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# ASYMPTOTIC EXPANSION OF TRANSMISSION EIGENVALUES FOR ANISOTROPIC THIN LAYERS

*by*

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**Abstract.** — We study the asymptotic expansion of transmission eigenvalues for anisotropic thin layers. We establish a rigorous second order expansion for simple transmission eigenvalues with respect to the thickness of the layer. The convergence analysis is based on a generalization of Osborn's Theorem to non-linear eigenvalue problems by Moskow [19]. We also provide formal derivation in more general cases validating the obtained theoretical result.

## 1. Introduction

Transmission eigenvalues are associated with the existence of an incident field that does not scatter [9]. They can be equivalently defined as the eigenvalues of a system of two coupled partial differential equations posed on the inclusion domain. One of these equations refer to the equation satisfied by the total field and the other one is satisfied by the incident field. The two equations are coupled on the boundary by imposing that the Cauchy data coincide. This eigenvalue problem can then be formulated as the non-linear eigenvalue problem a non-selfadjoint compact operator [9, 19]. These special frequencies can be identified from far field data as proved in [8, 9, 18]. Since they carry information on the material properties of the scatterer, transmission eigenvalues would be then of interest for the inverse problem of retrieving qualitative information on the material properties from measured multistatic data [16, 17].

This paper is a continuation of the study started in [3]. More specifically, we consider the transmission eigenvalue problem corresponding to anisotropic media with a thin coating. It corresponds in particular to inclusions where the contrast is present in the main operator. There is a structural difference between isotropic and anisotropic transmission eigenvalue problems. In the first case, one can reformulate the eigenvalue problem as a quadratic eigenvalue problem that can be transformed into a linear eigenvalue problem for a linear (non-selfadjoint) compact operator. This transformation is not possible in the anisotropic

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**Key words and phrases.** — transmission eigenvalues, asymptotic expansions, anisotropic thin layers.

one. Following [12] we are lead to transform the problem into a non linear eigenvalue problem where the compact operator depends itself on the eigenvalue. An additional structural difference comes from different functional spaces for solutions. This implies for instance a different treatment for the corrector terms in the asymptotic analysis. Asymptotic expansion of transmission eigenvalues has been addressed in the literature for small inclusion perturbations in [6, 12, 19]. Our analysis follows the procedure developed in [12] for the case of anisotropic media. The main theorem is the expression and error analysis for corrector terms associated with simple transmission eigenvalues. Simplicity assumption is mainly needed when applying Osborn's theorem for correctors [19]. The extension to eigenvalues with higher multiplicity is an open question. We consider two dimensional setting for the formal calculations but the procedure can be extended to three dimensional problems as well.

The paper is structured as follows. We introduce in Section 2 the transmission eigenvalue problem for anisotropic media in  $\mathbb{R}^2$  and we formulate it as a non-linear eigenvalue problem for a compact operator. In Section 3, we prove the convergence results needed to later obtain the asymptotic expansion for the transmission eigenvalues. The main result is proved in Section 4 using the perturbation formula for non linear eigenvalues given in [19]. Finally, we outline a classical formal procedure (as in [1, 2, 3] for instance) to obtain the expression of the asymptotic expansion which coincides with the one in Section 4 (independently from the multiplicity of the eigenvalue).

## 2. Formulation of the problem and main results

Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain with a smooth boundary  $\Gamma$ . For  $\epsilon > 0$  a small enough parameter, we denote

$$\Omega_\epsilon^0 = \{x \in \Omega, d(x, \Gamma) > \epsilon\}$$

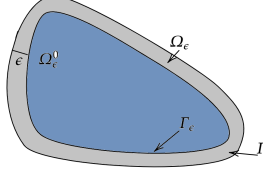
and its boundary

$$\Gamma_\epsilon = \{x \in \Omega, d(x, \Gamma) = \epsilon\} = \partial\Omega_\epsilon^0$$

where  $d(x, \Gamma)$  denotes the distance of a point  $x$  to the boundary  $\Gamma$ . Let  $\Omega_\epsilon = \Omega \setminus \overline{\Omega_\epsilon^0}$  be the layer of thickness  $\epsilon$  adjacent to  $\Omega_\epsilon^0$  (see Figure 1). Let  $\Theta_\epsilon, n_\epsilon$  be  $L^\infty(\Omega)$  functions such that

$$\Theta_\epsilon(x) = \begin{cases} \Theta_0(x) & \text{in } \Omega_\epsilon^0, \\ \Theta_1(x) & \text{in } \Omega_\epsilon, \end{cases} \quad n_\epsilon(x) = \begin{cases} n_0(x) & \text{in } \Omega_\epsilon^0, \\ n_1(x) & \text{in } \Omega_\epsilon, \end{cases}$$

where  $\Theta_0$  and  $\Theta_1$  are real valued positive definite matrices in  $L^\infty(\mathbb{R}^2)^{2 \times 2}$  and  $n_0$  and  $n_1$  are non negative real valued functions in  $L^\infty(\mathbb{R}^2)$  that are independent from  $\epsilon$ . For the sake of simplicity, we assume that the restriction of  $n_0$  to  $\Omega_\epsilon$  is constant function along the normal coordinate to  $\Gamma$  for  $\epsilon$  sufficiently small. We then consider the following interior

FIGURE 1. The scattering object  $\Omega$ .

transmission eigenvalue problem formulated inside  $\Omega$

$$\begin{cases} \operatorname{div}(\Theta_\epsilon \nabla w_\epsilon) + \lambda_\epsilon n_\epsilon w_\epsilon = 0 & \text{in } \Omega, \\ \Delta v_\epsilon + \lambda_\epsilon v_\epsilon = 0 & \text{in } \Omega, \\ w_\epsilon = v_\epsilon & \text{on } \Gamma, \\ \Theta_\epsilon \nabla w_\epsilon \cdot \nu = \nabla v_\epsilon \cdot \nu & \text{on } \Gamma, \end{cases} \quad (1)$$

where  $\nu$  the unitary normal to  $\Gamma$  directed to the interior of  $\Omega$ .

**Definition 2.1.** — *The values  $\lambda_\epsilon$  for which (1) has a non trivial solution  $(w_\epsilon, v_\epsilon) \in H^1(\Omega) \times H^1(\Omega)$  are called transmission eigenvalues.*

To justify the asymptotic expansion of transmission eigenvalues, we rewrite (1) as a non-linear eigenvalue problem. To this end, we define the variational space

$$X(\Omega) = \left\{ (w, v) \in H^1(\Omega) \times H^1(\Omega), w - v = 0 \text{ on } \Gamma \right\}$$

equipped with the  $H^1(\Omega) \times H^1(\Omega)$  inner product. It is clear that the variational form of (1) is given by:  $(w_\epsilon, v_\epsilon) \in X(\Omega)$  such that

$$\int_{\Omega} (\Theta_\epsilon \nabla w_\epsilon \nabla \phi_1 - \nabla v_\epsilon \nabla \phi_2 - \lambda_\epsilon (n_\epsilon w_\epsilon \phi_1 - v_\epsilon \phi_2)) dx = 0, \text{ for all } (\phi_1, \phi_2) \in X(\Omega). \quad (2)$$

We define the bounded sesquilinear forms:

$$\mathcal{A}_\epsilon((w, v); (\phi_1, \phi_2)) := \int_{\Omega} (\Theta_\epsilon \nabla w \nabla \phi_1 + w \phi_1) dx - \int_{\Omega} (\nabla v \nabla \phi_2 + v \phi_2) dx, \quad (3)$$

$$\mathcal{B}_\epsilon((w, v); (\phi_1, \phi_2)) := \int_{\Omega} (n_\epsilon w \phi_1 - v \phi_2) dx, \quad (4)$$

and

$$\mathcal{C}_\epsilon((w, v); (\phi_1, \phi_2)) := \int_{\Omega} (w \phi_1 - v \phi_2) dx \quad (5)$$

for all  $(w, v) \in X(\Omega)$ . Therefore (2) can be rewritten as

$$\mathcal{A}_\epsilon((w_\epsilon, v_\epsilon); (\phi_1, \phi_2)) - \lambda_\epsilon \mathcal{B}_\epsilon((w_\epsilon, v_\epsilon); (\phi_1, \phi_2)) - \mathcal{C}_\epsilon((w_\epsilon, v_\epsilon); (\phi_1, \phi_2)) = 0 \quad (6)$$

for all  $(\phi_1, \phi_2) \in X(\Omega)$ . Let us define by  $A_\epsilon, B_\epsilon$  and  $C_\epsilon : X(\Omega) \rightarrow X(\Omega)$  the bounded linear operators defined from  $\mathcal{A}_\epsilon(\cdot; \cdot), \mathcal{B}_\epsilon(\cdot; \cdot)$  and  $\mathcal{C}_\epsilon(\cdot; \cdot)$  by means of the Riesz representation theorem. Then (6) is equivalent to

$$A_\epsilon(w_\epsilon, v_\epsilon) - \lambda_\epsilon B_\epsilon(w_\epsilon, v_\epsilon) - C_\epsilon(w_\epsilon, v_\epsilon) = (0, 0). \quad (7)$$

To give the asymptotic expansion of transmission eigenvalues, we use the generalization of Osborn's Theorem to non-linear eigenvalue problem [19]. We rewrite (7) in the following form

$$(I - \lambda_\epsilon A_\epsilon^{-1} B_\epsilon - A_\epsilon^{-1} C_\epsilon)(w_\epsilon, v_\epsilon) = (0, 0).$$

Note that  $B_\epsilon, C_\epsilon : X(\Omega) \rightarrow X(\Omega)$  are compact operators by using the compact embedding of  $H^1(\Omega)$  in  $L^2(\Omega)$ . In order to prove the invertibility of  $\mathcal{A}_\epsilon$ , we can show that  $A_\epsilon$  is  $\mathbb{T}$ -coercive. The idea behind the  $\mathbb{T}$ -coercivity [15] is to consider an equivalent formulation of (6) where  $\mathcal{A}_\epsilon$  is replaced by  $\mathcal{A}_\epsilon^\mathbb{T}$  defined by

$$\mathcal{A}_\epsilon^\mathbb{T}\left((w_\epsilon, v_\epsilon); (\phi_1, \phi_2)\right) = \mathcal{A}_\epsilon\left((w_\epsilon, v_\epsilon); \mathbb{T}(\phi_1, \phi_2)\right)$$

with  $\mathbb{T}$  being an isomorphism of  $X(\Omega)$ . In particular, we consider the isomorphism  $\mathbb{T}(w, v) = (w, \alpha w + \beta v)$  with  $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$  such that  $\alpha + \beta = 1$ . Indeed  $\mathbb{T}$  continuously maps  $X(\Omega)$  into itself, thanks to  $\alpha + \beta = 1$  and the inverse can be explicitly expressed as  $\mathbb{T}^{-1}(w, v) = (w, (v - \alpha w)/\beta)$ . Then

$$\begin{aligned} & \left| \mathcal{A}_\epsilon\left((w, v); \mathbb{T}(w, v)\right) \right| \\ &= \left| \int_\Omega \left( \Theta_\epsilon |\nabla w|^2 + |w|^2 \right) dx - \beta \int_\Omega \left( |\nabla v|^2 + |v|^2 \right) dx - \alpha \int_\Omega \left( \nabla w \nabla v + wv \right) dx \right| \\ &\geq \int_\Omega \left( \Theta_\epsilon |\nabla w|^2 + |w|^2 \right) dx - \beta \left| \int_\Omega \left( |\nabla v|^2 + |v|^2 \right) dx \right| - \alpha \left| \int_\Omega \left( \nabla w \nabla v + wv \right) dx \right| \end{aligned}$$

