A posteriori error estimates for the time dependent convection-diffusion-reaction equation coupled with the Darcy system

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A POSTERIORI ERROR ESTIMATES FOR THE TIME DEPENDENT CONVECTION-DIFFUSION-REACTION EQUATION COUPLED WITH THE DARCY SYSTEM

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Abstract. In this article, we consider the time dependent convection–diffusion–reaction equation coupled with the Darcy equation. We propose a numerical scheme based on finite element methods for the discretization in space and the implicit Euler method for the discretization in time. We establish optimal *a posteriori* error estimates with two types of computable error indicators, the first one linked to the time discretization and the second one to the space discretization. Finally, numerical investigations are performed and presented.
Keywords. Darcy's equations; convection-diffusion-reaction equation; finite element method; *a posteriori* error estimates; adaptive methods.

1. Introduction

The main objective of the *a posteriori* error analysis is to give tools that allow the control of the overall discretization error of a problem by providing error indicators that are easy to compute. Once the error indicators are constructed, their efficiency can be proven by bounding each indicator by the local error. The *a posteriori* error analysis was first introduced by Babuška and Rheinboldt [5]. Later developments are presented, among other references, in Verfürth [29] and Ainsworth and Oden [1]. Error estimators for the Darcy equation discretized by mixed finite element methods have been studied in many works, see for instance [2], [10] and [28]. In [14], Chen and Wang establish optimal *a posteriori* error estimates for the $H(\mathrm{div},\Omega)$-conforming mixed finite element method applied to the coupled Darcy-Stokes system in two dimensions. For the Darcy equations with pressure dependent viscosity, we refer to [25] and the references therein. In [31], Vohralík establishes a residual *a posteriori* error estimate for lowest-order Raviart-Thomas mixed finite element discretizations of the convection–diffusion–reaction equation on simplicial meshes in two and three space dimensions, while in [32], general locally conservative methods on general meshes are also studied. In [22], Ern et al. proposed and studied *a posteriori* error estimates for the convection–diffusion–reaction problem with non-homogeneous and anisotropic diffusion approximated by weighted interior-penalty discontinuous Galerkin methods. For the time-dependent convection–diffusion–reaction equation, Chalhoub et al. [11] established an *a posteriori* error estimate for the cell-centered finite volume scheme in space and backward Euler scheme in time. In [33], the authors propose a simple *a posteriori* error estimator for porous media problems discretized by lowest-order locally conservative methods on meshes consisting of general polytopal elements, ranging from the steady linear Darcy equation to unsteady nonlinear systems describing multiphase Darcy flows.

For the coupled problem of Darcy's law with the heat equation, we can refer to Bernardi, Maarouf and Yakoubi [7] where the coupled problem is treated using the spectral method. Bernardi et al. [6] and Dib et al. [18] considered the same stationary system but coupled with a nonlinear viscosity that depends on the temperature. In [19], Dib et al. derived an optimal *a posteriori* error estimate for each of

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the numerical schemes proposed in [6]. We can also refer to Amaziane et al. in [3] where the authors used a vertex-centred finite volume method to discretize the coupled system. In [12], we studied the time-dependent convection–diffusion–reaction equation coupled with Darcy's equation, then we proposed and analysed two numerical schemes based on finite element methods for the discretization in space and the implicit Euler method for the discretization in time. One of these schemes will be considered in the present work for the *a posteriori* error studies.

Let Ω be a connected bounded open set in $\mathbb{R}^d, d = 2, 3$, with a Lipschitz-continuous boundary $\Gamma = \partial\Omega$, and let $[0, T]$ be an interval of $\mathbb{R}$. We consider the following system

$$
(\mathrm{P}) \left\{ \begin{array}{rcll}
\nu(C(\mathbf{x},t))\mathbf{u}(\mathbf{x},t) + \nabla\, p(\mathbf{x},t) & = & \mathbf{f}(\mathbf{x},t,C(\mathbf{x},t)) & \text{in}\,\Omega\times[0,T], \\
(\operatorname{div}\mathbf{u})(\mathbf{x},t) & = & 0 & \text{in}\,\Omega\times[0,T], \\
\dfrac{\partial C}{\partial t}(\mathbf{x},t) - \alpha\Delta C(\mathbf{x},t) + (\mathbf{u}(\mathbf{x},t)\cdot\nabla\, C)(\mathbf{x},t) + r_0 C(\mathbf{x},t) & = & g(\mathbf{x},t) & \text{in}\,\Omega\times]0,T[, \\
(\mathbf{u}\cdot\mathbf{n})(\mathbf{x},t) & = & 0 & \text{on}\,\Gamma\times[0,T], \\
C(\mathbf{x},t) & = & 0 & \text{on}\,\Gamma\times[0,T], \\
C(\mathbf{x},0) & = & 0 & \text{in}\,\Omega,
\end{array} \right.
$$

where $\mathbf{n}$ is the unit outward normal vector on Γ. The unknowns are the fluid velocity $\mathbf{u}$, the pressure p and the concentration C in the fluid. The function $\mathbf{f}$ represents an external force that depends on the concentration C and the function g represents an external concentration source. The viscosity ν also depends on the concentration C but the diffusion coefficient α and the parameter r_0 are positive constants. To simplify, a homogeneous Dirichlet boundary condition is prescribed on the concentration C, but the present analysis can be easily extended to a non-homogeneous boundary condition.
The outline of the paper is as follows:

- In Section 2, we introduce some notations and functional spaces that are useful for the study of the problem.
- In section 3, we introduce the variational formulation.
- In section 4, we introduce the discrete problem and we recall its main properties.
- In section 5, we study the *a posteriori* errors.
- Some numerical experiments are presented in section 6.

2. Preliminaries

In this section, we recall the main notations and results which we use later on. We introduce the Sobolev space

$$W^{m,r}(\Omega)^d = \left\{\mathbf{v}\in L^r(\Omega)^d;\ \partial^k\mathbf{v}\in L^r(\Omega)^d,\ \forall|k|\le m\right\},$$

where $k = \{k_1, \cdots, k_d\}$ is a vector of non negative integers, such that $|k| = k_1 + \cdots + k_d$ and

$$\partial^k\mathbf{v} = \frac{\partial^{|k|}\mathbf{v}}{\partial^{k_1}x_1 ... \partial^{k_d}x_d}.$$

This space is equipped with the semi-norm

$$|\mathbf{v}|_{W^{m,r}(\Omega)^d} = \left(\sum_{|k|=m}\int_\Omega |\partial^k\mathbf{v}|^r d\mathbf{x}\right)^{\frac{1}{r}},$$

and is a Banach space for the norm

$$\|\ \mathbf{v}\ \|_{W^{m,r}(\Omega)^d} = \left(\sum_{l=0}^{m}\int_\Omega |\mathbf{v}|^r_{W^{l,r}(\Omega)^d} d\mathbf{x}\right)^{\frac{1}{r}}.$$

When $r = 2$, this space is the Hilbert space $H^m(\Omega)^d$.
In particular, we consider the following spaces

$$H^1_0(\Omega)^d = \left\{\mathbf{v}\in H^1(\Omega)^d;\ \mathbf{v}_{|\partial\Omega} = 0\right\},$$

equipped with the norm

$$|\mathbf{v}|_{H_0^1(\Omega)^d} = |\mathbf{v}|_{1,\Omega} = \left(\int_\Omega |\nabla \mathbf{v}|^2 d\mathbf{x}\right)^{1/2}.$$

The dual of $H_0^1(\Omega)^d$ is denoted by $H^{-1}(\Omega)^d$.
We also introduce

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega); \int_\Omega q(\mathbf{x}) d\mathbf{x} = 0 \right\}.$$

We define the following scalar product in $L^2(\Omega)$:

$$(v, w) = \int_\Omega v(\mathbf{x}) w(\mathbf{x}) d\mathbf{x}, \quad \forall v, w \in L^2(\Omega).$$

We recall the following Poincaré and Sobolev inequalities:

Lemma 2.1. *For any $p \geq 1$ when $d = 1$ or 2, or $1 \leq p \leq \dfrac{2d}{d-2}$ when $d \geq 3$, there exist two positive constants S_p and S_p^0 such that*

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d, \quad \| \boldsymbol{v} \|_{L^p(\Omega)^d} \leq S_p^0 |\boldsymbol{v}|_{H_0^1(\Omega)^d},$$

and

$$\forall \boldsymbol{v} \in H^1(\Omega)^d \ , \| \boldsymbol{v} \|_{L^p(\Omega)^d} \leq S_p \| \boldsymbol{v} \|_{H^1(\Omega)^d} .$$

Finally, we recall the inf-sup condition between $H^1(\Omega) \cap L_0^2(\Omega)$ and $L^2(\Omega)^d$, (see [6]),

$$\inf_{q \in H^1(\Omega) \cap L_0^2(\Omega)} \sup_{\mathbf{v} \in L^2(\Omega)^d} \frac{\displaystyle\int_\Omega \mathbf{v}.\nabla\, q \, d\mathbf{x}}{\|\mathbf{v}\|_{L^2(\Omega)^d} |q|_{H^1(\Omega)}} \geq 1. \tag{2.1}$$

As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a, b[$ with values in a separable functional space W equipped with a norm $\| \cdot \|_W$. For all $r \geq 1$, we introduce the space

$$L^r(a, b; W) = \left\{ f \text{ measurable on }]a, b[; \int_a^b \| f(t) \|_W^r \, dt < \infty \right\};$$

equipped with the norm

$$\| f \|_{L^r(a,b;W)} = \left(\int_a^b \| f(t) \|_W^r \, dt \right)^{\frac{1}{r}}.$$

If $r = \infty$, then

$$L^\infty(a, b; W) = \left\{ f \text{ measurable on }]a, b[; \sup_{t \in [a,b]} \| f(t) \|_W < \infty \right\}.$$

In the next lemma we recall the Gronwall-Bellman inequality shown in [30, p. 292] and [17, p. 252].

Lemma 2.2. *Let*

(1) *$\bar{f}$, $\bar{g}$ and $\bar{k}$, be integrable functions defined from $\mathbb{R}^+$ to $\mathbb{R}$,*
(2) *$\bar{g} \geqslant 0$, $\bar{k} \geqslant 0$,*
(3) *$\bar{g} \in L^\infty(\mathbb{R}^+)$,*
(4) *$\bar{g} \times \bar{k}$ is an integrable function on $\mathbb{R}^+$.*

If $y : \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$\forall t \in \mathbb{R}^+, y(t) \leq \bar{f}(t) + \bar{g}(t) \int_0^t \bar{k}(\tau) y(\tau) d\tau, \tag{2.2}$$

then

$$y(t) \leq \bar{f}(t) + \bar{g}(t) \int_0^t \bar{k}(\tau) \bar{f}(\tau) \exp\left(\int_\tau^t \bar{k}(s) \bar{g}(s) ds \right) d\tau. \tag{2.3}$$

To end this section, we recall a useful bound which will be used in all the article:

Remark 2.3. *Let a and b be two real numbers. For any positive real number ξ, we have*

$$ab \leq \frac{1}{2\xi}a^2 + \frac{1}{2}\xi b^2. \tag{2.4}$$

3. Weak formulation

In this section, we introduce the weak corresponding to Problem (P).
We assume that the data of the problem verify the following conditions:

Assumption 3.1. *We assume that the data $\mathbf{f}$, g and ν verify:*

(1) $\mathbf{f}$ *can be written as follows:*

$$\forall \mathbf{x} \in \Omega, \forall t \in [0,T], \forall C \in \mathbb{R}, \qquad \mathbf{f}(\mathbf{x},t,C) = \mathbf{f}_0(\mathbf{x},t) + \mathbf{f}_1(\mathbf{x},C), \tag{3.1}$$

where $\mathbf{f}_0 \in C^0(0,T;L^2(\Omega)^d)$ and $\mathbf{f}_1$ is (uniformly in $\mathbf{x}$) $c^*_{\mathbf{f}_1}$-Lipschitz with respect to its second variable with values in $\mathbb{R}^d$. In addition, we suppose that

$$\forall \mathbf{x} \in \Omega, \forall \xi \in \mathbb{R}, \ \|\mathbf{f}_1(\mathbf{x},\xi)\|_{\mathbb{R}^d} \leq c_{\mathbf{f}_1}|\xi|, \tag{3.2}$$

where $c_{\mathbf{f}_1}$ is a positive constant.

(2) $g \in C^0\left(0,T,L^2(\Omega)\right)$.

(3) ν *is λ-Lipschitz for a certain number $\lambda \in \mathbb{R}$ and there exist two strictly positive constants ν_1 and ν_2 such that, for any $\theta \in \mathbb{R}$,*

$$\nu_1 \leq \nu(\theta) \leq \nu_2. \tag{3.3}$$

The choice of space for Darcy's velocity and pressure $(\mathbf{u},p)$ is $L^\infty(0,T;L^2(\Omega)^d)\times L^\infty(0,T;H^1(\Omega)\cap L^2_0(\Omega))$ and for the concentration C is in $L^2(0,T;H^1_0(\Omega))$. The works of Feng [24], Chen and Ewing [13] and Fabrie and Gallouet [23], Cheng et al. [15], Droniou et al. [20] study the existence of a weak solution associated to problems very closely related to Problem (P); moreover Feng [24] states uniqueness of sufficiently regular solutions. Following [24] and [26], we consider the following weak formulation: Find $(\mathbf{u},p,C)$ in $L^\infty(0,T;L^2(\Omega)^d) \times L^\infty(0,T;H^1(\Omega)\cap L^2_0(\Omega)) \times L^2(0,T;H^1_0(\Omega))$ such that

$$(W)\begin{cases} \forall\, \mathbf{v} \in \mathcal{D}(\Omega\times]0,T[)^d, & \displaystyle\int_0^T\int_\Omega \nu(C)\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x}\,dt + \int_0^T\int_\Omega \nabla p\cdot\mathbf{v}\,d\mathbf{x}\,dt = \int_0^T\int_\Omega \mathbf{f}(C)\cdot\mathbf{v}\,d\mathbf{x}\,dt, \\ \forall\, q \in C^\infty(\bar\Omega\times]0,T[), & \displaystyle\int_0^T\int_\Omega \nabla q\cdot\mathbf{u}\,d\mathbf{x}\,dt = 0, \\ \forall\, S \in \mathcal{D}(\Omega\times[0,T[), & \displaystyle\int_0^T\int_\Omega C\frac{\partial S}{\partial t}d\mathbf{x}\,dt + \alpha\int_0^T\int_\Omega \nabla C\cdot\nabla S\,d\mathbf{x}\,dt \\ & \displaystyle + \int_0^T\int_\Omega (\mathbf{u}\cdot\nabla C)S\,d\mathbf{x}\,dt + r_0\int_0^T\int_\Omega C\,S\,d\mathbf{x}\,dt = \int_0^T\int_\Omega g\,S\,d\mathbf{x}\,dt\,. \end{cases}$$

A straightforward argument shows that any triple of functions $(\mathbf{u},p,C)$ solution of (W) is solution of (P) in the sense of distributions. Conversely, any solution $(\mathbf{u},p,C)$ of problem (P) verifying $(\mathbf{u},p,C)$ in $L^\infty(0,T;L^2(\Omega)^d)\times L^\infty(0,T;H^1(\Omega)\cap L^2_0(\Omega))\times L^2(0,T;H^1_0(\Omega))$ solves problem (W) in the above sense.

In the sequel of the present article, we shall assume existence and uniqueness of a regular solution to Problem (P). The objective of this work is to establish an *a posteriori* error estimate under such assumptions. In particular we assume that the concentration is in $C^0(0,T;L^2(\Omega))$.

Remark 3.2. *Under assumption 3.1 and if $C \in C^0(0,T;L^2(\Omega))$, then the solution $(\mathbf{u},p)$ are also continuous in time.*

Remark 3.3. *Under assumption 3.1 and if the concentration is continuous at $t = 0$, the first two equations of (V) (which are the Darcy system) determine uniquely the velocity $\mathbf{u}(0)$ and the pressure $p(0)$. Since we have $C(0) = 0$, the corresponding system will be written as:*

$$\begin{cases} \text{Find } (\mathbf{u}(0), p(0)) \in L^2(\Omega)^d \times (H^1(\Omega) \cap L^2_0(\Omega)) \text{ such that} \\ \forall\, \mathbf{v} \in L^2(\Omega)^d, \quad \displaystyle\int_\Omega \nu(0)\mathbf{u}(0) \cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega \nabla\, p(0) \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f}(., 0, 0) \cdot \mathbf{v}\, d\mathbf{x}\,, \\ \forall\, q \in H^1(\Omega) \cap L^2_0(\Omega), \quad \displaystyle\int_\Omega \nabla\, q \cdot \mathbf{u}(0)\, d\mathbf{x} = 0, \end{cases}$$

which is a classical well-posed problem.

