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Abstract. We revisit the differential sampling method introduced in [13] for the identification of a periodic domain and some local perturbation. We provide a theoretical justification of the method that avoids assuming that the local perturbation is also periodic. Our theoretical framework uses functional spaces with continuous dependence with respect to the Floquet-Bloch variable. The corner stone of the analysis is the justification of the Generalized Linear Sampling Method in this setting for a single Floquet-Bloch mode.

Introduction

We consider in this work the inverse scattering problem for the reconstruction of a local perturbation in unknown periodic layers from near field measurements at fixed frequency. This problem has connections with many practical applications, such as non-destructive testing of photonic structures, antenna arrays... The presence of the perturbation does not allow us to reduce the problem to one-period cell and makes the analysis more challenging. We would like to develop so-called sampling methods to address the inverse problem of identifying the geometry of the defect. For the non perturbed inverse periodic problem we refer to [1, 19, 14, 24] and references therein. For the perturbed case, it is frequently assumed that the periodic background is known a priori. We refer for those cases for instance to [4, 11, 18, 20, 23, 10]. However, for some applications, this information is not available or cannot be obtained in an exact way. This is what we would like to consider in this work. More specifically we would like to study the so-called differential sampling indicator function introduced in [13]. For this algorithm only the periodicity size of the background is assumed to be known a priori. Combining sampling methods for a single Floquet-Bloch mode and the sampling method using the full measurement operator, one is able to design an indicator function that separates the perturbation from the periodic background. However, the analysis in [13] assumes that the defect is also periodic with a larger periodicity (equals to an integer multiple of the background periodicity).

Our goal here is to revisit the theoretical foundations of this method and remove this technical assumption on the defect. In order to do so, we analyze the scattering problem in spaces that include continuity with respect to the Floquet-Bloch variable. This allows for instance to consider the scattering problem at a fixed Floquet-Bloch mode. We first provide the theoretical justification of the so-called Generalized Linear Sampling Method (GLSM) [3, 2] for quasi-periodic incident waves. We remark that although a classical factorization of the near field operator can be obtained in this case, we are not able to apply the abstract framework of the factorization method as introduced in [16]. This is why for the GLSM method seems to be more adapted and this is why the penalty term that we use in our theory is different from the one used in the literature [3, 5]. For the justification of the method we assume that the local perturbation does not intersect the periodic background. The case where this intersection is not empty requires the study of an interior transmission problem that has a non standard structure similar to the one considered in [6]. For the sake of conciseness we leave this to future investigations.

The analysis of GLSM for quasi-periodic incident waves is by itself sufficient to derive an indicator function in the spirit of the differential linear sampling method of [13]. The principle consists in observing that we do not change the scattering problem if we redefine the periodicity of the background as an integer multiple of the original periodicity. The differential indicator function is build using a comparison of the GLSM indicator function when we use these different definitions of the periodicity of the background. This method is introduced and analyzed in Section 4.

We also provide a justification of a GLSM method using the whole near field operator associated with point sources. This method needs in particular a specific result related to the denseness of a single layer operator in the space of solutions to the Helmholtz equation that have continuous dependence with respect to the Floquet-Bloch variable. This is what mainly justifies the consideration of the scattering problem in half plane with Dirichlet boundary condition at the interface. Our analysis also assumes that the periodic index of refraction has a positive imaginary part in at least some open domain of the periodic background. We believe that this assumption can be removed using the analysis of the direct problem as in [17, 15]. Considering this case will be subject of future work. For the numerical validation of the GLSM and the differential sampling method we refer to [13, 21, 22].

The paper is organized as follows. Section 1 is dedicated to the introduction of the direct problems (with point source incident waves or quasi-periodic point source incident waves). In Section 2, we study the GLSM method for quasi-periodic incident waves. Section 3 is dedicated to the analysis of the GLSM for non quasi-periodic incident waves. In the last section we introduce and analyze the differential indicator function for the defect.

1. Setting of the direct problem

In this section, we introduce the direct scattering problem for a locally perturbed two dimensional periodic medium and the corresponding quasi-periodic problems.

1.1. The locally perturbed periodic scattering problem

Let U^0 be the upper half-space $\mathbb{R} \times \mathbb{R}_+$ in $\mathbb{R}^2$. We set $\Omega^R := \mathbb{R} \times [0, R]$ the domain delimited by $\Gamma^0 := \mathbb{R} \times \{0\}$ and $\Gamma^R := \mathbb{R} \times \{R\}$, with $R > R_0 > 0$ as shown in Figure 1. Let $n_p \in L^\infty(U^0)$ be the refractive index with non negative imaginary part, 2π-periodic with respect to the first component x_1 such that $n_p = 1$ outside a 2π periodic domain D^p included in Ω^R.
We consider $D := D^p \cup \tilde{D}$ where $\tilde{D}$ is a bounded domain included in $\Omega_0^R := [0, 2\pi] \times [0, R]$. We assume that the complement of D in $\mathbb{R}^2$ is connected. Let $n \in L^\infty(U^0)$ be the perturbed refractive index with non negative imaginary part verifying $n = n_p$ outside $\tilde{D}$.
Consider an incident field $v \in L^2(D)$, the direct scattering problem we are interested in can be formulated as: seek a scattered field $w \in H^2_{loc}(\Omega^R)$ verifying

$$(\mathcal{P}) \begin{cases} \Delta w + k^2 n w = k^2(1-n)v & \text{in } \Omega^R, \\ w = 0 & \text{on } \Gamma^0, \\ \frac{\partial w}{\partial x_2}(\cdot, R) = T^R(w|_{\Gamma^R}) & \text{on } \Gamma^R, \end{cases}$$

where $T^R : H^{1/2}(\Gamma^R) \longrightarrow H^{-1/2}(\Gamma^R)$ is exterior Dirichlet-to-Neumann map defined by

$$T^R(\varphi) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{k^2 - |\xi|^2} e^{ix_1 \cdot \xi} \hat{\varphi}(\xi) d\xi, \tag{1}$$

with $\hat{\varphi}$ is the Fourier transform defined as $\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x_1} \varphi(x_1, R) dx_1$ for L^1 functions on Γ^R.

Assumption 1.1. *Assume that in addition to the assumptions above, the set $\{\Im m(n_p) > 0\}$ contains a non empty open set $\mathcal{O}$ and $\Im m(n - n_p) \geqslant 0$.*

Under this assumption the above stated direct scattering problem has been studied in [18] and we hereafter state the main theorem.

Theorem 1.2. *If assumption 1.1 holds then there exists a unique solution $w \in H^2_{loc}(\Omega^R) \cap H^1(\Omega^R)$ satisfying $(\mathcal{P})$ and continuously depend on $v \in L^2(D)$.*

Remark 1.3. *Given the solution w to problem $(\mathcal{P})$ we extend w for $|x_2| \geqslant R$ by*

$$w(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\cdot\xi + i\sqrt{k^2-|\xi|^2}(x_2-R)} \widehat{w}(\xi, R) d\xi, \quad \textit{for} \ \ x_2 > R. \tag{2}$$

This provides the solution satisfying

$$\Delta w + k^2 n w = k^2(1-n)v \quad \textit{in } U^0. \tag{3}$$

The scattering problem $(\mathcal{P})$ can be equivalently formulated as (2)-(3) and the boundary conditions on Γ^0.

Let $\Phi(\cdot, y)$ be the fundamental solution of the homogeneous problem associated with $(\mathcal{P})$ given by

$$\Phi(\cdot, y) := \frac{i}{4}\left[H_0^{(1)}(k|\cdot - y|) - H_0^{(1)}(k|\cdot - y'|)\right], \tag{4}$$

where $y := (y_1, y_2)$, $y' := (y_1, -y_2)$ and $y_2 > 0$.

Theorem 1.4. *For $v \in L^2(D)$, the solution $w \in H^2_{loc}(\Omega^R)$ of problem $(\mathcal{P})$ can be represented as*

$$w(x) = k^2 \int_D \Phi(x, y)(n-1)(w+v)(y) dy. \tag{5}$$

Proof. Consider $N \in \mathbb{N}$ sufficiently large and let $D^N := ([-N, N] \times \mathbb{R}_+) \cap D$. Let us define a cut-off function $\chi_N \in C^\infty(U^0)$ such that $\chi_N(y) = 1$ for $y \in D^N$ and $\chi_N(y) = 0$ for $y \notin D^{N+1}$ and $\chi_N(y)$ only depends on y_1. We set

$$w_N := k^2 \int_D \Phi(x, y)(n-1)(w+v)\chi_N(y) dy \quad \text{for} \ \ x \in U^0,$$

with $w \in H^2_{loc}(\Omega^R)$ being the solution of $(\mathcal{P})$. Using the properties of the volume potentials [8] we deduce that $w_N \in H^2_{loc}(\Omega^R)$ satisfies

$$\begin{cases} \Delta w_N + k^2 w_N = k^2(1-n)(w+v)\chi_N(y) & \text{in } \Omega^R, \\ w_N = 0 & \text{on } \Gamma^0, \\ \frac{\partial w_N}{\partial x_2}(\cdot, R) = T^R(w_N|_{\Gamma^R}) & \text{on } \Gamma^R. \end{cases} \tag{6}$$

We set $u_N := w - w_N$, then $u_N \in H^2_{loc}(\Omega^R)$ satisfies (6) with the right hand side of the first equation is replaced by $k^2(1-n)(w+v)(1-\chi_N)(y)$. Since $(1-n)(w+v) \in L^2(\Omega^R)$, then

$$k^2(1-n)(w+v)(1-\chi_N) \underset{N\to\infty}{\longrightarrow} 0 \quad \text{in} \ \ L^2(\Omega^R).$$

Hence $\lim_{N\to\infty} u_N = 0$ in $H^2_{loc}(\Omega^R)$. On the other hand, since $(n-1)(w+v)(y) \in L^2(\Omega^R)$ and $\Phi(\cdot, y) \in L^2(\Omega^R)$[9], then we have

$$\lim_{N\to\infty} w_N(x) = k^2 \int_D \Phi(x,y)(n-1)(w+v)(y)dy,$$

almost everywhere in Ω^R by Lebesgue's dominated convergence theorem. Therefore $w \in H^2_{loc}(\Omega^R)$ satisfies (5) by the uniqueness of the limit. $\square$

For the study of the inverse problem we shall consider quasi-periodic solutions obtained by applying the Floquet-Bloch transform to the solution of $(\mathcal{P})$. We will need to restrict the set of admissible solutions to those with some continuity property with respect to the Floquet-Bloch variable. This is the subject of the following subsection.

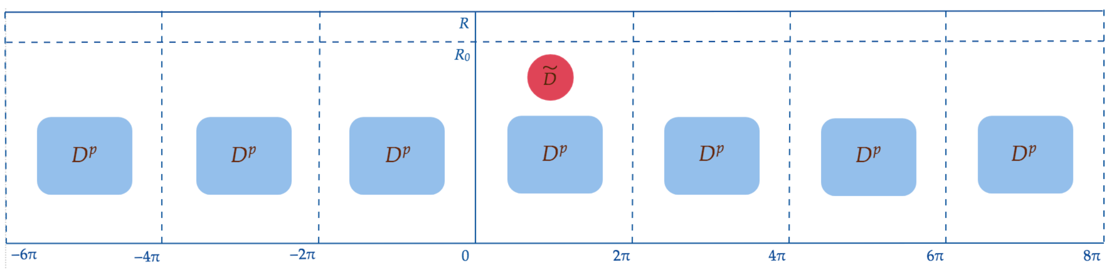

Figure 1. Sketch of the domain

1.2. The quasi-periodic scattering problem

A function u is called ξ-quasi-periodic with period 2π for some $\xi \in \mathbb{R}$ if it verifies

$$u(x_1 + 2\pi j, x_2) = e^{i2\pi\xi\cdot j} u(x_1, x_2) \quad \text{for all } j \in \mathbb{Z}, \tag{7}$$

and $(x_1, x_2) \in \mathbb{R}^2$. In the sequel we shall skip indicating the periodicity length 2π since it is kept fixed and periodicity or quasi-periodicity only apply to the first variable x_1.
In the following we denote by $L^2_\xi(\Omega^R)$ the set of ξ-quasi periodic functions in $L^2_{loc}(\Omega^R)$ and by $H^m_\xi(\Omega^R)$ the set of ξ-quasi periodic functions in $H^m_{loc}(\overline{\Omega^R})$. For $m \geqslant 1$ we denote by $\check{H}^m_\xi(\Omega^R)$ the subspace of functions in $H^m_\xi(\Omega^R)$ that vanish on Γ^0. We define $H^1_\xi(\Omega^R_0)$ as the restriction to Ω^R_0 of functions in $H^1_\xi(\Omega^R)$. In order to avoid notation confusion we denote by $H^1_\sharp(\Omega^R_0)$ the space $H^1_\xi(\Omega^R_0)$ for $\xi = 0$. We finally define $H^s_\xi(\Gamma^R_0)$ to be the restriction to Γ^R_0 of ξ-quasi periodic functions in $H^s_{loc}(\Gamma^R)$.

