ASYMPTOTIC EXPANSION OF TRANSMISSION EIGENVALUES FOR ANISOTROPIC THIN LAYERS

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by

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Abstract. — We study the asymptotic expansion of transmission eigenvalues for anisotropic thin layers. We establish a rigorous second order expansion for simple transmission eigenvalues with respect to the thickness of the layer. The convergence analysis is based on a generalization of Osborn's Theorem to non-linear eigenvalue problems by Moskow [**19**]. We also provide formal derivation in more general cases validating the obtained theoretical result.

1. Introduction

Transmission eigenvalues are associated with the existence of an incident field that does not scatter [**9**]. They can be equivalently defined as the eigenvalues of a system of two coupled partial differential equations posed on the inclusion domain. One of these equations refer to the equation satisfied by the total field and the other one is satisfied by the incident field. The two equations are coupled on the boundary by imposing that the Cauchy data coincide. This eigenvalue problem can then be formulated as the non-linear eigenvalue problem a non-selfadjoint compact operator [**9, 19**]. These special frequencies can be identified from far field data as proved in [**8, 9, 18**]. Since they carry information on the material properties of the scatterer, transmission eigenvalues would be then of interest for the inverse problem of retrieving qualitative information on the material properties from measured multistatic data [**16, 17**].

This paper is a continuation of the study started in [**3**]. More specifically, we consider the transmission eigenvalue problem corresponding to anisotropic media with a thin coating. It corresponds in particular to inclusions where the contrast is present in the main operator. There is a structural difference between isotropic and anisotropic transmission eigenvalue problems. In the first case, one can reformulate the eigenvalue problem as a quadratic eigenvalue problem that can be transformed into a linear eigenvalue problem for a linear (non-selfadjoint) compact operator. This transformation is not possible in the anisotropic

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one. Following [**12**] we are lead to transform the problem into a non linear eigenvalue problem where the compact operator depends itself on the eigenvalue. An additional structural difference comes from different functional spaces for solutions. This implies for instance a different treatment for the corrector terms in the asymptotic analysis. Asymptotic expansion of transmission eigenvalues has been addressed in the literature for small inclusion perturbations in [**6, 12, 19**]. Our analysis follows the procedure developed in [**12**] for the case of anisotropic media. The main theorem is the expression and error analysis for corrector terms associated with simple transmission eigenvalues. Simplicity assumption is mainly needed when applying Osborn's theorem for correctors [**19**]. The extension to eigenvalues with higher multiplicity is an open question. We consider two dimensional setting for the formal calculations but the procedure can be extended to three dimensional problems as well.

The paper is structured as follows. We introduce in Section 2 the transmission eigenvalue problem for anisotropic media in $\mathbb{R}^2$ and we formulate it as a non-linear eigenvalue problem for a compact operator. In Section 3, we prove the convergence results needed to later obtain the asymptotic for the transmission eigenvalues. The main result is proved in Section 4 using the perturbation formula for non linear eigenvalues given in [**19**]. Finally, we outline a classical formal procedure (as in [**1, 2, 3**] for instance) to obtain the expression of the asymptotic expansion which coincides with the one in Section 4 (independently from the multiplicity of the eigenvalue).

2. Formulation of the problem and main results

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ. For $\epsilon > 0$ a small enough parameter, we denote

$$\Omega_\epsilon^0 = \{x \in \Omega,\ d(x,\Gamma) > \epsilon\}$$

and its boundary

$$\Gamma_\epsilon = \{x \in \Omega,\ d(x,\Gamma) = \epsilon\} = \partial\Omega_\epsilon^0$$

where $d(x,\Gamma)$ denotes the distance of a point x to the boundary Γ. Let $\Omega_\epsilon = \Omega\backslash\overline{\Omega_\epsilon^0}$ be the layer of thickness ϵ adjacent to Ω_ϵ^0 (see Figure 1). Let Θ_ϵ, n_ϵ be $L^\infty(\Omega)$ functions such that

$$\Theta_\epsilon(x) = \begin{cases} \Theta_0(x) & \text{in } \Omega_\epsilon^0, \\ \Theta_1(x) & \text{in } \Omega_\epsilon, \end{cases} \qquad n_\epsilon(x) = \begin{cases} n_0(x) & \text{in } \Omega_\epsilon^0, \\ n_1(x) & \text{in } \Omega_\epsilon, \end{cases}$$

where Θ_0 and Θ_1 are real valued positive definite matrices in $L^\infty(\mathbb{R}^2)^{2\times 2}$ and n_0 and n_1 are non negative real valued functions in $L^\infty(\mathbb{R}^2)$ that are independent from ϵ. For the sake of simplicity, we assume that the restriction of n_0 to Ω_ϵ is constant function along the normal coordinate to Γ for ϵ sufficiently small. We then consider the following interior

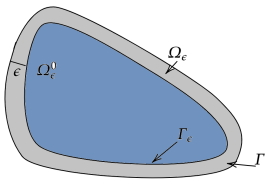

FIGURE 1. The scattering object Ω.

transmission eigenvalue problem formulated inside Ω

$$\begin{cases} div(\Theta_\epsilon \nabla w_\epsilon) + \lambda_\epsilon n_\epsilon w_\epsilon = 0 & \text{in} \quad \Omega, \\ \Delta v_\epsilon + \lambda_\epsilon v_\epsilon = 0 & \text{in} \quad \Omega, \\ w_\epsilon = v_\epsilon & \text{on} \quad \Gamma, \\ \Theta_\epsilon \nabla w_\epsilon \cdot \nu = \nabla v_\epsilon \cdot \nu & \text{on} \quad \Gamma, \end{cases} \tag{1}$$

where ν the unitary normal to Γ directed to the interior of Ω.

Definition 2.1. — *The values λ_ϵ for which (1) has a non trivial solution $(w_\epsilon, v_\epsilon) \in H^1(\Omega) \times H^1(\Omega)$ are called transmission eigenvalues.*

To investigate the analysis asymptotic expansion for transmission eigenvalues, we rewrite (1) as a non-linear eigenvalue problem. To this end, we define the variational space

$$X(\Omega) = \Big\{(w, v) \in H^1(\Omega) \times H^1(\Omega), w - v = 0 \text{ on } \Gamma\Big\}$$

equipped with the $H^1(\Omega) \times H^1(\Omega)$ inner product. It is clear that the variational form of (1) is given by ,$(w_\epsilon, v_\epsilon) \in X(\Omega)$ such that

$$\int_\Omega (\Theta_\epsilon \nabla w_\epsilon \nabla \phi_1 - \nabla v_\epsilon \nabla \phi_2 - \lambda_\epsilon (n_\epsilon w_\epsilon \phi_1 - v_\epsilon \phi_2)) dx = 0, \text{ for all } (\phi_1, \phi_2) \in X(\Omega). \tag{2}$$

We define the bounded sesquilinear forms:

$$\mathcal{A}_\epsilon\Big((w, v); (\phi_1, \phi_2)\Big) := \int_\Omega \Big(\Theta_\epsilon \nabla w \nabla \phi_1 + w\phi_1\Big) dx - \int_\Omega \Big(\nabla v \nabla \phi_2 + v\phi_2\Big) dx, \tag{3}$$

$$\mathcal{B}_\epsilon\Big((w, v); (\phi_1, \phi_2)\Big) := \int_\Omega \Big(n_\epsilon w\phi_1 - v\phi_2\Big) dx, \tag{4}$$

and

$$\mathcal{C}_\epsilon\Big((w, v); (\phi_1, \phi_2)\Big) := \int_\Omega \Big(w\phi_1 - v\phi_2\Big) dx. \tag{5}$$

for all $(w, v) \in X(\Omega)$. Therefore, we have that (2) can be written as for all $(\phi_1, \phi_2) \in X(\Omega)$

$$\mathcal{A}_\epsilon\Big((w_\epsilon, v_\epsilon); (\phi_1, \phi_2)\Big) - \lambda_\epsilon \mathcal{B}_\epsilon\Big((w_\epsilon, v_\epsilon); (\phi_1, \phi_2)\Big) - \mathcal{C}_\epsilon\Big((w_\epsilon, v_\epsilon); (\phi_1, \phi_2)\Big) = 0. \tag{6}$$

Let us define by A_ϵ, B_ϵ and $C_\epsilon : X(\Omega) \to X(\Omega)$ the bounded linear operators defined from $\mathcal{A}_\epsilon\Big(.;.\Big), \mathcal{B}_\epsilon\Big(.;.\Big)$ and $\mathcal{C}_\epsilon\Big(.;.\Big)$ by means of the Riesz representation theorem. Then (6) is equivalent to

$$A_\epsilon(w_\epsilon, v_\epsilon) - \lambda_\epsilon B_\epsilon(w_\epsilon, v_\epsilon) - C_\epsilon(w_\epsilon, v_\epsilon) = (0,0). \tag{7}$$

To give the asymptotic expansion of transmission eigenvalues, we use the generalization of Osborn's Theorem to non-linear eigenvalue problem [**19**]. We rewrite (7) as the following form

$$(I - \lambda_\epsilon A_\epsilon^{-1} B_\epsilon - A_\epsilon^{-1} C_\epsilon)(w_\epsilon, v_\epsilon) = (0,0).$$

Note that $B_\epsilon, C_\epsilon : X(\Omega) \to X(\Omega)$ are compact operators by using the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$. In order, to prove the invertibility of $\mathcal{A}_\epsilon$, we can show that A_ϵ is $\mathbb{T}$-coercive. The idea behind the $\mathbb{T}$-coercivity [**15**] is to consider an equivalent formulation of (6) where $\mathcal{A}_\epsilon$ is replaced by $\mathcal{A}_\epsilon^{\mathbb{T}}$ defined by

$$\mathcal{A}_\epsilon^{\mathbb{T}}\Big((w_\epsilon, v_\epsilon); (\phi_1, \phi_2)\Big) = \mathcal{A}_\epsilon\Big((w_\epsilon, v_\epsilon); \mathbb{T}(\phi_1, \phi_2)\Big)$$

with $\mathbb{T}$ being an isomorphism of $X(\Omega)$. In particular, we consider the isomorphism $\mathbb{T}(w,v) = (w, \alpha w + \beta v)$ with $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^*$ such that $\alpha + \beta = 1$. Then

$$\begin{aligned}
&\Big|\mathcal{A}_\epsilon\Big((w,v); \mathbb{T}(w,v)\Big)\Big| \\
&= \Big| \int_\Omega \Big(\Theta_\epsilon |\nabla w|^2 + |w|^2\Big) dx - \beta \int_\Omega \Big(|\nabla v|^2 + |v|^2\Big) dx - \alpha \int_\Omega \Big(\nabla w \nabla v + wv\Big) dx \Big| \\
&\geq \int_\Omega \Big(\Theta_\epsilon |\nabla w|^2 + |w|^2\Big) dx - \beta \Big| \int_\Omega \Big(|\nabla v|^2 + |v|^2\Big) dx \Big| - \alpha \Big| \int_\Omega \Big(\nabla w \nabla v + wv\Big) dx \Big|
\end{aligned}$$