Under the above assumptions, Problem (P) can be written for $0 \le t \le T$ under the following form:

$$(V) \begin{cases} \text{Find } (\mathbf{u}(t), p(t), C(t)) \in L^2(\Omega)^d \times (H^1(\Omega) \cap L^2_0(\Omega)) \times H^1_0(\Omega) \text{ such that } C(0) = 0 \text{ and} \\ \forall\, \mathbf{v} \in L^2(\Omega)^d, \quad \displaystyle\int_\Omega \nu(C(t))\mathbf{u}(t) \cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega \nabla\, p(t) \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f}(., t, C(t)) \cdot \mathbf{v}\, d\mathbf{x}\,, \\ \forall\, q \in H^1(\Omega) \cap L^2_0(\Omega), \quad \displaystyle\int_\Omega \nabla\, q \cdot \mathbf{u}(t)\, d\mathbf{x} = 0, \\ \forall\, S \in H^1_0(\Omega) \cap L^\infty(\Omega), \quad \displaystyle\frac{d}{dt}\int_\Omega C(t) S d\mathbf{x} + \alpha \int_\Omega \nabla\, C(t) \cdot \nabla\, S\, d\mathbf{x} + \int_\Omega (\mathbf{u}(t) \cdot \nabla\, C(t)) S\, d\mathbf{x} \\ \qquad\qquad + r_0 \displaystyle\int_\Omega C(t)\, S\, d\mathbf{x} = \int_\Omega g(t)\, S\, d\mathbf{x}\,. \end{cases}$$

4. The discrete problem

In this section, we consider the discrete problem associated to (P) which was introduced in our previous work [12], where we used the semi-implicit Euler method for the time discretization and the "mini-element" introduced by Arnold et al. in [4] for the space discretization of the velocity and pressure unknowns, and the standard first-order Lagrange finite element space for the concentration.
In order to describe the time discretization with an adaptive choice of local time steps, we introduce a partition of the interval $[0, T]$ into subintervals $[t_{n-1}, t_n]$, $1 \le n \le N$, with $0 = t_0 < t_1 < \cdots < t_N = T$.
For all $n \in [\![1, N]\!]$, we denote by τ_n the length of $[t_{n-1}, t_n]$, by τ the maximum of the τ_n and finally by σ_τ the regularity parameter

$$\sigma_\tau = \max_{2 \le n \le N} \frac{\tau_n}{\tau_{n-1}}.$$

We introduce the following operator π_τ:
Let X be a Banach space and g a continuous function from $]0, T]$ into X. We denote by $\pi_\tau g$ the piecewise constant function which is equal to $g(t_n)$ on each interval $]t_{n-1}, t_n]$, for all $[\![1, N]\!]$.
Similarly, for any sequence $(\phi_n)_{0 \le n \le N}$ in X, we associate the piecewise constant function $\pi_\tau \phi_\tau$ which is equal to ϕ_n on each interval $]t_{n-1}, t_n]$, $1 \le n \le N$.
Now, we describe the space discretization. For each step n with $0 \le n \le N$, let $(\mathcal{T}_{nh})_h$ be a regular family of triangulations of Ω by closed triangles (in dimension $d = 2$) or tetrahedra (in dimension $d = 3$), in the usual sense that:

- for each n, $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_{nh}$;
- the intersection of two different elements of $\mathcal{T}_{nh}$, if not empty, is a vertex or a whole edge or a whole face (in dimension $d = 3$) of both of them;
- the ratio of the diameter of an element K_n in $\mathcal{T}_{nh}$ to the diameter of its inscribed sphere is bounded by a constant independent of n and h.

For each $n \in [\![1, N]\!]$, h_n denotes the maximal diameter of the elements of $\mathcal{T}_{nh}$ and h denotes the maximal diameter of the elements of all $\mathcal{T}_{nh}$. For each K_n in $\mathcal{T}_{nh}$ and each nonnegative integer k, we denote by $\mathbb{P}_k(K_n)$ the space of restrictions to K_n of polynomials with d variables and total degree at most k.
In what follows, $c, c_1, C_1, \ldots$ stand for generic constants which may vary from line to line but are always

independent of h, τ and n.

We introduce the following inverse inequalities, (see [21], page 75): For any number $p \geq 2$, for any dimension d, and for any non negative integer r, there exist constants $c_I^0(p)$ such that for any polynomial function v_h of degree r on an element K_n of $\mathcal{T}_{nh}$,

$$\|v_h\|_{L^p(K_n)} \leq c_I^0(p) h_{K_n}^{\frac{d}{p}-\frac{d}{2}} \|v_h\|_{L^2(K_n)}. \tag{4.1}$$

From now on, we call finite element space associated with $\mathcal{T}_{nh}$ a space of functions such that their restrictions to any element K_n of $\mathcal{T}_{nh}$ belong to a space of polynomials of fixed degree.
For each n and h, we associate with $\mathcal{T}_{nh}$ three finite element spaces X_{nh}, M_{nh} and Y_{nh} which are contained in $L^2(\Omega)^d$, $L_0^2(\Omega) \cap H^1(\Omega)$ and $H_0^1(\Omega)$, respectively.
The velocity is discretized using the "Mini-Element" finite element method. The space associated to this discretisation is

$$X_{nh} = \{\mathbf{v}_h \in (\mathcal{C}^0(\bar{\Omega}))^d; \forall K_n \in \mathcal{T}_{nh}, \mathbf{v}_h|_{K_n} \in \mathcal{P}_b(K_n)^d\},$$

where the space $\mathcal{P}_b(K_n)$ is spanned by functions in $\mathbb{P}_1(K_n)$ and the bubble function on K_n (for each element K_n, the bubble function is equal to the product of the barycentric coordinates associated with the vertices of K_n).
The pressure is discretized with classical continuous finite elements of order one

$$M_{nh} = \{q_h \in L_0^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega});\ \forall\, K_n \in \mathcal{T}_{nh},\ q_h|_{K_n} \in \mathbb{P}_1(K_n)\}.$$

The spaces X_{nh} and M_{nh} verify the following discrete inf-sup condition for a certain constant $\beta > 0$, which is independent of n and h, (see [6]),

$$\forall\, q_h \in M_{nh},\ \sup_{\mathbf{v}_h \in X_{nh}} \frac{\displaystyle\int_\Omega \nabla\, q_h \cdot \mathbf{v}_h\, d\mathbf{x}}{\|\mathbf{v}_h\|_{L^2(\Omega)^d}} \geq \beta\, |q_h|_{1,\Omega}. \tag{4.2}$$

Finally, the concentration is discretized with classical continuous finite elements of order one

$$Y_{nh} = \{S_h \in H_0^1(\Omega) \cap \mathcal{C}^0(\bar{\Omega});\ \forall\, K_n \in \mathcal{T}_{nh},\ S_h|_{K_n} \in \mathbb{P}_1(K_n)\}.$$

As usual, we denote by V_{nh} the kernel of the divergence

$$V_{nh} = \left\{ \mathbf{v}_h \in X_{nh}; \forall q_h \in M_{nh}, \int_\Omega \mathbf{v}_h \cdot \nabla q_h d\mathbf{x} = 0 \right\}.$$

We approximate problem (V) by the following discrete scheme: $C_h^0 = 0$, for $n = 1, \ldots, N$,

$$(V_h) \begin{cases} \text{Having } C_h^{n-1} \in Y_{n-1h}, \text{Find } (\mathbf{u}_h^n, p_h^n) \in X_{nh} \times M_{nh} \text{ such as} \\ \forall\, \mathbf{v}_h \in X_{nh},\ \displaystyle\int_\Omega \nu(C_h^{n-1}) \mathbf{u}_h^n \cdot \mathbf{v}_h\, d\mathbf{x} + \int_\Omega \nabla\, p_h^n \cdot \mathbf{v}_h\, d\mathbf{x} = \int_\Omega \mathbf{f}^n(C_h^{n-1}) \cdot \mathbf{v}_h\, d\mathbf{x}\,, \\ \forall\, q_h \in M_{nh},\ \displaystyle\int_\Omega \nabla\, q_h \cdot \mathbf{u}_h^n\, d\mathbf{x} = 0, \\ \\ \text{Having } C_h^{n-1} \in Y_{n-1h}, \text{Find } C_h^n \in Y_{nh} \text{ such that} \\ \forall\, S_h \in Y_{nh},\ \displaystyle\int_\Omega \frac{C_h^n - C_h^{n-1}}{\tau_n} S_h\, d\mathbf{x} + \alpha \int_\Omega \nabla\, C_h^n \cdot \nabla\, S_h\, d\mathbf{x} + \int_\Omega (\mathbf{u}_h^n \cdot \nabla\, C_h^n) S_h\, d\mathbf{x} \\ \qquad\qquad + \frac{1}{2} \displaystyle\int_\Omega \text{div}\, (\mathbf{u}_h^n)\, C_h^n\, S_h\, d\mathbf{x} + r_0 \int_\Omega C_h^n\, S_h\, d\mathbf{x} = \int_\Omega g^n\, S_h\, d\mathbf{x}\,, \end{cases}$$

where $g^n = \dfrac{1}{\tau_n} \displaystyle\int_{t_{n-1}}^{t_n} g(t) dt$ and $\mathbf{f}^n(C_h^{n-1}) = \mathbf{f}_0^n + \mathbf{f}_1(C_h^{n-1})$, with $\mathbf{f}_0^n = \mathbf{f}_0(t_n)$.
The second nonlinear term in the last equation is added to compensate the fact that $\text{div}\, \mathbf{u}_h^n \neq 0$. It is well-known that Green's formula implies that

$$\int_\Omega (\mathbf{u}_h^n \cdot \nabla\, C_h^n) S_h\, d\mathbf{x} + \frac{1}{2} \int_\Omega (\text{div}\, \mathbf{u}_h^n) C_h^n\, S_h\, d\mathbf{x} = \frac{1}{2} \Big(\int_\Omega (\mathbf{u}_h^n \cdot \nabla\, C_h^n) S_h\, d\mathbf{x} - \int_\Omega (\mathbf{u}_h^n \cdot \nabla\, S_h) C_h^n\, d\mathbf{x} \Big), \tag{4.3}$$

hence the nonlinear term is antisymmetric.

Remark 4.1. *Problem (V_h) computes $\mathbf{u}_h^n$, p_h^n and C_h^n for each $n \in \{1, \dots, N\}$ starting from $C_h^0 = 0$. To complete this algorithm, we can calculate the initial values of the velocity $\mathbf{u}_h^0$ and the pressure p_h^0. These values can be determined as follows:*

$$\begin{cases} \text{Find } (\mathbf{u}_h^0, p_h^0) \in X_{0h} \times M_{0h} \text{ such as} \\ \forall\, \mathbf{v}_h \in X_{0h}, \quad \displaystyle\int_\Omega \nu(0)\mathbf{u}_h^0 \cdot \mathbf{v}_h \, d\mathbf{x} + \int_\Omega \nabla\, p_h^0 \cdot \mathbf{v}_h \, d\mathbf{x} = \int_\Omega \mathbf{f}^0(0) \cdot \mathbf{v}_h \, d\mathbf{x}\,, \\ \forall\, q_h \in M_{0h}, \quad \displaystyle\int_\Omega \nabla\, q_h \cdot \mathbf{u}_h^0 \, d\mathbf{x} = 0. \end{cases}$$

These values could be used to write algorithm (V_h) in a different way, in which C_h^n would be computed from C_h^{n-1} and $\mathbf{u}_h^{n-1}$, and then $(\mathbf{u}_h^n, p_h^n)$ from C_h^n. Since the numerical scheme is based on the Euler scheme and on explicit first-order linearizations of the non-linear terms, this alternative algorithm would have the same overall first-order accuracy as well as the same algorithmic complexity.

Existence and uniqueness of the solution of (V_h) was proved in [12] with a constant time step. These results can be easily extended to the case where the time step varies between the time iterations:

Theorem 4.2 (Existence and uniqueness of the solution of (V_h))**.** *At each time step n and for a given $C_h^{n-1} \in Y_{n-1h}$, Problem (V_h) has a unique solution $(\mathbf{u}_h^n, p_h^n, C_h^n) \in X_{nh} \times M_{nh} \times Y_{nh}$ which verifies, for $m = 1, \dots, N$, the following bounds*

$$\|\mathbf{u}_h^m\|_{L^2(\Omega)^d} \leq \frac{1}{\nu_1}\big(\|\mathbf{f}_0\|_{L^\infty(0,T;L^2(\Omega)^d)} + c_{\mathbf{f}_1}\|C_h^{m-1}\|_{L^2(\Omega)}\big) \tag{4.4}$$

and

$$\|C_h^m\|_{L^2(\Omega)}^2 + \alpha \sum_{n=1}^m \tau_n |C_h^n|_{H^1(\Omega)}^2 + 2r_0 \sum_{n=1}^m \tau_n \|C_h^n\|_{L^2(\Omega)}^2 \leq \frac{(S_2^0)^2}{\alpha}\|g\|_{L^2(0,T;L^2(\Omega))}^2. \tag{4.5}$$

In the next theorem, we recall the *a priori* error estimates already proven in [12] for a constant time step that can be easily extended to the case of variable time step as follows: we denote by $\mathbf{u}^n = \mathbf{u}(t_n)$, $p^n = p(t_n)$ and $C^n = C(t_n)$

Theorem 4.3 (*A priori* estimates)**.** *Let $(\mathbf{u}, p, C)$ be the solution to Problem (V) and $(\mathbf{u}_h^n, p_h^n, C_h^n)$ the solution to Problem (V_h). If $\mathbf{u} \in L^\infty(0,T;H^1(\Omega)^d) \cap L^\infty(0,T;L^\infty(\Omega)^d)$, $p \in L^\infty(0,T;H^2(\Omega))$, $C \in L^\infty(0,T;W^{2,4}(\Omega))$ and $\dfrac{\partial C}{\partial t} \in L^2(0,T;W^{1,4}(\Omega))$, we have,*

$$\sup_{0\leq n\leq N} \|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2(\Omega)^d} \leq c(h+\tau), \qquad \sup_{0\leq n\leq N} |p^n - p_h^n|_{H^1(\Omega)} \leq c'(h+\tau),$$

$$\sup_{0\leq n\leq N} \|C_h^n - C^n\|_{L^2(\Omega)}^2 + \alpha \sum_{n=1}^N \tau_n |C_h^n - C^n|_{1,\Omega}^2 \leq c''(h^2+\tau^2),$$

where c, c' and c'' are constants independent of h and τ, but that depend on the exact solution and the diffusion constant α.

Remark 4.4. *It is legitimate to consider values of the exact concentration of C in t_n since, under the hypothesis of Theorem 4.3, C is a continuous function of time with values in any functional space B such that $W^{2,4}(\Omega) \subset B \subset W^{1,4}(\Omega)$ with continuous embedding ([21, Lemma 6.2]). Continuity of C and of $\mathbf{f}$ imply from the first equation of (V) that $\mathbf{u}$ and p are continuous in time in their respective appropriate functional spaces, and it is thus also legitimate to consider the values of the exact velocity and pressure in t_n.*

5. A posteriori error analysis

In this section, we will prove *a posteriori* error estimates between the exact solution of problem (V) and the numerical solution of problem (V_h). We begin by constructing the indicators, then we prove the *a posteriori* error estimates and finally we establish the corresponding optimality.

5.1. Construction of the error indicators.

We denote by

- Γ_h^i the set of edges (when $d=2$) or faces (when $d=3$) of the mesh that are not contained in $\partial\Omega$,
- Γ_h^b the set of edges (when $d=2$) or faces (when $d=3$) of the mesh which are contained in $\partial\Omega$,
- $[\cdot]_{e_n}$ the jump through e_n (edge when $d=2$ or face when $d=3$) in Γ_h^i,
- Δ_{K_n} the union of elements of $\mathcal{T}_{nh}$ that intersect K_n,
- Δ_{e_n} the union of elements of $\mathcal{T}_{nh}$ that intersect the face e_n,
- $\mathbf{n}$ stands for the unit outward normal vector to K_n on ∂K_n.

In order to establish the *a posteriori* error estimates, we will need a supplementary condition of quasi-uniformity of the mesh. We suppose that we have the following assumption:

Assumption 5.1. *There exists a constant $\bar{C}>0$, independent of h and τ, such that*

$$\forall n \in [\![1,N]\!],\ \ \forall K_n \in \mathcal{T}_{nh},\quad h_{K_n} \geq \bar{C}h. \tag{5.1}$$

For the proofs of the next theorems, we introduce for each element K_n of $\mathcal{T}_{nh}$ (resp. each face $e_n \in K_n$), the bubble function ψ_{K_n} (resp. ψ_{e_n}) which is equal to the product of the $d+1$ barycentric coordinates associated with the vertices of K_n (resp. of the d barycentric coordinates associated with the vertices of e_n). We also consider a lifting operator $\mathcal{L}_{e_n}$ defined on polynomials on e_n vanishing on ∂e_n into polynomials on at most the two elements K_n containing e_n and vanishing on $\partial K_n \backslash e_n$, which is constructed by affine transformation from a fixed operator on the reference element.
We recall the following results from [29, Lemma 3.3].

Property 5.2. *Denoting by $P_r(K_n)$ the space of polynomials of degree smaller than r on K_n, we have, for all $v \in P_r(K_n)$*

$$\begin{cases} C_1\|v\|_{L^2(K_n)} \leq \|v\psi_{K_n}^{1/2}\|_{L^2(K_n)} \leq C_1'\|v\|_{L^2(K_n)}, \\ |v|_{1,K_n} \leq c_1 h_{K_n}^{-1}\|v\|_{L^2(K_n)}. \end{cases} \tag{5.2}$$

Property 5.3. *Denoting by $P_r(e_n)$ the space of polynomials of degree smaller than r on e_n, we have, for all $v \in P_r(e_n)$*

$$C_1\|v\|_{L^2(e_n)} \leq \|v\psi_{e_n}^{1/2}\|_{L^2(e_n)} \leq C_1'\|v\|_{L^2(e_n)}, \tag{5.3}$$

and, for all polynomials v in $P_r(e_n)$ vanishing on ∂e_n, if K_n is an element which contains e_n,

$$\|\mathcal{L}_{e_n}(v)\|_{L^2(K_n)} + h_{e_n}|\mathcal{L}_{e_n}(v)|_{1,K_n} \leq c_1 h_{e_n}^{1/2}\|v\|_{L^2(e_n)}. \tag{5.4}$$

We also introduce a Clément type regularization operator R_{nh} [16] which has the following properties: (see [8, section IX.3]): For each function S in $H_0^1(\Omega)$ (resp. each function q in $H^1(\Omega)$), $R_{nh}S$ belongs to the finite element space Y_{nh} (resp. M_{nh}) and satisfies for each K_n in $\mathcal{T}_{nh}$ and e_n in Γ_h^i,

$$\|S - R_{nh}S\|_{L^2(K_n)} \leq ch_{K_n}|S|_{1,\Delta_{K_n}} \text{ and } \|S - R_{nh}S\|_{L^2(e_n)} \leq ch_{e_n}^{1/2}|S|_{1,\Delta e_n}, \tag{5.5}$$

with identical relations for q instead of S.

