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Abstract: We discuss controller designs for asymmetric saturated discrete-time linear systems. Under the assumption that a locally stabilizing controller of the origin is known, we augment the original controller with an additional term that vanishes in a neighborhood of the origin. The augmented controller outperforms the original controller in terms of the estimate of the region of attraction. The paper translates the results discussed in Braun et al. (2022a,b), from the continuous-time setting to the discrete-time setting, and numerically verifies that the results derived for continuous-time systems are recovered if the discrete-time system is obtained through an Euler discretization of a continuous-time system with a sufficiently small sampling rate.

Keywords: Systems with saturation; Lyapunov methods; Asymptotic stabilization; discrete-time systems; semidefinite programming.

1. INTRODUCTION

Actuator saturation is widely present in control systems due to physical limitations or safety/compliance reasons. It may affect the closed-loop system stability and its performances. The saturated responses may exhibit diverse properties, including global stability, multiple equilibria, local stability or undesired limit cycles for instance. A large literature is dedicated to saturated control in both the continuous-time and discrete-time domains (see for instance the books (Hu and Lin, 2001; Tarbouriech et al., 2011; Benzaouia et al., 2017)).

When local stability is ensured, an important problem is providing nonconservative estimates of the region of attraction of the origin. A standard technique is to obtain an estimate of the region of attraction via the level set of a Lyapunov function or of a Gauge function (Fiacchini et al., 2013), considered as invariant sets, see (Blanchini, 1999). A large range of contributions is available by using different models for the saturation (polytopic representation (Hu and Lin, 2001), (generalized) cone bounded sector conditions (Tarbouriech et al., 2006)) and various classes of Lyapunov functions (polyhedral (da Silva and Tarbouriech, 1999), piecewise affine (Milani, 2002), piecewise quadratic, composite functions (Hu and Lin, 2003), for example). A frequent assumption is to impose (or restrict) the saturation in the symmetric case, nevertheless, in practice, the saturation has often asymmetric thresholds. Only a few contributions are focused on asymmetric saturation. One of the first result provides a non-symmetrical polyhedral Lyapunov function (Benzaouia and Burgat, 1988), which has been generalized to the union of preconstructed positively invariant sets (Benzaouia, 2005). Asymmetric partition of the space and piecewise quadratic Lyapunov functions have been proposed in (Groff et al., 2019). The paper (Li and Lin, 2016) introduces an asymmetric Lyapunov function that is contractively invariant. The idea in (Benhayoun et al., 2013) is to shift the asymmetric constraints to the symmetric framework. Finally the anti-windup approach used in (Gomes da Silva Jr. and Tarbouriech, 2006) can be underlined. In the continuous-time setting, the paper (Yuan and Wu, 2015) uses a switching strategy to deal with asymmetric saturation.

In this paper, we consider shifted coordinates for a linear system with asymmetric saturation and, by assuming that there exists a locally stabilizing controller of the origin associated with an ellipsoidal estimate of the region of attraction, we schedule a shifting parameter to obtain enlarged estimates of the region of attraction. This allows convergence to a shifted equilibrium, which will be afterwards steered to the origin (Benzaouia et al., 2017, Chapter 3). As a consequence the estimate of the region of attraction comprises the union of shifted ellipsoids. The paper translates results in (Braun et al., 2022a,b) from the continuous-time setting to the discrete-time setting.

The rest of the paper is organized as follows. Section 2 introduces the setting and the shifted stabilizers, while Section 3 provides an optimization-based control law that asymptotically stabilizes the origin associated with an estimate of the region of attraction as the union of shifted ellipsoids. A numerical illustration is discussed in Section 4

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to highlight the practical relevance of our approach. Finally, concluding remarks are provided in Section 5.

Notation. For $u^-, u^+ \in \mathbb{R}^m_{\geq 0}$, $m \in \mathbb{N}$, $\operatorname{sat}_{[u^-,u^+]}(u) = \max\{\min\{u^+, u\}, -u^-\}$ defines the saturation, where the maximum/minimum are to be understood componentwise. The deadzone is defined as $\operatorname{dz}_{[u^-,u^+]}(u) = u - \operatorname{sat}_{[u^-,u^+]}(u)$. For $Z \in \mathbb{R}^{n\times n}$, $\operatorname{He}(\cdot)$ denotes $\operatorname{He}(Z) = Z + Z^\top$. For $Z \in \mathbb{R}^{n\times m}$ and $z \in \mathbb{R}^n$, $Z_{[k]}$ and z_k denote the k-th row and the k-th entry, respectively. A vector $v \in \mathbb{R}^n$ satisfies $v \leq \min\{u^-, u^+\}$ if $v_k \leq \min\{u_k^-, u_k^+\}$ for all $k \in \{1, \dots, n\}$. For a vector $v \in \mathbb{R}^n$, $\min\{v\} = \min\{v_1, \dots, v_n\} \in \mathbb{R}$. In $\mathbb{R}^n$, we use the norms $|x| = \sqrt{x^\top x}$, $|x|_P = \sqrt{x^\top P x}$, $P \in \mathbb{R}^{n\times n}$ positive definite. Symbol I denotes the identity matrix of appropriate dimensions, and the vector $\mathbb{1} \in \mathbb{R}^n$ satisfies $\mathbb{1}_k = 1$, $k \in \{1, \dots, n\}$. Finally, $\operatorname{int}(\mathcal{A})$, $\overline{\mathcal{A}}$ denote the interior and the closure of a set $\mathcal{A} \subset \mathbb{R}^n$.