Using Young's inequality, for all $\sigma > 0$ we obtain

$$\Big| \int_\Omega \Big(\nabla w \nabla v + wv\Big) dx \Big| \leq \frac{1}{2\sigma} \|w\|^2_{H^1(\Omega)} + \frac{\sigma}{2} \|v\|^2_{H^1(\Omega)} \tag{8}$$

We have that

$$\Big|\mathcal{A}_\epsilon\Big((w,v); \mathbb{T}(w,v)\Big)\Big| \geq (\gamma - \frac{\alpha}{2\sigma}) \|w\|^2_{H^1(\Omega)} + (-\beta - \frac{\sigma\alpha}{2}) \|v\|^2_{H^1(\Omega)}$$

where $\gamma = \inf(\|\Theta_1\|_\infty, \|\Theta_0\|_\infty)$. Choosing $\alpha^2 < -4\beta\gamma$ and $\beta < 0$, to have for example $\sigma \in (\frac{\alpha}{2\gamma}, -\frac{2\beta}{\alpha})$, implying $\mathcal{A}_\epsilon$ is $\mathbb{T}-$coercive.

Therefore (7) can be noted as follows [**12**]

$$(I - \lambda_\epsilon A_\epsilon^{-1} B_\epsilon - A_\epsilon^{-1} C_\epsilon)(w_\epsilon, v_\epsilon) = (0,0). \tag{9}$$

Let us define $T_\epsilon : X(\Omega) \to X(\Omega)$

$$T_\epsilon(\lambda) := A_\epsilon^{-1} B_\epsilon + \frac{1}{\lambda} A_\epsilon^{-1} C_\epsilon \tag{10}$$

Now we can rephrase (9) as a non-linear eigenvalue problem

$$\lambda_\epsilon T_\epsilon(\lambda_\epsilon)(w_\epsilon, v_\epsilon) = (w_\epsilon, v_\epsilon)$$

Note that T_ϵ is compact operator for all $\epsilon \geq 0$ and depends analytically on λ in any subset of the complex plane that doesn't include the origin.
We set

$$T_0(\lambda) := A_0^{-1}B_0 + \frac{1}{\lambda}A_0^{-1}C_0, \tag{11}$$

where A_0, B_0 and C_0 are defined by (3), (4) and (5) for $n_\epsilon = n_0$ and $\Theta_\epsilon = \Theta_0$ in Ω, respectively. We notice that $C_\epsilon = C_0$.
In all the following we use the notation

$$\Big\langle (f_1, f_2); (g_1, g_2) \Big\rangle = \Big((f_1, f_2); (g_1, g_2) \Big)_{X(\Omega)} \text{ and } \|(f_1, f_2)\| = \Big((f_1, f_2); (f_1, f_2) \Big)^{\frac{1}{2}}_{X(\Omega)}.$$

To state and prove the main result we make the following important hypothesis on the anisotropy structure. We assume that inside a thin layer Ω_{ϵ_0} for some fixed ϵ_0, we can decompose Θ_1 and Θ_0 as

$$\Theta_1(x) \cdot \tau(s) = \theta_1^\tau(s)\tau(s) \text{ and } \Theta_1(x) \cdot \nu(s) = \theta_1^\nu(s)\nu(s),$$
$$\Theta_0(x) \cdot \tau(s) = \theta_0^\tau(s)\tau(s) \text{ and } \Theta_0(x) \cdot \nu(s) = \theta_0^\nu(s)\nu(s)$$

for a.e. $x \in \Omega_{\epsilon_0}$ where $\theta_1^\tau, \theta_1^\nu, \theta_0^\tau$ and θ_0^ν are real functions in $L^\infty(\mathbb{R})$ that depend only on the curvilinear abscissa s of the projection of x on $\partial\Omega$ and where τ and ν are respectively the tangent and normal vectors defined on $\partial\Omega$ (see Subsection 4.1).

Theorem 2.2. — *Let λ_0 be a simple transmission eigenvalue associated with Θ_0 and n_0 and let $(w_0, v_0) \in X(\Omega)$ be the corresponding eigenfunctions. Assume that*

$$\Big\langle (w_0, v_0); A_0(w_0, v_0) \Big\rangle = 1 \text{ and } \int_\Omega (n_0|w_0|^2 - |v_0|^2)dx \neq 0.$$

We assume in addition that (w_0, v_0) and $A_0^{-1}(w_0, v_0)$ are in $C^2(\overline{\Omega})$. Then for $\epsilon > 0$ small enough, there exists a transmission eigenvalue λ_ϵ corresponding to Θ_ϵ such that

$$\lambda_\epsilon = \lambda_0 + \epsilon\lambda_1 + O(\epsilon^{\frac{3}{2}}) \tag{12}$$

where λ_1 is given by the following expression when $n_1 = n_0$

$$\lambda_1 = \frac{-\displaystyle\int_\Gamma (\theta_0^\tau - \theta_1^\tau)|\frac{\partial w_0}{\partial s}|^2 ds - \int_\Gamma \theta_0^\nu \left(\frac{\theta_0^\nu - \theta_1^\nu}{\theta_1^\nu} \right) |\frac{\partial w_0}{\partial \nu}|^2 ds(x)}{\displaystyle\int_\Omega \Big(n_0|w_0|^2 - |v_0|^2 \Big) dx}.$$

For the convergence proof we restrict ourselves for to $n_1 = n_0$. The expression of λ_1 in the case $n_1 \neq n_0$ in given in the last Section using formal expansion.

3. Convergence of the transmission eigenvalue spectrum

We study in this section the convergence in norm of the operator T_ϵ to T_0. For an operator $A : H \to H$, $\|A\|$ denotes the operator norm. To simplify the writing, C will

denote a generic constant whose value may change, but remains independent from ϵ as $\epsilon \to 0$.

We start by proving the convergence of the operator B_ϵ to B_0 in norm .

Lemma 3.1. — *Let B_ϵ and B_0 be the operators defined by (4) for $\epsilon > 0$ and $\epsilon = 0$, respectively. Then, for all $\alpha \in (0,1)$ there exists $C > 0$ independent of ϵ*

$$\|B_\epsilon - B_0\| \leq C\epsilon^\alpha.$$

Proof. — From the definition of B_ϵ and B_0, we have

$$\begin{aligned}\Big|\Big\langle B_\epsilon(w,v);(\phi_1,\phi_2)\Big\rangle - \Big\langle B_0(w,v);(\phi_1,\phi_2)\Big\rangle\Big| &= \Big|\int_{\Omega_\epsilon}(n_1-n_0)w\phi_1 dx\Big| \\ &\leq \|(n_1-n_0)w\|_{L^2(\Omega_\epsilon)}\|(\phi_1,\phi_2)\|_{X(\Omega)}.\end{aligned}$$

Choosing $(\phi_1,\phi_2) = (w,v)$ we get

$$\|(B_\epsilon - B_0)(w,v)\|_{X(\Omega)} \leq \|(n_1-n_0)w\|_{L^2(\Omega_\epsilon)}.$$

Since $w \in H^1(\Omega)$ and from Sobolev's embedding, we have that $w \in L^p(\Omega), \forall p \geq 2$ (see [**4**]) which gives $|w|^2 \in L^{\frac{p}{2}}(\Omega)$.
Now let $q > 0$ be defined by $\dfrac{2}{p} + \dfrac{1}{q} = 1$ (i.e $q = \dfrac{p}{p-2}$). Using the duality between $L^{\frac{p}{2}}(\Omega)$ and $L^q(\Omega)$ with the Sobolev embedding Theorem, we have that

$$\begin{aligned}\|(B_\epsilon - B_0)(w,v)\|^2_{X(\Omega)} &\leq C\|(n_1-n_0)w\|^2_{L^\infty(\Omega)}\|w\|^2_{L^2(\Omega_\epsilon)} \\ &\leq C\|w\|^2_{L^2(\Omega_\epsilon)} \leq \||w|^2\|_{L^{\frac{p}{2}}(\Omega)}\|\mathbb{1}_{\Omega_\epsilon}\|_{L^q(\Omega)} \\ &\leq C\epsilon^{\frac{1}{q}}\|w\|^2_{L^p(\Omega)} \leq C\epsilon^{\frac{1}{q}}\|(w,v)\|^2_{X(\Omega)}.\end{aligned}$$

We can conclude that $\|B_\epsilon - B_0\| \leq C\epsilon^\alpha$ with $\alpha = \frac{1}{2q}$. □

Lemma 3.2. — *Let A_ϵ and A_0 be defined by (3) for $\epsilon > 0$ and $\epsilon = 0$, respectively and $(f,g) \in X(\Omega)$. If $A_0^{-1}(f,g) \in \mathcal{C}^1(\Omega) \times \mathcal{C}^1(\Omega)$, then for sufficiently small ϵ*

$$\|(A_\epsilon^{-1} - A_0^{-1})(f,g)\|_{X(\Omega)} \leq C\epsilon^{\frac{1}{2}}. \tag{13}$$

Proof. — For a fixed $(f,g) \in X(\Omega)$, we define (z_ϵ, t_ϵ) and (z_0,t_0) in $X(\Omega)$ as

$$(z_\epsilon,t_\epsilon) = A_\epsilon^{-1}(f,g) \text{ and } (z_0,t_0) = A_0^{-1}(f,g).$$

Since

$$\mathcal{A}_\epsilon\Big((z_\epsilon,t_\epsilon);(\phi_1,\phi_2)\Big) = \mathcal{A}_0\Big((z_0,t_0);(\phi_1,\phi_2)\Big) = \Big\langle (f,g);(\phi_1,\phi_2)\Big\rangle.$$

we have that, for $(\phi_1, \phi_2) \in X(\Omega)$

$$\mathcal{A}_\epsilon\Big((z_\epsilon - z_0, t_\epsilon - t_0); (\phi_1, \phi_2)\Big) = (\mathcal{A}_0 - \mathcal{A}_\epsilon)\Big((z_0, t_0); (\phi_1, \phi_2)\Big) = \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1)\nabla z_0 \nabla \phi_1 dx. \tag{14}$$

If $z_0 \in \mathcal{C}^1(\overline{\Omega})$, we get

$$\int_{\Omega_\epsilon} (\Theta_0 - \Theta_1)\nabla z_0 \nabla \phi_1 dx \leq C\epsilon^{\frac{1}{2}} \|(\phi_1, \phi_2)\|_{X(\Omega)}.$$

From the $\mathbb{T}$-coercivity and by plugging $(\phi_1, \phi_2) = \mathbb{T}(z_\epsilon - z_0, t_\epsilon - t_0)$, we obtain the desired convergence. □

Lemma 3.3. — *Let $A_\epsilon, B_\epsilon, C_\epsilon, A_0, B_0$ and C_0 be defined by (3), (4), (5) for $\epsilon > 0$ and $\epsilon = 0$, respectively. we have that*

$$A_\epsilon^{-1} B_\epsilon \to A_0^{-1} B_0 \text{ and } A_\epsilon^{-1} C_0 \to A_0^{-1} C_0$$

in norm as $\epsilon \to 0$.

