{-n} \in \{\beta_{\varepsilon} \leq \tau\}$  and Proposition 3.7 recalling that  $\hat{x}, \hat{y} \in W^u(\hat{z}, \mathcal{R}) \cap \Lambda_{\varepsilon}$ .

— Let us prove that the function  $C_n : f_{\hat{z}}^{-n}(B_0) \rightarrow \mathbb{C}$  is bounded by  $1/\mathcal{R}$ . We verify more generally that

$$|C_n^A|, |C_n^B| \leq 1/\mathcal{R} \text{ on } B_n. \quad (27)$$

By Theorem 8.2, the change of coordinates  $H := \xi_{\hat{y}_{-n}} \circ \xi_{\hat{x}_{-n}}^{-1}$  satisfies  $\|dH\| \leq 2\beta_{\varepsilon}(\hat{y}_{-n})$  on  $B_n$ . Since  $\hat{y}_{-n} \in \mathcal{A} \cap \Lambda_{\varepsilon}$ , Proposition 3.7 yields  $\|dH\| \leq 1/\mathcal{R}$ , which implies Equation (27) through the definition of  $C_n^Z$  and  $C_n^W$ .

— It remains to prove the upper estimate on  $|J_n|$ . Since  $f_{\hat{z}}^{-n}(B_0) \subset B_n$ , by definition of  $F_n^{AB}$  and from Equation (27), we have  $|F_n^{AB}| \leq 1/\mathcal{R}^2$  on  $B_0$ . Now let us observe (a similar inequality holds for  $\lambda_{WW}$ )

$$\lambda_{ZZ} = T \wedge dd^c |Z_{\hat{x}}|^2 \leq \|d\xi_{\hat{x}}\|^2 (T \wedge \omega_{\mathbb{P}^2}) \leq \beta_{\varepsilon}(\hat{x})^2 \sigma_T \text{ on } B(x_0, 2\eta_{\varepsilon}(\hat{x})).$$

The same estimate holds for  $|\lambda_{ZW}|$  and  $|\lambda_{WZ}|$ , since  $|\lambda_{ZW}| \leq \sqrt{\lambda_{ZZ}} \sqrt{\lambda_{WW}}$  by Cauchy-Schwarz inequality. Finally, Theorem 8.2 provides  $|\alpha_{n,\hat{x}} \cdot \beta_{n,\hat{x}}|/|\beta_{n,\hat{y}}|^2 \leq e^{-n(\lambda_1 - \lambda_2 - 4\varepsilon)}$  and one has (since  $\pi_0(\eta_{\hat{x}}) \subset B_0$ )

$$|J_n| \leq c_0 e^{-n(\lambda_1 - \lambda_2 - 4\varepsilon)} \sigma_T \circ \pi_0|_{\eta_{\hat{x}}}$$

by setting  $c_0 := \frac{M(\hat{y})^2 \beta_{\varepsilon}(\hat{x})^2}{L(\hat{y})\mathcal{R}^2}$ . This completes the proof of Proposition 7.2.  $\square$

## 7.4 Proof of Proposition 7.2 in the resonant case

When the Lyapunov exponents satisfy  $\lambda_1 = k\lambda_2$  for some  $k \geq 2$ , the first component of the map  $R_{n,\hat{x}}$  may be not linear (see Theorem 8.2) :

$$R_{n,\hat{x}}(z, w) = \left( \alpha_{n,\hat{x}}z + \gamma_{n,\hat{x}}w^k, \beta_{n,\hat{x}}w \right).$$

The proof of Proposition 7.2 can be modified as follows. Equation (25) becomes :

$$q_{\hat{y}} = u_n |C_n \circ f_{\hat{z}}^{-n} \circ \pi_0|^2 \times q_{\hat{x}} + \tilde{J}_n \text{ on } \eta_{\hat{x}},$$

where  $\tilde{J}_n$  is a new sequence satisfying  $|\tilde{J}_n| \leq \tilde{c}_0 e^{-n(\lambda_1 - \lambda_2 - 4\varepsilon)} \sigma_T \circ \pi_0|_{\eta_{\hat{x}}}$  for some  $\tilde{c}_0(\hat{x}, \hat{y}, \mathcal{R}) > 0$ . Indeed, Equation (24) is replaced by :