Let $\xi \in I := [0,1]$ and $\Gamma_0^R := [0,2\pi] \times \{R\}$. Consider an incident field $v_\xi \in L^2_\xi(\Omega^R)$, the quasi-periodic direct scattering problem is formulated as: seek a scattered field $w_\xi \in H^1_\xi(\Omega^R)$ verifying

$$\begin{cases} \Delta w_\xi + k^2 n_p w_\xi = -k^2(n_p - 1)v_\xi & \text{in } \Omega^R, \\ w_\xi = 0 & \text{on } \Gamma^0, \\ \frac{\partial w_\xi}{\partial x_2}(\cdot, R) = T^R_\xi(w_\xi|_{\Gamma_0^R}) & \text{on } \Gamma_0^R, \end{cases} \tag{8}$$

where $T^R_\xi : H^{1/2}_\xi(\Gamma_0^R) \longrightarrow H^{-1/2}_\xi(\Gamma_0^R)$ is the exterior quasi-periodic Dirichlet-to-Neumann map defined by

$$T^R_\xi(\varphi)(x_1) = i\sum_{j\in\mathbb{Z}} \beta_\xi(j)\hat{\varphi}_\xi(j)e^{i\alpha_\xi(j)\cdot x_1}, \tag{9}$$

where

$$\alpha_\xi(j) := \xi + j, \quad \beta_\xi(j) := \sqrt{k^2 - |\xi + j|^2}, \quad \Im m(\beta_\xi(j)) \geqslant 0, \ \text{ for } \ j \in \mathbb{Z},$$

and $\widehat{\varphi}_\xi(j)$ is the j-th Fourier coefficient of $e^{-i\xi x_1}\varphi(x_1, R)$ defined as $\widehat{\varphi}_\xi(j) := \frac{1}{2\pi}\int_0^{2\pi} e^{-i\alpha_\xi(j)x_1}\varphi(x_1, R)dx_1$. For the norm in $H^s_\xi(\Gamma_0^R)$ we shall use the following definition

$$\|\varphi\|^2_{H^s_\xi(\Gamma_0^R)} = \sum_{j\in\mathbb{Z}}(1 + j^2)^s|\hat{\varphi}_\xi(j)|^2.$$

Multiplying the first equation of (8) with $\overline{\psi_\xi} \in \check{H}^1_\xi(\Omega_0^R)$, integrating by parts and using the boundary conditions and the quasi periodicity we obtain the variational formulation given as

$$\int_{\Omega_0^R} \left(\nabla w_\xi \cdot \nabla\overline{\psi_\xi} - k^2 n_p w_\xi\overline{\psi_\xi}\right) dx - \left\langle T^R_\xi w_\xi, \psi_\xi\right\rangle_{\Gamma_0^R} = k^2\int_{\Omega_0^R}(n_p - 1)v_\xi\overline{\psi_\xi}, \quad \forall\ \psi_\xi \in \check{H}^1_\xi(\Omega_0^R). \tag{10}$$

where the notation $\langle\cdot,\cdot\rangle_{\Gamma_0^R}$ refers to the $H^{-1/2}(\Gamma_0^R) - H^{1/2}(\Gamma_0^R)$ duality product. Using the Riesz representation theorem we can define the operator $A_\xi : \check{H}^1_\xi(\Omega_0^R) \longrightarrow \check{H}^1_\xi(\Omega_0^R)$ such that

$$(A_\xi w_\xi, v_\xi)_{H^1_\xi(\Omega_0^R)} := \int_{\Omega_0^R} \left(\nabla w_\xi \cdot \nabla\overline{\psi_\xi} - k^2 n_p w_\xi\overline{\psi_\xi}\right) dx - \left\langle T^R_\xi w_\xi, \psi_\xi\right\rangle_{\Gamma_0^R} \quad \forall\ \psi_\xi, w_\xi \in \check{H}^1_\xi(\Omega_0^R).$$

Theorem 1.5. *Assume that $\{\Im m(n_p) > 0\}$ in a non empty open set of Ω_0^R, then problem (8) is well posed. Moreover, $\|A_\xi^{-1}\| \leqslant c$ with c is a constant independent of ξ.*

Remark 1.6. *Given the solution w_ξ to problem* (8) *we extend w_ξ for $|x_2| \geqslant R$ by*

$$w_\xi(x) := \sum_{j\in\mathbb{Z}}\widehat{(w_\xi|_{\Gamma_0^R})}(j)e^{i\alpha_\xi(j)\cdot x_1 + i\beta_\xi(j)(x_2 - R)} \qquad \text{for } x_2 > R. \tag{11}$$

This provides the solution satisfying

$$\Delta w_\xi + k^2 n_p w_\xi = k^2(1 - n_p)v_\xi \quad \text{in } U^0. \tag{12}$$

The scattering problem (8) *can be equivalently formulated as (11)-(12) and the boundary conditions on Γ^0.*

Defining now for $\phi \in C_0^\infty(U^0)$ the one dimensional Floquet-Bloch transform as the following

$$\mathcal{J}\phi(\xi, x_1, x_2) = \sum_{j\in\mathbb{Z}} \phi(x_1 + 2\pi j, x_2) e^{-i2\pi\xi\cdot j}, \quad \xi \in I,\ (x_1, x_2) \in U^0. \tag{13}$$

Recall that the Floquet-Bloch transform is an isomorphism between $H^s(\Omega^R)$ (respectively $H^s(\Gamma^R)$) and $L^2(I, H^s_\xi(\Omega^R_0))$ (respectively $L^2(I, H^s_\xi(\Gamma^R_0))$). Then, for $0 \leqslant \alpha < 1$, we denote by

$$C^{0,\alpha}_\sharp(I, H^s_\xi(\Omega^R_0)) := \{\varphi \in L^2(I, H^s_\xi(\Omega^R_0));\ e^{-i\xi\cdot x_1}\varphi \in C^{0,\alpha}_\sharp(I, H^s_\sharp(\Omega^R_0))\},$$

the space of periodic and α Hölderian functions on I with values in $H^s_\xi(\Omega^R_0)$. The norm of $\varphi \in C^{0,\alpha}_\sharp(I, H^s_\sharp(\Omega^R_0))$ is defined as

$$\sup_{\xi\in I} \|\varphi(\xi,\cdot)\|_{H^s(\Omega^R_0)} + \sup_{\xi_1\neq\xi_2\in I} \left(\frac{\|\tilde{\varphi}(\xi_1,\cdot) - \tilde{\varphi}(\xi_2,\cdot)\|_{H^s(\Omega^R_0)}}{|\xi_1 - \xi_2|^\alpha} \right),$$

with $\tilde{\varphi} := e^{-i\xi\cdot x_1}\varphi$. We then set

$$\tilde{H}^{s,\alpha}(\Omega^R) := \left\{ u \in H^s(\Omega^R) / \mathcal{J}u \in C^{0,\alpha}_\sharp(I, H^s_\xi(\Omega^R_0)) \right\}, \tag{14}$$

$$\tilde{H}^{s,\alpha}(\Gamma^R) := \left\{ u \in H^s(\Gamma^R) / \mathcal{J}u \in C^{0,\alpha}_\sharp(I, H^s_\xi(\Gamma^R_0)) \right\}. \tag{15}$$

Then we have the following theorem complementing the result of Theorem 1.5.

Theorem 1.7. *Assume that hypothesis of Theorem 1.5 holds and consider $v \in \tilde{L}^{2,\alpha}(\Omega^R)$ for $0 \leqslant \alpha < 1$. Let $w_\xi \in H^1_\xi(\Omega^R_0)$ be the solution of (8) with $v_\xi = (\mathcal{J}v)(\xi,\cdot)$. Then $w = \mathcal{J}^{-1}w_\xi := \int_I w_\xi d\xi$ belongs to $\tilde{H}^{1,\tilde{\alpha}}(\Omega^R)$ with $\tilde{\alpha} = \min(\alpha, \frac{1}{2})$ and*

$$\|w\|_{\tilde{H}^{1,\alpha}(\Omega^R)} \leqslant c\|v\|_{\tilde{L}^{2,\alpha}(D^p)}$$

with c independent from v.

Proof. Set $\tilde{w}_\xi := e^{-i\xi\cdot x_1}w_\xi$ and $\check{v}_\xi = -k^2(n_p - 1)v_\xi$. Then we have $\tilde{w}_\xi \in H^1_\sharp(\Omega^R_0)$ and verifies

$$\Delta\tilde{w}_\xi = e^{-i\xi\cdot x_1}\check{v}_\xi - 2i\xi\frac{\partial\tilde{w}_\xi}{\partial x_1} + \xi^2\tilde{w}_\xi - k^2 n_p\tilde{w}_\xi \quad \text{in } \Omega^R.$$

Let $\xi_1, \xi_2 \in I$, and set $e := e^{i\xi_1\cdot x_1}(\tilde{w}_{\xi_1} - \tilde{w}_{\xi_2})$. Then $e \in H^1_{\xi_1}(\Omega^R_0)$ and

$$\Delta e + k^2 n_p e = L_{\xi_1,\xi_2} \quad \text{in } \Omega^R, \tag{16}$$

with

$$L_{\xi_1,\xi_2} := e^{i\xi_1\cdot x_1}\left[(e^{-i\xi_1\cdot x_1} - e^{-i\xi_2 x_1})v_{\xi_1} + e^{-i\xi_2\cdot x_1}(\check{v}_{\xi_1} - \check{v}_{\xi_2}) - 2i(\xi_1 - \xi_2)\frac{\partial\tilde{w}_{\xi_2}}{\partial x_1} + (\xi_1^2 - \xi_2^2)\tilde{w}_{\xi_2}\right].$$

By the Cauchy-Schwartz inequality and $\check{v} \in L^{2,\alpha}(\Omega_0^R)$ we get the existence of a constant $c > 0$ independent from ξ_1 and ξ_2 such that

$$\|L_{\xi_1,\xi_2}\|_{L^2(\Omega_0^R)} \leqslant c|\xi_1 - \xi_2|^{\alpha} \left(\|v\|_{\tilde{L}^{2,\alpha}(D^p)} + \|w_{\xi_2}\|_{H^1(\Omega_0^R)} \right). \tag{17}$$

On the other hand, using the third equation in (8) we have on Γ^R

$$\frac{\partial e}{\partial x_2} = T^R_{\xi_1}(w_{\xi_1}) - e^{i(\xi_1-\xi_2)x_1} T^R_{\xi_2}(w_{\xi_2}) = T^R_{\xi_1}(e) + i \sum_{j\in\mathbb{Z}} (\beta_{\xi_1}(j) - \beta_{\xi_2}(j)) \widehat{w}_{\xi_2}(j) e^{i\alpha_{\xi_1}(j)}. \tag{18}$$