Using Young's inequality, we obtain

$$\left| \int_\Omega \left( \nabla w \nabla v + wv \right) dx \right| \leq \frac{1}{2\sigma} \|w\|_{H^1(\Omega)}^2 + \frac{\sigma}{2} \|v\|_{H^1(\Omega)}^2 \quad (8)$$

for all  $\sigma > 0$ . We then get

$$\left| \mathcal{A}_\epsilon\left((w, v); \mathbb{T}(w, v)\right) \right| \geq \left( \gamma - \frac{\alpha}{2\sigma} \right) \|w\|_{H^1(\Omega)}^2 + \left( -\beta - \frac{\sigma\alpha}{2} \right) \|v\|_{H^1(\Omega)}^2$$

where  $\gamma = \inf(\|\Theta_1\|_\infty, \|\Theta_0\|_\infty)$ . Choosing  $\alpha^2 < -4\beta\gamma$  and  $\beta < 0$ , to have for example  $\sigma \in \left( \frac{\alpha}{2\gamma}, -\frac{2\beta}{\alpha} \right)$ , implies that  $\mathcal{A}_\epsilon$  is  $\mathbb{T}$ -coercive.

Therefore (7) can be equivalently written as follows [12]

$$(I - \lambda_\epsilon A_\epsilon^{-1} B_\epsilon - A_\epsilon^{-1} C_\epsilon)(w_\epsilon, v_\epsilon) = (0, 0). \quad (9)$$

Let us consider  $T_\epsilon(\lambda) : X(\Omega) \rightarrow X(\Omega)$  defined as

$$T_\epsilon(\lambda) := A_\epsilon^{-1} B_\epsilon + \frac{1}{\lambda} A_\epsilon^{-1} C_\epsilon. \quad (10)$$

Now we can rephrase (9) as a non-linear eigenvalue problem

$$\lambda_\epsilon T_\epsilon(\lambda_\epsilon)(w_\epsilon, v_\epsilon) = (w_\epsilon, v_\epsilon).$$

Note that  $T_\epsilon$  is a compact operator for all  $\epsilon \geq 0$  and depends analytically on  $\lambda$  in any subset of the complex plane that does not include the origin. We set

$$T_0(\lambda) := A_0^{-1}B_0 + \frac{1}{\lambda}A_0^{-1}C_0, \quad (11)$$

where  $A_0$ ,  $B_0$  and  $C_0$  are defined by (3), (4) and (5) for  $n_\epsilon = n_0$  and  $\Theta_\epsilon = \Theta_0$  in  $\Omega$ , respectively. We notice that  $C_\epsilon = C_0$ .

In all the following we use the notation

$$\left\langle (f_1, f_2); (g_1, g_2) \right\rangle = \left( (f_1, f_2); (g_1, g_2) \right)_{X(\Omega)} \quad \text{and} \quad \|(f_1, f_2)\| = \left( (f_1, f_2); (f_1, f_2) \right)_{X(\Omega)}^{\frac{1}{2}}.$$

To state and prove the main result we make the following important hypothesis on the anisotropy structure. We assume that inside a thin layer  $\Omega_{\epsilon_0}$  for some fixed  $\epsilon_0$ , we can decompose  $\Theta_1$  and  $\Theta_0$  as

$$\begin{cases} \Theta_1(x) \cdot \tau(s) = \theta_1^\tau(s)\tau(s) \text{ and } \Theta_1(x) \cdot \nu(s) = \theta_1^\nu(s)\nu(s), \\ \Theta_0(x) \cdot \tau(s) = \theta_0^\tau(s)\tau(s) \text{ and } \Theta_0(x) \cdot \nu(s) = \theta_0^\nu(s)\nu(s) \end{cases} \quad (12)$$

for a.e.  $x \in \Omega_{\epsilon_0}$  where  $\theta_1^\tau, \theta_1^\nu, \theta_0^\tau$  and  $\theta_0^\nu$  are real functions in  $L^\infty(\mathbb{R})$  that depend only on the curvilinear abscissa  $s$  of the projection of  $x$  on  $\partial\Omega$  and where  $\tau$  and  $\nu$  are respectively the tangent and normal vectors defined on  $\partial\Omega$  (see Subsection 4.1).

**Theorem 2.2.** — *Assume that  $\Theta_0$  and  $\Theta_1$  are bounded real valued positive definite matrices that satisfy (12) and  $n_0$  and  $n_1$  are bounded non negative real valued functions. Let  $\lambda_0$  be a simple transmission eigenvalue associated with  $\Theta_0$  and  $n_0$  and let  $(w_0, v_0) \in X(\Omega)$  be the corresponding eigenfunctions. We suppose that*

$$\left\langle (w_0, v_0); A_0(w_0, v_0) \right\rangle = 1 \quad \text{and} \quad \int_{\Omega} (n_0|w_0|^2 - |v_0|^2)dx \neq 0$$

and that  $\Theta_0$  and  $n_0$  are sufficiently regular (at least  $C^1$ ) so that  $(w_0, v_0)$  and  $A_0^{-1}(w_0, v_0)$  are in  $C^2(\overline{\Omega})$ . Then for  $\epsilon > 0$  small enough, there exists a transmission eigenvalue  $\lambda_\epsilon$  corresponding to  $\Theta_\epsilon$  such that

$$\lambda_\epsilon = \lambda_0 + \epsilon\lambda_1 + O(\epsilon^{\frac{3}{2}}) \quad (13)$$

where  $\lambda_1$  is given by the following expression when  $n_1 = n_0$

$$\lambda_1 = \frac{- \int_{\Gamma} (\theta_0^\tau - \theta_1^\tau) \left| \frac{\partial w_0}{\partial s} \right|^2 ds(x) - \int_{\Gamma} \theta_0^\nu \left( \frac{\theta_0^\nu - \theta_1^\nu}{\theta_1^\nu} \right) \left| \frac{\partial w_0}{\partial \nu} \right|^2 ds(x)}{\int_{\Omega} (n_0|w_0|^2 - |v_0|^2)dx}.$$

For the convergence proof we restrict ourselves to  $n_1 = n_0$ . The expression of  $\lambda_1$  in the case  $n_1 \neq n_0$  is given in the last Section using formal expansion.

**Remark 2.3.** — *The technical assumption (12) is mainly used to obtain explicit expressions for  $\lambda_1$  as in the Theorem above. It corresponds for instance to (macroscopic or homogenized) anisotropy generated by (microscopic) mixture of isotropic materials distributed periodically along  $\Gamma$ .*

From Section 4.1 and following the formal expansion in Section 5, one can be easily convinced that explicit, but much lengthier, calculations can be still carried out if we only assume that  $\theta'_1 := (\Theta_1 \cdot \nu) \cdot \nu$  is only a function of the curvilinear abscissa  $s$ . If not, then non explicit expressions would also be possible in terms of the formal solution associated with the Cauchy problem (40).

### 3. Convergence of the transmission eigenvalue spectrum

We study in this section the convergence in norm of the operator  $T_\epsilon$  to  $T_0$ . For an operator  $A : H \rightarrow H$ ,  $\|A\|$  denotes the operator norm. To simplify the writing,  $C$  will denote a generic constant whose value may change, but remains independent from  $\epsilon$  as  $\epsilon \rightarrow 0$ . The assumptions of Theorem 2.2 are supposed to be verified for all the results stated in this section.

We start by proving the convergence of the operator  $B_\epsilon$  to  $B_0$  in norm .

**Lemma 3.1.** — *Let  $B_\epsilon$  and  $B_0$  be the operators defined by (4) for  $\epsilon > 0$  and  $\epsilon = 0$ , respectively. Then, for all  $\alpha \in (0, 1)$  there exists  $C > 0$  independent of  $\epsilon$*

$$\|B_\epsilon - B_0\| \leq C\epsilon^\alpha.$$

*Proof.* — From the definition of  $B_\epsilon$  and  $B_0$ , we have

$$\begin{aligned} \left| \left\langle B_\epsilon(w, v); (\phi_1, \phi_2) \right\rangle - \left\langle B_0(w, v); (\phi_1, \phi_2) \right\rangle \right| &= \left| \int_{\Omega_\epsilon} (n_1 - n_0)w\phi_1 dx \right| \\ &\leq \|(n_1 - n_0)w\|_{L^2(\Omega_\epsilon)} \|(\phi_1, \phi_2)\|_{X(\Omega)}. \end{aligned}$$

Choosing  $(\phi_1, \phi_2) = (B_\epsilon - B_0)(w, v)$  we get

$$\|(B_\epsilon - B_0)(w, v)\|_{X(\Omega)} \leq \|(n_1 - n_0)w\|_{L^2(\Omega_\epsilon)}.$$

Since  $w \in H^1(\Omega)$  and from Sobolev's embedding, we have that  $w \in L^p(\Omega)$ ,  $\forall p \geq 2$  (see [4]) which gives  $|w|^2 \in L^{\frac{p}{2}}(\Omega)$ .