Proof. — From (14) we have that

$$\mathcal{A}_\epsilon\Big((z_\epsilon - z_0, t_\epsilon - t_0); (\phi_1, \phi_2)\Big) = \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1)\nabla z_0 \nabla \phi_1 dx.$$

By using the $\mathbb{T}$-coercivity of $\mathcal{A}_\epsilon$, we obtain

$$\|(A_\epsilon^{-1} - A_0^{-1})(f, g)\|_{X(\Omega)} \leq C\|(\Theta_0 - \Theta_1)\nabla z_0\|_{L^2(\Omega_\epsilon)}.$$

If $B_0(f, g) = (p, q)$ then $(p, q) \in H^1(\Omega) \times H^1(\Omega)$ and

$$\begin{cases} -\Delta p + p = n_0 f & \text{in} \quad \Omega, \\ -\Delta q + q = -g & \text{in} \quad \Omega, \\ p = q & \text{on} \quad \Gamma, \\ \dfrac{\partial p}{\partial \nu} = \dfrac{\partial q}{\partial \nu} & \text{on} \quad \Gamma, \end{cases} \tag{15}$$

Then $p - q \in H_0^1(\Omega)$ and verifies

$$-\Delta(p - q) + (p - q) = n_0 f + g$$

Elliptic regularity implies that if $\partial\Omega$ is $\mathcal{C}^\infty(\Omega)$, we get

$$\|p - q\|_{H^3(\Omega)} \leq C'\Big(\|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)}\Big).$$

(we here need $n_0 \in \mathcal{C}^1(\Omega)$). Similarly, $p + q \in H_0^1(\Omega)$ and verifies

$$\begin{cases} -\Delta(p + q) + (p + q) = n_0 f - g & \text{in} \quad \Omega, \\ \dfrac{\partial(p + q)}{\partial \nu} = 0 & \text{on} \quad \Gamma, \end{cases} \tag{16}$$

Therefore, regularity results for the Newmann problem implies that, if $\partial\Omega$ is $\mathcal{C}^3(\Omega)$ then

$$\|p+q\|_{H^3(\Omega)} \leq C''\Big(\|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)}\Big).$$

Hence, we have

$$\|p\|_{H^3(\Omega)} + \|q\|_{H^3(\Omega)} \leq C\Big(\|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)}\Big).$$

Note that $A_0(z_0,t_0) = (p,q)$ and due to the variational form of A_0 satisfies

$$-div(\Theta_0 \nabla z_0) + z_0 = -\Delta p + p \text{ and } \Delta t_0 - t_0 = -\Delta q + q \text{ in } \Omega.$$

Next we have that $A_0^{-1}B_0(f,g) = (z_0,t_0)$, by elliptic regularity we have

$$\|z_0\|_{H^3(\Omega)} + \|t_0\|_{H^3(\Omega)} \leq C\Big(\|p\|_{H^3(\Omega)} + \|q\|_{H^3(\Omega)}\Big) \leq C\|(f,g)\|_{X(\Omega)}.$$

For all ϵ sufficently small, we have the following estimates

$$\|(A_\epsilon^{-1} - A_0^{-1})B_0(f,g)\|_{X(\Omega)} \leq C\|z_0\|_{C^1(\Omega)}\|\mathbb{1}_{\Omega_\epsilon}\|_{L^2(\Omega)} \leq C\epsilon^{\frac{1}{2}}\|z_0\|_{C^1(\Omega)}.$$

From the Sobolev embedding Theorem of $H^3(\Omega)$ into $C^1(\Omega)$, we have that

$$\|(A_\epsilon^{-1} - A_0^{-1})B_0(f,g)\|_{X(\Omega)} \leq C\epsilon^{\frac{1}{2}}\|z_0\|_{H^3(\Omega)} \leq C\epsilon^{\frac{1}{2}}\|(f,g)\|_{X(\Omega)}.$$

Since

$$\|A_\epsilon^{-1}B_\epsilon - A_0^{-1}B_0\| \leq \|A_\epsilon^{-1}(B_\epsilon - B_0)\| + \|(A_\epsilon^{-1} - A_0^{-1})B_0\|,$$

using the fact that $\|A_\epsilon^{-1}\|$ is uniformly bounded with respect to ϵ and the convergence in norm of B_ϵ to B_0, implies

$$\|A_\epsilon^{-1}B_\epsilon - A_0^{-1}B_0\| \to 0 \text{ as } \epsilon \to 0.$$

The second convergence result follows from similar arguments. □

From the previous results, we have the following lemma:

Lemma 3.4. — *Let $A_\epsilon, B_\epsilon, C_\epsilon, A_0, B_0$ and C_0 be defined by* (3), (4), (5), *for $\epsilon > 0$ and $\epsilon = 0$, respectively. Then we have that*

$$\|(A_\epsilon^{-1} - A_0^{-1})B_0\| \leq C\epsilon^{\frac{1}{2}},\ \|(A_\epsilon^{-1} - A_0^{-1})C_0\| \leq C\epsilon^{\frac{1}{2}}$$

and $\|A_\epsilon^{-1}(B_\epsilon - B_0)\| \leq C\epsilon^\alpha$, for some $0 < \alpha < 1$.

Combining the above results and using the definition of T_ϵ and T_0 we have:

Theorem 3.5. — *Let T_ϵ and T_0 be defined in* (10) *for $\epsilon > 0$ and $\epsilon = 0$, respectively, then we have that*

$$\|T_\epsilon(\lambda) - T_0(\lambda)\| \to 0 \text{ as } \epsilon \to 0.$$

To prove the convergence of the real transmission eigenvalues, using the following abstract result from [**19**] (see also [**5**] for a more general framework allowing for weaker convergence). For the convenience of the reader, we restate it here:

Theorem 3.6. — *Let λ_0 be a non-linear eigenvalue of T_0 and assume that $R_\epsilon(\lambda)$ and $R_0(\lambda)$ are both meromorphic in some region $U \subset \mathbb{C}$ containing λ_0. Also assume that for any $\lambda \in U$, $T_\epsilon(\lambda)$ converge in norm to $T_0(\lambda)$. Then for any ball B around λ_0 there exists a $\epsilon_0 > 0$ such that T_ϵ has a non-linear eigenvalue in B for all $\epsilon < \epsilon_0$. Conversely if $\{\lambda_\epsilon\}_\epsilon$ is a sequence of a non-linear eigenvalues of T_ϵ that converges as $\epsilon \to 0$ then the limit is a non-linear eigenvalue of T_0 .*

4. Asymptotic formula for the transmission eigenvalue

Having proved the convergence of the transmission eigenvalues, we would like to obtain an asymptotic formula for the simple real transmission eigenvalue of the form:

$$\lambda_\epsilon = \lambda_0 + \epsilon\lambda_1 + O(\epsilon^2),$$

where λ_0 is a non-linear eigenvalue to T_0 associated with the eigenfunction (w_0, v_0) i.e

$$\lambda_0 T_0(w_0, v_0) = (w_0, v_0).$$

and the expression for λ_1 is as explicit as possible, and depends only on λ_0 and (w_0, v_0). We refer to λ_1 as the eigenvalue correction. To this end, we define explicit formula for the corrector term associated with $T_\epsilon - T_0$, in particular $A_\epsilon^{-1} - A_0^{-1}$. Then, we need a classical technique for thin layers asymptotics based on rescaling.

4.1. Scaling. — We assume that the boundary Γ is $\mathcal{C}^{k+2}-$smooth for $k \geq 1$, we start by parametrizing Γ as follows

$$\Gamma = \{x_\Gamma(s), s \in [0, L[\},$$

with L being the length of Γ and s is the curvilinear abscissa. At the point $x_\Gamma(s)$, the unit tangent vector is $\tau(s) := \dfrac{dx_\Gamma(s)}{ds}$, the curvature $\kappa(s)$ is defined by:

$$\frac{d\tau(s)}{ds} = -\kappa(s)\nu(s) \text{ or } \frac{d\nu(s)}{ds} = \kappa(s)\tau(s).$$

Within these notations, the boundary of Ω_ϵ^0 is parametrized as

$$\Gamma_\epsilon = \{x_\Gamma(s) + \epsilon\nu(s), s \in [0, L[\}.$$

This parametrization of the surface Γ_ϵ is equivalent to the definition of Γ_ϵ, for $\epsilon > 0$ a small enough parameter. For the function u defined in Ω_ϵ, we consider $\tilde{u}$ defined on $[0, L[\times]0, \epsilon[$ by

$$\tilde{u}(s, \eta) = u(\varphi(s, \eta)) \text{ where } \varphi(s, \eta) := x_\Gamma(s) + \eta\nu(s). \tag{17}$$

then the gradient operator of u in the local coordinates writes as:

$$\nabla u = \frac{1}{(1 + \eta\kappa(s))}\frac{\partial\tilde{u}}{\partial s}\tau(s) + \frac{\partial\tilde{u}}{\partial\eta}\nu(s).$$

Using integration by parts we have that the divergence of a vector field: $\overrightarrow{u} = u_\tau\tau + u_\nu\nu$

$$div\,\overrightarrow{u} = \frac{1}{(1+\eta\kappa(s))}\frac{\partial u_\tau}{\partial s} + \frac{1}{(1+\eta\kappa(s))}\frac{\partial}{\partial \eta}((1+\eta\kappa(s))u_\nu)$$

Let Θ be a function defined as follows

$$\Theta(x).\tau = \Theta_\tau(s)\tau \text{ and } \Theta(x).\nu = \Theta_\nu(s)\nu$$