The variational formulation of (16)-(18) can be written as

$$\int_{\Omega_0^R} \left(\nabla e \cdot \nabla \bar{\psi} - k^2 n_p e \bar{\psi} \right) - \left\langle T^R_{\xi_1} e, \psi \right\rangle_{\Gamma_0^R} = \int_{\Omega_0^R} L_{\xi_1,\xi_2} \bar{\psi} dx - \langle g_{\xi_1,\xi_2}, \psi \rangle_{\Gamma_0^R}, \quad \forall\, \psi \in \check{H}^1_{\xi_1}(\Omega_0^R), \tag{19}$$

with $g_{\xi_1,\xi_2} := \sum_{j\in\mathbb{Z}} (\beta_{\xi_1}(j) - \beta_{\xi_2}(j)) \widehat{w}_{\xi_2}(j) e^{i\alpha_{\xi_1}(j)}$. Using the definition of $H^{-1/2}(\Gamma_0^R)$ norm in terms of Fourier coefficients we get

$$\|g_{\xi_1,\xi_2}(x_1)\|^2_{H^{-1/2}_{\xi_1}(\Gamma_0^R)} = \sum_{j\in\mathbb{Z}} (1+|j|^2)^{1/2} |\widehat{w}_{\xi_2}(j)|^2 C^2_{\xi_1,\xi_2}(j),$$

with $C_{\xi_1,\xi_2}(j) := \frac{|\beta_{\xi_1}(j) - \beta_{\xi_2}(j)|}{(1+|j|^2)^{1/2}}$. For $j \in \mathbb{Z}^* := \mathbb{Z}\backslash\{0\}$ such that $\beta_{\xi_1}(j) = 0$, i.e $k^2 = |\xi_1 + j|^2$

$$C_{\xi_1,\xi_2}(j) = \frac{||j+\xi_1|^2 - |j+\xi_2|^2|^{1/2}}{(1+|j|^2)^{1/2}} \leqslant (k+3)^{1/2} |\xi_1 - \xi_2|^{1/2},$$

since $|j| \leqslant |j + \xi_1| + |\xi_1| \leqslant k + 1$. For the case $j = 0$, if $(k^2 - \xi_2^2)(k^2 - \xi_1^2) \leqslant 0$

$$C_{\xi_1,\xi_2}(0) = |\sqrt{k^2 - \xi_2^2} - i\sqrt{\xi_1^2 - k^2}| = |\xi_1^2 - \xi_2^2|^{1/2} \leqslant 2|\xi_1 - \xi_2|^{1/2},$$

while if $(k^2 - \xi_2^2)(k^2 - \xi_1^2) > 0$

$$C_{\xi_1,\xi_2}(0) = \frac{|\xi_1^2 - \xi_2^2|}{|\sqrt{k^2 - \xi_2^2} + \sqrt{k^2 - \xi_1^2}|^{1/2}} \leqslant \sqrt{2} |\xi_1 - \xi_2|^{1/2}.$$

Finally for the case where $k^2 \neq |\xi_1 + j|^2$ and $j \neq 0$ we write

$$C_{\xi_1,\xi_2}(j) = \frac{||\xi_2 + j|^2 - |\xi_1 + j|^2|}{(1+|j|^2)^{1/2} |\sqrt{k^2 - |\xi_1 + j|^2} + \sqrt{k^2 - |\xi_2 + j|^2}|}.$$

Since $|\beta_{\xi_1}(j)| > 0$, then there exists $\delta > 0$ independent of j such that $|\beta_{\xi_1}(j) + \beta_{\xi_2}(j)| \geqslant \delta$, therefore

$$\sup_{j\in\mathbb{Z}^*, k^2 \neq |\xi_1+j|^2} C_{\xi_1,\xi_2}(j) \leqslant \sup_{j\in\mathbb{Z}^*, k^2 \neq |\xi_1+j|^2} \frac{|\xi_2 - \xi_1||\xi_2 + \xi_1 + 2j|}{|j|\delta} \leqslant \frac{4}{\delta} |\xi_2 - \xi_1|.$$

Summarizing, there exist a constant β independent from j, ξ_1 and ξ_2 such that

$$\sup_{j\in\mathbb{Z}} C_{\xi_1,\xi_2}(j) \leqslant \beta|\xi_2 - \xi_1|^{1/2}.$$

Consequently

$$\|g_{\xi_1,\xi_2}\|^2_{H^{-1/2}_{\xi_1}(\Gamma_0^R)} \leqslant \beta^2|\xi_1 - \xi_2|\|w_{\xi_2}\|^2_{H^{1/2}_{\xi_2}(\Gamma_0^R)}. \tag{20}$$

We observe that

$$\|w_{\xi_2}\|_{H^{1/2}_{\xi_2}(\Gamma_0^R)} \leqslant c\|e^{-i\xi_2\cdot x_1}w_{\xi_2}\|_{H^1_\sharp(\Omega_0^R)} \leqslant 2c\|w_{\xi_2}\|_{H^1(\Omega_0^R)},$$

with c being the continuity constant of the trace operator on $H^1_\sharp(\Omega_0^R)$. From Theorem 1.5 we have that

$$\|w_{\xi_2}\|_{H^1(\Omega_0^R)} \leqslant \tilde{c}\|v_{\xi_2}\|_{L^2(\Omega_0^R)} \leqslant \|v\|_{\tilde{L}^{2,\alpha}(D^p)}, \tag{21}$$

where $\tilde{c}$ is independent of ξ_2. Applying Theorem 1.5 to the variational formulation (19) proves, using (17), (20) and (21), the existence of a constant c independent of ξ_1 and ξ_2 such that

$$\|e\|_{H^1(\Omega_0^R)} \leqslant c|\xi_1 - \xi_2|^{\tilde{\alpha}}\|v\|_{\tilde{L}^{2,\alpha}(D^p)},$$

which ends the proof. $\square$

Theorem 1.8. *Assume that hypothesis 1.1 holds and that $v \in \tilde{L}^{2,\alpha}(\Omega^R)$ for $0 \leqslant \alpha < 1$. Then the solution $w \in H^2_{loc}(\Omega^R)$ of problem $(\mathcal{P})$ belongs to $\tilde{H}^{1,\alpha}(\Omega^R)$ and*

$$\|w\|_{\tilde{H}^{1,\alpha}(\Omega^R)} \leqslant c(\|v\|_{\tilde{L}^{2,\alpha}(D^p)} + \|v\|_{L^2(\tilde{D})}),$$

with c independent from v.

Proof. Let $w_\xi := (\mathcal{J}w)(\xi,\cdot)$. Since the support of $n - n_p$ is included in Ω_0^R, then we have $w_\xi \in H^1_\xi(\Omega^R)$ and satisfies

$$\begin{cases} \Delta w_\xi + k^2 n_p w_\xi = -k^2(n_p - 1)v_\xi - k^2\mathcal{R}_\xi((n - n_p)(w + v)) & \text{in } \Omega^R, \\ w_\xi = 0 & \text{on } \Gamma^0, \\ \frac{\partial w_\xi}{\partial x_2}(\cdot, R) = T^R_\xi(w_\xi|_{\Gamma_0^R}) & \text{on } \Gamma_0^R, \end{cases}$$

with $v_\xi := (\mathcal{J}v)(\xi,\cdot)$ and $\mathcal{R}_\xi(\varphi)$ for a function φ compactly supported in Ω_0^R denotes the extension by ξ-quasi periodicity of φ to all of Ω^R. Indeed $\mathcal{R}_\xi(\varphi) = \mathcal{J}(\varphi\chi_{\Omega_0^R})$ where $\chi_{\Omega_0^R}$ indicates the indicator function of the domain Ω_0^R. We decompose w_ξ as

$$w_\xi = w_\xi^p + \tilde{w}_\xi,$$

with w_ξ^p being the solution of (8) and $\breve{w}_\xi := w_\xi - w_\xi^p$ satisfying

$$\begin{cases} \Delta \breve{w}_\xi + k^2 n_p \breve{w}_\xi = -k^2 \mathcal{R}_\xi((n-n_p)(w+v)) & \text{in } \Omega^R, \\ \breve{w}_\xi = 0 & \text{on } \Gamma^0, \\ \frac{\partial \breve{w}_\xi}{\partial x_2}(\cdot, R) = T_\xi^R(\breve{w}_\xi|_{\Gamma_0^R}) & \text{on } \Gamma_0^R. \end{cases}$$

Denoting by $w^p := \mathcal{J}^{-1} w_\xi^p$. From Theorem 1.7 we have that $w^p \in \tilde{H}^{1,\tilde{\alpha}}(\Omega^R)$. Moreover, since $\mathcal{R}_\xi((n-n_p)(w+v)) \in C_\sharp^{0,\alpha}(I, L_\xi^2(\Omega_0^R))$, then using Theorem 1.7 we deduce that $\breve{w} := \mathcal{J}^{-1}\breve{w}_\xi \in \tilde{H}^{1,\tilde{\alpha}}(\Omega^R)$ and then $w = w^p + \breve{w} \in \tilde{H}^{1,\tilde{\alpha}}(\Omega^R)$. The estimate follows also from application of Theorem 1.7. $\square$

For the sake of studying the inverse problem, for fixed $\xi_0 \in I$, we consider $v_{\xi_0} \in L_{\xi_0}^2(D)$, where

$$L_{\xi_0}^2(D) := \{v \in L_{loc}^2(D) / \; v|_{D^p} \in L_{\xi_0}^2(D^p)\},$$

where $L_{\xi_0}^2(D^p)$ denotes the set of ξ_0 quasi-periodic functions that are in $L_{loc}^2(D^p)$. We would like to define a solution w_{ξ_0} to problem $(\mathcal{P})$ associated with $v = v_{\xi_0}$. Indeed, since $v_{\xi_0} \notin L^2(D)$, the solution can not be defined as in Theorem 1.2. We rather define the solution in this case as

$$w_{\xi_0} := w_{\xi_0}^p + \tilde{w}_{\xi_0}, \tag{22}$$

with $w_{\xi_0}^p \in H_{\xi_0}^1(\Omega_0^R)$ verifying

$$\begin{cases} \Delta w_{\xi_0}^p + k^2 n_p w_{\xi_0}^p = k^2(1-n_p) v_{\xi_0} & \text{in } \Omega^R, \\ w_{\xi_0}^p = 0 & \text{on } \Gamma_0^R, \\ \frac{\partial w_{\xi_0}^p}{\partial x_2}(\cdot, R) = T_\xi^R(w_{\xi_0}^p|_{\Gamma_0^R}) & \text{on } \Gamma_0^R, \end{cases} \tag{23}$$

and $\tilde{w}_{\xi_0} \in \tilde{H}^1(\Omega^R)$ satisfying

$$\begin{cases} \Delta \tilde{w}_{\xi_0} + k^2 n \tilde{w}_{\xi_0} = k^2(n_p - n)(v_{\xi_0} + w_{\xi_0}^p) & \text{in } \Omega^R, \\ \tilde{w}_{\xi_0} = 0 & \text{on } \Gamma^0, \\ \frac{\partial \tilde{w}_{\xi_0}}{\partial x_2}(\cdot, R) = T^R(\tilde{w}_{\xi_0}|_{\Gamma^R}) & \text{on } \Gamma^R. \end{cases} \tag{24}$$

The solutions of (23) and (24) are respectively defined by Theorems 1.5 and 1.8. Multiplying the first equation of (24) with $\overline{\psi} \in \tilde{H}^1(\Omega^R)$, integrating by parts and using the boundary conditions we obtain the variational formulation given as

$$\int_{\Omega^R} \left(\nabla \tilde{w}_{\xi_0} \cdot \nabla \overline{\psi} - k^2 n \tilde{w}_{\xi_0} \overline{\psi}\right) dx - \left\langle T^R \tilde{w}_{\xi_0}, \psi \right\rangle_{\Gamma^R} = k^2 \int_{\Omega^R} (n-n_p)(v_{\xi_0} + w_{\xi_0}^p)\overline{\psi}, \; \forall \, \psi \in \tilde{H}^1(\Omega^R), \tag{25}$$

where the notation $\langle \cdot, \cdot \rangle_{\Gamma^R}$ refers to the $H^{-1/2}(\Gamma^R) - H^{1/2}(\Gamma^R)$ duality product.

Remark 1.9. *Let $v \in \tilde{L}^2(\Omega^R)$ and set $v_\xi = \mathcal{J}(v)(\xi, \cdot)$. We define $w_\xi^p \in H_\xi^1(\Omega_0^R)$ verifying* (23) *and $\tilde{w}_\xi \in \tilde{H}^1(\Omega^R)$ the solution of (24). Then $w^p := \int_I w_\xi^p d\xi$ is solution to $(\mathcal{P})$ with $n = n_p$ and $\tilde{w} := \int_I \tilde{w}_\xi d\xi \in \tilde{H}^1(\Omega^R)$ is solution of*

$$\begin{cases} \Delta \tilde{w} + k^2 n_p \tilde{w} = k^2(n_p - n)(w^p + v + \tilde{w}) & \textit{in } \Omega^R, \\ \tilde{w} = 0 & \textit{on } \Gamma^0, \\ \frac{\partial \tilde{w}}{\partial x_2}(\cdot, R) = T^R(\tilde{w}|_{\Gamma^R}) & \textit{on } \Gamma^R. \end{cases} \tag{26}$$

Consequently $\tilde{w} + w^p \in \tilde{H}^1(\Omega^R)$ and is the solution of problem $(\mathcal{P})$.