Now let  $q > 0$  be defined by  $\frac{2}{p} + \frac{1}{q} = 1$  (i.e  $q = \frac{p}{p-2}$ ). Using the duality between  $L^{\frac{p}{2}}(\Omega)$  and  $L^q(\Omega)$  with the Sobolev embedding Theorem, we have that

$$\begin{aligned} \|(B_\epsilon - B_0)(w, v)\|_{X(\Omega)}^2 &\leq C \|(n_1 - n_0)\|_{L^\infty(\Omega)}^2 \|w\|_{L^2(\Omega_\epsilon)}^2 \\ &\leq C \|w\|_{L^2(\Omega_\epsilon)}^2 \leq \| |w|^2 \|_{L^{\frac{p}{2}}(\Omega)} \| \mathbf{1}_{\Omega_\epsilon} \|_{L^q(\Omega)} \\ &\leq C \epsilon^{\frac{1}{q}} \|w\|_{L^p(\Omega)}^2 \leq C \epsilon^{\frac{1}{q}} \|(w, v)\|_{X(\Omega)}^2. \end{aligned}$$

We can conclude that  $\|B_\epsilon - B_0\| \leq C\epsilon^\alpha$  with  $\alpha = \frac{1}{2q}$ .  $\square$

**Lemma 3.2.** — Let  $A_\epsilon$  and  $A_0$  be defined by (3) for  $\epsilon > 0$  and  $\epsilon = 0$ , respectively and  $(f, g) \in X(\Omega)$ . If  $A_0^{-1}(f, g) \in \mathcal{C}^1(\Omega) \times \mathcal{C}^1(\Omega)$ , then for sufficiently small  $\epsilon$

$$\|(A_\epsilon^{-1} - A_0^{-1})(f, g)\|_{X(\Omega)} \leq C\epsilon^{\frac{1}{2}}. \quad (14)$$

*Proof.* — For a fixed  $(f, g) \in X(\Omega)$ , we define  $(z_\epsilon, t_\epsilon)$  and  $(z_0, t_0)$  in  $X(\Omega)$  as

$$(z_\epsilon, t_\epsilon) = A_\epsilon^{-1}(f, g) \text{ and } (z_0, t_0) = A_0^{-1}(f, g).$$

Since

$$\mathcal{A}_\epsilon\left((z_\epsilon, t_\epsilon); (\phi_1, \phi_2)\right) = \mathcal{A}_0\left((z_0, t_0); (\phi_1, \phi_2)\right) = \left\langle (f, g); (\phi_1, \phi_2) \right\rangle.$$

we have that, for  $(\phi_1, \phi_2) \in X(\Omega)$

$$\mathcal{A}_\epsilon\left((z_\epsilon - z_0, t_\epsilon - t_0); (\phi_1, \phi_2)\right) = (\mathcal{A}_0 - \mathcal{A}_\epsilon)\left((z_0, t_0); (\phi_1, \phi_2)\right) = \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla z_0 \nabla \phi_1 dx. \quad (15)$$

If  $z_0 \in \mathcal{C}^1(\overline{\Omega})$ , we get

$$\int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla z_0 \nabla \phi_1 dx \leq C\epsilon^{\frac{1}{2}} \|(\phi_1, \phi_2)\|_{X(\Omega)}.$$

From the  $\mathbb{T}$ -coercivity and by plugging  $(\phi_1, \phi_2) = \mathbb{T}(z_\epsilon - z_0, t_\epsilon - t_0)$ , we obtain the desired convergence.  $\square$

**Lemma 3.3.** — Let  $A_\epsilon, B_\epsilon, C_\epsilon, A_0, B_0$  and  $C_0$  be defined by (3), (4), (5) for  $\epsilon > 0$  and  $\epsilon = 0$ , respectively. we have that

$$A_\epsilon^{-1}B_\epsilon \rightarrow A_0^{-1}B_0 \text{ and } A_\epsilon^{-1}C_0 \rightarrow A_0^{-1}C_0$$

in norm as  $\epsilon \rightarrow 0$ .

*Proof.* — From (15) we have that

$$\mathcal{A}_\epsilon\left((z_\epsilon - z_0, t_\epsilon - t_0); (\phi_1, \phi_2)\right) = \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla z_0 \nabla \phi_1 dx.$$

By using the  $\mathbb{T}$ -coercivity of  $\mathcal{A}_\epsilon$ , we obtain

$$\|(A_\epsilon^{-1} - A_0^{-1})(f, g)\|_{X(\Omega)} \leq C \|(\Theta_0 - \Theta_1) \nabla z_0\|_{L^2(\Omega_\epsilon)}.$$

If  $B_0(f, g) = (p, q)$  then  $(p, q) \in H^1(\Omega) \times H^1(\Omega)$  and

$$\begin{cases} -\Delta p + p = n_0 f & \text{in } \Omega, \\ -\Delta q + q = -g & \text{in } \Omega, \\ p = q & \text{on } \Gamma, \\ -\frac{\partial p}{\partial \nu} = \frac{\partial q}{\partial \nu} & \text{on } \Gamma. \end{cases} \quad (16)$$



Then  $p - q \in H_0^1(\Omega)$  and verifies

$$-\Delta(p - q) + (p - q) = n_0 f + g \text{ in } \Omega.$$

Elliptic regularity implies that if  $\partial\Omega$  is  $\mathcal{C}^3$ , then  $(p - q) \in H^3(\Omega)$  and

$$\|p - q\|_{H^3(\Omega)} \leq C' \left( \|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)} \right).$$

(We here need  $n_0 \in C^1(\bar{\Omega})$ ). Similarly,  $p + q \in H^1(\Omega)$  and verifies

$$\begin{cases} -\Delta(p + q) + (p + q) = n_0 f - g & \text{in } \Omega, \\ \frac{\partial(p + q)}{\partial\nu} = 0 & \text{on } \Gamma, \end{cases} \quad (17)$$

Therefore, regularity results for the Neumann problem implies that, if  $\partial\Omega$  is  $\mathcal{C}^3$  then

$$\|p + q\|_{H^3(\Omega)} \leq C'' \left( \|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)} \right).$$

Hence, we have

$$\|p\|_{H^3(\Omega)} + \|q\|_{H^3(\Omega)} \leq C \left( \|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)} \right).$$

Note that  $A_0(z_0, t_0) = (p, q)$  and due to the variational form of  $A_0$  satisfies

$$-div(\Theta_0 \nabla z_0) + z_0 = -\Delta p + p \text{ and } \Delta t_0 - t_0 = -\Delta q + q \text{ in } \Omega.$$

Next we have that  $A_0^{-1}B_0(f, g) = (z_0, t_0)$ , by elliptic regularity we have

$$\|z_0\|_{H^3(\Omega)} + \|t_0\|_{H^3(\Omega)} \leq C \left( \|p\|_{H^3(\Omega)} + \|q\|_{H^3(\Omega)} \right) \leq C \|(f, g)\|_{X(\Omega)}.$$

For all  $\epsilon$  sufficiently small, we have the following estimates

$$\|(A_\epsilon^{-1} - A_0^{-1})B_0(f, g)\|_{X(\Omega)} \leq C \|z_0\|_{C^1(\Omega)} \|\mathbb{1}_{\Omega_\epsilon}\|_{L^2(\Omega)} \leq C\epsilon^{\frac{1}{2}} \|z_0\|_{C^1(\Omega)}.$$

From the Sobolev embedding Theorem of  $H^3(\Omega)$  into  $C^1(\Omega)$ , we have that

$$\|(A_\epsilon^{-1} - A_0^{-1})B_0(f, g)\|_{X(\Omega)} \leq C\epsilon^{\frac{1}{2}} \|z_0\|_{H^3(\Omega)} \leq C\epsilon^{\frac{1}{2}} \|(f, g)\|_{X(\Omega)}.$$

Since

$$\|A_\epsilon^{-1}B_\epsilon - A_0^{-1}B_0\| \leq \|A_\epsilon^{-1}(B_\epsilon - B_0)\| + \|(A_\epsilon^{-1} - A_0^{-1})B_0\|,$$

using the fact that  $\|A_\epsilon^{-1}\|$  is uniformly bounded with respect to  $\epsilon$  and the convergence in norm of  $B_\epsilon$  to  $B_0$ , implies

$$\|A_\epsilon^{-1}B_\epsilon - A_0^{-1}B_0\| \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

The second convergence result follows from similar arguments.  $\square$

From the previous results, we have the following lemma:

**Lemma 3.4.** — Let  $A_\epsilon, B_\epsilon, C_\epsilon, A_0, B_0$  and  $C_0$  be defined by (3), (4), (5), for  $\epsilon > 0$  and  $\epsilon = 0$ , respectively. Then we have that

$$\|(A_\epsilon^{-1} - A_0^{-1})B_0\| \leq C\epsilon^{\frac{1}{2}}, \|(A_\epsilon^{-1} - A_0^{-1})C_0\| \leq C\epsilon^{\frac{1}{2}}$$

and  $\|A_\epsilon^{-1}(B_\epsilon - B_0)\| \leq C\epsilon^\alpha$ , for some  $0 < \alpha < 1$ .

Combining the above results and using the definition of  $T_\epsilon$  and  $T_0$  we have:

**Theorem 3.5.** — Let  $T_\epsilon$  and  $T_0$  be defined in (10) for  $\epsilon > 0$  and  $\epsilon = 0$ , respectively, then we have that

$$\|T_\epsilon(\lambda) - T_0(\lambda)\| \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

To prove the convergence of real transmission eigenvalues, we use the following abstract result from [19] (see also [5] for a more general framework allowing for weaker convergence) that we restate here for the reader's convenience.

**Theorem 3.6.** — Let  $\lambda_0$  be a non-linear eigenvalue of  $T_0$  and assume that  $R_\epsilon(\lambda)$  and  $R_0(\lambda)$  are both meromorphic in some region  $U \subset \mathbb{C}$  containing  $\lambda_0$ . Also assume that for any  $\lambda \in U$ ,  $T_\epsilon(\lambda)$  converge in norm to  $T_0(\lambda)$ . Then for any ball  $B$  around  $\lambda_0$  there exists a  $\epsilon_0 > 0$  such that  $T_\epsilon$  has a non-linear eigenvalue in  $B$  for all  $\epsilon < \epsilon_0$ . Conversely if  $\{\lambda_\epsilon\}_\epsilon$  is a sequence of a non-linear eigenvalues of  $T_\epsilon$  that converges as  $\epsilon \rightarrow 0$  then the limit is a non-linear eigenvalue of  $T_0$ .

#### 4. Asymptotic formula for the transmission eigenvalue

Having proved the convergence of the transmission eigenvalues, we would like to obtain an asymptotic formula for the simple real transmission eigenvalue of the form:

$$\lambda_\epsilon = \lambda_0 + \epsilon\lambda_1 + O(\epsilon^2),$$

where  $\lambda_0$  is a non-linear eigenvalue to  $T_0$  associated with the eigenfunction  $(w_0, v_0)$  i.e

$$\lambda_0 T_0(w_0, v_0) = (w_0, v_0).$$

and the expression for  $\lambda_1$  is as explicit as possible, and depends only on  $\lambda_0$  and  $(w_0, v_0)$ . We refer to  $\lambda_1$  as the eigenvalue correction. To this end, we define explicit formula for the corrector term associated with  $T_\epsilon - T_0$ , in particular  $A_\epsilon^{-1} - A_0^{-1}$ . Then, we need a classical technique for thin layers asymptotics based on rescaling. Again, the assumptions of Theorem 2.2 are supposed to be verified for all the results stated in this section.