Then the $div(\Theta\cdot\nabla)$ operator is expressed in (s,η) as:

$$div(\Theta\cdot\nabla u) = \frac{1}{(1+\eta\kappa)}\frac{\partial}{\partial s}\Big(\Theta_\tau\frac{1}{(1+\eta\kappa)}\frac{\partial \tilde{u}}{\partial s}\Big) + \frac{1}{(1+\eta\kappa)}\frac{\partial}{\partial \eta}\Big((1+\eta\kappa)\Theta_\nu\frac{\partial \tilde{u}}{\partial \eta}\Big). \tag{18}$$

We denote by $\xi = \dfrac{\eta}{\epsilon}$ the stretched normal variable inside Ω_ϵ and define as

$$\begin{array}{rccc} \varphi_\epsilon & : & \mathcal{G} = [0,L[\times]0,1[& \to & \Omega_\epsilon \\ & & (s,\xi) & \mapsto & \varphi_\epsilon(s,\xi) = x_\Gamma(s) + \epsilon\xi\nu(s), \end{array}$$

then the expression of the $div(\Theta\cdot\nabla)$ operator in the scaled layer noted by $\Delta_{s,\xi}$ is:

$$\Delta_{s,\xi}\hat{u} := \frac{1}{(1+\epsilon\xi\kappa)}\frac{\partial}{\partial s}\Big(\Theta_\tau\frac{1}{(1+\epsilon\xi\kappa)}\frac{\partial \tilde{u}}{\partial s}\Big) + \frac{1}{\epsilon^2(1+\epsilon\xi\kappa)}\frac{\partial}{\partial \xi}\Big((1+\epsilon\xi\kappa)\Theta_\nu\frac{\partial \tilde{u}}{\partial \xi}\Big) \tag{19}$$

for

$$\hat{u}(s,\xi) := u(\varphi_\epsilon(s,\xi)).$$

4.2. Correction for the operator $A_\epsilon^{-1} - A_0^{-1}$. — We assume in the following that $n_1 = n_0$. We construct a corrector term and use it to estimate the convergence rate of $(z_\epsilon, t_\epsilon) = A_\epsilon^{-1}(w,v)$ for $(w,v)\in X(\Omega)$. Let $(z_0,t_0) = A_0^{-1}(w,v)\in X(\Omega)$ solution of

$$\begin{cases} div(\Theta_0\nabla z_0) - z_0 = 0 & \text{in} \quad \Omega, \\ \Delta t_0 - t_0 = 0 & \text{in} \quad \Omega. \\ z_0 - t_0 = 0 & \text{on} \quad \Gamma. \\ \Theta_0\dfrac{\partial z_0}{\partial \nu} - \dfrac{\partial t_0}{\partial \nu} = 0 & \text{on} \quad \Gamma. \end{cases} \tag{20}$$

Inspired by the formal calculations of the next section, we define (z_1,t_1) solution of

$$\begin{cases} div(\Theta_0\nabla z_1) - z_1 = 0 & \text{in} \quad \Omega, \\ \Delta t_1 - t_1 = 0 & \text{in} \quad \Omega. \\ z_1 - t_1 = \left(\dfrac{\theta_0^\nu - \theta_1^\nu}{\theta_1^\nu}\right)\dfrac{\partial z_0}{\partial \eta} & \text{on} \quad \Gamma \\ \Theta_0\dfrac{\partial z_1}{\partial \nu} - \dfrac{\partial t_1}{\partial \nu} = \dfrac{\partial}{\partial s}\left((\theta_0^\tau - \theta_1^\tau)\dfrac{\partial z_0}{\partial s}\right) & \text{on} \quad \Gamma \end{cases} \tag{21}$$

We extend z_1 in Ω_ϵ as $\tilde{z}_1^\epsilon$ defined by

$$\tilde{z}_1^\epsilon = \begin{cases} z_1 & \text{in } \Omega_\epsilon^0, \\ \psi & \text{in } \Omega_\epsilon, \end{cases} \tag{22}$$

where ψ is a polynomial of order ≤ 1 and satisfying the boundary conditions:

$$\begin{cases} \psi = t_1 & \text{on } \Gamma, \\ \psi = z_1 & \text{on } \Gamma_\epsilon. \end{cases} \tag{23}$$

This gives the following expression of ψ in the local coordinates as:

$$\hat{\psi}(s,\xi) = z_1(\varphi(s,\epsilon))\xi + t_1(\varphi(s,0))(1-\xi)$$

The choice of ψ ensures in particular that $(\tilde{z}_1^\epsilon, t_1) \in X(\Omega)$. Using Taylor's expansion for z_1 and the third equation of (21), we then get

$$\begin{aligned} \hat{\psi}(s,\xi) &= z_1(\varphi(s,0))\xi + t_1(\varphi(s,0))(1-\xi) + O(\epsilon) \\ &= \left(\frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)}\right) \frac{\partial z_0}{\partial \eta}(\varphi(s,0))\xi + t_1(\varphi(s,0)) + O(\epsilon) \end{aligned}$$

which indicates that up to order 1 in ϵ, $\hat{\psi}$ coincides with the first term of the formal expansion of z_ϵ inside Ω_ϵ (See the last section). Now we have the following Lemma that will be useful in the proof of Lemma 4.2.

Lemma 4.1. — *Let $(w,v) \in X(\Omega)$ then set*

$$(z_\epsilon, t_\epsilon) = A_\epsilon^{-1}(w,v) and (z_0,t_0) = A_0^{-1}(w,v). \tag{24}$$

Assume that $(z_0,t_1) \in C^2(\overline{\Omega}) \times C^1(\overline{\Omega})$ and we define $\tilde{z}_1^\epsilon$ as in (22), then we have for sufficiently small ϵ

$$\|z_\epsilon - z_0 - \epsilon\tilde{z}_1^\epsilon\|_{H^1(\Omega)} + \|t_\epsilon - t_0 - \epsilon t_1\|_{H^1(\Omega)} \leq C\epsilon^{\frac{3}{2}}.$$

Proof. — We define the error function in $X(\Omega)$ as

$$z_2 = z_\epsilon - z_0 - \epsilon\tilde{z}_1^\epsilon \text{ and } t_2 = t_\epsilon - t_0 - \epsilon t_1.$$

Now for any $(\phi_1,\phi_2) \in X(\Omega)$, we have that

$$\begin{aligned} \Big\langle A_\epsilon(z_2,t_2);(\phi_1,\phi_2)\Big\rangle &= \int_\Omega \Big(\Theta_\epsilon \nabla z_2 \nabla\phi_1 + z_2\phi_1\Big)dx - \int_\Omega \Big(\nabla t_2\nabla\phi_2 + t_2\phi_2\Big)dx \\ &= \int_\Omega \Big(\Theta_\epsilon\nabla(z_\epsilon - z_0)\nabla\phi_1 + (z_\epsilon - z_0)\phi_1\Big)dx \\ &- \int_\Omega \Big(\nabla(t_\epsilon - t_0)\nabla\phi_2 + (t_\epsilon - t_0)\phi_2\Big)dx \\ &+ \epsilon\int_\Omega \Big(\nabla t_1\nabla\phi_2 + t_1\phi_2\Big)dx \\ &- \epsilon\int_\Omega \Big(\Theta_\epsilon\nabla\tilde{z}_1^\epsilon\nabla\phi_1 + \Theta_0\tilde{z}_1^\epsilon\phi_1\Big)dx \end{aligned}$$

Using(24) we get

$$\begin{aligned}\Big\langle A_\epsilon(z_\epsilon - z_0, t_\epsilon - t_0); (\phi_1, \phi_2)\Big\rangle &= \int_\Omega \Big(\Theta_\epsilon \nabla(z_\epsilon - z_0)\nabla\phi_1 + (z_\epsilon - z_0)\phi_1\Big)dx \\ &- \int_\Omega \Big(\nabla(t_\epsilon - t_0)\nabla\phi_2 + (t_\epsilon - t_0)\phi_2\Big)dx \\ &= \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1)\nabla z_0 \nabla\phi_1 dx\end{aligned}$$

Using the fact that $div(\Theta_0\nabla z_1) - z_1 = 0$, $\Delta t_1 - t_1 = 0$ and the Green formula yield

$$\begin{aligned}\Big\langle A_\epsilon(z_2, t_2); (\phi_1, \phi_2)\Big\rangle &= \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1)\nabla z_0 \nabla\phi_1 dx \\ &- \epsilon \int_{\Omega_\epsilon} \Theta_1 \nabla\psi\nabla\phi_1 + \psi\phi_1 dx \\ &- \epsilon \int_\Gamma \frac{\partial t_1}{\partial\nu}\phi_2 ds(x) + \epsilon \int_{\Gamma_\epsilon} \Theta_0 \frac{\partial z_1}{\partial\nu}\phi_1 ds(x).\end{aligned}$$

To estimate $\displaystyle\int_{\Omega_\epsilon} \psi\phi_1 dx$, we use Cauchy Schwartz to obtain

$$\int_{\Omega_\epsilon} \psi\phi_1 dx \leq \|\psi \mathbb{1}_{\Omega_\epsilon}\|_{L^2(\Omega)} \|\phi_1\|_{L^2(\Omega)}$$

From the definition of ψ and if $(z_0, t_1) \in \mathcal{C}^2(\overline{\Omega}) \times \mathcal{C}^1(\overline{\Omega})$, we then get

$$\|\psi\mathbb{1}_{\Omega_\epsilon}\|^2_{L^2(\Omega)} = \int_0^L \int_0^1 \left((\frac{\theta_0^\nu(s) - \theta_1^\nu(s)}{\theta_1^\nu(s)})\frac{\partial z_0}{\partial\eta}(\varphi(s,0))\xi + t_1(x_\Gamma(s))\right)^2 \epsilon(1 + \epsilon\xi\kappa)ds d\xi \leq C\epsilon$$

Hence

$$\epsilon \int_{\Omega_\epsilon} \psi\phi_1 dx \leq C\epsilon^{\frac{3}{2}} \|\phi_1\|_{H^1(\Omega)}$$

We then deduce

$$\begin{aligned}\Big\langle A_\epsilon(z_2, t_2); (\phi_1, \phi_2)\Big\rangle &= \int_{\Omega_\epsilon} (\Theta_0 - \Theta_1)\nabla z_0 \nabla\phi_1 dx - \epsilon \int_{\Omega_\epsilon} \Theta_1 \nabla\psi\nabla\phi_1 dx \\ &- \epsilon \int_\Gamma \frac{\partial t_1}{\partial\nu}\phi_2 ds(x) + \epsilon \int_{\Gamma_\epsilon} \Theta_0 \frac{\partial z_1}{\partial\nu}\phi_1 ds(x) \\ &+ O(\epsilon^{\frac{3}{2}})\|(\phi_1, \phi_2)\|_{X(\Omega)}.\end{aligned}$$