2. SYMMETRIC AND SHIFTED STABILIZERS

We consider linear saturated discrete-time systems

$$x^+ = Ax + B \operatorname{sat}_{[u^-,u^+]}(u) \tag{1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and saturation limits $u^-, u^+ \in \mathbb{R}^m_{>0}$. We define the average saturation range and the average saturation center as

$$\bar{u} = \tfrac{1}{2}(u^+ + u^-), \quad u_\circ = \tfrac{1}{2}(u^+ - u^-). \tag{2}$$

We assume that the average saturation range $\bar{u}_k$ satisfies $\bar{u}_k = 1$ for all $k \in \{1, \dots, m\}$, which can be assumed without loss of generality by scaling the columns of B.

Assumption 1. It holds that $\bar{u} = \mathbb{1} \in \mathbb{R}^m$ and the pair (A, B) is stabilizable. ⋄

Of particular interest is the subspace of induced equilibria

$$\Gamma = \{x_e \in \mathbb{R}^n : \ Ax_e + Bu_e = x_e, \ u_e \in \mathbb{R}^m\}. \tag{3}$$

Alternatively, Γ can be represented through the kernel of the matrix $[\, A - I \ B \,]$ (of dimension $q \in \mathbb{N}$), which we denote by

$$M = \begin{bmatrix} M_A \\ M_B \end{bmatrix}.$$

Here $M_A \in \mathbb{R}^{n\times q}$, $M_B \in \mathbb{R}^{m\times q}$, $q \in \mathbb{N}$, and it holds that $[A - I \ \ B] \cdot M = 0$ by the definition of the kernel. It can be underlined that $q \geq 1$ due to the rectangular structure of $[A - I \ \ B]$. Thus, Γ, defined in (3) is not empty and can be written as

$$\Gamma = \{x_e = M_A\delta \in \mathbb{R}^n : \delta \in \mathbb{R}^q\}$$

and an equilibrium pair (x_e, u_e) is uniquely defined through $\delta \in \mathbb{R}^q$, i.e.,

$$\delta \mapsto x_e(\delta) = M_A\delta, \quad \delta \mapsto u_e(\delta) = M_B\delta.$$

To be able to stabilize an equilibrium pair (x_e, u_e) additionally the condition

$$\delta \in \Phi = \left\{\delta \in \mathbb{R}^q : \ -u^- \leq M_B\delta \leq u^+\right\}$$

needs to be satisfied. As in Braun et al. (2022a,b) we define the function $\beta : \mathbb{R}^q \to \mathbb{R}$,

$$\beta(\delta) = \min\{\min\{u^- + M_B\delta, u^+ - M_B\delta\}\}, \tag{4}$$

which satisfies $\beta : \Phi \to [0, 1]$ under Assumption 1.

In a neighborhood of the origin (away from the origin suitably shifted versions will be constructed), we propose to use the feedback law

$$u = Kx + L \operatorname{dz}_{[u^-,u^+]}(u), \tag{5}$$

with $K \in \mathbb{R}^{m\times n}$, $L \in \mathbb{R}^{m\times m}$, asymptotically stabilizing the origin. Combining (1) and (5), the closed-loop dynamics can be written (close to the origin) as

$$\begin{aligned} x^+ &= (A + BK)x - (B - BL) \operatorname{dz}_{[u^-,u^+]}(u) \\ u &= Kx + L \operatorname{dz}_{[u^-,u^+]}(u). \end{aligned} \tag{6}$$

Well-posedness of the algebraic loop in control law (5) is addressed in Proposition 1 below.

For designing our stabilizer away from the origin, for any $\delta \in \Phi$, consider the coordinate transformation

$$\tilde{x}_\delta = x - x_e(\delta) \qquad \text{and} \qquad \tilde{u}_\delta = u - u_e(\delta).$$

It holds that

$$\begin{aligned} \tilde{x}_\delta^+ &= (x - x_e(\delta))^+ = x^+ - x_e(\delta) \\ &= Ax + B \operatorname{sat}_{[u^-,u^+]}(u) - Ax_e(\delta) - Bu_e(\delta) \\ &= A\tilde{x}_\delta + B(u_e(\delta) + \operatorname{sat}_{[u^- + u_e(\delta), u^+ - u_e(\delta)]}(u - u_e(\delta))) \\ &\quad - Bu_e(\delta) \\ &= A\tilde{x} + B \operatorname{sat}_{[u^- + u_e(\delta), u^+ - u_e(\delta)]}(\tilde{u}_\delta), \end{aligned} \tag{7}$$

with the shifted input $\tilde{u}_\delta$ selected as follows

$$\tilde{u}_\delta = K\tilde{x}_\delta + L \operatorname{dz}_{[u^- + u_e(\delta), u^+ - u_e(\delta)]}(\tilde{u}_\delta). \tag{8}$$

