



HAL
open science

Point Motion in Flat Spaces: An Ample Starting Point

Luciano da Fontoura Costa

► **To cite this version:**

Luciano da Fontoura Costa. Point Motion in Flat Spaces: An Ample Starting Point. 2022. hal-03877552

HAL Id: hal-03877552

<https://hal.science/hal-03877552v1>

Preprint submitted on 29 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Point Motion in Flat Spaces: An Ample Starting Point

Luciano da F. Costa
luciano@ifsc.usp.br

São Carlos Institute of Physics – DFCM/USP

15th Nov. 2022

Abstract

In this work, we approach the interesting problem of representing and studying the position, velocity, acceleration and arc-length of trajectories of points defined in a two-dimensional geometrical space, which has been chosen for being simpler than three-dimensional spaces while retaining much of their structure, and richer than the one-dimensional line. In addition to providing an integrated revision of several aspects from linear algebra, differential calculus, and analytic geometry, the present work can also be understood as a starting point for studies of several areas in physics, especially particle dynamics, but also including electromagnetism, fluid dynamics and statistical physics. The presentation starts with mathematical concepts of spaces, bases and vectors, as well as the Dirac delta ‘function’ and its derivative, as a means for approaching motion discontinuity. The text then proceeds by presenting the concepts of position, motion (or trajectory) of points in terms of parametric curves, also addressing. The concept of arc-length of a trajectory is then presented, followed by a discussion of how point motion can interact with scalar and vector fields (including the concepts of gradient and line integrals). Brief presentations of the the concepts of relative position and speed, as well as moving frames conclude this work. Several examples and illustrations of the covered concepts have also been provided.

“... the intrinsic affinity between mathematics and physics has intrigued humans from at least as early as Pythagora’s time...”

Excerpt from the present work.

1 Introduction

One of the most basic concepts developed by humans while trying to model nature (e.g. [1]) is that of a *point*, corresponding to a minute sphere whose radius can be as small as possible. It is probably not by chance that Euclid’s *Elements of Geometry* (e.g. [2, 3]) starts with the definition of a point, which is simply understood as ‘*having no part*’. Interestingly, Euclid’s geometric understanding of a point was directly related to Democritus (c.460bC – c.370bC) definition of an *atom* as being *indivisible*. The fact that Euclid was active around 300bC suggests that his definition may have been inspired by Democritus’s concept of atom.

Be that as it may, the fact that the point has long been motivated attention from the geometrical and physical perspective is of particular interest to the present work (see Fig. 1), as this concept represents one of the earliest

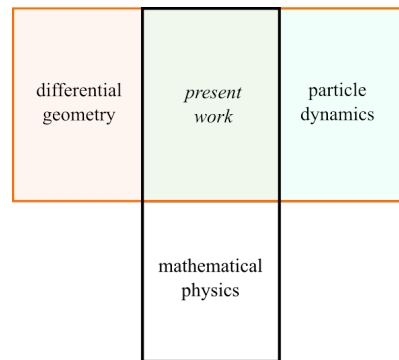


Figure 1: Even though approaching point motion mostly from the mathematic-geometric perspective, the present work can be understood as being at the intersection between *differential geometry* (meaning also the areas of linear algebra, calculus, analytical geometry, and differential equations) and *particle dynamics*.

and most fundamental *links* between *mathematics* and *physics*, two areas which have since then been actively developed and merged into areas such as *mathematical physics* and *applied mathematics*.

The continuing importance of mathematical physics stems from the success of using mathematical concepts

and constructs to model and better explain the physical world. Indeed, the intrinsic affinity between mathematics and physics has intrigued humans from at least as early as Pythagora's time (c.570bC –c.495bC), to the point that it has been suggested that mathematics could be the very language underlying the physical world. Though that possibility could be at least partially accounted for by the fact that mathematics was actually developed mostly to remain compatible with the real-world, its success an effectivity in describing and predicting physical phenomena remains as being particularly remarkable.

Despite the great interest motivated by better understanding the nature of the relationship between mathematics and physics, this captivating discussion shall not constitute the main focus of the present work, as it involves many areas including *logic*, *ontology*, *foundations of mathematics*, as well as *philosophy*. Our objective here is much simpler and specific, consisting of approaching the *motion of a single massless point*, more specifically a *point*, from a purely mathematical/geometrical perspective, i.e. without expressly considering mass (and therefore physical force, work, energy, etc.).

The separation between the mathematical and physical aspects of motion underlying the present work has the two following motivations. First, by being mostly independent of the physical related aspects, the concepts and methods presented here can provide an entry point not only to all areas of physics involving particle motion, but also many other related areas involving point motion, geometry of trajectories and scalar and vector fields. Examples of such areas include, but are not limited to: fluid dynamics, computer graphics, image processing/analysis, optimization, pattern recognition, and even artificial intelligence.

A more mathematically focused approach also allows the subjects of motion and fields to be approached from a broader perspective, since there are not specific physical constraints to be satisfied, especially Newton's laws of motion establishing how particles with mass interact with force fields (gravitational). Therefore, the present work provides an ampler perspective that can be subsequently contrasted with the physical world specificities. The adopted approach also helps realizing that some important physical concepts such as inertial reference frames and conservative fields originate directly from the mathematical point of view, without additional physical requirements.

In addition, though the initial section, addressing vector spaces, develops in \mathbb{R}^3 , the remainder of this work develops exclusively on the \mathbb{R}^2 space. This choice has been taken as a compromise between dealing on the ampler \mathbb{R}^3 space (more equations would have been involved, also involving more challenging visualizations in \mathbb{R}^3) and the

much simpler \mathbb{R} space of the line. At the same time, the concepts and results in \mathbb{R}^2 are almost straightforwardly extensible to higher dimensional metric spaces. It is arguable that generalizing from \mathbb{R}^2 to \mathbb{R}^3 is simpler than from \mathbb{R} into \mathbb{R}^2 , not to mention \mathbb{R}^3 .

Special attention is throughout given to the concepts of *parametric curves* (here understood as specifying the point *motion*), *arc-length*, as well as respective *derivatives*. Though we shall refer to the first and second derivative of a parametric curve also as *instantaneous velocity* and *instantaneous acceleration* respectively, these terms should not be necessarily related to their physical counterparts, reflecting the fact that we shall not be expressly considering mass. The adopted approach is therefore more closely related to the mathematical areas (e.g. [4]) of *linear algebra* (e.g. [5]), *calculus* (e.g. [6, 7, 8, 9]), *analytic geometry* (e.g. [10]), *differential Equations* (e.g. [11, 12]), as well as *differential geometry* (e.g. [13]).

To complement our mathematical study of the properties of the motion of a single massless point, we will also consider how it can be influenced by *scalar* and *vector fields*, as well as how motion can be approached in *relative* terms and from distinct *moving frames* of reference.

Before we proceed with the presentation, it should be observed that it is expected that the reader his preliminary acquainted with basic mathematics (including some background on trigonometry, vectors, and matrices), up to rudiments of analytical geometry and differential calculus level. Some references are also suggested for complementary and subsequent studies.

2 Spaces, Bases, and Vectors

As studied in linear algebra (e.g. []), vector spaces are mathematical spaces whose elements, known as *vectors*, satisfy several conditions, including but not being limited to vector addition closure, associativity, commutativity, as well as existence of inverse. These properties refer to both the respective *vectors* in those spaces, but also to product with scalar values, which obey the distributive property, among others. Vector spaces are much broader than the traditional geometrical vector, also encompassing matrices and functions. The present work will focus on Cartesian spaces, especially \mathbb{R}^2 and \mathbb{R}^3 . These spaces have an additional interesting property that they incorporated the concept of *inner product* $\langle \vec{u}, \vec{v} \rangle$ between any two respective vectors \vec{u} and \vec{v} and respectively induced *distance* between vectors, therefore constituting *geometrical spaces*, or *metric spaces*.

The importance of vector spaces is that they have a respective *structure*, specified by the respective general properties as summarized above, that allows us to handle

vectors in well-posed manners. For instance, this structure ensures that the sum of two vectors will always exist.

Given a vector space, it is possible to obtain one or more respective *basis*, namely a subset of vectors from which all other vectors can be obtained by respective *linear combinations*.

Let B be a basis of some 3D vector space, any vector \vec{u}_B in this space should be uniquely written as the following linear combination of its basis vectors:

$$\vec{u}_B = u_{x,B} \vec{B}_1|_B + u_{y,B} \vec{B}_2|_B + u_{z,B} \vec{B}_3|_B \quad (1)$$

where $u_{x,B}$, $u_{y,B}$, and $u_{z,B}$, are the *coordinates* of the vector \vec{u}_B in basis B and $\vec{B}_i|_B$ are the components of that same basis. We can represent the vector in this basis more compactly as:

$$\vec{u}_B = \begin{bmatrix} u_{x,B} \\ u_{y,B} \\ u_{z,B} \end{bmatrix} = (u_{x,B}, u_{y,B}, u_{z,B}) \quad (2)$$

Because there is an infinite number of possible basis, it is convenient to identify one of them that can be understood as being *canonical* (e.g. simplest, including that the basis elements have magnitude one and are mutually orthogonal). In the case of \mathbb{R}^3 , its canonical basis consists of the three *versors* \vec{i} , \vec{j} , and \vec{k} given as:

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (3)$$

so that:

$$\|\hat{i}\| = \|\hat{j}\| = \|\hat{k}\| = 1 \quad (4)$$

$$\langle \hat{i}, \hat{j} \rangle = \hat{i}^T \hat{j} = 0 \quad (5)$$

$$\langle \hat{i}, \hat{k} \rangle = \hat{i}^T \hat{k} = 0 \quad (6)$$

$$\langle \hat{j}, \hat{k} \rangle = \hat{j}^T \hat{k} = 0 \quad (7)$$

where T stands for the transpose of a vector (or matrix).

Observe that, though there is an infinite number of alternative basis composed by vectors that are mutually orthogonal and have unit magnitude (e.g. any *rotation* of the canonical basis above), they are not usually considered to be canonic. It is interesting to observe that there is no more precise definition of a canonical basis.

Now, any vector \vec{u} of \mathbb{R}^3 can be uniquely expressed in terms of the following *linear combination* of the canonical basis elements:

$$\begin{aligned} \vec{u} &= x \hat{i} + y \hat{j} + z \hat{k} = \\ &= x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \\ &= (x, y, z) \end{aligned} \quad (8)$$

Thus, we can represent the canonical basis in terms of the following matrix:

$$C_C = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \hat{i} & \hat{j} & \hat{k} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \quad (9)$$

where I is the identity matrix.