Using the expression of ψ we have

$$\Theta_1\nabla_{s,\xi}\hat{\psi} = \frac{1}{\epsilon}(\theta_0^\nu(s)-\theta_1^\nu(s))\frac{\partial z_0}{\partial\eta}(\varphi(s,0)) + \frac{\theta_1^\tau(s)}{1+\epsilon\xi\kappa}\left(\frac{\partial}{\partial s}\Big(\frac{\theta_0^\nu(s)-\theta_1^\nu(s)}{\theta_1^\nu(s)}\frac{\partial z_0}{\partial\eta}(\varphi(s,0))\Big)\xi + \Theta_\tau\frac{\partial}{\partial s}t_1(\varphi(s,0))\right)+O(\epsilon)$$

we then get after substitution of these expressions

$$\begin{aligned}
&\epsilon\int_{\Omega_\epsilon}\Theta_1\nabla\psi\nabla\phi_1 dx\\
&=\epsilon\int_0^L\int_0^1\frac{1}{\epsilon}\left(\frac{\theta_0^\nu(s)-\theta_1^\nu(s)}{\theta_1^\nu(s)}\right)\frac{\partial z_0}{\partial\eta}(\varphi(s,0))\nabla\phi_1(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi\\
&+\epsilon\int_0^L\int_0^1\frac{\theta_1^\tau(s)}{1+\epsilon\xi\kappa}\frac{\partial}{\partial s}\Big(\frac{\theta_0^\nu(s)-\theta_1^\nu(s)}{\theta_1^\nu(s)}\frac{\partial z_0}{\partial\eta}(\varphi(s,0))\Big)\nabla\phi_1(\varphi(s,\epsilon\xi))\epsilon\xi(1+\epsilon\xi\kappa)dsd\xi.\\
&+\epsilon\int_0^L\int_0^1\frac{1}{1+\epsilon\xi\kappa}\left(\theta_1^\tau(s)\frac{\partial}{\partial s}t_1(x_\Gamma(s))\right)\nabla\phi_1(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi.
\end{aligned}$$

For the second term, if $z_0\in\mathcal{C}^2(\overline{\Omega})$ we have

$$\begin{aligned}
&\epsilon\int_0^L\int_0^1\frac{\theta_1^\tau(s)\xi}{1+\epsilon\xi\kappa}\frac{\partial}{\partial s}\Big(\frac{\theta_0^\nu(s)-\theta_1^\nu(s)}{\theta_1^\nu(s)}\frac{\partial z_0}{\partial\eta}(\varphi(s,0))\Big)\nabla\phi_1(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi\\
&\le C\epsilon\|\theta_1^\tau(s)\|_\infty\|\frac{\partial}{\partial s}\Big(\frac{\theta_0^\nu(s)-\theta_1^\nu(s)}{\theta_1^\nu(s)}\frac{\partial z_0}{\partial\eta}(\varphi(s,0))\Big)\|_\infty\int_0^L\int_0^1\nabla\phi_1(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi\\
&\le C\epsilon\|\mathbb{1}_{\Omega_\epsilon}\|_{L^2(\Omega)}\|\nabla\phi_1\|_{L^2(\Omega)}\le C\epsilon^{\frac{3}{2}}\|\phi_1\|_{H^1(\Omega)}
\end{aligned}$$

Similarly if $t_1\in\mathcal{C}^1(\overline{\Omega})$ we obtain

$$\epsilon\int_0^L\int_0^1\frac{\theta_1^\tau(s)}{1+\epsilon\xi\kappa}\frac{\partial}{\partial s}t_1(x_\Gamma(s))\nabla\phi_1(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi = O(\epsilon^{\frac{3}{2}})\|\phi_1\|_{H^1(\Omega)}$$

Thus

$$\begin{aligned}
\Big\langle A_\epsilon(z_2,t_2);(\phi_1,\phi_2)\Big\rangle &= \int_0^L\int_0^1(\Theta_0-\Theta_1)\nabla z_0(\varphi(s,\epsilon\xi))\nabla\phi_1(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi\\
&-\int_0^L\int_0^1(\theta_0^\nu(s)-\theta_1^\nu(s))\frac{\partial z_0}{\partial\eta}(\varphi(s,0))\nabla\phi_1(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi\\
&-\epsilon\int_\Gamma\frac{\partial t_1}{\partial\nu}\phi_2 ds(x)+\epsilon\int_{\Gamma_\epsilon}\Theta_0\frac{\partial z_1}{\partial\nu}\phi_1 ds(x)\\
&+O(\epsilon^{\frac{3}{2}})\|(\phi_1,\phi_2)\|_{X(\Omega)}
\end{aligned}$$

We use the expression of the gradient operator in the local coordinates

$$(\Theta_1 - \Theta_0)\nabla z_0(\varphi(s,\epsilon\xi)) = \frac{(\theta_1^\tau(s) - \theta_0^\tau(s))}{(1+\epsilon\xi\kappa(s))}\frac{\partial z_0}{\partial s}(\varphi(s,\epsilon\xi))\tau(s) + (\theta_1^\nu(s) - \theta_0^\nu(s))\frac{\partial z_0}{\partial \eta}(\varphi(s,\epsilon\xi))\nu(s)$$

to make the decomposition

$$\begin{aligned}
&\Big\langle A_\epsilon(z_2,t_2);(\phi_1,\phi_2)\Big\rangle \\
&= \int_0^L\int_0^1 (\theta_0^\nu(s)-\theta_1^\nu(s))\Big(\frac{\partial z_0}{\partial \eta}(\varphi(s,\epsilon\xi) - \frac{\partial z_0}{\partial \eta}(\varphi(s,0))\Big)\nabla\phi_1(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi \\
&+ \int_0^L\int_0^1 (\theta_0^\tau(s)-\theta_1^\tau(s))\frac{\partial z_0}{\partial s}(\varphi(s,\epsilon\xi))\frac{\partial \phi_1}{\partial s}(\varphi(s,\epsilon\xi))\epsilon dsd\xi \\
&- \epsilon\int_\Gamma \frac{\partial t_1}{\partial \nu}\phi_2 ds(x) + \epsilon\int_{\Gamma_\epsilon}\Theta_0\frac{\partial z_1}{\partial\nu}\phi_1 ds(x) \\
&+ O(\epsilon^{\frac{3}{2}})\|(\phi_1,\phi_2)\|_{X(\Omega)}
\end{aligned}$$

Using the parametrization of the curve Γ_ϵ, $s\mapsto\varphi(s,\epsilon)$ we have:

$$\int_{\Gamma_\epsilon}\Theta_0\frac{\partial z_1}{\partial\nu}\phi_1 ds(x) = \int_0^L \theta_0^\nu(s)\frac{\partial z_1}{\partial\eta}(\varphi(s,\epsilon))\phi_1(\varphi(s,\epsilon))(1+\epsilon\kappa)ds$$

we use Taylor expansion to obtain

$$\begin{aligned}
\int_{\Gamma_\epsilon}\Theta_0\frac{\partial z_1}{\partial\nu}\phi_1 ds(x) &= \int_0^L \theta_0^\nu(s)\frac{\partial z_1}{\partial\eta}(\varphi(s,0))\phi_1(\varphi(s,0))ds + O(\epsilon^{\frac{1}{2}})\|\phi_1\|_{H^1(\Omega)} &(25)\\
&= \int_\Gamma \Theta_0\frac{\partial z_1}{\partial\nu}\phi_1 ds(x) + O(\epsilon^{\frac{1}{2}})\|\phi_1\|_{H^1(\Omega)} &(26)
\end{aligned}$$

The last equation of problem (21) then gives

$$\begin{aligned}
\epsilon\int_{\Gamma_\epsilon}\Theta_0\frac{\partial z_1}{\partial\nu}\phi_1 ds(x) - \epsilon\int_\Gamma\frac{\partial t_1}{\partial\nu}\phi_2 ds(x) &= \epsilon\int_\Gamma(\Theta_0\frac{\partial z_1}{\partial\nu} - \frac{\partial t_1}{\partial\nu})\phi_1 ds(x) + O(\epsilon^{\frac{3}{2}})\|\phi_1\|_{H^1(\Omega)} \\
&= \epsilon\int_0^L \frac{\partial}{\partial s}\left((\theta_0^\tau(s)-\theta_1^\tau(s))\frac{\partial z_0}{\partial s}(\varphi(s,0))\right)\phi_1(\varphi(s,0))ds \\
&\quad + O(\epsilon^{\frac{3}{2}})\|\phi_1\|_{H^1(\Omega)}
\end{aligned}$$

One ends up with

$$\begin{aligned}
&\Big\langle A_\epsilon(z_\epsilon - z_0 - \epsilon\tilde{z}_1^\epsilon, t_\epsilon - t_0); (\phi_1,\phi_2)\Big\rangle \\
&= \int_0^L \int_0^1 (\theta_0^\nu(s) - \theta_1^\nu(s))\Big(\frac{\partial z_0}{\partial \eta}(\varphi(s,\epsilon\xi) - \frac{\partial z_0}{\partial \eta}(\varphi(s,0))\Big)\nabla\phi_1(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi \\
&+ \epsilon\int_0^L \frac{\partial}{\partial s}\left((\theta_0^\tau(s) - \theta_1^\tau(s))\frac{\partial z_0}{\partial s}(\varphi(s,0))\right)\phi_1(\varphi(s,0))ds \\
&+ \epsilon\int_0^L \int_0^1 (\theta_0^\tau(s) - \theta_1^\tau(s))\frac{\partial z_0}{\partial s}(\varphi(s,\epsilon\xi))\frac{\partial \phi_1}{\partial s}(\varphi(s,\epsilon\xi))dsd\xi + O(\epsilon^{\frac{3}{2}})\|(\phi_1,\phi_2)\|_{X(\Omega)}
\end{aligned}$$

For the last term, we integrate by parts and we use the periodicity of $(\theta_0^\tau - \theta_1^\tau)\dfrac{\partial z_0}{\partial s}$ and ϕ_1 on $[0, L[$ we get

$$\begin{aligned}
&\epsilon\int_0^L \int_0^1 (\theta_0^\tau(s) - \theta_1^\tau(s))\frac{\partial z_0}{\partial s}(\varphi(s,\epsilon\xi))\frac{\partial \phi_1}{\partial s}(\varphi(s,\epsilon\xi))dsd\xi \\
&= -\epsilon\int_0^L \frac{\partial}{\partial s}\left((\theta_0^\tau(s) - \theta_1^\tau(s))\frac{\partial z_0}{\partial s}(\varphi(s,0))\right)\phi_1(\varphi(s,0))ds + O(\epsilon^{\frac{3}{2}})\|\phi_1\|_{H^1(\Omega)}
\end{aligned}$$