In the original coordinates, the input (8) is defined as $u = \nu(x, \delta)$, with

$$\begin{aligned} \nu(x,\delta) &= M_B\delta + K(x - M_A\delta) \\ &\quad + L \operatorname{dz}_{[u^- + M_B\delta, u^+ - M_B\delta]}(\nu(x,\delta) - M_B\delta) \\ &= M_B\delta + K(x - M_A\delta) + L \operatorname{dz}_{[u^-,u^+]}(\nu(x,\delta)), \end{aligned} \tag{9}$$

which allows us to parametrically represent the closed-loop dynamics (1), (9) for any selection of the parameter $\delta \in \Phi$.

To characterize regions of attraction of asymptotically stable (induced) equilibria $x_e(\delta)$, we consider sublevel sets of quadratic functions. In particular, for any $\delta \in \operatorname{int}(\Phi)$ and $P \in \mathbb{R}^{n\times n}$ positive definite, we define the set

$$\mathcal{E}_\delta(P) = \{x \in \mathbb{R}^n : \ |x - x_e(\delta)|_P \leq \beta(\delta)\}, \tag{10}$$

where β is defined in (4).

Proposition 1. (Symmetric Stabilizer). Given the plant-controller pair (1), (9) for $\delta \in \operatorname{int}(\Phi)$ let Assumption 1 be satisfied and let $\alpha \in \mathbb{R}_{>0}$. Moreover, let $Q \in \mathbb{R}^{n\times n}$, $W, Y \in \mathbb{R}^{m\times n}$, $U, X \in \mathbb{R}^{m\times m}$ be a solution of the optimization problem

$$\max_{Q,Y,U,W,X} \quad \log\det(Q) \tag{11a}$$

$$\text{subject to} \quad U > 0 \ \text{ diagonal}, \quad Q = Q^\top > 0$$

$$\operatorname{He}\begin{bmatrix} -\frac{\alpha}{2}Q & 0 & Y^\top \\ AQ + BW & -\frac{\alpha}{2}Q & -BU + BX \\ W & 0 & X - U \end{bmatrix} < 0 \tag{11b}$$

$$\begin{bmatrix} 1 & Y_{[k]} \\ Y_{[k]}^\top & Q \end{bmatrix} \geq 0, \quad k = 1, \dots, m. \tag{11c}$$

Then, selecting

$$K = WQ^{-1}, \quad L = XU^{-1}, \quad P = Q^{-1}, \tag{12}$$

for any $\delta \in \operatorname{int}(\Phi)$ the nonlinear algebraic loop in (9) is well posed (i.e., its solution is unique and Lipschitz) and the Lyapunov function

$$V_\delta(x) := |x - M_A\delta|_P^2 \tag{13}$$

exponentially decreases with rate larger than α^2 within the set $\mathcal{E}_\delta(P)$, i.e.,

$$V_\delta(x^+) < \alpha^2 V_\delta(x) \tag{14}$$

for all $x \in \mathcal{E}_\delta(P)\backslash\{0\}$. ⌟

Proposition 1 for discrete-time systems is the analogue result to (Braun et al., 2022b, Proposition 1) (see also (Mariano et al., 2020, Theorem 1)), both stating parallel properties for the continuous-time setting, as clarified in the next remark.

Remark 1. For continuous-time systems

$$\dot{x}_c = (A_c + B_c K_c)x_c$$

and a quadratic Lyapunov functions $V_c(x) = x^\top P_c x$, $P_c > 0$, when stabilizing the origin, the decrease condition (14) generalizes to

$$\dot{V}_c(x_c(t)) = \langle 2Px_c(t), Ax_c(t)\rangle \leq -2\alpha_c V_c(x_c(t))$$

(see (Braun et al., 2022b, Proposition 1)). The ensuing exponential decrease property for the continuous-time solutions $t \mapsto x_c(t)$ can be rewritten in terms of the upper bound

$$|x_c(t)| \leq \sqrt{\tfrac{\lambda_{\max}(P_c)}{\lambda_{\min}(P_c)}} e^{-\alpha_c t}|x_c(0)|,$$

where $\lambda_{\max}(P_c)$ and $\lambda_{\max}(P_c)$ denote the largest and the smallest eigenvalue of P_c, respectively. Similarly, setting $\delta = 0$ for simplicity, the discrete-time system $x^+ = (A + BK)x$, with Lyapunov function $V_0(x) = x^\top Px$ obtained through Proposition 1, locally satisfies the bound

$$|x(k)| \leq \sqrt{\tfrac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \alpha^k |x(0)|,$$

which can be derived from (14).

