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A *POSTERIORI* ERROR ESTIMATES FOR THE LARGE EDDY SIMULATION APPLIED TO STATIONARY NAVIER-STOKES EQUATIONS

GHINA NASSREDDINE^{1,2}, PASCAL OMNES^{2,3}, AND TONI SAYAH¹

ABSTRACT. In this paper, we study in two and three space dimensions, the *a posteriori* error estimates for the Large Eddy Simulation applied to the Navier-Stokes system. We begin by introducing the Navier-Stokes and the corresponding Large Eddy Simulation (LES) equations. Then we introduce the corresponding discrete problem based on the finite element method. We establish an *a posteriori* error estimation with three types of error indicators related to the filter of the LES method, to the discretization and to the linearization. Finally, numerical investigations are shown and discussed.

Keywords : Large Eddy Simulation, Navier-Stokes, finite element method, *a posteriori* error estimation.

1. INTRODUCTION.

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be a bounded connected open domain, with a Lipschitz-continuous connected boundary $\Gamma = \partial\Omega$. We consider for a positive constant viscosity ν , the following incompressible Navier-Stokes system:

$$(P) \begin{cases} -\nu\Delta\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} + \nabla p & = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} & = 0 & \text{in } \Omega, \\ \mathbf{u} & = \mathbf{0} & \text{on } \Gamma, \end{cases}$$

where \mathbf{f} represents a density of body forces and belongs to $X' = H^{-1}(\Omega)^d$, the dual of the Sobolev space $X = H_0^1(\Omega)^d$. The unknowns are the velocity \mathbf{u} and the pressure p of the fluid.

In this paper, we establish an *a posteriori* error estimate corresponding to the Large Eddy Simulation (LES) applied to the Navier-Stokes problem (P). It is known that for small values of the viscosity coefficient, the flow may become turbulent and may thus develop short length scale details that would be computationally expensive to resolve. To reduce the cost of the computation, the LES method consists in focusing on the description of the largest length scales of the flow, while modeling the effect of the shortest length scales with a supplementary non-linear diffusion term in the Navier-Stokes momentum equation. The idea of this method is based on filtering the Navier-Stokes system to obtain the corresponding large scale behavior. But filtering the non-linear convective term leads to a tensor, called the Reynolds stress tensor, in which the entries are not related to the large scale components of the velocity and pressure, and which thus have to be modeled. There is a variety of ways in the literature to model the Reynolds stress tensor. Based on mathematical considerations, Leonard [27] and Clark et al. [9] transform the Reynolds stress tensor to the Fourier space where the Fourier transform of the filter function is approximated by a second order Taylor polynomial approximation. Another kind of transform is used by Galdi and Layton[17], where the Fourier transform of the filter function is approximated by a second-order rational Padé development. In addition, the Smagorinsky model [31] is used to describe the sub-grid scale effects on the resolved scales, where a non-linear eddy viscosity is added. Besides the Smagorinsky

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model, other models are proposed by Iliescu and Layton [21] to describe the sub-grid scale tensor. In the present article, we consider the Smagorinsky model which, due to its simplicity, remains widely used in industrial LES. The existence of the weak solution of the filtered problem is proved in [25, 26] and more general existence and uniqueness results are given in [13]. One can find in [30] a thorough introduction to LES, while in [24], mathematical aspects of LES of turbulent incompressible flows are gathered.

On the other hand, *a posteriori* error estimation consists in bounding the error between the exact solution of (P) and a numerical approximation of the filtered problem with a sum of local indicators, depending only on the computed approximate solution. Each local indicator is related to a given cell of the mesh and, in order to obtain a more accurate solution with a low additional CPU cost, the goal is to adapt the mesh by refining or coarsening it, according to the local values of the indicators. The *a posteriori* error estimate is optimal if each indicator can be bounded by the local error in the solution around the corresponding element. This kind of analysis was first introduced by Babuška and Rheinboldt [3] and then further developed, among others, by Verfürth [33] or Ainsworth and Oden [1]. In [14], El Akkad, El Khalifi and Guessous proposed a discretization of the incompressible Navier-Stokes equations by mixed finite elements with *a posteriori* error estimation of the computed solutions. Other works on *a posteriori* estimations for the stationary Navier-Stokes system are proposed in [22, 23] and for the non-stationary Navier-Stokes system in [7, 28]. In [5], Bernardi, Hecht and Verfürth considered a variational formulation of the three-dimensional Navier-Stokes equations with mixed boundary conditions for which they proved *a priori* and *a posteriori error estimates*. In [4], the authors discretized and linearized the Navier-Stokes problem by using an iterative fixed-point algorithm, established corresponding *a posteriori* error estimates and showed numerical investigations for academic applications and the Lid-Driven cavity test. Furthermore, *a posteriori* analysis of the Newton method applied to the Navier-Stokes problem were treated in [11].

In the present paper, we discretize the LES formulation of the Navier-Stokes system by finite elements and use a fixed-point linearization algorithm to approximate the solution of the resulting non-linear algebraic system. We establish the corresponding *a posteriori* error estimates based on three types of computable error indicators, the first one being linked to the discretization, the second one to the filter of the LES method and the last one to the linearization. The outline of the paper is as follows:

- In Section 2, we present the variational formulation of Navier-Stokes problem (P).
- Section 3 is devoted to the application of the LES method to the Navier-Stokes equation.
- In Section 4, we introduce the discrete filtered problem and we recall its main properties.
- In Section 5, we study the *a posteriori* errors and derive quasi-optimal estimates.
- Section 6 is devoted to the numerical results.
- We conclude in Section 7.

2. PRELIMINARIES

In this section, we introduce the variational problem corresponding to Problem (P). We begin by introducing some notations, definitions and properties which will be used later on.

Let $\alpha = (\alpha_1, \dots, \alpha_d)$ be non negative integers and $|\alpha| = \alpha_1 + \dots + \alpha_d$. We define the partial derivative ∂^α by

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}.$$

Then, for any positive integer m and real number $p \geq 1$, we recall the classical Sobolev space

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega), \partial^\alpha v \in L^p(\Omega), \quad \forall |\alpha| \leq m\}$$

equipped with the following semi-norm and norm:

$$|v|_{m,p,\Omega} = \left(\sum_{|\alpha|=m} \int_{\Omega} |\partial^\alpha v(\mathbf{x})|^p d\mathbf{x} \right)^{1/p} \quad \text{and} \quad \|v\|_{m,p,\Omega} = \left(\sum_{k \leq m} |v|_{k,p,\Omega}^p \right)^{1/p}.$$

We denote by $H^m(\Omega) = W^{m,2}(\Omega)$ and the corresponding semi-norm and norm by

$$|v|_{H^m(\Omega)} = |v|_{m,2,\Omega} \quad \text{and} \quad \|v\|_{H^m(\Omega)} = \|v\|_{m,2,\Omega}.$$

As usual, we shall omit p when $p = 2$ and denote by (\cdot, \cdot) the scalar product in $L^2(\Omega)$. Let \mathbf{v} be a vector valued function, we set

$$\|\mathbf{v}\|_{L^p(\Omega)^d} = \left(\int_{\Omega} |\mathbf{v}(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}}.$$

In view of the boundary conditions in system (P), we thus consider the space

$$H_0^1(\Omega) = \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma\}.$$

We denote by $X = H_0^1(\Omega)^d$ and by $M = L_0^2(\Omega)$ the space of functions in $L^2(\Omega)$ with a zero mean-value on Ω . We notice that the dual space of M is itself.

Lemma 2.1. *For any $p \leq 6$, there exists a positive constant S_p only depending on Ω such that*

$$\forall \mathbf{v} \in X, \quad \|\mathbf{v}\|_{L^p(\Omega)^d} \leq S_p \|\mathbf{v}\|_X. \quad (1)$$

We introduce the kernel

$$V = \left\{ \mathbf{v} \in X; \forall q \in M, \int_{\Omega} q(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x}) d\mathbf{x} = 0 \right\},$$

which is a closed subspace of X and coincides with

$$V = \{ \mathbf{v} \in X; \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \}.$$

Definition 2.2. *We introduce the trilinear form c defined by:*

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{v}) \cdot \mathbf{w} d\mathbf{x}.$$

Lemma 2.3. *For every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in X$ we have*

$$\begin{aligned} |c(\mathbf{u}, \mathbf{v}, \mathbf{w})| &\leq \|\mathbf{u}\|_{L^4(\Omega)^d} \|\mathbf{v}\|_X \|\mathbf{w}\|_{L^4(\Omega)^d} \\ &\leq S_4^2 \|\mathbf{u}\|_X \|\mathbf{v}\|_X \|\mathbf{w}\|_X. \end{aligned}$$

Lemma 2.4. *For $\mathbf{u}, \mathbf{v} \in X$ and $\operatorname{div} \mathbf{u} = 0$, we have $c(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$.*

Remark 2.5. *The spaces M and X satisfy a uniform inf-sup condition (see [18]): There exists a constant $\beta_* > 0$ such that*

$$\forall q \in M, \quad \sup_{\mathbf{v} \in X} \frac{\int_{\Omega} q(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x}) d\mathbf{x}}{\|\mathbf{v}\|_X} \geq \beta_* \|q\|_{L^2(\Omega)}.$$

System (P) is equivalent to the following variational problem:

$$(FV) \begin{cases} \text{find } \mathbf{u} \in X \text{ and } p \in M \text{ such that:} \\ \forall \mathbf{v} \in X, \quad \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle, \\ \forall q \in M, \quad b(\mathbf{u}, q) = 0, \end{cases}$$

where the bilinear form is defined by:

$$b(\mathbf{v}, q) = - \int_{\Omega} q \operatorname{div} \mathbf{v} d\mathbf{x}.$$

The existence and the conditional uniqueness of the solution (\mathbf{u}, p) of problem (FV) is given in [18, Chapter IV, §2]. Furthermore, the solution verifies the bound (with a constant $c > 0$):

$$\|\mathbf{u}\|_X \leq \frac{c}{\nu} \|\mathbf{f}\|_{X'}. \quad (2)$$

For the development of the *a posteriori* error estimate, we introduce the Stokes problem defined as follows: For a given $(\mathbf{f}, g) \in X' \times M$, look for (\mathbf{u}, p) in $X \times M$ such that

$$(S) \begin{cases} -\nu \Delta \mathbf{u}(\mathbf{x}) + \nabla p(\mathbf{x}) & = \mathbf{f}(\mathbf{x}) & \text{in } \Omega, \\ \operatorname{div} \mathbf{u}(\mathbf{x}) & = g & \text{in } \Omega, \\ \mathbf{u}(\mathbf{x}) & = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

More precisely, Problem (S) is set under the following variational formulation:

$$(FVS) \begin{cases} \text{Find } \mathbf{u} \in X \text{ and } p \in M \text{ such that:} \\ \forall \mathbf{v} \in X, \quad \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (\operatorname{div} \mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle, \\ \forall q \in M, \quad (\operatorname{div} \mathbf{u}, q) = (g, q). \end{cases}$$

We refer to the general abstract framework given by [18, Chapter I, §4] for the existence and the uniqueness of the solution $(\mathbf{u}, p) \in X \times M$ of Problem (FVS).

We introduce the following Stokes operator \mathcal{S} :

$$\begin{aligned} \mathcal{S} : X' \times M &\rightarrow X \times M \\ (\mathbf{f}, g) &\mapsto \mathcal{S}(\mathbf{f}, g) = (\mathbf{w}, \xi) \end{aligned}$$

where (\mathbf{w}, ξ) is the solution of the Stokes problem (FVS). We have the following bound (see [18, Chapter I, §4])

$$\|\mathcal{S}(\mathbf{f}, g)\|_{X \times M} \leq \frac{c}{\nu} \|(\mathbf{f}, g)\|_{X' \times M}. \quad (3)$$

We define also the function G given by

$$\begin{aligned} G : X &\rightarrow X' \\ \mathbf{w} &\mapsto G(\mathbf{w}) = (\mathbf{w} \cdot \nabla) \mathbf{w} + \frac{1}{2} (\operatorname{div} \mathbf{w}) \mathbf{w} - \mathbf{f}, \end{aligned}$$

and we introduce the map F on $X \times M$ such that for all $\mathbf{V} = (\mathbf{v}, q) \in X \times M$, we have

$$F(\mathbf{V}) = \mathbf{V} + \mathcal{S}(G(\mathbf{v}), 0).$$

Then, Problem (FV) can be equivalently written as

$$F(\mathbf{U}) = \mathbf{0}, \quad (4)$$

where $\mathbf{U} = (\mathbf{u}, p)$.

Definition 2.6. We define a non singular solution \mathbf{U} of Problem (FV) in the following way:

- (1) $F(\mathbf{U}) = \mathbf{0}$.
- (2) $DF(\mathbf{U})$ is an isomorphism of $X \times M$,

where $DF(\mathbf{U})$ is the Fréchet-differential of F in \mathbf{U} .

Proposition 2.7. There exists $L > 0$ such that the application $DF(\mathbf{U}) = Id + \mathcal{S}(DG(\mathbf{u}), 0)$ is L -Lipschitz-continuous: For all $\mathbf{U} = (\mathbf{u}, p)$ and $\mathbf{V} = (\mathbf{v}, q)$ in $X \times M$, we have

$$\|DF(\mathbf{U}) - DF(\mathbf{V})\|_{\mathcal{L}(X \times M)} \leq L \|\mathbf{U} - \mathbf{V}\|_{X \times M}. \quad (5)$$

Proof. For all $\mathbf{U} = (\mathbf{u}, p)$ and $\mathbf{V} = (\mathbf{v}, q)$ in $X \times M$, we have:

$$\|DF(\mathbf{U}) - DF(\mathbf{V})\|_{\mathcal{L}(X \times M)} = \|\mathcal{S}(DG(\mathbf{u}), 0) - \mathcal{S}(DG(\mathbf{v}), 0)\|_{\mathcal{L}(X \times M)}.$$

To obtain the desired result, we have by using (3) and for all $\mathbf{Z} = (\mathbf{z}, \xi) \in X \times M$,

$$\|\mathcal{S}((DG(\mathbf{u}), 0) \cdot \mathbf{Z}) - \mathcal{S}((DG(\mathbf{v}), 0) \cdot \mathbf{Z})\|_{X \times M} \leq \frac{c}{\nu} \|DG(\mathbf{u}) \cdot \mathbf{z} - DG(\mathbf{v}) \cdot \mathbf{z}\|_{X'}.$$

By using the definition of G , we get the following equality:

$$DG(\mathbf{u}) \cdot \mathbf{z} - DG(\mathbf{v}) \cdot \mathbf{z} = \mathbf{z} \cdot \nabla(\mathbf{u} - \mathbf{v}) + (\mathbf{u} - \mathbf{v}) \cdot \nabla \mathbf{z} + \frac{1}{2} \operatorname{div}(\mathbf{u} - \mathbf{v}) \mathbf{z} + \frac{1}{2} (\operatorname{div} \mathbf{z})(\mathbf{u} - \mathbf{v})$$

and then

$$\|DG(\mathbf{u}) \cdot \mathbf{z} - DG(\mathbf{v}) \cdot \mathbf{z}\|_{X'} \leq 4S_4^2 \|\mathbf{u} - \mathbf{v}\|_X \|\mathbf{z}\|_X.$$

Thus, combining the previous inequalities, we obtain (5) with $L = 4S_4^2$. \square