However, it is also possible to uniquely express the vector \vec{u}_B respectively to the canonical basis, which can be done by translating the basis vectors B represented into the canonical basis, i.e.:

$$\begin{aligned} \vec{u}_C &= u_{x,B} \vec{B}_1|_C + u_{y,B} \vec{B}_2|_C + u_{z,B} \vec{B}_3|_C = \\ &= u_{x,B} \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ b_{3,1} \end{bmatrix} + u_{y,B} \begin{bmatrix} b_{1,2} \\ b_{2,2} \\ b_{3,2} \end{bmatrix} + u_{z,B} \begin{bmatrix} b_{1,3} \\ b_{2,3} \\ b_{3,3} \end{bmatrix} = \\ &= \begin{bmatrix} u_{x,B} b_{1,1} + u_{y,B} b_{1,2} + u_{z,B} b_{1,3} \\ u_{x,B} b_{2,1} + u_{y,B} b_{2,2} + u_{z,B} b_{2,3} \\ u_{x,B} b_{3,1} + u_{y,B} b_{3,2} + u_{z,B} b_{3,3} \end{bmatrix} = \\ &= \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix} \begin{bmatrix} u_{x,B} \\ u_{y,B} \\ u_{z,B} \end{bmatrix} = \\ &\implies \vec{u}_C = B_C \vec{u}_B \end{aligned} \quad (10)$$

where:

$$B_B = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{B}_1|_C & \vec{B}_2|_C & \vec{B}_3|_C \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix} \quad (11)$$

Observe that the matrices representing basis are square. So, we have that:

$$\begin{aligned} \vec{u}_C &= B_C \vec{u}_B \implies \\ \implies \vec{u}_B &= B_C^{-1} \vec{u}_C \end{aligned} \quad (12)$$

where B_C is assumed to be invertible (i.e. $\det(B_C) = |B_C| \neq 0$) in order to allow any vector expressed in the canonical basis to be uniquely translated into basis B (and vice-versa). As there is an infinite number of invertible real-valued matrices, it becomes plain that a same vector space can have an infinite number of respective bases.

Going back to our discussion of basis transformation, let us illustrate Equations 12 and 13 respectively to Figure 2, which assumes the vector space \mathbb{R}^2 . Here, we have a vector \vec{u} represented in terms of the canonical basis C as well as another basis B .

More specifically, we have that:

$$B_C = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \quad \vec{u}_B = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix}$$

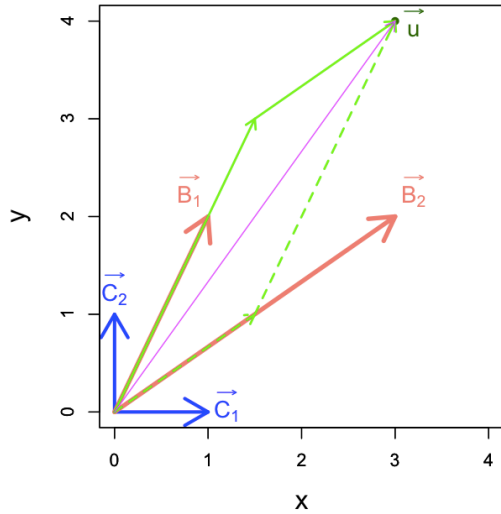


Figure 2: Example of basis transformation (or coordinates exchange). Observe that the plot coordinates refer to the canonical basis C , shown in blue. The green vectors illustrate the linear combination of the coordinates of vector \vec{u} in basis B . It corresponds to the parallelogram law for vector addition after respective scalings of the basis elements specified by the coordinates of \vec{u}_B . A similar construction (not shown) also holds respectively to the canonical basis. The vector in any valid coordinate system is shown in magenta. Please refer to the text for additional information.

The canonical coordinates of the vector \vec{u}_B can be obtained by Equation 13 as:

$$\vec{u}_C = B_C \vec{u}_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

It is interesting to observe that Equations 12 and 13 correspond to *linear transformations* of vectors \vec{u}_B and \vec{u}_C , respectively. Expressed more generally respectively to a vector \vec{x} and a real-valued matrix A , a linear transformation of the elements of \vec{x} can be written as:

$$\vec{y} = A \vec{x} \quad (14)$$

A closely related more general type of transformation consists of *affine transformations* of a vector \vec{u} by a matrix A and a displacement vector \vec{v} :

$$\vec{y} = A \vec{x} + \vec{v} \quad (15)$$

Therefore, linear transforms can be understood as a special case of affine transformations. As affine transformations are known to preserve parallelism and straight lines, so are linear transforms.

Given that nothing specific about the canonical basis has been taken in consideration in the above developments and results, Equations 12 and 13 also hold for any other generic basis D other than the canonical, provided B_D is

invertible. Thus, we can generalize this result as:

$$\vec{u}_D = B_D \vec{u}_B \quad (16)$$

$$\vec{u}_B = B_D^{-1} \vec{u}_D \quad (17)$$

Equations 12 and 13 (as well as Equations 16 and 17) can be immediately modified to translate two (or more) vectors from basis B to basis C in “parallel” fashion:

$$V_B = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{u}_{B,1} & \vec{u}_{B,2} & \vec{u}_{B,3} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \quad (18)$$

$$V_C = B_C V_B \quad (19)$$

In particular, if V_B is the triple of vectors of the basis B expressed respectively to itself, we have:

$$V_B = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{B}_1|_B & \vec{B}_2|_B & \vec{B}_3|_B \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = B_B$$

$$B_C = B_C B_B \implies \quad (20)$$

$$\implies \boxed{B_B = I = C_C} \quad (21)$$

Thus, we have the interesting (and sometimes unnoticed) fact that a generic valid basis B becomes the canonical basis when its vectors are represented respectively to itself. At the same time, the vectors of B represented in terms of the canonical basis (i.e. the columns of B_C), in general, are not guaranteed to be orthogonal or to have unit magnitude.

Now, let us consider the previous vector \vec{u} translated into another basis D of the same vector space. We already know that:

$$\vec{u}_C = D_C \vec{u}_D \implies \quad (22)$$

$$\implies \vec{u}_D = D_C^{-1} \vec{u}_C \quad (23)$$

By combining Equations 23 and 12, it follows that:

$$\begin{aligned} \vec{u}_D &= D_C^{-1} \vec{u}_C = D_C^{-1} B_C \vec{u}_B = \\ &= (D_C^{-1} B_C) \vec{u}_B = W \vec{u}_B \end{aligned} \quad (24)$$

which provides a means for exchanging the coordinates of a vector from basis B to basis D , while matrix W^{-1} specifies the coordinate changes from frame D to B . These results can be summarized as:

$$\boxed{\begin{cases} \vec{u}_D = W \vec{u}_B = D_C^{-1} B_C \vec{u}_B \\ \vec{u}_B = W^{-1} \vec{u}_D = B_C^{-1} D_C \vec{u}_D \end{cases}} \quad (25)$$

These equations remain valid when the canonical basis C is replaced by any valid other basis E .

An important additional aspect of vectors in a vector space concerns their respective *position*. Consider the

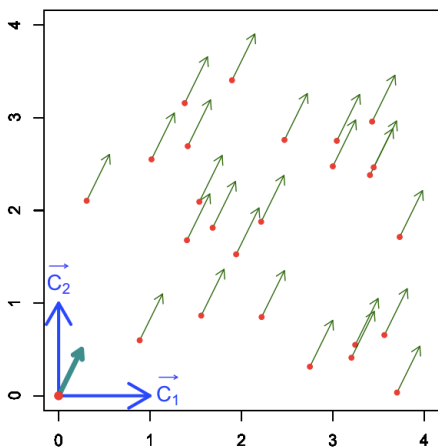


Figure 3: Though all the displayed vectors have the same magnitude and orientation, their respective varying positions cannot be determined from their coordinates.

situation shown in Figure 3. There are at least two possible respective understandings. One of them is that these vectors (except for that at the coordinate system origin) are not actually valid because their positions in the respective space cannot be specified by their coordinates, which only establish position of the arrowed extremity of a vector respectively to the coordinate systems origin $(0,0)$. In this sense, the only vector valid in Figure 3 would be that starting at $(0,0)$.

Another possible is that the vectors represent *relative displacement*, which allows them to be shifted in the respective space. However, this requires the definition an *additional property* of each vector, specifying its *point of application*, which can be done by associating an additional vector expressing the coordinates (x_p, y_p, z_p) of the non-arrowed extremity of the vector.

There is an additional interesting issue regarding vector spaces, and it concerns *specifying the position of the reference coordinate system* or *frame*. Typically, this choice is performed respectively to some useful purpose. For instance, the coordinate system in Figure 3 is placed at a specific coordinate relatively to the respective so that is reasonably close to where it is discussed (i.e. *the present paragraph*). The figure remains at its absolute position until the page is moved (e.g. scrolled), while its position relative to the page is maintained.

In several situations, as the just mentioned example, the position of the coordinate system may vary with time, which constitutes yet another interesting issue that will be briefly approached in Section 11.

To conclude this section, we observe that the *dot product* (also *inner product* or *scalar product*) between any two vectors \vec{u} and \vec{v} in a vector space provided with this operation (i.e. a *Hilbert space*) can be expressed respec-

tively to a same basis as:

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= \langle \vec{v}, \vec{u} \rangle = x v_x + y v_y + z v_z = \\ &= \|\vec{u}\| \|\vec{v}\| \cos \theta \end{aligned} \quad (26)$$

where θ is the smallest angle between the two vectors. Observe that, in particular:

$$\langle \vec{u}, \vec{u} \rangle = x^2 + y^2 + z^2 \implies \|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} \quad (27)$$

where the *norm* $\|\vec{u}\|$ of \vec{u} allows us to define the *distance* between two points specified by respective vectors \vec{u} and \vec{v} as:

$$d(\vec{u}, \vec{v}) = \|\vec{v} - \vec{u}\| \quad (28)$$

3 The Dirac Delta “Function” and its Derivative

Strictly speaking, the Dirac delta “function” is not a function, but a distribution (e.g. [14]). However, this function (for simplicity, the quotation marks will not be indicated henceforth) can be approximated in terms of the following limit of the normal function (statistical distribution, with unit area):

$$\delta(t) = \lim_{\sigma \rightarrow 0} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}(\frac{t}{\sigma})^2} \quad (29)$$

we then necessarily have that:

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (30)$$

Figure 4(b) illustrates the function $c \delta(t - t_0)$.

As will be informally discussed in the present section, the Dirac delta function provides a convenient way to represent discontinuities. We start by integrating the more general Dirac delta function $c \delta(t - t_0)$ for $t \in (-\infty, T)$, with $a \in \mathbb{R}$:

$$g(t) = \int_{-\infty}^t c \delta(u - t_0) du = \begin{cases} 0 & t < t_0 \\ c & t \geq t_0 \end{cases} \quad (31)$$

The constant c in $c \delta(t)$ will be henceforth referred to as the *amplitude* or *area* of the Dirac delta function $\delta(t)$.

The result of the above integration is illustrated in Figure 4. Observe that, as could be expected, the Dirac delta has implied a discontinuity at t_0 with amplitude determined by the Dirac delta intensity (or area) c .

Interestingly, it is possible to express the first “derivative” of the Dirac delta function can be expressed as:

$$\Pi(t) = \frac{d \delta(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\delta(t) - \delta(t + \Delta t)}{\Delta t} \quad (32)$$

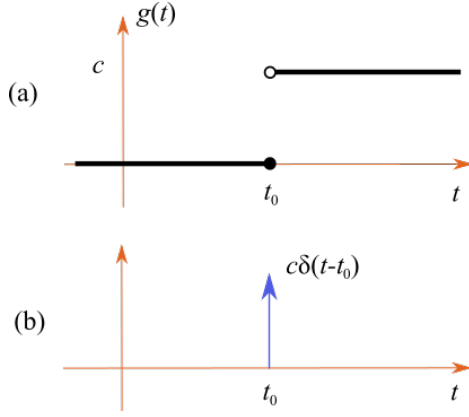


Figure 4: The integral function $g(t)$ of the Dirac delta function with intensity (or area) c at position t_0 , namely $g(t) = \int_0^t a\delta(t-t_0) dt$, consists of a discontinuous function characterized by being zero before t_0 , and becoming the constant function 1 afterwards. This property often provides a convenient means for expressing discontinuous functions.