Thus, if the discrete-time dynamics stems from an Euler discretization of the continuous-time system, i.e.,

$$A = \tau A_c + I, \qquad B = \tau B_c, \qquad \tau > 0,$$

then α_c and α can be related through the condition

$$\alpha = e^{-\alpha_c \tau}. \tag{15}$$

Additionally, note that the subspace of induced equilibria Γ is invariant under the Euler discretization, i.e., a pair (x_e, u_e) satisfies $0 = A_c x_e + B_c u_e$ if and only if

$$x_e = Ax_e + Bu_e = \tau A_c x_e + x_e + \tau B_c$$

independent of the sampling time $\tau > 0$. ∘

Proposition 1 can be proven by using the results in Massimetti et al. (2009), for example. Here we give a sketch of these derivations.

Proof of Proposition 1. An immediate consequence of (Massimetti et al., 2009, Thm. 1) [1] is that for a generic system

$$x^+ = A_{\mathrm{cl}} x + B_{\mathrm{q}}\,\mathrm{dz}(u), \quad u = Kx + L\,\mathrm{dz}(u), \tag{16}$$

where the deadzone range is larger than 1, and for any scalar $\alpha \in (0, 1]$, the Lyapunov function $V(x) = x^\top Q^{-1} x$ satisfies $(V(x) \leq 1) \Rightarrow (V(x^+) \leq \alpha^2 V(x))$ if (11c) holds together with the linear matrix inequality

$$\mathrm{He}\begin{bmatrix} -\frac{\alpha}{2}Q & 0 & Y^\top \\ A_{\mathrm{cl}}Q & -\frac{\alpha}{2}Q & B_{\mathrm{q}}U \\ KQ & 0 & LU - U \end{bmatrix} < 0 \tag{17}$$

where U is diagonal positive definite. The linear matrix inequality in (11b) then stems from noticing that (6) can be written as (16) with the selection $A_{\mathrm{cl}} = A + BK$, $B_{\mathrm{q}} = -B + BL$ and exploiting the choices in (12).

The remainder of the proof follows the same steps as those in (Braun et al., 2022b, Cor. 2) for dealing with the shifted coordinates. ■

Remark 2. As in the continuous-time setting, expression (9) specifies the control input u only implicitly, even though Proposition 1 ensures that the corresponding solution is Lipschitz. Proceeding as in (Mariano et al., 2020, Lemma 3), for the single-input case $m = 1$, the selection

$$\begin{aligned}\nu(x, \delta) = {} & M_B\delta + K(x - M_A\delta) \\ & + L(I - L)^{-1}\,\mathrm{dz}_{[u^-, u^+]}(M_B\delta + K(x - M_A\delta))\end{aligned} \tag{18}$$

can be proven to be the explicit solution to (9). ∘

3. OPTIMIZATION-BASED SHIFTED STABILIZER

3.1 Main result

In this section we propose a scheduled control law and prove that it asymptotically stabilizes the origin of plant (1) with region of attraction containing the union of the sublevel sets generated by all possible values of $\delta \in \mathrm{int}(\Phi)$,

$$\mathcal{R} = \bigcup_{\delta \in \mathrm{int}(\Phi)} \mathcal{E}_\delta(P). \tag{19}$$

To define the feedback law, for $x \in \mathrm{int}(\mathcal{R})$ and $c \geq 0$ we consider the following optimization problem

$$\begin{aligned} \delta^\star(x) \in \underset{\delta \in \Phi}{\mathrm{argmin}}\ & |M_A\delta|^2 + c|\delta|^2 \\ \text{subject to}\ \ & |x - M_A\delta|_P \leq \beta(\delta). \end{aligned} \tag{20}$$

Remark 3. The first term in the objective function of the optimization problem (20) minimizes the norm of the induced equilibrium $x_e(\delta) = M_A\delta$ while the second term minimizes δ. For $c > 0$, the second term in the objective function ensures that $|M_A\delta|^2 + c|\delta|^2$ is strongly convex. If $M_A^\top M_A$ is full rank, then the second term is not necessary to ensure that $|M_A\delta|^2 + c|\delta|^2 = 0$ if and only if $\delta = 0$. ∘

It follows from the results in (Braun et al., 2022a,b, Lemma 1) that for $c > 0$, $\delta^\star(x) \in \mathrm{int}(\Phi)$ is unique and $\delta^\star(\cdot) : \mathcal{R} \to \mathrm{int}(\Phi)$ is Lipschitz continuous. For completeness, we report the result here again.