$$\begin{aligned} |\beta_{n,\hat{y}}|^2 \frac{i}{2} (dW_{\hat{y}} \wedge d\overline{W}_{\hat{y}}) &= F_n^{ZZ} |\alpha_{n,\hat{x}}|^2 \frac{i}{2} dZ_{\hat{x}} \wedge d\overline{Z}_{\hat{x}} \\ &+ 2\text{Re} \left[ \left( k F_n^{ZZ} \alpha_{n,\hat{x}} \overline{\gamma}_{n,\hat{x}} \overline{W}_{\hat{x}}^{k-1} + F_n^{ZW} \alpha_{n,\hat{x}} \overline{\beta}_{n,\hat{x}} \right) \frac{i}{2} dZ_{\hat{x}} \wedge d\overline{W}_{\hat{x}} \right] \\ &+ \left( 2\text{Re} \left[ k F_n^{ZW} \gamma_{n,\hat{x}} \overline{\beta}_{n,\hat{x}} W_{\hat{x}}^{k-1} \right] + k^2 F_n^{ZZ} |\gamma_{n,\hat{x}}|^2 |W_{\hat{x}}|^{2(k-1)} \right) \frac{i}{2} dW_{\hat{x}} \wedge d\overline{W}_{\hat{x}} \\ &+ F_n^{WW} |\beta_{n,\hat{x}}|^2 \frac{i}{2} dW_{\hat{x}} \wedge d\overline{W}_{\hat{x}}. \end{aligned}$$

From this new expression one can compute  $\tilde{J}_n$  as we did for Equation (26), and get on  $\eta_{\hat{x}}$  :

$$\begin{aligned} \frac{L(\hat{y}) |\beta_{n,\hat{y}}|^2}{M(\hat{y})^2} \tilde{J}_n &= (F_n^{ZZ} \circ \pi_0) |\alpha_{n,\hat{x}}|^2 \lambda_{ZZ} \circ \pi_0 \\ &+ 2\text{Re} \left[ \left( k (F_n^{ZZ} \circ \pi_0) \alpha_{n,\hat{x}} \overline{\gamma}_{n,\hat{x}} (\overline{W}_{\hat{x}}^{k-1} \circ \pi_0) + (F_n^{ZW} \circ \pi_0) \alpha_{n,\hat{x}} \overline{\beta}_{n,\hat{x}} \right) \lambda_{ZW} \circ \pi_0 \right] \\ &+ 2\text{Re} \left[ k (F_n^{ZW} \circ \pi_0) \gamma_{n,\hat{x}} \overline{\beta}_{n,\hat{x}} (W_{\hat{x}}^{k-1} \circ \pi_0) \right] \lambda_{WW} \circ \pi_0 \\ &+ k^2 (F_n^{ZZ} \circ \pi_0) |\gamma_{n,\hat{x}}|^2 |W_{\hat{x}} \circ \pi_0|^{2(k-1)} \lambda_{WW} \circ \pi_0. \end{aligned}$$

We get a constant  $\tilde{c}_0$  by  $|\beta_{n,\hat{y}}|^{-1} \leq e^{n(\lambda_2 + \varepsilon)}$ ,  $|\alpha_{n,\hat{x}}| \leq e^{-n(\lambda_1 - \varepsilon)}$ ,  $|\beta_{n,\hat{x}}| \leq e^{-n(\lambda_2 - \varepsilon)}$ ,  $|\gamma_{n,\hat{x}}| \leq M_\varepsilon(\hat{x}) e^{-n(\lambda_1 - \varepsilon)}$  (Theorem 8.2) and the fact that  $W_{\hat{x}}$  is bounded on  $B(x_0, \eta_\varepsilon(\hat{x})) \supset \pi_0(\eta_{\hat{x}})$ , and  $|F_n^{AB}| \leq 1/\mathcal{R}^2$  by Equation (27).