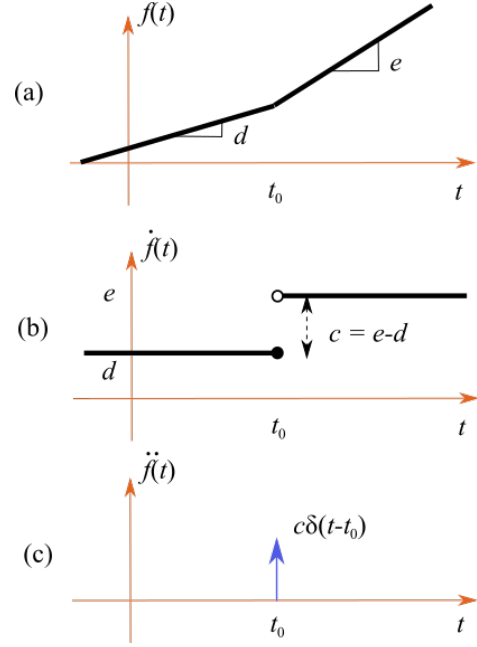


Figure 6: Illustration of how the first and second derivatives of a function $f(t)$ with a discontinuity at t_0 can be understood in terms of the Dirac delta function. As with derivatives in general, the constant d cannot be directly recovered from $\dot{f}(t)$.

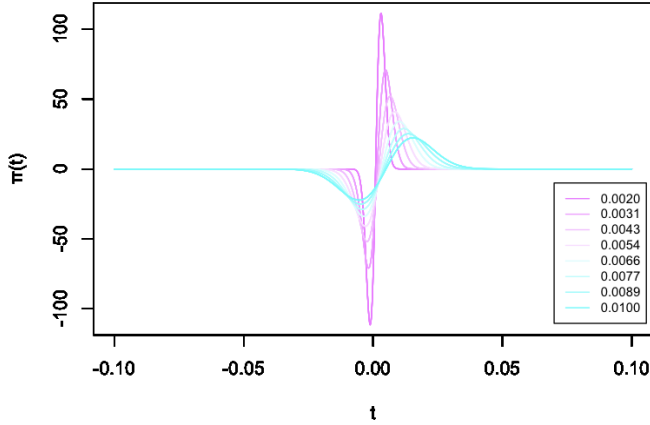


Figure 5: Illustration of the derivative of the Dirac delta, namely $\Pi(t)$, can be understood as the limit of a difference of normal distributions. The values of Δt are shown in the inset. Observe that any of the functions in this plot have area equal to zero.

with the understanding that:

$$\int \Pi(t-t_0) dt = \delta(t-t_0) + c \quad (33)$$

where c is a constant implied by the indefinite integral.

The “function” $\Pi(t)$ can be expressed in terms of the limit of the normal distribution as follows:

$$\Pi(t) = \frac{d\delta(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\delta(t+\Delta t) - \delta(t)}{\Delta t} \quad (34)$$

Therefore, $\Pi(t)$ has null area. Figure 5 illustrates the function $\Pi(t)$ as the limit of the difference of normal distributions as specified in Equation 32.

The Dirac delta provides a convenient resource for understanding discontinuities not of a function itself, but of

its first (or higher) derivatives. Consider the situation depicted in Figure 6(a), consisting of a function $f(t)$ with a first derivative discontinuity at $t = t_0$, shown in Figure 6(b). The slope of the first linear part of $f(t)$ is d , becoming e after t_0 . The difference of slopes therefore is $c = e - d$. The first derivative of $f(x)$, namely $g(t) = \dot{f}(t)$, is shown in Figure 6(b), while the second derivative of $f(t)$ is presented in Figure 6(c), corresponding to a single Dirac delta with amplitude c at t_0 .

Let us now consider a function that is continuous and has at least its two first derivatives also being continuous, such as is the case with $h(t) = 0.5t^2 + b$. In case we want to incorporate a discontinuity with amplitude c at t_0 , we can simply write:

$$\begin{aligned} h(t) &= \int_0^t g(u) + c\delta(u-t_0) du + b = \\ h(t) &= \int_0^t u + c\delta(u-t_0) du + b = \\ &= \int_0^t \left[\int_0^u 1 + c\Pi(v-t_0) dv \right] du + b \end{aligned} \quad (35)$$

where $g(t) = \frac{dh(t)}{dt} = t$. Observe that b has been considered externally to the integral representations.

Figure 7 illustrates the above integral representations of the discontinuity of $h(t)$ in terms of an integral involving the Dirac delta function, or a double integral incorporating the $\Pi(t)$ function, which is represented in the present work as a double arrow.

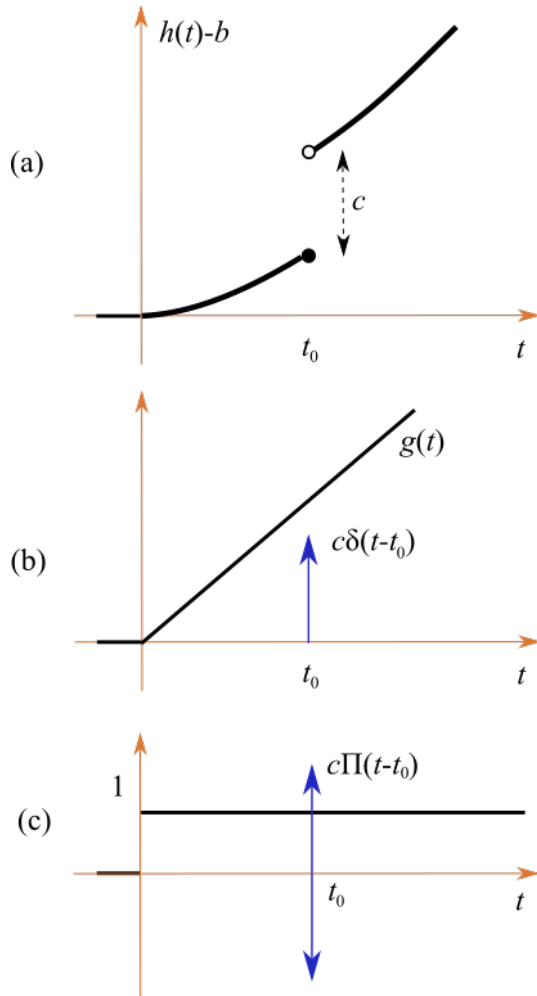


Figure 7: The parabola $h(t) = 0.5t^2 + b$ with a discontinuity of amplitude c at $t = t_0$ can be expressed as the integral (with null initial condition) of the identity function $u(t) = t$ plus one Dirac delta with amplitude c at t_0 , or as the double integral (also with null initial condition) of the constant function 1 plus the Pi function with amplitude c at that same position. Therefore, this figure also can also be understood as illustrating the first two derivatives of the function in (a).

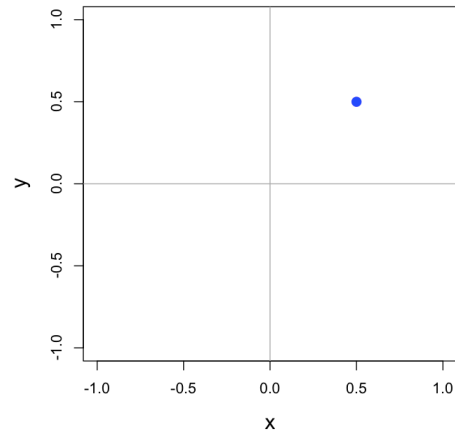


Figure 8: The position of a point in \mathbb{R}^2 can be specified in terms of its Cartesian coordinates (x, y) , which can be associated to a position vector $\vec{u} = [x \ y]^T$ represented in the canonical basis. More specifically, in this case we have $(x, y) = (0.5, 0.5)$.

4 Position, Parametric Curves, and Motion

We start our approach considering a single, static infinitesimal point remaining in an arbitrary position in \mathbb{R}^2 , as illustrated in Figure 8, respective to the Cartesian (orthogonal) coordinate system $x \times y$. The natural manner to mathematically specify the position of the point is in terms of its coordinates (x, y) , which can also be associated to a respective position (column) vector as follows:

$$\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} = (x, y) \quad (36)$$

Now, consider that the position of the point may change along time t . Mathematically, this can be expressed in terms of the following mapping Γ from a free variable t in real interval $[a, b]$ into the space where the point is being contained, in our case \mathbb{R}^2 :

$$\Gamma : t \in [a, b] \subset \mathbb{R} \longrightarrow \vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \in \mathbb{R}^2 \quad (37)$$

with $a, b \in \mathbb{R}$.

For simplicity's sake, we shall henceforth consider that the position of a point in \mathbb{R}^2 along the variable t as specified by the mapping Γ . Therefore, the sequence of points or vectors $(x(t), y(t))$ can be understood as the *motion*, *displacement* or *trajectory* of the point, while the parameter t corresponds to 'time'. Curves as Γ are often referred to as being *parametric*.

However, from the mathematical point of view the variable t has not physical interpretation other than being a *parameter*. The set of points $(x(t), y(t))$ defined by point

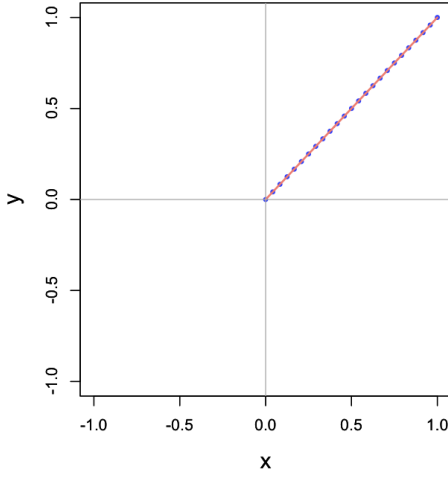


Figure 9: The *motion* or *trajectory*, shown in salmon, specified by Eq. 38 considering $t \in [0, 1]$. The blue points, sampled at $\Delta t = 2$ intervals along the trajectory, have been included in order to provide more information about the motion which, in this case, is *linear uniform*, the constant speed magnitude being equal to $\sqrt{v_x^2 + v_y^2}$.

through the mapping Γ and respective motion is henceforth associated to the *motion* of the point according to the time parameter t .

An important point to bear in mind is that t varies continuously in the real interval $[a, b]$, going from a to b , except at both of its extremities a and b , where the mapping is discontinuous. Thus, the parameter t intrinsically imposes a *sequence* or *order* along the single point positions $\vec{u}(t) = (x(t), y(t))$.

In principle, parameters can be transformed in any continuous manner, e.g. by introducing a respective relationship $r = g(t)$, such as $r = 2t + 3$. Then, it becomes possible to refer to the *reversion* of the parameter, namely considering the transformed parameter $q = -r$ instead of r .

Let us consider a type of motion, defined as:

$$\vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 + v_x t \\ y_0 + v_y t \end{bmatrix} \quad (38)$$

with $x_0, y_0 \in \mathbb{R}$, $v_x, v_y \in \mathbb{R}^+$.

Figure 9 illustrates this trajectory for $x_0 = 0$, $y_0 = 0$, $v_x = 1$ and $v_y = 1$ considering $t \in [0, 1]$. The points in blue have been sampled at $\Delta t = 2$ intervals along the trajectory.