2. The inverse problem for quasi-periodic incident fields

2.1. Setting for the inverse problem

Consider $\xi_0 \in I$ fixed, and let $\Phi_{\xi_0}(x, y) := (\mathcal{J}\Phi(\cdot, y))(\xi_0, x)$ be the ξ_0-quasi-periodic Green function having the following expression [20]

$$\Phi_{\xi_0}(x, y) := \frac{i}{4\pi} \sum_{j \in \mathbb{Z}} e^{i\alpha_{\xi_0}(j)(x_1 - y_1)} \theta_{\xi_0}(j, x_2, y_2), \quad y_2 < x_2, \tag{27}$$

with

$$\theta_{\xi_0}(j, x_2, y_2) := e^{i\beta_{\xi_0}(j)x_2} \left[\frac{e^{-i\beta_{\xi_0}(j)y_2} - e^{i\beta_{\xi_0}(j)y_2}}{\beta_{\xi_0}(j)} \right]. \tag{28}$$

Let $y \in \Gamma_0^R$. We define $u_{\xi_0}^s(\cdot, y) = w_{\xi_0}$ given by (22) with $v_{\xi_0} = \overline{\Phi_{\xi_0}(y, \cdot)} \in L_{\xi_0}^2(D)$. From (22) we decompose $u_{\xi_0}^s(\cdot, y) = u_{\xi_0}^{s,p}(\cdot, y) + \tilde{u}_{\xi_0}^s(\cdot, y)$ with $u_{\xi_0}^{s,p}(\cdot, y) = w_{\xi_0}^p$ solution of (23) and $\tilde{u}_{\xi_0}^s(\cdot, y) = \tilde{w}_{\xi_0}$ solution of (24). We introduce the ξ_0-quasi periodic near field operator $N_{\xi_0} : L_{\xi_0}^2(\Gamma^R) \longrightarrow L_{\xi_0}^2(\Gamma^R)$ as

$$N_{\xi_0} g_{\xi_0}(x) := \int_{\Gamma_0^R} g_{\xi_0}(y) u_{\xi_0}^{s,p}(x, y) ds(y) + \int_{\Gamma_0^R} g_{\xi_0}(y) \mathcal{J}(\tilde{u}_{\xi_0}^s(\cdot, y))(\xi_0, x) ds(y). \tag{29}$$

Define $S_{\xi_0} : L_{\xi_0}^2(\Gamma^R) \longrightarrow L_{\xi_0}^2(D)$ as

$$S_{\xi_0} g_{\xi_0}(x) := \int_{\Gamma_0^R} g_{\xi_0}(y) \overline{\Phi_{\xi_0}(y, x)} ds(y). \tag{30}$$

Then, obviously the operator N_{ξ_0} can be decomposed as

$$N_{\xi_0} = G_{\xi_0}(S_{\xi_0}), \tag{31}$$

where $G_{\xi_0} : L_{\xi_0}^2(D) \longrightarrow L_{\xi_0}^2(\Gamma^R)$ is the operator defined by

$$G_{\xi_0}(v_{\xi_0}) = (w_{\xi_0}^p + \tilde{w}_{\xi_0}^p)|_{\Gamma_0^R}, \tag{32}$$

with $w^p_{\xi_0}$ being the solution of (23) and $\tilde{w}^p_{\xi_0} = \mathcal{J}(\tilde{w}_{\xi_0})(\xi_0, \cdot)$ with $\tilde{w}_{\xi_0}$ is the solution of (24). We observe that $\tilde{w}^p_{\xi_0} \in H^1_{\xi_0}(\Omega^R_0)$ and verifies

$$\begin{cases} \Delta \tilde{w}^p_{\xi_0} + k^2 n_p \tilde{w}^p_{\xi_0} = k^2(n_p - n)(w^p_{\xi_0} + v_{\xi_0} + \tilde{w}_{\xi_0}) & \text{in } \Omega^R, \\ \tilde{w}^p_{\xi_0} = 0 & \text{on } \Gamma^R_0, \\ \frac{\partial \tilde{w}^p_{\xi_0}}{\partial x_2}(\cdot, R) = T^R_\xi(\tilde{w}^p_{\xi_0}|_{\Gamma^R_0}) & \text{on } \Gamma^R_0. \end{cases} \tag{33}$$

Multiplying the first equation of (33) with $\overline{\psi_{\xi_0}} \in \check{H}^1_{\xi_0}(\Omega^R_0)$, integrating by parts and using the boundary conditions we obtain the variational formulation given as

$$\int_{\Omega^R_0} \left(\nabla \tilde{w}^p_{\xi_0} \cdot \nabla \overline{\psi_{\xi_0}} - k^2 n_p \tilde{w}^p_{\xi_0} \overline{\psi_{\xi_0}} \right) dx - \left\langle T^R_{\xi_0} \tilde{w}^p_{\xi_0}, \psi_{\xi_0} \right\rangle_{\Gamma^R_0} = k^2 \int_{\Omega^R_0} (n - n_p)(w^p_{\xi_0} + v_{\xi_0} + \tilde{w}_{\xi_0}) \overline{\psi_{\xi_0}}, \tag{34}$$

for all $\psi_{\xi_0} \in \check{H}^1_{\xi_0}(\Omega^R_0)$. For later use we decompose $N_{\xi_0} = N^p_{\xi_0} + \tilde{N}^p_{\xi_0}$ where $N^p_{\xi_0} : L^2_{\xi_0}(\Gamma^R) \longrightarrow L^2_{\xi_0}(\Gamma^R)$ and $\tilde{N}^p_{\xi_0} : L^2_{\xi_0}(\Gamma^R) \longrightarrow L^2_{\xi_0}(\Gamma^R)$ are respectively defined as

$$N^p_{\xi_0} g_{\xi_0}(x) := \int_{\Gamma^R_0} g_{\xi_0}(y) u^{s,p}_{\xi_0}(x, y) ds(y), \quad \tilde{N}^p_{\xi_0} g_{\xi_0}(x) := \int_{\Gamma^R_0} g_{\xi_0}(y) \mathcal{J}(\tilde{u}^s_{\xi_0}(\cdot, y))(\xi_0, x) ds(y). \tag{35}$$

Lemma 2.1. *The operators $N^p_{\xi_0}$ and $\tilde{N}^p_{\xi_0}$ can be respectively factorized as*

$$N^p_{\xi_0} = S^*_{\xi_0} T^p_{\xi_0} S_{\xi_0} \quad \text{and} \quad \tilde{N}^p_{\xi_0} = S^*_{\xi_0} \tilde{T}^p_{\xi_0} S_{\xi_0}, \tag{36}$$

with $T^p_{\xi_0} : L^2_{\xi_0}(D^p) \longrightarrow L^2_{\xi_0}(D^p)$ and $\tilde{T}^p_{\xi_0} : L^2_{\xi_0}(D) \longrightarrow L^2_{\xi_0}(D)$ are respectively defined by

$$T^p_{\xi_0} v_{\xi_0} = k^2(1 - n_p)(v_{\xi_0} + w^p_{\xi_0}), \tag{37}$$

$$\tilde{T}^p_{\xi_0} v_{\xi_0} = k^2(1 - n_p)\tilde{w}^p_{\xi_0} + k^2(n_p - n)(w^p_{\xi_0} + v_{\xi_0} + \tilde{w}_{\xi_0}), \tag{38}$$

where $w^p_{\xi_0}$ being the solution of (23) and $\tilde{w}^p_{\xi_0} = \mathcal{J}(\tilde{w}_{\xi_0})(\xi_0, \cdot)$ with $\tilde{w}_{\xi_0}$ is the solution of (24).

Proof. The proof of (36) is classical and we here outline the main steps. The solution $w^p_{\xi_0}$ of (23) with $v_{\xi_0} = S_{\xi_0} g_{\xi_0}$ can be represented as [12]

$$w^p_{\xi_0}(x) = \int_{D^p_0} k^2 \Phi_{\xi_0}(x, y)(1 - n_p)(w^p_{\xi_0} + S_{\xi_0} g_{\xi_0})(y) dy \quad \text{for } x \in \Omega^R_0.$$

Then

$$\begin{aligned} (N^p_{\xi_0} g_{\xi_0}, \tilde{g}_{\xi_0})_{L^2(\Gamma^R_0)} &= \int_{D^p_0} k^2(1 - n_p)(w^p_{\xi_0} + S_{\xi_0} g_{\xi_0})(y) \int_{\Gamma^R_0} \Phi_{\xi_0}(x, y) \overline{\tilde{g}_{\xi_0}(x)} ds(x) dy, \\ &= \int_{D^p_0} k^2(1 - n_p)(w^p_{\xi_0} + S_{\xi_0} g_{\xi_0})(y) \overline{S_{\xi_0} \tilde{g}_{\xi_0}(y)} dy, \end{aligned}$$

$$= (T^p_{\xi_0} S_{\xi_0} g_{\xi_0}, S_{\xi_0} \tilde{g}_{\xi_0})_{L^2(D^p_0)},$$

which proves the first of factorization in (36).
The second factorization is obtained in the same way based on the fact that $w^p_{\xi_0}$ solution of (33) can be represented as

$$\tilde{w}^p_{\xi_0} = \int_{D^p_0} k^2 \Phi_{\xi_0}(\cdot, y)(1-n_p)\tilde{w}^p_{\xi_0}(y)dy + \int_{\tilde{D}} k^2 \Phi_{\xi_0}(\cdot, y)(n_p - n)(w^p_{\xi_0} + S_{\xi_0} g_{\xi_0} + \tilde{w}_{\xi_0})dy.$$

□

From Lemma 2.1 we conclude the following factorization

$$N_{\xi_0} = S^*_{\xi_0} T_{\xi_0} S_{\xi_0}$$

with $T_{\xi_0} : L^2_{\xi_0}(D) \longrightarrow L^2_{\xi_0}(D)$ is defined by $T_{\xi_0}(v_{\xi_0}) = T^p_{\xi_0}(v_{\xi_0}|_{D^p}) + \tilde{T}^p_{\xi_0}(v_{\xi_0})$ or equivalently

$$T_{\xi_0} v_{\xi_0} = k^2(1-n_p)(w^p_{\xi_0} + v_{\xi_0} + \tilde{w}^p_{\xi_0}) + k^2(n_p - n)(w^p_{\xi_0} + v_{\xi_0} + \tilde{w}_{\xi_0}). \tag{39}$$

2.2. Some properties of the operators defined in the previous section

In order to study the inverse problem we need to prove some properties of the operators defined in the previous sections.

Lemma 2.2. *The operator $S_{\xi_0} : L^2_{\xi_0}(\Gamma^R) \longrightarrow L^2_{\xi_0}(D)$ is injective. The closure of its range is*

$$H^{inc}_{\xi_0}(D) := \{v \in L^2_{\xi_0}(D),\ \Delta v + k^2 v = 0 \textit{ in } D\}. \tag{40}$$

Proof. Let $g_{\xi_0} \in L^2_{\xi_0}(\Gamma^R)$ such that $S_{\xi_0} g_{\xi_0} = 0$ in D_0, where $D_0 := \Omega^R_0 \cap D$. Using the unique continuation principle we obtain $S_{\xi_0} g_{\xi_0} = 0$ in Ω^R_0. Let $U^R := \mathbb{R} \times [R, \infty[$. Using the continuity and regularity of single layer potentials we have that $S_{\xi_0} \in H^2_{loc}(U^R)$, is ξ_0-quasi periodic, verifies

$$\begin{cases} \Delta S_{\xi_0} g_{\xi_0} + k^2 S_{\xi_0} g_{\xi_0} = 0 & \text{in } U^R, \\ S_{\xi_0} g_{\xi_0} = 0 & \text{on } \Gamma^R, \end{cases} \tag{41}$$

and the upper going radiation condition (11) with Γ^R replaced by $\Gamma^{R'}$ with $R' > R$. The uniqueness for this Dirichlet quasi-periodic scattering problem [18] implies that $S_{\xi_0} g_{\xi_0} = 0$ in U^R. Therefore, using the jump relations for the normal derivative of S_{ξ_0} we obtain $g_{\xi_0} = 0$ which proves the injectivety of S_{ξ_0}.
Let $S^*_{\xi_0} : L^2_{\xi_0}(D) \longrightarrow L^2_{\xi_0}(\Gamma^R)$ be the adjoint of S_{ξ_0} given by

$$S^*_{\xi_0} v_{\xi_0}(y) := \int_{D_0} \Phi_{\xi_0}(y, x) v_{\xi_0}(x) dx.$$

Let $v_{\xi_0} \in H^{inc}_{\xi_0}(D)$, we set

$$w_{\xi_0} := \int_{D_0} \Phi_{\xi_0}(\cdot, x) v_{\xi_0}(x) dx.$$

Using the properties of the volume potential we deduce that $w_{\xi_0} \in H^2_{\xi_0}(\Omega^R)$ satisfies

$$\begin{cases} \Delta w_{\xi_0} + k^2 w_{\xi_0} = -v_{\xi_0} & \text{in } D_0, \\ \Delta w_{\xi_0} + k^2 w_{\xi_0} = 0 & \text{in } \Omega^R_0 \backslash D_0, \\ \frac{\partial w_{\xi_0}}{\partial x_2}(\cdot, R) = T^R_\xi(w_{\xi_0}|_{\Gamma^R_0}) & \text{on } \Gamma^R_0. \end{cases} \tag{42}$$

Assume that $w_{\xi_0} = 0$ on Γ^R_0. Then w_{ξ_0} vanishes in U^R. Using the unique continuation principle we obtain that $w_{\xi_0} = 0$ in $\Omega^R_0 \backslash D_0$. We then have $w_{\xi_0} \in H^2_0(D_0)$. Therefore, since $\Delta v_{\xi_0} + k^2 v_{\xi_0} = 0$ in D,

$$0 = \int_{D_0} v_{\xi_0}(y) \overline{(\Delta w_{\xi_0} + k^2 w_{\xi_0})}(y) dy = -\|v_{\xi_0}\|^2_{L^2(D_0)}. \tag{43}$$

This proves that $v_{\xi_0} = 0$ and S_{ξ_0} has a dense range in $H^{inc}_{\xi_0}(D)$. □

For the analysis below we need to assume the well posedness of the following two Interior Transmission Problems (ITP).