## 8 Classical results

Let  $f$  be an endomorphism of  $\mathbb{P}^2$  of degree  $d \geq 2$ . Let  $\mu$  be its equilibrium measure and let us denote  $\lambda_1 \geq \lambda_2$  the Lyapunov exponents of  $\mu$ . Let  $\text{Crit}(f)$  denote the critical set of  $f$  and  $\mathcal{C} := \bigcup_{n \in \mathbb{Z}} f^n(\text{Crit}(f))$ . We know that  $\mu(\mathcal{C}) = 0$ .

### 8.1 Oseledec Theorem

**Theorem 8.1.** *Assume that  $\lambda_1 > \lambda_2$ . There exists a totally invariant borel set  $A_{os}$  of full  $\mu$ -measure disjoint from  $\mathcal{C}$  and a measurable map  $v_s : A_{os} \rightarrow \mathbb{P}(T\mathbb{P}^2)$  such that*

$$\forall x \in A_{os}, \forall \vec{v} \in v_s(x) \setminus \{0\}, \lim_{n \rightarrow \infty} \frac{1}{n} \text{Log} \|d_x f^n \cdot \vec{v}\| = \lambda_2.$$

Moreover  $d_x f^n(v_s(x)) = v_s(f^n(x))$ . If  $\vec{v} \in T_x \mathbb{P}^2 \setminus v_s(x)$  then the limit above is equal to  $\lambda_1$ . We denote

$$\chi_f(x) := \text{Log} \|d_x f \cdot \vec{v}_s(x)\|,$$

where  $\vec{v}_s(x)$  is any vector of norm 1 in the complex line  $v_s(x)$ . The following points hold :

1.  $\chi_f \in L^1(\mu)$ ,
2. for every  $x \in A_{os}$ ,  $\frac{1}{n} \sum_{i=0}^{n-1} \chi_f(f^i(x)) = \frac{1}{n} \chi_{f^n}(x)$ . In particular,  $\int \chi_f d\mu = \lambda_2$ .
3.  $\chi_{f^n} \circ \pi_0 \in L^1(\widehat{\mu})$  and

$$\int_{\widehat{\mathbb{P}^2}} \chi_{f^n}(x_0) d\widehat{\mu}(\widehat{x}) = n\lambda_2.$$

We will denote  $\Delta(x, n) := \|d_x f^n \cdot \vec{v}_s(x)\|$ .

The proof of the first item can be found in [24, Section 3.7]. The second one comes from the chain rule formula and Birkhoff ergodic theorem. The third item relies on  $(\pi_0)_* \widehat{\mu} = \mu$ .

## 8.2 Normal forms for inverse branches

Let  $\widehat{\mathbb{P}^2}$  be the set of orbits  $\{\widehat{x} = (x_n)_{n \in \mathbb{Z}}, f(x_n) = x_{n+1}\}$ . We denote  $\pi_0 : \widehat{\mathbb{P}^2} \rightarrow \mathbb{P}^2$  the projection  $\pi_0 : \widehat{x} \mapsto x_0$  and  $\widehat{f}$  the left shift. By Kolmogorov extension theorem, there exists a unique  $\widehat{f}$ -invariant probability measure on  $\widehat{\mathbb{P}^2}$  such that  $(\pi_0)_* \widehat{\mu} = \mu$ . Let  $X := \mathbb{P}^2 \setminus \mathcal{C}$  and  $\widehat{X} := \{\widehat{x} \in \widehat{\mathbb{P}^2} : x_n \in X, \forall n \in \mathbb{Z}\}$ , this is a  $\widehat{f}$ -invariant subset of  $\widehat{\mathbb{P}^2}$  of full  $\widehat{\mu}$ -measure.

The following theorem was proved by Berteloot-Dupont-Molino [4], it gives normal forms for the iterated inverse branches of the dynamical system. We refer to the articles [18] by Jonsso-Varolin and [3] by Berteloot-Dupont for related results. A real-valued function  $\varphi$  on  $\widehat{\mathbb{P}^2}$  is  $\varepsilon$ -tempered if  $e^{-|n|\varepsilon} \varphi \leq \varphi \circ \widehat{f}^n \leq e^{|n|\varepsilon} \varphi$  for every  $n \in \mathbb{Z}$ .