**4.1. Scaling.** — We assume that the boundary  $\Gamma$  is  $\mathcal{C}^3$  and parametrize  $\Gamma$  as

$$\Gamma = \{x_\Gamma(s), s \in [0, L]\},$$

with  $L$  being the length of  $\Gamma$  and  $s$  is the curvilinear abscissa. At the point  $x_\Gamma(s)$ , the unit tangent vector is  $\tau(s) := \frac{dx_\Gamma(s)}{ds}$ , the curvature  $\kappa(s)$  is defined by:

$$\frac{d\tau(s)}{ds} = -\kappa(s)\nu(s) \text{ or } \frac{d\nu(s)}{ds} = \kappa(s)\tau(s).$$

With these notations, the boundary of  $\Omega_\epsilon^0$  is parametrized as

$$\Gamma_\epsilon = \{x_\Gamma(s) + \epsilon\nu(s), s \in [0, L]\}.$$

This parametrization of  $\Gamma_\epsilon$  is equivalent to the definition of  $\Gamma_\epsilon$ , for  $\epsilon > 0$  small enough. For a function  $u$  defined in  $\Omega_\epsilon$  we associate  $\tilde{u}$  defined on  $[0, L[\times]0, \epsilon[$  by

$$\tilde{u}(s, \eta) = u(\varphi(s, \eta)) \text{ where } \varphi(s, \eta) := x_\Gamma(s) + \eta\nu(s). \quad (18)$$

Then, the gradient of  $u$  in the local coordinates  $(s, \eta)$  can be expressed as

$$\nabla u = \frac{1}{(1 + \eta\kappa(s))} \frac{\partial \tilde{u}}{\partial s} \tau(s) + \frac{\partial \tilde{u}}{\partial \eta} \nu(s).$$

Using integration by parts one can verify that the divergence of a vector field  $\vec{u} = u_\tau \tau + u_\nu \nu$  can be expressed as

$$\operatorname{div} \vec{u} = \frac{1}{(1 + \eta\kappa(s))} \frac{\partial \tilde{u}_\tau}{\partial s} + \frac{1}{(1 + \eta\kappa(s))} \frac{\partial}{\partial \eta} ((1 + \eta\kappa(s)) \tilde{u}_\nu)$$

Let  $\Theta$  be a 2 by 2 matrix that satisfies

$$\Theta(x) \cdot \tau = \theta^\tau(s) \tau \text{ and } \Theta(x) \cdot \nu = \theta^\nu(s) \nu.$$

Then the expression of the operator  $\operatorname{div}(\Theta \cdot \nabla)$  in  $(s, \eta)$  coordinates is

$$\operatorname{div}(\Theta \cdot \nabla u) = \frac{1}{(1 + \eta\kappa)} \frac{\partial}{\partial s} \left( \theta^\tau \frac{1}{(1 + \eta\kappa)} \frac{\partial \tilde{u}}{\partial s} \right) + \frac{1}{(1 + \eta\kappa)} \frac{\partial}{\partial \eta} \left( (1 + \eta\kappa) \theta^\nu \frac{\partial \tilde{u}}{\partial \eta} \right). \quad (19)$$

We denote by  $\xi = \frac{\eta}{\epsilon}$  the stretched normal variable inside  $\Omega_\epsilon$  and define

$$\begin{aligned} \varphi_\epsilon : \mathcal{G} = [0, L[\times]0, 1[ &\rightarrow \Omega_\epsilon \\ (s, \xi) &\mapsto \varphi_\epsilon(s, \xi) = x_\Gamma(s) + \epsilon\xi\nu(s). \end{aligned}$$

Then  $\operatorname{div}(\Theta \cdot \nabla u) = \Delta_{s, \xi} \hat{u}$  with  $\hat{u}(s, \xi) := u(\varphi_\epsilon(s, \xi))$  and

$$\Delta_{s, \xi} \hat{u} := \frac{1}{(1 + \epsilon\xi\kappa)} \frac{\partial}{\partial s} \left( \theta^\tau \frac{1}{(1 + \epsilon\xi\kappa)} \frac{\partial \hat{u}}{\partial s} \right) + \frac{1}{\epsilon^2(1 + \epsilon\xi\kappa)} \frac{\partial}{\partial \xi} \left( (1 + \epsilon\xi\kappa) \theta^\nu \frac{\partial \hat{u}}{\partial \xi} \right). \quad (20)$$

**4.2. Correction for the operator  $A_\epsilon^{-1} - A_0^{-1}$ .** — We assume in the following that  $n_1 = n_0$ . We construct a corrector term and use it to estimate the convergence rate of  $(z_\epsilon, t_\epsilon) = A_\epsilon^{-1}(w, v)$  for  $(w, v) \in X(\Omega)$ . Let  $(z_0, t_0) = A_0^{-1}(w, v) \in X(\Omega)$  solution of

$$\begin{cases} \operatorname{div}(\Theta_0 \nabla z_0) - z_0 = 0 & \text{in } \Omega, \\ \Delta t_0 - t_0 = 0 & \text{in } \Omega, \\ z_0 - t_0 = 0 & \text{on } \Gamma, \\ \Theta_0 \frac{\partial z_0}{\partial \nu} - \frac{\partial t_0}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases} \quad (21)$$

Inspired by the formal calculations in Section 5, we define  $(z_1, t_1)$  solution of

$$\begin{cases} \operatorname{div}(\Theta_0 \nabla z_1) - z_1 = 0 & \text{in } \Omega, \\ \Delta t_1 - t_1 = 0 & \text{in } \Omega, \\ z_1 - t_1 = \left( \frac{\theta_0^\nu - \theta_1^\nu}{\theta_1^\nu} \right) \frac{\partial z_0}{\partial \eta} & \text{on } \Gamma, \\ \Theta_0 \frac{\partial z_1}{\partial \nu} - \frac{\partial t_1}{\partial \nu} = \frac{\partial}{\partial s} \left( (\theta_0^\tau - \theta_1^\tau) \frac{\partial z_0}{\partial s} \right) & \text{on } \Gamma. \end{cases} \quad (22)$$

We extend  $z_1$  in  $\Omega_\epsilon$  as  $\tilde{z}_1^\epsilon$  defined by

$$\tilde{z}_1^\epsilon = \begin{cases} z_1 & \text{in } \Omega_\epsilon^0, \\ \psi & \text{in } \Omega_\epsilon, \end{cases} \quad (23)$$

where  $\psi$  is a polynomial of order  $\leq 1$  and satisfying the boundary conditions:

$$\begin{cases} \psi = t_1 & \text{on } \Gamma, \\ \psi = z_1 & \text{on } \Gamma_\epsilon. \end{cases} \quad (24)$$

This gives the following expression of  $\hat{\psi}(s, \xi) = \psi(x)$  in the local coordinates:

$$\hat{\psi}(s, \xi) = z_1(\varphi(s, \epsilon))\xi + t_1(\varphi(s, 0))(1 - \xi).$$

The choice of  $\psi$  ensures in particular that  $(\tilde{z}_1^\epsilon, t_1) \in X(\Omega)$ . Using Taylor's expansion for  $z_1$  and the third equation of (22) **and using the fact that  $\theta_j^\nu = \Theta_j \cdot \nu$  and  $\theta_j^\tau = \Theta_j \cdot \tau$ ,  $j = 0, 1$** , we then get

$$\begin{aligned} \hat{\psi}(s, \xi) &= z_1(\varphi(s, 0))\xi + t_1(\varphi(s, 0))(1 - \xi) + O(\epsilon) \\ &= \left( \frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)} \right) \frac{\partial z_0}{\partial \eta}(\varphi(s, 0))\xi + t_1(\varphi(s, 0)) + O(\epsilon) \end{aligned}$$

which indicates that up to order 1 in  $\epsilon$ ,  $\hat{\psi}$  coincides with the first term of the formal expansion of  $z_\epsilon$  inside  $\Omega_\epsilon$  (See Section 5). Now we have the following Lemma that will be useful in the proof of Lemma 4.2.

**Lemma 4.1.** — *Let  $(w, v) \in X(\Omega)$  then set*

$$(z_\epsilon, t_\epsilon) = A_\epsilon^{-1}(w, v) \text{ and } (z_0, t_0) = A_0^{-1}(w, v). \quad (25)$$

*Assume that  $(z_0, t_1) \in C^2(\bar{\Omega}) \times C^1(\bar{\Omega})$  and we define  $\tilde{z}_1^\epsilon$  as in (23), then we have for sufficiently small  $\epsilon$*

$$\|z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon\|_{H^1(\Omega)} + \|t_\epsilon - t_0 - \epsilon t_1\|_{H^1(\Omega)} \leq C\epsilon^{\frac{3}{2}}.$$

*Proof.* — We define the error function in  $X(\Omega)$  as

$$z_2 = z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon \text{ and } t_2 = t_\epsilon - t_0 - \epsilon t_1.$$

Now for any  $(\phi_1, \phi_2) \in X(\Omega)$ , we have that

$$\begin{aligned}
\langle A_\epsilon(z_2, t_2); (\phi_1, \phi_2) \rangle &= \int_{\Omega} \left( \Theta_\epsilon \nabla z_2 \nabla \phi_1 + z_2 \phi_1 \right) dx - \int_{\Omega} \left( \nabla t_2 \nabla \phi_2 + t_2 \phi_2 \right) dx \\
&= \int_{\Omega} \left( \Theta_\epsilon \nabla (z_\epsilon - z_0) \nabla \phi_1 + (z_\epsilon - z_0) \phi_1 \right) dx \\
&\quad - \int_{\Omega} \left( \nabla (t_\epsilon - t_0) \nabla \phi_2 + (t_\epsilon - t_0) \phi_2 \right) dx \\
&\quad + \epsilon \int_{\Omega} \left( \nabla t_1 \nabla \phi_2 + t_1 \phi_2 \right) dx \\
&\quad - \epsilon \int_{\Omega} \left( \Theta_\epsilon \nabla \tilde{z}_1^\epsilon \nabla \phi_1 + \Theta_0 \tilde{z}_1^\epsilon \phi_1 \right) dx.
\end{aligned}$$

Using(25) we get

$$\begin{aligned}
\langle A_\epsilon(z_\epsilon - z_0, t_\epsilon - t_0); (\phi_1, \phi_2) \rangle &= \int_{\Omega} \left( \Theta_\epsilon \nabla (z_\epsilon - z_0) \nabla \phi_1 + (z_\epsilon - z_0) \phi_1 \right) dx \\
&\quad - \int_{\Omega} \left( \nabla (t_\epsilon - t_0) \nabla \phi_2 + (t_\epsilon - t_0) \phi_2 \right) dx \\
&= \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla z_0 \nabla \phi_1 dx.
\end{aligned}$$

Using the fact that  $\operatorname{div}(\Theta_0 \nabla z_1) - z_1 = 0$ ,  $\Delta t_1 - t_1 = 0$  and the Green formula yield

$$\begin{aligned}
\langle A_\epsilon(z_2, t_2); (\phi_1, \phi_2) \rangle &= \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla z_0 \nabla \phi_1 dx \\
&\quad - \epsilon \int_{\Omega_\epsilon} \Theta_1 \nabla \psi \nabla \phi_1 + \psi \phi_1 dx \\
&\quad - \epsilon \int_{\Gamma} \frac{\partial t_1}{\partial \nu} \phi_2 ds(x) + \epsilon \int_{\Gamma_\epsilon} \Theta_0 \frac{\partial z_1}{\partial \nu} \phi_1 ds(x).
\end{aligned}$$

To estimate  $\int_{\Omega_\epsilon} \psi \phi_1 dx$ , we use Cauchy-Schwarz inequality to obtain

$$\int_{\Omega_\epsilon} \psi \phi_1 dx \leq \|\psi \mathbf{1}_{\Omega_\epsilon}\|_{L^2(\Omega)} \|\phi_1\|_{L^2(\Omega)}.$$