For $t = 0$, we have that the point is at position $(0, 0)$. As t increases, the point initiates a straight displacement along the orientation specified by vector $\vec{v} = (1, 1)$, up to the other extremity $\vec{u}(t = 1) = (1, 1)$. The type of motion expressed by Equation 38 is typically called *linear*.

Let us now consider the following modified version of

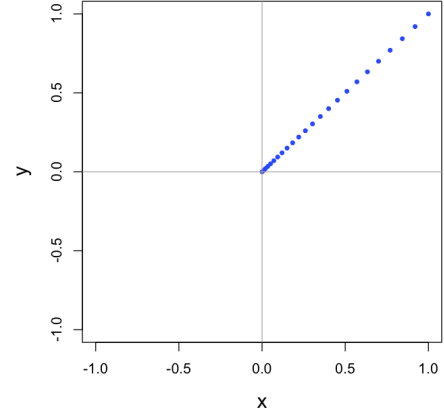


Figure 10: Example of a uniformly accelerated linear motion, with the points being equally sampled along time. Observe that progressively larger displacements (increasing speed) implied by the acceleration.

the previous type of motion:

$$\vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 + a t^2 \\ y_0 + b t^2 \end{bmatrix} \quad (39)$$

with $x_0, y_0 \in \mathbb{R}$, $v_x, v_y \in \mathbb{R}^+$.

Figure 10 illustrates this type of motion considering $x_0 = y_0 = 0$, $a = b = 1$, and $\Delta t = 0.04$. Only the sampled points in blue are shown for the sake of enhanced visualization. Though the respectively defined set of points is identical to those obtained in the previous example, we do not necessarily have *pointwise* identity: except for $t = 0$ in the case of the present example, $(x(t), y(t)) \neq (\tilde{x}(t), \tilde{y}(t))$ for each time instant t (in the particular case of the current example, pointwise identity is verified only at both extremities of the trajectory).

Also of interest in the trajectory in Figure 10 is the fact that the distance between the points along the trajectory, and therefore along time, successively grow, implying the points to be further apart. As it will be further discussed in Sections 5 and 6, this means that the velocity of the motion is not constant, implying in respective acceleration.

However, for any given $t \in [0, 1]$, it is possible to establish a bijective correspondence between the two trajectories by solving the system:

$$\begin{cases} x(t) = \tilde{x}(t) \implies x_0 + v_x t = x_0 + a t^2 \implies v_x = a t \\ y(t) = \tilde{y}(t) \implies y_0 + v_y t = y_0 + b t^2 \implies v_y = b t \end{cases} \quad (40)$$

Another possible type of movement, known as *circular*, can be specified as:

$$\vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \rho \cos(\omega t) + x_0 \\ \rho \sin(\omega t) + y_0 \end{bmatrix} \quad (41)$$

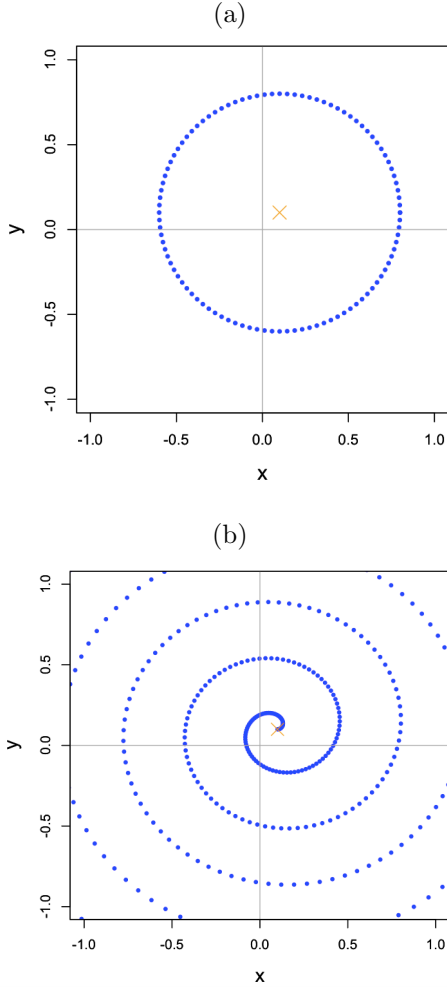


Figure 11: (a): The *circular* trajectory obtained respectively to Eq. 41 assuming $x_0 = y_0 = 0.1$ and $\rho = 0.7$. The blue points are equally spaced with $\Delta t = 0.005$. Observe that this trajectory corresponds to a circle with radius ρ centered at the point (x_0, y_0) , shown as a red 'x'. As its angular speed $\omega = 2\pi f_0$ is constant, this motion is often referred to as being *circular uniform*. (b): A spiral trajectory as described in Eq. eq:spiral, adopting $x_0 = y_0 = 0.1$, $f_0 = 2$, and $\rho = 0.7$. The blue points are sampled at equal intervals $\Delta t = 0.006$.

for $t \in \mathbb{R}$, with $x_0, y_0, f_0 \in \mathbb{R}$ and $\omega = 2\pi f_0$, where $f_0 = 2$ is the frequency of the motion. This type of trajectory yields an orbit of radius ρ centered at the position (x_0, y_0) , as illustrated in Figure 11(a) for $x_0 = y_0 = 0.1$ and $\rho = 0.7$. The blue points have been sampled along the trajectory at intervals $\Delta t = 0.005$.

In the case of the above type of uniform circular motion, it is possible to link the angular and the magnitude of the Cartesian speed \vec{v} , which is known as scalar speed v_s , which in the uniform case is also a constant, as:

$$v_s = \omega \rho \quad (42)$$

The orientation of \vec{v} always points toward the center of the defined circle.

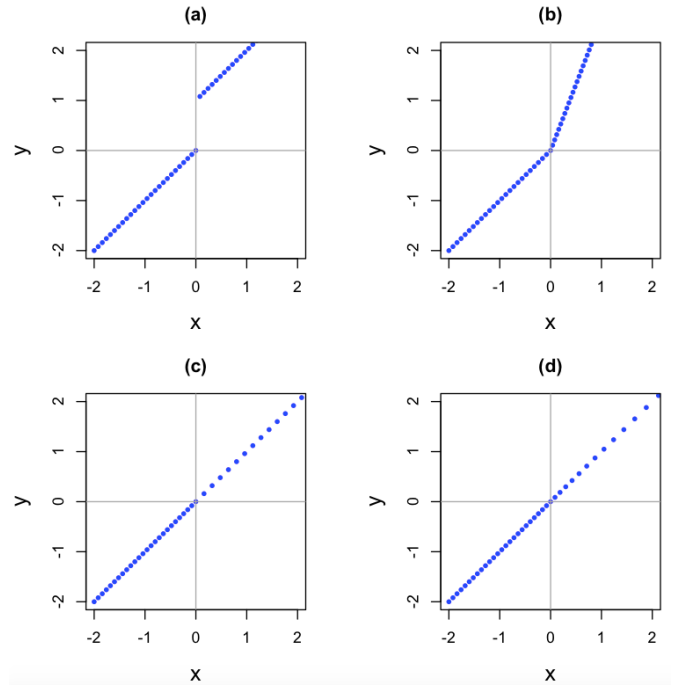


Figure 12: Four types of linear motions with discontinuities at $t = 0$. The case shown in (a) has a position discontinuity; the situation in (b) has speed orientation discontinuity; the motion in (c) is characterized speed magnitude discontinuity, while (b) concerns acceleration discontinuity.

Yet another type of motion, closely related to the circular case, can be specified as:

$$\vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \rho t \cos(\omega t) + x_0 \\ \rho t \sin(\omega t) + y_0 \end{bmatrix} \quad (43)$$

with $x_c, y_c, f_0 \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. We shall refer to this type of motion, illustrated in Figure 11(b), as *spiral*. Interestingly, the velocity along this motion varies not only in terms of its orientation, but its magnitude also progressively increases along time. Unlike the uniform circular motion, the velocity is not orthogonal to the acceleration in this type of spiral motion.

Now, consider the following type of motion, assuming $t \in [-1, 1]$:

$$(x(t), y(t)) = \begin{cases} (t, t) & t < 0 \\ (t, t+1) & t \geq 0 \end{cases} \quad (44)$$

Figure 12(a) depicts the trajectory defined by this type of motion.

The previous example related to a position discontinuity, but there are other possible types of discontinuities. For instance:

$$(x(t), y(t)) = \begin{cases} (t, t) & t < 0 \\ (0.5, \sqrt{2-0.5^2}) & t \geq 0 \end{cases} \quad (45)$$

is characterized by orientation discontinuity (see Fig. 12b).

Let us now consider the motion:

$$(x(t), y(t)) = \begin{cases} (t, t) & t < 0 \\ (2t, 2t) & t \geq 0 \end{cases} \quad (46)$$

which is characterized by a discontinuity at the first derivative of the position at $t = 0$. Yet another type of discontinuity is as follows:

$$(x(t), y(t)) = \begin{cases} (t, t) & t < 0 \\ (t^2 + t, t^2 + t) & t \geq 0 \end{cases} \quad (47)$$

in which case the second derivative of the position is discontinuous at $t = 0$. This sequence of examples can extend to discontinuities at any successive derivative orders. These two types of motion are illustrated in Figures 12(b) and (c), respectively.

A completely discontinuous displacement can be specified as:

$$(x(t), y(t)) = (u(t), v(t)) \quad (48)$$

where both $u(t)$ and $v(t)$ are random values uniformly distributed in the interval $[0, 1]$, drawn at each time instant t . Though the respectively obtained trajectory is extremely dense and discontinuous, to a point that it cannot even be graphically discerned, it can be shown to cover the whole square $(0 \leq x \leq 1, 0 \leq y \leq 1) \subset \mathbb{R}^2$. One manner to start approaching this mapping is by considering that all $[a, b]$, \mathbb{R} and \mathbb{R}^2 have *Cantor cardinality* equal to $2^{\aleph_0} = \aleph_1$, therefore having the ‘‘same number of points’’.

5 First Derivative of Position

Given a motion specified parametrically, it is possible to consider its *first derivative*, in case it exists, which is also called *instantaneous velocity* or *speed*, which can be expressed in many ways:

$$\begin{aligned} \vec{v}(t) = \dot{\vec{u}}(t) &= \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt} \right) = \\ &= (\dot{x}(t), \dot{y}(t)) = (x'(t), y'(t)) = (v_x(t), v_y(t)) = \end{aligned} \quad (49)$$

Observe that $\vec{v}(t)$ is a *vector* quantity. The following notation will be predominantly adopted in the present work:

$$\boxed{\vec{v}(t) = \dot{\vec{u}}(t) = \begin{bmatrix} v_x(t) \\ v_y(t) \end{bmatrix} = \begin{bmatrix} \dot{u}_x(t) \\ \dot{u}_y(t) \end{bmatrix}} \quad (50)$$

As an example, let us calculate the speeds of the examples of motion provided in Section 4. Let us start by the speeds corresponding to the previously considered *continuous* motions, respectively to Equations 38, 39, 41,

and 43:

$$\vec{v}(t) = (v_x, v_y) \quad (51)$$

$$\vec{v}(t) = (2t v_x, 2t v_y) \quad (52)$$

$$\vec{v}(t) = \rho \omega (-\sin(\omega t), \cos(\omega t)) \quad (53)$$

$$\vec{v}(t) = \rho (\cos(\omega t) - \omega t \sin(\omega t), \sin(\omega t) + \omega t \cos(\omega t)) \quad (54)$$

In both Equations 51 and 52, the obtained velocity is not a function of time, being therefore *constant*. As these motions are also linear, we have a *uniform linear motion*.