For the *a posteriori* error studies, we need to define time-dependent functions from the discrete solutions $(C_h^n, \mathbf{u}_h^n, p_h^n)$.

Definition 5.4. *For each $n \in [\![1,N]\!]$ we consider the time-piecewise affine function C_h defined on the time interval $[t_{n-1}, t_n]$ by*

$$\forall t \in [t_{n-1}, t_n],\quad C_h(t) = \frac{t-t_{n-1}}{\tau_n}(C_h^n - C_h^{n-1}) + C_h^{n-1} = \frac{t-t_n}{\tau_n}(C_h^n - C_h^{n-1}) + C_h^n. \tag{5.6}$$

Moreover, for each $n \in [\![1,N]\!]$, we consider the piecewise constant function $\mathbf{u}_h$ equal to $\mathbf{u}_h^n$ on the time interval $]t_{n-1}, t_n]$ and the piecewise constant function p_h equal to p_h^n on the interval $]t_{n-1}, t_n]$.

Following the ideas of [9], we will prove optimal *a posteriori* error estimates using the norm:

$$\begin{aligned}[\![C-C_h]\!](t_n) =& \Big(\|C(t_n)-C_h(t_n)\|^2_{L^2(\Omega)} \\ &+ \alpha \max\Big(\int_0^{t_n} |C(t)-C_h(t)|^2_{1,\Omega}\,dt, \int_0^{t_n} |C(t)-\pi_\tau C_h(t)|^2_{1,\Omega}\,dt \Big)\Big).\end{aligned} \tag{5.7}$$

Remark 5.5. *In view of the efficiency bounds obtained in Section 5.3, it is important to keep in mind that we could also have used in definition* (5.7) *the equivalent full $H^1(\Omega)$ norm instead of the $H^1_0(\Omega)$ semi-norm since both C and C_h belongs to $H^1_0(\Omega)$.*

We introduce, for each edge e_n of the mesh, the function

$$\forall n \in [\![1,N]\!], \quad \phi_h^{e_n}|_{]t_{n-1},t_n]} = \begin{cases} 0 & \text{if } e_n \in \Gamma_h^i, \\ \mathbf{u}_h^n \cdot \mathbf{n} & \text{if } e_n \in \Gamma_h^b. \end{cases} \tag{5.8}$$

A standard calculation shows that the solutions of problems (V) and (V_h) verify for all $(\mathbf{v},q,S) \in L^2(\Omega)^d \times (H^1(\Omega)\cap L^2_0(\Omega)) \times (H^1_0(\Omega)\cap L^\infty(\Omega))$ and $q_h \in M_{nh}$, for each $t \in]t_{n-1},t_n]$:

$$\begin{aligned}&\int_\Omega \nu(C(t))\mathbf{u}(t)\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega \nabla p(t)\cdot\mathbf{v}\,d\mathbf{x} - \int_\Omega \nu(C_h(t))\mathbf{u}_h^n\cdot\mathbf{v}d\mathbf{x} - \int_\Omega \nabla p_h^n\cdot\mathbf{v}d\mathbf{x} \\ &= \int_\Omega \big(\mathbf{f}(t,C(t)) - \mathbf{f}^n(C_h^{n-1})\big)\cdot\mathbf{v}d\mathbf{x} + \int_\Omega \big(\nu(C_h^{n-1}) - \nu(C_h(t))\big)\mathbf{u}_h^n\cdot\mathbf{v}d\mathbf{x} \\ &\quad + \int_\Omega \big(\mathbf{f}^n(C_h^{n-1}) - \nu(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n\big)\cdot\mathbf{v}d\mathbf{x},\end{aligned} \tag{5.9}$$

$$\int_\Omega \nabla q\cdot(\mathbf{u}(t)-\mathbf{u}_h^n)d\mathbf{x} = \sum_{K_n\in\mathcal{T}_{nh}} \left[\int_{K_n} (q-q_h)\text{div }\mathbf{u}_h^n d\mathbf{x} - \sum_{e_n\in\partial K_n\cap\Gamma_h^b} \int_{e_n} (\mathbf{u}_h^n\cdot\mathbf{n})(q-q_h)ds \right], \tag{5.10}$$

and

$$\begin{aligned}&\left(\frac{\partial}{\partial t}(C-C_h)(t),S\right) + \alpha(\nabla(C-C_h)(t),\nabla S) + (\mathbf{u}(t)\cdot\nabla C(t) - \mathbf{u}_h(t)\cdot\nabla C_h(t),S) \\ &\quad + r_0((C-C_h)(t),S) - \frac{1}{2}(\text{div }\mathbf{u}_h(t)C_h(t),S) \\ &= (g(t),S) - \left(\frac{\partial}{\partial t}C_h(t),S\right) - (\mathbf{u}_h(t)\cdot\nabla C_h(t),S) - \frac{1}{2}(\text{div }\mathbf{u}_h(t)C_h(t),S) \\ &\quad - r_0(C_h(t),S) - \alpha(\nabla C_h(t),\nabla S).\end{aligned} \tag{5.11}$$

We introduce the residual $R(C_h) \in L^2(0,T;H^{-1}(\Omega))$ given by: for each $n \in [\![1,N]\!]$, for each $t \in]t_{n-1},t_n]$ and for all $S \in H^1_0(\Omega)$

$$\begin{aligned}\langle R(C_h)(t),S\rangle =& (g(t),S) - \frac{1}{\tau_n}(C_h^n - C_h^{n-1},S) - (\mathbf{u}_h(t)\cdot\nabla C_h(t),S) \\ &- \frac{1}{2}(\text{div }\mathbf{u}_h(t)C_h(t),S) - r_0(C_h(t),S) - \alpha(\nabla C_h(t),\nabla S).\end{aligned} \tag{5.12}$$

The residual $R(C_h)$ is now decomposed into a space residual $R^h \in L^2(0,T;H^{-1}(\Omega))$ as well as a time residual $R^\tau \in L^2(0,T;H^{-1}(\Omega))$ defined in the following way: For $n \in [\![1,N]\!]$, for $t \in]t_{n-1},t_n]$, for all $S \in H^1_0(\Omega)$

$$\begin{aligned}\langle R^h(C_h)(t),S\rangle =& \sum_{K_n\in\mathcal{T}_{nh}} \Big\{ \int_{K_n} \Big(g^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) - \mathbf{u}_h^n\cdot\nabla C_h^n - \frac{1}{2}\text{div }\mathbf{u}_h^n C_h^n - r_0 C_h^n \\ &+ \alpha\Delta C_h^n\Big)(\mathbf{x})\times S(\mathbf{x})d\mathbf{x} - \frac{1}{2}\sum_{e_n\subset\partial K_n\cap\Gamma_h^i} \int_{e_n} [\alpha\nabla C_h^n\cdot\mathbf{n}]_{e_n}(\sigma)\cdot S(\sigma)d\sigma \Big\},\end{aligned} \tag{5.13}$$

and

$$\langle R^\tau(C_h)(t), S\rangle = \frac{t_n - t}{\tau_n} \sum_{K_n \in \mathcal{T}_{nh}} \Big\{ \frac{1}{2} \int_{K_n} \mathbf{u}_h^n \cdot \nabla (C_h^n - C_h^{n-1})(\mathbf{x}) S(\mathbf{x}) d\mathbf{x} + r_0 \int_{K_n} (C_h^n - C_h^{n-1})(\mathbf{x}) S(\mathbf{x}) d\mathbf{x} + \int_{K_n} \alpha \nabla (C_h^n - C_h^{n-1})(\mathbf{x}) \cdot \nabla S(\mathbf{x}) d\mathbf{x} - \frac{1}{2} \int_{K_n} \mathbf{u}_h^n \cdot \nabla S(\mathbf{x}) (C_h^n - C_h^{n-1})(\mathbf{x}) d\mathbf{x} \Big\}. \tag{5.14}$$

Using (V_h), we remark that

$$\langle R^h(C_h)(t), S_h\rangle = 0 \ , \ \forall S_h \in Y_{nh}. \tag{5.15}$$

This leads to

$$\langle R(C_h)(t), S\rangle = \langle g - g^n, S\rangle + \langle R^h(C_h)(t), S - S_h\rangle + \langle R^\tau(C_h)(t), S\rangle. \tag{5.16}$$

We define now the error indicators:

Definition 5.6. *For each $K_n \in \mathcal{T}_{nh}$, $\forall n \in [\![1, N]\!]$,*

$$(\eta^h_{n,K_n,1})^2 = \|\mathbf{f}^n(C_h^{n-1}) - \nu(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n\|^2_{L^2(K_n)^d} + h^2_{K_n} \|\text{div } \mathbf{u}_h^n\|^2_{L^2(K_n)} + \sum_{e_n \subset \partial K_n, e_n \in \Gamma_h^b} h_{e_n} \|\phi_h^{e_n}\|^2_{L^2(e_n)},$$

$$(\eta^h_{n,K_n,2})^2 = h^2_{K_n} \left\| g^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\text{div } \mathbf{u}_h^n C_h^n - r_0 C_h^n + \alpha \Delta C_h^n \right\|^2_{L^2(K_n)} + \sum_{e_n \subset \partial K_n, e_n \in \cap \Gamma_h^i} h_{e_n} \|[\alpha \nabla C_h^n \cdot \mathbf{n}]_{e_n}\|^2_{L^2(e_n)},$$

and

$$(\eta^\tau_{n,K_n})^2 = \tau_n ||C_h^n - C_h^{n-1}||^2_{1,K_n}.$$

These indicators are easy to compute since they only depend on the discrete solution and they involve polynomials.

We shall need the following Lemma to bound the residuals involving C_h.

Lemma 5.7. *Under the assumptions of Theorem 4.3, the following estimates hold for each $n \in [\![1, N]\!]$ and $t \in]t_{n-1}, t_n]$,*

(1) *For all $S \in H_0^1(\Omega)$ let $S_h = R_{nh} S$,*

$$|\langle R^h(C_h)(t), S - S_h\rangle| \le c \Big(\sum_{K_n \in \mathcal{T}_{nh}} (\eta^h_{n,K_n,2})^2 \Big)^{1/2} |S|_{1,\Omega}. \tag{5.17}$$

(2) *We suppose that $\tau \le ch$. For all $S \in H_0^1(\Omega)$, we have:*

$$|\langle R^\tau(C_h)(t), S\rangle| \le c \frac{t_n - t}{\tau_n^{3/2}} \Big(\sum_{K_n \in \mathcal{T}_{nh}} (\eta^\tau_{n,K_n})^2 \Big)^{1/2} |S|_{1,\Omega}. \tag{5.18}$$

Proof. Let $n \in [\![1, N]\!]$ and $t \in]t_{n-1}, t_n]$. To derive inequality (5.17), we use formula (5.13) with $S_h = R_{nh} S$, the Cauchy-Schwarz inequality, the properties of R_{nh} (see (5.5)), the discrete Cauchy-Schwarz inequality and the fact that, for each K_n it holds that

$$\sum_{e_n \subset \partial K_n \cap \Gamma_h^i} |S|^2_{1,\Delta e_n} \le c |S|^2_{1,\Delta K_n}$$

and that, due to the regularity of $\mathcal{T}_{nh}$ we have

$$\sum_{K_n \in \mathcal{T}_{nh}} |S|^2_{1,\Delta K_n} \le c |S|^2_{1,\Omega}.$$

with a constant c that does not depend on h.

Let us now show inequality (5.18). The second and third terms of (5.14) can be bounded using the Cauchy-Schwarz inequality on each K_n and then the discrete Cauchy-Schwarz inequality. As far as the

second term is concerned, the Poincaré inequality must also be used to bound the $L^2(\Omega)$ norm of S by its $H_0^1(\Omega)$ semi-norm. We shall now treat the first and last terms in (5.14). Let us start by denoting by b the first term of (5.14). Then, inserting $\displaystyle\int_{K_n} \mathbf{u}^n \cdot \nabla(C_h^n - C_h^{n-1}) S d\mathbf{x}$, we obtain the following equality:

$$\begin{aligned} b &= \sum_{K_n \in \mathcal{T}_{nh}} \int_{K_n} \mathbf{u}_h^n \cdot \nabla(C_h^n - C_h^{n-1}) S d\mathbf{x} \\ &= \sum_{K_n \in \mathcal{T}_{nh}} \int_{K_n} (\mathbf{u}_h^n - \mathbf{u}^n) \cdot \nabla(C_h^n - C_h^{n-1}) S d\mathbf{x} + \sum_{K_n \in \mathcal{T}_{nh}} \int_{K_n} \mathbf{u}^n \cdot \nabla(C_h^n - C_h^{n-1}) S d\mathbf{x}. \end{aligned} \tag{5.19}$$

We denote by b_1 and b_2 the terms in the right-hand side of (5.19) respectively.

Using the $L^2 - L^3 - L^6$ generalized Cauchy-Schwarz inequality and the inverse inequality (4.1) in b_1, we obtain:

$$|b_1| \leq c \sum_{K_n \in \mathcal{T}_{nh}} \|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2(K_n)^d} |C_h^n - C_h^{n-1}|_{1,K_n} h_{K_n}^{-d/6} \|S\|_{L^6(K_n)}. \tag{5.20}$$

We recall the *a priori* error estimate of $\mathbf{u}$ in Theorem 4.3

$$\|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2(K_n)^d} \leq \|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2(\Omega)^d} \leq c(h + \tau) \tag{5.21}$$

Using the condition $\tau \leq ch$, we get the following bound:

$$|b_1| \leq c_1 \sum_{K_n \in \mathcal{T}_{nh}} h h_{K_n}^{-d/6} |C_h^n - C_h^{n-1}|_{1,K_n} \|S\|_{L^6(K_n)}. \tag{5.22}$$

Using inequality (5.1), the discrete Cauchy-Schwarz inequality and the Poincaré inequality, we get:

$$|b_1| \leq c \left(\sum_{K_n \in \mathcal{T}_{nh}} |C_h^n - C_h^{n-1}|_{1,K_n}^2 \right)^{1/2} |S|_{1,\Omega}. \tag{5.23}$$

We use the L^∞-L^2-L^2 generalized Cauchy-Schwarz inequality, then the discrete Cauchy-Schwarz inequality and finally the Poincaré inequality in b_2 to obtain:

$$\begin{aligned} |b_2| &\leq \sum_{K_n \in \mathcal{T}_{nh}} \|\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega)^d)} |C_h^n - C_h^{n-1}|_{1,K_n} \|S\|_{L^2(K_n)} \\ &\leq c_1 \left(\sum_{K_n \in \mathcal{T}_{nh}} |C_h^n - C_h^{n-1}|_{1,K_n}^2 \right)^{1/2} |S|_{1,\Omega}. \end{aligned} \tag{5.24}$$

Finally, gathering (5.19), (5.23) and (5.24), we deduce that

$$|b| \leq c_2 \left(\sum_{K_n \in \mathcal{T}_{nh}} |C_h^n - C_h^{n-1}|_{1,K_n}^2 \right)^{1/2} |S|_{1,\Omega}. \tag{5.25}$$

Now, let the last term of (5.14) be denoted by $\widetilde{b}$. We insert $\displaystyle\int_{K_n} \mathbf{u}^n \cdot \nabla S(C_h^n - C_h^{n-1}) d\mathbf{x}$ to get:

$$\widetilde{b} = \sum_{K_n \in \mathcal{T}_{nh}} \int_{K_n} (\mathbf{u}_h^n - \mathbf{u}^n) \cdot \nabla S(C_h^n - C_h^{n-1}) d\mathbf{x} + \sum_{K_n \in \mathcal{T}_{nh}} \int_{K_n} \mathbf{u}^n \cdot \nabla S(C_h^n - C_h^{n-1}) d\mathbf{x}. \tag{5.26}$$

We denote by $\widetilde{b}_1$ and $\widetilde{b}_2$ the terms in the right-hand side of (5.26) respectively.

Using the $L^2 - L^2 - L^\infty$ generalized Cauchy-Schwarz inequality, we obtain:

$$|\widetilde{b}_1| \leq \sum_{K_n \in \mathcal{T}_{nh}} \|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2(K_n)^d} |S|_{1,K_n} \|C_h^n - C_h^{n-1}\|_{L^\infty(\Omega)}.$$

Using the inverse inequality for $p = \infty$ (cf. Ern and Guermond [21, Lemma 1.142]) and then the Poincaré inequality, we get

$$\|C_h^n - C_h^{n-1}\|_{L^\infty(\Omega)} \leq \mathcal{K}(h_n) |C_h^n - C_h^{n-1}|_{1,\Omega},$$

where,

$$\mathcal{K}(h_n) = \begin{cases} c\,(1 + |\log(h_n)|) & \text{if } d = 2 \\ c\,h_n^{-1/2} & \text{if } d = 3. \end{cases} \tag{5.27}$$

The *a priori* error estimate (Theorem 4.3) and the condition $\tau \leq ch$, yield the following bound:

$$|\widetilde{b}_1| \leq c_1 \left(\sum_{K_n \in \mathcal{T}_{nh}} |C_h^n - C_h^{n-1}|^2_{1,K_n} \right)^{1/2} h\mathcal{K}(h_n)|S|_{1,\Omega}. \tag{5.28}$$

Condition (5.1) allows us to get

$$|\widetilde{b}_1| \leq c \left(\sum_{K_n \in \mathcal{T}_{nh}} |C_h^n - C_h^{n-1}|^2_{1,K_n} \right)^{1/2} |S|_{1,\Omega}. \tag{5.29}$$

The term $\widetilde{b}_2$ can be bounded like the term b_2, the only difference being that the Poincaré inequality is now used on $(C_h^n - C_h^{n-1})$. This gives

$$|\widetilde{b}_2| \leq c \left(\sum_{K_n \in \mathcal{T}_{nh}} |C_h^n - C_h^{n-1}|^2_{1,K_n} \right)^{1/2} |S|_{1,\Omega}. \tag{5.30}$$

Equality (5.26) as well as inequalities (5.29) and (5.30) allow us to obtain the following bound:

$$|\widetilde{b}| \leq c \left(\sum_{K_n \in \mathcal{T}_{nh}} |C_h^n - C_h^{n-1}|^2_{1,K_n} \right)^{1/2} |S|_{1,\Omega}. \tag{5.31}$$

Finally, gathering all the inequalities and multiplying and dividing by $\tau_n^{1/2}$, we obtain (5.18). ■

5.2. **Upper bound of the error.** In this section, we establish the *a posteriori* error estimates where we will bound the error between the exact solution $(\mathbf{u}, p, C)$ of Problem (V) and the numerical solution $(\mathbf{u}_h, p_h, C_h)$ of Problem (V_h) using the indicators given in Definition 5.6.