Lemma 1. ((Braun et al., 2022a, Lemma 1)). Let $c > 0$ and let Assumption 1 be satisfied. Consider the optimization problem (20) where β and $\mathcal{R}$ are defined in (4) and (19), respectively, and matrix P is defined through Proposition 1. Then the following properties are satisfied:

(1) for each $x \in \overline{\mathcal{R}}$, (20) is feasible, and the feasible set is closed and convex. Moreover, for $x \in \mathrm{int}(\mathcal{R})$ the interior of the feasible set is nonempty;
(2) the set-valued map $F : \overline{\mathcal{R}} \rightrightarrows \Phi$,
$$F(x) = \{\delta \in \Phi :\ |x - M_A\delta|_P \leq \beta(\delta)\},$$
defining the feasible set of (20), is continuous;

[1] The result in (Massimetti et al., 2009, Thm. 1) is given for the case with $\alpha = 1$. Its extension to a general $\alpha \in (0, 1]$ is straightforward by imposing $V(x^+) \leq \alpha^2 V(x) \Leftrightarrow \alpha^{-1} V(x^+) \leq \alpha V(x)$.

(3) $\delta^\star(x) = 0$ for all $x \in \overline{\mathcal{R}}$ such that $|x|_P \leq \beta(0)$;
(4) $|x - M_A\delta^\star(x)|_P = \beta(\delta^\star(x))$ for all $x \in \overline{\mathcal{R}}$ such that $|x|_P > \beta(0)$;
(5) $\delta^\star(x) \in \Phi$ is unique for all $x \in \overline{\mathcal{R}}$;
(6) $\delta^\star(\cdot) : \text{int}(\mathcal{R}) \to \Phi$ is Lipschitz continuous; and
(7) $\delta^\star(x) \in \text{int}(\Phi)$ for all $x \in \text{int}(\mathcal{R})$. ⌟

The result (Braun et al., 2022a, Lemma 1) is derived for a different objective function. However, since only strict convexity of the objective function is used in the proof, the result and its derivation remain unchanged. Additionally, item 6 in (Braun et al., 2022a, Lemma 1) only states continuity of $\delta^\star(\cdot)$. Lipschitz continuity follows from (Hager, 1979, Appendix D).

With $\delta^\star(\cdot)$ as in (20), the following state dependent feedback law is obtained:

$$u = \mu(x) = \nu(x, \delta^\star(x)), \tag{21}$$

where $\nu(\cdot,\cdot)$ is given implicitly in (9) and (for the single-input case) explicitly in (18).

Remark 4. If δ is one dimensional (i.e., $q = 1$), $\delta^\star(x)$ can be computed explicitly (see (Braun et al., 2022b, Section IV)). Since here the objective function is slightly different, the calculations need to be adapted, but the derivation does not change. For $q > 1$, convex optimization algorithms can be used to solve (20) efficiently. Additionally, suboptimal solutions can be used to update $\delta^\star(x)$ at every time step following an approach similar to (Braun et al., 2022a, Section V). ∘

With (20) the following properties of the closed-loop dynamics with shifted reference points can be shown for solutions starting in $x \in \text{int}(\mathcal{R})$.

Proposition 2. Consider the discrete-time plant (1) and let Assumption 1 be satisfied. In addition, let $P > 0$ be obtained through Proposition 1 for $\alpha > 0$ fixed. Let $c > 0$ correspond to the parameter in (20) and consider the controller (21) defined through (18) and (20) for $x \in \text{int}(\mathcal{R})$. Then the following properties are satisfied.

For any initial condition $x(0) = x_0 \in \text{int}(\mathcal{R})$ the solution $x(\cdot) : \mathbb{N} \to \text{int}(\mathcal{R})$ using controller (21) is well-defined and

1) the sequence $(f_k)_{k\in\mathbb{N}}$ with
$$f_k = |M_A\delta^\star(x(k))|^2 + c|\delta^\star(x(k))|^2$$
is monotonically decreasing;
2) there exists $K \in \mathbb{N}$ such that $\delta^\star(x(k)) = 0$ for all $k \geq K$. ⌟

The proof of Proposition 2 is given below in Section 3.2 to avoid breaking the flow of the exposition. From Propositions 1 and 2 the main result of this paper in terms of asymptotic stability properties of the origin of the closed-loop system (1), (21) defined through (18), (20) follows.

Theorem 1. Consider the discrete-time system (1) and let Assumption 1 be satisfied. In addition, let $P > 0$ be obtained through Proposition 1 for $\alpha > 0$ fixed. Let $c > 0$ correspond to the parameter in (20) and consider the controller (21) defined through (18) and (20) for $x \in \text{int}(\mathcal{R})$.

Then, the origin of the closed-loop system is asymptotically stable and the region of attraction contains the set $\text{int}(\mathcal{R})$ defined in (19). ⌟

Proof. First note that asymptotic stability with estimate of the region of attraction given by $\mathcal{E}_0(P)$ follows from Proposition 1 and Lemma 1, item 3. Then asymptotic stability of the origin with estimate of the region of attraction given by $\text{int}(\mathcal{R})$ follows from Proposition 2, item 2, where we have established that all solutions starting in $\text{int}(\mathcal{R})$ reach the set $\mathcal{E}_0(P)$ in finite time. ■

We emphasize that, not only the stabilizer (21) ensures an enlarged estimate of the region of attraction, but it also preserves locally the "local" feedback law (5), which, by Proposition 1, ensures (local) α-exponential convergence.