(ITP1): Seek $(u, v) \in L^2_{\xi_0}(D^p) \times L^2_{\xi_0}(D^p)$ such that $(u - v) \in H^2_{\xi_0}(D^p)$ satisfying

$$\begin{cases} \Delta u + k^2 n_p u = 0 & \text{in } D^p_0, \\ \Delta v + k^2 v = 0 & \text{in } D^p_0, \\ u - v = \varphi & \text{on } \partial D^p_0, \\ \frac{\partial (u-v)}{\partial \nu} = \psi & \text{on } \partial D^p_0, \end{cases} \tag{44}$$

for given $(\varphi, \psi) \in H^{3/2}_{\xi_0}(\partial D^p) \times H^{1/2}_{\xi_0}(\partial D^p)$. The spaces $H^m_{\xi_0}(D^p)$ and $H^s_{\xi_0}(D^p)$ are defined similarly to $H^m_{\xi_0}(\Omega^R)$ and $H^s_{\xi_0}(\Gamma^R)$.

(ITP2): Seek $(u, v) \in L^2(\tilde{D}) \times L^2(\tilde{D})$ such that $(u - v) \in H^2(\tilde{D})$ satisfying

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } \tilde{D}, \\ \Delta v + k^2 v = 0 & \text{in } \tilde{D}, \\ u - v = \varphi & \text{on } \partial \tilde{D}, \\ \frac{\partial (u-v)}{\partial \nu} = \psi & \text{on } \partial \tilde{D}, \end{cases} \tag{45}$$

for given $(\varphi, \psi) \in H^{3/2}(\partial \tilde{D}) \times H^{1/2}(\partial \tilde{D})$.

Assumption 2.3. *Assume that k, n_p are such as the **(ITP1)** is well posed.*

Assumption 2.4. *Assume that k, n_p and n are such that **(ITP2)** is well posed.*

Moreover, we need first to prove the following Lemma

Lemma 2.5. *For all $v^1_{\xi_0}, v^2_{\xi_0} \in L^2_{\xi_0}(D)$ we have*

$$(T_{\xi_0} v^1_{\xi_0}, \overline{v^2_{\xi_0}})_{L^2(D_0)} = (T_{\xi_0} v^2_{\xi_0}, \overline{v^1_{\xi_0}})_{L^2(D_0)}. \tag{46}$$

Proof. For $i = 1, 2$, consider $w^{p,i}_{\xi_0} \in \check{H}^1_{\xi_0}(\Omega^R_0)$ solution of (10) with $\xi = \xi_0$ and $v_\xi = v^i_{\xi_0}$. Define $\tilde{w}^{p,i}_{\xi_0} := \mathcal{J}(\tilde{w}^i_{\xi_0})(\xi_0, \cdot)$ where $\tilde{w}^i_{\xi_0} \in \tilde{H}^1(\Omega^R)$ satisfies (24) with $v_{\xi_0} = v^i_{\xi_0}$. We set

$$w^i_{\xi_0} := w^{p,i}_{\xi_0} + \tilde{w}^{p,i}_{\xi_0}$$

which verifies

$$\int_{\Omega^R_0} \left(\nabla w^i_{\xi_0} \cdot \nabla \overline{\psi_{\xi_0}} - k^2 n_p w^i_{\xi_0} \overline{\psi_{\xi_0}}\right) dx - \left\langle T^R_\xi w^i_{\xi_0}, \psi_{\xi_0} \right\rangle_{\Gamma^R_0} = -L^i_{\xi_0}(\psi_{\xi_0}) \tag{47}$$

for all $\psi_{\xi_0} \in \check{H}^1_{\xi_0}(\Omega^R_0)$ with

$$L^i_{\xi_0}(\psi_{\xi_0}) := \int_{D^p_0} k^2(1 - n_p) v^i_{\xi_0} \overline{\psi_{\xi_0}} dy + k^2 \int_{\Omega^R_0} (n_p - n)(w^{p,i}_{\xi_0} + v^i_{\xi_0} + \tilde{w}^i_{\xi_0}) \overline{\psi_{\xi_0}}.$$

Taking $\overline{\psi_{\xi_0}} = w^2_{\xi_0}$ and $\overline{\psi_{\xi_0}} = w^1_{\xi_0}$ respectively in the variational formulation (47) satisfied by $w^1_{\xi_0}$ and $w^2_{\xi_0}$ we obtain by taking the difference

$$\begin{aligned}\left\langle T^R_{\xi_0} w^1_{\xi_0}, \overline{w^2_{\xi_0}} \right\rangle_{\Gamma^R_0} - \left\langle T^R_{\xi_0} w^2_{\xi_0}, \overline{w^1_{\xi_0}} \right\rangle_{\Gamma^R_0} &= \int_{D^p_0} k^2(1 - n_p) v^1_{\xi_0} w^2_{\xi_0} dy + \int_{\tilde{D}} k^2(n_p - n)(w^{p,1}_{\xi_0} + \tilde{w}^1_{\xi_0} + v^1_{\xi_0}) w^2_{\xi_0} dy \\ &- \int_{D^p_0} k^2(1 - n_p) v^2_{\xi_0} w^1_{\xi_0} dy - \int_{\tilde{D}} k^2(n_p - n)(w^{p,2}_{\xi_0} + \tilde{w}^2_{\xi_0} + v^2_{\xi_0}) w^1_{\xi_0} dy.\end{aligned}$$

Since $T^R_{\xi_0}$ is symmetric, the left hand side in the previous equality vanishes and therefore

$$\begin{aligned}\int_{D^p_0} k^2(1 - n_p) v^2_{\xi_0} (w^{p,1}_{\xi_0} + \tilde{w}^{p,1}_{\xi_0} + v^1_{\xi_0}) dy &= \int_{D^p_0} k^2(1 - n_p) v^1_{\xi_0} (w^{p,2}_{\xi_0} + \tilde{w}^{p,2}_{\xi_0} + v^2_{\xi_0}) dy \\ + \int_{\tilde{D}} k^2(n_p - n)(w^{p,1}_{\xi_0} + \tilde{w}^1_{\xi_0} + v^1_{\xi_0})(w^{p,2}_{\xi_0} + \tilde{w}^{p,2}_{\xi_0}) dy &- \int_{\tilde{D}} k^2(n_p - n)(w^{p,2}_{\xi_0} + \tilde{w}^2_{\xi_0} + v^2_{\xi_0})(w^{p,1}_{\xi_0} + \tilde{w}^{p,1}_{\xi_0}) dy.\end{aligned} \tag{48}$$

On the other hand, taking $\overline{\psi_{\xi_0}} = \tilde{w}^{p,2}_{\xi_0}$ and $\overline{\psi_{\xi_0}} = \tilde{w}^{p,1}_{\xi_0}$ respectively in the variational formulations (34) satisfied by $\tilde{w}^{p,1}_{\xi_0}$ and $\tilde{w}^{p,2}_{\xi_0}$ we obtain by taking the difference and the symmetry of $T^R_{\xi_0}$

$$\int_{\tilde{D}} k^2(n_p - n)(w^{p,1}_{\xi_0} + v^1_{\xi_0} + \tilde{w}^1_{\xi_0}) \tilde{w}^{p,2}_{\xi_0} dy = \int_{\tilde{D}} k^2(n_p - n)(w^{p,2}_{\xi_0} + v^2_{\xi_0} + \tilde{w}^2_{\xi_0}) \tilde{w}^{p,1}_{\xi_0} dy. \tag{49}$$

Moreover, taking $\overline{\psi_{\xi_0}} = \tilde{w}^2_{\xi_0}$ and $\overline{\psi_{\xi_0}} = \tilde{w}^1_{\xi_0}$ respectively in the variational formulation (25) satisfied by $\tilde{w}^1_{\xi_0}$ and $\tilde{w}^2_{\xi_0}$ we obtain after taking the difference and using the symmetry of the operator T^R

$$\int_{\tilde{D}} k^2(n_p - n)v^2_{\xi_0}\tilde{w}^1_{\xi_0}dy = -\int_{\tilde{D}} k^2(n_p - n)w^{p,2}_{\xi_0}\tilde{w}^1_{\xi_0}dy + \int_{\tilde{D}} k^2(n_p - n)\tilde{w}^2_{\xi_0}(v^1_{\xi_0} + w^{p,1}_{\xi_0})dy. \quad (50)$$

Now, using (39) we have

$$T_{\xi_0}v^i_{\xi_0} = k^2(1 - n_p)(w^{p,i}_{\xi_0} + v^i_{\xi_0} + \tilde{w}^{p,i}_{\xi_0}) + k^2(n_p - n)(w^{p,i}_{\xi_0} + v^i_{\xi_0} + \tilde{w}^i_{\xi_0}). \quad (51)$$

Using (48) to substitute the first term in the right hand side of (51) we get

$$(T_{\xi_0}v^1_{\xi_0}, \overline{v^2_{\xi_0}})_{L^2(D_0)} = \int_{D^p_0} k^2(1 - n_p)v^1_{\xi_0}(w^{p,2}_{\xi_0} + \tilde{w}^{p,2}_{\xi_0} + v^2_{\xi_0}) + \int_{\tilde{D}} k^2(n_p - n)(w^{p,1}_{\xi_0} + \tilde{w}^1_{\xi_0} + v^1_{\xi_0})(w^{p,2}_{\xi_0} + \tilde{w}^{p,2}_{\xi_0})$$
$$-\int_{\tilde{D}} k^2(n_p - n)(w^{p,2}_{\xi_0} + \tilde{w}^2_{\xi_0} + v^2_{\xi_0})(w^{p,1}_{\xi_0} + \tilde{w}^{p,1}_{\xi_0}) + \int_{\tilde{D}} k^2(n_p - n)v^2_{\xi_0}(w^{p,1}_{\xi_0} + v^1_{\xi_0} + \tilde{w}^1_{\xi_0}). \quad (52)$$

Finally, using (49), (50) to simplify the previous expression we obtain

$$(T_{\xi_0}v^1_{\xi_0}, \overline{v^2_{\xi_0}})_{L^2(D_0)} = \int_{D^p_0} k^2(1-n_p)v^1_{\xi_0}(w^{p,2}_{\xi_0}+\tilde{w}^{p,2}_{\xi_0}+v^2_{\xi_0})dy + \int_{\tilde{D}} k^2(n_p-n)v^1_{\xi_0}(w^{p,2}_{\xi_0}+\tilde{w}^2_{\xi_0}+v^2_{\xi_0})dy,$$

where the right hand side coincides with the expression of $(T_{\xi_0}v^2_{\xi_0}, \overline{v^1_{\xi_0}})_{L^2(D_0)}$. This ends the proof. □

Lemma 2.6. *Assume that the assumptions of Theorem 1.5 hold and that Assumptions 2.3, 2.4 hold. Assume in addition that $D^p_0 \cap \tilde{D} = \emptyset$. Then, the operator $G_{\xi_0} : H^{inc}_{\xi_0}(D) \longrightarrow L^2_{\xi_0}(\Gamma^R)$ given by (32) is injective with dense range.*