From the definition of  $\psi$  and if  $(z_0, t_1) \in \mathcal{C}^2(\bar{\Omega}) \times \mathcal{C}^1(\bar{\Omega})$ , we then get

$$\|\psi \mathbf{1}_{\Omega_\epsilon}\|_{L^2(\Omega)}^2 = \int_0^L \int_0^1 \left( \left( \frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)} \right) \frac{\partial z_0}{\partial \eta}(\varphi(s, 0)) \xi + t_1(x_\Gamma(s)) \right)^2 \epsilon (1 + \epsilon \xi \kappa) ds d\xi \leq C\epsilon.$$

Hence

$$\epsilon \int_{\Omega_\epsilon} \psi \phi_1 dx \leq C\epsilon^{\frac{3}{2}} \|\phi_1\|_{H^1(\Omega)}.$$

We then deduce

$$\begin{aligned} \langle A_\epsilon(z_2, t_2); (\phi_1, \phi_2) \rangle &= \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla z_0 \nabla \phi_1 dx - \epsilon \int_{\Omega_\epsilon} \Theta_1 \nabla \psi \nabla \phi_1 dx \\ &\quad - \epsilon \int_\Gamma \frac{\partial t_1}{\partial \nu} \phi_2 ds(x) + \epsilon \int_{\Gamma_\epsilon} \Theta_0 \frac{\partial z_1}{\partial \nu} \phi_1 ds(x) \\ &\quad + O(\epsilon^{\frac{3}{2}}) \|(\phi_1, \phi_2)\|_{X(\Omega)}. \end{aligned}$$

Using the expression of  $\psi$  we have

$$\begin{aligned} \Theta_1 \nabla_{s,\xi} \hat{\psi} &= \frac{1}{\epsilon} (\theta_0^\nu(s) - \theta_1^\nu(s)) \frac{\partial z_0}{\partial \eta} (\varphi(s, 0)) \\ &\quad + \frac{\theta_1^\tau(s)}{1 + \epsilon \xi \kappa} \left( \frac{\partial}{\partial s} \left( \frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)} \frac{\partial z_0}{\partial \eta} (\varphi(s, 0)) \right) \xi + \Theta_\tau \frac{\partial}{\partial s} t_1(\varphi(s, 0)) \right) + O(\epsilon). \end{aligned}$$

We then get after substitution of these expressions

$$\begin{aligned} &\epsilon \int_{\Omega_\epsilon} \Theta_1 \nabla \psi \nabla \phi_1 dx \\ &= \epsilon \int_0^L \int_0^1 \frac{1}{\epsilon} \left( \frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)} \right) \frac{\partial z_0}{\partial \eta} (\varphi(s, 0)) \nabla \phi_1(\varphi(s, \epsilon \xi)) \epsilon (1 + \epsilon \xi \kappa) ds d\xi \\ &\quad + \epsilon \int_0^L \int_0^1 \frac{\theta_1^\tau(s)}{1 + \epsilon \xi \kappa} \frac{\partial}{\partial s} \left( \frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)} \frac{\partial z_0}{\partial \eta} (\varphi(s, 0)) \right) \nabla \phi_1(\varphi(s, \epsilon \xi)) \epsilon \xi (1 + \epsilon \xi \kappa) ds d\xi \\ &\quad + \epsilon \int_0^L \int_0^1 \frac{1}{1 + \epsilon \xi \kappa} \left( \theta_1^\tau(s) \frac{\partial}{\partial s} t_1(x_\Gamma(s)) \right) \nabla \phi_1(\varphi(s, \epsilon \xi)) \epsilon (1 + \epsilon \xi \kappa) ds d\xi. \end{aligned}$$

For the second term, if  $z_0 \in C^2(\bar{\Omega})$  we have

$$\begin{aligned} &\epsilon \int_0^L \int_0^1 \frac{\theta_1^\tau(s) \xi}{1 + \epsilon \xi \kappa} \frac{\partial}{\partial s} \left( \frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)} \frac{\partial z_0}{\partial \eta} (\varphi(s, 0)) \right) \nabla \phi_1(\varphi(s, \epsilon \xi)) \epsilon (1 + \epsilon \xi \kappa) ds d\xi \\ &\leq C \epsilon \|\theta_1^\tau(s)\|_\infty \left\| \frac{\partial}{\partial s} \left( \frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)} \frac{\partial z_0}{\partial \eta} (\varphi(s, 0)) \right) \right\|_\infty \int_0^L \int_0^1 \nabla \phi_1(\varphi(s, \epsilon \xi)) \epsilon (1 + \epsilon \xi \kappa) ds d\xi \\ &\leq C \epsilon \|\mathbf{1}_{\Omega_\epsilon}\|_{L^2(\Omega)} \|\nabla \phi_1\|_{L^2(\Omega)} \leq C \epsilon^{\frac{3}{2}} \|\phi_1\|_{H^1(\Omega)} \end{aligned}$$

Similarly if  $t_1 \in C^1(\bar{\Omega})$  we obtain

$$\epsilon \int_0^L \int_0^1 \frac{\theta_1^\tau(s)}{1 + \epsilon \xi \kappa} \frac{\partial}{\partial s} t_1(x_\Gamma(s)) \nabla \phi_1(\varphi(s, \epsilon \xi)) \epsilon (1 + \epsilon \xi \kappa) ds d\xi = O(\epsilon^{\frac{3}{2}}) \|\phi_1\|_{H^1(\Omega)}$$

Thus

$$\begin{aligned}
\left\langle A_\epsilon(z_2, t_2); (\phi_1, \phi_2) \right\rangle &= \int_0^L \int_0^1 (\Theta_0 - \Theta_1) \nabla z_0(\varphi(s, \epsilon\xi)) \nabla \phi_1(\varphi(s, \epsilon\xi)) \epsilon(1 + \epsilon\xi\kappa) ds d\xi \\
&- \int_0^L \int_0^1 (\theta_0^\nu(s) - \theta_1^\nu(s)) \frac{\partial z_0}{\partial \eta}(\varphi(s, 0)) \nabla \phi_1(\varphi(s, \epsilon\xi)) \epsilon(1 + \epsilon\xi\kappa) ds d\xi \\
&- \epsilon \int_\Gamma \frac{\partial t_1}{\partial \nu} \phi_2 ds(x) + \epsilon \int_{\Gamma_\epsilon} \Theta_0 \frac{\partial z_1}{\partial \nu} \phi_1 ds(x) \\
&+ O(\epsilon^{\frac{3}{2}}) \|(\phi_1, \phi_2)\|_{X(\Omega)}.
\end{aligned}$$

We use the expression of the gradient operator in the local coordinates

$$\begin{aligned}
&(\Theta_1 - \Theta_0) \nabla z_0(\varphi(s, \epsilon\xi)) = \\
&\frac{(\theta_1^\tau(s) - \theta_0^\tau(s))}{(1 + \epsilon\xi\kappa(s))} \frac{\partial z_0}{\partial s}(\varphi(s, \epsilon\xi)) \tau(s) + (\theta_1^\nu(s) - \theta_0^\nu(s)) \frac{\partial z_0}{\partial \eta}(\varphi(s, \epsilon\xi)) \nu(s)
\end{aligned}$$

to make the decomposition

$$\begin{aligned}
&\left\langle A_\epsilon(z_2, t_2); (\phi_1, \phi_2) \right\rangle \\
&= \int_0^L \int_0^1 (\theta_0^\nu(s) - \theta_1^\nu(s)) \left( \frac{\partial z_0}{\partial \eta}(\varphi(s, \epsilon\xi)) - \frac{\partial z_0}{\partial \eta}(\varphi(s, 0)) \right) \nabla \phi_1(\varphi(s, \epsilon\xi)) \epsilon(1 + \epsilon\xi\kappa) ds d\xi \\
&+ \int_0^L \int_0^1 (\theta_0^\tau(s) - \theta_1^\tau(s)) \frac{\partial z_0}{\partial s}(\varphi(s, \epsilon\xi)) \frac{\partial \phi_1}{\partial s}(\varphi(s, \epsilon\xi)) \epsilon ds d\xi \\
&- \epsilon \int_\Gamma \frac{\partial t_1}{\partial \nu} \phi_2 ds(x) + \epsilon \int_{\Gamma_\epsilon} \Theta_0 \frac{\partial z_1}{\partial \nu} \phi_1 ds(x) \\
&+ O(\epsilon^{\frac{3}{2}}) \|(\phi_1, \phi_2)\|_{X(\Omega)}.
\end{aligned}$$

Using the parametrization of the curve  $\Gamma_\epsilon$ ,  $s \mapsto \varphi(s, \epsilon)$  we have:

$$\int_{\Gamma_\epsilon} \Theta_0 \frac{\partial z_1}{\partial \nu} \phi_1 ds(x) = \int_0^L \theta_0^\nu(s) \frac{\partial z_1}{\partial \eta}(\varphi(s, \epsilon)) \phi_1(\varphi(s, \epsilon)) (1 + \epsilon\kappa) ds$$

We use Taylor expansion to obtain

$$\int_{\Gamma_\epsilon} \Theta_0 \frac{\partial z_1}{\partial \nu} \phi_1 ds(x) = \int_0^L \theta_0^\nu(s) \frac{\partial z_1}{\partial \eta}(\varphi(s, 0)) \phi_1(\varphi(s, 0)) ds + O(\epsilon^{\frac{1}{2}}) \|\phi_1\|_{H^1(\Omega)} \quad (26)$$

$$= \int_\Gamma \Theta_0 \frac{\partial z_1}{\partial \nu} \phi_1 ds(x) + O(\epsilon^{\frac{1}{2}}) \|\phi_1\|_{H^1(\Omega)}. \quad (27)$$

The last equation of problem (22) then gives

$$\begin{aligned} \epsilon \int_{\Gamma_\epsilon} \Theta_0 \frac{\partial z_1}{\partial \nu} \phi_1 ds(x) - \epsilon \int_{\Gamma} \frac{\partial t_1}{\partial \nu} \phi_2 ds(x) &= \epsilon \int_{\Gamma} \left( \Theta_0 \frac{\partial z_1}{\partial \nu} - \frac{\partial t_1}{\partial \nu} \right) \phi_1 ds(x) + O(\epsilon^{\frac{3}{2}}) \|\phi_1\|_{H^1(\Omega)} \\ &= \epsilon \int_0^L \frac{\partial}{\partial s} \left( (\theta_0^\tau(s) - \theta_1^\tau(s)) \frac{\partial z_0}{\partial s}(\varphi(s, 0)) \right) \phi_1(\varphi(s, 0)) ds \\ &\quad + O(\epsilon^{\frac{3}{2}}) \|\phi_1\|_{H^1(\Omega)} \end{aligned}$$