First, we establish the upper bound corresponding to the Darcy equation where we bound the errors of the velocity and the pressure with respect to the error of the concentration.

Lemma 5.8. *Let $(\mathbf{u}, p, C)$ be the solution of (V) and $(\mathbf{u}_h, p_h, C_h)$ be the functions constructed by definition 5.4 from the solution of (V_h). Under the assumptions of Theorem 4.3, the following error estimate holds for all $m = 1, \ldots, N$:*

$$\begin{aligned} &\|\mathbf{u}-\mathbf{u}_h\|_{L^2(0,t_m;L^2(\Omega)^d)} + \|p - p_h\|_{L^2(0,t_m;H_0^1(\Omega))} \\ &\quad \leq c\|C - C_h\|_{L^2(0,t_m,L^2(\Omega))} + c' \Bigg(\sum_{n=1}^{m} \sum_{K_n \in \mathcal{T}_{nh}} \left(\tau_n \left((\eta^h_{n,K_n,1})^2 + (\eta^\tau_{n,K_n})^2 \right) \right)^{1/2} \\ &\qquad + \left(\int_0^{t_m} \|\mathbf{f}_0(t) - \pi_\tau \mathbf{f}_0\|^2_{L^2(\Omega)^d} dt \right)^{1/2}, \end{aligned} \tag{5.32}$$

where c and c' are constants independent from h and τ.

Proof. We start with the velocity error.

Let $n \in [\![1, N]\!]$ and $t \in]t_{n-1}, t_n]$. As usual, a bound for the error on the velocity is derived by eliminating the pressure from (5.9). This is obtained from (5.10) in the following way: it follows from the inf-sup condition (2.1) that there exists a velocity $\mathbf{z}$ in $L^2(\Omega)^d$ that solves for all $q_h \in M_{nh}$

$$\forall q \in H^1(\Omega) \cap L^2_0(\Omega), \quad \int_\Omega \nabla q \cdot \mathbf{z} d\mathbf{x} = \sum_{K_n \in \mathcal{T}_{nh}} \left[\int_{K_n} (q - q_h) \operatorname{div} \mathbf{u}_h^n d\mathbf{x} - \sum_{e_n \in \partial K_n \cap \Gamma_h^b} \int_{e_n} (\mathbf{u}_h^n \cdot \mathbf{n})(q - q_h) ds \right], \tag{5.33}$$

and satisfies

$$\|\mathbf{z}\|_{L^2(\Omega)^d} \leq \sup_{q\in H^1(\Omega)\cap L^2_0(\Omega)} \frac{1}{|q|_{1,\Omega}} \left| \sum_{K_n\in\mathcal{T}_{nh}} \Big[\int_{K_n} (q-q_h)\,\mathrm{div}\,\mathbf{u}_h^n d\mathbf{x} - \sum_{e_n\in\partial K_n\cap\Gamma_h^b} \int_{e_n} (\mathbf{u}_h^n\cdot\mathbf{n})(q-q_h) ds \Big] \right|.$$

By taking $q_h = R_{nh}q$ and using the Cauchy-Schwarz inequality and the properties (5.5) of the Clément operator R_{nh}, we infer that

$$\begin{aligned} \|\mathbf{z}\|_{L^2(\Omega)^d} &\leq c\Big(\sum_{K_n\in\mathcal{T}_{nh}} \Big[h_{K_n}^2 \|\mathrm{div}\ \mathbf{u}_h^n\|^2_{L^2(K_n)} + \sum_{e_n\in\partial K_n\cap\mathcal{E}_{nh}^b} h_{e_n}\|\phi_h^{e_n}\|^2_{L^2(e_n)} \Big] \Big)^{1/2} \\ &\leq c\Big(\sum_{K_n\in\mathcal{T}_{nh}} (\eta^h_{n,K_n,1})^2 \Big)^{1/2}. \end{aligned} \tag{5.34}$$

To simplify the notations, we set $\mathbf{z}_0 = \mathbf{u} - \mathbf{u}_h^n - \mathbf{z}$. Equations (5.10) and (5.33) give for $q = p - p_h^n$

$$\int_\Omega \nabla(p - p_h^n)\cdot\mathbf{z}_0 d\mathbf{x} = 0.$$

Now, we take $\mathbf{v} = \mathbf{z}_0$ in (5.9), we get

$$\begin{aligned} &\int_\Omega \nu(C(t))\mathbf{u}\cdot\mathbf{z}_0 d\mathbf{x} - \int_\Omega \nu(C_h(t))\mathbf{u}_h^n\cdot\mathbf{z}_0 d\mathbf{x} \\ &= \int_\Omega \big(\mathbf{f}(t,C(t)) - \mathbf{f}^n(C_h^{n-1})\big)\cdot\mathbf{z}_0 d\mathbf{x} + \int_\Omega \big(\nu(C_h^{n-1}) - \nu(C_h(t))\big)\mathbf{u}_h^n\cdot\mathbf{z}_0 d\mathbf{x} \\ &\quad + \int_\Omega \big(\mathbf{f}^n(C_h^{n-1}) - \nu(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n\big)\cdot\mathbf{z}_0 d\mathbf{x}. \end{aligned} \tag{5.35}$$

We simplify by $\int_\Omega \nu(C_h(t))\mathbf{u}_h^n\cdot\mathbf{z}_0 d\mathbf{x}$ and insert $\displaystyle\int_\Omega \nu(C_h^{n-1})\mathbf{u}\cdot\mathbf{z}_0 d\mathbf{x}$ in (5.35) and we get

$$\begin{aligned} &\int_\Omega (\nu(C(t)) - \nu(C_h^{n-1}))\mathbf{u}\cdot\mathbf{z}_0 d\mathbf{x} + \int_\Omega \nu(C_h^{n-1})(\mathbf{u}(t) - \mathbf{u}_h^n)\cdot\mathbf{z}_0 d\mathbf{x} \\ &= \int_\Omega \big(\mathbf{f}(t,C(t)) - \mathbf{f}^n(C_h^{n-1})\big)\cdot\mathbf{z}_0 d\mathbf{x} + \int_\Omega \big(\mathbf{f}^n(C_h^{n-1}) - \nu(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n\big)\cdot\mathbf{z}_0 d\mathbf{x}. \end{aligned} \tag{5.36}$$

Since we have $\mathbf{u} - \mathbf{u}_h^n = \mathbf{z}_0 + \mathbf{z}$, we obtain

$$\begin{aligned} \int_\Omega \nu(C_h^{n-1})\mathbf{z}_0\cdot\mathbf{z}_0 d\mathbf{x} &= \int_\Omega (\nu(C_h^{n-1}) - \nu(C(t)))\mathbf{u}\cdot\mathbf{z}_0 d\mathbf{x} + \int_\Omega \big(\mathbf{f}(t,C(t)) - \mathbf{f}^n(C_h^{n-1})\big)\cdot\mathbf{z}_0 d\mathbf{x} \\ &\quad + \int_\Omega \big(\mathbf{f}^n(C_h^{n-1}) - \nu(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n\big)\cdot\mathbf{z}_0 d\mathbf{x} - \int_\Omega \nu(C_h^{n-1})\mathbf{z}\cdot\mathbf{z}_0 d\mathbf{x} \end{aligned} \tag{5.37}$$

By inserting $\nu(C_h(t))$ in the first term and $\mathbf{f}(t,C_h(t))$ in the second one in the right-hand side of (5.37), we get, using Assumption 3.1

$$\begin{aligned} \nu_1\|\mathbf{z}_0\|_{L^2(\Omega)^d} \leq& \lambda\|C_h(t) - C(t)\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega)^d)} \\ &+ \lambda\frac{t-t_{n-1}}{\tau_n}\|C_h^n - C_h^{n-1}\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega)^d)} + c^*_{\mathbf{f}_1}\|C(t) - C_h(t)\|_{L^2(\Omega)} \\ &+ c^*_{\mathbf{f}_1}\frac{t-t_{n-1}}{\tau_n}\|C_h^n - C_h^{n-1}\|_{L^2(\Omega)} + \|\mathbf{f}_0(t) - \mathbf{f}_0(t_n)\|_{L^2(\Omega)^d} \\ &+ \|\mathbf{f}^n(C_h^{n-1}) - \nu(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n\|_{L^2(\Omega)^d} + \nu_2\|\mathbf{z}\|_{L^2(\Omega)^d}. \end{aligned} \tag{5.38}$$

We gather (5.34) and (5.38) and use the fact that $\mathbf{u} - \mathbf{u}_h^n = \mathbf{z}_0 + \mathbf{z}$ to obtain the following bound:

$$\begin{aligned} \|\mathbf{u}(t) - \mathbf{u}_h(t)\|^2_{L^2(\Omega)^d} \leq& c_1\Big(\|C(t) - C_h(t)\|^2_{L^2(\Omega)} + \sum_{K_n\in\mathcal{T}_{nh}} (\eta^h_{n,K_n,1})^2 \\ &+ \|\mathbf{f}_0(t) - \mathbf{f}_0(t_n)\|^2_{L^2(\Omega)^d} + \frac{(t-t_{n-1})^2}{\tau_n^2}\|C_h^n - C_h^{n-1}\|^2_{L^2(\Omega)} \Big). \end{aligned} \tag{5.39}$$

Finally, we integrate inequality (5.39) between t_{n-1} and t_n and we sum over $n \in \{1, \dots, m\}$, we get

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(0,t_m;L^2(\Omega)^d)} \leq & c_2 \Bigg(\|C - C_h\|^2_{L^2(0,t_m;L^2(\Omega))} + \sum_{n=1}^{m} \sum_{K_n \in \mathcal{T}_{nh}} \left(\tau_n (\eta^h_{n,K_n,1})^2 + (\eta^\tau_{n,K_n})^2\right) \\ & + \int_0^{t_m} \|\mathbf{f}_0(t) - \pi_\tau \mathbf{f}_0\|^2_{L^2(\Omega)^d} dt \Bigg).\end{aligned} \tag{5.40}$$

For the pressure error, we consider equation (5.9). We insert $\displaystyle\int_\Omega \nu(C_h^{n-1})\mathbf{u}(t) \cdot \mathbf{v} d\mathbf{x}$ and we get

$$\begin{aligned}\int_\Omega \nabla(p - p_h^n) \cdot \mathbf{v} d\mathbf{x} = & \int_\Omega (\mathbf{f}(t, C(t)) - \mathbf{f}^n(C_h^{n-1})) \cdot \mathbf{v} d\mathbf{x} + \int_\Omega (\nu(C_h^{n-1}) - \nu(C(t)))\mathbf{u}(t) \cdot \mathbf{v} d\mathbf{x} \\ & + \int_\Omega \nu(C_h^{n-1})(\mathbf{u}_h^n - \mathbf{u}(t)) \cdot \mathbf{v} d\mathbf{x} + \int_\Omega (\mathbf{f}^n(C_h^{n-1}) - \nu(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n) \cdot \mathbf{v} d\mathbf{x}.\end{aligned} \tag{5.41}$$

We insert $\mathbf{f}(t, C_h(t))$ in the first term and $\nu(C_h(t))$ in the second one of the previous equality and we obtain the following bound using the inf-sup condition (2.1):

$$\begin{aligned}|p(t) - p_h(t)|_{1,\Omega} \leq & \tilde{c}_1 \|C(t) - C_h(t)\|_{L^2(\Omega)} + \tilde{c}_2 \frac{t - t_{n-1}}{\tau_n} \|C_h^n - C_h^{n-1}\|_{L^2(\Omega)} + \tilde{c}_5 \|\mathbf{f}_0(t) - \mathbf{f}_0(t_n)\|_{L^2(\Omega)^d} \\ & + \tilde{c}_3 \|\mathbf{f}^n(C_h^{n-1}) - \nu(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n\|_{L^2(\Omega)^d} + \tilde{c}_4 \|\mathbf{u}_h^n - \mathbf{u}(t)\|_{L^2(\Omega)^d}.\end{aligned} \tag{5.42}$$

Finally, we integrate between t_{n-1} and t_n, and we sum over $n \in [\![1, N]\!]$ to get

$$\begin{aligned}\|p - p_h\|^2_{L^2(0,t_m;H^1_0(\Omega))} \leq & \tilde{c}_1 \|C - C_h\|^2_{L^2(0,t_m;L^2(\Omega))} + \tilde{c}_2 \sum_{n=1}^{m} \sum_{K_n \in \mathcal{T}_{nh}} \left(\tau_n (\eta^h_{n,K_n,1})^2 + (\eta^\tau_{n,K_n})^2\right) \\ & + \tilde{c}_3 \|\mathbf{u}_h - \mathbf{u}\|^2_{L^2(0,t_m;L^2(\Omega)^d)} + \tilde{c}_4 \int_0^{t_m} \|\mathbf{f}_0(t) - \pi_\tau \mathbf{f}_0\|^2_{L^2(\Omega)^d} dt.\end{aligned} \tag{5.43}$$

To get the final result, we gather (5.40) and (5.43).

■

Now, we derive and prove an upper bound on the error corresponding to the concentration that depends on the velocity error.

Lemma 5.9. *Let $(\mathbf{u}, p, C)$ be the solution of Problem (V) and $(\mathbf{u}_h, p_h, C_h)$ be the solution of (V_h). Under the assumptions of Theorem 4.3, the following error estimate holds between the exact and numerical concentration: $\forall n \in [\![1, N]\!]$, $\forall t \in]t_{n-1}, t_n]$,*

$$\begin{aligned}& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|C(t) - C_h(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}|C(t) - C_h(t)|^2_{1,\Omega} + r_0 \|C(t) - C_h(t)\|^2_{L^2(\Omega)} \\ & \leq c_1 \|g(t) - g^n\|^2_{L^2(\Omega)} + c_3 \sum_{K_n \in \mathcal{T}_{nh}} (\eta^h_{n,K_n,2})^2 \\ & \quad + c_4 \frac{(t_n - t)^2}{\tau_n^3} \sum_{K_n \in \mathcal{T}_{nh}} (\eta^\tau_{n,K_n})^2 + c_5 \|\mathbf{u}_h(t) - \mathbf{u}(t)\|^2_{L^2(\Omega)^d}.\end{aligned} \tag{5.44}$$

Proof. Let $n \in [\![1, N]\!]$ and $t \in]t_{n-1}, t_n]$. Using (5.11) and (5.12), we get

$$\begin{aligned}& \left(\frac{\partial}{\partial t}(C - C_h)(t), S\right) + \alpha(\nabla(C - C_h)(t), \nabla S) + r_0((C - C_h)(t), S) \\ & = \left(R(C_h(t)), S\right) + (\mathbf{u}_h(t) \cdot \nabla C_h(t) - \mathbf{u}(t) \cdot \nabla C(t), S) + \frac{1}{2}(\text{div } \mathbf{u}_h(t) C_h(t), S).\end{aligned} \tag{5.45}$$

We take $S = C - C_h$ in (5.45) then we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|C - C_h\|^2_{L^2(\Omega)} + \alpha|C - C_h|^2_{1,\Omega} + r_0 \|C - C_h\|^2_{L^2(\Omega)} = \langle R(C_h(t)), S\rangle + T. \tag{5.46}$$

where

$$T = (\mathbf{u}_h(t)\cdot\nabla C_h(t) - \mathbf{u}(t)\cdot\nabla C(t), S) + \frac{1}{2}(\text{div } \mathbf{u}_h(t)C_h(t), S). \tag{5.47}$$

We start by bounding the term T.