**Theorem 8.2.** *Let  $f$  be an endomorphism of  $\mathbb{P}^2$  of degree  $d \geq 2$ . Let  $\varepsilon$  be small with respect to the Lyapunov exponents  $\lambda_1 > \lambda_2$  of  $\mu$ . There exist a totally  $\widehat{f}$ -invariant borel set  $\mathcal{FN}_\varepsilon \subset \widehat{X}$  of full  $\widehat{\mu}$ -measure,  $\varepsilon$ -tempered functions  $\rho_\varepsilon, \eta_\varepsilon : \mathcal{FN}_\varepsilon \rightarrow ]0, 1]$ ,  $\beta_\varepsilon, M_\varepsilon : \mathcal{FN}_\varepsilon \rightarrow [1, +\infty[$  and a function  $N : \mathcal{FN}_\varepsilon \rightarrow \mathbb{N}$  satisfying the following properties for every  $\widehat{x} \in \mathcal{FN}_\varepsilon$  :*

1. *there is an injective holomorphic map  $\xi_{\widehat{x}} : B(x_0, 2\eta_\varepsilon(\widehat{x})) \rightarrow \mathbb{D}^2(\rho_\varepsilon(\widehat{x}))$  such that :*
  - (a)  $\xi_{\widehat{x}}(x_0) = 0$  and  $v_s(x_0) = (d_0 \xi_{\widehat{x}}^{-1} \cdot (0, 1)) \mathbb{C}$ ,
  - (b)  $\forall p, q \in B(x_0, 2\eta_\varepsilon(\widehat{x}))$ ,  $\frac{1}{2} \text{dist}(p, q) \leq \|\xi_{\widehat{x}}(p) - \xi_{\widehat{x}}(q)\| \leq \beta_\varepsilon(\widehat{x}) \text{dist}(p, q)$ .
2. *there is a sequence of holomorphic maps  $(f_{\widehat{x}}^{-n})_n$  such that  $f^n \circ f_{\widehat{x}}^{-n} = \text{Id}$  on  $B(x_0, 2\eta_\varepsilon(\widehat{x}))$ ,  $f_{\widehat{x}}^{-n}(x_0) = x_{-n}$  and  $\text{Lip}(f_{\widehat{x}}^{-n}) \leq \beta_\varepsilon(\widehat{x}) e^{-n(\lambda_2 - \varepsilon)}$ ,*
3. *the diagram (9) in Section 2.2 commutes for any  $n \geq N(\widehat{x})$ .*

The map  $R_{n, \widehat{x}}$  is equal to

$$R_{n, \widehat{x}}(z, w) = \left( \alpha_{n, \widehat{x}} z + \gamma_{n, \widehat{x}} w^k, \beta_{n, \widehat{x}} w \right),$$

where  $k = \lambda_1 / \lambda_2$ . Moreover  $\gamma_{n, \widehat{x}} = 0$  if  $k \notin \{2, 3, \dots\}$ , and

1.  $e^{-n(\lambda_1 + \varepsilon)} \leq |\alpha_{n, \widehat{x}}| \leq e^{-n(\lambda_1 - \varepsilon)}$  and  $|\gamma_{n, \widehat{x}}| \leq M_\varepsilon(\widehat{x}) e^{-n(\lambda_1 - \varepsilon)}$ .
2.  $e^{-n(\lambda_2 + \varepsilon)} \leq |\beta_{n, \widehat{x}}| \leq e^{-n(\lambda_2 - \varepsilon)}$ .

**Remark 8.3.** *The functions  $\eta_\varepsilon, \rho_\varepsilon, \beta_\varepsilon$  are related by  $2\eta_\varepsilon = \rho_\varepsilon / \beta_\varepsilon \leq 1 / \beta_\varepsilon$ , see for instance [3].*

## 9 Results in higher dimensions

Let  $f$  be a holomorphic mapping of degree  $d \geq 2$  on  $\mathbb{P}^k$ , let  $\mu = T^{\wedge k}$  be the measure of maximal entropy and  $\lambda_1 \geq \dots \geq \lambda_k$  be its Lyapunov exponents. Assume that  $k \geq 2$  and that there exists  $r \in \{1, \dots, k-1\}$  such that :

$$\lambda_1 \geq \dots \geq \lambda_r > \lambda_{r+1} \geq \dots \geq \lambda_k. \quad (28)$$

The case  $\lambda_{r+1} = \dots = \lambda_r = \frac{1}{2} \text{Log } d$  generalizes the notion of semi-extremal maps on  $\mathbb{P}^2$ . To extend our method on  $\mathbb{P}^k$ , we have to consider a new stable cocycle  $\Delta(x, n)$  and to consider several submersions  $W_{\hat{x}}$ .