3. LARGE EDDY SIMULATION

In fluid mechanics, one would ideally use a direct numerical simulation (DNS) of the flow at all significant length scales since it is necessary to capture the dynamics of the fields having an impact on the solution. But DNS is computationally very expensive, and a cheaper method is to use the LES method, which consists in solving the large scales and modelling the influence of small scales by adding a supplementary non-linear diffusion term in the momentum equation. In LES, the large scales are defined by a spatial average of the velocity, the pressure and the external force. A common method is to define this spatial average by convolution of these quantities with an appropriate filter function of width denoted by δ . The velocity field will be decomposed as:

$$\mathbf{u}(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(\mathbf{x}),$$

where $\bar{\mathbf{u}}(\mathbf{x})$ is the filtered part of the velocity field and $\mathbf{u}'(\mathbf{x}) = \mathbf{u}(\mathbf{x}) - \bar{\mathbf{u}}(\mathbf{x})$ is the residue intended to be modeled. Since the convolution of the non-linear term $\mathbf{u} \cdot \nabla \mathbf{u}$ is not equal to the non-linear term applied to the convolution $\bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}$, the difference between these two terms, called the Reynolds stress tensor $\tilde{\mathbf{S}}(\mathbf{u}, p)$, needs to be modeled. The Smagorinsky model is the simplest model of LES that uses the assumption of local balance between the production and the dissipation of turbulent kinetic energy to express the turbulent viscosity according to the large scales (see [24]). Moreover, we consider the variant of this model in which the additional diffusion term is expressed in terms of $\nabla \bar{\mathbf{u}}$ instead of the more traditional $\frac{1}{2}(\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$, as discussed in [24], and as considered in [12, 13, 25, 26]. We consider the filtered problem:

$$(PL) \begin{cases} -\operatorname{div}((\nu + \nu_t(\bar{\mathbf{u}}))\nabla \bar{\mathbf{u}}(\mathbf{x})) + \bar{\mathbf{u}}(\mathbf{x}) \cdot \nabla \bar{\mathbf{u}}(\mathbf{x}) + \nabla \bar{p}(\mathbf{x}) & = \bar{\mathbf{f}}(\mathbf{x}) & \text{in } \Omega, \\ \operatorname{div} \bar{\mathbf{u}}(\mathbf{x}) & = \mathbf{0} & \text{in } \Omega, \\ \bar{\mathbf{u}}(\mathbf{x}) & = \mathbf{0} & \text{on } \Gamma, \end{cases}$$

where $\nu_t(\bar{\mathbf{u}}) = c_s \delta^2 \|\nabla \bar{\mathbf{u}}\|_F$ and $\|\nabla \bar{\mathbf{u}}\|_F = [\sum_{i,j=1}^d (\partial_i \bar{u}_j)^2]^{1/2}$. We suppose that $\bar{\mathbf{f}} \in X'$ and we note $Z = W_0^{1,3}(\Omega)^d$ and $W = \{\mathbf{v} \in Z / \operatorname{div} \mathbf{v} = 0\}$. The variational formulation associated to the problem (PL) is given by:

$$(FVL) \begin{cases} \text{Find } \bar{\mathbf{u}} \in Z \text{ and } \bar{p} \in M \text{ such that:} \\ \forall \bar{\mathbf{v}} \in Z, \quad ((\nu + \nu_t(\bar{\mathbf{u}}))\nabla \bar{\mathbf{u}}, \nabla \bar{\mathbf{v}}) + c(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \bar{\mathbf{v}}) + b(\bar{\mathbf{v}}, \bar{p}) = \langle \bar{\mathbf{f}}, \bar{\mathbf{v}} \rangle, \\ \forall \bar{q} \in M, \quad b(\bar{\mathbf{u}}, \bar{q}) = 0. \end{cases}$$

For the existence of the solution of Problem (FVL), we refer to [25] and [26]. Its uniqueness was established in [12, 13] under the following condition (notations are identical to those of [13]):

$$N\Psi_3(C_f) \leq \nu,$$

where Ψ_3 is defined as the inverse function of Φ_3 given by

$$\begin{aligned} \Phi_3 &: [0, \infty[\rightarrow \mathbb{R} \\ x &\mapsto \Phi_3(x) = \nu x + c_s \delta^2 \gamma_3^{-3} x^2, \end{aligned} \tag{6}$$

and

$$N := \sup_{\bar{\mathbf{u}}, \bar{\mathbf{v}} \in V} \frac{|c(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \bar{\mathbf{v}})|}{\|\bar{\mathbf{u}}\|_X^2 \|\bar{\mathbf{v}}\|_X}, \quad C_f = \sup_{\bar{\mathbf{v}} \in V} \frac{|\langle \bar{\mathbf{f}}, \bar{\mathbf{v}} \rangle|}{\|\bar{\mathbf{v}}\|_X} \quad \text{and} \quad \gamma_3 = \sup_{\bar{\mathbf{v}} \in W} \frac{\|\bar{\mathbf{v}}\|_X}{\|\bar{\mathbf{v}}\|_{1,3,\Omega}}.$$

Furthermore, the solution $\bar{\mathbf{u}}$ of Problem (FVL) verifies the following bound:

$$\|\bar{\mathbf{u}}\|_X \leq \Psi_3(C_f). \tag{7}$$

Remark 3.1. *In the sequel, we denote by $c, C, C_1, C_2 \dots$ constants independent of all discretization parameters.*

4. THE DISCRETE PROBLEM

In this section we introduce the discrete problem and some corresponding properties. We begin by collecting some notations concerning the discrete setting. We assume that Ω is a polygon when $d = 2$ or polyhedron when $d = 3$, so that it can be completely meshed with simplices. For the space discretization, we consider a regular family of triangulations (\mathcal{T}_h) of Ω , each of which is a set of closed non degenerate triangles for $d = 2$ or tetrahedra for $d = 3$, called elements. As usual, h stands for the maximum of the diameters of elements $\kappa \in \mathcal{T}_h$. The triangulations are supposed to satisfy:

- for each \mathcal{T}_h , $\bar{\Omega}$ is the union of all elements of \mathcal{T}_h ;
- The intersection of two distinct elements of \mathcal{T}_h is either empty, a common vertex, or an entire common edge (or face when $d = 3$);
- the ratio of the diameter h_κ of an element κ in \mathcal{T}_h to the diameter of its inscribed circle or ball is bounded by a constant independent of h .

Let (X_h, M_h) be the pair of discrete spaces corresponding to (Z, M) defined as follow

$$\begin{aligned} X_h &= \{ \mathbf{v}_h \in C_0(\bar{\Omega}); \forall \kappa \in \mathcal{T}_h, \mathbf{v}_h|_\kappa \in P_b(\kappa)^d; \mathbf{v}_h|_\Gamma = \mathbf{0} \}, \\ M_h &= \{ q_h \in M \cap C_0(\bar{\Omega}); \forall \kappa \in \mathcal{T}_h, q_h|_\kappa \in P_1(\kappa) \}, \end{aligned} \quad (8)$$

where $P_1(\kappa)$ stands for the space of restrictions to κ of affine functions. Moreover, $P_b(\kappa)$ is the sum of a polynomial of $P_1(\kappa)$ and a "bubble" function b_κ . Denoting the vertices of κ by $a_i, 1 \leq i \leq d+1$, and its corresponding barycentric coordinates by λ_i , the basic bubble function b_κ is the polynomial of degree $d+1$ which is defined by

$$b_\kappa(\mathbf{x}) = \lambda_1(\mathbf{x}) \cdots \lambda_d(\mathbf{x}).$$

We observe that $b_\kappa(\mathbf{x}) = 0$ on $\partial\kappa$ and that $b_\kappa(\mathbf{x}) > 0$ on κ . The graph of b_κ looks like a bubble attached to the boundary of κ , hence its name.

X_h and M_h are known [2] to satisfy the following uniform discrete inf-sup condition: $\exists \beta > 0$, independent of h such that:

$$\forall q_h \in M_h, \quad \sup_{\mathbf{v}_h \in X_h} \frac{\int_\Omega q_h(\mathbf{x}) \operatorname{div} \mathbf{v}_h(\mathbf{x}) \, d\mathbf{x}}{\|\mathbf{v}_h\|_{H_0^1(\Omega)^d}} \geq \beta \|q_h\|_{L^2(\Omega)}. \quad (9)$$

We introduce the following discrete kernel:

$$V_h = \{ \mathbf{v}_h \in X_h; \forall q_h \in M_h, (\operatorname{div} \mathbf{v}_h, q_h) = 0 \}.$$

Definition 4.1. We introduce the trilinear form:

$$d(\mathbf{u}, \mathbf{v}, \mathbf{w}) = c(\mathbf{u}, \mathbf{v}, \mathbf{w}) + \frac{1}{2} \int_\Omega (\operatorname{div} \mathbf{u}) \mathbf{v} \cdot \mathbf{w}.$$

Remark 4.2. the stabilisation term $\frac{1}{2} \int_\Omega (\operatorname{div} \mathbf{u}) \mathbf{v} \cdot \mathbf{w}$ vanishes when $\mathbf{u} \in V$ and doesn't vanish when \mathbf{u} is replaced by $\mathbf{u}_h \in X_h$. Moreover we have: $d(\mathbf{u}_h, \mathbf{v}_h, \mathbf{v}_h) = 0, \forall \mathbf{u}_h, \mathbf{v}_h \in X_h$.

Remark 4.3. In fact, for the discrete problem, we consider $\delta = h_\kappa$ in each element $\kappa \in \mathcal{T}_h$, which means that $\nu_t(\bar{\mathbf{u}}_h)|_\kappa = c_S h_\kappa^2 \|\nabla \bar{\mathbf{u}}_h\|_F$. This choice is common in LES and may be interpreted as the fact that the filter of the LES method is locally adapted to the diameter of each element of the mesh.

Proposition 4.4. There exists an operator P_h in $\mathcal{L}(V, V_h)$ (see [19, page 35]), such that for each $\mathbf{v} \in H^s(\Omega)^d$, we have

$$\|\mathbf{v} - P_h \mathbf{v}\|_{H^1(\Omega)^d} \leq C_P h^{s-1} \|\mathbf{v}\|_{H^s(\Omega)^d},$$

where C_P is a constant independent of h , and s is a real number such that $s \in [1, 2]$.

Proposition 4.5. We denote by \mathcal{F}_h the Scott-Zhang operator associated with \mathcal{T}_h and recall that for each $h \leq 1$ and $p \in H^s(\Omega)$, we have

$$\|p - \mathcal{F}_h p\|_{H^t(\Omega)} \leq C_z h^{s-t} \|p\|_{H^s(\Omega)},$$

where C_z is a constant independent of h , and s and t are real numbers such that $s \in]\frac{1}{2}, 1[$ and $0 \leq t \leq s$ (see [8]). However, $\mathcal{F}_h p$ does not necessarily have a vanishing mean-value over Ω , but an easy modification of \mathcal{F}_h yields an operator (that we still denote by \mathcal{F}_h) which is in $\mathcal{L}(H^s(\Omega) \cap L_0^2(\Omega), M_h)$ and has the same approximation properties as the original one.

Proposition 4.6. *For any dimension d and for any non negative integer r , there exists a constant \tilde{c} such that for any polynomial function \mathbf{v}_h of degree r on κ*

$$\|\mathbf{v}_h\|_{L^\infty(\kappa)} \leq \tilde{c} h_\kappa^{-d/2} \|\mathbf{v}_h\|_{L^2(\kappa)}. \quad (10)$$

We consider the following discrete problem associated to Problem (FVL) obtained by the Galerkin method:

$$(FVL_h) \left\{ \begin{array}{l} \text{find } (\bar{\mathbf{u}}_h, \bar{p}_h) \in X_h \times M_h \text{ solution of :} \\ \forall \mathbf{v}_h \in X_h, \quad \nu(\nabla \bar{\mathbf{u}}_h, \nabla \mathbf{v}_h) + (\nu_t(\bar{\mathbf{u}}_h) \nabla \bar{\mathbf{u}}_h, \nabla \mathbf{v}_h) + d(\bar{\mathbf{u}}_h, \bar{\mathbf{u}}_h, \mathbf{v}_h) \\ \qquad \qquad \qquad + b(\mathbf{v}_h, \bar{p}_h) = \langle \bar{\mathbf{f}}, \mathbf{v}_h \rangle, \\ \forall q_h \in M_h, \quad b(\bar{\mathbf{u}}_h, q_h) = 0. \end{array} \right.$$

The existence of the solution of Problem (FVL_h) is given in [12, page 6]. Its uniqueness was also established in [12] under the following condition:

$$N_h \Psi_3(C_{fh}) \leq \nu,$$

where Ψ_3 is defined in (6) and the following definitions hold:

$$N_h := \sup_{\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}} \in V_h} \frac{|d(\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}})|}{\|\bar{\mathbf{u}}\|_X \|\bar{\mathbf{v}}\|_X \|\bar{\mathbf{w}}\|_X} \quad \text{and} \quad C_{fh} = \sup_{\bar{\mathbf{v}} \in V_h} \frac{|\langle \bar{\mathbf{f}}, \bar{\mathbf{v}} \rangle|}{\|\bar{\mathbf{v}}\|_X}.$$

In order to compute the solution of the nonlinear problem (FVL_h), we introduce the following associated linear discrete problem: ($\bar{\mathbf{u}}_h^0$ is an arbitrary given value)

$$(FVL_{i,h}) \left\{ \begin{array}{l} \text{For any } i \geq 1 \text{ with a known } \bar{\mathbf{u}}_h^{i-1} \text{ in } X_h, \text{ find } (\bar{\mathbf{u}}_h^i, \bar{p}_h^i) \in X_h \times M_h \text{ solution of :} \\ \forall \mathbf{v}_h \in X_h, \quad \nu(\nabla \bar{\mathbf{u}}_h^i, \nabla \mathbf{v}_h) + (\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i, \nabla \mathbf{v}_h) \\ \qquad \qquad \qquad + d(\bar{\mathbf{u}}_h^{i-1}, \bar{\mathbf{u}}_h^i, \mathbf{v}_h) + b(\mathbf{v}_h, \bar{p}_h^i) = \langle \bar{\mathbf{f}}, \mathbf{v}_h \rangle, \\ \forall q_h \in M_h, \quad b(\bar{\mathbf{u}}_h^i, q_h) = 0. \end{array} \right.$$

Theorem 4.7. *For an iteration $i \geq 1$, with a known $\bar{\mathbf{u}}_h^{i-1} \in X_h$, Problem (FVL_{i,h}) admits a unique solution $(\bar{\mathbf{u}}_h^i, \bar{p}_h^i) \in X_h \times M_h$ verifying the following bound:*

$$\|\bar{\mathbf{u}}_h^i\|_X \leq \frac{1}{\nu} \|\bar{\mathbf{f}}\|_{X'}. \quad (11)$$

Proof. In finite dimension, in order to prove existence and uniqueness of the solution $(\bar{\mathbf{u}}_h^i, \bar{p}_h^i)$ of the linear problem (FVL_{i,h}), it suffices to prove the uniqueness. Thus, we consider System (FVL_{i,h}) with a vanishing right-hand side (i.e for $\bar{\mathbf{f}} = 0$) and we choose $\mathbf{v}_h = \bar{\mathbf{u}}_h^i$. Using Remark 4.2, we obtain:

$$\nu \|\nabla \bar{\mathbf{u}}_h^i\|_{L^2(\Omega)^d}^2 + (\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i, \nabla \bar{\mathbf{u}}_h^i) = 0.$$

As $(\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i, \nabla \bar{\mathbf{u}}_h^i) \geq 0$, we deduce that $\nu \|\nabla \bar{\mathbf{u}}_h^i\|_{L^2(\Omega)^d}^2 = 0$, then $\bar{\mathbf{u}}_h^i = 0$ and finally we obtain the uniqueness of the solution. The uniqueness of the pressure can be deduced by using the inf-sup condition (9).

To prove (11), we choose $\bar{\mathbf{v}}_h = \bar{\mathbf{u}}_h^i$ in (FVL_{i,h}) and we apply the Cauchy-Schwarz Inequality to obtain:

$$\nu \|\bar{\mathbf{u}}_h^i\|_X^2 + (\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i, \nabla \bar{\mathbf{u}}_h^i) = \langle \bar{\mathbf{f}}, \bar{\mathbf{u}}_h^i \rangle \leq \|\bar{\mathbf{f}}\|_{X'} \|\bar{\mathbf{u}}_h^i\|_X$$

Remarking that $(\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i, \nabla \bar{\mathbf{u}}_h^i) \geq 0$ we obtain (11). \square

4.1. Convergence to the continuous solution. In this section, we prove a convergence result for the solution $\bar{\mathbf{u}}_h^i$ of Problem $(FVL_{i,h})$ towards the solution \mathbf{u} of the exact problem (FV) , provided that \mathbf{f} is "small enough" or/and that ν is "large enough". We suppose that $\bar{\mathbf{f}}$ tends to \mathbf{f} when h and jointly δ tend to 0.