It is also interesting to consider the circular motion speed obtained in Equation 53 after transforming the Cartesian coordinates $(x(t), y(t))$ into respective polar coordinates. For simplicity’s sake, we assume that $x_0 = y_0 = 0$, which leads to:

$$\begin{cases} \rho = \sqrt{x(t)^2 + y(t)^2} = \rho \\ \theta = \arctan\left(\frac{y(t)}{x(t)}\right) = \omega t \end{cases} \quad (55)$$

from which we can obtain the respective *angular velocity*:

$$\dot{\theta} = \omega \quad (56)$$

which turns out to be constant. Therefore, the motion specified by Equation 41 can be called *circular with constant angular velocity*, or simply *uniform circular motion*, being illustrated in Figure 13(a). It can be observed that the speed vectors, shown in red in that figure, are always tangent to the vectors pointing toward the respective center of the circle defined by this specific type of motion, while their respective magnitude and angular speed remain both constant.

The speed of the spiral motion, given in Equation 43, are depicted in Figure 13(b). Interestingly, as the parameter t increases, both the magnitudes of the speed and acceleration also increase.

To conclude this section, we now provide the speed of the *discontinuous* motions specified by Equations 44, 45, 46, and 47:

$$\vec{v}(t) = \begin{cases} (1, 1) & t \neq 0 \\ (0, \delta(t)) & t = 0 \end{cases} \quad (57)$$

$$\vec{v}(t) = \begin{cases} (1, 1) & t < 0 \\ (0.5, \sqrt{2 - 0.5^2}) & t \geq 0 \end{cases} \quad (58)$$

$$\vec{v}(t) = \begin{cases} (1, 1) & t < 0 \\ (2, 2) & t \geq 0 \end{cases} \quad (59)$$

$$\vec{v}(t) = \begin{cases} (t, t) & t < 0 \\ (2t + 1, 2t + 1) & t \geq 0 \end{cases} \quad (60)$$

6 Second Derivative of Position

The second derivative of the position of a motion $\vec{u}(t)$ can be expressed as:

$$\begin{aligned}\vec{a}(t) &= \dot{\vec{v}}(t) = \ddot{\vec{u}}(t) = \left(\frac{d^2x(t)}{dt^2}, \frac{d^2y(t)}{dt^2} \right) = \\ &= (\ddot{x}(t), \ddot{y}(t)) = (x''(t), y''(t))\end{aligned}\quad (61)$$

as well as:

$$\vec{a}(t) = \begin{bmatrix} a_x(t) \\ a_y(t) \end{bmatrix} = \begin{bmatrix} \dot{v}_x(t) \\ \dot{v}_y(t) \end{bmatrix} = \begin{bmatrix} \ddot{u}_x(t) \\ \ddot{u}_y(t) \end{bmatrix}\quad (62)$$

The accelerations obtained for the motions defined by Equations 38, 39, 41, and 43 are presented respectively as follows:

$$\vec{a}(t) = (v_x, v_y)\quad (63)$$

$$\vec{a}(t) = (2t v_x, 2t v_y)\quad (64)$$

$$\vec{a}(t) = \rho \omega^2 (-\cos(\omega t), -\sin(\omega t))\quad (65)$$

$$\vec{a}(t) = \rho \omega (a_x(t), a_y(t))\quad (66)$$

$$\text{where: } \begin{cases} a_x(t) = -2 \sin(\omega t) - \omega t \cos(\omega t) \\ a_y(t) = 2 \cos(\omega t) + \omega t \sin(\omega t) \end{cases}$$

In the case of the uniform circular motion, we can learn more about the relationship between the respective speed and acceleration vectors by performing their respective inner product:

$$\begin{aligned}\langle (v_x(t), v_y(t)), (a_x(t), a_y(t)) \rangle &= \\ &= \langle (\dot{x}(t), \dot{y}(t)), (\ddot{x}(t), \ddot{y}(t)) \rangle = \\ &= \rho^2 \omega^3 [\cos(\omega t) \sin(\omega t) - \sin(\omega t) \cos(\omega t)] = 0\end{aligned}\quad (67)$$

thus indicating that circular motion with constant angular speed is characterized by having the velocity vector being always orthogonal to the acceleration vector for any value of t , which can be observed in Figure 13.

The second derivatives of the motions in Equations 44, 45, 46, and 47 can be obtained as:

$$\vec{v}(t) = \begin{cases} (0, 0) & t \neq 0 \\ (0, \Pi(t)) & t = 0 \end{cases}\quad (68)$$

$$\vec{v}(t) = \begin{cases} (0, 0) & t < 0 \\ (-0.5\delta(t), (\sqrt{2 - 0.5^2}) \delta(t)) & t = 0 \\ (0, 0) & t \geq 0 \end{cases}\quad (69)$$

$$\vec{v}(t) = \begin{cases} (0, 0) & t < 0 \\ (\delta(t), \delta(t)) & t = 0 \\ (0, 0) & t > 0 \end{cases}\quad (70)$$

$$\vec{v}(t) = \begin{cases} (1, 1) & t < 0 \\ (2, 2) & t \geq 0 \end{cases}\quad (71)$$

Though we shall constrain our more detailed study of point motion to its first and second derivative, higher derivatives can be obtained and analyzed in similar manners.

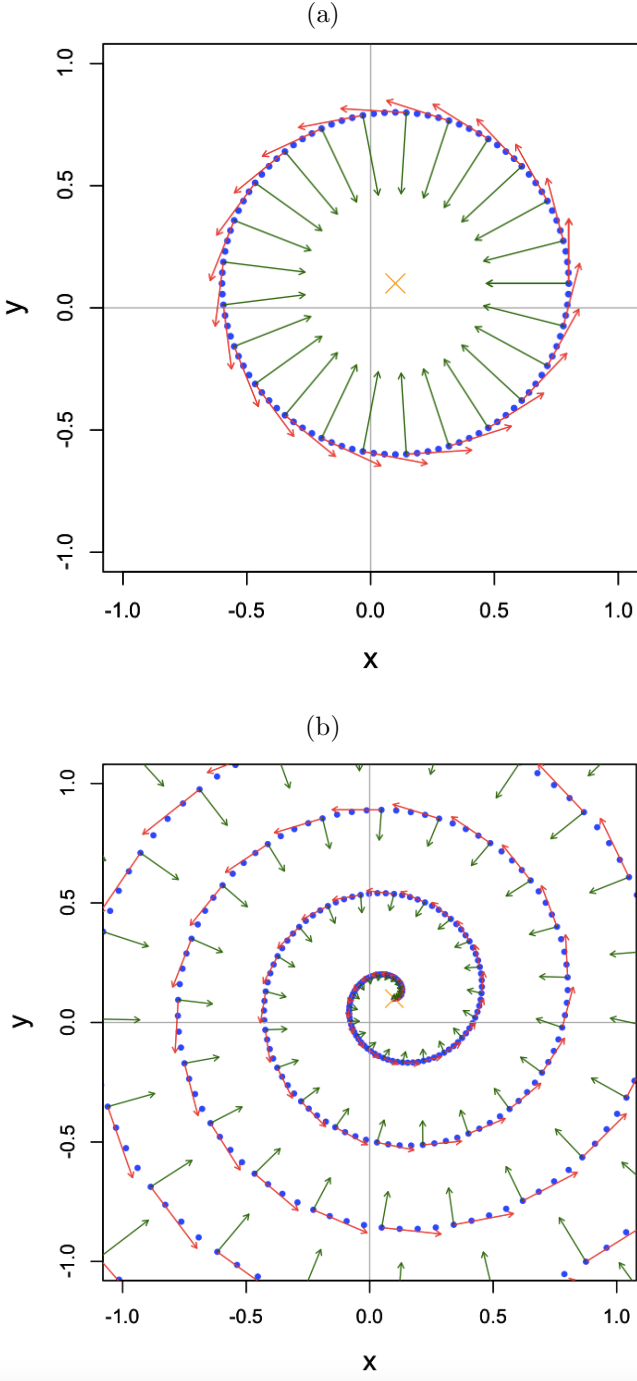


Figure 13: (a): The velocity (red) and acceleration (dark green) fields associated to a uniform circular motion with radius $\rho = 0.7$ and center $(x_0, y_0) = (0.1, 0.1)$. (b): The velocity (red) and acceleration (green) vectors respective to spiral motion obtained by using Eq. 54, with $(x_0, y_0) = (0.1, 0.1)$. In both cases, the magnitudes of the velocity and acceleration vectors are not shown to the original scale for improved visualization. It can be verified that the velocity is not orthogonal to the acceleration in this type of motion.

7 Arc-Length of Motion

Given a motion $\vec{u}(t)$ starting at t_0 , with respective speed $\vec{v}(t) = \dot{\vec{x}}(t)$, it is often interesting to know the total *arc-length* of its trajectory from the initial time t_0 until a subsequent time t , which can be formally defined as:

$$s(t) = \int_{t_0}^t \|d\vec{u}(r)\| dr = \int_{t_0}^t \|\dot{\vec{u}}(r)\| dr \quad (72)$$

where:

$$\|d\vec{u}(t)\| = \|\dot{\vec{u}}(t)\| = \sqrt{[v_x(t)]^2 + [v_y(t)]^2} \quad (73)$$

Conceptually, the arc-length between two distinct points of a trajectory can be understood as representing the trajectory as a thread, cutting it at those two points, extending it, and measuring the respective *length*.

As an example, let us calculate the arc-length of the trajectory defined by Equation 38:

$$\begin{cases} x(t) = x_0 + v_x t \\ y(t) = y_0 + v_y t \end{cases} \quad (74)$$

We thus have that:

$$\begin{cases} \frac{dx(t)}{dt} = \dot{x}(t) = v_x \\ \frac{dy(t)}{dt} = \dot{y}(t) = v_y \end{cases} \quad (75)$$

Assuming that $t_0 = 0$, we have:

$$\begin{aligned} s(t) &= \int_0^t \|d\vec{u}(v)\| dt = \int_0^t \sqrt{(v_x)^2 + (v_y)^2} dv = \\ &= \int_0^t c dv = ct \end{aligned} \quad (76)$$

with $c = \sqrt{(v_x)^2 + (v_y)^2}$.

Once we obtain the expression of the arc-length of a motion $\vec{u}(t)$ as a function of the free parameter t , i.e. $s(t)$, we obtain t as a function of s , i.e.:

$$t = s^{-1}(s) = h(s) \quad (77)$$

By replacing t by $h(s)$ in the original parametric motion $\vec{u}(t)$, we get $\vec{u}(s)$, which therefore is parametrized by the arc-length s . This type of parametric curve will be necessarily characterized by:

$$\|\dot{\vec{u}}(s)\| = \|\vec{v}(s)\| = 1 \quad (78)$$

In other words, the curve $\vec{u}(s)$ will have constant speed magnitude equal to 1, though the speed vector can undergo orientation changes.