3.2 Proof of Proposition 2

To the end of proving Proposition 2, we first note that the condition $x \in \text{int}(\mathcal{R})$ implies that the optimization problem (20) is feasible and $\delta^\star(x) \in \text{int}(\Phi)$ according to Lemma 1, item 7. Then $\mu(x)$ in (21), and in particular (9), is well defined according to the well-posedness of the algebraic loop established in Proposition 1. Moreover, Proposition 1 implies that $x^+ \in \text{int}(\mathcal{E}_{\delta(x))}(P)) \subset \text{int}(\mathcal{R})$ for $x \in \text{int}(\mathcal{R})$. Thus, x^+ and $\delta^\star(x^+) \in \text{int}(\Phi)$ are well defined. Since this argument can be applied iteratively, it follows that $x(\cdot) : \mathbb{N} \to \text{int}(\mathcal{R})$ is well defined.

Item 1. The first item follows from the definition of the control law (8) stabilizing the reference point $x_e(\delta) = M_A\delta$ and the definition of the optimization problem (20). In particular, under the assumption $x \in \mathcal{E}_{\delta^\star(x)}(P)$ it follows that $x^+ \in \text{int}(\mathcal{E}_{\delta^\star(x)}(P))$ according to the stabilizing properties established in Proposition 1. This implies that $|x^+ - M_A\delta^\star(x)|_P < \beta(\delta^\star(x))$, i.e., the constraints in (20) are satisfied for x^+ and $\delta^\star(x)$. Since, δ is optimized with respect to the objective function $f(\delta) = |M_A\delta|^2 + c|\delta|^2$, monotonicity in the form

$$|M_A\delta^\star(x^+)|^2 + c|\delta^\star(x^+)|^2 \leq |M_A\delta^\star(x)|^2 + c|\delta^\star(x)|^2$$

holds (according to Lemma 1, items 3 and 4).

Item 2. To simplify the notation, we use $\delta_k = \delta^\star(x(k))$ in the following. From Item 1 we know that f_k is monotonically decreasing. Furthermore, $|M_A\delta|^2 + c|\delta|^2 = 0$ if and only if $\delta = 0$ since $c > 0$ by assumption. Let $\lambda_{\min} > 0$ denote the smallest eigenvalue of $(M_A)^\top M_A + cI$. Then it holds that

$$\lambda_{min}|\delta_k|^2 \leq \delta_k^\top (M_A^\top M_A + cI)\delta_k = f_k \leq f_0.$$

Hence, $(f_k)_{k\in\mathbb{N}}$ as well as $(\delta_k)_{k\in N}$ are bounded sequences.

Due to the monotonicity of $f_k \geq 0$, the sequence f_k is convergent. Since $(\delta_k)_{k\in N}$ is bounded, there exists a convergent subsequence $(\delta_{k_j})_{j\in\mathbb{N}}$, $j \in \mathbb{N}$, satisfying $\delta_{k_j} \to \delta^\#$ for $k \to \infty$, and $\delta^\#$ denotes an accumulation point of $(\delta_k)_{k\in\mathbb{N}}$ (in view of the Bolzano-Weierstrass theorem).

To show that $\delta_k \to 0$ for $k \to \infty$ assume for the sake of a contradiction that there exists an accumulation point $\delta^\# \neq 0$. For $k \in \mathbb{N}$, let $\varepsilon_k \in \mathbb{R}^q$ be defined such that $\delta_k = \delta^\# + \varepsilon_k$. It holds that

$$\begin{aligned} V_{\delta_k}(x) &= |x - M_A\delta_k|_P^2 = |x - M_A(\delta^\# + \varepsilon_k)|_P^2 \\ &= |x - M_A\delta^\#|_P^2 - 2(x - M_A\delta^\#)^\top P M_A\varepsilon_k \\ &\quad + (M_A\varepsilon_k)^\top P(M_A\varepsilon_k) \\ &= V_{\delta^\#}(x) + \mu_x(\varepsilon_k) \end{aligned}$$

where

$$\mu_x(\varepsilon) = -2(x - M_A\delta^\#)^\top P M_A\varepsilon + (M_A\varepsilon)^\top P(M_A\varepsilon)$$

by definition. For all $x \in \mathbb{R}^n$ it holds that $\mu_x(\varepsilon) \to 0$ for $|\varepsilon| \to 0$.