By classical ergodic theory, Theorem 8.1 extends as follows. Assuming (28), there exists  $A_{os} \subset \mathbb{P}^k$  a totally invariant borel set of full  $\mu$ -measure, such that for every  $x \in A_{os}$  there exists a complex sub-vector space  $V_s(x) \subset T_x \mathbb{P}^k$  of dimension  $k-r$  satisfying :

- $d_x f^n : V_s(x) \rightarrow V_s(f^n x)$  is an isomorphism  $\mathbb{C}$ -linear.
- $\forall \vec{v} \in V_s(x) \setminus \{0\}$ ,  $\lim_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n \cdot \vec{v}\| \in \{\lambda_{r+1}, \dots, \lambda_k\}$ .
- $\forall \vec{v} \in T_x \mathbb{P}^k \setminus V_s(x)$ ,  $\lim_{n \rightarrow +\infty} \frac{1}{n} \text{Log} \|d_x f^n \cdot \vec{v}\| \in \{\lambda_1, \dots, \lambda_r\}$ .

By setting

$$\Delta(x, n) := |\det_{\mathbb{C}}(d_x f^n : V_s(x) \rightarrow V_s(f^n x))|, \quad (29)$$

we also have :

$$\int_{\mathbb{P}^k} \text{Log } \Delta(x, n) \, d\mu(x) = n(\lambda_{r+1} + \dots + \lambda_k).$$

Theorem 8.2 extends on  $\mathbb{P}^k$  with the commutative diagram

$$\begin{array}{ccc} B(x_{-n}, 2\eta_\varepsilon(\hat{x}_{-n})) & \xleftarrow{f_{\hat{x}}^{-n}} & B(x_0, 2\eta_\varepsilon(\hat{x})) \\ \xi_{\hat{x}_{-n}} \downarrow & & \downarrow \xi_{\hat{x}} \\ \mathbb{D}^k(\rho_\varepsilon(\hat{x}_{-n})) & \xleftarrow{R_{n, \hat{x}}} & \mathbb{D}^k(\rho_\varepsilon(\hat{x})) \end{array}$$

Assuming that there is no resonance between  $\lambda_{r+1} \geq \dots \geq \lambda_k$ , the map  $R_{n, \hat{x}}$  has the form

$$\left( \alpha_{n, \hat{x}}^1 z_k + P_1(z_2, \dots, w_k), \dots, \alpha_{n, \hat{x}}^r z_r + P_r(w_{r+1}, \dots, w_k), \beta_{n, \hat{x}}^{r+1} w_{r+1}, \dots, \beta_{n, \hat{x}}^k w_k \right), \quad (30)$$

where  $e^{-n(\lambda_j + \varepsilon)} \leq |\alpha_{n, \hat{x}}^j|, |\beta_{n, \hat{x}}^j| \leq e^{-n(\lambda_j - \varepsilon)}$ . The polynomials  $P_j$  are sums of

$$\gamma z_{j+1}^{\alpha_{j+1}} \dots z_r^{\alpha_r} w_{r+1}^{\beta_{r+1}} \dots w_k^{\beta_k},$$

where the length of  $(\alpha_{j+1}, \dots, \beta_k) \in \mathbb{N}^{k-j}$  is  $\geq 2$ , and  $\gamma$  is bounded as  $\gamma_{n, \hat{x}}$  in Theorem 8.2. If  $\lambda_{j+1}^{\alpha_{j+1}} \dots \lambda_k^{\beta_k} \neq \lambda_j$ , then  $\gamma = 0$ . If we write  $\xi_{\hat{x}} : B(x_0, \eta_\varepsilon(\hat{x})) \rightarrow \mathbb{D}^k(\rho_\varepsilon(\hat{x}))$  as

$$\xi_{\hat{x}} = (Z_{\hat{x}}^1, \dots, Z_{\hat{x}}^r, W_{\hat{x}}^{r+1}, \dots, W_{\hat{x}}^k).$$

then it satisfies

$$d_0 \xi_{\hat{x}}^{-1} \left( \{0\}^r \times \mathbb{C}^{k-r} \right) = V_s(x).$$