Let us illustrate the above procedure respectively to the case in Equation 38. We already know that the arc-length of this motion is given by Equation 76, so that:

$$s(t) = ct \implies t = \frac{1}{c} s = h(s) \quad (79)$$

with:

$$c = \sqrt{(v_x)^2 + (v_y)^2}. \quad (80)$$

Substituting this expression into Equation 38, we obtain:

$$\begin{aligned} \vec{u}(s) &= (x(s), y(s)) = \frac{1}{c} (v_x s + x_0, v_y s + y_0) \implies \\ \implies \dot{\vec{u}}(s) &= (\dot{x}(s), \dot{y}(s)) = \frac{1}{c} (v_x, v_y) \end{aligned}$$

From which it follows that:

$$\|\dot{\vec{u}}(s)\| = \frac{\sqrt{(v_x)^2 + (v_y)^2}}{\sqrt{(v_x)^2 + (v_y)^2}} = 1$$

Now, let us proceed to the circular motion in Equation 41. Assuming that $t_0 = 0$, we have:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{[-\rho\omega \sin(\omega t)]^2 + [\rho\omega \cos(\omega t)]^2} dt = \\ &= \rho\omega \int_0^t \left([\sin(\omega t)]^2 + [\cos(\omega t)]^2 \right) dt = \rho\omega t \end{aligned} \quad (81)$$

Recalling that $\omega = 2\pi f_0$, if we take $t = 1/f_0 = T_0$, corresponding to a complete period of this motion, the arc-length results equal to $s(T_0) = \rho\omega T_0 = \rho 2\pi f_0 T_0 = 2\pi\rho$, which indeed coincides with the perimeter of a circle with radius ρ .

8 Probing Fields

A *scalar field* $\psi(x, y)$ in \mathbb{R}^2 can be understood as the mapping that associates a *scalar* value to each point (x, y) of \mathbb{R}^2 , i.e.:

$$\boxed{\psi(x, y) : (x, y) \in \mathbb{R}^2 \rightarrow \psi(x, y) = c \in \mathbb{R}} \quad (82)$$

A simple example of scalar field is:

$$\psi(x, y) = axy, \quad a \in \mathbb{R} \quad (83)$$

A *vector field* is a mapping that associates a *vector* to each point (x, y) of \mathbb{R}^2 , i.e.:

$$\boxed{\vec{\phi}(x, y) : (x, y) \in \mathbb{R}^2 \rightarrow \vec{\phi}(x, y) = \begin{bmatrix} \phi_x(x, y) \\ \phi_y(x, y) \end{bmatrix} \in \mathbb{R}^2} \quad (84)$$

The *gradient* of a scalar field on \mathbb{R}^2 is defined as:

$$\boxed{\vec{\nabla}\psi(x, y) = \frac{\partial\psi(x, y)}{\partial x} \vec{i} + \frac{\partial\psi(x, y)}{\partial y} \vec{j}} \quad (85)$$

from which we conclude that the gradient defines a vector field on \mathbb{R}^2 .

The gradient of a scalar field $\psi(x, y)$ at any point (x, y) can be understood as pointing to the direction of maximum variation of that field at that point.

As a simple example of a scalar field, we have:

$$\psi(x, y) = a x y \quad (86)$$

which induces the following respective gradient vector field:

$$\vec{\nabla}\psi(x, y) = a y \hat{i} + a x \hat{j} \quad (87)$$

Given a scalar field $\psi(x, y)$, and a generic small (differential) displacement vector $d\vec{u}$ at point (x, y) :

$$d\vec{u} = dx \vec{i} + dy \vec{j} \quad (88)$$

the *total derivative* of $\psi(x, y)$ implied by $d\vec{u}$ can be expressed as follows:

$$\begin{aligned} d\psi(x, y)|_{d\vec{u}} &= \langle \psi(x, y), d\vec{u} \rangle = \vec{\nabla}\psi(x, y) \cdot d\vec{u} = \\ &= \frac{\partial\psi(x, y)}{\partial x} dx + \frac{\partial\psi(x, y)}{\partial y} dy \end{aligned} \quad (89)$$

As an example, consider the small displacement vector placed at $(x = 1, y = 2)$:

$$d\vec{u} = 0.02 \vec{i} - 0.003 \vec{j} \quad (90)$$

and the scalar field defined as:

$$\psi(x, y) = 5 x y \quad (91)$$

We have from Equation 89 that the total variation of the scalar field associated to the displacement $d\vec{u}$ is:

$$\begin{aligned} d\psi(x = 1, y = 2)|_{d\vec{u}} &= \\ &= 5(0.02)(2) + 5(0.003)(1) = 0.215 \approx \\ &\approx \psi(x + dx, y + dy) - \psi(x, y) = 0.2153 \end{aligned}$$

A particular important aspect regarding the possible relationship between trajectories and scalar and vector fields concerns the concept of *line integrals*.

Given a scalar field $\psi(x, y)$ and a continuous trajectory $\vec{u}(t)$ sharing the same domain (e.g. R^2), the line integral of this scalar field from t_i to t_f can be written as:

$$\int_{t_i}^{t_f} \psi(x(t), y(t)) \|\dot{\vec{u}}(t)\| dt \quad (92)$$

The resulting scalar value can be understood as the area beneath the scalar field sliced by the considered trajectory. It is sometimes interesting to think of the moving point as a probe that is used to measure and analyze the vector field, in terms of its integration, along the respectively defined trajectory.

As an example, let us consider the line integral from $t_i = 0$ to $t_f = 1$ of the scalar field:

$$\psi(x, y) = 2 x y$$

considering the trajectory:

$$\vec{u}(t) = (x(t), y(t)) = \begin{bmatrix} t \\ -t + 2 \end{bmatrix}$$

so that:

$$\dot{\vec{u}}(t) = (\dot{x}(t), \dot{y}(t)) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The scalar field restricted by the considered trajectory can be written as:

$$\psi(x(t), y(t)) = 2(t)(-t + 2) \quad (93)$$

It follows from Equation 92 that:

$$\begin{aligned} &\int_0^1 \psi(x(t), y(t)) \|\dot{\vec{u}}(t)\| dt = \\ &= \int_0^1 2t(-t + 2) \sqrt{(1)^2 + (-1)^2} dt = \\ &= 2 \int_0^1 [-t^2 + 2t] \sqrt{2} dt = \\ &= 2\sqrt{2} \left[-\frac{t^3}{3} + \frac{2t^2}{2} \right] \Big|_0^1 = \\ &= 2\sqrt{2} \left[-\frac{1}{3} + 1 \right] = \frac{4}{3}\sqrt{2} \end{aligned} \quad (94)$$

Given a *vector field* $\vec{\phi}(x, y)$, it is possible to define its line integral, by a continuous trajectory $\vec{u}(t)$ from t_i to t_f as:

$$\int_{t_i}^{t_f} \vec{\phi}(x(t), y(t)) \cdot \dot{\vec{u}}(t) dt \quad (95)$$

Therefore, the line integral of a scalar field along a given trajectory can be thought of as probing the ‘net agreement’ between the relative orientations of the gradient and the first derivative of the point motion along its respective trajectory.

As an example, let us calculate the line integral of the vector field corresponding to the gradient of the scalar field used in the previous example and that same trajectory considering $t_i = 0$ and $t_f = 1$. We then have that:

$$\vec{\phi}(x, y) = \vec{\nabla}\psi(x, y) = \begin{bmatrix} 2y \\ 2x \end{bmatrix} \quad (96)$$

The gradient field restricted by the considered trajectory can be expressed as:

$$\vec{\phi}(x(t), y(t)) = \begin{bmatrix} 2(-t + 2) \\ 2t \end{bmatrix} \quad (97)$$

No, by using Equation 95, it follows that:

$$\begin{aligned} & \int_0^1 \vec{\phi}(x(t), y(t)) \cdot \dot{\vec{u}}(t) dt = \\ &= \int_0^1 (-2t + 4, 2t) \cdot (1, -1) dt = \\ &= 2 \int_0^1 (-2t + 2) dt = \\ &= 2 \left[-t^2 + 2t \right]_0^1 = 2 \end{aligned}$$

Interestingly, it can be shown that the line integral between two points $\vec{a} = (x(t_i), y(t_i))$ and $\vec{b} = (x(t_f), y(t_f))$ of a vector field $\vec{\phi}(x, y)$ corresponding to the gradient of a scalar field $\psi(x, y)$ does not depend on the specific trajectory between these two points, provided it is continuous. That is why this type of vector field is called *conservative*. We also have that:

$$\begin{aligned} & \int_{t_i}^{t_f} \vec{\phi}(x(t), y(t)) \cdot \dot{\vec{u}}(t) dt = \\ &= \psi(x(t_f), y(t_f)) - \psi(x(t_i), y(t_i)) = \psi(\vec{b}) - \psi(\vec{a}) \quad (98) \end{aligned}$$

Interestingly, it follows that even though the scalar field $\psi(x, y)$ is not directly taken into account, it is still possible to calculate its relative change from its gradient scalar field by using the property above.

The line integral over a closed trajectory $\vec{u}(t)$ is called *circulation* of that field by the given trajectory. It follows from the previous property that:

$$\oint_{\vec{u}(t)} \vec{\phi}(x(t), y(t)) \cdot \dot{\vec{u}}(t) dt = 0 \quad (99)$$

Let us now switch \mathbb{R}^3 , so that we can define the *vector product* between two vectors, as this will be used in the section about interaction between fields and trajectories. Thus, the *vector product* or *cross product* between two vectors $\vec{u} = (u_x, u_y, u_z)$ and $\vec{v} = (v_x, v_y, v_z)$ can be defined in terms of the following determinant:

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \\ &= u_y v_z \hat{i} + u_z v_x \hat{j} + u_x v_y \hat{k} - \\ &- u_z v_y \hat{i} - u_x v_z \hat{j} - u_y v_x \hat{k} = \\ &= (u_y v_z - u_z v_y) \hat{i} + (u_z v_x - u_x v_z) \hat{j} + (u_x v_y - u_y v_x) \hat{k} \quad (100) \end{aligned}$$

We also have that the magnitude of the cross-product can be obtained as:

$$|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta \hat{n} \quad (101)$$

where θ is the smallest angle between the two vectors, and \hat{n} is the unit magnitude vector that is normal to both

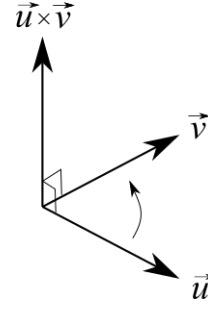


Figure 14: The corkscrew rule for the cross product $\vec{u} \times \vec{v}$ between two vectors \vec{u} and \vec{v} .

vectors. Observe that, unlike the dot product, the cross product is not commutative.

The direction of the cross product between \vec{u} and \vec{v} can be obtained by the *right-hand rule*, in which we use our right hand, aligning vector \vec{u} with the pointer finger, vector \vec{v} with the middle finger, so that the resulting direction is given by the thumb. It is also possible to consider the *corkscrew rule*, in which vector \vec{u} is rotated towards \vec{v} , with the direction of the respective cross product being provided by the direction of a corkscrew rotating in that manner, as illustrated in Figure 14.

9 Field-Motion Interactions

So far in the present work, we have concentrated our attention on how motions/trajectories can be understood as providing means for better understanding and probing scalar and vector fields, which can be done in terms of respective line integrals.

Now, we turn our attention to the particularly interesting question of how a given vector field $\vec{\phi}(t)$ can be made to influence the motion of a point. Although there is an unlimited number of ways in which this can be done, here we first focus our interest on directly proportional relationships, through proportionality constants $b, c \in \mathbb{R}, c \neq 0$, between the field vectors and the vectors obtained by time-differentiating the point location function $\vec{u}(t)$, i.e.:

$$b \frac{d^n \vec{u}(t)}{dt^n} = \vec{u}^{[n]}(t) = c \vec{\phi}(x, y, t) \quad (102)$$

where the non-negative integer n indicates the order of the considered derivative at which the field/point motion relationship takes place.