Theorem 4.8. *We suppose that the solution \mathbf{u} of (FV) is in $H^{1+\varepsilon^*}(\Omega)^d$ and the pressure $p \in H^{1/2+\varepsilon^*}(\Omega)$ for a given positive ε_* . We assume that \mathbf{f} and $\bar{\mathbf{f}}$ verify the following condition:*

$$\frac{cS_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{(4 + \sqrt{d})S_4^2}{2\nu} \|\bar{\mathbf{f}}\|_{X'} < \nu, \quad (12)$$

where S_4 is the constant of (1) and c is the constant of (2). Then, the solution $\bar{\mathbf{u}}_h^i$ of Problem $(FVL_{i,h})$ converges to the solution \mathbf{u} of Problem (FV) , in the following sense:

$$\forall \varepsilon > 0, \exists h_\varepsilon > 0 \text{ such that } \forall (\mathcal{T}_h) \text{ with } h \leq h_\varepsilon, \exists I_{\mathcal{T}_h} \in \mathbb{N} \text{ such that } \forall i \geq I_{\mathcal{T}_h}, \text{ we have } \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X \leq \varepsilon.$$

Proof. We denote by

$$D = \left(\nu - \frac{cS_4^2}{\nu} \|\mathbf{f}\|_{X'} - \frac{(4 + \sqrt{d})S_4^2}{2\nu} \|\bar{\mathbf{f}}\|_{X'} \right)$$

and

$$g(h) = \frac{1}{D} \left(C_P h^{\varepsilon_*} \left(2\nu + \frac{2cS_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{(1 + \sqrt{d})S_4^2}{\nu} \|\bar{\mathbf{f}}\|_{X'} \right) \|\mathbf{u}\|_{H^{1+\varepsilon_*}(\Omega)^d} \right. \\ \left. + h^{2-d/2} \frac{c_s \tilde{c}}{\nu^2} \|\bar{\mathbf{f}}\|_{X'}^2 + \|\mathbf{f} - \bar{\mathbf{f}}\|_{X'} + C_z \sqrt{d} h^{1/2+\varepsilon_*} \|p\|_{H^{1/2+\varepsilon_*}(\Omega)} \right),$$

where S_4 is the constant in (1), c is the constant in (2), C_p is the constant in Proposition 4.4, C_z is the constant in Proposition 4.5, \tilde{c} is the constant in Proposition 4.6 and c_s is the Smagorinsky constant.

Let $\varepsilon > 0$, then there exists a real h_ε sufficiently small such that $\forall h \leq h_\varepsilon$, $g(h) \leq \frac{\varepsilon}{2}$. This can be obtained by using the fact that $\bar{\mathbf{f}}$ tends to \mathbf{f} when h and jointly δ (size of the filter) tend to 0.

Next, we consider a mesh \mathcal{T}_h such that $h \leq h_\varepsilon$, we take $\mathbf{v} = \mathbf{v}_h \in V_h$ and we consider the difference between (FV) and $(FVL_{i,h})$. Then we obtain:

$$\nu(\nabla(\mathbf{u} - \bar{\mathbf{u}}_h^i), \nabla \mathbf{v}_h) + (\mathbf{u} \cdot \nabla \mathbf{u} - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i, \mathbf{v}_h) - (\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i, \nabla \mathbf{v}_h) \\ - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1} \bar{\mathbf{u}}_h^i, \mathbf{v}_h) - (p, \operatorname{div} \mathbf{v}_h) = \langle \mathbf{f} - \bar{\mathbf{f}}, \mathbf{v}_h \rangle. \quad (13)$$

We insert $\nabla P_h \mathbf{u}$ in the first term of (13), $\bar{\mathbf{u}}_h^{i-1} \cdot \nabla \mathbf{u}$ in the second term and $\mathcal{F}_h p$ in the fourth term, where P_h is the operator defined in Proposition 4.4 and \mathcal{F}_h is the modified Scott-Zhang operator defined in Proposition 4.5. We obtain:

$$\nu(\nabla(P_h \mathbf{u} - \bar{\mathbf{u}}_h^i), \nabla \mathbf{v}_h) - ((\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}) \cdot \nabla \mathbf{u}, \mathbf{v}_h) - (\bar{\mathbf{u}}_h^{i-1} \cdot \nabla(\bar{\mathbf{u}}_h^i - \mathbf{u}), \mathbf{v}_h) - (\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i, \nabla \mathbf{v}_h) \\ - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1} \bar{\mathbf{u}}_h^i, \mathbf{v}_h) - (p - \mathcal{F}_h p, \operatorname{div} \mathbf{v}_h) = \langle \mathbf{f} - \bar{\mathbf{f}}, \mathbf{v}_h \rangle + \nu(\nabla(P_h \mathbf{u} - \mathbf{u}), \nabla \mathbf{v}_h). \quad (14)$$

Note that $(\mathcal{F}_h p, \operatorname{div} \mathbf{v}_h)$ vanishes since $\mathcal{F}_h p$ is in M_h and \mathbf{v}_h in the discrete kernel V_h .

Next, we add and subtract $P_h \mathbf{u}$ in the second and third terms of (14), add the vanishing term $-\frac{1}{2}(\operatorname{div} \mathbf{u} \bar{\mathbf{u}}_h^i, \mathbf{v}_h)$ and insert the term $\operatorname{div} P_h \mathbf{u}$ in the fifth term of (14) to obtain:

$$\nu(\nabla(P_h \mathbf{u} - \bar{\mathbf{u}}_h^i), \nabla \mathbf{v}_h) - ((\bar{\mathbf{u}}_h^{i-1} - P_h \mathbf{u}) \cdot \nabla \mathbf{u}, \mathbf{v}_h) - ((P_h \mathbf{u} - \mathbf{u}) \cdot \nabla \mathbf{u}, \mathbf{v}_h) - (\bar{\mathbf{u}}_h^{i-1} \cdot \nabla(\bar{\mathbf{u}}_h^i - P_h \mathbf{u}), \mathbf{v}_h) \\ - (\bar{\mathbf{u}}_h^{i-1} \cdot \nabla(P_h \mathbf{u} - \mathbf{u}), \mathbf{v}_h) - (\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i, \nabla \mathbf{v}_h) - \frac{1}{2}(\operatorname{div}(\bar{\mathbf{u}}_h^{i-1} - P_h \mathbf{u}) \bar{\mathbf{u}}_h^i, \mathbf{v}_h) - \frac{1}{2}(\operatorname{div}(P_h \mathbf{u} - \mathbf{u}) \bar{\mathbf{u}}_h^i, \mathbf{v}_h) \\ - (p - \mathcal{F}_h p, \operatorname{div} \mathbf{v}_h) = \langle \mathbf{f} - \bar{\mathbf{f}}, \mathbf{v}_h \rangle + \nu(\nabla(P_h \mathbf{u} - \mathbf{u}), \nabla \mathbf{v}_h). \quad (15)$$

Since both $\bar{\mathbf{u}}_h^i$ and $P_h \mathbf{u}$ are in V_h , we may choose $\mathbf{v}_h = P_h \mathbf{u} - \bar{\mathbf{u}}_h^i$ in (15); we get the following inequality:

$$\begin{aligned}
\nu \|\mathbf{v}_h\|_X^2 &\leq \|P_h \mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{L^4(\Omega)^d} \|\mathbf{u}\|_X \|\mathbf{v}_h\|_{L^4(\Omega)^d} + \|P_h \mathbf{u} - \mathbf{u}\|_{L^4(\Omega)^d} \|\mathbf{u}\|_X \|\mathbf{v}_h\|_{L^4(\Omega)^d} \\
&\quad + \|\bar{\mathbf{u}}_h^{i-1}\|_{L^4(\Omega)^d} \|\bar{\mathbf{u}}_h^i - P_h \mathbf{u}\|_X \|\mathbf{v}_h\|_{L^4(\Omega)^d} \\
&\quad + \|\bar{\mathbf{u}}_h^{i-1}\|_{L^4(\Omega)^d} \|P_h \mathbf{u} - \mathbf{u}\|_X \|\mathbf{v}_h\|_{L^4(\Omega)^d} \\
&\quad + \|\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\Omega)^{d \times d}} \|\mathbf{v}_h\|_X + \frac{1}{2} \|\operatorname{div}(\bar{\mathbf{u}}_h^{i-1} - P_h \mathbf{u})\|_{L^2(\Omega)} \|\bar{\mathbf{u}}_h^i\|_{L^4(\Omega)} \|\mathbf{v}_h\|_{L^4(\Omega)^d} \\
&\quad + \frac{1}{2} \|\operatorname{div}(P_h \mathbf{u} - \mathbf{u})\|_{L^2(\Omega)} \|\bar{\mathbf{u}}_h^i\|_{L^4(\Omega)} \|\mathbf{v}_h\|_{L^4(\Omega)^d} + \sqrt{d} \|p - \mathcal{F}_h p\|_{L^2(\Omega)} \|\mathbf{v}_h\|_X \\
&\quad + \|\mathbf{f} - \bar{\mathbf{f}}\|_{X'} \|\mathbf{v}_h\|_X + \nu \|P_h \mathbf{u} - \mathbf{u}\|_X \|\mathbf{v}_h\|_X.
\end{aligned} \tag{16}$$

Moreover, taking into account the expression of ν_t and applying the inverse inequality (10), we have:

$$\begin{aligned}
\|\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\Omega)^{d \times d}} &\leq c_s \sum_{\kappa \in \mathcal{T}_h} h_\kappa^2 \|\nabla \bar{\mathbf{u}}_h^{i-1}\|_{L^\infty(\kappa)^d} \|\nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)} \\
&\leq c_s \tilde{c} \sum_{\kappa \in \mathcal{T}_h} h_\kappa^{2-d/2} \|\nabla \bar{\mathbf{u}}_h^{i-1}\|_{L^2(\kappa)} \|\nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)} \\
&\leq c_s \tilde{c} h^{2-d/2} \|\bar{\mathbf{u}}_h^{i-1}\|_X \|\bar{\mathbf{u}}_h^i\|_X.
\end{aligned} \tag{17}$$

Then by using (1), (11) and (2), we obtain:

$$\begin{aligned}
\left(\nu - \frac{S_4^2}{\nu} \|\bar{\mathbf{f}}\|_{X'}\right) \|P_h \mathbf{u} - \bar{\mathbf{u}}_h^i\|_X &\leq \left(\nu + \frac{c S_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{(2 + \sqrt{d}) S_4^2}{2\nu} \|\bar{\mathbf{f}}\|_{X'}\right) \|P_h \mathbf{u} - \mathbf{u}\|_X \\
&\quad + \left(\frac{c S_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{(2 + \sqrt{d}) S_4^2}{2\nu} \|\bar{\mathbf{f}}\|_{X'}\right) \|P_h \mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_X \\
&\quad + \frac{c_s \tilde{c}}{\nu^2} h^{2-d/2} \|\bar{\mathbf{f}}\|_{X'}^2 + \|\mathbf{f} - \bar{\mathbf{f}}\|_{X'} + \sqrt{d} \|p - \mathcal{F}_h p\|_{L^2(\Omega)}.
\end{aligned} \tag{18}$$

According to (12), the term $C_{\mathbf{f}} = \nu - \frac{S_4^2}{\nu} \|\bar{\mathbf{f}}\|_{X'}$ is positive. Therefore, we have by using the triangle inequality:

$$\begin{aligned}
C_{\mathbf{f}} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X &\leq \left(2\nu + \frac{c S_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{\sqrt{d} S_4^2}{2\nu} \|\bar{\mathbf{f}}\|_{X'}\right) \|P_h \mathbf{u} - \mathbf{u}\|_X \\
&\quad + \left(\frac{c S_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{(2 + \sqrt{d}) S_4^2}{2\nu} \|\bar{\mathbf{f}}\|_{X'}\right) \|P_h \mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_X \\
&\quad + \frac{c_s \tilde{c}}{\nu^2} h^{2-d/2} \|\bar{\mathbf{f}}\|_{X'}^2 + \|\mathbf{f} - \bar{\mathbf{f}}\|_{X'} + \sqrt{d} \|p - \mathcal{F}_h p\|_{L^2(\Omega)}.
\end{aligned} \tag{19}$$

Using now the triangle inequality on $\|P_h \mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_X$, we get:

$$\begin{aligned}
C_{\mathbf{f}} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X &\leq \left(2\nu + \frac{2c S_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{(1 + \sqrt{d}) S_4^2}{2\nu} \|\bar{\mathbf{f}}\|_{X'}\right) \|P_h \mathbf{u} - \mathbf{u}\|_X \\
&\quad + \left(\frac{c S_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{(2 + \sqrt{d}) S_4^2}{2\nu} \|\bar{\mathbf{f}}\|_{X'}\right) \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_X \\
&\quad + \frac{c_s \tilde{c}}{\nu^2} h^{2-d/2} \|\bar{\mathbf{f}}\|_{X'}^2 + \|\mathbf{f} - \bar{\mathbf{f}}\|_{X'} + \sqrt{d} \|p - \mathcal{F}_h p\|_{L^2(\Omega)}.
\end{aligned} \tag{20}$$

For the given mesh \mathcal{T}_h , we now consider two cases:

- (1) There exists an index i_0 such that

$$\|\mathbf{u} - \bar{\mathbf{u}}_h^{i_0-1}\|_X \leq \|\mathbf{u} - \bar{\mathbf{u}}_h^{i_0}\|_X,$$

then

$$\sup_{i \geq i_0} \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_X = \sup_{i \geq i_0+1} \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_X = \sup_{i \geq i_0} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X.$$

We consider now (20) for $i \geq i_0$, take the sup over $i \geq i_0$ in the right-hand side and we obtain that for all $i \geq i_0$

$$\begin{aligned} C_{\mathbf{f}} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X &\leq (2\nu + \frac{2cS_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{(1 + \sqrt{d})S_4^2}{\nu} \|\bar{\mathbf{f}}\|_{X'}) \|P_h \mathbf{u} - \mathbf{u}\|_X \\ &\quad + (\frac{cS_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{(2 + \sqrt{d})S_4^2}{2\nu} \|\bar{\mathbf{f}}\|_{X'}) \sup_{i \geq i_0} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X \\ &\quad + \frac{c_s \tilde{c}}{\nu^2} h^{2-d/2} \|\bar{\mathbf{f}}\|_{X'}^2 + \|\mathbf{f} - \bar{\mathbf{f}}\|_{X'} + \sqrt{d} \|p - \mathcal{F}_h p\|_{L^2(\Omega)}. \end{aligned} \quad (21)$$

Now the right-hand side of (21) does not depend on i , and we may take the sup in its left-hand side for $i \geq i_0$; we obtain:

$$\begin{aligned} (\nu - \frac{cS_4^2}{\nu} \|\mathbf{f}\|_{X'} - \frac{(4 + \sqrt{d})S_4^2}{2\nu} \|\bar{\mathbf{f}}\|_{X'}) \sup_{i \geq i_0} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X \\ \leq (2\nu + \frac{2cS_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{(1 + \sqrt{d})S_4^2}{\nu} \|\bar{\mathbf{f}}\|_{X'}) \|P_h \mathbf{u} - \mathbf{u}\|_X \\ + \frac{c_s \tilde{c}}{\nu^2} h^{2-d/2} \|\bar{\mathbf{f}}\|_{X'}^2 + \|\mathbf{f} - \bar{\mathbf{f}}\|_{X'} + \sqrt{d} \|p - \mathcal{F}_h p\|_{L^2(\Omega)}. \end{aligned} \quad (22)$$

Then, by applying the properties of the operator P_h to the first term in the right-hand side of (22) and those of the Scott-Zhang operator to its last term, we have :

$$\sup_{i \geq i_0} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X \leq g(h).$$

Then, for all $i \geq i_0$ we have :

$$\|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X \leq g(h) \leq \frac{\varepsilon}{2} \leq \varepsilon,$$

which proves the result in the first case with $I_{\mathcal{T}_h} = i_0$.

(2) If such i_0 doesn't exist, then for all i ,

$$\|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_X > \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X,$$

In this case, $(\|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X)_{i \geq 0}$ is a decreasing sequence bounded below by 0, so it is convergent and, from (20), its limit verifies:

$$\begin{aligned} (\nu - \frac{cS_4^2}{\nu} \|\mathbf{f}\|_{X'} - \frac{(4 + \sqrt{d})S_4^2}{2\nu} \|\bar{\mathbf{f}}\|_{X'}) \lim_{i \rightarrow +\infty} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X \\ \leq (2\nu + \frac{2cS_4^2}{\nu} \|\mathbf{f}\|_{X'} + \frac{(1 + \sqrt{d})S_4^2}{\nu} \|\bar{\mathbf{f}}\|_{X'}) \|P_h \mathbf{u} - \mathbf{u}\|_X \\ + \frac{c_s \tilde{c}}{\nu^2} h^{2-d/2} \|\bar{\mathbf{f}}\|_{X'}^2 + \|\mathbf{f} - \bar{\mathbf{f}}\|_{X'} + \sqrt{d} \|p - \mathcal{F}_h p\|_{L^2(\Omega)}. \end{aligned} \quad (23)$$

Just like in the first case, this leads to

$$\lim_{i \rightarrow \infty} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X \leq g(h) \leq \frac{\varepsilon}{2}.$$

Then, there exists an integer i_1 such that for all $i \geq i_1$ we have

$$\|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X - \lim_{i \rightarrow \infty} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X \leq \frac{\varepsilon}{2},$$

and then

$$\|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X \leq \lim_{i \rightarrow \infty} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X + \frac{\varepsilon}{2} \leq \varepsilon,$$

which proves the result in the second case with $I_{\mathcal{T}_h} = i_1$.