One ends up with

$$\begin{aligned} &\left\langle A_\epsilon(z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon, t_\epsilon - t_0); (\phi_1, \phi_2) \right\rangle \\ &= \int_0^L \int_0^1 (\theta_0^\nu(s) - \theta_1^\nu(s)) \left( \frac{\partial z_0}{\partial \eta}(\varphi(s, \epsilon \xi)) - \frac{\partial z_0}{\partial \eta}(\varphi(s, 0)) \right) \nabla \phi_1(\varphi(s, \epsilon \xi)) \epsilon (1 + \epsilon \xi \kappa) ds d\xi \\ &\quad + \epsilon \int_0^L \frac{\partial}{\partial s} \left( (\theta_0^\tau(s) - \theta_1^\tau(s)) \frac{\partial z_0}{\partial s}(\varphi(s, 0)) \right) \phi_1(\varphi(s, 0)) ds \\ &\quad + \epsilon \int_0^L \int_0^1 (\theta_0^\tau(s) - \theta_1^\tau(s)) \frac{\partial z_0}{\partial s}(\varphi(s, \epsilon \xi)) \frac{\partial \phi_1}{\partial s}(\varphi(s, \epsilon \xi)) ds d\xi + O(\epsilon^{\frac{3}{2}}) \|(\phi_1, \phi_2)\|_{X(\Omega)}. \end{aligned}$$

For the last term, we integrate by parts and we use the periodicity of  $(\theta_0^\tau - \theta_1^\tau) \frac{\partial z_0}{\partial s}$  and  $\phi_1$  on  $[0, L[$  to get

$$\begin{aligned} &\epsilon \int_0^L \int_0^1 (\theta_0^\tau(s) - \theta_1^\tau(s)) \frac{\partial z_0}{\partial s}(\varphi(s, \epsilon \xi)) \frac{\partial \phi_1}{\partial s}(\varphi(s, \epsilon \xi)) ds d\xi \\ &= -\epsilon \int_0^L \frac{\partial}{\partial s} \left( (\theta_0^\tau(s) - \theta_1^\tau(s)) \frac{\partial z_0}{\partial s}(\varphi(s, 0)) \right) \phi_1(\varphi(s, 0)) ds + O(\epsilon^{\frac{3}{2}}) \|\phi_1\|_{H^1(\Omega)} \end{aligned}$$

To conclude, we just use Taylor expansion for  $\frac{\partial z_0}{\partial \eta}$  and  $\frac{\partial z_0}{\partial s}$  and obtain

$$\left\langle A_\epsilon(z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon, t_\epsilon - t_0 - \epsilon t_1); (\phi_1, \phi_2) \right\rangle \leq C \epsilon^{\frac{3}{2}} \|(\phi_1, \phi_2)\|_{X(\Omega)}$$

Choosing  $(\phi_1, \phi_2) = \mathbb{T}(z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon, t_\epsilon - t_0 - \epsilon t_1)$  and since the  $\mathbb{T}$ -coercivity constant associated with  $\mathcal{A}_\epsilon$  is independent from  $\epsilon$ , we get

$$\|z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon\|_{H^1(\Omega)} + \|t_\epsilon - t_0 - \epsilon t_1\|_{H^1(\Omega)} \leq C \epsilon^{\frac{3}{2}} \quad (28)$$

which ends the proof.  $\square$

**Lemma 4.2.** — *If  $(w, v) \in X(\Omega)$ , then for sufficiently small  $\epsilon$ ,*

$$\|B_0(A_\epsilon^{-1} - A_0^{-1})(w, v)\| = O(\epsilon) \text{ and } \|C_0(A_\epsilon^{-1} - A_0^{-1})(w, v)\| = O(\epsilon).$$

*Proof.* — For all  $(\phi_1, \phi_2) \in X(\Omega)$ , we have

$$\left\langle B_0(A_\epsilon^{-1} - A_0^{-1})(w, v); (\phi_1, \phi_2) \right\rangle = \mathcal{B}_0\left((z_\epsilon - z_0, t_\epsilon - t_0); (\phi_1, \phi_2)\right).$$



Since the sesquilinear  $\mathcal{B}_0$  only has  $L^2(\Omega)$  terms, we have

$$\left| \left\langle B_0(A_\epsilon^{-1} - A_0^{-1})(w, v); (\phi_1, \phi_2) \right\rangle \right| \leq C \| (z_\epsilon - z_0, t_\epsilon - t_0) \|_{L^2(\Omega)} \| (\phi_1, \phi_2) \|_{X(\Omega)}$$

From (28) and  $\epsilon \| \tilde{z}_1^\epsilon \|_{L^2(\Omega)} = O(\epsilon)$ ,  $\epsilon \| t_1 \|_{L^2(\Omega)} = O(\epsilon)$  we get

$$\left| \left\langle B_0(A_\epsilon^{-1} - A_0^{-1})(w, v); (\phi_1, \phi_2) \right\rangle \right| \leq C \epsilon \| (\phi_1, \phi_2) \|_{X(\Omega)}.$$

Therefore

$$\| B_0(A_\epsilon^{-1} - A_0^{-1})(w, v) \| = O(\epsilon).$$

A similar argument gives the second result.  $\square$

**4.3. Asymptotic formulas.** — We have proved that the operator  $T_\epsilon(\lambda) = A_\epsilon^{-1}B_0 + \frac{1}{\lambda}A_\epsilon^{-1}C_0$  converges in norm to  $T_0(\lambda)$ . Using the results of the previous section we are now in a position to derive the expansion for the simple real transmission eigenvalues under the assumption  $n_1 = n_0$ . We employ the following abstract result of [19] which is a generalization of Osborn's Theorem to non-linear eigenvalues problems.

**Theorem 4.3.** — *Let  $Y$  be a Hilbert space and let  $\{K_\epsilon : Y \rightarrow Y\}$  be a set of compact linear operator valued functions of  $\lambda$  which are analytic in a region  $U \subset \mathbb{C}$ . Assume that  $K_\epsilon(\lambda)$  converge to  $K_0(\lambda)$  in the operator norm for all  $\lambda \in U$ . Now suppose that  $\lambda_0$  is a simple non-linear eigenvalue of  $K_0$  with eigenfunction  $\phi$ . Then if*

$$\lambda_0^2 \left\langle DK_0(\lambda_0)\phi, \phi^* \right\rangle \neq -1,$$

*we have the following formula*

$$\begin{aligned} \lambda_\epsilon = \lambda_0 + & \frac{\lambda_0^2 \left\langle (K_0(\lambda_0) - K_\epsilon(\lambda_0))\phi, \phi^* \right\rangle}{1 + \lambda_0^2 \left\langle DK_0(\lambda_0)\phi, \phi^* \right\rangle} \\ & + O\left( \sup_{\lambda \in U} \| (K_\epsilon(\lambda) - K_0(\lambda))\phi \| \| (K_\epsilon^*(\lambda) - K_0^*(\lambda))\phi^* \| \right) \end{aligned}$$

*with  $\lambda_\epsilon$  is a non-linear eigenvalue for  $K_\epsilon(\lambda)$ .*

Now, we have all the ingredients to prove the main result of this paper by applying this theorem to  $K_\epsilon = T_\epsilon$ ,  $K_0 = T_0$  and  $Y = X(\Omega) \times X(\Omega)$ . By Theorem 3.5, we have that  $T_\epsilon \rightarrow T_0$  in the operator norm as  $\epsilon \rightarrow 0$ .

Let  $\lambda_0$  be a non-linear eigenvalue of  $T_0$ , then for  $\epsilon$  small enough, some non-linear eigenvalue  $\lambda_\epsilon$  of  $T_\epsilon$  is such that  $\lambda_\epsilon \rightarrow \lambda_0$  as  $\epsilon \rightarrow 0$ . Let  $(w_0, v_0)$  and  $A_0(w_0, v_0)$  are eigenfunctions respectively of the operators  $T_0$  and  $T_0^*$  corresponding to  $\lambda_0$ .

*Proof.* — of Theorem 2.2

Using Theorem 4.3, we have that

$$\begin{aligned} \lambda_\epsilon - \lambda_0 &= \frac{\lambda_0^2 \langle (T_0(\lambda_0) - T_\epsilon(\lambda_0))(w_0, v_0), A_0(w_0, v_0) \rangle}{1 + \lambda_0^2 \langle DT_0(\lambda_0)(w_0, v_0), A_0(w_0, v_0) \rangle} \\ &\quad + O\left(\sup_{\lambda \in U} \|(T_\epsilon(\lambda) - T_0(\lambda))(w_0, v_0)\| \|(T_\epsilon^*(\lambda) - T_0^*(\lambda))A_0(w_0, v_0)\|\right). \end{aligned}$$

From Lemma 3.4, we obtain

$$\|(T_\epsilon(\lambda) - T_0(\lambda))(w_0, v_0)\| = O(\epsilon^{\frac{1}{2}}).$$

Since  $T_\epsilon^*(\lambda) = B_0 A_\epsilon^{-1} + \frac{1}{\lambda} C_0 A_\epsilon^{-1}$  and from Lemma 4.2, we have

$$\|(T_\epsilon^*(\lambda) - T_0^*(\lambda))A_0(w_0, v_0)\| = O(\epsilon).$$

Next we have that

$$\lambda_\epsilon - \lambda_0 = \frac{\lambda_0^2 \langle (T_0(\lambda_0) - T_\epsilon(\lambda_0))(w_0, v_0), A_0(w_0, v_0) \rangle}{1 + \lambda_0^2 \langle DT_0(\lambda_0)(w_0, v_0), A_0(w_0, v_0) \rangle} + O(\epsilon^{\frac{3}{2}}).$$

From the definition of  $T_\epsilon$  and  $T_0$  and using the fact that

$$B_0(w_0, v_0) + \frac{1}{\lambda_0} C_0(w_0, v_0) = \frac{1}{\lambda_0} A_0(w_0, v_0),$$

we get

$$\langle (T_0(\lambda_0) - T_\epsilon(\lambda_0))(w_0, v_0); A_0(w_0, v_0) \rangle = -\frac{1}{\lambda_0} \langle (A_\epsilon^{-1} - A_0^{-1})A_0(w_0, v_0); A_0(w_0, v_0) \rangle \quad (29)$$

Applying Lemma 4.4 with  $(w, v) = A_0(w_0, v_0)$  and  $(\phi_1, \phi_2) = (w_0, v_0)$ , we have

$$\begin{aligned} \langle (A_\epsilon^{-1} - A_0^{-1})A_0(w_0, v_0); A_0(w_0, v_0) \rangle &= \epsilon \int_0^L (\theta_0^\tau(s) - \theta_1^\tau(s)) \left| \frac{\partial w_0}{\partial s}(\varphi(s, 0)) \right|^2 ds \\ &\quad + \epsilon \int_0^L \theta_0^\nu(s) \left( \frac{\theta_0^\nu(s) - \theta_\nu^1(s)}{\theta_\nu^1(s)} \right) \left| \frac{\partial w_0}{\partial \eta}(\varphi(s, 0)) \right|^2 ds + O(\epsilon^{\frac{3}{2}}). \end{aligned} \quad (30)$$

From(30), we have

$$\begin{aligned} \langle (T_0(\lambda_0) - T_\epsilon(\lambda_0))(w_0, v_0); A_0(w_0, v_0) \rangle &= -\frac{\epsilon}{\lambda_0} \int_0^L (\theta_0^\tau(s) - \theta_1^\tau(s)) \left| \frac{\partial w_0}{\partial s}(\varphi(s, 0)) \right|^2 ds \\ &\quad - \frac{\epsilon}{\lambda_0} \int_0^L \theta_0^\nu(s) \left( \frac{\theta_0^\nu(s) - \theta_\nu^1(s)}{\theta_\nu^1(s)} \right) \left| \frac{\partial w_0}{\partial \eta}(\varphi(s, 0)) \right|^2 ds + O(\epsilon^{\frac{3}{2}}). \end{aligned}$$

For the denominator in the correction theorem, we must compute the derivative of  $T_0(\lambda)$  with respect to  $\lambda$ . In fact this derivative is  $DT_0(\lambda_0) = -\frac{1}{\lambda_0^2}A_0^{-1}C_0$ .