Observe that the relationship specified by Equation 102 necessarily implies that the n -th derivative of the position vector $\vec{x}(t)$ will have the same orientation as that of the field or, in other words, will be necessarily *parallel* to that field. Therefore, only changes in the magnitude of the n -th derivative of $\vec{x}(t)$ can take place.

The establishment of a fully determined relationship between the point trajectory \vec{u} and the field $\vec{\phi}(t)$ therefore requires the definition of two parameters: (i) the proportionality constant c ; and (ii) the order of the derivative n .

The specification of n implies some important consequences, including: (i) the derivatives from n upwards are determined by the vector field irrespectively to initial conditions; and (ii) the n derivatives $n - 1, n - 2, 1, 0$ of the point position are determined by the action of the the vector field over respective the initial conditions.

Given a specified differentiation of order n , we have that when $\vec{\phi}(x, y) = \vec{0}$ throughout, it will follow the $\vec{u}^{[n-1]}$ down to $\vec{u}^{[0]}$ can be non-zero vectors. For instance, if $\vec{u}^{[n]}(t)$ is a constant vector, $\vec{u}^{[n-1]}(t) = \int x^{[n]} dt$ will be linearly increasing, etc.

Assuming continuity of the derivatives, the choice of n determines the extent of the ‘memory’ of the motion along time t after the force field $\vec{f}(x)$ is removed or abruptly vanishes. Therefore, if a point is moving under influence of a field as discussed above respectively to a given value of n , the point will henceforth keep its derivative $\vec{u}^{[n-1]}$ constant as of at the instant when the field is completely removed, while the other lower order derivatives, as well as the point position, are progressive changed by the remaining constant derivative $\vec{u}^{[n-1]}$.

The above discussion is illustrated in Figure 15 respectively to a point moving under the influence of the following time-constant field:

$$\vec{\phi}(x, y) = \begin{cases} \phi_x(x, y) = 0.1 y \\ \phi_y(x, y) = 0.2 x \end{cases} \quad (103)$$

For simplicity’s sake, we henceforth take $c = 1$, $t \in [0, 10]$, and that the initial point position in all cases is $(x_0, y_0) = (3, 2)$. We shall assume two possible situations: (i) the field remains during all considered time values, defining the trajectory shown in blue; and (ii) that the vector field $\phi(x, y)$ is completely removed at $t_r = 1.616162$, shown in green. The share portion of the respective trajectories for t varying from 0 to t_r are shown in red. The presented points have been equally sampled at time intervals $\delta t = 0.1010101$.

The situation depicted in Figure 15(a) concerns point motion with $n = 1$, in which case the tangent of the point velocity, at each time instant t , simply follows the imposed vector field until, when the latter is removed, the trajectory simply stops at its current position (last point along the trajectory shown in red).

Now, we consider the case in which the field interaction takes place for $n = 2$, i.e. the action of the field implies directly proportional changes in the velocity of the point motion, which is illustrated in Figure 15(b). The initial velocity $(v_x(0), v_y(0))$ is shown as the blue vector

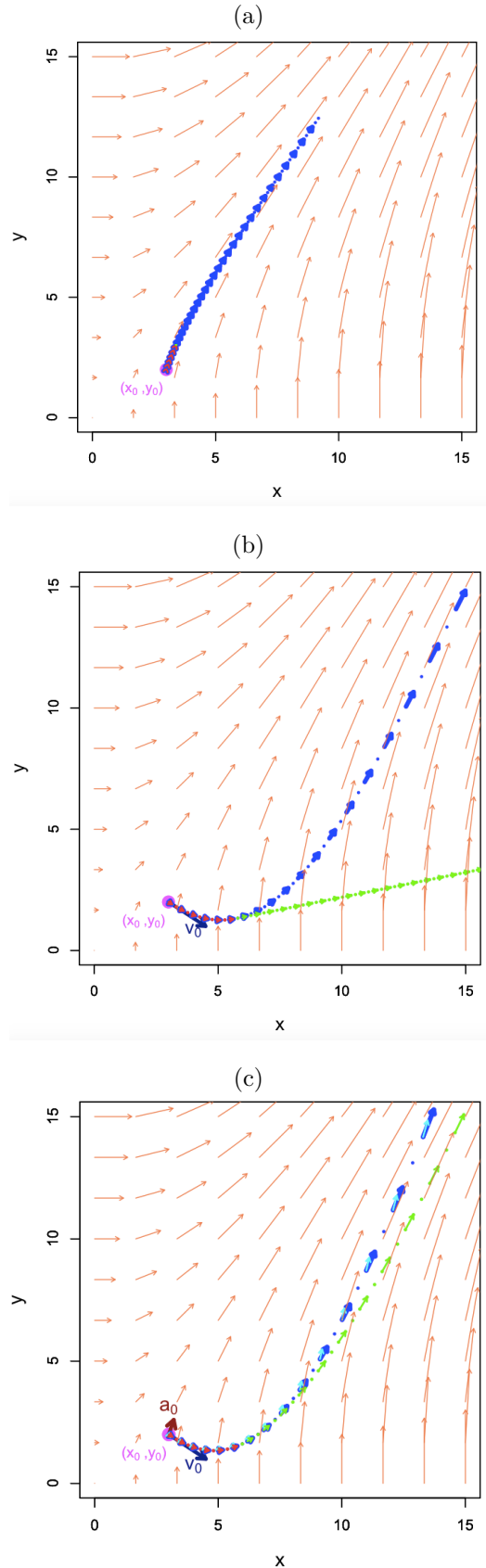


Figure 15: Different types of interaction between a point and a field. Motion of a point with $(x_0, y_0) = (3, 2)$ under the vector field $\vec{\phi}(x, y)$ considering $n = 0$ (a), $n = 1$ (b) and $n = 2$ (c).

at the initial position. In case the field is kept along time, the point has its velocity (blue vectors) continuously changed, becoming progressively aligned with the field vectors. However, in the situation where the force is nullified at t_r , the point continues its motion with constant vector speed (in green) equal to what it had when of the ceasing of the field .

Figure 15(c) depicts the case when $n = 3$. The initial velocity and acceleration vectors are shown at $(v_x(0), y_y(0))$ respectively in blue and dark red. The blue and green vectors along the two trajectories refer to the vector velocity, while the acceleration vectors are presented in cyan. When the field is kept constant, the point has its respective velocity and acceleration progressively aligned with the field vectors (blue trajectory). At the same time, the acceleration itself is increased along time. Interesting, and in marked contrast with the previous example, the point retains its acceleration when the field is removed at t_r , which then acts so as to progressively change the velocity it had at that same time, while the velocity of the acceleration remains constant. Though not shown in the figure as a consequence of the considered spatial region, the green trajectory will thereon substantially diverge from the blue trajectory, while keeping its velocity vectors not necessarily aligned to the originally existing field.

Given a second vector field $\vec{\psi}(x, y, t)$ also influencing the point motion as specified in Equation 102 for a certain proportionality constant e and the same differentiation order n , in case these two effects can be *linearly combined*, we have:

$$b \frac{d^n \vec{u}(t)}{dt^n} = \vec{u}^{[n]}(t) = c \vec{\phi}(x, y, t) + e \vec{\psi}(x, y, t) \quad (104)$$

This equation therefore provides an example of the *linear superimposition of effects*.

Another interesting basic manner in which the motion of a point can be affected by a vector field corresponds to considering a vector field $\vec{\tau}$ defined as a consequence of another vector field $\vec{\beta}$ acting on some of the derivatives m of the point position, such as in the following example, which necessarily assumes \mathbb{R}^3 because of the cross product:

$$\vec{\tau}(x, y, z, t) = a \frac{d^m \vec{u}(t)}{dt^m} \times \vec{\beta}(x, y, z, t) \quad (105)$$

The vector field $\vec{\tau}(x, y, x, t)$ will therefore be orthogonal to both the considered derivative of the point position and the vector field $\vec{\beta}(x, y, x, t)$. The resulting vector field $\vec{\tau}(x, y, z, t)$ can then influence the n -th derivative of the point motion as:

$$b \frac{d^n \vec{u}(t)}{dt^n} = a \frac{d^m \vec{u}(t)}{dt^m} \times \vec{\beta}(x, y, z, t) \quad (106)$$

In case all the above vector fields $\phi(x, y, x, t)$, $\psi(x, y, x, t)$ and $\tau(x, y, x, t)$ influence the point motion by acting on the same derivative n of its position by linear superimposition, the following single equation incorporating the interactions of the point motion with the three considered vector fields:

$$b \frac{d^n \vec{u}(t)}{dt^n} = c \vec{\phi}(x, y, z, t) + e \vec{\psi}(x, y, z, t) + a \frac{d^m \vec{u}(t)}{dt^m} \times \vec{\beta}(x, y, z, t) \quad (107)$$

10 Relative Position and Speed

Let two, possibly moving, points P and Q in \mathbb{R}^2 be specified by respective canonical coordinates $(x_P(t), x_Q(t))$ and $(x_Q(t), y_Q(t))$. We can define the *relative position* of Q respectively to P as:

$$\begin{cases} x_{Q,P}(t) = x_Q(t) - x_P(t) \\ y_{Q,P}(t) = y_Q(t) - y_P(t) \end{cases} \quad (108)$$

which allows us to determine the respective *instantaneous relative velocity* as corresponding to:

$$\begin{cases} \dot{x}_{Q,P}(t) = \dot{x}_Q(t) - \dot{x}_P(t) \\ \dot{y}_{Q,P}(t) = \dot{y}_Q(t) - \dot{y}_P(t) \end{cases} \quad (109)$$

The *instantaneous relative acceleration* then corresponds to:

$$\begin{cases} \ddot{x}_{Q,P}(t) = \ddot{x}_Q(t) - \ddot{x}_P(t) \\ \ddot{y}_{Q,P}(t) = \ddot{y}_Q(t) - \ddot{y}_P(t) \end{cases} \quad (110)$$

In case we translated these two points equally to another position $(X, Y) \in \mathbb{R}^2$, we will have:

$$\begin{cases} x_P(t) = x_P(t) + X \\ y_P(t) = y_P(t) + Y \end{cases} \quad (111)$$

$$\begin{cases} x_Q(t) = x_Q(t) + X \\ y_Q(t) = y_Q(t) + Y \end{cases} \quad (112)$$

It can be immediately verified that this translation will have no effect whatsoever on the relative position, velocity or acceleration, because the constant terms X and Y are respectively canceled by the subtraction involved in the definition of the relative position. Actually, any other function derived from the relative position will be preserved.