The result is thus proved with $I_{\mathcal{T}_h} = i_0$ in the first case or $I_{\mathcal{T}_h} = i_1$ in the second case. \square

Theorem 4.9. *Under the assumption of Theorem 4.8, we have the following bound between the numerical pressure \bar{p}_h^i solution of Problem (FVL $_{i,h}$) and the exact pressure p solution of (FV) :*

$$\|\bar{p}_h^i - p\|_{L^2(\Omega)} \leq 2C_2 h^{1/2+\varepsilon_*} \|p\|_{H^{1/2+\varepsilon_*}(\Omega)} + C_1(\mathbf{f}, \bar{\mathbf{f}}) (\|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X + \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_X) + C_2 h^{2-d/2} \|\bar{\mathbf{f}}\|_{X'}^2 + C_3 \|\mathbf{f} - \bar{\mathbf{f}}\|_{X'}, \quad (24)$$

where C_z is the constant of Proposition 4.5, C_1 is a positive constant depending on \mathbf{f} and $\bar{\mathbf{f}}$, C_2 and C_3 are positive constants.

Proof. We consider the difference between (FV) and (FVL_{i,h}), and we choose $\mathbf{v} = \mathbf{v}_h \in X_h$ to get

$$\begin{aligned} (\bar{p}_h^i - p, \operatorname{div} \mathbf{v}_h) &= \nu(\nabla(\bar{\mathbf{u}}_h^i - \mathbf{u}), \nabla \mathbf{v}_h) + (\bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}_h) + (\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i, \nabla \mathbf{v}_h) \\ &\quad + \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1} \bar{\mathbf{u}}_h^i, \mathbf{v}_h) + \langle \mathbf{f} - \bar{\mathbf{f}}, \mathbf{v}_h \rangle. \end{aligned} \quad (25)$$

We add and subtract $\mathcal{F}_h p$ in the left-hand side and $\bar{\mathbf{u}}_h^{i-1} \cdot \nabla \mathbf{u}$ in the right-hand side, and we insert the term $-\frac{1}{2}((\operatorname{div} \mathbf{u}) \bar{\mathbf{u}}_h^i, \mathbf{v}_h) = 0$ in (25) to obtain

$$\begin{aligned} (\bar{p}_h^i - \mathcal{F}_h p, \operatorname{div} \mathbf{v}_h) &= (p - \mathcal{F}_h p, \operatorname{div} \mathbf{v}_h) + \nu(\nabla(\bar{\mathbf{u}}_h^i - \mathbf{u}), \nabla \mathbf{v}_h) + (\bar{\mathbf{u}}_h^{i-1} \cdot \nabla(\bar{\mathbf{u}}_h^i - \mathbf{u}), \mathbf{v}_h) \\ &\quad + ((\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}) \cdot \nabla \mathbf{u}, \mathbf{v}_h) + (\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i, \nabla \mathbf{v}_h) + \frac{1}{2}(\operatorname{div}(\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}) \bar{\mathbf{u}}_h^i, \mathbf{v}_h) \\ &\quad + \langle \mathbf{f} - \bar{\mathbf{f}}, \mathbf{v}_h \rangle. \end{aligned} \quad (26)$$

It follows from the inf-sup condition (9) and by applying the Cauchy-Schwarz inequality to the right-hand side of (26) that

$$\begin{aligned} \|\bar{p}_h^i - \mathcal{F}_h p\|_{L^2(\Omega)} &\leq \|p - \mathcal{F}_h p\|_{L^2(\Omega)} + \frac{\nu}{\beta} \|\bar{\mathbf{u}}_h^i - \mathbf{u}\|_X + \frac{S_4^2}{\beta} \|\bar{\mathbf{u}}_h^{i-1}\|_X \|\bar{\mathbf{u}}_h^i - \mathbf{u}\|_X \\ &\quad + \frac{S_4^2}{\beta} \|\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}\|_X \|\mathbf{u}\|_X + \frac{1}{\beta} \|\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\Omega)^{d \times d}} + \frac{\sqrt{d} S_4^2}{2\beta} \|\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}\|_X \|\bar{\mathbf{u}}_h^i\|_X \\ &\quad + \frac{1}{\beta} \|\mathbf{f} - \bar{\mathbf{f}}\|_{X'}. \end{aligned}$$

Using (2), (11), (17), the triangular inequality $\|p - \bar{p}_h^i\|_{L^2(\Omega)} \leq \|\bar{p}_h^i - \mathcal{F}_h p\|_{L^2(\Omega)} + \|\mathcal{F}_h p - p\|_{L^2(\Omega)}$ and Proposition 4.5 we obtain the result. \square

Remark 4.10. *Theorem 4.8 shows the convergence of the numerical velocity $\bar{\mathbf{u}}_h^i$ solution of Problem (FVL_{i,h}) to the exact velocity \mathbf{u} solution of Problem (FV) in the sense that it suffices to first let h be "small enough", and then the index i "large enough" to have $\bar{\mathbf{u}}_h^i$ arbitrarily close to \mathbf{u} . Furthermore, Theorem 4.9 shows the convergence of the numerical pressure \bar{p}_h^i , solution of Problem (FVL_{i,h}), to the exact pressure p , solution of Problem (FV).*

5. A posteriori ERROR ANALYSIS

We start this section by introducing some additional notations and properties that will be useful in order to establish an *a posteriori* estimate.

For any element κ in \mathcal{T}_h , we denote by:

- ε_κ the set of edges (when $d = 2$) or faces (when $d = 3$) of κ that are not contained in Γ ,
- h_κ the diameter of the element κ and h_e the diameter of edge (or face) e ,
- $[\cdot]_e$ the jump through e on each edge (or face) e on ε_κ ,
- \mathbf{n}_κ stands for the unit outward normal vector to κ on $\partial\kappa$.

We now recall the following definitions and properties (see R. Verfürth, [32, Chapter 1]): for an element κ of \mathcal{T}_h , the bubble function ψ_κ (resp. ψ_e for the face e) is defined as the product of the $d + 1$ barycentric coordinates associated with the vertices of κ (resp. of the d barycentric coordinates associated with the vertices of e). We also consider a lifting operator \mathcal{L}_e defined on polynomials on e vanishing on ∂e into polynomials on the at most two elements κ containing e and vanishing on $\partial\kappa \setminus e$, which is constructed by affine transformation from a fixed operator on the reference element. As a result of the regularity of triangulation, The constants in the following properties are independent of κ .

Property 5.1. Denoting by $P_r(\kappa)$ the space of polynomials of degree smaller than or equal to r on κ , the following properties hold:

$$\forall v \in P_r(\kappa), \quad \begin{cases} c\|v\|_{L^2(\kappa)} \leq \|v\psi_\kappa^{1/2}\|_{L^2(\kappa)} \leq c'\|v\|_{L^2(\kappa)}, \\ |v|_{H^1(\kappa)} \leq ch_\kappa^{-1}\|v\|_{L^2(\kappa)}. \end{cases} \quad (27)$$

Property 5.2. Denoting by $P_r(e)$ the space of polynomials of degree smaller than or equal to r on e , we have

$$\forall v \in P_r(e), \quad c\|v\|_{L^2(e)} \leq \|v\psi_e^{1/2}\|_{L^2(e)} \leq c'\|v\|_{L^2(e)},$$

and, for all polynomials v in $P_r(e)$ vanishing on ∂e , if κ is an element which contains e ,

$$\|\mathcal{L}_e v\|_{L^2(\kappa)} + h_e |\mathcal{L}_e v|_{H^1(\kappa)} \leq ch_e^{1/2}\|v\|_{L^2(e)}.$$

We also introduce the Clément type regularization operator C_h which has the following properties, see [6, section IX.3] and [10]: For any function \mathbf{w} in $H^1(\Omega)^d$, $C_h \mathbf{w}$ belongs to the continuous affine finite element space and satisfies for any κ in \mathcal{T}_h and e in ε_κ ,

$$\|\mathbf{w} - C_h \mathbf{w}\|_{L^2(\kappa)^d} \leq ch_\kappa \|\mathbf{w}\|_{H^1(\Delta_\kappa)^d} \quad \text{and} \quad \|\mathbf{w} - C_h \mathbf{w}\|_{L^2(e)^d} \leq ch_e^{1/2} \|\mathbf{w}\|_{H^1(\Delta_e)^d}, \quad (28)$$

where Δ_κ and Δ_e are the following sets:

$$\Delta_\kappa = \bigcup \left\{ \kappa' \in \mathcal{T}_h; \kappa' \cap \kappa \neq \emptyset \right\} \quad \text{and} \quad \Delta_e = \bigcup \left\{ \kappa' \in \mathcal{T}_h; \kappa' \cap e \neq \emptyset \right\}.$$

Note that we use the variant of C_h which ensures that $C_h \mathbf{w}$ belongs to $H_0^1(\Omega)^d$ (see [10]).

We first introduce averaged values for the turbulent diffusion coefficient and for the data; these approximations will be useful to prove the optimality of the indicators.

First, $\nu_{th}(\bar{\mathbf{v}})$ is the piecewise constant approximation of $\nu_t(\bar{\mathbf{v}})$ defined for each element $\kappa \in \mathcal{T}_h$ as follows: for all $\bar{\mathbf{v}} \in Z$,

$$\nu_{th}(\bar{\mathbf{v}})|_\kappa = \frac{1}{|\kappa|} \int_\kappa \nu_t(\bar{\mathbf{v}}(\mathbf{x})) d\mathbf{x}. \quad (29)$$

Next, $\bar{\mathbf{f}}_h$ is the piecewise constant approximation of the data $\bar{\mathbf{f}}$:

$$\bar{\mathbf{f}}_h|_\kappa = \frac{1}{|\kappa|} \int_\kappa \bar{\mathbf{f}}(\mathbf{x}) d\mathbf{x}.$$

Next, we distinguish the discretization and linearization estimators; we also introduce an estimator related to the filtering process and thus to the turbulent viscosity term. For this, we first write the residual equation. The difference between (FV) and $(FVL_{i,h})$ gives the following relations for all $\mathbf{v} \in X$ and all $\mathbf{v}_h \in X_h$:

$$\begin{aligned} & \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} d\mathbf{x} + \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} d\mathbf{x} - \int_\Omega p \operatorname{div} \mathbf{v} d\mathbf{x} - \nu \int_\Omega \nabla \bar{\mathbf{u}}_h^i : \nabla \mathbf{v} d\mathbf{x} - \int_\Omega (\bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i) \cdot \mathbf{v} d\mathbf{x} \\ & - \frac{1}{2} \int_\Omega (\operatorname{div} \bar{\mathbf{u}}_h^{i-1}) \bar{\mathbf{u}}_h^i \cdot \mathbf{v} d\mathbf{x} - \int_\Omega \nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i : \nabla \mathbf{v} d\mathbf{x} + \int_\Omega \bar{p}_h^i \operatorname{div} \mathbf{v} d\mathbf{x} \\ & = \langle \mathbf{f}, \mathbf{v} \rangle - \langle \bar{\mathbf{f}}, \mathbf{v}_h \rangle - \nu \int_\Omega \nabla \bar{\mathbf{u}}_h^i : \nabla (\mathbf{v} - \mathbf{v}_h) d\mathbf{x} - \int_\Omega (\bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i) \cdot (\mathbf{v} - \mathbf{v}_h) d\mathbf{x} \\ & + \int_\Omega \bar{p}_h^i \operatorname{div} (\mathbf{v} - \mathbf{v}_h) d\mathbf{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \bar{\mathbf{u}}_h^{i-1}) \bar{\mathbf{u}}_h^i \cdot (\mathbf{v} - \mathbf{v}_h) d\mathbf{x} - \int_\Omega \nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i : \nabla (\mathbf{v} - \mathbf{v}_h) d\mathbf{x} \end{aligned} \quad (30)$$

and

$$b(\mathbf{u} - \bar{\mathbf{u}}_h^i, q) = \int_\Omega q \operatorname{div} \bar{\mathbf{u}}_h^i d\mathbf{x}. \quad (31)$$

Adding and subtracting $\int_{\Omega} (\bar{\mathbf{u}}_h^i \cdot \nabla \bar{\mathbf{u}}_h^i) \cdot \mathbf{v} d\mathbf{x}$, $\frac{1}{2} \int_{\Omega} (\operatorname{div} \bar{\mathbf{u}}_h^i) \bar{\mathbf{u}}_h^i \cdot \mathbf{v} d\mathbf{x}$ and $\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i$ in (30) and using the Green formula on each $\kappa \in \mathcal{T}_h$, we obtain for all $\mathbf{v} \in X$ and all $\mathbf{v}_h \in X_h$ the following equality:

$$\begin{aligned}
& \nu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} d\mathbf{x} + \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} d\mathbf{x} - \int_{\Omega} p \operatorname{div} \mathbf{v} d\mathbf{x} - \nu \int_{\Omega} \nabla \bar{\mathbf{u}}_h^i : \nabla \mathbf{v} d\mathbf{x} - \int_{\Omega} (\bar{\mathbf{u}}_h^i \cdot \nabla \bar{\mathbf{u}}_h^i) \cdot \mathbf{v} d\mathbf{x} \\
& - \frac{1}{2} \int_{\Omega} (\operatorname{div} \bar{\mathbf{u}}_h^i) \bar{\mathbf{u}}_h^i \cdot \mathbf{v} d\mathbf{x} + \int_{\Omega} \bar{p}_h^i \operatorname{div} \mathbf{v} d\mathbf{x} \\
& = \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} (\mathbf{f} - \bar{\mathbf{f}}) \cdot \mathbf{v} d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} (\bar{\mathbf{f}} - \bar{\mathbf{f}}_h) \cdot (\mathbf{v} - \mathbf{v}_h) d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_{\kappa} (\bar{\mathbf{f}}_h + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i) \right. \\
& \quad - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{i-1}) \bar{\mathbf{u}}_h^i - \nabla \bar{p}_h^i) \cdot (\mathbf{v} - \mathbf{v}_h) d\mathbf{x} \\
& \quad \left. - \frac{1}{2} \sum_{e \in \varepsilon_{\kappa}} \int_e [(\nu \nabla \bar{\mathbf{u}}_h^i + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i - \bar{p}_h^i \mathbb{I})(\sigma) \cdot \mathbf{n}] \cdot (\mathbf{v} - \mathbf{v}_h) d\sigma \right\} \\
& - \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} (\nu_t(\bar{\mathbf{u}}_h^{i-1}) - \nu_{th}(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i : \nabla (\mathbf{v} - \mathbf{v}_h) d\mathbf{x} + \int_{\Omega} ((\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i) \cdot \nabla \bar{\mathbf{u}}_h^i) \cdot \mathbf{v} d\mathbf{x} \\
& + \frac{1}{2} \int_{\Omega} (\operatorname{div}(\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i)) \bar{\mathbf{u}}_h^i \cdot \mathbf{v} d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} \nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i : \nabla \mathbf{v} d\mathbf{x},
\end{aligned} \tag{32}$$

We define the local linearization indicator $(\eta_{i,\kappa}^L)$, the local discretization indicator $(\eta_{i,\kappa}^D)$ and the local filter indicator $(\eta_{i,\kappa}^F)$ corresponding to an element $\kappa \in \mathcal{T}_h$, by:

$$(\eta_{i,\kappa}^L) = |\bar{\mathbf{u}}_h^i - \bar{\mathbf{u}}_h^{i-1}|_{H^1(\kappa)^d}, \tag{33}$$

$$\begin{aligned}
(\eta_{i,\kappa}^D) &= h_{\kappa} \|\bar{\mathbf{f}}_h + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{i-1}) \bar{\mathbf{u}}_h^i - \nabla \bar{p}_h^i\|_{L^2(\kappa)^d} \\
&+ \frac{1}{2} \sum_{e \in \varepsilon_{\kappa}} h_e^{1/2} \| [(\nu \nabla \bar{\mathbf{u}}_h^i + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i - \bar{p}_h^i \mathbb{I}) \cdot \mathbf{n}] \|_{L^2(e)^d} + \|\operatorname{div} \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)} \\
&+ \|(\nu_t(\bar{\mathbf{u}}_h^{i-1}) - \nu_{th}(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d \times d}}
\end{aligned} \tag{34}$$

and

$$(\eta_{i,\kappa}^F) = \|\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d \times d}}. \tag{35}$$

We now prove that the error between the solution of step i of the fixed-point algorithm used to calculate an approximation of the solution of the full non-linear finite element approximation of the LES system may be bounded by a combination of the estimators, with some other terms related to the data.