Therefore, we have that

$$\lambda_0^2 \left\langle DT_0(\lambda_0)(w_0, v_0); A_0(w_0, v_0) \right\rangle = - \left\langle C_0(w_0, v_0); (w_0, v_0) \right\rangle = - \int_{\Omega} (|w_0|^2 - |v_0|^2) dx.$$

Since

$$1 = \left\langle A_0(w_0, v_0); (w_0, v_0) \right\rangle,$$

and from the definition of  $A_0$ , we get

$$1 = \int_{\Omega} (\Theta_0 |\nabla w_0|^2 + |w_0|^2) dx - \int_{\Omega} (|\nabla v_0|^2 + |v_0|^2) dx.$$

We then deduce

$$1 + \lambda_0^2 \left\langle DT_0(\lambda_0)(w_0, v_0); (w_0, v_0) \right\rangle = \int_{\Omega} (\Theta_0 |\nabla w_0|^2 - |\nabla v_0|^2) dx.$$

The couple  $(w_0, v_0)$  satisfies

$$\begin{cases} \operatorname{div}(\Theta_0 \nabla w_0) + \lambda_0 n_0 w_0 = 0 & \text{in } \Omega, \\ \Delta v_0 + \lambda_0 v_0 = 0 & \text{in } \Omega, \\ w_0 - v_0 = 0 & \text{on } \Gamma, \\ \Theta_0 \frac{\partial w_0}{\partial \nu} - \frac{\partial v_0}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases} \quad (31)$$

We multiply the first equation of (31) with  $\bar{w}_0$  and the second equation with  $\bar{v}_0$ , taking the difference then integrating by parts, we have

$$\int_{\Omega} (\Theta_0 |\nabla w_0|^2 - |\nabla v_0|^2) dx - \lambda_0 \int_{\Omega} (n_0 |w_0|^2 - |v_0|^2) dx = 0,$$

which gives

$$1 + \lambda_0^2 \left\langle DT_0(\lambda_0)(w_0, v_0); A_0(w_0, v_0) \right\rangle = \lambda_0 \int_{\Omega} (n_0 |w_0|^2 - |v_0|^2) dx.$$

Finally, we obtain

$$\begin{aligned} \lambda_{\epsilon} - \lambda_0 &= \frac{\lambda_0^2 \left\langle (T_0(\lambda_0) - T_{\epsilon}(\lambda_0))(w_0, v_0), A_0(w_0, v_0) \right\rangle}{1 + \lambda_0^2 \left\langle DT_0(\lambda_0)(w_0, v_0), A_0(w_0, v_0) \right\rangle} + O(\epsilon^{\frac{3}{2}}) \\ &= \epsilon \lambda_1 + O(\epsilon^{\frac{3}{2}}) \end{aligned}$$

with

$$\lambda_1 = \frac{-\int_0^L (\theta_0^\tau(s) - \theta_1^\tau(s)) \left| \frac{\partial w_0}{\partial s}(\varphi(s, 0)) \right|^2 ds - \int_\Gamma \theta_0^\nu(s) \left( \frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)} \right) \left| \frac{\partial w_0}{\partial \nu} \right|^2 ds(x)}{\int_\Omega (n_0 |w_0|^2 - |v_0|^2) dx}$$

which corresponds with the formula announced in the theorem and concludes the proof.  $\square$

**Lemma 4.4.** — *Let  $A_\epsilon$  and  $A_0$  be defined by (3) for  $\epsilon > 0$  and  $\epsilon = 0$ , respectively. Then for any  $(\phi_1, \phi_2) \in X(\Omega)$*

$$\begin{aligned} \left\langle A_0(A_\epsilon^{-1} - A_0^{-1})(w, v); (\phi_1, \phi_2) \right\rangle &= \epsilon \int_0^L (\theta_0^\tau(s) - \theta_1^\tau(s)) \frac{\partial z_0}{\partial s}(\varphi(s, 0)) \frac{\partial \phi_1}{\partial s}(\varphi(s, 0)) ds \\ &+ \epsilon \int_0^L \left( \theta_0^\nu(s) \left( \frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)} \right) \right) \frac{\partial z_0}{\partial \eta}(\varphi(s, 0)) \frac{\partial \phi_1}{\partial \eta}(\varphi(s, 0)) ds + O(\epsilon^{\frac{3}{2}}). \end{aligned} \quad (32)$$

*Proof.* — With  $(z_\epsilon, t_\epsilon) = A_\epsilon^{-1}(w, v)$

$$\begin{aligned} \left\langle A_0(A_\epsilon^{-1} - A_0^{-1})(w, v); (\phi_1, \phi_2) \right\rangle &= \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla z_\epsilon \nabla \phi_1 dx \\ &= \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla (z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon) \nabla \phi_1 dx \\ &+ \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla (z_0 + \epsilon \tilde{z}_1^\epsilon) \nabla \phi_1 dx. \end{aligned}$$

Applying Lemma 4.1, we obtain

$$\begin{aligned} \left\langle A_0(A_\epsilon^{-1} - A_0^{-1})(w, v); (\phi_1, \phi_2) \right\rangle &= \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla z_0 \nabla \phi_1 dx \\ &+ \epsilon \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla \tilde{z}_1^\epsilon \nabla \phi_1 dx + O(\epsilon^{\frac{3}{2}}). \end{aligned} \quad (33)$$

Making use the local coordinates, we show

$$\begin{aligned} \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla z_0 \nabla \phi_1 dx &= \int_0^L (\theta_0^\tau(s) - \theta_1^\tau(s)) \frac{\partial z_0}{\partial s}(\varphi(s, 0)) \frac{\partial \phi_1}{\partial s}(\varphi(s, 0)) ds \\ &+ \epsilon \int_0^L (\theta_0^\nu(s) - \theta_1^\nu(s)) \frac{\partial z_0}{\partial \eta}(\varphi(s, 0)) \frac{\partial \phi_1}{\partial \eta}(\varphi(s, 0)) ds + O(\epsilon^{\frac{3}{2}}). \end{aligned}$$

For the second term of (33)

$$\begin{aligned} \epsilon \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla \tilde{z}_1^\epsilon \nabla \phi_1 dx &= \epsilon \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1) \nabla \psi \nabla \phi_1 dx \\ &= \epsilon \int_0^L \int_0^1 (\theta_0^\tau(s) - \theta_1^\tau(s)) \nabla_{s,\xi} \hat{\psi}(s, \xi) \nabla \phi_1(\varphi(s, \epsilon \xi)) \epsilon (1 + \epsilon \xi \kappa) ds d\xi \\ &= \epsilon \int_0^L (\theta_0^\nu(s) - \theta_1^\nu(s)) \frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)} \frac{\partial z_0}{\partial \eta}(\varphi(s, 0)) \frac{\partial \phi_1}{\partial \eta}(\varphi(s, 0)) ds + O(\epsilon^{\frac{3}{2}}). \end{aligned}$$

which implies

$$\begin{aligned} \left\langle A_0(A_\epsilon^{-1} - A_0^{-1})(w, v); (\phi_1, \phi_2) \right\rangle &= \epsilon \int_0^L (\theta_0^\tau(s) - \theta_1^\tau(s)) \frac{\partial z_0}{\partial s}(\varphi(s, 0)) \frac{\partial \phi_1}{\partial s}(\varphi(s, 0)) ds \\ &+ \epsilon \int_0^L \theta_0^\nu(s) \frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)} \frac{\partial z_0}{\partial \eta}(\varphi(s, 0)) \frac{\partial \phi_1}{\partial \eta}(\varphi(s, 0)) ds + O(\epsilon^{\frac{3}{2}}) \end{aligned}$$

and concludes the proof.  $\square$

## 5. Validation of the asymptotic formula

Our main goal in this section is to validate formally the asymptotic formula given in Theorem 2.2. To this end, we propose the following expansion for the transmission eigenvalues:

$$\lambda_\epsilon = \sum_{j=0}^{\infty} \epsilon^j \lambda_j. \quad (34)$$

To make the formal calculations, we need to separate the thin layer and scaled it with respect to the thickness so that the first equation of (35) is posed on a domain independent from  $\epsilon$ . We therefore rewrite the transmission eigenvalue problem (1) in the following equivalent form

$$\left\{ \begin{array}{ll} \operatorname{div}(\Theta_1 \nabla w_\epsilon^+) + \lambda_\epsilon n_1 w_\epsilon^+ = 0 & \text{in } \Omega_\epsilon, \\ \operatorname{div}(\Theta_0 \nabla w_\epsilon^-) + \lambda_\epsilon n_0 w_\epsilon^- = 0 & \text{in } \Omega_\epsilon^0, \\ \Delta v_\epsilon + \lambda_\epsilon v_\epsilon = 0 & \text{in } \Omega, \\ w_\epsilon^+ = w_\epsilon^-, \Theta_1 \frac{\partial w_\epsilon^+}{\partial \nu} = \Theta_0 \frac{\partial w_\epsilon^-}{\partial \nu} & \text{on } \Gamma_\epsilon, \\ w_\epsilon^+ = v_\epsilon & \text{on } \Gamma, \\ \Theta_1 \frac{\partial w_\epsilon^+}{\partial \nu} = \frac{\partial v_\epsilon}{\partial \nu} & \text{on } \Gamma. \end{array} \right. \quad (35)$$

Now we write the equation for  $w_\epsilon^+$  in the scaled domain and we solve for the asymptotic expansion of  $w_\epsilon^+$  in terms of the boundary values on  $\Gamma$ . Setting  $\hat{w}_\epsilon(s, \xi) := w_\epsilon^+(\varphi_\epsilon(s, \xi))$ , we have that

$$\Delta_{s,\xi} \hat{w}_\epsilon + \lambda_\epsilon n_1 \hat{w}_\epsilon = 0 \text{ in } \mathcal{G} = [0, L[ \times ]0, 1[, \quad (36)$$

together with the boundary conditions

$$\left\{ \begin{array}{ll} \hat{w}_\epsilon(s, 0) = v_\epsilon(x_\Gamma(s)) & s \in [0, L[, \\ \theta_1^\nu(s) \frac{1}{\epsilon} \frac{\partial \hat{w}_\epsilon}{\partial \xi}(s, 0) = \frac{\partial v_\epsilon}{\partial \nu}(x_\Gamma(s)) & s \in [0, L[. \end{array} \right. \quad (37)$$