11 Moving Frames

Though we have so far studied the motion of a point respectively to a fixed Cartesian coordinate system, it is also interesting to consider how a same motion can be understood from different coordinate systems. One main

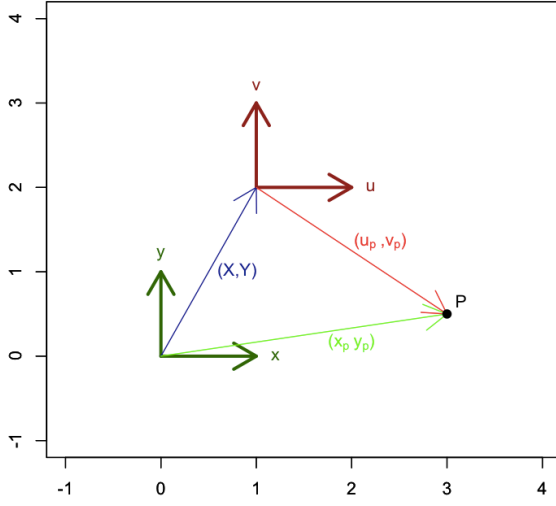


Figure 16: Two coordinate systems, or frames, with respective origins at different positions, are used to specify the position of the point p . Observe that we chose to express the position of the origin of the (u, x) frame in terms of the relative vector (X, Y) .

motivation for this study is that there is no absolute position where to place our coordinate system in the real world. For simplicity's sake, we shall also refer to these coordinate systems as *frames*.

We start with the situation illustrated in Figure 16, where two Cartesian frames (x, y) and (u, v) are placed a distinct positions, namely $(0, 0)$ and $(1, 2)$.

More specifically, we have that the origin of the frame (u, v) is here specified in terms of the relative vector (X, Y) , which allows us to write:

$$\begin{cases} x = X + u \\ y = Y + v \end{cases} \quad (113)$$

therefore relating the coordinates from both frames. We can now express the relationship between the position of the point P in the two considered frames as:

$$\begin{cases} x_P = X + u_P \\ y_P = Y + v_P \end{cases} \quad (114)$$

It immediately follows that any motion observed in terms of the frame (x, y) will be identical to that tracked by using frame (u, v) , except for a translation by the relative vector (X, Y) .

Let us now consider the situation in which frame (u, v) is moving with *constant speed* respectively to frame (x, y) , as illustrated in Figure 17. In this case, we can write:

$$\begin{cases} X(t) = (at + b) \\ Y(t) = (ct + d) \end{cases} \quad (115)$$

In case point P is performing a generic motion, so that its coordinates become a function of t . In this case, it

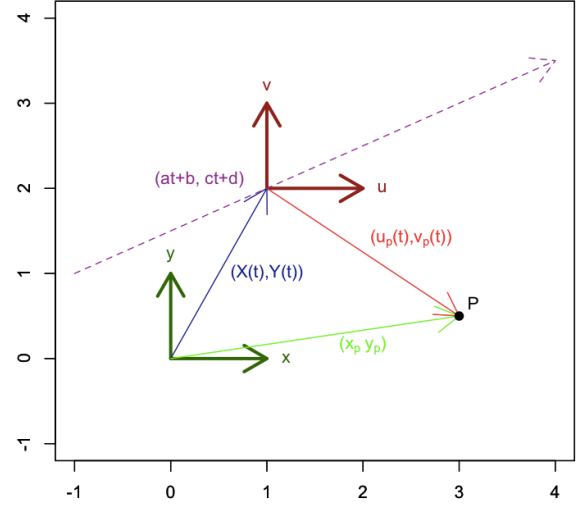


Figure 17: Two distinct frames are considered for observing a point, but one is moving with constant speed relatively to the other.

follows that:

$$\begin{cases} x_P(t) = (at + b) + u_P(t) \\ y_P(t) = (ct + d) + v_P(t) \end{cases} \quad (116)$$

The coordinates relative to the (u, v) frame become functions of time. As the motion of the frames is relative one another, we can rewrite the previous equations as:

$$\begin{cases} u_P(t) = x_P(t) - (at + b) \\ v_P(t) = y_P(t) - (ct + d) \end{cases} \quad (117)$$

We therefore also have that:

$$\begin{cases} \dot{u}_P(t) = \dot{x}_P(t) - a \\ \dot{v}_P(t) = \dot{y}_P(t) - c \end{cases} \quad (118)$$

Thus, if (\dot{x}_P, \dot{y}_P) is constant, so will (\dot{u}_P, \dot{v}_P) be. This implies that a uniform linear motion in any of the frames will also be perceived as such by the other frame, though with possibly distinct constant speed implied by the relative motion between the two frames. Figure 18 illustrates this important property, showing a uniform motion as seen from frames (x, y) and (u, v) , respectively.

Now, let us consider that frame (u, v) is undergoing an *accelerated* motion relatively to frame (x, y) , which can be expressed as:

$$\begin{cases} u_P(t) = x_P(t) - (at^2 + bt + c) \\ v_P(t) = y_P(t) - (dt^2 + et + f) \end{cases} \quad (119)$$

In this case, if point P is moving with constant speed respectively to the frame (x, y) , the following motion speed will be perceived from frame (u, v) :

$$\begin{cases} \dot{u}_P(t) = \dot{x}_P(t) - (2at + b) \\ \dot{v}_P(t) = \dot{y}_P(t) - (2dt + e) \end{cases} \quad (120)$$

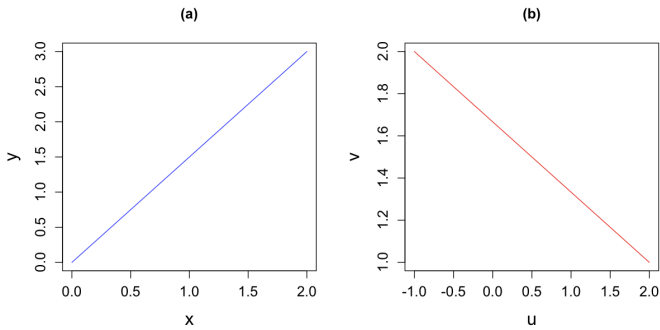


Figure 18: A uniform linear motion in frame (x, y) will be necessarily perceived as a uniform motion when observed from the frame (u, v) undergoing uniform linear motion relatively to the other frame.

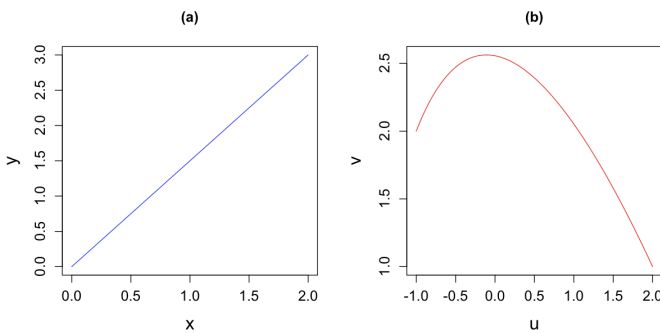


Figure 19: A uniform linear motion in frame (x, y) can be perceived as an accelerated motion when observed from a relatively accelerated frame (u, v) .

Therefore indicating that, unless $a = d = 0$, the uniform linear motion respectively to frame (x, y) will be understood as being accelerated (changing speed) when observed from frame (u, v) . Figure 19 presents how a uniform linear motion in frame (x, y) will become when observed from frame (u, v) .

Important implication of the above developments is that a relative uniform linear motion between two frames will transform linear motions into linear motions. In case one of the frames is accelerating relatively to the other, this will no longer be guaranteed, and it will be impossible for a observed placed in the accelerating frame to decide if the observed non-uniform motion is indeed non-uniform, of whether the observed variation of speed is a consequence of its frameworks not moving with uniform linear motion.

12 Concluding Remarks

Mathematics and physics are possibly the two most basic areas in the physical sciences. That there is an intrinsic, effective relationship underlying these two areas, es-

pecially in the sense that the former provides a suitable main backbone for the latter, has not only paved the way to continuing discoveries and advances, but also intrigued scientists for a long time.

The present work has been conceived and developed from the perspective of approaching the interesting problem of point motion, as well as its properties including interaction with fields, from an almost exclusively general mathematical perspective. The choice of \mathbb{R}^2 instead of \mathbb{R}^3 has been motivated as a manner to reach a balance between the extreme simplicity of \mathbb{R} and the substantially denser development considering \mathbb{R}^3 . Thus, even though the present work is in a sense less general regarding the adopted space dimensionality, the extension to \mathbb{R}^3 should be mostly direct, especially because the present approach focuses on differential aspects, and not integration, which would be intrinsically less directly extensible to \mathbb{R}^3 .

The main reasons for the mathematics-based presentation is that it could provide useful background for subsequent studies in any area involving point motion, of which there are many, as well as for obtaining a treatment of motion that is more general than that in physics, as there is not need to satisfy constraints such as Newton's lass of motion. Therefore, in a sense the present work could be understood as approaching the issue of particle point motion, but without considering any related physical property such as mass or electric charge. Of additional interest is the fact that the derivation, along the current work, of important physics concepts such as inertial frames of reference and conservative fields while considering only mathematical aspects.

The enhanced generality of treating point motion as a preparation for approaching the physics of particle motion provides an additional interesting feature: it creates an ampler perspective that can then be made more specific when the physical constraints are brought in. For instance, the studied interaction between a moving point and scalar and vector fields considered a linear relationship taken at any possible motion derivative order. Thus, when Newton's law of motion are considered, it becomes clear that they implement one choice among a much wider range of possibilities, emphasizing the importance of the discovery and application of these laws. The possibility of instantiating the broader concepts presented in the present work by incorporation of physical constraints is expected to be developed in a subsequent work.

Acknowledgments

Luciano da F. Costa thanks CNPq (grant no. 307085/2018-0) and FAPESP (grant 15/22308-2).

Observations

As all other preprints by the author, the present work contains preliminary work subject to further revision and validation. Respective modification, commercial use, or distribution of any of its parts are not possible, as this work is under author copyright. Many of the preprints by the author are also available in Hal and/or arXiv. This work can also be cited by using the DOI number or article identification link. Thanks for reading.

References

- [1] L. da F. Costa. Modeling: The human approach to science. Researchgate, 2019. https://www.researchgate.net/publication/333389500_Modeling_The_Human_Approach_to_Science_CDT-8. [Online; accessed 1-Oct-2020].
- [2] F. Cajori. *A history of mathematics*, volume 303. American Mathematical Soc., 1999.
- [3] C. B. Boyer and U. C. Merzbach. *A history of mathematics*. John Wiley & Sons, 2011.
- [4] E. Kreyszig. *Advanced Engineering Mathematics*. Wiley and Sons, 2015.
- [5] K. Hoffman. *Linear Algebra*. Pearson India, 2015.
- [6] T. M. Apostol. *Calculus*. Wiley and Sons, 2019.
- [7] R. Larson and B. E. Edwards. *Calculus*. Cengage Learning, 2013.
- [8] J. Stewart. *Multivariable Calculus*. Cengage Learning, 2015.
- [9] L. da F. Costa. A mosaic of multivariate calculus. Researchgate, 2020. https://www.researchgate.net/publication/340570370_A_Mosaic_of_Multivariate_Calculus_CDT-27. [Online; accessed 18-Apr-2020].
- [10] R. Larson, R. P. Hostetler, B. H. Edwards, and D. E. Heyd. *Calculus with analytic geometry*. DC Heath Lexington, Massachusetts, USA, 1986.
- [11] R. K. Nagle, E. B. Saff, and A. D. Snider. *Fundamentals of Differential Equations*. Pearson, 2017.
- [12] L. da F. Costa. Visualizing the content of differential equations. https://www.researchgate.net/publication/336839290_Visualizing_the_Content_of_Differential_Equations_CDT-16, 2019.
- [13] M. P. do Carmo. *Differential geometry of curves and surfaces*. Prentice Hall, 1976.
- [14] E. O. Brigham. *Fast Fourier Transform and its Applications*. Pearson, 1988.