Theorem 5.3. *Let $\mathbf{U} = (\mathbf{u}, p)$ be a non singular solution of Problem (FV). Then, there exists a neighborhood Θ of \mathbf{U} in $X \times M$ such that any solution $\bar{\mathbf{U}}_h^i = (\bar{\mathbf{u}}_h^i, \bar{p}_h^i) \in X_h \times M_h$ in Θ of Problem (FVL $_{i,h}$) satisfies the following a posteriori error estimate:*

$$\begin{aligned}
& \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_X + \|p - \bar{p}_h^i\|_M \\
& \leq C \left(\sum_{\kappa \in \mathcal{T}_h} (h_{\kappa}^2 \|\bar{\mathbf{f}} - \bar{\mathbf{f}}_h\|_{L^2(\kappa)^d}^2 + \|\mathbf{f} - \bar{\mathbf{f}}\|_{L^2(\kappa)^d}^2) \right)^{1/2} + C' \left(\sum_{\kappa \in \mathcal{T}_h} ((\eta_{i,\kappa}^D)^2 + (\eta_{i,\kappa}^F)^2 + (\eta_{i,\kappa}^L)^2) \right)^{1/2},
\end{aligned}$$

where C is a constant depending on \mathbf{U} but independent of h .

Proof. Let $\mathbf{U} = (\mathbf{u}, p)$ be a non singular solution of Problem (FV) and $\bar{\mathbf{U}}_h^i = (\bar{\mathbf{u}}_h^i, \bar{p}_h^i) \in X_h \times M_h$ be the solution of the iterative problem (FVL $_{i,h}$). Having proved Proposition 2.7, and owing to [29] and [33, Prop. 2.2], there exists a neighborhood Θ of \mathbf{U} in $X \times M$ such that if $\bar{\mathbf{U}}_h^i$ is in Θ , we have the following bound:

$$\begin{aligned}
\|\mathbf{U} - \bar{\mathbf{U}}_h^i\|_{X \times M} &\leq c \|F(\mathbf{U}) - F(\bar{\mathbf{U}}_h^i)\|_{X \times M} \\
&\leq c \|\bar{\mathbf{U}}_h^i + \mathcal{S}(G(\bar{\mathbf{u}}_h^i), 0) - \mathbf{U} - \mathcal{S}(G(\mathbf{u}), 0)\|_{X \times M} \\
&\leq c \|\mathcal{S}(\mathcal{S}^{-1}(\bar{\mathbf{U}}_h^i) + (G(\bar{\mathbf{u}}_h^i), 0) - \mathcal{S}^{-1}(\mathbf{U}) - (G(\mathbf{u}), 0))\|_{X \times M}.
\end{aligned}$$

Using (3), we have:

$$\|\mathbf{U} - \bar{\mathbf{U}}_h^i\|_{X \times M} \leq c' \|\mathcal{S}^{-1}(\bar{\mathbf{U}}_h^i) + (G(\bar{\mathbf{u}}_h^i), 0) - \mathcal{S}^{-1}(\mathbf{U}) - (G(\mathbf{u}), 0)\|_{X' \times M'}$$

Relations (32) and (31) allow us to get: for all $\mathbf{v}_h \in X_h$,

$$\begin{aligned} \|\mathbf{U} - \bar{\mathbf{U}}_h^i\|_{X \times M} \leq & C_1 \left(\sup_{\substack{\mathbf{v} \in X \\ \mathbf{v} \neq 0}} \frac{\langle \mathbf{f} - \bar{\mathbf{f}}, \mathbf{v} \rangle + \langle \bar{\mathbf{f}} - \bar{\mathbf{f}}_h, \mathbf{v} - \mathbf{v}_h \rangle + \langle \mathcal{R}_h^1, \mathbf{v} - \mathbf{v}_h \rangle + \langle \mathcal{R}_h^2, \mathbf{v} \rangle}{\|\mathbf{v}\|_X} \right. \\ & + \sup_{\substack{\mathbf{v} \in X \\ \mathbf{v} \neq 0}} \frac{\int_{\Omega} ((\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i) \cdot \nabla \bar{\mathbf{u}}_h^i) \cdot \mathbf{v} d\mathbf{x}}{\|\mathbf{v}\|_X} + \sup_{\substack{q \in M \\ q \neq 0}} \frac{\int_{\Omega} q \operatorname{div} \bar{\mathbf{u}}_h^i d\mathbf{x}}{\|q\|_{L^2(\Omega)}} \\ & \left. + \sup_{\substack{\mathbf{v} \in X \\ \mathbf{v} \neq 0}} \frac{\frac{1}{2} \int_{\Omega} (\operatorname{div}(\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i)) \bar{\mathbf{u}}_h^i \cdot \mathbf{v} d\mathbf{x}}{\|\mathbf{v}\|_X} \right), \end{aligned} \quad (36)$$

where

$$\begin{aligned} \langle \mathcal{R}_h^1, \mathbf{v} - \mathbf{v}_h \rangle = & \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_{\kappa} (\bar{\mathbf{f}}_h + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^i)) \nabla \bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2} (\operatorname{div} \bar{\mathbf{u}}_h^{i-1}) \bar{\mathbf{u}}_h^i - \nabla \bar{p}_h^i) \cdot (\mathbf{v} - \mathbf{v}_h) d\mathbf{x} \right. \\ & - \frac{1}{2} \sum_{e \in \mathcal{E}_\kappa} \int_e [(\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i - \bar{p}_h^i \mathbb{I}](\sigma) \cdot \mathbf{n}] \cdot (\mathbf{v} - \mathbf{v}_h) d\sigma \\ & \left. - \int_{\kappa} (\nu_t(\bar{\mathbf{u}}_h^{i-1}) - \nu_{th}(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i : \nabla (\mathbf{v} - \mathbf{v}_h) d\mathbf{x} \right\}, \end{aligned}$$

and

$$\langle \mathcal{R}_h^2, \mathbf{v} \rangle = \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} \nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i : \nabla \mathbf{v} d\mathbf{x}.$$

The second term in the right-hand side of (36) can be bounded as:

$$\begin{aligned} \int_{\Omega} ((\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i) \cdot \nabla \bar{\mathbf{u}}_h^i) \cdot \mathbf{v} d\mathbf{x} & \leq \|\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i\|_{L^4(\Omega)^d} \|\bar{\mathbf{u}}_h^i\|_X \|\mathbf{v}\|_{L^4(\Omega)^d} \\ & \leq S_4^2 \|\bar{\mathbf{u}}_h^i\|_X \left(\sum_{\kappa \in \mathcal{T}_h} |\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i|_{H^1(\kappa)^d}^2 \right)^{1/2} \|\mathbf{v}\|_X. \end{aligned}$$

By using (11), we obtain

$$\sup_{\substack{\mathbf{v} \in X \\ \mathbf{v} \neq 0}} \frac{\int_{\Omega} ((\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i) \cdot \nabla \bar{\mathbf{u}}_h^i) \cdot \mathbf{v} d\mathbf{x}}{\|\mathbf{v}\|_X} \leq c_1 \left(\sum_{\kappa \in \mathcal{T}_h} (\eta_{i,\kappa}^L)^2 \right)^{1/2}.$$

In the same way, the last term in the right-hand side of (36) can be bounded as follows:

$$\begin{aligned} \frac{1}{2} \int_{\Omega} (\operatorname{div}(\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i)) \bar{\mathbf{u}}_h^i \cdot \mathbf{v} d\mathbf{x} & \leq \frac{1}{2} \|\operatorname{div}(\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i)\|_{L^2(\Omega)} \|\bar{\mathbf{u}}_h^i\|_{L^4(\Omega)^d} \|\mathbf{v}\|_{L^4(\Omega)^d} \\ & \leq \frac{\sqrt{d} S_4^2}{2} \|\bar{\mathbf{u}}_h^i\|_X \left(\sum_{\kappa \in \mathcal{T}_h} |\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i|_{H^1(\kappa)^d}^2 \right)^{1/2} \|\mathbf{v}\|_X. \end{aligned}$$

Hence, we obtain

$$\sup_{\substack{\mathbf{v} \in X \\ \mathbf{v} \neq 0}} \frac{\frac{1}{2} \int_{\Omega} (\operatorname{div}(\bar{\mathbf{u}}_h^{i-1} - \bar{\mathbf{u}}_h^i)) \bar{\mathbf{u}}_h^i \cdot \mathbf{v} d\mathbf{x}}{\|\mathbf{v}\|_X} \leq c_2 \left(\sum_{\kappa \in \mathcal{T}_h} (\eta_{i,\kappa}^L)^2 \right)^{1/2}.$$

We choose $\mathbf{v}_h = C_h \mathbf{v}$ and we use (28), to obtain the desired result. \square

Now, we address the efficiency of the previous indicators. We begin by estimating $(\eta_{i,\kappa}^F)$, the filter indicator; this will also be useful to estimate the discretization indicator $(\eta_{i,\kappa}^D)$.

Theorem 5.4. *We assume that the solution \mathbf{u} of Problem (FV) is such that $\nabla \mathbf{u} \in L^\infty(\Omega)^{d \times d}$. For each $\kappa \in \mathcal{T}_h$, the following estimates hold for the indicator $(\eta_{i,\kappa}^F)^2$ defined in (35):*

$$(\eta_{i,\kappa}^F)^2 \leq C \left(h_\kappa^{4-d} |\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}|_{H^1(\kappa)^d}^2 + h_\kappa^4 |\bar{\mathbf{u}}_h^i - \mathbf{u}|_{H^1(\kappa)^d}^2 + \|\nu_t(\mathbf{u}) \nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}}^2 \right), \quad (37)$$

where C is a positive constant independent of h .

Proof. We add and subtract $\nu_t(\mathbf{u}) \nabla \mathbf{u}$ in $(\eta_{i,\kappa}^F)^2$ and we use the inequality $(a+b)^2 \leq 2(a^2 + b^2)$. We obtain:

$$\|\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d \times d}}^2 \leq 2\|\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i - \nu_t(\mathbf{u}) \nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}}^2 + 2\|\nu_t(\mathbf{u}) \nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}}^2. \quad (38)$$

Adding and subtracting $\nu_t(\mathbf{u}) \nabla \bar{\mathbf{u}}_h^i$ in the first term in the right hand side of (38) gives:

$$\begin{aligned} \|\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d \times d}}^2 &\leq \\ &4\|(\nu_t(\bar{\mathbf{u}}_h^{i-1}) - \nu_t(\mathbf{u})) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d \times d}}^2 + 4\|\nu_t(\mathbf{u}) \nabla (\bar{\mathbf{u}}_h^i - \mathbf{u})\|_{L^2(\kappa)^{d \times d}}^2 + 2\|\nu_t(\mathbf{u}) \nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}}^2. \end{aligned}$$

We use the inverse inequality to get:

$$\begin{aligned} \|(\nu_t(\bar{\mathbf{u}}_h^{i-1}) - \nu_t(\mathbf{u})) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d \times d}}^2 &\leq \int_\kappa \left(c_s h_\kappa^2 \|\nabla (\bar{\mathbf{u}}_h^{i-1} - \mathbf{u})\|_F \|\nabla \bar{\mathbf{u}}_h^i\| \right)^2 dx \\ &\leq c_s^2 h_\kappa^4 \|\nabla \bar{\mathbf{u}}_h^i\|_{L^\infty(\kappa)^{d \times d}}^2 \int_\kappa \|\nabla (\bar{\mathbf{u}}_h^{i-1} - \mathbf{u})\|_F^2 dx \\ &\leq c_s^2 \tilde{c} h_\kappa^{4-d} \|\nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^d}^2 |\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}|_{H^1(\kappa)^d}^2. \end{aligned}$$

Moreover, using the fact that the solution \mathbf{u} of Problem (FV) is such that $\nabla \mathbf{u} \in L^\infty(\Omega)$, gives

$$\begin{aligned} \|\nu_t(\mathbf{u}) \nabla (\bar{\mathbf{u}}_h^i - \mathbf{u})\|_{L^2(\kappa)^{d \times d}}^2 &\leq \int_\kappa \left(c h_\kappa^2 \|\nabla \mathbf{u}\|_F \|\nabla (\bar{\mathbf{u}}_h^i - \mathbf{u})\| \right)^2 dx \\ &\leq c_s^2 h_\kappa^4 \|\nabla \mathbf{u}\|_{L^\infty(\kappa)^{d \times d}}^2 \int_\kappa \|\nabla (\bar{\mathbf{u}}_h^i - \mathbf{u})\|^2 dx \\ &\leq c_s^2 h_\kappa^4 \|\nabla \mathbf{u}\|_{L^\infty(\kappa)^{d \times d}}^2 |\bar{\mathbf{u}}_h^i - \mathbf{u}|_{H^1(\kappa)^d}^2. \end{aligned}$$

By using (11) and the fact that $\nabla(\mathbf{u}) \in L^\infty(\Omega)^{d \times d}$, we obtain (37). \square

Remark 5.5. *The term $\|\nu_t(\mathbf{u}) \nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}}^2$ which appears in (37), can be bounded as follows:*

$$\begin{aligned} \|\nu_t(\mathbf{u}) \nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}}^2 &\leq 2\|\nu_t(\mathbf{u}) \nabla (\mathbf{u} - \bar{\mathbf{u}})\|_{L^2(\kappa)^{d \times d}}^2 + 2\|\nu_t(\mathbf{u}) \nabla \bar{\mathbf{u}}\|_{L^2(\kappa)^{d \times d}}^2 \\ &\leq 2\|\nu_t(\mathbf{u}) \nabla (\mathbf{u} - \bar{\mathbf{u}})\|_{L^2(\kappa)^{d \times d}}^2 + 4\|\nu_t(\mathbf{u} - \bar{\mathbf{u}}) \nabla \bar{\mathbf{u}}\|_{L^2(\kappa)^{d \times d}}^2 + 4\|\nu_t(\bar{\mathbf{u}}) \nabla \bar{\mathbf{u}}\|_{L^2(\kappa)^{d \times d}}^2 \\ &\leq 2c_S^2 h_\kappa^4 \|\nabla \mathbf{u}\|_{L^\infty(\kappa)^{d \times d}}^2 |\mathbf{u} - \bar{\mathbf{u}}|_{H^1(\kappa)^d}^2 + 4c_S^2 h_\kappa^4 \|\nabla \bar{\mathbf{u}}\|_{L^\infty(\kappa)^{d \times d}}^2 |\mathbf{u} - \bar{\mathbf{u}}|_{H^1(\kappa)^d}^2 \\ &\quad + 4\|\nu_t(\bar{\mathbf{u}}) \nabla \bar{\mathbf{u}}\|_{L^2(\kappa)^{d \times d}}^2. \end{aligned}$$

Supposing $\nabla \mathbf{u}$ and $\nabla \bar{\mathbf{u}}$ in $L^\infty(\Omega)^{d \times d}$, the term $\|\nu_t(\mathbf{u}) \nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}}^2$ can be bounded by the error between \mathbf{u} and $\bar{\mathbf{u}}$ and by the term $\|\nu_t(\bar{\mathbf{u}}) \nabla \bar{\mathbf{u}}\|_{L^2(\kappa)^{d \times d}}^2$ which comes from the filtering process.

Theorem 5.6. *We assume that the solution \mathbf{u} of Problem (FV) is such that $\nabla \mathbf{u} \in L^\infty(\Omega)^{d \times d}$. For each $\kappa \in \mathcal{T}_h$, we have the following estimations:*

$$(\eta_{i,\kappa}^L)^2 \leq 2|\mathbf{u} - \bar{\mathbf{u}}_h^i|_{H^1(\kappa)^d}^2 + 2|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}|_{H^1(\kappa)^d}^2, \quad (39)$$

$$(\eta_{i,\kappa}^D)^2 \leq CL(w_\kappa), \quad (40)$$

where w_κ denotes the set of elements of \mathcal{T}_h that share at least one edge (or face when $d = 3$) with κ and where, for any set W , we define

$$\begin{aligned} L(W) &= \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_{H^1(W)^d}^2 + \sum_{\kappa_1 \in W} h_{\kappa_1}^2 \|\mathbf{f} - \bar{\mathbf{f}}_h\|_{L^2(\kappa_1)^d}^2 + \|p - \bar{p}_h^i\|_{L^2(W)}^2 + \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{H^1(W)^d}^2 \\ &\quad + \|(\nu_{th}(\mathbf{u}) - \nu_t(\mathbf{u})) \nabla \mathbf{u}\|_{L^2(W)^{d \times d}}^2 + \|\nu_t(\mathbf{u}) \nabla \mathbf{u}\|_{L^2(W)^{d \times d}}^2. \end{aligned} \quad (41)$$

Proof. The estimation of the linearization indicator follows easily from the triangle inequality by introducing \mathbf{u} in $(\eta_{i,\kappa}^L)^2$.