From the decrease condition (14) it follows that

$$\begin{aligned} V_{\delta_k}(x(k+1)) &= V_{\delta^\#}(x(k+1)) + \mu_{x(k+1)}(\varepsilon_k) \\ &\leq (1-2\alpha)\left(V_{\delta^\#}(x(k)) + \mu_{x(k)}(\varepsilon_k)\right) \end{aligned}$$

and thus

$$\begin{aligned} V_{\delta^\#}(x(k+1)) \leq (1-\alpha)V_{\delta^\#}(x(k)) - \alpha V_{\delta^\#}(x(k)) \\ + |\mu_{x(k+1)}(\varepsilon_k)| + (1-2\alpha)|\mu_{x(k)}(\varepsilon_k)|. \end{aligned} \tag{22}$$

Since

$$f_k = |M_A\delta_k|^2 + c|\delta_k|^2 \to |M_A\delta^\#|^2 + c|\delta^\#|^2$$

for $k \to \infty$, $f_k \geq |M_A\delta^\#|^2 + c|\delta^\#|^2$ and since $\delta_k = \delta^\star(x_k)$ is optimal with respect to the objective function, it holds that $V_{\delta^\#}(x(k)) \geq \beta(\delta^\#)^2 > 0$.

However, from the continuity of $\beta(\cdot)$ and the continuity of $V_\delta(x)$ in δ and x, and from the convergence of a subsequence of $(\delta_k)_{k\in\mathbb{N}}$ to $\delta^\#$, it follows that for all $K \in N$ there exists $k > K$ with the property

$$(1-\alpha)V_{\delta^\#}(x(k)) \leq \beta(\delta^\#)^2.$$

Moreover, for $k \in \mathbb{N}$ sufficiently large, it holds that

$$\alpha V_{\delta^\#}(x(k)) \geq \alpha\beta(\delta^\#)^2 > |\mu_{x(k+1)}(\varepsilon_k)| + (1-2\alpha)|\mu_{x(k)}(\varepsilon_k)|.$$

Combining the last two inequalities with (22) implies

$$V_{\delta^\#}(x(k+1)) = |x(k+1) - M_A\delta^\#|_P^2 < \beta(\delta^\#)^2.$$

However, this leads to a contradiction of the update δ_{k+1} since

$$V_{\delta_{k+1}}(x(k+1)) = |x(k+1) - M_A\delta_{k+1}|_P^2 = \beta(\delta_{k+1})^2$$

according to Lemma 1, item 4, and

$$|M_A\delta_{k+1}|^2 + c|\delta_{k+1}|^2 < |M_A\delta^\#|^2 + c|\delta^\#|^2.$$

Thus, we can conclude that $\delta^\# = 0$, which in particular implies that $(\delta_k)_{k\in\mathbb{N}}$ is converging.

Finally, since $\beta(0) > 0$, there exists $K \in N$ such that $M_A\delta^\star(x(K)) \in \mathcal{E}_0(P)$ and $x(K) \in \mathcal{E}_0(P)$ (since $x(k) \to M_A\delta^\star(x(k))$ for $k \to \infty$). This completes the proof of item 2 and the proof of Proposition 2.

4. NUMERICAL ILLUSTRATION

To illustrate the results derived in this paper and to illustrate the connection with the continuous-time setting discussed in Braun et al. (2022a,b), with reference to the notation in Remark 1, we consider the continuous-time counterpart of (1) defined through the matrices

$$A_c = \begin{bmatrix} 0.6 & -0.5 \\ 0.3 & 1 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

and its Euler discretization $A = \tau A_c + I$, $B = \tau B_c$, for different values of $\tau > 0$. Additionally, we consider the saturation limits $u_{\min} = 1.5$, $u_{\max} = 0.5$.

Figure 1 shows on the left the set $\mathcal{R}$ obtained through the continuous-time results in Braun et al. (2022b) (see in particular (Braun et al., 2022b, Section 5)), and on the right the set $\mathcal{R}$ defined in (19) for the discrete-time dynamics with $\tau = 0.01$. For the computation of $\mathcal{R}$, the parameter $\alpha_c = 0.1$ is used, and α is defined

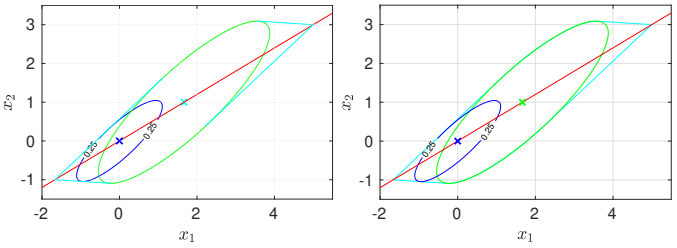

Fig. 1. Comparison of the set $\mathcal{R}$ for the continuous-time setting (left) and the discrete-time setting (right). The discrete-time dynamics are derived from the continuous-time dynamics using an Euler discretization with sampling time $\tau = 0.01$.

according to Remark 1, Equation (15). The two sets are almost indistinguishable. This is also reflected through the matrices

$$P_c = \begin{bmatrix} 0.7399 & -0.6654 \\ -0.6654 & 0.8266 \end{bmatrix}, \quad P = \begin{bmatrix} 0.7476 & -0.6745 \\ -0.6745 & 0.8376 \end{bmatrix},$$

(see Remark 1) defining the two quadratic Lyapunov functions in (Braun et al., 2022a, Proposition 1) and in Proposition 1, respectively. [2]