Using the following antisymmetry property

$$(\mathbf{u}_h(t)\cdot\nabla(C(t)-C_h(t)), S) + \frac{1}{2}(\text{div } \mathbf{u}_h(t)(C(t)-C_h(t)), S) = (\mathbf{u}_h(t)\cdot\nabla S, S) + \frac{1}{2}(\text{div } \mathbf{u}_h(t)S, S) = 0,$$

and adding it to (5.47), we get

$$T = (\mathbf{u}_h(t)\cdot\nabla C(t), S) - (\mathbf{u}(t)\cdot\nabla C(t), S) + \frac{1}{2}(\text{div } \mathbf{u}_h(t)C(t), S). \tag{5.48}$$

Integrating by parts the last term and one-half of the second term in (5.48) and using the incompressibility condition $\text{div}\,\mathbf{u} = 0$, we get:

$$T = \frac{1}{2}((\mathbf{u}_h - \mathbf{u})(t)\cdot\nabla C(t), S) - \frac{1}{2}((\mathbf{u}_h - \mathbf{u})(t)\cdot\nabla S, C(t)). \tag{5.49}$$

Now, we use the $L^2 - L^3 - L^6$ and $L^2 - L^2 - L^\infty$ Cauchy-Schwarz inequalities and the Poincaré inequality to obtain the following inequality:

$$|T| \le \frac{1}{2}S_6^0\|(\mathbf{u}_h - \mathbf{u})(t)\|_{L^2(\Omega)^d}|C(t)|_{W^{1,3}(\Omega)}|S|_{1,\Omega} + \frac{1}{2}\|(\mathbf{u}_h - \mathbf{u})(t)\|_{L^2(\Omega)^d}|S|_{1,\Omega}\|C(t)\|_{L^\infty(\Omega)}. \tag{5.50}$$

Now, we treat the term $\langle R(C_h)(t), S\rangle$. Using (5.16), the Cauchy-Schwarz inequality and inequalities (5.17) and (5.18), we get:

$$\begin{aligned}|\langle R(C_h)(t), S\rangle| \le& \|g - g^n\|_{L^2(\Omega)}\|S\|_{L^2(\Omega)}\\ &+ c\Big(\sum_{K_n\in\mathcal{T}_{nh}}(\eta^h_{n,K_n,2})^2\Big)^{1/2}|S|_{1,\Omega} + c\frac{t_n - t}{\tau_n^{3/2}}\Big(\sum_{K_n\in\mathcal{T}_{nh}}(\eta^\tau_{n,K_n})^2\Big)^{1/2}|S|_{1,\Omega}.\end{aligned} \tag{5.51}$$

We gather (5.46), (5.50) and (5.51), and, using the Poincaré inequality for the first term in the right-hand side of (5.51) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|C - C_h\|^2_{L^2(\Omega)} + \alpha|C - C_h|^2_{1,\Omega} + r_0\|C - C_h\|^2_{L^2(\Omega)}\\ &\quad\le S_2^0\|g(t) - g^n\|_{L^2(\Omega)}|C - C_h|_{1,\Omega}\\ &\qquad + c\Big(\sum_{K_n\in\mathcal{T}_{nh}}(\eta^h_{n,K_n,2})^2\Big)^{1/2}|C - C_h|_{1,\Omega} + c\frac{t_n - t}{\tau_n^{3/2}}\Big(\sum_{K_n\in\mathcal{T}_{nh}}(\eta^\tau_{n,K_n})^2\Big)^{1/2}|C - C_h|_{1,\Omega}\\ &\qquad + \frac{1}{2}S_6^0\|(\mathbf{u}_h - \mathbf{u})(t)\|_{L^2(\Omega)^d}|C(t)|_{W^{1,3}(\Omega)}|C - C_h|_{1,\Omega}\\ &\qquad + \frac{1}{2}\|(\mathbf{u}_h - \mathbf{u})(t)\|_{L^2(\Omega)^d}\|C(t)\|_{L^\infty(\Omega)}|C - C_h|_{1,\Omega}.\end{aligned} \tag{5.52}$$

Finally, we get the result using inequality (2.4) for all the terms in the right-hand side of (5.52), where we systematically use $b = |C - C_h|_{1,\Omega}$ and $\xi = \frac{\alpha}{6}$. ■

To find the error bound of the concentration, we have to bound the last term of the concentration error given by (5.7).

Lemma 5.10. *The exact concentration C solution of Problem (V) and the numerical concentration C_h solution of (V_h) verify the following inequality:*

$$\sum_{n=1}^m\int_{t_{n-1}}^{t_n}|C(s) - \pi_\tau C_h(s)|^2_{1,\Omega}ds \le c\Big(\int_0^{t_m}|C(s) - C_h(s)|^2_{1,\Omega}ds + \sum_{n=1}^m\sum_{K_n\in\mathcal{T}_{nh}}(\eta^\tau_{n,K_n})^2\Big),$$

where c is a positive constant independent of h and τ.

Proof. For each $n \in [\![1, N]\!]$ and for each $t \in]t_{n-1}, t_n]$, we have

$$\begin{aligned} |C(t) - \pi_\tau C_h(t)|^2_{1,\Omega} &\leq \Big(|C(t) - C_h(t)|_{1,\Omega} + |C_h(t) - C_h^n|_{1,\Omega} \Big)^2 \\ &\leq 2\Big(|C(t) - C_h(t)|^2_{1,\Omega} + \big(\frac{t - t_n}{\tau_n}\big)^2 |C_h^n - C_h^{n-1}|^2_{1,\Omega} \Big). \end{aligned} \tag{5.53}$$

We obtain the desired result by integrating (5.53) between t_{n-1} and t_n, and then summing over n. ■

Lemmas 5.8, 5.9 and 5.10 allow us to show in the next theorem the total error bound corresponding to our *a posteriori* error estimates.

Theorem 5.11. *Let $(\mathbf{u}, p, C)$ be the solution of Problem (V) and $(\mathbf{u}_h, p_h, C_h)$ be the solution of (V_h). Under the assumptions of Theorem 4.3, the following error estimate holds for all $m = 1, \ldots, N$:*

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_{L^2(0,t_m;L^2(\Omega)^d)} + \|p - p_h\|_{L^2(0,t_m;H^1_0(\Omega))} + [\![C - C_h]\!](t_m) \\ &\quad \leq C_1 \|g - \pi_\tau g\|_{L^2(0,t_m;L^2(\Omega))} \\ &\qquad + C_1' \Bigg(\sum_{n=1}^{m} \sum_{K_n \in \mathcal{T}_{nh}} \tau_n \big((\eta^h_{n,K_n,1})^2 + (\eta^h_{n,K_n,2})^2 \big) + (\eta^\tau_{n,K_n})^2 \Bigg)^{1/2} \\ &\qquad + \Bigg(\int_0^{t_m} \|\mathbf{f}_0(t) - \pi_\tau \mathbf{f}_0\|^2_{L^2(\Omega)^d} dt \Bigg)^{1/2}, \end{aligned} \tag{5.54}$$

where C_1 and C_1' are constants independent of h and τ.

Proof. Let us consider each $n \in [\![1, N]\!]$ and each $t \in]t_{n-1}, t_n]$. We replace (5.39) in (5.44) then we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|C - C_h\|^2_{L^2(\Omega)} + \frac{\alpha}{2}|C - C_h|^2_{1,\Omega} + r_0\|C - C_h\|^2_{L^2(\Omega)} \leq f(t) + \bar{c}\|C(t) - C_h(t)\|^2_{L^2(\Omega)}, \tag{5.55}$$

where we have defined the function $f : \mathbb{R}^+ \to \mathbb{R}$ such that, for all $n \in [\![1, N]\!]$,

$$\begin{aligned} f(t)|_{]t_{n-1},t_n]} = c\big(&\|g(t) - g^n\|^2_{L^2(\Omega)} + \sum_{K_n \in \mathcal{T}_{nh}} \Big((\eta^h_{n,K_n,1})^2 + (\eta^h_{n,K_n,2})^2 \Big) \\ &+ \frac{(t_n - t)^2}{\tau_n^3} \sum_{K_n \in \mathcal{T}_{nh}} (\eta^\tau_{n,K_n})^2 + \frac{(t - t_{n-1})^2}{\tau_n^2}\|C_h^n - C_h^{n-1}\|^2_{L^2(\Omega)} + \|\mathbf{f}_0(t) - \mathbf{f}_0(t_n)\|^2_{L^2(\Omega)^d} \big). \end{aligned}$$

We integrate inequality (5.55) between 0 and t and we get

$$\begin{aligned} &\frac{1}{2}\|C(t) - C_h(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_0^t |C(\tau) - C_h(\tau)|^2_{1,\Omega} d\tau + r_0 \int_0^t \|C(\tau) - C_h(\tau)\|^2_{L^2(\Omega)} d\tau \\ &\quad \leq \int_0^t f(\tau) d\tau + \bar{c}\int_0^t \|C(\tau) - C_h(\tau)\|^2_{L^2(\Omega)} d\tau. \end{aligned}$$

We designate by $\bar{f}$ the function

$$\bar{f} : t \to \bar{f}(t) = \int_0^t f(\tau) d\tau.$$

We apply Gronwall's lemma 2.2 to

$$y(t) = \frac{1}{2}\|C(t) - C_h(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_0^t |C(\tau) - C_h(\tau)|^2_{1,\Omega} d\tau + r_0 \int_0^t \|C(\tau) - C_h(\tau)\|^2_{L^2(\Omega)} d\tau, \tag{5.56}$$

and

$$\bar{g}(t) = 1 \ , \ \bar{k}(\tau) = \bar{c}.$$

We get

$$y(t) \leq \bar{f}(t) + \bar{c}\exp(\bar{c}T)\int_0^t \bar{f}(\tau) d\tau. \tag{5.57}$$

Since $\bar{f}$ is an increasing function, we have for all τ between 0 and t, $\bar{f}(\tau) \leq \bar{f}(t)$.
Hence, (5.57) can be written

$$y(t) \leq \bar{f}(t) + t\bar{c}\exp(\bar{c}T)\bar{f}(t).$$

For $t = t_m$, we get

$$y(t_m) \leq (1 + T\bar{c}\exp(\bar{c}T))\bar{f}(t_m) \leq \bar{C}_1 \bar{f}(t_m). \tag{5.58}$$

The definition of $\bar{f}$ implies

$$\bar{f}(t_m) = \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} f(\tau) d\tau.$$

Thus, we have

$$\begin{aligned} \bar{f}(t_m) = c\big(&\|g - \pi_\tau g\|^2_{L^2(0,t_m;L^2(\Omega))} \\ &+ \sum_{n=1}^{m} \sum_{K_n \in \mathcal{T}_{nh}} \tau_n \Big((\eta^h_{n,K_n,1})^2 + (\eta^h_{n,K_n,2})^2 \Big) + \frac{1}{3} \sum_{n=1}^{m} \sum_{K_n \in \mathcal{T}_{nh}} (\eta^\tau_{n,K_n})^2 \\ &+ \frac{1}{3} \sum_{n=1}^{m} \tau_n \|C^n_h - C^{n-1}_h\|^2_{L^2(\Omega)} + \|\mathbf{f}_0 - \pi_\tau \mathbf{f}_0\|^2_{L^2(0,t_m;L^2(\Omega)^d)} \big). \end{aligned} \tag{5.59}$$

Now, using (5.56), (5.58) and (5.59), we obtain the following bound:

$$\begin{aligned} &\|C(t_m) - C_h(t_m)\|^2_{L^2(\Omega)} + \alpha \|C - C_h\|^2_{L^2(0,t_m;H^1_0(\Omega))} + 2r_0 \|C - C_h\|^2_{L^2(0,t_m;L^2(\Omega))} \\ &\leq c \Bigg(\|g - \pi_\tau g\|^2_{L^2(0,t_m;L^2(\Omega))} + \sum_{n=1}^{m} \sum_{K_n \in \mathcal{T}_{nh}} (\eta^\tau_{n,K_n})^2 \\ &\quad + \sum_{n=1}^{m} \sum_{K_n \in \mathcal{T}_{nh}} \tau_n \Big((\eta^h_{n,K_n,1})^2 + (\eta^h_{n,K_n,2})^2 \Big) + \|\mathbf{f}_0 - \pi_\tau \mathbf{f}_0\|^2_{L^2(0,t_m;L^2(\Omega)^d)} \Bigg). \end{aligned} \tag{5.60}$$

Finally, we gather (5.32) and (5.60), and we use Lemma 5.10 to get the desired result. ■

Theorem 5.11 gives us a bound of the error between the exact and approximate solutions, in terms of the error estimators and oscillations of the data. In Lemma 5.12 below, we additionally bound the norm of a residual that appears in the efficiency study which will be led in Section 5.3.

Lemma 5.12. *Under the assumptions of Theorem 4.3, the solutions* $(\mathbf{u}, C)$ *and* $(\mathbf{u}_h, C_h)$ *of problems* (V) *and* (V_h) *respectively, verify the following inequality for all* $m = 1, \ldots, N$*:*

$$\begin{aligned} &\Big\| \frac{\partial}{\partial t}(C - C_h) + \mathbf{u}(t) \cdot \nabla C - \mathbf{u}_h \cdot \nabla \pi_\tau C_h - \frac{1}{2} \operatorname{div} \mathbf{u}_h \pi_\tau C_h + r_0(C - \pi_\tau C_h) \Big\|_{L^2(0,t_m;H^{-1}(\Omega))} \\ &\leq c_1 \Bigg(\sum_{n=1}^{m} \sum_{K_n \in \mathcal{T}_{nh}} [\tau_n (\eta^h_{n,K_n,2})^2 + (\eta^\tau_{n,K_n})^2] + \|g - \pi_\tau g\|^2_{L^2(0,t_m;L^2(\Omega))} \\ &\quad + \int_0^{t_m} |C(s) - C_h(s)|^2_{1,\Omega} ds \Bigg)^{1/2}, \end{aligned}$$

where c_1 *is a constant independent of* h *and* τ*.*

Proof. For each $n \in [\![1, N]\!]$ and for each $t \in]t_{n-1}, t_n]$, we consider the third equation of Problem (V):

$$\left(\frac{\partial}{\partial t} C(t), S \right) + \alpha(\nabla C(t), \nabla S) + (\mathbf{u}(t) \cdot \nabla C(t), S) + r_0(C(t), S) = (g(t), S), \tag{5.61}$$

where $S \in H^1_0(\Omega) \cap L^\infty(\Omega)$. The assumption of Theorem 4.3 implies that this formula also makes sense for all $S \in H^1_0(\Omega)$.

We recall that $\pi_\tau C_h(t)|_{]t_{n-1},t_n]} = C^n_h$ and we insert $C_h(t)$ in the first term of (5.61), $\pi_\tau C_h(t)$ in the second term, $\mathbf{u}_h(t) \cdot \nabla \pi_\tau C_h(t)$ in the third term and $\pi_\tau C_h(t)$ in the fourth one, and we get, also adding the term

$\frac{1}{2}(\text{div } \mathbf{u}_h(t)\pi_\tau C_h(t), S)$:

$$\begin{aligned}\left(\frac{\partial}{\partial t}(C-C_h)(t),S\right)+\alpha(\nabla(C-\pi_\tau C_h)(t),\nabla S)+(\mathbf{u}(t)\cdot\nabla C(t)-\mathbf{u}_h(t)\cdot\nabla\pi_\tau C_h(t),S)\\ +r_0((C-\pi_\tau C_h)(t),S)-\frac{1}{2}(\text{div}\,\mathbf{u}_h(t)\pi_\tau C_h(t),S)\\ =(g(t),S)-\left(\frac{\partial}{\partial t}C_h(t),S\right)-\alpha(\nabla\pi_\tau C_h(t),\nabla S)-(\mathbf{u}_h(t)\cdot\nabla\pi_\tau C_h(t),S)\\ -r_0(\pi_\tau C_h(t),S)-\frac{1}{2}(\text{div}\,\mathbf{u}_h(t)\pi_\tau C_h(t),S).\end{aligned}$$

We use the definition of $R^h(C_h)$ given by (5.13) to get the following equality:

$$\begin{aligned}\left(\frac{\partial}{\partial t}(C-C_h)(t),S\right)+(\mathbf{u}(t)\cdot\nabla C(t)-\mathbf{u}_h\cdot\nabla\pi_\tau C_h(t),S)+r_0(C-\pi_\tau C_h)(t),S)\\ -\frac{1}{2}(\text{div}\,\mathbf{u}_h(t)\pi_\tau C_h(t),S)\\ =-\alpha(\nabla(C-\pi_\tau C_h)(t),\nabla S)+\langle g(t)-g^n+R^h(C_h)(t),S\rangle.\end{aligned}$$

The equality above as well as (5.15) allow us to get the following inequality: for any $S_h\in Y_{nh}$,

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}(C-C_h)(t)+\mathbf{u}(t)\cdot\nabla C(t)-\mathbf{u}_h(t)\cdot\nabla\pi_\tau C_h(t)+r_0(C-\pi_\tau C_h)(t)-\frac{1}{2}\,\text{div}\,\mathbf{u}_h(t)\pi_\tau C_h(t)\right\|_{H^{-1}(\Omega)}\\ &\le\sup_{S\in H_0^1(\Omega)}\frac{1}{\|S\|_{H_0^1(\Omega)}}\left[\left(\frac{\partial}{\partial t}(C-C_h)(t)+\mathbf{u}(t)\cdot\nabla C(t)-\mathbf{u}_h(t)\cdot\nabla\pi_\tau C_h(t)+r_0(C-\pi_\tau C_h)(t)\right.\right.\\ &\qquad\left.\left.-\frac{1}{2}\,\text{div}\,\mathbf{u}_h(t)\pi_\tau C_h(t),S\right)\right]\\ &\le\sup_{S\in H_0^1(\Omega)}\frac{-\alpha(\nabla(C-\pi_\tau C_h)(t),\nabla S)+\langle g-g^n,S\rangle+\langle R^h(C_h)(t),S-S_h\rangle}{\|S\|_{H_0^1(\Omega)}}.\end{aligned}$$

Using the Cauchy-Schwarz inequality, choosing $S_h=R_{nh}S$ and using properties (5.5) and (5.17), we obtain

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}(C-C_h)(t)+\mathbf{u}(t)\cdot\nabla C(t)-\mathbf{u}_h(t)\cdot\nabla\pi_\tau C_h(t)+r_0(C-\pi_\tau C_h)(t)-\frac{1}{2}\,\text{div}\,\mathbf{u}_h(t)\pi_\tau C_h(t)\right\|_{H^{-1}(\Omega)}\\ &\le c\left(|C-\pi_\tau C_h|_{1,\Omega}^2+\|g(t)-g^n\|_{L^2(\Omega)}^2+\sum_{K_n\in\mathcal{T}_{nh}}(\eta_{n,K_n,2}^h)^2\right)^{1/2}.\end{aligned}$$

Finally, we obtain the desired result by integrating over t between t_{n-1} and t_n, and then by summing over $n\in\{0,\dots,m\}$. ■

Remark 5.13. *Theorem 5.11 and Lemma 5.12 summarize our a posteriori error estimate which give our global upper bound using the indicators defined by Definition 5.6*

Remark 5.14. *The statement of the a priori error estimates given in Theorem 4.3 show an L^∞-time error bound for the velocity and the pressure, while the a posteriori error estimates given in Theorem 5.11 shows an L^2-time error. In fact, to get the L^∞-time estimate for the velocity and the pressure, we can consider Inequality* (5.39) *for $t=t_n$ and get the following bound*

$$\|\mathbf{u}(t_n)-\mathbf{u}_h(t_n)\|_{L^2(\Omega)^d}^2\le c_1\left(\|C(t_n)-C_h(t_n)\|_{L^2(\Omega)}^2+\sum_{K_n\in\mathcal{T}_{nh}}(\eta_{n,K_n,1}^h)^2+\|C_h^n-C_h^{n-1}\|_{L^2(\Omega)}^2\right)\quad(5.62)$$

which can be combined with (5.60) *to obtain the corresponding L^∞-time a posteriori error estimate. The pressure estimate can then be obtained by using Relation* (5.42).