Now, we aim to bound $(\eta_{i,\kappa}^D)^2$. We proceed in 4 steps:

(1) We add and subtract $(\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) - \nu_t(\bar{\mathbf{u}}_h^{i-1}))\nabla\mathbf{u}$ in the last term of $\eta_{i,\kappa}^D$:

$$\begin{aligned} \|(\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) - \nu_t(\bar{\mathbf{u}}_h^{i-1}))\nabla\bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d\times d}} &\leq \|(\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) - \nu_t(\bar{\mathbf{u}}_h^{i-1}))\nabla(\bar{\mathbf{u}}_h^i - \mathbf{u})\|_{L^2(\kappa)^{d\times d}} \\ &\quad + \|(\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) - \nu_t(\bar{\mathbf{u}}_h^{i-1}))\nabla\mathbf{u}\|_{L^2(\kappa)^{d\times d}}. \end{aligned} \quad (42)$$

Adding and subtracting $\nu_t(\mathbf{u})\nabla\mathbf{u}$ and $\nu_{th}(\mathbf{u})\nabla\mathbf{u}$ in the second term in the right-hand side of (42), gives:

$$\begin{aligned} \|(\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) - \nu_t(\bar{\mathbf{u}}_h^{i-1}))\nabla\bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d\times d}} &\leq \|(\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) - \nu_t(\bar{\mathbf{u}}_h^{i-1}))\nabla(\bar{\mathbf{u}}_h^i - \mathbf{u})\|_{L^2(\kappa)^{d\times d}} \\ &\quad + \|(\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) - \nu_{th}(\mathbf{u}))\nabla\mathbf{u}\|_{L^2(\kappa)^{d\times d}} \\ &\quad + \|(\nu_t(\mathbf{u}) - \nu_t(\bar{\mathbf{u}}_h^{i-1}))\nabla\mathbf{u}\|_{L^2(\kappa)^{d\times d}} \\ &\quad + \|(\nu_{th}(\mathbf{u}) - \nu_t(\mathbf{u}))\nabla\mathbf{u}\|_{L^2(\kappa)^{d\times d}}. \end{aligned} \quad (43)$$

We bound the first term in the right-hand side of (43) as follows:

$$\begin{aligned} &\|(\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) - \nu_t(\bar{\mathbf{u}}_h^{i-1}))\nabla(\bar{\mathbf{u}}_h^i - \mathbf{u})\|_{L^2(\kappa)^{d\times d}} \\ &= \left\{ \int_{\kappa} \left[\left(\frac{1}{|\kappa|} \int_{\kappa} c_s h_{\kappa}^2 \|\nabla\bar{\mathbf{u}}_h^{i-1}\|_F d\mathbf{x} - c_s h_{\kappa}^2 \|\nabla\bar{\mathbf{u}}_h^{i-1}\|_F \right) \nabla(\bar{\mathbf{u}}_h^i - \mathbf{u}) \right]^2 d\mathbf{x} \right\}^{1/2} \\ &\leq 2c_s h_{\kappa}^2 \|\nabla(\bar{\mathbf{u}}_h^{i-1})\|_{L^{\infty}(\kappa)^{d\times d}} |\bar{\mathbf{u}}_h^i - \mathbf{u}|_{H^1(\kappa)^d}. \end{aligned}$$

By using the inverse inequality, we have:

$$\|(\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) - \nu_t(\bar{\mathbf{u}}_h^{i-1}))\nabla(\bar{\mathbf{u}}_h^i - \mathbf{u})\|_{L^2(\kappa)^{d\times d}} \leq 2c_s \tilde{c} h_{\kappa}^{2-d/2} |\bar{\mathbf{u}}_h^{i-1}|_{H^1(\kappa)^d} |\bar{\mathbf{u}}_h^i - \mathbf{u}|_{H^1(\kappa)^d}.$$

Using the fact that $\|\nabla\mathbf{u}\|_{L^{\infty}(\Omega)^{d\times d}}$ is bounded, we treat the second term in the right-hand side of (43) as:

$$\|(\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) - \nu_{th}(\mathbf{u}))\nabla\mathbf{u}\|_{L^2(\kappa)^{d\times d}} \leq c' h_{\kappa}^2 \|\nabla\mathbf{u}\|_{L^{\infty}(\Omega)^{d\times d}} |\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}|_{H^1(\kappa)^d}.$$

Similarly,

$$\|(\nu_t(\mathbf{u}) - \nu_t(\bar{\mathbf{u}}_h^{i-1}))\nabla\mathbf{u}\|_{L^2(\kappa)^{d\times d}} \leq c' h_{\kappa}^2 \|\nabla\mathbf{u}\|_{L^{\infty}(\Omega)^{d\times d}} |\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}|_{H^1(\kappa)^d}.$$

By regrouping all the previous inequalities, using (11) and the fact that $\|\nabla\mathbf{u}\|_{L^{\infty}(\Omega)^{d\times d}}$ is bounded, we obtain:

$$\begin{aligned} \|(\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) - \nu_t(\bar{\mathbf{u}}_h^{i-1}))\nabla\bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d\times d}} &\leq C_2 \left(h_{\kappa}^{2-d/2} |\bar{\mathbf{u}}_h^i - \mathbf{u}|_{H^1(\kappa)^d} + h_{\kappa}^2 |\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}|_{H^1(\kappa)^d} \right. \\ &\quad \left. + \|(\nu_{th}(\mathbf{u}) - \nu_t(\mathbf{u}))\nabla\mathbf{u}\|_{L^2(\kappa)^{d\times d}} \right). \end{aligned} \quad (44)$$

(2) We take $\mathbf{v}_h = \mathbf{0}$, add and subtract $\int_{\Omega} (\mathbf{u} \cdot \nabla\bar{\mathbf{u}}_h^i) \cdot \mathbf{v} d\mathbf{x}$ and add the vanishing term $\frac{1}{2} \int_{\Omega} (\operatorname{div} \mathbf{u}) \bar{\mathbf{u}}_h^i \cdot \mathbf{v} d\mathbf{x}$ in (32), we obtain:

$$\begin{aligned} &\sum_{\kappa_1 \in \mathcal{T}_h} \int_{\kappa_1} \left(\bar{\mathbf{f}}_h + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla\bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla\bar{\mathbf{u}}_h^i - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1})\bar{\mathbf{u}}_h^i - \nabla\bar{p}_h^i \right) \cdot \mathbf{v} d\mathbf{x} \\ &= \nu \int_{\Omega} \nabla(\mathbf{u} - \bar{\mathbf{u}}_h^i) : \nabla\mathbf{v} d\mathbf{x} + \int_{\Omega} (\mathbf{u} \cdot \nabla(\mathbf{u} - \bar{\mathbf{u}}_h^i)) \cdot \mathbf{v} d\mathbf{x} - \int_{\Omega} (p - \bar{p}_h^i) \operatorname{div} \mathbf{v} d\mathbf{x} \\ &\quad + \int_{\Omega} ((\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}) \cdot \nabla\bar{\mathbf{u}}_h^i) \cdot \mathbf{v} d\mathbf{x} + \frac{1}{2} \int_{\Omega} (\operatorname{div}(\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}))\bar{\mathbf{u}}_h^i \cdot \mathbf{v} d\mathbf{x} - \sum_{\kappa_1 \in \mathcal{T}_h} \int_{\kappa_1} (\mathbf{f} - \bar{\mathbf{f}}_h) \cdot \mathbf{v} d\mathbf{x} \\ &\quad + \frac{1}{2} \sum_{\kappa_1 \in \mathcal{T}_h} \sum_{e \in \varepsilon_{\kappa_1}} \int_e [(\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla\bar{\mathbf{u}}_h^i - \bar{p}_h^i \mathbb{I}(\sigma) \cdot \mathbf{n}] \cdot \mathbf{v} d\sigma - \int_{\Omega} \nu_{th}(\bar{\mathbf{u}}_h^{i-1})\nabla\bar{\mathbf{u}}_h^i : \nabla\mathbf{v} d\mathbf{x}. \end{aligned} \quad (45)$$

We add and subtract $\int_{\Omega} ((\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}) \cdot \nabla \mathbf{u}) \cdot \mathbf{v} d\mathbf{x}$ in the fourth term in the right-hand side of (45)

$$\begin{aligned}
& \sum_{\kappa_1 \in \mathcal{T}_h} \int_{\kappa_1} \left(\bar{\mathbf{f}}_h + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1}) \bar{\mathbf{u}}_h^i - \nabla \bar{p}_h^i \right) \cdot \mathbf{v} d\mathbf{x} \\
&= \nu \int_{\Omega} \nabla(\mathbf{u} - \bar{\mathbf{u}}_h^i) : \nabla \mathbf{v} d\mathbf{x} + \int_{\Omega} (\mathbf{u} \cdot \nabla(\mathbf{u} - \bar{\mathbf{u}}_h^i)) \cdot \mathbf{v} d\mathbf{x} - \int_{\Omega} (p - \bar{p}_h^i) \operatorname{div} \mathbf{v} d\mathbf{x} \\
&+ \int_{\Omega} ((\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}) \cdot \nabla(\bar{\mathbf{u}}_h^i - \mathbf{u})) \cdot \mathbf{v} d\mathbf{x} + \int_{\Omega} ((\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}) \cdot \nabla \mathbf{u}) \cdot \mathbf{v} d\mathbf{x} + \frac{1}{2} \int_{\Omega} (\operatorname{div}(\mathbf{u} - \bar{\mathbf{u}}_h^{i-1})) \bar{\mathbf{u}}_h^i \cdot \mathbf{v} d\mathbf{x} \\
&- \sum_{\kappa_1 \in \mathcal{T}_h} \int_{\kappa_1} (\mathbf{f} - \bar{\mathbf{f}}_h) \cdot \mathbf{v} d\mathbf{x} + \frac{1}{2} \sum_{\kappa_1 \in \mathcal{T}_h} \sum_{e \in \varepsilon_{\kappa_1}} \int_e [(\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i - \bar{p}_h^i \mathbb{I}](\sigma) \cdot \mathbf{n}] \cdot \mathbf{v} d\sigma \\
&- \int_{\Omega} \nu_{th}(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i : \nabla \mathbf{v} d\mathbf{x}.
\end{aligned} \tag{46}$$

For a given $\kappa \in \mathcal{T}_h$, we choose \mathbf{v} as follows:

$$\mathbf{v} = \mathbf{v}_{\kappa} = \begin{cases} (\bar{\mathbf{f}}_h - \nabla \bar{p}_h^i + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1}) \bar{\mathbf{u}}_h^i) \psi_{\kappa} & \text{on } \kappa, \\ 0 & \text{on } \Omega \setminus \kappa, \end{cases}$$

where ψ_{κ} is the bubble function of the element κ . Since this function vanishes outside κ as well as on its boundaries, all volume integrals in (46) reduce to integrals on κ , and the edge (or face) integrals vanish.

By using the Cauchy-Schwarz inequality and the fact that $\nabla \mathbf{u} \in L^{\infty}(\Omega)^{d \times d}$, we obtain:

$$\begin{aligned}
& \|(\bar{\mathbf{f}}_h - \nabla \bar{p}_h^i + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1}) \bar{\mathbf{u}}_h^i) \psi_{\kappa}^{1/2}\|_{L^2(\kappa)^d}^2 \\
&\leq C_3 \left(\nu \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_{H^1(\kappa)^d} \|\mathbf{v}_{\kappa}\|_{H^1(\kappa)^d} + \|\mathbf{u}\|_{L^4(\kappa)^d} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_{H^1(\kappa)^d} \|\mathbf{v}_{\kappa}\|_{L^4(\kappa)^d} + \|p - \bar{p}_h^i\|_{L^2(\kappa)} \|\mathbf{v}_{\kappa}\|_{H^1(\kappa)^d} \right. \\
&\quad + \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{L^4(\kappa)^d} \|\bar{\mathbf{u}}_h^i - \mathbf{u}\|_{H^1(\kappa)^d} \|\mathbf{v}_{\kappa}\|_{L^4(\kappa)^d} + \|\nabla \mathbf{u}\|_{L^{\infty}(\Omega)^{d \times d}} \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{L^2(\kappa)^d} \|\mathbf{v}_{\kappa}\|_{L^2(\kappa)^d} \\
&\quad + \frac{1}{2} \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{H^1(\kappa)^d} \|\bar{\mathbf{u}}_h^i\|_{L^4(\kappa)^d} \|\mathbf{v}_{\kappa}\|_{L^4(\kappa)^d} + \|\mathbf{f} - \bar{\mathbf{f}}_h\|_{L^2(\kappa)^d} \|\mathbf{v}_{\kappa}\|_{L^2(\kappa)^d} \\
&\quad \left. + \|\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d \times d}} \|\mathbf{v}_{\kappa}\|_{H^1(\kappa)^d} \right).
\end{aligned} \tag{47}$$

We add and subtract $\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i$ in the last term in the right-hand side of (47) to obtain:

$$\|\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d \times d}} \leq \|\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i - \nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d \times d}} + \|\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d \times d}}.$$

We use (44) for the first term in the right-hand side of the above inequality, as well as definition (35) and estimation (37) for the second term, and finally the fact that h_{κ} is bounded by the diameter of Ω ; we obtain:

$$\begin{aligned}
\|\nu_{th}(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^{d \times d}} &\leq C_4 \left(\|\bar{\mathbf{u}}_h^i - \mathbf{u}\|_{H^1(\kappa)^d} + \|\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}\|_{H^1(\kappa)^d} \right. \\
&\quad \left. + \|(\nu_{th}(\mathbf{u}) - \nu_t(\mathbf{u})) \nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}} + \|\nu_t(\mathbf{u}) \nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}} \right).
\end{aligned} \tag{48}$$

Inequality (47) can then be bounded as follows:

$$\begin{aligned}
& \|(\bar{\mathbf{f}}_h - \nabla \bar{p}_h^i + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla \bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1})\bar{\mathbf{u}}_h^i)\psi_\kappa^{1/2}\|_{L^2(\kappa)^d}^2 \\
& \leq C_5 \left(\nu \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_{H^1(\kappa)^d} \|\mathbf{v}_\kappa\|_{H^1(\kappa)^d} + \|\mathbf{u}\|_{L^4(\kappa)^d} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_{H^1(\kappa)^d} \|\mathbf{v}_\kappa\|_{L^4(\kappa)^d} \right. \\
& \quad + \|p - \bar{p}_h^i\|_{L^2(\kappa)} \|\mathbf{v}_\kappa\|_{H^1(\kappa)^d} + \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{L^4(\kappa)^d} \|\bar{\mathbf{u}}_h^i - \mathbf{u}\|_{H^1(\kappa)^d} \|\mathbf{v}_\kappa\|_{L^4(\kappa)} + \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{L^2(\kappa)^d} \|\mathbf{v}_\kappa\|_{L^2(\kappa)^d} \\
& \quad + \frac{1}{2} \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{H^1(\kappa)^d} \|\bar{\mathbf{u}}_h^i\|_{L^4(\kappa)^d} \|\mathbf{v}_\kappa\|_{L^4(\kappa)^d} + \|\mathbf{f} - \bar{\mathbf{f}}_h\|_{L^2(\kappa)^d} \|\mathbf{v}_\kappa\|_{L^2(\kappa)^d} \\
& \quad + C_4 \left(\|\bar{\mathbf{u}}_h^i - \mathbf{u}\|_{H^1(\kappa)^d} + \|\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}\|_{H^1(\kappa)^d} + \|(\nu_{th}(\mathbf{u}) - \nu_t(\mathbf{u}))\nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}} \right. \\
& \quad \left. + \|\nu_t(\mathbf{u})\nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}} \right) \|\mathbf{v}_\kappa\|_{H^1(\kappa)^d} \Big). \tag{49}
\end{aligned}$$

We multiply this inequality by h_κ^2 ; we then use the second inequality in (27) to bound $h_\kappa \|\mathbf{v}_\kappa\|_{H^1(\kappa)^d}$. Moreover, we use the fact that \mathbf{v}_κ is in $H_0^1(\Omega)$ and that its support is κ to show that $\|\mathbf{v}_\kappa\|_{L^4(\kappa)^d} = \|\mathbf{v}_\kappa\|_{L^4(\Omega)^d} \leq S_4 \|\mathbf{v}_\kappa\|_{H^1(\Omega)^d} = S_4 \|\mathbf{v}_\kappa\|_{H^1(\kappa)^d}$. We then use (1), (2) and Theorem 4.7 to obtain:

$$\begin{aligned}
& h_\kappa^2 \|(\bar{\mathbf{f}}_h - \nabla \bar{p}_h^i + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla \bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1})\bar{\mathbf{u}}_h^i)\psi_\kappa^{1/2}\|_{L^2(\kappa)^d}^2 \\
& \leq C_6 \left(\|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_{H^1(\kappa)^d} h_\kappa \|\bar{\mathbf{v}}_\kappa\|_{L^2(\kappa)^d} + h_\kappa \|\mathbf{f} - \bar{\mathbf{f}}_h\|_{L^2(\kappa)^d} h_\kappa \|\mathbf{v}_\kappa\|_{L^2(\kappa)^d} \right. \\
& \quad + \|p - \bar{p}_h^i\|_{L^2(\kappa)} h_\kappa \|\mathbf{v}_\kappa\|_{L^2(\kappa)^d} + \|\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}\|_{H^1(\kappa)^d} h_\kappa \|\mathbf{v}_\kappa\|_{L^2(\kappa)^d} \\
& \quad \left. + \|(\nu_{th}(\mathbf{u}) - \nu_t(\mathbf{u}))\nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}} h_\kappa \|\mathbf{v}_\kappa\|_{L^2(\kappa)^d} + \|\nu_t(\mathbf{u})\nabla \mathbf{u}\|_{L^2(\kappa)^{d \times d}} h_\kappa \|\mathbf{v}_\kappa\|_{L^2(\kappa)^d} \right).
\end{aligned}$$

We replace \mathbf{v}_κ by its value, simplify by $h_\kappa \|\mathbf{v}_\kappa\|_{L^2(\kappa)^d}$, square the resulting inequality and use Property 5.1 to get:

$$h_\kappa^2 \|(\bar{\mathbf{f}}_h - \nabla \bar{p}_h^i + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla \bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1})\bar{\mathbf{u}}_h^i)\|_{L^2(\kappa)^d}^2 \leq CL(\kappa), \tag{50}$$

where L is given by (41).