Proof. Let $v_{\xi_0} \in H^{inc}_{\xi_0}(D)$ such that $G_{\xi_0}v_{\xi_0} = 0$ on Γ^R_0, i.e

$$w_{\xi_0} := w^p_{\xi_0} + \tilde{w}^p_{\xi_0} = 0 \quad \text{on } \Gamma^R_0,$$

where $w^p_{\xi_0}$ is the solution of (23) and $\tilde{w}^p_{\xi_0} = \mathcal{J}(\tilde{w}_{\xi_0})(\xi_0, \cdot)$ with $\tilde{w}_{\xi_0}$ being the solution of (24). Therefore, $w_{\xi_0} \in H^2_{loc}(U^R)$, is ξ_0-quasi periodic and verifies

$$\begin{cases} \Delta w_{\xi_0} + k^2 w_{\xi_0} = 0 & \text{in } U^R, \\ w_{\xi_0} = 0 & \text{on } \Gamma^R, \end{cases}$$

and the upper going radiation condition (11) with Γ^R replaced by $\Gamma^{R'}$ with $R' > R$. The uniqueness for this Dirichlet quasi-periodic scattering problem implies that $w_{\xi_0} = 0$ in U^R. Using the unique continuation principle we obtain that $w_{\xi_0} = 0$ in $U^0 \backslash D_0$. Moreover, $w_{\xi_0} \in H^2_{\xi_0}(\Omega^R)$ and satisfies

$$\Delta w_{\xi_0} + k^2 n_p w_{\xi_0} + k^2(n - n_p)(w^p_{\xi_0} + \tilde{w}_{\xi_0}) = k^2(1 - n)v_{\xi_0} \quad \text{in } \Omega^R_0. \quad (53)$$

Since $D_0^p \cap \tilde{D} = \varnothing$, then we have in particular $w_{\xi_0} \in H_0^2(D_0^p)$ and

$$\Delta w_{\xi_0} + k^2 n_p w_{\xi_0} = k^2(1-n_p)v_{\xi_0} \quad \text{in } D_0^p.$$

Setting $u_{\xi_0} := w_{\xi_0} + v_{\xi_0}$ we observe that the couple (u_{ξ_0}, v_{ξ_0}) verifies **(ITP1)** with zero data. Therefore, $w_{\xi_0}|_{D_0^p} = v_{\xi_0}|_{D_0^p} = 0$. The latter implies in particular that $w_{\xi_0}^p = 0$ by well posedness of the periodic direct scattering problem. On the other hand, since $n_p = 1$ in $\tilde{D}$, then $w_{\xi_0} \in H_0^2(\tilde{D})$ and satisfies

$$\Delta w_{\xi_0} + k^2 w_{\xi_0} + k^2(n-1)\tilde{w}_{\xi_0} = k^2(1-n)v_{\xi_0} \quad \text{in } \tilde{D}. \tag{54}$$

Since $w_{\xi_0} \in H_0^2(\tilde{D})$, then we have

$$\int_{\tilde{D}} (\Delta w_{\xi_0} + k^2 w_{\xi_0})\theta = 0 \quad \text{for all } \theta \in H^{inc}(\tilde{D}),$$

where $H^{inc}(\tilde{D}) := \{v \in L^2(\tilde{D}) / \ \Delta v + k^2 v = 0 \text{ in } \tilde{D}\}$. Therefore, taking the L^2 scalar product of (54) with θ we get

$$\int_{\tilde{D}} (k^2(1-n)v_{\xi_0} + k^2(1-n)\tilde{w}_{\xi_0})\bar{\theta} = 0 \quad \forall\, \theta \in H^{inc}(\tilde{D}). \tag{55}$$

From Theorem 1.4 we have that $\tilde{w}_{\xi_0} \in \tilde{H}^1(\Omega^R)$ can be represented as

$$\tilde{w}_{\xi_0}(x) = k^2 \int_{D^p} \Phi(x,y)(1-n_p)\tilde{w}_{\xi_0}(y)dy + k^2 \int_{\tilde{D}} \Phi(x,y)((1-n)v_{\xi_0} + (1-n)\tilde{w}_{\xi_0})dy \quad \text{for } x \in \Omega^R.$$

Since $y \to \Phi(x,y) \in H^{inc}(\tilde{D})$ for $x \notin \tilde{D}$, we obtain from (55) that

$$\tilde{w}_{\xi_0}(x) = \int_{D^p} k^2(1-n_p)\tilde{w}_{\xi_0}(y)\Phi(x,y)dy \quad \text{for } x \notin \tilde{D}.$$

Let us define $\breve{w}_{\xi_0} \in H^2(\tilde{D})$ by

$$\breve{w}_{\xi_0}(x) := \int_{D^p} k^2(1-n_p)\tilde{w}_{\xi_0}(y)\Phi(x,y)dy \quad \text{for } x \in \tilde{D}.$$

We set $u_{\xi_0} := \tilde{w}_{\xi_0} + v_{\xi_0}$. Then the couple (u_{ξ_0}, v_{ξ_0}) satisfies **(ITP2)** with $(\varphi, \psi) = (\breve{w}_{\xi_0}, \frac{\partial \breve{w}_{\xi_0}}{\partial \nu})$. Moreover, since

$$\Delta \breve{w}_{\xi_0} + k^2 \breve{w}_{\xi_0} = 0 \quad \text{in } \tilde{D},$$

and **(ITP2)** is well posed then $v_{\xi_0} + \breve{w}_{\xi_0} = 0$ and $\tilde{w}_{\xi_0} + v_{\xi_0} = 0$ in $\tilde{D}$. We then deduce that $\tilde{w}_{\xi_0} = \breve{w}_{\xi_0}$ in $\tilde{D}$. Consequently

$$\tilde{w}_{\xi_0}(x) = \int_{D^p} k^2(1-n_p)\tilde{w}_{\xi_0}(y)\Phi(x,y)dy \quad \text{for } x \in \Omega^R,$$

which implies that $\tilde{w}_{\xi_0}$ satisfies $(\mathcal{P})$ with $n = n_p$ and $v = 0$. We then conclude that $\tilde{w}_{\xi_0} = 0$ by uniqueness of the solution to problem $(\mathcal{P})$ with $n = n_p$. Therefore $v_{\xi_0} = 0$ in $\tilde{D}$ which, together with $v_{\xi_0} = 0$ in D^p prove the injectivety of G_{ξ_0}.
Now, we prove the denseness of the range of G_{ξ_0}. Let $g_{\xi_0} \in \overline{R(G_{\xi_0})}^{\perp}$, then

$$(G_{\xi_0} v_{\xi_0}, g_{\xi_0})_{L^2(D_0)} = 0 \quad \forall\, v_{\xi_0} \in H^{inc}_{\xi_0}(D).$$

Let $f_{\xi_0} \in L^2_{\xi_0}(\Gamma^R)$ and consider $v_{\xi_0} = \overline{S_{\xi_0} f_{\xi_0}}$. Using Lemma 2.1 we have

$$(T_{\xi_0} \overline{(S_{\xi_0} f_{\xi_0})}, S_{\xi_0} g_{\xi_0}) = 0, \quad \forall\, f_{\xi_0} \in L^2_{\xi_0}(\Gamma^R). \tag{56}$$

Moreover, using Lemma 2.5 we get

$$(T_{\xi_0} \overline{(S_{\xi_0} f_{\xi_0})}, S_{\xi_0} g_{\xi_0}) = (T_{\xi_0} \overline{(S_{\xi_0} g_{\xi_0})}, S_{\xi_0} f_{\xi_0}) \quad \forall\, f_{\xi_0} \in L^2_{\xi_0}(\Gamma^R).$$

Therefore, (56) implies that

$$(G_{\xi_0}(\overline{S_{\xi_0} g_{\xi_0}}), f_{\xi_0})_{L^2(\Gamma^R_0)} = 0, \quad \forall\, f_{\xi_0} \in L^2_{\xi_0}(\Gamma^R).$$

Then $G_{\xi_0}(\overline{S_{\xi_0} g_{\xi_0}}) = 0$. The injectivety of G_{ξ_0} gives that $S_{\xi_0} g_{\xi_0} = 0$ and then $g_{\xi_0} = 0$ by Lemma 2.2. $\square$

Lemma 2.7. *Under the same assumptions of Lemma 2.6 we have that*

$$(z \in D_0) \quad \Longleftrightarrow \quad (\Phi_{\xi_0}(\cdot, z) \in Range(G_{\xi_0}))\,.$$

Proof. Let $z \in D^p$. We consider $v_{\xi_0} \in H^{inc}_{\xi_0}(D)$ such that $v_{\xi_0}|_{\tilde{D}} = -\Phi_{\xi_0}(\cdot, z)$. Let $(u_{\xi_0}, v_{\xi_0}) \in L^2_{\xi_0}(D^p) \times L^2_{\xi_0}(D^p)$ be the solution of **(ITP1)** with $(\varphi, \psi) = (\Phi_{\xi_0}(\cdot, z), \frac{\partial \Phi_{\xi_0}(\cdot, z)}{\partial \nu})$. We set

$$w^p_{\xi_0} := \begin{cases} u_{\xi_0} - v_{\xi_0} & \text{in } D^p, \\ \Phi_{\xi_0}(\cdot, z) & \text{in } \Omega^R \backslash D^p. \end{cases}$$

We observe that $w^p_{\xi_0} \in H^2_{\xi_0}(\Omega^R)$ and satisfies (23). Moreover, let $\tilde{w}_{\xi_0}$ be the solution of (24). Since $(n_p - n)(w^p_{\xi_0} + v_{\xi_0}) = 0$, then $\tilde{w}_{\xi_0} = 0$ and consequently $G_{\xi_0}(v_{\xi_0}) = \Phi_{\xi_0}(\cdot, z)$.
Consider now the case where $z \in \tilde{D}$. Since $\tilde{D} \cap D^p = \varnothing$ Recall that the Green function $\Phi(\cdot, z)$ defined by (4) belongs to $L^2(\Omega^R)$ [9]. Let $u \in H^2_{loc}(\Omega^R)$ be the solution of $(\mathcal{P})$ with $n = n_p$ and $v = \Phi(\cdot, z)$. Let us define $\Phi_{n_p}(\cdot, z) := u + \Phi(\cdot, z)$ that satisfies in particular, $\Phi_{n_p}(\cdot, z) \in L^2(\Omega^R) \cap H^2_{loc}(\Omega^R \backslash \tilde{D})$

$$\Delta \Phi_{n_p}(\cdot, z) + k^2 n_p \Phi_{n_p}(\cdot, z) = -\delta_z \quad \text{in } \Omega^R$$

together with the upper going radiation condition (2). Consider $v_{\xi_0} \in H^{inc}_{\xi_0}(D)$ such that $v_{\xi_0}|_{D^p} = 0$ and let $(\tilde{u}_{\xi_0}, v_{\xi_0}) \in L^2(\tilde{D}) \times L^2(\tilde{D})$ be the solution of **(ITP2)** with $(\varphi, \psi) = (\Phi_{n_p}(\cdot, z), \frac{\partial \Phi_{n_p}(\cdot,z)}{\partial \nu})$. We set

$$\tilde{w}_{\xi_0} := \begin{cases} \tilde{u}_{\xi_0} - v_{\xi_0} & \text{in } \tilde{D}, \\ \Phi_{n_p}(\cdot, z) & \text{in } \Omega^R \backslash \tilde{D}. \end{cases}$$

We observe that $\tilde{w}_{\xi_0} \in H^2_{loc}(\Omega^R)$ satisfies (24). Moreover, since $v_{\xi_0}|_{D^p} = 0$, then the right hand side of the first equation of (23) vanishes and therefore the solution $w^p_{\xi_0}$ of (23) vanishes. Consequently

$$G_{\xi_0}(v_{\xi_0}) = \mathcal{J}(\Phi_{n_p}(\cdot, z))(\xi_0) =: \Phi_{n_p,\xi_0}(\cdot, z).$$

On the other hand, we consider $u_{\xi_0} := \mathcal{J}(u)(\xi_0, \cdot) \in H^1_{\xi_0}(\Omega^R_0)$. Then u_{ξ_0} satisfies (23) with $v_{\xi_0} = \Phi_{\xi_0}(\cdot, z)$. We set

$$\tilde{v}_{\xi_0} := \begin{cases} \Phi_{\xi_0}(\cdot, z) & \text{in } D^p, \\ -u_{\xi_0} & \text{in } \tilde{D}. \end{cases}$$