To conclude, we just use Taylor expansion for $\dfrac{\partial z_0}{\partial \eta}$ and $\dfrac{\partial z_0}{\partial s}$, we have

$$\Big\langle A_\epsilon(z_\epsilon - z_0 - \epsilon\tilde{z}_1^\epsilon, t_\epsilon - t_0 - \epsilon t_1); (\phi_1,\phi_2)\Big\rangle \leq C\epsilon^{\frac{3}{2}}\|(\phi_1,\phi_2)\|_{X(\Omega)}$$

Choosing $(\phi_1,\phi_2) = \mathbb{T}(z_\epsilon - z_0 - \epsilon\tilde{z}_1^\epsilon, t_\epsilon - t_0 - \epsilon t_1)$, since the $\mathbb{T}-$coercivity constant associated with $\mathcal{A}_\epsilon$ from ϵ, we get

$$\|z_\epsilon - z_0 - \epsilon\tilde{z}_1^\epsilon\|_{H^1(\Omega)} + \|t_\epsilon - t_0 - \epsilon t_1\|_{H^1(\Omega)} \leq C\epsilon^{\frac{3}{2}}. \tag{27}$$

which ends the proof. □

Lemma 4.2. — *If $(w,v) \in X(\Omega)$, then for sufficiently small ϵ,*

$$\|B_0(A_\epsilon^{-1} - A_0^{-1})(w,v)\| = O(\epsilon) \text{ and } \|C_0(A_\epsilon^{-1} - A_0^{-1})(w,v)\| = O(\epsilon)$$

Proof. — For all $(\phi_1,\phi_2) \in X(\Omega)$, we have

$$\Big\langle B_0(A_\epsilon^{-1} - A_0^{-1})(w,v); (\phi_1,\phi_2)\Big\rangle = \mathcal{B}_0\Big((z_\epsilon - z_0, t_\epsilon - t_0); (\phi_1,\phi_2)\Big).$$

Since the sesquilinear $\mathcal{B}_0$ only has $L^2(\Omega)$ terms, we have

$$\Big|\Big\langle B_0(A_\epsilon^{-1} - A_0^{-1})(w,v); (\phi_1,\phi_2)\Big\rangle\Big| \leq C\|(z_\epsilon - z_0, t_\epsilon - t_0)\|_{L^2(\Omega)}\|(\phi_1,\phi_2)\|_{X(\Omega)}$$

From (27) and $\epsilon\|\tilde{z}_1^\epsilon\|_{L^2(\Omega)} = O(\epsilon)$, $\epsilon\|t_1\|_{L^2(\Omega)} = O(\epsilon)$ we get

$$\Big|\Big\langle B_0(A_\epsilon^{-1} - A_0^{-1})(w,v); (\phi_1,\phi_2)\Big\rangle\Big| \leq C\epsilon\|(\phi_1,\phi_2)\|_{X(\Omega)}$$

Therefore

$$\|B_0(A_\epsilon^{-1} - A_0^{-1})(w, v)\| = O(\epsilon).$$

A similar argument gives the second result. □

4.3. Asymptotic formulas. — In this section, we assume that $n_\epsilon = n_0,\ \forall \epsilon > 0$. Under this assumption, we have that the operator $T_\epsilon(\lambda) = A_\epsilon^{-1}B_0 + \frac{1}{\lambda}A_\epsilon^{-1}C_0$ converge in norm to T_0. Finally, we are in a position to derive the expansion for the simple real transmission eigenvalues using the results of [**19**] which is a generalization of Osborn's Theorem to non-linear eigenvalues problems:

Theorem 4.3. — *Let Y be a Hilbert space and let $\{K_\epsilon : Y \to Y\}$ be a set of compact linear operator valued functions of λ which are analytic in a region $U \subset \mathbb{C}$. Assume that $K_\epsilon(\lambda)$ converge to $K_0(\lambda)$ in the operator norm for all $\lambda \in U$. Now suppose that λ_0 is a simple non-linear eigenvalue of K_0 with eigenfunction ϕ. Then if*

$$\lambda_0^2\Big\langle DK_0(\lambda_0)\phi, \phi^*\Big\rangle \neq -1,$$

we have the following formula

$$\lambda_\epsilon = \lambda_0 + \frac{\lambda_0^2\Big\langle (K_0(\lambda_0) - K_\epsilon(\lambda_0)\phi, \phi^*\Big\rangle}{1 + \lambda_0^2\Big\langle DK_0(\lambda_0)\phi, \phi^*\Big\rangle} + O\Big(\sup_{\lambda\in U}\|(K_\epsilon(\lambda) - K_0(\lambda))\phi\|\|(K_\epsilon^*(\lambda) - K_0^*(\lambda))\phi^*\|\Big)$$

with λ_ϵ is a non-linear eigenvalue for $K_\epsilon(\lambda)$.

Now, we have all the ingredients to prove the main result of this paper by applying this theorem to $K_\epsilon = T_\epsilon$, $K_0 = T_0$ and $Y = X(\Omega) \times X(\Omega)$. By Theorem 3.5, we have that $T_\epsilon \to T_0$ in the operator norm as $\epsilon \to 0$.
Let λ_0 be a non-linear eigenvalue of T_0, then for ϵ small enough, some non-linear eigenvalue λ_ϵ of T_ϵ is such that $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$. Let (w_0, v_0) and $A_0(w_0, v_0)$ are eigenfunctions respectively of the operators T_0 and T_0^* corresponding to λ_0.

Proof. — of Theorem 2.2
Using Theorem 4.3, we have that

$$\lambda_\epsilon - \lambda_0 = \frac{\lambda_0^2\Big\langle (T_0(\lambda_0) - T_\epsilon(\lambda_0)(w_0, v_0), A_0(w_0, v_0)\Big\rangle}{1 + \lambda_0^2\Big\langle DT_0(\lambda_0)(w_0, v_0), A_0(w_0, v_0)\Big\rangle} + O\Big(\sup_{\lambda\in U}\|(T_\epsilon(\lambda) - T_0(\lambda))(w_0, v_0)\|\|(T_\epsilon^*(\lambda) - T_0^*(\lambda))A_0(w_0^-, v_0)\|\Big)$$

From Lemma 3.4, we obtain

$$\|(T_\epsilon(\lambda) - T_0(\lambda))(w_0, v_0)\| = O(\epsilon^{\frac{1}{2}}).$$

Since $T^*_\epsilon(\lambda) = B_0 A^{-1}_\epsilon + \frac{1}{\lambda} C_0 A^{-1}_\epsilon$ and from Lemma 4.2, we have

$$\|(T^*_\epsilon(\lambda) - T^*_0(\lambda))A_0(w_0, v_0)\| = O(\epsilon).$$

Next we have that

$$\lambda_\epsilon - \lambda_0 = \frac{\lambda_0^2 \Big\langle (T_0(\lambda_0) - T_\epsilon(\lambda_0)(w_0, v_0), A_0(w_0, v_0) \Big\rangle}{1 + \lambda_0^2 \Big\langle DT_0(\lambda_0)(w_0, v_0), A_0(w_0, v_0) \Big\rangle} + O(\epsilon^{\frac{3}{2}})$$

From the definition of T_ϵ and T_0 and using the fact that

$$B_0(w_0, v_0) + \frac{1}{\lambda_0} C_0(w_0, v_0) = \frac{1}{\lambda_0} A_0(w_0, v_0),$$

we then get

$$\Big\langle (T_0(\lambda_0) - T_\epsilon(\lambda_0))(w_0, v_0); A_0(w_0, v_0) \Big\rangle = -\frac{1}{\lambda_0} \Big\langle (A^{-1}_\epsilon - A^{-1}_0) A_0(w_0, v_0); A_0(w_0, v_0) \Big\rangle \tag{28}$$

Applying Lemma 4.4 with $(w, v) = A_0(w_0, v_0)$ and $(\phi_1, \phi_2) = (w_0, v_0)$, we have

$$\Big\langle (A^{-1}_\epsilon - A^{-1}_0) A_0(w_0, v_0); A_0(w_0, v_0) \Big\rangle = \epsilon \int_0^L (\theta_0^\tau(s) - \theta_1^\tau(s)) |\frac{\partial w_0}{\partial s}(\varphi(s, 0))|^2 ds$$
$$+ \epsilon \int_0^L \theta_0^\nu(s) (\frac{\theta_0^\nu(s) - \theta_\nu^1(s)}{\theta_\nu^1(s)}) |\frac{\partial w_0}{\partial \eta}(\varphi(s, 0))|^2 ds + O(\epsilon^{\frac{3}{2}}). \tag{29}$$

From(29), we have

$$\Big\langle (T_0(\lambda_0) - T_\epsilon(\lambda_0))(w_0, v_0); A_0(w_0, v_0) \Big\rangle = -\frac{\epsilon}{\lambda_0} \int_0^L (\theta_0^\tau(s) - \theta_1^\tau(s)) |\frac{\partial w_0}{\partial s}(\varphi(s, 0))|^2 ds$$
$$- \frac{\epsilon}{\lambda_0} \int_0^L \theta_0^\nu(s) (\frac{\theta_0^\nu(s) - \theta_\nu^1(s)}{\theta_\nu^1(s)}) |\frac{\partial w_0}{\partial \eta}(\varphi(s, 0))|^2 ds + O(\epsilon^{\frac{3}{2}}).$$

For the denominator in the correction theorem, we must compute the derivative of $T_0(\lambda)$ with respect to λ. In fact this derivative is $DT_0(\lambda_0) = -\frac{1}{\lambda_0^2} A_0^{-1} C_0$.
Therefore, we have that

$$\lambda_0^2 \Big\langle DT_0(\lambda_0)(w_0, v_0); A_0(w_0, v_0) \Big\rangle = -\Big\langle C_0(w_0, v_0); (w_0, v_0) \Big\rangle = -\int_\Omega \Big(|w_0|^2 - |v_0|^2 \Big) dx.$$