(3) Now we estimate the second term in $(\eta_{i,\kappa}^D)^2$. Rewriting (46), we infer:

$$\begin{aligned}
& \frac{1}{2} \sum_{\kappa_1 \in \mathcal{T}_h} \sum_{e \in \varepsilon_{\kappa_1}} \int_e [(\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla \bar{\mathbf{u}}_h^i - \bar{p}_h^i \mathbb{I}](\sigma) \mathbf{n} \cdot \mathbf{v} \, d\sigma \\
& = \sum_{\kappa_1 \in \mathcal{T}_h} \int_{\kappa_1} (\bar{\mathbf{f}}_h + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla \bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2} \operatorname{div}(\bar{\mathbf{u}}_h^{i-1})\bar{\mathbf{u}}_h^i - \nabla \bar{p}_h^i) \cdot \mathbf{v} \, dx \\
& \quad + \nu \int_\Omega \nabla(\bar{\mathbf{u}}_h^i - \mathbf{u}) : \nabla \mathbf{v} \, dx + \int_\Omega (\mathbf{u} \cdot \nabla(\bar{\mathbf{u}}_h^i - \mathbf{u})) \cdot \mathbf{v} \, dx + \int_\Omega (p - \bar{p}_h^i) \operatorname{div} \mathbf{v} \, dx \\
& \quad + \int_\Omega ((\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}) \cdot \nabla(\bar{\mathbf{u}}_h^i - \mathbf{u})) \cdot \mathbf{v} \, dx + \int_\Omega ((\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}) \cdot \nabla \mathbf{u}) \cdot \mathbf{v} \, dx + \frac{1}{2} \int_\Omega (\operatorname{div}(\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}))\bar{\mathbf{u}}_h^i \cdot \mathbf{v} \, dx \\
& \quad + \sum_{\kappa_1 \in \mathcal{T}_h} \int_{\kappa_1} (\mathbf{f} - \bar{\mathbf{f}}_h) \cdot \mathbf{v} \, dx + \int_\Omega \nu_{th}(\bar{\mathbf{u}}_h^{i-1})\nabla \bar{\mathbf{u}}_h^i : \nabla \mathbf{v} \, dx. \tag{51}
\end{aligned}$$

For a given mesh edge (or face) $e \not\subset \partial\Omega$, we denote by $(\kappa, \kappa') \in (\mathcal{T}_h)^2$ the two elements that share e and set

$$\mathbf{v} = \mathbf{v}_e = \begin{cases} \mathcal{L}_e \left([(\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla \bar{\mathbf{u}}_h^i - \bar{p}_h^i \mathbb{I}](\sigma) \mathbf{n} \right) \psi_e & \text{on } \{\kappa, \kappa'\}, \\ 0 & \text{on } \Omega \setminus \{\kappa \cup \kappa'\}, \end{cases}$$

where ψ_e is the edge-bubble (or face-bubble) function and \mathcal{L}_e the lifting operator defined at the beginning of Section 5.

Using (48), replacing \mathbf{v}_e by its value, using the fact that $\nabla \mathbf{u} \in L^\infty(\Omega)$ and applying the Cauchy-Schwarz inequality, we get:

$$\begin{aligned} & \left\| [(\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla \bar{\mathbf{u}}_h^i - \bar{p}_h^i \mathbb{I}](\sigma) \mathbf{n} \psi_e^{1/2} \right\|_{L^2(e)^d}^2 \\ & \leq C_4 \left(\|\bar{\mathbf{f}}_h - \nabla \bar{p}_h^i + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla \bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1})\bar{\mathbf{u}}_h^i\|_{L^2(\kappa \cup \kappa')^d} \|\mathbf{v}_e\|_{L^2(\kappa \cup \kappa')^d} \right. \\ & \quad + \nu \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_{H^1(\kappa \cup \kappa')^d} \|\mathbf{v}_e\|_{H^1(\kappa \cup \kappa')^d} + \|\mathbf{u}\|_{L^4(\kappa \cup \kappa')^d} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_{H^1(\kappa \cup \kappa')^d} \|\mathbf{v}_e\|_{L^4(\kappa \cup \kappa')^d} \\ & \quad + \|p - \bar{p}_h^i\|_{L^2(\kappa \cup \kappa')} \|\mathbf{v}_e\|_{H^1(\kappa \cup \kappa')^d} + \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{L^4(\kappa \cup \kappa')^d} \|\bar{\mathbf{u}}_h^i - \mathbf{u}\|_{H^1(\kappa \cup \kappa')^d} \|\mathbf{v}_e\|_{L^4(\kappa \cup \kappa')^d} \\ & \quad + \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{L^2(\kappa \cup \kappa')^d} \|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \|\mathbf{v}_e\|_{L^2(\kappa \cup \kappa')^d} \\ & \quad + \frac{1}{2} \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{H^1(\kappa \cup \kappa')^d} \|\bar{\mathbf{u}}_h^i\|_{L^4(\kappa \cup \kappa')^d} \|\mathbf{v}_e\|_{L^4(\kappa \cup \kappa')^d} + \|\mathbf{f} - \bar{\mathbf{f}}_h\|_{L^2(\kappa \cup \kappa')^d} \|\mathbf{v}_e\|_{L^2(\kappa \cup \kappa')^d} \\ & \quad + C''' \left(\|\bar{\mathbf{u}}_h^i - \mathbf{u}\|_{H^1(\kappa \cup \kappa')^d} + \|\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}\|_{H^1(\kappa \cup \kappa')^d} + \|(\nu_{th}(\mathbf{u}) - \nu_t(\mathbf{u}))\nabla \mathbf{u}\|_{L^2(\kappa \cup \kappa')^{d \times d}} \right. \\ & \quad \left. + \|\nu_t(\mathbf{u})\nabla \mathbf{u}\|_{L^2(\kappa \cup \kappa')^{d \times d}} \right) \|\mathbf{v}_e\|_{H^1(\kappa \cup \kappa')^d} \Big). \end{aligned}$$

We multiply by h_e and we use (2) and Theorem 4.7 to obtain:

$$\begin{aligned} & h_e \left\| [(\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla \bar{\mathbf{u}}_h^i - \bar{p}_h^i \mathbb{I}](\sigma) \mathbf{n} \psi_e^{1/2} \right\|_{L^2(e)^d}^2 \\ & \leq C_5 \left(h_e \|\bar{\mathbf{f}}_h - \nabla \bar{p}_h^i + \operatorname{div}((\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla \bar{\mathbf{u}}_h^i) - \bar{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2}(\operatorname{div} \bar{\mathbf{u}}_h^{i-1})\bar{\mathbf{u}}_h^i\|_{L^2(\kappa \cup \kappa')^d} \|\mathbf{v}_e\|_{L^2(\kappa \cup \kappa')^d} \right. \\ & \quad + \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_{H^1(\kappa \cup \kappa')^d} h_e \|\mathbf{v}_e\|_{H^1(\kappa \cup \kappa')^d} + (\|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_{H^1(\kappa \cup \kappa')^d} + \|\mathbf{u} - \bar{\mathbf{u}}_h^{i-1}\|_{H^1(\kappa \cup \kappa')^d}) h_e \|\mathbf{v}_e\|_{L^4(\kappa \cup \kappa')^d} \\ & \quad + \|p - \bar{p}_h^i\|_{L^2(\kappa \cup \kappa')} h_e \|\mathbf{v}_e\|_{H^1(\kappa \cup \kappa')^d} + \|\bar{\mathbf{u}}_h^{i-1} - \mathbf{u}\|_{H^1(\kappa \cup \kappa')^d} h_e \|\mathbf{v}_e\|_{L^2(\kappa \cup \kappa')^d} \\ & \quad + \|\mathbf{f} - \bar{\mathbf{f}}_h\|_{L^2(\kappa \cup \kappa')^d} h_e \|\mathbf{v}_e\|_{L^2(\kappa \cup \kappa')^d} + \|(\nu_{th}(\mathbf{u}) - \nu_t(\mathbf{u}))\nabla \mathbf{u}\|_{L^2(\kappa \cup \kappa')^{d \times d}} h_e \|\mathbf{v}_e\|_{H^1(\kappa \cup \kappa')^d} \\ & \quad \left. + \|\nu_t(\mathbf{u})\nabla \mathbf{u}\|_{L^2(\kappa \cup \kappa')^{d \times d}} h_e \|\mathbf{v}_e\|_{H^1(\kappa \cup \kappa')^d} \right). \end{aligned}$$

As was done above for \mathbf{v}_κ , we have the following bound: $\|\mathbf{v}_e\|_{L^4(\kappa \cup \kappa')^d} \leq S_4 \|\mathbf{v}_e\|_{H^1(\kappa \cup \kappa')^d}$. Next, we use both items of Property 5.2, inequality (50), simplify by $h_e^{1/2} \|\mathbf{v}_e\|_{L^2(\kappa \cup \kappa')^d}$, square the resulting inequality and, for a given κ , we sum over $e \in \partial\kappa$; we obtain:

$$\sum_{e \in \varepsilon_\kappa} h_e \left\| [(\nu + \nu_{th}(\bar{\mathbf{u}}_h^{i-1}))\nabla \bar{\mathbf{u}}_h^i - \bar{p}_h^i \mathbb{I}](\sigma) \mathbf{n} \right\|_{L^2(e)^d}^2 \leq L(w_\kappa), \quad (52)$$

where w_κ is defined in Theorem 5.6.

(4) Finally, as $\operatorname{div} \mathbf{u} = 0$, we have:

$$\int_\Omega q(\mathbf{x}) \operatorname{div}(\mathbf{u}(\mathbf{x}) - \bar{\mathbf{u}}_h^i(\mathbf{x})) d\mathbf{x} = - \int_\Omega q(\mathbf{x}) \operatorname{div} \bar{\mathbf{u}}_h^i(\mathbf{x}) d\mathbf{x}. \quad (53)$$

We choose for a given $\kappa \in \mathcal{T}_h$

$$q = q_\kappa = \operatorname{div} \bar{\mathbf{u}}_h^i \xi_\kappa.$$

where ξ_κ denotes the characteristic function of κ . We have

$$\int_\kappa \operatorname{div} \bar{\mathbf{u}}_h^i \operatorname{div}(\mathbf{u} - \bar{\mathbf{u}}_h^i) d\mathbf{x} = - \int_\kappa (\operatorname{div} \bar{\mathbf{u}}_h^i)^2 d\mathbf{x}.$$

Applying the Cauchy-Schwarz inequality to the left-hand side, we obtain

$$\|\operatorname{div} \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)} \leq \sqrt{d} \|\mathbf{u} - \bar{\mathbf{u}}_h^i\|_{H^1(\kappa)}. \quad (54)$$

Collecting (44), (50), (52) and (54), we get the final result. \square

6. NUMERICAL RESULTS

In this section, we show numerical simulations using the FreeFem++ code (see [20]). We consider the domain Ω given by Figure 1 with the following properties: $AB = 1.5$, $BC = DE = 1$, $CD = 0.25$, $EF = 2.45$, $FG = AL = 1.5$, $GH = LK = 0.4$, $HI = KJ = 1.1$, $LG = KH = JI = 4.2$.

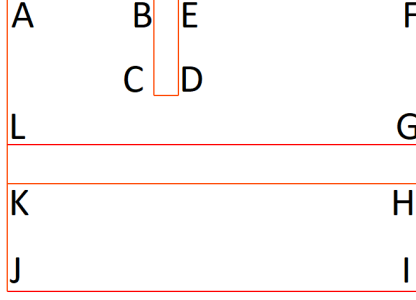


FIGURE 1. The domain

Concerning uniform meshes, we divide the edges of $\partial\Omega$ into segments of equal lengths, and we define N to be the number of such mesh segments per unit length.

We consider $\nu = \frac{1}{Re}$ where Re is the Reynolds number and we choose the density of body forces \mathbf{f} equal to $(-1, 0)$ in the rectangle $LGHK$ and to $(0, 0)$ elsewhere. This implies that the flow will move from the right to the left at the center of the domain; when the flow hits the left boundary, it splits into an upper flow and a lower flow; the $BCDE$ obstacle will cause recirculations in the upper flow on each side of the obstacle. This will in turn generate strong turbulent interactions with the main flow at the center of the domain; the advantage of the LES method is that we can describe the large vortices without having to resolve all the small turbulent details of the flow.

In this section, the initial guess \mathbf{u}_h^0 is considered to be the solution of the Stokes problem with the same right-hand side and boundary conditions as Problem (P).

We remind that the finite element spaces X_h and M_h , for the velocity and the pressure, are given by (8). We begin the numerical simulations by considering the following iterative Navier-Stokes algorithm:

$$\left\{ \begin{array}{l} \text{For any } i \geq 1, \text{ with a known } \mathbf{u}_h^{i-1} \in X_h, \text{ find } (\mathbf{u}_h^i, p_h^i) \in X_h \times M_h \text{ solution of :} \\ \forall \mathbf{v}_h \in X_h, \quad \nu(\nabla \mathbf{u}_h^i, \nabla \mathbf{v}_h) + d(\mathbf{u}_h^{i-1}, \mathbf{u}_h^i, \mathbf{v}_h) - (p_h^i, \text{div } \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle, \\ \forall q_h \in M_h, \quad (\text{div } \mathbf{u}_h^i, q_h) = 0. \end{array} \right. \quad (55)$$

We introduce the global linearization indicator $\tilde{\eta}_i^L$ as follows:

$$\tilde{\eta}_i^L = \left(\sum_{\kappa \in \mathcal{T}_h} (\eta_{i,\kappa}^L)^2 \right)^{\frac{1}{2}}, \quad (56)$$

where $\eta_{i,\kappa}^L$ is given by (33).

Algorithm (55) is stopped by using the classical global stopping criterion:

$$\eta_i^L \leq 10^{-6}. \quad (57)$$

The first numerical simulations corresponding to Problem (55) with $N = 5, 10, 20, 30$ show that the algorithm converges for $Re \leq 135$. These results are coherent with the literature (see for instance [4] for more details) which announces that the convergence depends on the Reynolds number for the cavity domain.