Let $w^p_{\xi_0}$ and $\tilde{w}_{\xi_0}$ be respectively the solutions of (23) and (24) with $v_{\xi_0} = \tilde{v}_{\xi_0}$. By uniqueness of the solution of problem (23) we have $w^p_{\xi_0} = u_{\xi_0}$. Moreover, $\tilde{w}_{\xi_0} = 0$ since $k^2(n_p - n)(\tilde{v}_{\xi_0} + u_{\xi_0}) = 0$. Therefore

$$G_{\xi_0}(\tilde{v}_{\xi_0}) = u_{\xi_0}.$$

Consequently

$$G_{\xi_0}(v_{\xi_0} - \tilde{v}_{\xi_0}) = \Phi_{\xi_0}(\cdot, z).$$

Consider finally the case where $z \notin D_0$. Assume that there exists $v_{\xi_0} \in H^{inc}_{\xi_0}(D)$ such that $G_{\xi_0}(v_{\xi_0}) = \Phi_{\xi_0}(\cdot, z)$. Using the unique continuation principle we get $w_{\xi_0} := w^p_{\xi_0} + \tilde{w}^p_{\xi_0} = \Phi_{\xi_0}(\cdot, z)$ in $U^0 \backslash D_0$ which is a contradiction since $w_{\xi_0} \in H^2_{\xi_0}(\Omega^R \backslash D_0)$ while $\Phi_{\xi_0}(\cdot, z) \notin H^2_{\xi_0}(\Omega^R \backslash D_0)$. $\square$

Let us define now the following norm for $g_{\xi_0} \in L^2_{\xi_0}(\Gamma^R)$

$$I_{\xi_0} g_{\xi_0} := \left| (N^p_{\xi_0} g_{\xi_0}, g_{\xi_0})_{L^2(\Gamma^R_0)} \right| + \left| (\tilde{N}^p_{\xi_0} g_{\xi_0}, g_{\xi_0})_{L^2(\Gamma^R_0)} \right|. \tag{57}$$

From the factorizations (36) we have the equivalent expression

$$I_{\xi_0} g_{\xi_0} := \left| (T^p_{\xi_0} S_{\xi_0} g_{\xi_0}, S_{\xi_0} g_{\xi_0})_{L^2_{\xi_0}(D^p)} \right| + \left| (\tilde{T}^p_{\xi_0} S_{\xi_0} g_{\xi_0}, S_{\xi_0} g_{\xi_0})_{L^2_{\xi_0}(D)} \right|. \tag{58}$$

Lemma 2.8. *Assume that Assumption 1.1 and 2.4 hold. Assume in addition that $\Re e(1-n_p) \geqslant \gamma > 0$ in D^p and $\Re e(n_p - n) \geqslant \gamma_1 > 0$ in $\tilde{D}$ or $\Re e(1-n_p) \leqslant -\gamma < 0$ in D^p and $\Re e(n_p - n) \leqslant -\gamma_1 < 0$. Then, there exists a constant $c > 0$ independent from ξ_0 such that*

$$I_{\xi_0} g_{\xi_0} \geqslant c \left(\|S_{\xi_0} g_{\xi_0}\|^2_{L^2(D^p_0)} + \|S_{\xi_0} g_{\xi_0}\|^2_{L^2(\tilde{D})} \right) \quad \forall\, g_{\xi_0} \in L^2_{\xi_0}(\Gamma^R),$$

with $I_{\xi_0} g_{\xi_0}$ is the norm given by (57).

Proof. We prove the Lemma in the case $\Re e(1-n_p) \geqslant \gamma > 0$ in D^p and $\Re e(n_p - n) \geqslant \gamma_1 > 0$ in $\tilde{D}$. The other case can be proved in the same way. We shall use a contradiction argument. We consider $g_{\xi_0} \in L^2_{\xi_0}(\Gamma^R)$ and we denote by $v_{\xi_0} = S_{\xi_0} g_{\xi_0}$. Assume that there exists a sequence $v^\ell_{\xi_0}$ such that

$$\frac{1}{\ell}\|v^\ell_{\xi_0}\|^2_{L^2(D_0)} \geqslant \left|(T^p_{\xi_0} v^\ell_{\xi_0}, v^\ell_{\xi_0})_{L^2(D^p_0)}\right| + \left|(\tilde{T}^p_{\xi_0} v^\ell_{\xi_0}, v^\ell_{\xi_0})_{L^2(D_0)}\right|. \tag{59}$$

We set $\hat{v}^\ell_{\xi_0} = \frac{v^\ell_{\xi_0}}{\|v^\ell_{\xi_0}\|_{L^2(D_0)}}$. Let $w^{p,\ell}_{\xi_0}$ be the solution of (10) and $\tilde{w}^{p,\ell}_{\xi_0} = \mathcal{J}(\tilde{w}^\ell_{\xi_0})(\xi_0,\cdot)$ with $v_\xi = \hat{v}^\ell_{\xi_0}$ and $\tilde{w}_{\xi_0}$ is the solution of (25). Since $\|\hat{v}^\ell_{\xi_0}\|_{L^2(D_0)}$ is bounded, then we can extract a subsequence (that we still denote the same) $\hat{v}^\ell_{\xi_0}$ that converge weakly to $\hat{v}_{\xi_0}$ in $L^2(D_0)$. Moreover, $w^{p,\ell}_{\xi_0}$ and $\tilde{w}^{p,\ell}_{\xi_0}$ converge weakly in $H^1_{\xi_0}(\Omega^R_0)$ and strongly in $L^2(D_0)$ respectively to some $w^p_{\xi_0}$ and $\tilde{w}^p_{\xi_0} \in H^1_{\xi_0}(\Omega^R_0)$. On the other hand, taking $\psi_{\xi_0} = w^{p,\ell}_{\xi_0}$ in the variational formulation (10) satisfied by $w^{p,\ell}_{\xi_0}$ we obtain

$$\int_{\Omega^R_0}(|\nabla w^{p,\ell}_{\xi_0}|^2 - k^2|w^{p,\ell}_{\xi_0}|^2)dy = -k^2\int_{D^p_0}(1-n_p)(w^{p,\ell}_{\xi_0} + \hat{v}^\ell_{\xi_0})\overline{w^{p,\ell}_{\xi_0}}dy + \left\langle T^R_{\xi_0} w^{p,\ell}_{\xi_0}, w^{p,\ell}_{\xi_0}\right\rangle_{\Gamma^R_0}.$$

Therefore, decomposing $(\hat{v}^\ell_{\xi_0} + w^{p,\ell}_{\xi_0})\overline{\hat{v}^\ell_{\xi_0}} = |\hat{v}^\ell_{\xi_0} + w^{p,\ell}_{\xi_0}|^2 - (\hat{v}^\ell_{\xi_0} + w^{p,\ell}_{\xi_0})\overline{w^{p,\ell}_{\xi_0}}$ we get

$$(T^p_{\xi_0}\hat{v}^\ell_{\xi_0}, \hat{v}^\ell_{\xi_0})_{L^2(D^p_0)} = \int_{D^p_0} k^2(1-n_p)|\hat{v}^\ell_{\xi_0} + w^{p,\ell}_{\xi_0}|^2 dy + \int_{\Omega^R_0}(|\nabla w^{p,\ell}_{\xi_0}|^2 - k^2|w^{p,\ell}_{\xi_0}|^2)dy - \left\langle T^R_{\xi_0} w^{p,\ell}_{\xi_0}, w^{p,\ell}_{\xi_0}\right\rangle_{\Gamma^R_0}.$$

Taking the imaginary part we obtain

$$\Im m(T^p_{\xi_0}\hat{v}^\ell_{\xi_0}, \hat{v}^\ell_{\xi_0})_{L^2(D^p_0)} = -\int_{D^p_0}\Im m(n_p)|\hat{v}^\ell_{\xi_0} + w^{p,\ell}_{\xi_0}|^2 dy - \Im m\left\langle T^R_{\xi_0} w^{p,\ell}_{\xi_0}, w^{p,\ell}_{\xi_0}\right\rangle_{\Gamma^R_0}.$$

From (59) we have

$$\left|\Im m(T^p_{\xi_0}\hat{v}^\ell_{\xi_0}, \hat{v}^\ell_{\xi_0})_{L^2(D^p_0)}\right| \underset{\ell\to\infty}{\longrightarrow} 0.$$

Therefore, using the fact that $\Im m\left\langle T^R_{\xi_0} w^{p,\ell}_{\xi_0}, w^{p,\ell}_{\xi_0}\right\rangle_{\Gamma^R_0} \geqslant 0$,

$$\int_{D^p_0}\Im m(n_p)|\hat{v}^\ell_{\xi_0} + w^{p,\ell}_{\xi_0}|^2 dy \underset{\ell\to\infty}{\longrightarrow} 0 \geqslant \int_{D^p_0}\Im m(n_p)|\hat{v}_{\xi_0} + w^p_{\xi_0}|^2 dy.$$

Since $\Im m(n_p) > 0$ in $\mathcal{O}$ we obtain that $u^p_{\xi_0} := \hat{v}_{\xi_0} + w^p_{\xi_0} = 0$ in $\mathcal{O}$. Observing that $\Delta u^p_{\xi_0} + k^2 n_p u^p_{\xi_0} = 0$ in D^p_0, by unique continuation principle we deduce that $u^p_{\xi_0} = 0$ in D^p_0. Therefore, $w^p_{\xi_0}$ satisfies (23) with $n_p = 1$ and $\hat{v}_{\xi_0} = 0$, which implies that $w^p_{\xi_0}$ vanishes in Ω^R_0. Moreover, since $w^{p,\ell}_{\xi_0}$ converges strongly to $w^p_{\xi_0}$ in $L^2(D^p_0)$, then we have

$$\int_{D^p_0} k^2(1-n_p)w^{p,\ell}_{\xi_0}\overline{\hat{v}^\ell_{\xi_0}}dy \longrightarrow \int_{D^p_0} k^2(1-n_p)w^p_{\xi_0}\overline{\hat{v}_{\xi_0}}dy = 0.$$

On the other hand, we have that

$$\left|(T^p_{\xi_0}\hat{v}^\ell_{\xi_0},\hat{v}^\ell_{\xi_0})_{L^2(D^p_0)}\right| \geqslant k^2\left|\int_{D^p_0}(1-n_p)|\hat{v}^\ell_{\xi_0}|^2dy\right| - k^2\left|\int_{D^p_0}k^2(1-n_p)w^{p,\ell}_{\xi_0}\overline{\hat{v}^\ell_{\xi_0}}dy\right|.$$

From the hypothesis $\Re e(1-n_p)\geqslant\gamma>0$ in D^p we then have

$$\lim_{\ell\to\infty}\left|(T^p_{\xi_0}\hat{v}^\ell_{\xi_0},\hat{v}^\ell_{\xi_0})_{L^2(D^p_0)}\right| \geqslant \gamma\|\hat{v}_{\xi_0}\|^2_{L^2(D^p_0)}.$$

Therefore $\hat{v}_{\xi_0}|_{D^p_0}=0$.
Now, taking $\psi=\tilde{w}^\ell_{\xi_0}$ in the variational formulation (25) satisfied by $\tilde{w}^\ell_{\xi_0}$ we obtain

$$-k^2\int_{\tilde{D}}(n_p-n)(w^{p,\ell}_{\xi_0}+\hat{v}^\ell_{\xi_0}+\tilde{w}^\ell_{\xi_0})\overline{\tilde{w}^\ell_{\xi_0}}dy = k^2\int_{\Omega^R}(1-n_p)|\tilde{w}^\ell_{\xi_0}|^2dy+\int_{\Omega^R}(|\nabla\tilde{w}^\ell_{\xi_0}|^2-k^2|\tilde{w}^\ell_{\xi_0}|^2)dy - \left\langle T^R\tilde{w}^\ell_{\xi_0},\tilde{w}^\ell_{\xi_0}\right\rangle_{\Gamma^R}. \tag{60}$$