Since

$$1 = \Big\langle A_0(w_0, v_0); (w_0, v_0) \Big\rangle,$$

and from the definition of A_0, we get

$$1 = \int_\Omega \Big(\Theta_0|\nabla w_0|^2 + |w_0|^2\Big)dx - \int_\Omega \Big(|\nabla v_0|^2 + |v_0|^2\Big)dx$$

we deduce then

$$1 + \lambda_0^2\langle DT_0(\lambda_0)(w_0, v_0); (w_0, v_0)\rangle = \int_\Omega \Big(\Theta_0|\nabla w_0|^2 - |\nabla v_0|^2\Big)dx$$

Since (w_0, v_0) solution

$$\begin{cases} div(\Theta_0\nabla w_0) + \lambda_0 n_0 w_0 = 0 & \text{in} \quad \Omega, \\ \Delta v_0 + \lambda_0 v_0 = 0 & \text{in} \quad \Omega, \\ w_0 - v_0 = 0 & \text{on} \quad \Gamma, \\ \Theta_0\dfrac{\partial w_0}{\partial \nu} - \dfrac{\partial v_0}{\partial \nu} = 0 & \text{on} \quad \Gamma, \end{cases} \tag{30}$$

We multiply the first equation of (30) with $\overline{w}_0$ and the second equation with $\overline{v}_0$, taking the difference then integrating by parts, we have

$$\int_\Omega \Big(\Theta_0|\nabla w_0|^2 - |\nabla v_0|^2\Big)dx - \lambda_0\int_\Omega (n_0|w_0|^2 - |v_0|^2)dx = 0,$$

which gives

$$1 + \lambda_0^2\Big\langle DT_0(\lambda_0)(w_0, v_0); A_0(w_0, v_0)\Big\rangle = \lambda_0\int_\Omega \Big(n_0|w_0|^2 - |v_0|^2\Big)dx$$

Finally, we obtain

$$\begin{aligned} \lambda_\epsilon - \lambda_0 &= \frac{\lambda_0^2\Big\langle (T_0(\lambda_0) - T_\epsilon(\lambda_0)(w_0, v_0), A_0(w_0, v_0)\Big\rangle}{1 + \lambda_0^2\Big\langle DT_0(\lambda_0)(w_0, v_0), A_0(w_0, v_0)\Big\rangle} + O(\epsilon^{\frac{3}{2}}) \\ &= \epsilon\lambda_1 + O(\epsilon^{\frac{3}{2}}) \end{aligned}$$

with

$$\lambda_1 = \frac{-\displaystyle\int_0^L (\theta_0^\tau(s) - \theta_1^\tau(s))|\frac{\partial w_0}{\partial s}(\varphi(s,0))|^2 ds - \int_\Gamma \theta_0^\nu(s)(\frac{\theta_0^\nu(s) - \theta_\nu^1(s)}{\theta_\nu^1(s)}|\frac{\partial w_0}{\partial \nu}|^2 ds(x)}{\displaystyle\int_\Omega \Big(n_0|w_0|^2 - |v_0|^2\Big)dx}.$$

which corresponds with the formula announced in the theorem and concludes the proof. □

Lemma 4.4. — *Let A_ϵ and A_0 be defined by* (3) *for $\epsilon > 0$ and $\epsilon = 0$, respectively. Then for any $(\phi_1, \phi_2) \in X(\Omega)$*

$$\Big\langle A_0(A_\epsilon^{-1} - A_0^{-1})(w, v); (\phi_1, \phi_2)\Big\rangle = \epsilon\int_0^L (\theta_0^\tau(s) - \theta_1^\tau(s))\frac{\partial z_0}{\partial s}(\varphi(s,0))\frac{\partial \phi_1}{\partial s}(\varphi(s,0))ds$$

$$+\epsilon \int_0^L \left(\theta_0^\nu(s)(\frac{\theta_0^\nu(s)-\theta_\nu^1(s)}{\theta_\nu^1(s)} \right) \frac{\partial z_0}{\partial \eta}(\varphi(s,0)) \frac{\partial \phi_1}{\partial \eta}(\varphi(s,0))ds + O(\epsilon^{\frac{3}{2}}). \tag{31}$$

Proof. — With $(z_\epsilon, t_\epsilon) = A_\epsilon^{-1}(w,v)$

$$\begin{aligned}\Big\langle A_0(A_\epsilon^{-1} - A_0^{-1})(w,v);(\phi_1,\phi_2)\Big\rangle &= \int_{\Omega_\epsilon} (\Theta_0-\Theta_1)\nabla z_\epsilon \nabla \phi_1 dx \\ &= \int_{\Omega_\epsilon} (\Theta_0-\Theta_1)\nabla (z_\epsilon - z_0 - \epsilon \tilde{z}_1^\epsilon)\nabla \phi_1 dx \\ &+ \int_{\Omega_\epsilon} (\Theta_0-\Theta_1)\nabla (z_0 + \epsilon \tilde{z}_1^\epsilon)\nabla \phi_1 dx.\end{aligned}$$

Applying Lemma 4.1, we obtain

$$\begin{aligned}\Big\langle A_0(A_\epsilon^{-1} - A_0^{-1})(w,v);(\phi_1,\phi_2)\Big\rangle &= \int_{\Omega_\epsilon} (\Theta_0-\Theta_1)\nabla z_0 \nabla \phi_1 dx \\ &+ \epsilon \int_{\Omega_\epsilon} (\Theta_0-\Theta_1)\nabla \tilde{z}_1^\epsilon \nabla \phi_1 dx + O(\epsilon^{\frac{3}{2}}).\end{aligned} \tag{32}$$

Making use the local coordinates, we show

$$\begin{aligned}\int_{\Omega_\epsilon} (\Theta_0-\Theta_1)\nabla z_0 \nabla \phi_1 dx &= \int_0^L (\theta_0^\tau(s)-\theta_1^\tau(s))\frac{\partial z_0}{\partial s}(\varphi(s,0))\frac{\partial \phi_1}{\partial s}(\varphi(s,0))ds \\ &+ \epsilon \int_0^L (\theta_0^\nu(s)-\theta_1^\nu(s))\frac{\partial z_0}{\partial \eta}(\varphi(s,0))\frac{\partial \phi_1}{\partial \eta}(\varphi(s,0))ds + O(\epsilon^{\frac{3}{2}}).\end{aligned}$$

For the second term of (32)

$$\begin{aligned}&\epsilon \int_{\Omega_\epsilon} (\Theta_0-\Theta_1)\nabla \tilde{z}_1^\epsilon \nabla \phi_1 dx = \epsilon \int_{\Omega_\epsilon} (\Theta_0-\Theta_1)\nabla \psi \nabla \phi_1 dx \\ &= \epsilon \int_0^L \int_0^1 (\theta_0^\tau(s)-\theta_1^\tau(s))\nabla_{s,\xi}\hat{\psi}(s,\xi)\nabla\phi_1(\varphi(s,\epsilon\xi))\epsilon(1+\epsilon\xi\kappa)dsd\xi \\ &= \epsilon \int_0^L (\theta_0^\nu(s)-\theta_1^\nu(s))\frac{\theta_0^\nu(s)-\theta_1^\nu(s)}{\theta_1^\nu(s)}\frac{\partial z_0}{\partial \eta}(\varphi(s,0))\frac{\partial \phi_1}{\partial \eta}(\varphi(s,0))ds + O(\epsilon^{\frac{3}{2}}).\end{aligned}$$

which implies,

$$\begin{aligned}\Big\langle A_0(A_\epsilon^{-1} - A_0^{-1})(w,v);(\phi_1,\phi_2)\Big\rangle &= \epsilon \int_0^L (\theta_0^\tau(s)-\theta_1^\tau(s))\frac{\partial z_0}{\partial s}(\varphi(s,0))\frac{\partial \phi_1}{\partial s}(\varphi(s,0))ds \\ &+ \epsilon \int_0^L \theta_0^\nu(s)\frac{\theta_0^\nu(s)-\theta_1^\nu(s)}{\theta_1^\nu(s)}\frac{\partial z_0}{\partial \eta}(\varphi(s,0))\frac{\partial \phi_1}{\partial \eta}(\varphi(s,0))ds + O(\epsilon^{\frac{3}{2}}).\end{aligned}$$

and concludes the proof. □

5. Validation of the asymptotic formula

Our main goal in this section is to validate formally the asymptotic formula given in Theorem 2.2. To this end, we propose the following expansion for the transmission eigenvalues:

$$\lambda_\epsilon = \sum_{j=0}^{\infty} \epsilon^j \lambda_j. \tag{33}$$

To make the formal calculations, we need to separate the thin layer and scaled it with respect to the thickness so that the first equation of (34) is posed on a domain independent from ϵ. We therefore rewrite the transmission eigenvalue problem (1) in the following equivalent form

$$\begin{cases} div(\Theta_1 \nabla w_\epsilon^+) + \lambda_\epsilon n_1 w_\epsilon^+ = 0 & \text{in} \quad \Omega_\epsilon, \\ div(\Theta_0 \nabla w_\epsilon^-) + \lambda_\epsilon n_0 w_\epsilon^- = 0 & \text{in} \quad \Omega_\epsilon^0, \\ \Delta v_\epsilon + \lambda_\epsilon v_\epsilon = 0 & \text{in} \quad \Omega, \\ w_\epsilon^+ = w_\epsilon^-, \ \Theta_1 \dfrac{\partial w_\epsilon^+}{\partial \nu} = \Theta_0 \dfrac{\partial w_\epsilon^-}{\partial \nu} & \text{on} \quad \Gamma_\epsilon, \\ w_\epsilon^+ = v_\epsilon & \text{on} \quad \Gamma, \\ \Theta_1 \dfrac{\partial w_\epsilon^+}{\partial \nu} = \dfrac{\partial v_\epsilon}{\partial \nu} & \text{on} \quad \Gamma. \end{cases} \tag{34}$$

Now we write the equation for w_ϵ^+ in the scaled domain and we solve for the asymptotic expansion of w_ϵ^+ in terms of the boundary values on Γ. Setting $\hat{w}_\epsilon(s,\xi) := w_\epsilon^+(\varphi_\epsilon(s,\xi))$, we have that

$$\Delta_{s,\xi} \hat{w}_\epsilon + \lambda_\epsilon n_1 \hat{w}_\epsilon = 0 \text{ in } \mathcal{G} = [0, L[\times]0, 1[, \tag{35}$$

together with the boundary conditions

$$\begin{cases} \hat{w}_\epsilon(s,0) = v_\epsilon(x_\Gamma(s)) & s \in [0, L[, \\ \theta_1^\nu(s) \dfrac{1}{\epsilon} \dfrac{\partial \hat{w}_\epsilon}{\partial \xi}(s,0) = \dfrac{\partial v_\epsilon}{\partial \nu}(x_\Gamma(s)) & s \in [0, L[. \end{cases} \tag{36}$$