5.3. **Upper bounds of the indicators.** In this section, we show the efficiency of the established *a posteriori* error estimates. This can be done by locally bounding each indicator with the error between the exact and numerical solutions, these bounds are called lower bounds of the error.

To accomplish the desired efficiency proof, we introduce approximations $\mathbf{f}_h^n$ of $\mathbf{f}^n$ and ν_h of ν as follows: for any function $\xi \in L^p(K_n)$ and for each element $K_n \in \mathcal{T}_{nh}$ we set

$$\mathbf{f}_h^n(\xi)|_{K_n} = \frac{1}{|K_n|}\int_{K_n} \mathbf{f}^n(\xi(\mathbf{x}))d\mathbf{x}, \tag{5.63}$$

and

$$\nu_h(\xi)|_{K_n} = \frac{1}{|K_n|}\int_{K_n} \nu(\xi(\mathbf{x}))d\mathbf{x}. \tag{5.64}$$

We also define g_h^n in a similar way

$$g_h^n|_{K_n} = \frac{1}{|K_n|}\int_{K_n} g^n(\mathbf{x})d\mathbf{x}. \tag{5.65}$$

A straightforward consequence of the properties of $\mathbf{f}^n$ and ν is that $\mathbf{f}_h^n$ and ν_h are Lipschitz with respect to ξ.

Remark 5.15. *It is important to note that $\mathbf{f}_h^n$, ν_h and g_h^n are in $P_0(K_n)$ on each K_n and we shall be able to use properties 5.2 and 5.3 which hold on polynomial functions.*

We begin by bounding the indicator η^τ_{n,K_n}:

Theorem 5.16. *For each $n \in \{1,\dots,N\}$, we have the following estimate:*

$$\left(\eta^\tau_{n,K_n}\right)^2 \leq \tilde{c}\big(\|C - C_h\|^2_{L^2(t_{n-1},t_n;H^1(K_n))} + \|C - \pi_\tau C_h\|^2_{L^2(t_{n-1},t_n;H^1(K_n))}\big), \tag{5.66}$$

where $\tilde{c}$ is a constant independent of n, h and τ.

Proof. The definition of C_h given in (5.6) and that of $\pi_\tau C_h$ give for any $n \in \{1,\dots,N\}$,

$$\forall t \in [t_{n-1}, t_n], \quad C_h(t) - \pi_\tau C_h(t) = \frac{t - t_n}{\tau_n}(C_h^n - C_h^{n-1}).$$

We insert $C(t)$ and we get

$$\left(\frac{t-t_n}{\tau_n}\right)^2 ||C_h^n - C_h^{n-1}||^2_{1,K_n} \leq 2\left(||C(t) - C_h(t)||^2_{1,K_n} + ||C(t) - \pi_\tau C_h(t)||^2_{1,K_n}\right).$$

We integrate between t_{n-1} and t_n to obtain the result. ∎

We will now bound sequentially the indicators $\eta^h_{n,K_n,1}$ and $\eta^h_{n,K_n,2}$. We first start with $\eta^h_{n,K_n,1}$. Note that in Theorem 5.17 below the term $\tau_n\|C_h^n - C_h^{n-1}\|^2_{L^2(K_n)}$ can itself be bounded by $\left(\eta^\tau_{n,K_n}\right)^2$, which was estimated in Theorem 5.16.

Theorem 5.17. *We assume that $\mathbf{u} \in L^\infty(0,T;L^\infty(\Omega)^d)$. The following estimate holds*

$$\begin{aligned}\tau_n\Big(\eta^h_{n,K_n,1}\Big)^2 \leq \tilde{c}\big(&||C - C_h||^2_{L^2(t_{n-1},t_n,H^1(K_n))} + \tau_n\|C_h^n - C_h^{n-1}\|^2_{L^2(K_n)} \\ &+ \|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(t_{n-1},t_n,L^2(K_n)^d)} + \|\nu(C) - \nu_h(C)\|^2_{L^2(t_{n-1},t_n,L^2(K_n))} \\ &+ \|p - p_h\|^2_{L^2(t_{n-1},t_n,H^1(K_n))} + \|\mathbf{f}_0 - \pi_\tau \mathbf{f}_0\|^2_{L^2(t_{n-1},t_n,L^2(K_n)^d)} \\ &+ \|\mathbf{f}_h^n(C) - \mathbf{f}^n(C)\|^2_{L^2(t_{n-1},t_n,L^2(K_n)^d)}\Big),\end{aligned} \tag{5.67}$$

where $\bar{c}$ is a positive constant independent of h and τ.

Proof. Let us start with the first term of $\left(\eta^h_{n,K_n,1}\right)^2$. We consider equation (5.9) and we insert the terms $\int_\Omega \nu(C(t))\mathbf{u}_h^n \cdot \mathbf{v}d\mathbf{x}$, $\int_\Omega \mathbf{f}_h^n(C_h^{n-1}) \cdot \mathbf{v}d\mathbf{x}$ and $\int_\Omega \nu_h(C_h^{n-1})\mathbf{u}_h^n \cdot \mathbf{v}d\mathbf{x}$, we get for all $t \in [t_{n-1}, t_n]$,

$$\begin{aligned}\int_\Omega (\mathbf{f}_h^n(C_h^{n-1}) - \nu_h(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n) \cdot \mathbf{v}d\mathbf{x} &= \int_\Omega (\mathbf{f}^n(C_h^{n-1}) - \mathbf{f}(t, C(t))) \cdot \mathbf{v}d\mathbf{x} + \int_\Omega \nabla(p - p_h)(t) \cdot \mathbf{v}d\mathbf{x} \\ &+ \int_\Omega (\mathbf{f}_h^n(C_h^{n-1}) - \mathbf{f}^n(C_h^{n-1})) \cdot \mathbf{v}d\mathbf{x} + \int_\Omega (\nu(C(t)) - \nu(C_h(t)))\mathbf{u}_h^n \cdot \mathbf{v}d\mathbf{x} \\ &+ \int_\Omega \nu(C(t))(\mathbf{u} - \mathbf{u}_h)(t) \cdot \mathbf{v}d\mathbf{x} + \int_\Omega (\nu(C_h^{n-1}) - \nu_h(C_h^{n-1}))\mathbf{u}_h^n \cdot \mathbf{v}d\mathbf{x} \\ &+ \int_\Omega (\nu(C_h(t)) - \nu(C_h^{n-1}))\mathbf{u}_h^n \cdot \mathbf{v}d\mathbf{x}.\end{aligned} \tag{5.68}$$

We take $\mathbf{v} = \mathbf{v}_{K_n}$ in (5.68), where

$$\mathbf{v}_{K_n} = \begin{cases} (\mathbf{f}_h^n(C_h^{n-1}) - \nu_h(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n)\psi_{K_n} & \text{in } K_n \\ 0 & \text{in } \Omega \backslash K_n. \end{cases} \tag{5.69}$$

It is important to note that $\mathbf{v}_{K_n}$ is a polynomial function thanks to the definitions of $\mathbf{f}_h^n$ and ν_h (see Remark 5.15). We get

$$\begin{aligned}\int_{K_n} (\mathbf{f}_h^n(C_h^{n-1}) - \nu_h(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n)^2 \cdot \psi_{K_n} d\mathbf{x} = \\ \int_{K_n} (\mathbf{f}^n(C_h^{n-1}) - \mathbf{f}(t, C(t))) \cdot \mathbf{v}_{K_n} d\mathbf{x} + \int_{K_n} \nabla(p - p_h)(t) \cdot \mathbf{v}_{K_n} d\mathbf{x} \\ + \int_{K_n} (\mathbf{f}_h^n(C_h^{n-1}) - \mathbf{f}^n(C_h^{n-1})) \cdot \mathbf{v}_{K_n} d\mathbf{x} + \int_{K_n} (\nu(C(t)) - \nu(C_h(t)))\mathbf{u}_h^n \cdot \mathbf{v}_{K_n} d\mathbf{x} \\ + \int_{K_n} \nu(C(t))(\mathbf{u} - \mathbf{u}_h)(t) \cdot \mathbf{v}_{K_n} d\mathbf{x} + \int_{K_n} (\nu(C_h^{n-1}) - \nu_h(C_h^{n-1}))\mathbf{u}_h^n \cdot \mathbf{v}_{K_n} d\mathbf{x} \\ + \int_{K_n} (\nu(C_h(t)) - \nu(C_h^{n-1}))\mathbf{u}_h^n \cdot \mathbf{v}_{K_n} d\mathbf{x}.\end{aligned} \tag{5.70}$$

The second and fifth terms in the right-hand side of (5.70) can easily be bounded using the Cauchy-Schwarz inequality. For the fourth and last terms, we insert $\mathbf{u}(t)$ then we apply the Cauchy-Schwarz inequality. In order to bound the first and third terms of (5.70), we use the Cauchy-Schwarz inequality and we set

$$\tilde{\varepsilon}_{K_n} = \|\mathbf{f}^n(C_h^{n-1}) - \mathbf{f}(t, C(t))\|_{L^2(K_n)^d} + \|\mathbf{f}_h^n(C_h^{n-1}) - \mathbf{f}^n(C_h^{n-1})\|_{L^2(K_n)^d}.$$

Using the triangle inequality we get

$$\begin{aligned}\tilde{\varepsilon}_{K_n} \leq &\|\mathbf{f}^n(C_h^{n-1}) - \mathbf{f}(t, C(t))\|_{L^2(K_n)^d} + \|\mathbf{f}_h^n(C_h^{n-1}) - \mathbf{f}_h^n(C(t))\|_{L^2(K_n)^d} \\ &+ \|\mathbf{f}_h^n(C(t)) - \mathbf{f}^n(C(t))\|_{L^2(K_n)^d} + \|\mathbf{f}^n(C(t)) - \mathbf{f}^n(C_h^{n-1})\|_{L^2(K_n)^d}.\end{aligned} \tag{5.71}$$

We deduce from the properties of $\mathbf{f}$, $\mathbf{f}^n$ and $\mathbf{f}_h^n$ that

$$\tilde{\varepsilon}_{K_n} \leq c_1 \|C_h^{n-1} - C(t)\|_{L^2(K_n)} + \|\mathbf{f}_0 - \pi_\tau \mathbf{f}_0\|_{L^2(K_n)^d} + \|\mathbf{f}_h^n(C(t)) - \mathbf{f}^n(C(t))\|_{L^2(K_n)^d}. \tag{5.72}$$

In the first term of the right-hand side of (5.72), we insert $C_h(t)$ and we obtain

$$\begin{aligned}\tilde{\varepsilon}_{K_n} \leq c\Bigg(&\|C(t) - C_h(t)\|^2_{L^2(K_n)} + \frac{(t_{n-1} - t)^2}{\tau_n^2} \|C_h^n - C_h^{n-1}\|^2_{L^2(K_n)} \\ &+ \|\mathbf{f}_0 - \pi_\tau \mathbf{f}_0\|_{L^2(K_n)^d} + \|\mathbf{f}_h^n(C(t)) - \mathbf{f}^n(C(t))\|^2_{L^2(K_n)^d} \Bigg).\end{aligned} \tag{5.73}$$

We will now bound the final term of equality (5.70) as follows:

$$\left| \int_{K_n} (\nu(C_h^{n-1}) - \nu_h(C_h^{n-1}))\mathbf{u}_h^n \cdot \mathbf{v}_{K_n} d\mathbf{x} \right| \leq \|(\nu(C_h^{n-1}) - \nu_h(C_h^{n-1}))\mathbf{u}_h^n\|_{L^2(K_n)^d} \|\mathbf{v}_{K_n}\|_{L^2(K_n)^d}.$$

We denote by

$$\varepsilon_{K_n} = \|(\nu(C_h^{n-1}) - \nu_h(C_h^{n-1}))\mathbf{u}_h^n\|^2_{L^2(K_n)^d}$$

and we use the triangle inequality to get

$$\varepsilon_{K_n} \leq \|(\nu(C_h^{n-1}) - \nu_h(C_h^{n-1}))(\mathbf{u}_h^n - \mathbf{u}(t))\|^2_{L^2(K_n)^d} + \|(\nu(C_h^{n-1}) - \nu_h(C_h^{n-1}))\mathbf{u}(t)\|^2_{L^2(K_n)^d}. \tag{5.74}$$

The first term of the right-hand side of (5.74) can be bounded by $(\nu_2 - \nu_1)\|\mathbf{u}_h^n - \mathbf{u}(t)\|_{L^2(K_n)^d}$. For the second, we insert $\nu_h(C)\mathbf{u}$ and $\nu(C)\mathbf{u}$. We have

$$\begin{aligned}\|(\nu(C_h^{n-1}) - \nu_h(C_h^{n-1}))\mathbf{u}(t)\|^2_{L^2(K_n)^d} \leq{}& \|(\nu_h(C(t)) - \nu_h(C_h^{n-1}))\mathbf{u}(t)\|^2_{L^2(K_n)^d} \\ &+ \|(\nu(C(t)) - \nu_h(C(t)))\mathbf{u}(t)\|^2_{L^2(K_n)^d} \\ &+ \|(\nu(C_h^{n-1}) - \nu(C(t)))\mathbf{u}(t)\|^2_{L^2(K_n)^d}.\end{aligned} \tag{5.75}$$

Inserting $\nu_h(C_h(t))$ in the first term and $\nu(C_h(t))$ in the last one in the right-hand side of (5.75), using the fact that ν and ν_h are Lipschitz and since $\mathbf{u} \in L^\infty(0,T;L^\infty(\Omega)^d)$, we get

$$\begin{aligned}\varepsilon_{K_n} \leq{}& \tilde{c}\bigg(||C(t) - C_h(t)||^2_{L^2(K_n)} + \left(\frac{t - t_{n-1}}{\tau_n}\right)^2 \|C_h^n - C_h^{n-1}\|^2_{L^2(K_n)} + \|\mathbf{u}(t) - \mathbf{u}_h(t)\|^2_{L^2(K_n)^d} \\ &+ \|\nu(C(t)) - \nu_h(C(t))\|^2_{L^2(K_n)}\bigg).\end{aligned} \tag{5.76}$$

For the left-hand side of (5.70), we use Property 5.2 to obtain

$$\|\mathbf{f}_h^n(C_h^{n-1}) - \nu_h(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n\|^2_{L^2(K_n)^d} \leq c\|(\mathbf{f}_h^n(C_h^{n-1}) - \nu_h(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n)\psi_{K_n}^{1/2}\|^2_{L^2(K_n)^d}. \tag{5.77}$$

Therefore, to get the bound of the first term of $(\eta^h_{n,K_n,1})^2$, we begin by applying the triangle inequality to obtain

$$\begin{aligned}\|\mathbf{f}^n(C_h^{n-1}) - \nu(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n\|^2_{L^2(K_n)^d} \leq{}& \|\mathbf{f}_h^n(C_h^{n-1}) - \nu_h(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n\|^2_{L^2(K_n)^d} \\ &+ \|(\nu(C_h^{n-1}) - \nu_h(C_h^{n-1}))\mathbf{u}_h^n\|^2_{L^2(K_n)^d} \\ &+ \|\mathbf{f}_h^n(C_h^{n-1}) - \mathbf{f}^n(C_h^{n-1})\|^2_{L^2(K_n)^d},\end{aligned} \tag{5.78}$$

then we use relations (5.70), (5.73), (5.76), (5.77) and (5.78) and we get the following bound of the first term of $(\eta^h_{n,K_n,1})^2$:

$$\begin{aligned}\|\mathbf{f}^n(C_h^{n-1}) - \nu(C_h^{n-1})\mathbf{u}_h^n - \nabla p_h^n\|^2_{L^2(K_n)^d} \leq{}& \tilde{c}\big(||C(t) - C_h(t)||^2_{L^2(K_n)} + \left(\frac{t_{n-1} - t}{\tau_n}\right)^2 \|C_h^n - C_h^{n-1}\|^2_{L^2(K_n)} \\ &+ \|\mathbf{u}(t) - \mathbf{u}_h(t)\|^2_{L^2(K_n)^d} + \|\nu(C(t)) - \nu_h(C(t))\|^2_{L^2(K_n)} \\ &+ |p(t) - p_h(t)|^2_{1,K_n} + \|\mathbf{f}_0(t) - \mathbf{f}_0(t_n)\|^2_{L^2(K_n)^d} \\ &+ \|\mathbf{f}_h^n(C(t)) - \mathbf{f}^n(C(t))\|^2_{L^2(\Omega)^d}\big).\end{aligned} \tag{5.79}$$

Next, to bound the second term of $(\eta^h_{n,K_n,1})^2$, we take $q = q_{K_n}$ and $q_h = 0$ in (5.10) where

$$q_{K_n} = \begin{cases} (\operatorname{div}\mathbf{u}_h^n)\psi_{K_n} & \text{in } K_n \\ 0 & \text{in } \Omega\backslash K_n. \end{cases} \tag{5.80}$$

Using the fact that q_{K_n} has it support limited to K_n and that it vanishes on all edges, applying the Cauchy-Schwarz inequality and using relations (5.2) and the fact that $\psi_{K_n} \leq \psi_{K_n}^{1/2}$ since $\psi_{K_n} \leq 1$, we get

$$h^2_{K_n}\|\operatorname{div}\mathbf{u}_h^n\|^2_{L^2(K_n)} \leq c\|\mathbf{u}(t) - \mathbf{u}_h(t)\|^2_{L^2(K_n)^d}. \tag{5.81}$$