In [4], the authors proposed a simple modification of the numerical scheme (55) allowing to get convergence for a larger range of Reynolds numbers; they introduce the following relaxed numerical scheme:

$$\left\{ \begin{array}{l} \text{For any } i \geq 1, \text{ with a known } \tilde{\mathbf{u}}_h^{i-1} \in X_h, \text{ find } (\mathbf{u}_h^i, p_h^i) \in X_h \times M_h \text{ solution of :} \\ \forall \mathbf{v}_h \in X_h, \quad \nu(\nabla \mathbf{u}_h^i, \nabla \mathbf{v}_h) + d(\tilde{\mathbf{u}}_h^{i-1}, \mathbf{u}_h^i, \mathbf{v}_h) - (p_h^i, \text{div } \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle, \\ \forall q_h \in M_h, \quad (\text{div } \mathbf{u}_h^i, q_h) = 0, \end{array} \right. \quad (58)$$

and then set

$$\tilde{\mathbf{u}}_h^i = \frac{\mathbf{u}_h^i + \tilde{\mathbf{u}}_h^{i-1}}{2}.$$

In our case, Scheme (58) with $N = 5, 10, 20, 30$ converges for $Re \leq 2000$. Thus, we adopt this relaxed method for our LES algorithm that we rewrite as follows:

$$\left\{ \begin{array}{l} \text{For any } i \geq 1, \text{ with a known } \tilde{\mathbf{u}}_h^{i-1} \in X_h, \text{ find } (\bar{\mathbf{u}}_h^i, \bar{p}_h^i) \in X_h \times M_h \text{ solution of :} \\ \forall \mathbf{v}_h \in X_h, \quad \nu(\nabla \bar{\mathbf{u}}_h^i, \nabla \mathbf{v}_h) + (\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i, \nabla \mathbf{v}_h) + d(\tilde{\mathbf{u}}_h^{i-1}, \bar{\mathbf{u}}_h^i, \mathbf{v}_h) \\ \quad \quad \quad - (\bar{p}_h^i, \operatorname{div} \mathbf{v}_h) = \langle \bar{\mathbf{f}}, \mathbf{v}_h \rangle, \\ \forall q_h \in M_h, \quad (\operatorname{div} \bar{\mathbf{u}}_h^i, q_h) = 0, \end{array} \right. \quad (59)$$

and then set

$$\tilde{\mathbf{u}}_h^i = \frac{\bar{\mathbf{u}}_h^i + \tilde{\mathbf{u}}_h^{i-1}}{2}.$$

We introduce the global linearization error indicator $\bar{\eta}_i^L$ corresponding to (59) as follows:

$$\bar{\eta}_i^L = \left(\sum_{\kappa \in \mathcal{T}_h} (\bar{\eta}_{i,\kappa}^L)^2 \right)^{\frac{1}{2}}, \quad (60)$$

where

$$(\bar{\eta}_{i,\kappa}^L) = |\bar{\mathbf{u}}_h^i - \tilde{\mathbf{u}}_h^{i-1}|_{H^1(\kappa)^d}.$$

In our tests, Scheme (59) for $N = 20$ and with the stopping criterion $\bar{\eta}_i^L \leq 10^{-6}$, converges for large values of Re (we tested up to $Re = 50000$).

To compare Schemes (58) and (59), we introduce the global discretization error indicator associated to:

(1) Problem (58):

$$\tilde{\eta}_i^D = \left(\sum_{\kappa \in \mathcal{T}_h} (\tilde{\eta}_{i,\kappa}^D)^2 \right)^{\frac{1}{2}}, \quad (61)$$

where

$$\begin{aligned} (\tilde{\eta}_{i,\kappa}^D) &= h_\kappa \|\mathbf{f}_h + \operatorname{div}(\nu \nabla \mathbf{u}_h^i) - \tilde{\mathbf{u}}_h^{i-1} \cdot \nabla \mathbf{u}_h^i - \frac{1}{2} \operatorname{div}(\tilde{\mathbf{u}}_h^{i-1}) \mathbf{u}_h^i - \nabla p_h^i\|_{L^2(\kappa)^d} \\ &\quad + \frac{1}{2} \sum_{e \in \varepsilon_\kappa} h_e^{1/2} \|[(\nu \nabla \mathbf{u}_h^i - p_h^i \mathbb{I}) \cdot \mathbf{n}]\|_{0,e} + \|\operatorname{div} \mathbf{u}_h^i\|_{L^2(\kappa)}. \end{aligned}$$

(2) Problem (59):

$$\bar{\eta}_i^D = \left(\sum_{\kappa \in \mathcal{T}_h} ((\bar{\eta}_{i,\kappa}^D)^2 + (\bar{\eta}_{i,\kappa}^F)^2) \right)^{\frac{1}{2}}, \quad (62)$$

where

$$\begin{aligned} (\bar{\eta}_{i,\kappa}^D) &= h_\kappa \|\bar{\mathbf{f}}_h + \operatorname{div}((\nu + \nu_t(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i) - \tilde{\mathbf{u}}_h^{i-1} \cdot \nabla \bar{\mathbf{u}}_h^i - \frac{1}{2} \operatorname{div}(\tilde{\mathbf{u}}_h^{i-1}) \bar{\mathbf{u}}_h^i - \nabla \bar{p}_h^i\|_{L^2(\kappa)^d} \\ &\quad + \frac{1}{2} \sum_{e \in \varepsilon_\kappa} h_e^{1/2} \|[(\nu + \nu_t(\bar{\mathbf{u}}_h^{i-1})) \nabla \bar{\mathbf{u}}_h^i - \bar{p}_h^i \mathbb{I}] \cdot \mathbf{n}\|_{L^2(e)^d} + \|\operatorname{div} \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)} \end{aligned}$$

and

$$(\bar{\eta}_{i,\kappa}^F) = \|\nu_t(\bar{\mathbf{u}}_h^{i-1}) \nabla \bar{\mathbf{u}}_h^i\|_{L^2(\kappa)^d}.$$

Figures 2 and 3 show a comparison in logarithmic scale of the error indicators as a function of the iteration counts between the NS algorithm (58) and LES algorithm (59) for $Re = 1000$ on a uniform mesh with $N = 20$. We remark that the LES scheme (59) converges with fewer iterations compared to the NS scheme, and we attribute this to the higher viscosity induced by the turbulence model. Figure 4 shows a comparison in logarithmic scale of the global error indicators between the uniform NS (58) and uniform LES (59) Schemes for $Re = 1000$ with respect to the number of degrees of freedom for various values of N , multiplied by the number of iterations needed to reach the prescribed convergence tolerance. From these figures, we could conclude that the LES scheme (59) produces more accurate results than the NS scheme (58) for a given computational effort; however, we have to be very careful since the indicators are equivalent to, but are not the actual errors, and since unknown constants are involved in the equivalence inequalities. The fact that both curves have the same final trend was expected, since refining the mesh in LES implies that the turbulent viscosity tends to 0 and thus the total viscosity is close to the standard NS physical viscosity.

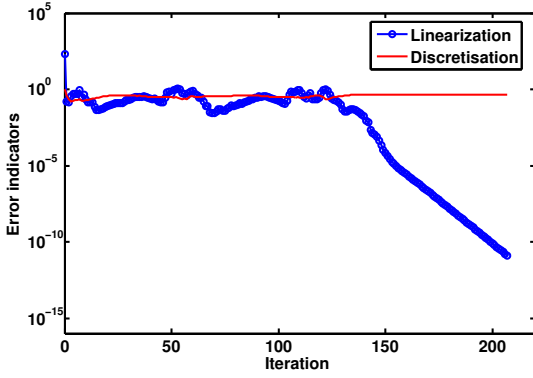


FIGURE 2. Uniform NS

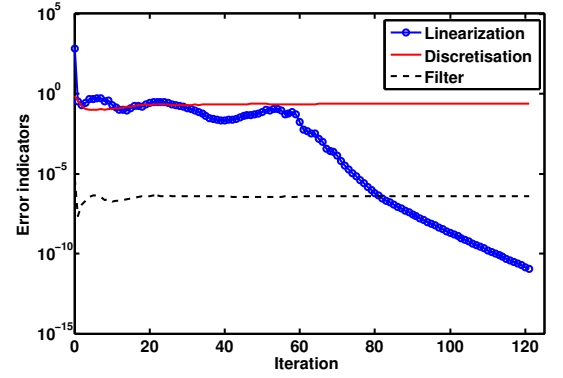


FIGURE 3. Uniform LES

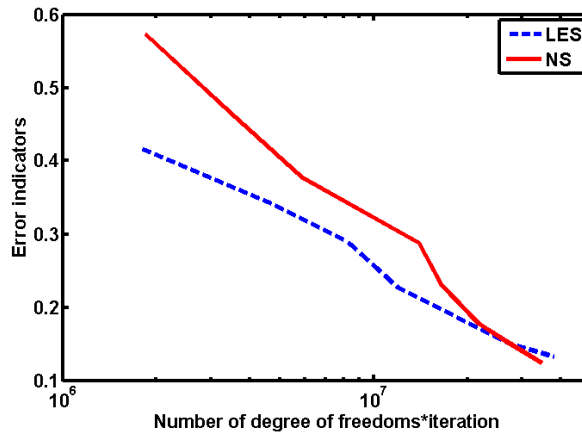


FIGURE 4. Comparison of the global error indicators between the NS and LES problems on uniform meshes with respect to the numbers of degrees of freedom multiplied by the number of iterations needed to reach convergence, in logarithmic scale.

Now, we aim to compare the LES method on uniform meshes with the LES method on meshes which are locally refined based on the indicators (62) and in which the iterative stopping criterion is based on (60). We refer to the algorithm proposed in [4] for the strategy of the adapted method: the authors used the following adaptive stopping criterion:

$$\eta_i^L \leq \gamma \eta_i^D, \quad (63)$$

where γ is a positive parameter which balances the discretization and linearization errors. They chose $\gamma = 0.01$ for their numerical experiments. This adaptive stopping criterion was introduced and studied in [15, 16]. In the present work, we choose $\gamma = 0.01$ (same value as in [4]) for our numerical applications.

Figures 5-8 show the evolution of the mesh during the refinement levels of the algorithm for $Re = 1000$ and a starting uniform mesh produced with $N = 20$. We remark that, from an iteration to another, the concentration of the refinement is on the complex vorticity region.

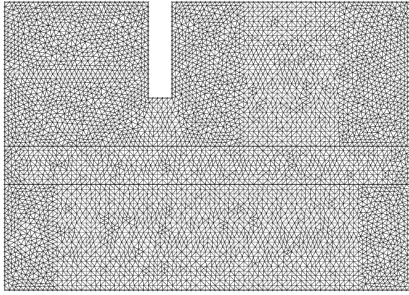


FIGURE 5. Initial mesh (11114 triangles)

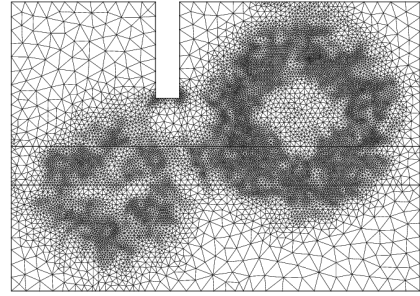


FIGURE 6. First level mesh (23238 triangles)

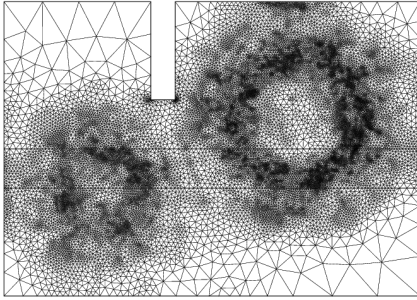


FIGURE 7. Second level mesh (37814 triangles)

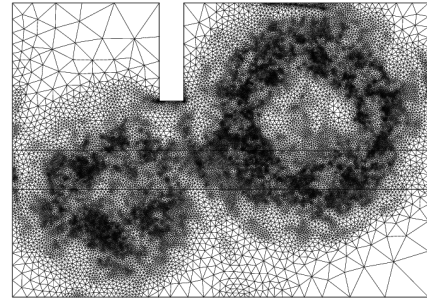


FIGURE 8. Third level mesh (67086 triangles)

For $Re = 1000$, figure 9 shows a comparison, in logarithmic scale, of the error indicators for the uniform LES problems with the classical criterion (57) and the new one (63) and adapted LES problem with respect to the number of degrees of freedom. This figure shows the effectiveness of the adapted algorithm versus the uniform one. Another way to measure the improvement with adaptive computations is by looking at the results obtained on uniform meshes and adapted meshes with approximately the same number of elements; this is what is done on Figures 10 and 11 on the one hand, on which the velocity norm is plotted and on Figures 12 and 13 on the other hand, on which a vertical cut of the velocity norm is shown. Both comparisons show a better accuracy of the results on adapted meshes. Indeed, as one refines the meshes, the velocity modulus tends to increase (because the turbulent viscosity decreases), and we observe that it increases more rapidly on adapted meshes than on uniform ones.

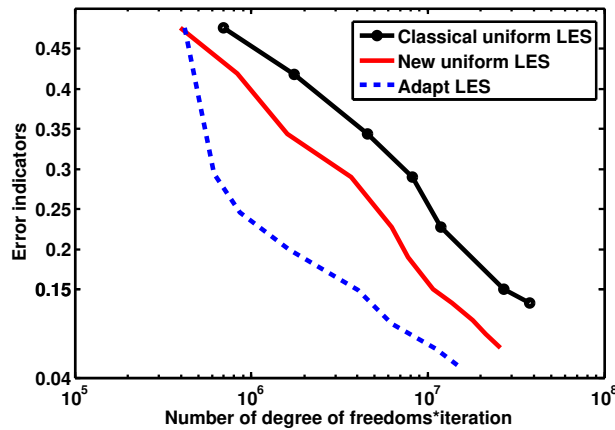


FIGURE 9. Comparison between uniform and adaptive LES estimators with respect to the numbers of degrees of freedom in logarithmic scale.

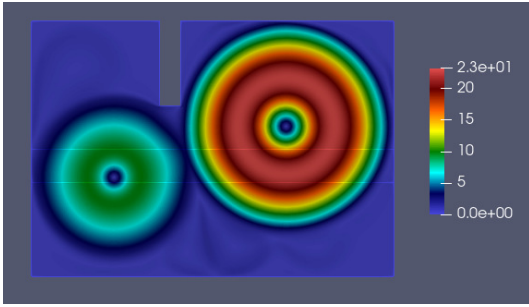


FIGURE 10. Velocity norm on a uniform mesh with $N = 160$

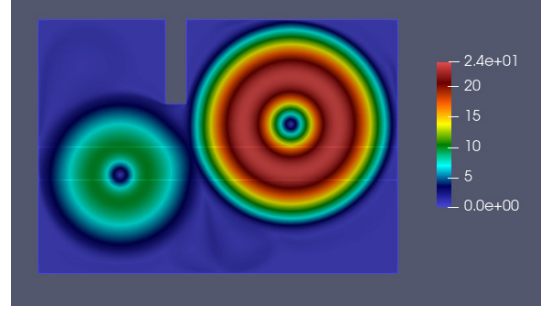


FIGURE 11. Velocity norm on the eighth level mesh

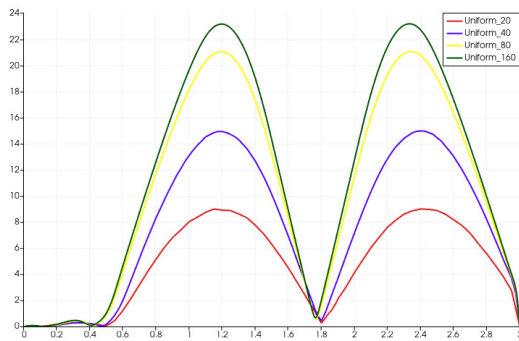


FIGURE 12. Velocity norm: Vertical cut (constant x) through the right-hand vortex center on uniform meshes with $N = 20, 40, 80$ and 160 .

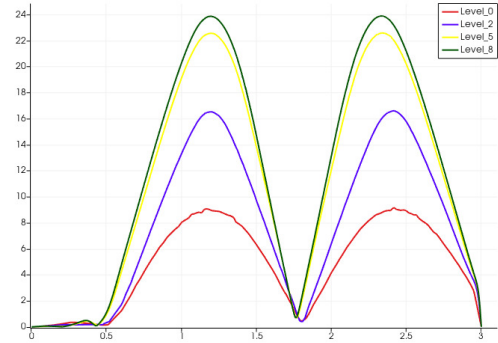


FIGURE 13. Velocity norm: Vertical cut (constant x) through the right-hand vortex center on adapted meshes with level 0 (uniform with $N = 20$), levels 2, 5 and 8.

7. CONCLUSION

In this work, we have derived *a posteriori* error estimates for the finite element discretization of the Large Eddy Simulation applied to the Navier-Stokes system. These estimates yield an upper bound of the error which is computable up to unknown constants and allows to distinguish the discretization, the linearization and the filtering errors. In this work, we show the advantages of the LES method combined to an adaptive mesh refinement and an adaptive linearization stopping criterion.

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