We assume that

$$\hat{w}_\epsilon(s, \xi) = \sum_{j=0}^{\infty} \epsilon^j \hat{w}_j(s, \xi), \quad (s, \xi) \in \mathcal{G} \quad \text{and} \quad v_\epsilon(x) = \sum_{j=0}^{\infty} \epsilon^j v_j(x), \quad x \in \Omega \quad (38)$$

for some functions  $\hat{w}_j$  defined on  $\mathcal{G}$  and  $v_j$  defined on  $\Omega$  that are independent from  $\epsilon$ . Multiplying (36) by  $\epsilon^2(1 + \epsilon\xi\kappa)^3$ , we obtain

$$\sum_{k=0}^{\infty} \epsilon^k D_k \hat{w}_\epsilon = 0 \quad (39)$$

where  $D_k, k = 0, \dots, 5$  are differential operators with the following expressions for the first four terms:

$$\begin{aligned} D_0 &= \frac{\partial}{\partial \xi} (\theta_1^\nu \frac{\partial}{\partial \xi}), \\ D_1 &= 3\xi\kappa \frac{\partial}{\partial \xi} (\theta_1^\nu \frac{\partial}{\partial \xi}) + \kappa \theta_1^\nu \frac{\partial}{\partial \xi}, \\ D_2 &= \frac{\partial}{\partial s} (\theta_1^\tau \frac{\partial}{\partial s}) + 3\xi^2 \kappa^2 \frac{\partial}{\partial \xi} (\theta_1^\nu \frac{\partial}{\partial \xi}) + 2\xi \kappa^2 \theta_1^\nu \frac{\partial}{\partial \xi} + \lambda_0 n_1, \\ D_3 &= \xi^3 \kappa^3 \frac{\partial}{\partial \xi} (\theta_1^\nu \frac{\partial}{\partial \xi}) + \xi^2 \kappa^3 \theta_1^\nu \frac{\partial}{\partial \xi} - \xi \frac{\partial \kappa}{\partial s} \theta_1^\nu \frac{\partial}{\partial s} + \xi \kappa \frac{\partial}{\partial s} (\theta_1^\tau \frac{\partial}{\partial s}) + 3\lambda_0 n_1 \xi \kappa + \lambda_1 n_1. \end{aligned}$$

Inserting the expansion (38) in (39) and equating the terms of the same order in  $\epsilon$ , we obtain

$$\begin{cases} \frac{\partial}{\partial \xi} (\theta_1^\nu \frac{\partial \hat{w}_j}{\partial \xi}) = - \sum_{k=1}^5 D_k \hat{w}_{j-k} & \text{in } \mathcal{G}, \\ \hat{w}_j(s, 0) = v_j(x_\Gamma(s)) & s \in [0, L[, \\ \theta_1^\nu \frac{\partial \hat{w}_j}{\partial \xi}(s, 0) = \frac{\partial v_{j-1}}{\partial \nu}(x_\Gamma(s)) & s \in [0, L[. \end{cases} \quad (40)$$

These equations can be solved inductively to get the expressions of  $\hat{w}_j$  in terms of the boundary values of  $v_l, l \leq j$ . One gets for  $j = 0, 1, 2$

$$\begin{aligned} \hat{w}_0(s, \xi) &= v_0(x_\Gamma(s)), \\ \hat{w}_1(s, \xi) &= \frac{1}{\theta_1^\nu} \frac{\partial v_0}{\partial \nu}(x_\Gamma(s)) \xi + v_1(x_\Gamma(s)), \end{aligned} \quad (41)$$

and

$$\begin{aligned} \hat{w}_2(s, \xi) &= -\frac{1}{2\theta_1^\nu} \left( \kappa(s) \theta_0^\nu \frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) + \frac{\partial}{\partial s} (\theta_1^\tau \frac{\partial w_0^-}{\partial s})(x_\Gamma(s)) + \lambda_0 n_1 w_0^-(x_\Gamma(s)) \right) \xi^2 \\ &\quad + \frac{1}{\theta_1^\nu} \frac{\partial v_1}{\partial \nu}(x_\Gamma(s)) \xi + v_2(x_\Gamma(s)). \end{aligned} \quad (42)$$

Now we also postulate the following expansion for  $w_\epsilon^-$ :

$$w_\epsilon^-(x) = \sum_{j=0}^{\infty} \epsilon^j w_j^-(x), \quad x \in \Omega \quad (43)$$

with  $w_j^- : \Omega \rightarrow \mathbb{R}$  are functions independent of  $\epsilon$ . Then  $(w_j^-, v_j)$  satisfies

$$\begin{cases} \Theta_0 \Delta w_j^- + \lambda_0 n_0 w_j^- = - \sum_{l=1}^j \lambda_l n_0 w_{j-l}^- & \text{dans } \Omega, \\ \Delta v_j + \lambda_0 v_j = - \sum_{l=1}^j \lambda_l v_{j-l} & \text{dans } \Omega, \end{cases} \quad (44)$$

with the following continuity conditions at  $\Gamma$

$$\tilde{w}_\epsilon^-(s, \epsilon) = \hat{w}_\epsilon(s, 1), \quad \theta_0^\nu \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s, \epsilon) = \theta_1^\nu \frac{1}{\epsilon} \frac{\partial \hat{w}_\epsilon}{\partial \xi}(s, 1),$$

where  $\tilde{w}_\epsilon^-$  is defined from  $w_\epsilon^-$  using the local change of variables (18) in a neighborhood of  $\Gamma$ . Using Taylor's expansion we get

$$\tilde{w}_\epsilon^-(s, \epsilon) = \tilde{w}_\epsilon^-(s, 0) + \epsilon \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s, 0) + \frac{\epsilon^2}{2} \frac{\partial^2 \tilde{w}_\epsilon^-}{\partial \eta^2}(s, 0) + o(\epsilon^2) = \hat{w}_\epsilon(s, 1) \quad (45)$$

and

$$\frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s, \epsilon) = \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s, 0) + \epsilon \frac{\partial^2 \tilde{w}_\epsilon^-}{\partial \eta^2}(s, 0) + \frac{\epsilon^2}{2} \frac{\partial^3 \tilde{w}_\epsilon^-}{\partial \eta^3}(s, 0) + o(\epsilon^2) = \frac{1}{\epsilon} \frac{\partial \hat{w}_\epsilon}{\partial \xi}(s, 1). \quad (46)$$

Injecting (38) and (43) into (45) and (46), we respectively obtain the following continuity conditions on  $\Gamma$ ,

$$\begin{aligned} w_0^-(x_\Gamma(s)) &= \hat{w}_0(s, 1), \\ w_1^-(x_\Gamma(s)) + \frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) &= \hat{w}_1(s, 1), \\ w_2^-(x_\Gamma(s)) + \frac{\partial w_1^-}{\partial \nu}(x_\Gamma(s)) + \frac{1}{2} \frac{\partial^2 w_0^-}{\partial \nu^2}(x_\Gamma(s)) &= \hat{w}_2(s, 1), \end{aligned} \quad (47)$$

and

$$\begin{aligned} 0 &= \frac{\partial \hat{w}_0}{\partial \xi}(s, 1), \\ \theta_0^\nu \frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) &= \theta_1^\nu \frac{\partial \hat{w}_1}{\partial \xi}(s, 1), \\ \theta_0^\nu \left( \frac{\partial w_1^-}{\partial \nu}(x_\Gamma(s)) + \frac{\partial^2 w_0^-}{\partial \nu^2}(x_\Gamma(s)) \right) &= \theta_1^\nu \frac{\partial \hat{w}_2}{\partial \xi}(s, 1). \end{aligned} \quad (48)$$

System (44) coupled with the boundary conditions (48) and (47) provide an inductive way to determine  $(w_j^-, v_j)$ . We obtain the set of equations satisfied by these terms after substituting the expressions of  $\hat{w}_j(s, 1)$  given by (41),(42). We hereafter summarize the set of equations obtained for  $(w_j^-, v_j)$  and how to use them to get the expressions of  $\lambda_j$ ,

$j = 0, 1$ . We first obtain that the couple  $(w_0^-, v_0)$  satisfies

$$\begin{cases} \Theta_0 \Delta w_0^- + \lambda_0 n_0 w_0^- = 0 & \text{in } \Omega, \\ \Delta v_0 + \lambda_0 v_0 = 0 & \text{in } \Omega, \\ w_0^- - v_0 = 0 & \text{on } \Gamma, \\ \Theta_0 \frac{\partial w_0^-}{\partial \nu} - \frac{\partial v_0}{\partial \nu} = 0 & \text{on } \Gamma, \end{cases} \quad (49)$$

with  $\lambda_0$  is a transmission eigenvalues. We then obtain that the couple  $(w_1^-, v_1)$  solves

$$\begin{cases} \Theta_0 \Delta w_1^- + \lambda_0 n_0 w_1^- = -\lambda_1 n_0 w_0^- & \text{in } \Omega, \\ \Delta v_1 + \lambda_0 v_1 = -\lambda_1 v_0 & \text{in } \Omega, \\ w_1^- - v_1 = \left( \frac{\theta_0^\nu - \theta_1^\nu}{\theta_1^\nu} \right) \frac{\partial w_0^-}{\partial \nu} & \text{on } \Gamma, \\ \Theta_0 \frac{\partial w_1^-}{\partial \nu} - \frac{\partial v_1}{\partial \nu} = \lambda_0 (n_0 - n_1) w_0^- + \frac{\partial}{\partial s} \left( (\theta_0^\tau - \theta_1^\tau) \frac{\partial w_0^-}{\partial s} \right) & \text{on } \Gamma. \end{cases} \quad (50)$$

To calculate the value of  $\lambda_1$ , we multiply the first equation with  $\overline{w_0^-}$  and the second equation with  $\overline{v_0}$ , taking the difference then integrating by parts and using (49), we have that

$$\begin{aligned} \lambda_1 \int_{\Omega} n_0 |w_0^-|^2 - |v_0|^2 dx &= \int_{\Gamma} \lambda_0 (n_0 - n_1) |w_0^-|^2 dx - \int_{\Gamma} \theta_0^\nu \frac{\theta_0^\nu - \theta_1^\nu}{\theta_1^\nu} \left| \frac{\partial w_0^-}{\partial \nu} \right|^2 ds(x) \\ &\quad - \int_{\Gamma} (\theta_0^\tau - \theta_1^\tau) \left| \frac{\partial w_0^-}{\partial s} \right|^2 ds(x), \end{aligned} \quad (51)$$

which coincides with the expression of  $\lambda_1$  given in Theorem (2.2) with  $n_0 = n_1$ .

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