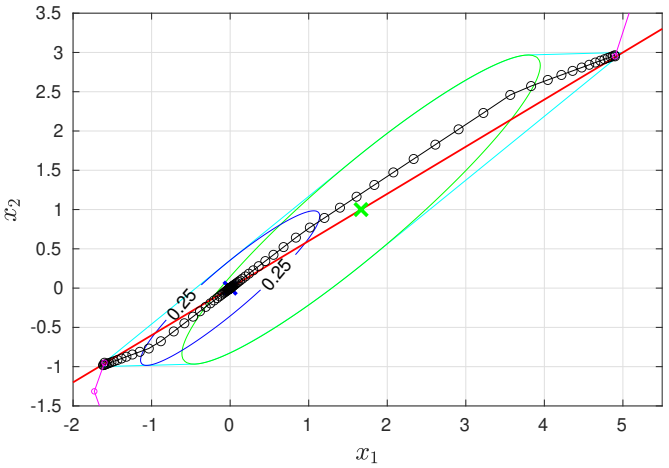

Fig. 2. Visualization of $\mathcal{E}_0(P)$ (blue), the set $\mathcal{R}$ (cyan), the set Γ (red) and two solutions starting in $x_0 = -[1.6, 0.95]^\top$ and $x_0 = -[4.9, 2.95]^\top$ for the discretized system with sampling time $\tau = 1$. The solutions corresponding to the shifted stabilizer (21) are converging to the origin (black) while the solutions corresponding to (5) are diverging (magenta).

For $\tau = 1$, the set $\mathcal{R}$ changes slightly (see Figure 2). Figure 2 additionally contains the closed-loop solutions starting in $x_0 = -[1.6, 0.95]^\top$ and $x_0 = -[4.9, 2.95]^\top$, respectively, using shifted stabilizer (21) (black) and the control law (5) (magenta). The convergence/divergence of the solutions numerically verifies the statements of Theorem 1 and shows the superiority of the control law (21) compared to (5). The solutions corresponding to the initial conditions $x_0 = -[1.6, 0.95]^\top$ and $x_0 = -[4.9, 2.95]^\top$ together with the evolution of the input u and $\delta^\star(x)$ are also shown in Figure 3 for completeness.

5. CONCLUSIONS

In this paper we have shown how the shifting-based controller design for continuous-time linear input-saturated

[2] The corresponding linear matrix inequalities are solved using CVX Grant and Boyd (2014) in Matlab.

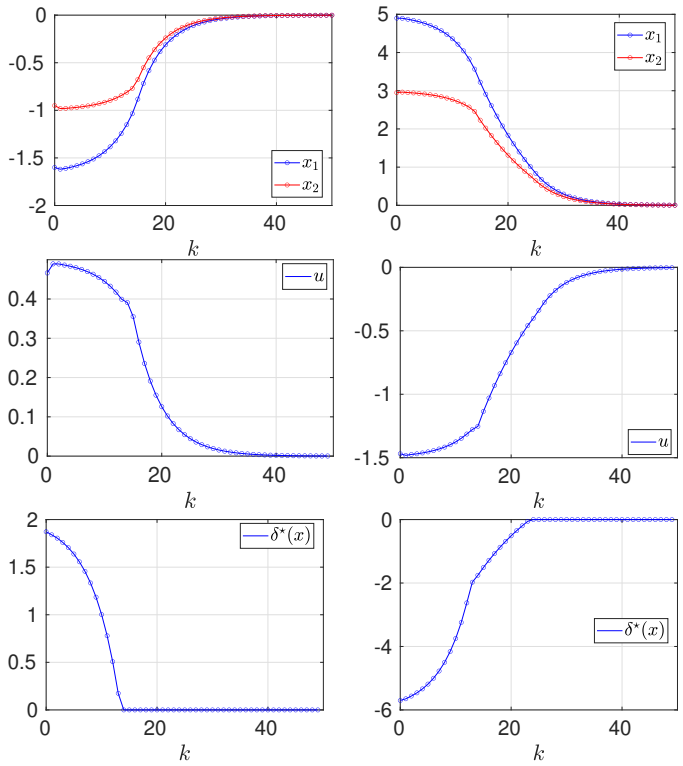

Fig. 3. Closed-loop solution as well as input u and $\delta^\star(x)$ corresponding to the initial conditions $x_0 = -[1.6, 0.95]^\top$ (left) and $x_0 = -[4.9, 2.95]^\top$ (right), and the control law (21).

systems derived in Braun et al. (2022a,b) can equivalently be applied and derived in the discrete-time setting. In particular, we have shown how a locally stabilizing controller can be augmented with an additional state-dependent term to increase the estimate of the region of attraction of the origin of the closed-loop system using the modified controller. In particular, by leveraging on Lyapunov arguments and forward invariance and by gradually shifting the additional term in the controller to the origin, a closed-loop system with enlarged estimate of the region of attraction of the origin has been derived.

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