We assume that

$$\hat{w}_\epsilon(s,\xi) = \sum_{j=0}^{\infty} \epsilon^j \hat{w}_j(s,\xi), \ (s,\xi) \in \mathcal{G} \quad \text{and} \quad v_\epsilon(x) = \sum_{j=0}^{\infty} \epsilon^j v_j(x), \ x \in \Omega \tag{37}$$

for some functions $\hat{w}_j$ defined on $\mathcal{G}$ and v_j defined on Ω that are independent from ϵ. Multiplying (35) by $\epsilon^2(1+\epsilon\xi\kappa)^3$, we obtain

$$\sum_{k=0}^{\infty} \epsilon^k D_k \hat{w}_\epsilon = 0 \tag{38}$$

where $D_k, k = 0, ..., 5$ are differential operators with the following expressions for the first four terms:

$$\begin{aligned}
D_0 =&\frac{\partial}{\partial\xi}(\theta_1^\nu\frac{\partial}{\partial\xi}),\\
D_1 =&3\xi\kappa\frac{\partial}{\partial\xi}(\theta_1^\nu\frac{\partial}{\partial\xi})+\kappa\theta_1^\nu\frac{\partial}{\partial\xi},\\
D_2 =&\frac{\partial}{\partial s}(\theta_1^\tau\frac{\partial}{\partial s})+3\xi^2\kappa^2\frac{\partial}{\partial\xi}(\theta_1^\nu\frac{\partial}{\partial\xi})+2\xi\kappa^2\theta_1^\nu\frac{\partial}{\partial\xi})+\lambda_0 n_1,\\
D_3 =&\xi^3\kappa^3\frac{\partial}{\partial\xi}(\theta_1^\nu\frac{\partial}{\partial\xi})+\xi^2\kappa^3\theta_1^\nu\frac{\partial}{\partial\xi}-\xi\frac{\partial\kappa}{\partial s}\theta_1^\nu\frac{\partial}{\partial s}+\xi\kappa\frac{\partial}{\partial s}(\theta_1^\tau\frac{\partial}{\partial s})+3\lambda_0 n_1\xi\kappa+\lambda_1 n_1.
\end{aligned}$$

Inserting the expansion (37) in (38) and equating the terms of the same order in ϵ, we obtain

$$\begin{cases}
\dfrac{\partial}{\partial\xi}(\theta_1^\nu\dfrac{\partial\hat{w}_j}{\partial\xi})=-\displaystyle\sum_{k=1}^{5}D_k\hat{w}_{j-k} & \text{in} \qquad \mathcal{G},\\
\hat{w}_j(s,0)=v_j(x_\Gamma(s)) & s\in[0,L[,\\
\theta_1^\nu\dfrac{\partial\hat{w}_j}{\partial\xi}(s,0)=\dfrac{\partial v_{j-1}}{\partial\nu}(x_\Gamma(s)) & s\in[0,L[.
\end{cases}\tag{39}$$

These equations can be solved inductively to get the expressions of $\hat{w}_j$ in terms of the boundary values of v_l, $l\le j$. One gets for $j=0,1,2$

$$\begin{aligned}
\hat{w_0}(s,\xi)&=v_0(x_\Gamma(s)),\\
\hat{w_1}(s,\xi)&=\frac{1}{\theta_1^\nu}\frac{\partial v_0}{\partial\nu}(x_\Gamma(s))\xi+v_1(x_\Gamma(s)),
\end{aligned}\tag{40}$$

and

$$\begin{aligned}
\hat{w_2}(s,\xi)=&-\frac{1}{2\theta_1^\nu}\Big(\kappa(s)\theta_0^\nu\frac{\partial w_0^-}{\partial\nu}(x_\Gamma(s))+\frac{\partial}{\partial s}(\theta_1^\tau\frac{\partial w_0^-}{\partial s})(x_\Gamma(s))+\lambda_0 n_1 w_0^-(x_\Gamma(s))\Big)\xi^2\\
&+\frac{1}{\theta_1^\nu}\frac{\partial v_1}{\partial\nu}(x_\Gamma(s))\xi+v_2(x_\Gamma(s)).
\end{aligned}\tag{41}$$

Now we also postulate the following expansion for w_ϵ^-:

$$w_\epsilon^-(x)=\sum_{j=0}^{\infty}\epsilon^j w_j^-(x),\ x\in\Omega\tag{42}$$

with $w_j^-:\Omega\to\mathbb{R}$ are functions independent of ϵ. Then (w_j^-, v_j) satisfies

$$\begin{cases}
\Theta_0\Delta w_j^-+\lambda_0 n_0 w_j^-=-\displaystyle\sum_{l=1}^{j}\lambda_l n_0 w_{j-l}^- & \text{dans} \qquad \Omega,\\
\Delta v_j+\lambda_0 v_j=-\displaystyle\sum_{l=1}^{j}\lambda_l v_{j-l} & \text{dans} \qquad \Omega,
\end{cases}\tag{43}$$

with the following continuity conditions at Γ

$$\tilde{w}_\epsilon^-(s,\epsilon) = \hat{w}_\epsilon(s,1),\ \theta_0^\nu \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s,\epsilon) = \theta_1^\nu \frac{1}{\epsilon}\frac{\partial \hat{w}_\epsilon}{\partial \xi}(s,1),$$

where $\tilde{w}_\epsilon^-$ is defined from w_ϵ^- using the local change of variables (17) in a neighborhood of Γ. Using Taylor's expansion we get

$$\tilde{w}_\epsilon^-(s,\epsilon) = \tilde{w}_\epsilon^-(s,0) + \epsilon \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s,0) + \frac{\epsilon^2}{2}\frac{\partial^2 \tilde{w}_\epsilon^-}{\partial \eta^2}(s,0) + o(\epsilon^2) = \hat{w}_\epsilon(s,1) \tag{44}$$

and

$$\frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s,\epsilon) = \frac{\partial \tilde{w}_\epsilon^-}{\partial \eta}(s,0) + \epsilon \frac{\partial^2 \tilde{w}_\epsilon^-}{\partial \eta^2}(s,0) + \frac{\epsilon^2}{2}\frac{\partial^3 \tilde{w}_\epsilon^-}{\partial \eta^3}(s,0) + o(\epsilon^2) = \frac{1}{\epsilon}\frac{\partial \hat{w}_\epsilon}{\partial \xi}(s,1). \tag{45}$$

Injecting (37) and (42) into (44) and (45), we respectively obtain the following continuity conditions on Γ,

$$\begin{aligned} & w_0^-(x_\Gamma(s)) = \hat{w}_0(s,1), \\ & w_1^-(x_\Gamma(s)) + \frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) = \hat{w}_1(s,1), \\ & w_2^-(x_\Gamma(s)) + \frac{\partial w_1^-}{\partial \nu}(x_\Gamma(s)) + \frac{1}{2}\frac{\partial^2 w_0^-}{\partial \nu^2}(x_\Gamma(s)) = \hat{w}_2(s,1), \end{aligned} \tag{46}$$

and

$$\begin{aligned} & 0 = \frac{\partial \hat{w}_0}{\partial \xi}(s,1), \\ & \theta_0^\nu \frac{\partial w_0^-}{\partial \nu}(x_\Gamma(s)) = \theta_1^\nu \frac{\partial \hat{w}_1}{\partial \xi}(s,1), \\ & \theta_0^\nu \Big(\frac{\partial w_1^-}{\partial \nu}(x_\Gamma(s)) + \frac{\partial^2 w_0^-}{\partial \nu^2}(x_\Gamma(s)) \Big) = \theta_1^\nu \frac{\partial \hat{w}_2}{\partial \xi}(s,1). \end{aligned} \tag{47}$$

System (43) coupled with the boundary conditions (47) and (46) provide an inductive way to determine (w_j^-, v_j). We obtain the set of equations satisfied by these terms after substituting the expressions of $\hat{w}_j(s,1)$ given by (40),(41). We hereafter summarize the set of equations obtained for (w_j^-, v_j) and how to use them to get the expressions of λ_j, $j = 0, 1$. We first obtain that the couple (w_0^-, v_0) satisfies

$$\begin{cases} \Theta_0 \Delta w_0^- + \lambda_0 n_0 w_0^- = 0 & \text{in} \quad \Omega, \\ \Delta v_0 + \lambda_0 v_0 = 0 & \text{in} \quad \Omega, \\ w_0^- - v_0 = 0 & \text{on} \quad \Gamma, \\ \Theta_0 \dfrac{\partial w_0^-}{\partial \nu} - \dfrac{\partial v_0}{\partial \nu} = 0 & \text{on} \quad \Gamma, \end{cases} \tag{48}$$

with λ_0 is a transmission eigenvalues. We then obtain that the couple (w_1^-, v_1) solves

$$\begin{cases} \Theta_0 \Delta w_1^- + \lambda_0 n_0 w_1^- = -\lambda_1 n_0 w_0^- & \text{in} \quad \Omega, \\ \Delta v_1 + \lambda_0 v_1 = -\lambda_1 v_0 & \text{in} \quad \Omega, \\ w_1^- - v_1 = (\dfrac{\theta_0^\nu - \theta_1^\nu}{\theta_1^\nu}) \dfrac{\partial w_0^-}{\partial \nu} & \text{on} \quad \Gamma, \\ \Theta_0 \dfrac{\partial w_1^-}{\partial \nu} - \dfrac{\partial v_1}{\partial \nu} = \lambda_0 (n_0 - n_1) w_0^- + \dfrac{\partial}{\partial s}((\theta_0^\tau - \theta_1^\tau) \dfrac{\partial w_0^-}{\partial s}) & \text{on} \quad \Gamma. \end{cases} \tag{49}$$

To calculate the value of λ_1, we multiply the first equation with $\overline{w_0^-}$ and the second equation with $\overline{v_0}$, taking the difference then integrating by parts and using (48), we have that

$$\begin{aligned} \lambda_1 \int_\Omega n_0 |w_0^-|^2 - |v_0|^2 \Big) dx = \int_\Gamma \Big(\lambda_0 (n_0 - n_1) |w_0^-|^2 dx - \int_\Gamma \theta_0^\nu \frac{\theta_0^\nu - \theta_1^\nu}{\theta_1^\nu} |\frac{\partial w_0^-}{\partial \nu}|^2 ds(x) \\ - \int_\Gamma (\theta_0^\tau - \theta_1^\tau) |\frac{\partial w_0^-}{\partial s}|^2 ds(x), \end{aligned} \tag{50}$$

which coincides with the expression of λ_1 given in Theorem (2.2) with $n_0 = n_1$.

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