Now we set

$$M(\hat{x}) := \left| \det_{\mathbb{C}} \left( d_0 \xi_{\hat{x}}^{-1} : \{0\}^r \times \mathbb{C}^{k-r} \rightarrow V_s(x) \right) \right|$$

and we recall that the cocycle  $\Delta(x, n)$  is defined in (29).

**Definition 9.1.** Assume  $\mathcal{H}_1$ . For every  $\widehat{x} \in \Lambda_\varepsilon$ , we define

$$\tilde{q}_{\widehat{x}} := (d^r)^p (\Delta(x_{-p}, p))^2 M(\widehat{x}_{-p})^2 \times \left( T^r \wedge dd^c |W_{\widehat{x}_{-p}}^{r+1}|^2 \wedge \cdots \wedge dd^c |W_{\widehat{x}_{-p}}^k|^2 \right) \circ \pi_0 \circ \widehat{f}^{-p}|_{\eta_{\widehat{x}}},$$

where  $p \geq 0$  is such that  $\widehat{x}_{-p}$  belongs to  $\mathcal{A}$ . Then  $q_{\widehat{x}} := \frac{1}{L(\widehat{x})} \tilde{q}_{\widehat{x}}$ , where  $L(\widehat{x}) := \tilde{q}_{\widehat{x}}(\eta_{\widehat{x}})$ .

Since there is no resonance between the  $k-r$  smallest exponents, the  $k-r$  last components of  $R_{n,\widehat{x}}$  are linear (see (30)), and one can verify as in Proposition 4.7 that the definition of  $\tilde{q}_{\widehat{x}}$  does not depend on  $p$ . The following formula, which extends Lemma 4.9, holds true

$$\Delta(x_{-n}, n) \prod_{i=r+1}^k |\beta_{n,\widehat{x}}^i| = \frac{M(\widehat{x})}{M(\widehat{x}_{-n})},$$

(note that it does not require the linearity for the last components of  $R_{n,\widehat{x}}$ ). Using similar arguments as for  $k=2$ , Theorems A and B extends to  $\mathbb{P}^k$  as follows.

**Theorem 9.2.** Let  $f$  be an endomorphism of degree  $d \geq 2$  on  $\mathbb{P}^k$  with  $k \geq 2$ . Assume that the Lyapunov exponents of the measure of maximal entropy  $\mu$  satisfy  $\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} \geq \cdots \geq \lambda_k$  for some  $r \in \{1, \dots, k-1\}$ . Assume that there is no resonance between the  $k-r$  smallest exponents.

1. If  $\mathcal{H}_1$  holds, then there exist a measurable partition  $\eta$  and a measurable family  $\widehat{x} \mapsto q_{\widehat{x}}$  of probability measures on  $\widehat{\mathbb{P}}^2$  supported on  $\eta_{\widehat{x}}$  which satisfy

$$(\pi_0)_* q_{\widehat{x}} \ll T^r \wedge \omega_{\mathbb{P}^k}^{k-r},$$

$$\forall n \geq 0, \int_{\widehat{\mathbb{P}}^k} -\text{Log } q_{\widehat{x}}(\widehat{f}^{-n}\eta)_{\widehat{x}} d\widehat{\mu}(x) = \text{Log } (d^r)^n + 2n(\lambda_{r+1} + \cdots + \lambda_k).$$

2. If  $\mathcal{H}_2$  holds and if  $\lambda_{r+1} = \cdots = \lambda_k = \frac{1}{2} \text{Log } d$ , then  $\mu \ll T^r \wedge \omega_{\mathbb{P}^k}^{k-r}$ .

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