We finish this proof by bounding the last term of $(\eta^h_{n,K_n,1})^2$. This term is not equal to zero if and only if the edge (for $d = 2$) or face (for $d = 3$) e_n of K_n belongs to Γ_h^b. In this case, we define

$$q_{e_n} = \begin{cases} \mathcal{L}_{e_n}(\phi_h^{e_n}\psi_{e_n}) & \text{in } K_n \\ 0 & \text{in } \Omega\backslash K_n, \end{cases} \tag{5.82}$$

where $\phi_h^{e_n}$ is given by (5.8).
Using (5.10) with $q_h = 0$ and $q = q_{e_n}$, and using the fact that the definition of $\mathcal{L}_{e_n}$ implies that the support of q_{e_n} is restricted to K_n and that it vanishes on the other edges or faces of K_n, we obtain

$$\int_{K_n} \nabla q_{e_n} \cdot (\mathbf{u} - \mathbf{u}_h)(t) d\mathbf{x} = \int_{K_n} q_{e_n} \operatorname{div} \mathbf{u}_h^n d\mathbf{x} - \int_{e_n} \phi_h^{e_n} q_{e_n} d\mathbf{x}.$$

Using inequality (5.4) and applying the Cauchy-Schwarz inequality we get

$$\|\phi_h^{e_n} \psi_{e_n}^{1/2}\|^2_{L^2(e_n)} \leq \tilde{c} \left[h_{e_n}^{1/2} \| \operatorname{div} \mathbf{u}_h^n \|_{L^2(K_n)} + h_{e_n}^{-1/2} \|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{L^2(K_n)^d} \right] \|\phi_h^{e_n} \psi_{e_n}\|_{L^2(e_n)}. \tag{5.83}$$

We use the left inequality in (5.3) to get a lower bound for the left-hand side of (5.83). Using that $|\psi_{e_n}| \leq 1$ we can bound $\|\phi_h^{e_n} \psi_{e_n}\|_{L^2(e_n)}$ by $\|\phi_h^{e_n} \psi_{e_n}^{1/2}\|_{L^2(e_n)}$ and then use the right inequality in (5.3) to get an upper bound for the right-hand side of (5.83). This allows us to get:

$$h_{e_n} \|\phi_h^{e_n}\|^2_{L^2(e_n)} \leq c \left(h_{e_n}^2 \| \operatorname{div} \mathbf{u}_h^n \|^2_{L^2(K_n)} + \|\mathbf{u}(t) - \mathbf{u}_h(t)\|^2_{L^2(K_n)^d} \right). \tag{5.84}$$

We gather (5.79), (5.81) and (5.84), then we integrate between t_{n-1} and t_n to obtain:

$$\begin{aligned} \tau_n \big(\eta^h_{n,K_n,1}\big)^2 \leq c \Big(& ||C - C_h||^2_{L^2(t_{n-1},t_n,H^1(K_n))} + \tau_n \|C_h^n - C_h^{n-1}\|^2_{L^2(K_n)} \\ & + \|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(t_{n-1},t_n,L^2(K_n)^d)} + \|\nu(C) - \nu_h(C)\|^2_{L^2(t_{n-1},t_n,L^2(K_n))} \\ & + \|p - p_h\|^2_{L^2(t_{n-1},t_n,H^1(K_n))} + \|\mathbf{f}_0 - \pi_\tau \mathbf{f}_0\|^2_{L^2(t_{n-1},t_n,L^2(K_n)^d)} \\ & + \|\mathbf{f}_h^n(C) - \mathbf{f}^n(C)\|^2_{L^2(t_{n-1},t_n,L^2(K_n)^d)} \Big). \end{aligned} \tag{5.85}$$

Hence we deduce the desired result. ■

Now, we complete the efficiency study by obtaining an upper bound for the estimator $\eta^h_{n,K_n,2}$.

Theorem 5.18. *For each $n \in \{1, \ldots, N\}$ and for each $K_n \in \mathcal{T}_{nh}$, we have*

$$\begin{aligned} & \tau_n (\eta^h_{n,K_n,2})^2 \\ & \leq c \Bigg(\left\| \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}(t) \cdot \nabla C(t) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2} \operatorname{div} \mathbf{u}_h^n C_h^n + r_0 (C(t) - C_h^n) \right\|^2_{L^2(t_{n-1},t_n;H^{-1}(\Delta_{K_n}))} \\ & \quad + \int_{t_{n-1}}^{t_n} |C - \pi_\tau C_h|^2_{1,\Delta_{K_n}} + h^2_{K_n} \|g - \pi_\tau g\|^2_{L^2(t_{n-1},t_n;L^2(\Delta_{K_n}))} + \tau_n h^2_{K_n} \|g^n - g_h^n\|^2_{L^2(\Delta_{K_n})} \Bigg), \end{aligned} \tag{5.86}$$

where c is a constant independent of h and τ.

Proof. We take the third equation of problem (V). We insert $\alpha(\nabla C_h^n, \nabla S)$ and $(-g^n + g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2} \operatorname{div} \mathbf{u}_h^n C_h^n - r_0 C_h^n, S)$, then we apply the Green formula on $\alpha(\nabla C_h^n, \nabla S)$ to obtain

$$\begin{aligned} & \int_\Omega \frac{\partial}{\partial t}(C - C_h)(t) S d\mathbf{x} + \alpha \int_\Omega \nabla (C(t) - C_h^n) \cdot \nabla S d\mathbf{x} + \int_\Omega \mathbf{u}(t) \cdot \nabla C(t) S d\mathbf{x} \\ & \quad - \int_\Omega \mathbf{u}_h^n \cdot \nabla C_h^n S d\mathbf{x} - \frac{1}{2} \int_\Omega \operatorname{div} \mathbf{u}_h^n C_h^n S d\mathbf{x} + r_0 \int_\Omega (C(t) - C_h^n) S d\mathbf{x} \\ & = \sum_{K_n \in \mathcal{T}_{nh}} \Bigg[\int_{K_n} (g(t) - g^n) S d\mathbf{x} + \int_{K_n} (g^n - g_h^n) S d\mathbf{x} \\ & \quad + \int_{K_n} (g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2} \operatorname{div} \mathbf{u}_h^n C_h^n - r_0 C_h^n + \alpha \Delta C_h^n) S d\mathbf{x} \\ & \quad - \frac{\alpha}{2} \sum_{e_n \in \partial K_n \cap \Gamma_h^i} \int_{e_n} [\nabla C_h^n \cdot \mathbf{n}]_{e_n} S ds \Bigg]. \end{aligned} \tag{5.87}$$

In 5.87, for a fixed K_n, we choose $S = S_{K_n}$ where

$$S_{K_n} = \begin{cases} (g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\operatorname{div}\mathbf{u}_h^n C_h^n - r_0 C_h^n + \alpha\Delta C_h^n)\psi_{K_n} & \text{in } K_n \\ 0 & \text{in } \Omega\backslash K_n, \end{cases} \tag{5.88}$$

and we get the following equality

$$\begin{aligned} &\int_{K_n} (g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\operatorname{div}\mathbf{u}_h^n C_h^n - r_0 C_h^n + \alpha\Delta C_h^n)^2 \psi_{K_n} d\mathbf{x} \\ &= \int_{K_n} \frac{\partial}{\partial t}(C - C_h)(t) S_{K_n} d\mathbf{x} + \alpha \int_{K_n} \nabla(C(t) - C_h^n) \cdot \nabla S_{K_n} d\mathbf{x} + \int_{K_n} \mathbf{u}(t) \cdot \nabla C(t) S_{K_n} d\mathbf{x} \\ &\quad - \int_{K_n} \mathbf{u}_h^n \cdot \nabla C_h^n S_{K_n} d\mathbf{x} - \frac{1}{2}\int_{K_n} \operatorname{div}\mathbf{u}_h^n C_h^n S_{K_n} d\mathbf{x} + r_0 \int_{K_n} (C(t) - C_h^n) S_{K_n} d\mathbf{x} \\ &\quad - \int_{K_n} (g(t) - g^n) S_{K_n} d\mathbf{x} - \int_{K_n} (g^n - g_h^n) S_{K_n} d\mathbf{x}. \end{aligned} \tag{5.89}$$

Since $S_{K_n} \in H_0^1(K_n)$, we can apply the definition of the $H^{-1}(K_n)$ norm; moreover, we apply the usual Cauchy-Schwarz inequality and we get

$$\begin{aligned} &\left\| \left(g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\operatorname{div}\mathbf{u}_h^n C_h^n - r_0 C_h^n + \alpha\Delta C_h^n \right) \psi_{K_n}^{1/2} \right\|_{L^2(K_n)}^2 \\ &\leq c_1 \Bigg(\left\| \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}(t) \cdot \nabla C(t) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\operatorname{div}\mathbf{u}_h^n C_h^n + r_0(C(t) - C_h^n) \right\|_{H^{-1}(K_n)} \|\nabla S_{K_n}\|_{L^2(K_n)} \\ &+ \alpha |C(t) - C_h^n|_{1,K_n} \|\nabla S_{K_n}\|_{L^2(K_n)} + \|g(t) - g^n\|_{L^2(K_n)} \|S_{K_n}\|_{L^2(K_n)} + \|g^n - g_h^n\|_{L^2(K_n)} \|S_{K_n}\|_{L^2(K_n)} \Bigg). \end{aligned} \tag{5.90}$$

In order to obtain a lower bound for the left-hand side of (5.90), we use the left inequality in the upper row of (5.2). In order to find an upper bound for the right-hand side of (5.90), we use the second line of (5.2) and then use the fact that $\psi_{K_n} \leq 1$. Then we multiply both sides of the resulting inequality by h_{K_n} and we get the following bound:

$$\begin{aligned} &h_{K_n}^2 \left\| g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\operatorname{div}\mathbf{u}_h^n C_h^n - r_0 C_h^n + \alpha\Delta C_h^n \right\|_{L^2(K_n)}^2 \\ &\leq c_2 \Bigg(\left\| \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}(t) \cdot \nabla C(t) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2}\operatorname{div}\mathbf{u}_h^n C_h^n + r_0(C(t) - C_h^n) \right\|_{H^{-1}(K_n)}^2 \\ &\quad + \alpha^2 |C(t) - C_h^n|_{1,K_n}^2 + h_{K_n}^2 \|g(t) - g^n\|_{L^2(K_n)}^2 + h_{K_n}^2 \|g^n - g_h^n\|_{L^2(K_n)}^2 \Bigg). \end{aligned} \tag{5.91}$$

Inserting $\pm g^n$ and using the triangular inequality, this yields a bound for the first term of $(\eta_{n,K_n,2}^h)^2$. To obtain a bound for the second term of $(\eta_{n,K_n,2}^h)^2$, we consider an element $K_n \in \mathcal{T}_{nh}$ and $e_n \subset \partial K_n \cap \Gamma_h^i$; we denote by K_n' the other cell of the mesh that shares e_n and we take $S = S_{e_n}$ in (5.87) where

$$S_{e_n} = \begin{cases} \mathcal{L}_{e_n}(\alpha[\nabla C_h^n \cdot \mathbf{n}]_{e_n} \psi_{e_n}) & \text{in } K_n \cup K_n' \\ 0 & \text{in } \Omega\backslash K_n \cup K_n'. \end{cases} \tag{5.92}$$

Using the fact that the support of S_{e_n} is limited to $K_n \cup K'_n$, we get

$$\begin{aligned}
&\int_{e_n} [\alpha \nabla C_h^n \cdot \mathbf{n}]_{e_n}^2 \psi_{e_n} ds \\
&= \int_{K_n \cup K'_n} \left(g_h^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2} \operatorname{div} \mathbf{u}_h^n C_h^n - r_0 C_h^n + \alpha \Delta C_h^n \right) S_{e_n} d\mathbf{x} \\
&\quad - \int_{K_n \cup K'_n} \left(\frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}(t) \cdot \nabla C(t) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2} \operatorname{div} \mathbf{u}_h^n C_h^n + r_0 (C(t) - C_h^n) \right) S_{e_n} d\mathbf{x} \quad (5.93) \\
&\quad + \int_{K_n \cup K'_n} (g(t) - g^n) S_{e_n} d\mathbf{x} + \int_{K_n \cup K'_n} (g^n - g_h^n) S_{e_n} d\mathbf{x} \\
&\quad - \alpha \int_{K_n \cup K'_n} \nabla (C(t) - C_h^n) \cdot \nabla S_{e_n} d\mathbf{x}.
\end{aligned}$$

Since $S_{e_n} \in H_0^1(K_n \cup K'_n)$, we can apply the definition of the $H^{-1}(K_n \cup K'_n)$ norm; we also apply the Cauchy-Schwarz inequality, we use both inequalities of Property 5.3 and the fact that $\psi_{e_n} \leq 1$ and we multiply by h_{e_n} and use (5.91) to obtain the following inequality:

$$\begin{aligned}
&h_{e_n} \|\alpha [\nabla C_h^n \cdot \mathbf{n}]_{e_n}\|_{L^2(e_n)}^2 \\
&\leq C_1 \Bigg[\left\| \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}(t) \cdot \nabla C(t) - \mathbf{u}_h^n \cdot \nabla C_h^n - \frac{1}{2} \operatorname{div} \mathbf{u}_h^n C_h^n + r_0 (C(t) - C_h^n) \right\|_{H^{-1}(K_n \cup K'_n)}^2 \quad (5.94) \\
&\qquad + h_{e_n}^2 \|g(t) - g^n\|_{L^2(K_n \cup K'_n)}^2 + h_{e_n}^2 \|g^n - g_h^n\|_{L^2(K_n \cup K'_n)}^2 + \alpha^2 |C(t) - C_h^n|_{1, K_n \cup K'_n}^2 \Bigg].
\end{aligned}$$

Finally, we gather (5.91) and (5.94), then we integrate on $[t_{n-1}, t_n]$ to get the result of Theorem 5.18. ■

6. Numerical results

The results proved in this paper are validated using numerical simulations on Freefem++ (see [27]). We consider a square domain Ω where each edge is divided into N segments of equal length. Hence Ω is divided into $2N^2$ triangles.

We define the following relative total error between the exact and numerical solutions:

$$err = \left(\frac{\displaystyle\sum_{n=1}^{N} \tau_n \big(\|\mathbf{u}(t_n) - \mathbf{u}_h(t_n)\|_{L^2(\Omega)^2}^2 + |p(t_n) - p_h(t_n)|_{1,\Omega}^2 + |C(t_n) - C_h(t_n)|_{1,\Omega}^2 \big)}{\displaystyle\sum_{n=1}^{N} \tau_n \big(\|\mathbf{u}(t_n)\|_{L^2(\Omega)^2}^2 + |p(t_n)|_{1,\Omega}^2 + |C(t_n)|_{1,\Omega}^2 \big)} \right)^{1/2}, \quad (6.1)$$

where $(\mathbf{u}, p, C)$ is a solution of the problem (V) and $(\mathbf{u}_h, p_h, C_h)$ is a solution of the problem (V_h).
The *a posteriori* error estimates between the exact and numerical solutions obtained in the previous sections will be used in this part to show numerical results based on mesh adaptation. For this objective, we introduce the relative time indicator E_τ defined by

$$E_\tau = \left(\frac{\sum_{n=0}^{N} (\eta_n^\tau)^2}{D} \right)^{1/2}, \quad (6.2)$$

where

$$(\eta_n^\tau)^2 = \sum_{K_n \in \mathcal{T}_{nh}} \left(\eta_{n,K_n}^\tau \right)^2 \quad \text{and} \quad D = \sum_{n=1}^{N} \tau_n \left(\|\mathbf{u}_h(t_n)\|_{L^2(\Omega)^2}^2 + |p_h(t_n)|_{1,\Omega}^2 + |C_h(t_n)|_{1,\Omega}^2 \right), \quad (6.3)$$

and the relative space indicators defined by

$$E_{h_1} = \left(\frac{\sum_{n=0}^{N} (\eta_{1n}^h)^2}{D} \right)^{1/2} \quad \text{and} \quad E_{h_2} = \left(\frac{\sum_{n=0}^{N} (\eta_{2n}^h)^2}{D} \right)^{1/2}$$

where η_{1n}^h and η_{2n}^h are given by

$$(\eta_{1n}^h)^2 = \sum_{K_n \in \mathcal{T}_{nh}} \tau_n \left(\eta_{n,K_n,1}^h \right)^2 \quad \text{and} \quad (\eta_{2n}^h)^2 = \sum_{K_n \in \mathcal{T}_{nh}} \tau_n \left(\eta_{n,K_n,2}^h \right)^2. \tag{6.4}$$

We introduce also the relative total error indicator given by

$$E_{total} = E_\tau + E_{h_1} + E_{h_2}. \tag{6.5}$$

First test: We consider the case where the exact solution of problem (V) is given by $(\mathbf{u}, p, C) = (\mathbf{curl}\,\psi, p, C)$ where

$$\left\{ \begin{array}{lcl} \psi(x,y) & = & e^{-30[(x-1-t/2)^2+(y-1)^2]}, \\ p(x,y,t) & = & (1+t)\cos(\pi x/3)\cos(\pi y/3), \\ C(x,y,t) & = & te^{-30[(x-1-t/2)^2+(y-1)^2]}. \end{array} \right.$$

We can compare the exact and numerical solutions and we can compute the error between the exact and numerical solutions.

We consider $\Omega =]0,3[^2$, $T = 2$, $\alpha = 1$, $r_0 = 1$, $\mathbf{f}_1(C) = C + 1$ and $\nu(C) = 1 + 0.2\sin(C)$. First, we consider the adaptive algorithm with an initial time step $\tau_1 = \frac{3\sqrt{2}}{N}$ and an initial mesh corresponding to $N = 30$. Figures 1-4 show the evolution of the mesh during the time steps for $t = 0, 0.141, 1.027$ and 2.

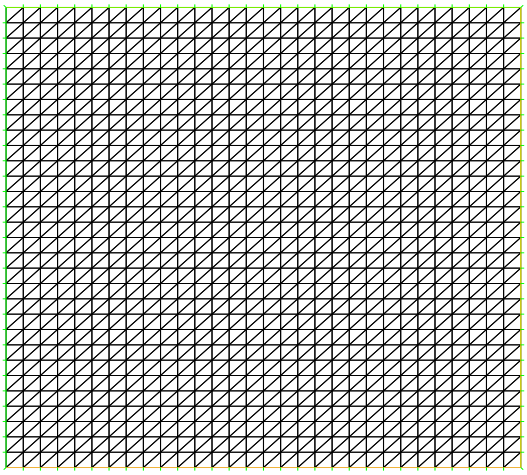

FIGURE 1. Initial mesh

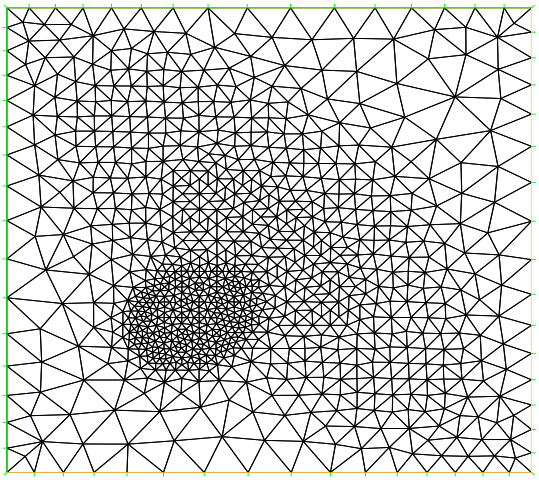

FIGURE 2. Mesh at $t = 0.141$

Next, in the left part of Figure 5, we compare the relative total error with respect to the total number of space-time unknowns in logarithmic scale, for both uniform and adaptive numerical algorithms. In the right part of Figure 5, we show a comparison of the global error for the indicators E_{total} with respect to the total number of unknowns in logarithmic scale for the adaptive and uniform refinement methods. Both parts of Figure 5 show clearly the advantage of the adaptive method versus the uniform one since the total errors are smaller.

We define the efficiency index by

$$EI = \left(\frac{\displaystyle\sum_{n=1}^{N} \sum_{K_n \in \mathcal{T}_{nh}} (\tau_n (\eta_{n,K_n,1}^h)^2 + \tau_n (\eta_{n,K_n,2}^h)^2 + (\eta_{n,K_n}^\tau)^2)}{\displaystyle\sum_{n=1}^{N} \tau_n (\|\mathbf{u}_h^n - \mathbf{u}(t_n)\|_{L^2(\Omega)}^2 + |p_h^n - p(t_n)|_{1,\Omega}^2 + |C_h^n - C(t_n)|_{1,\Omega}^2)} \right)^{1/2}. \tag{6.6}$$

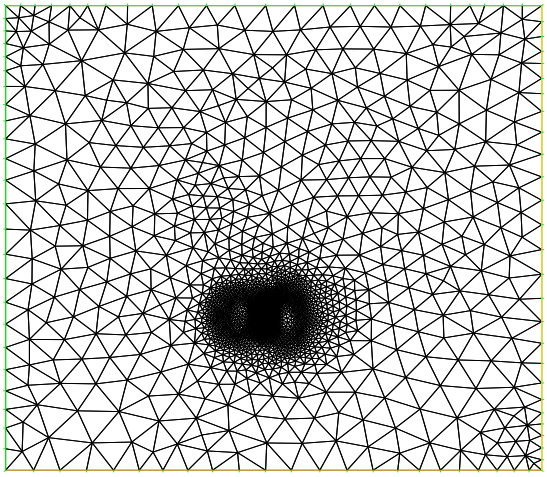

FIGURE 3. Mesh at $t = 1.027$

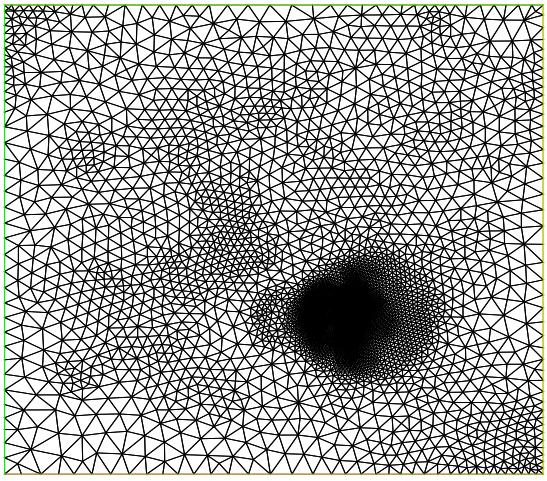

FIGURE 4. Mesh at $t = 2$

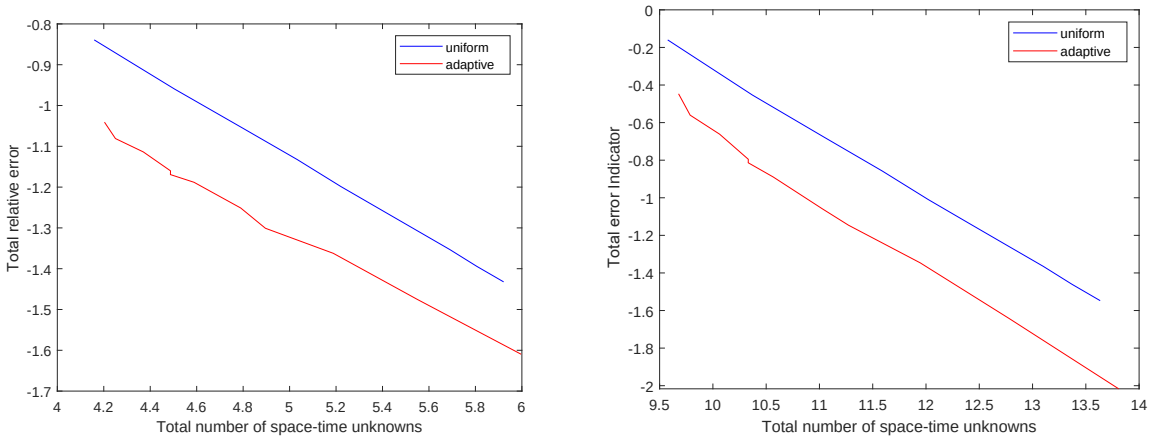

FIGURE 5. left: Total relative errors err. Right: Total error indicators E_{total}.

In Table 1, we can see the value of the efficiency index for different values of space-time unknowns STU:

STU	9694	12575	16685	25491	29281	38304	58668	88053	142265	309398	959966
EI	6.78	6.55	6.43	6.28	6.20	6.04	5.92	5.95	5.88	5.60	5.15

TABLE 1. Repartition of EI with respect to the total number of space-time unknowns STU.

We can see that the coefficient EI decreases from 6.78 to 5.15 when the STU number increases from 9694 to 959966.

Second test: In this case, we consider the coupling problem without any external forces ($\mathbf{f} = 0$ and $g = 0$). We take $T = 3$ and $\Omega =]0, 4[^2$.

For the Darcy problem, we take the viscosity $\nu = 1$ (independent of the concentration). We assume that the velocity of the fluid verifies the following boundary conditions:

$$\varphi|_{[0,4]\times\{0\}} = (0, -0.4), \quad \varphi|_{\{4\}\times[0,4]} = (0.4, 0), \quad \varphi|_{[0,4]\times\{4\}} = (0.4, 0), \quad \varphi|_{\{0\}\times[0,4]} = (0, -0.4).$$

For the concentration equation, we take $\alpha = 10^{-4}$, $r_0 = 10^{-6}$ and the following initial condition:

$$C(x, y, 0) = e^{-50((x-1)^2+(y-1)^2)},$$

which is represented by a ball centered at the point $(1,1)$.

In this case, it is easy to check that the indicators are as follows: for each $K_n \in \mathcal{T}_{nh}$, $\forall n \in [\![1,N]\!]$,

$$(\eta^h_{n,K_n,1})^2 = \| - \mathbf{u}^n_h - \nabla p^n_h\|^2_{L^2(K_n)^d} + h^2_{K_n}\|\text{div } \mathbf{u}^n_h\|^2_{L^2(K_n)} + \sum_{e_n\in\partial K_n\cap\Gamma^b_h} h_{e_n}\|\phi^{e_n}_h + \varphi\cdot\mathbf{n}\|^2_{L^2(e_n)},$$

$$(\eta^h_{n,K_n,2})^2 = h^2_{K_n}\left\|-\frac{1}{\tau_n}(C^n_h - C^{n-1}_h) - \mathbf{u}^n_h\cdot\nabla C^n_h - \frac{1}{2}\text{div } \mathbf{u}^n_h C^n_h - r_0 C^n_h + \alpha\Delta C^n_h\right\|^2_{L^2(K_n)}$$

$$+\frac{1}{2}\sum_{e_n\in\partial K_n\cap\Gamma^i_h} h_{e_n}\|[\alpha\nabla C^n_h\cdot\mathbf{n}]_{e_n}\|^2_{L^2(e_n)},$$

and

$$(\eta^\tau_{n,K_n})^2 = \tau_n|C^n_h - C^{n-1}_h|^2_{1,K_n}.$$

The algorithm is tested with an initial time step $\tau_1 = \frac{4\sqrt{2}}{N}$ and an initial mesh corresponding to $N = 40$.

In Figures 6, 7, 8 and 9, we show the evolution of the mesh during the time iterations.

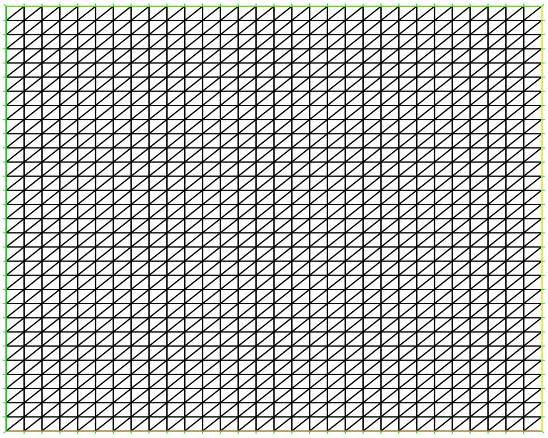

FIGURE 6. Mesh at $t = 0$

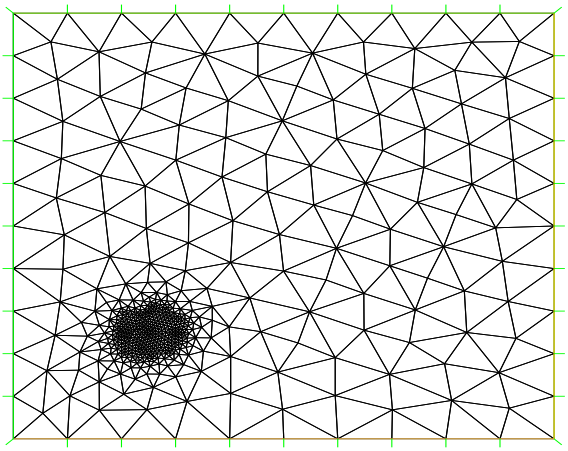

FIGURE 7. Mesh at $t = 0.128$

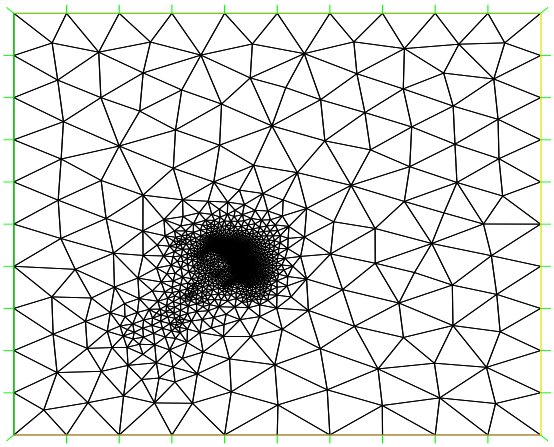

FIGURE 8. Mesh at $t = 1.767$

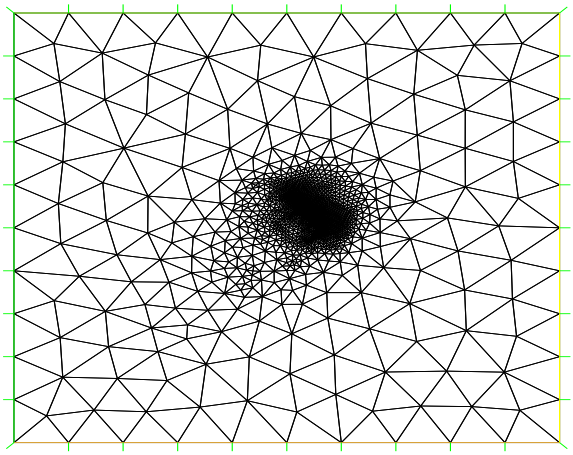

FIGURE 9. Mesh at $t = 3$

We notice from Figures 6-9 and Figures 10-13 that the ball of the concentration is moving across the diagonal during the time steps.

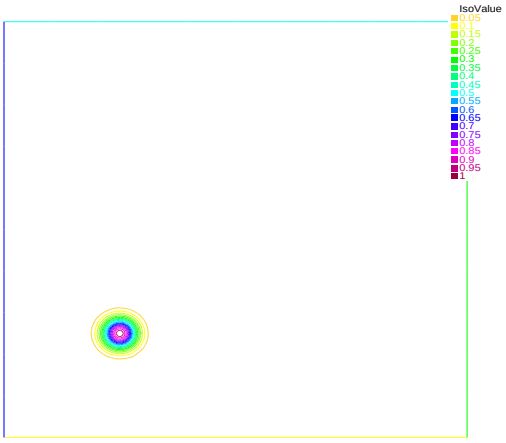

FIGURE 10. Concentration at $t = 0$

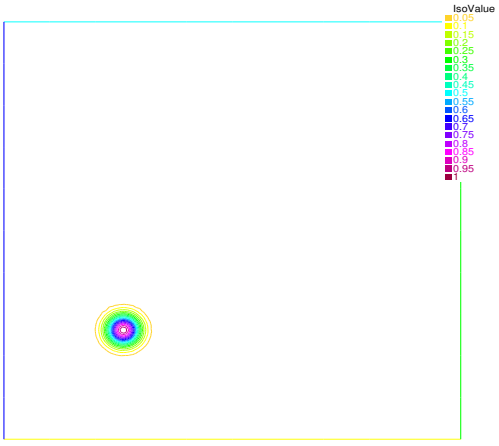

FIGURE 11. Concentration at $t = 0.128$

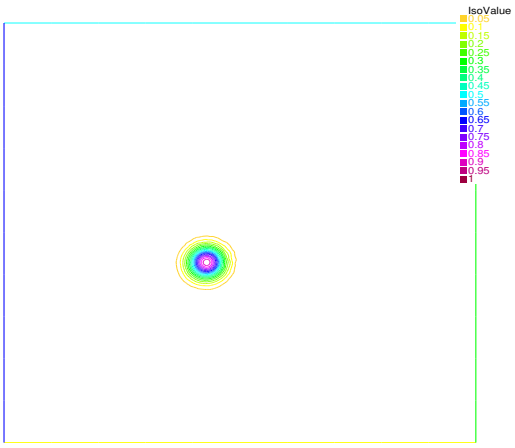

FIGURE 12. Concentration at $t = 1.767$

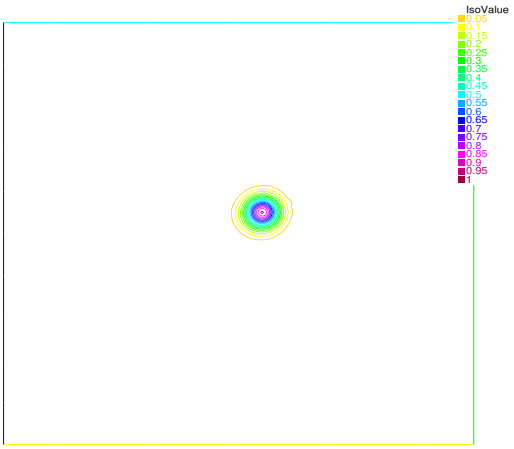

FIGURE 13. Concentration at $t = 3$

Figure 14 shows comparisons of the global error indicator E_{total} with respect to the number of unknowns in logarithmic scale for the adaptive and uniform refinement methods. We can clearly see the advantages of the adaptive mesh refinement versus the uniform one since the error indicators are much smaller in the case of the adaptive method.

7. Conclusion

In this work, we have derived *a posteriori* error estimates for a fully discrete scheme associated to the time dependent convection-diffusion-reaction equation coupled with the Darcy system. The obtained upper bound of the error is computable, up to data errors and based on two kinds of indicators: space discretization and time discretization errors. Furthermore, numerical investigations show the advantages of the adaptive mesh and time step method versus the uniform method.

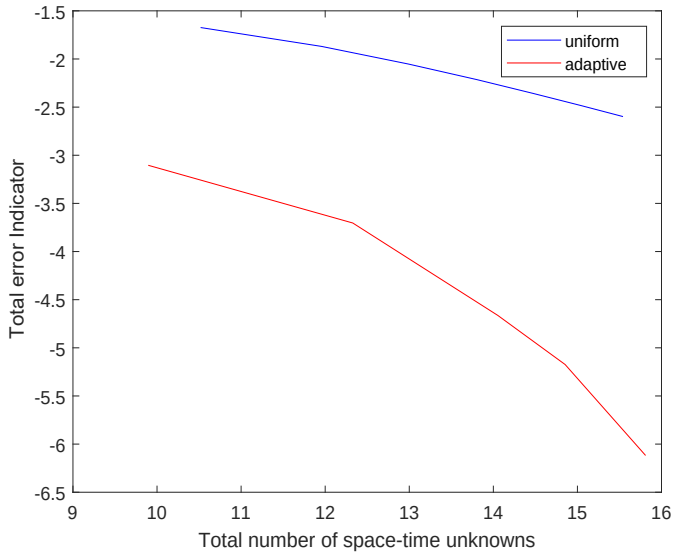
uniform
adaptive
Total error Indicator
Total number of space-time unknowns
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
-5.5
-6
-6.5
9
10
11
12
13
14
15
16

FIGURE 14

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