Moreover, taking $\psi=\tilde{w}^{p,\ell}_{\xi_0}$ in the variational formulation (34) satisfied by $\tilde{w}^{p,\ell}_{\xi_0}$ we get

$$-k^2\int_{\tilde{D}}(n_p-n)(w^{p,\ell}_{\xi_0}+\hat{v}^\ell_{\xi_0}+\tilde{w}^\ell_{\xi_0})\overline{\tilde{w}^{p,\ell}_{\xi_0}}dy = \int_{\Omega^R_0}(|\nabla\tilde{w}^{p,\ell}_{\xi_0}|^2-k^2|\tilde{w}^{p,\ell}_{\xi_0}|^2)dy+k^2\int_{D^p_0}(1-n_p)|\tilde{w}^{p,\ell}_{\xi_0}|^2dy - \left\langle T^R_{\xi_0}\tilde{w}^{p,\ell}_{\xi_0},\tilde{w}^{p,\ell}_{\xi_0}\right\rangle_{\Gamma^R_0}. \tag{61}$$

On the other hand, we have

$$(\tilde{T}^p_{\xi_0}\hat{v}^\ell_{\xi_0},\hat{v}^\ell_{\xi_0})_{L^2(D_0)} = \int_{\Omega^R_0}k^2(1-n_p)\tilde{w}^{p,\ell}_{\xi_0}\overline{\hat{v}^\ell_{\xi_0}}dy+\int_{\Omega^R_0}k^2(n_p-n)|w^{p,\ell}_{\xi_0}+\hat{v}^\ell_{\xi_0}+\tilde{w}^\ell_{\xi_0}|^2dy - \int_{\Omega^R}k^2(n_p-n)(w^{p,\ell}_{\xi_0}+\hat{v}^\ell_{\xi_0}+\tilde{w}^\ell_{\xi_0})\overline{\tilde{w}^{p,\ell}_{\xi_0}}dy - \int_{\Omega^R}k^2(n_p-n)(w^{p,\ell}_{\xi_0}+\hat{v}^\ell_{\xi_0}+\tilde{w}^\ell_{\xi_0})\overline{\tilde{w}^\ell_{\xi_0}}dy.$$

Then, using (60) and (61) we obtain by taking the imaginary part

$$\Im m(\tilde{T}^p_{\xi_0}\hat{v}^\ell_{\xi_0},\hat{v}^\ell_{\xi_0})_{L^2(D_0)} = \int_{\Omega^R_0}k^2\Im m(n_p-n)|w^{p,\ell}_{\xi_0}+\hat{v}^\ell_{\xi_0}+\tilde{w}^\ell_{\xi_0}|^2dy - k^2\int_{\Omega^R}\Im m(n_p)|\tilde{w}^\ell_{\xi_0}|^2dy + k^2\int_{\Omega^R_0}\Im m((1-n_p)\tilde{w}^{p,\ell}_{\xi_0}\overline{\hat{v}^\ell_{\xi_0}})dy - \int_{\Omega^R}k^2\Im m((n_p-n)(w^{p,\ell}_{\xi_0}+\hat{v}^\ell_{\xi_0}+\tilde{w}^\ell_{\xi_0})\overline{w^{p,\ell}_{\xi_0}})dy - \Im m\left\langle T^R\tilde{w}^\ell_{\xi_0},\tilde{w}^\ell_{\xi_0}\right\rangle_{\Gamma^R}. \tag{62}$$

On the other hand, the application $v_{\xi_0}\longrightarrow(1-n_p)\tilde{w}^p_{\xi_0}$ with $\tilde{w}^p_{\xi_0}$ is solution of (33) is compact from $L^2(D_0)$ into $L^2(D^p_0)$ using the compactness of the injection of $H^1_{\xi_0}(\Omega^R_0)$ into $L^2(D^p_0)$. Similarly, the application $v_{\xi_0}\longrightarrow(n_p-n)w^p_{\xi_0}$ with $w^p_{\xi_0}$ is solution of (23) is compact from $L^2(D_0)$ into $L^2(\tilde{D})$. Therefore we have, using that $\hat{v}_{\xi_0}|_{D^p_0}=0$ and $w^p_{\xi_0}=0$,

$$k^2\int_{\Omega^R_0}\Im m((1-n_p)\tilde{w}^{p,\ell}_{\xi_0}\overline{\hat{v}^\ell_{\xi_0}})dy\underset{\ell\to\infty}{\longrightarrow}0,$$

$$\int_{\Omega^R} k^2 \Im m((n_p - n)(w^{p,\ell}_{\xi_0} + \hat{v}^\ell_{\xi_0} + \tilde{w}^\ell_{\xi_0})\overline{w^{p,\ell}_{\xi_0}})dy \underset{\ell\to\infty}{\longrightarrow} 0. \tag{63}$$

From (59) we have

$$\left|\Im m(\tilde{T}^p_{\xi_0}\hat{v}^\ell_{\xi_0}, \hat{v}^\ell_{\xi_0})_{L^2(D_0)}\right| \underset{\ell\to\infty}{\longrightarrow} 0. \tag{64}$$

We observe that $\Im m \left\langle T^R \tilde{w}^\ell_{\xi_0}, \tilde{w}^\ell_{\xi_0}\right\rangle_{\Gamma^R} \geqslant 0$. Consequently, using (62) , (63) and (64) we get

$$\int_{\tilde{D}} \Im m(n - n_p)|w^{p,\ell}_{\xi_0} + \hat{v}^\ell_{\xi_0} + \tilde{w}^\ell_{\xi_0}|^2 dy + \int_{D^p} \Im m(n_p)|\tilde{w}^\ell_{\xi_0}|^2 dy \underset{\ell\to\infty}{\longrightarrow} 0.$$

In particular, by Assumption 1.1 we deduce that $\int_{\mathcal{O}} \Im m(n_p)|\tilde{w}^\ell_{\xi_0}|^2 dy \underset{\ell\to\infty}{\longrightarrow} 0$. This implies that $\tilde{w}_{\xi_0} = 0$ in $\mathcal{O}$. We remark that $w^p_{\xi_0} = 0$ implies in particular that $\Delta\tilde{w}_{\xi_0} + k^2 n\tilde{w}_{\xi_0} = 0$ in $\Omega^R\backslash\tilde{D}$. Consequently $\tilde{w}_{\xi_0} = 0$ in $\Omega^R\backslash\tilde{D}$ by unique continuation principle. This proves that the couple $(\tilde{w}_{\xi_0} + \hat{v}_{\xi_0}, \hat{v}_{\xi_0}) \in L^2(\tilde{D}) \times L^2(\tilde{D})$ is solution of **(ITP2)** with zero data. Hence $\tilde{w}_{\xi_0} = 0$ in $\tilde{D}$. On the other hand, the application $v_{\xi_0} \longrightarrow (n_p - n)\tilde{w}_{\xi_0}$ with $\tilde{w}_{\xi_0}$ is solution of (24) is compact from $H^{inc}_{\xi_0}(D)$ into $L^2(\tilde{D})$ thanks to the compactness of the injection of $H^1(\tilde{D})$ into $L^2(\tilde{D})$. Therefore

$$\int_{D^p_0} k^2(1 - n_p)\tilde{w}^{p,\ell}_{\xi_0}\overline{\hat{v}^\ell_{\xi_0}} + \int_{\tilde{D}} k^2(n_p - n)(w^{p,\ell}_{\xi_0} + \tilde{w}^\ell_{\xi_0})\overline{\hat{v}^\ell_{\xi_0}} dy \longrightarrow 0.$$

Moreover, we have

$$\begin{aligned}\left|(\tilde{T}^p_{\xi_0}\hat{v}^\ell_{\xi_0}, \hat{v}^\ell_{\xi_0})_{L^2(D_0)}\right| \geqslant \left|\int_{\tilde{D}} k^2(n_p - n)|\hat{v}^\ell_{\xi_0}|^2\right| - \Bigg|\int_{D^p_0} & k^2(1 - n_p)\tilde{w}^{p,\ell}_{\xi_0}\overline{\hat{v}^\ell_{\xi_0}} \\ & + \int_{\tilde{D}} k^2(n_p - n)(w^{p,\ell}_{\xi_0} + \tilde{w}^\ell_{\xi_0})\overline{\hat{v}^\ell_{\xi_0}} dy\Bigg|.\end{aligned}$$

Using the hypothesis $\Re e(n_p - n) \geqslant \gamma_1 > 0$ in $\tilde{D}$ we conclude

$$0 = \lim_{\ell\to\infty} \left|(\tilde{T}^p_{\xi_0}\hat{v}^\ell_{\xi_0}, \hat{v}^\ell_{\xi_0})\right| \geqslant \gamma_1 \|\hat{v}_{\xi_0}\|^2_{L^2(\tilde{D})},$$

which gives $\hat{v}_{\xi_0} = 0$ in $\tilde{D}$. Combined with the result above we have that $\hat{v}_{\xi_0} = 0$ in D_0 which contradicts with $\|\hat{v}_{\xi_0}\|_{L^2(D_0)} = 1$. □

2.3. Application of the Generalized Linear Sampling Method (GLSM)

We present the free noise version of the GLSM. For fixed $\xi_0 \in I$, introducing the functional $J^\alpha_{\xi_0} : L^2_{\xi_0}(\Gamma^R) \longrightarrow \mathbb{R}$ given as

$$J^\alpha_{\xi_0}(\phi; g_{\xi_0}) = \alpha I_{\xi_0}(g_{\xi_0}) + \|(N^p_{\xi_0} + \tilde{N}^p_{\xi_0})g_{\xi_0} - \phi\|^2,$$

We denote by $j^\alpha_{\xi_0}(\phi) = \inf_{g_{\xi_0}\in L^2_{\xi_0}(\Gamma^R)} J^\alpha_{\xi_0}(\phi; g_{\xi_0})$. Moreover, let $c(\alpha) > 0$ verifying $\frac{c(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$.

Theorem 2.9. *Assume that Assumptions 2.3 and 2.4 hold. Assume in addition that the hypothesis of Theorem 1.5 and Lemma 2.8 hold. Consider $z \in \Omega^R$, and let $g^\alpha_{\xi_0} \in L^2_{\xi_0}(\Gamma^R)$ such that*

$$J^\alpha_{\xi_0}(\Phi_{\xi_0}(\cdot, z), g^\alpha_{\xi_0}(z)) \leqslant j^\alpha_{\xi_0}(\Phi_{\xi_0}(\cdot, z)) + c(\alpha),$$

then

$$z \in D_0 \iff \lim_{\alpha \to 0} I_{\xi_0}(g^\alpha_{\xi_0}(z)) < \infty.$$

*Moreover, if $z \in D_0$ then $S_{\xi_0} g_{\xi_0}|_{\tilde{D}}$ converges to some $\tilde{v}$ in $L^2(\tilde{D})$ and $S_{\xi_0} g_{\xi_0}|_{D^p_0}$ converges to some v_{ξ_0} in $L^2(D^p_0)$ where $\tilde{v}$ is solution of **(ITP2)** and v_{ξ_0} is solution of **(ITP1)**.*

Proof. The proof of this theorem is an application of the abstract framework of GLSM given by Theorem 2.7 in [5] and the series of Lemmas (2.1)-(2.8). Lemma 2.1, Lemma 2.2 and Lemma 2.6 prove that the operator $N_{\xi_0} = N^p_{\xi_0} + \tilde{N}^p_{\xi_0}$ can be factorized as

$$N_{\xi_0} = G_{\xi_0} S_{\xi_0} = S^*_{\xi_0} T_{\xi_0} S_{\xi_0}, \tag{65}$$

and has dense range. Moreover, we need to verify that the norm $I_{\xi_0} g_{\xi_0}$ is an equivalent norm to $\|S_{\xi_0} g_{\xi_0}\|_{L^2(D_0)}$ for all $g_{\xi_0} \in L^2_{\xi_0}(\Gamma^R)$. Theorem 1.5, Theorem 1.8 and the expression of the operator T_{ξ_0} prove the existence of a constant $c_1 > 0$ (independent from ξ_0) such that

$$I_{\xi_0} g_{\xi_0} \leqslant c_1 \left(\|S_{\xi_0} g_{\xi_0}\|^2_{L^2(D^p_0)} + \|S_{\xi_0} g_{\xi_0}\|^2_{L^2(\tilde{D})} \right) \quad \forall\, g_{\xi_0} \in L^2_{\xi_0}(\Gamma^R). \tag{66}$$

Therefore, Lemma 2.8 and (66) prove this norm equivalence. The results of the theorem are then a straightforward application of Theorem 2.7 in [5] and Lemma 2.7. □

3. Inverse problem for non-periodic incident fields

3.1. Setting of the inverse problem

Let $y \in \Gamma^R$. One can deduce from (93)-(97) that $\Phi(\cdot, y) \in \tilde{L}^2(\Omega^R)$ (see also [7, 25]). We then define $u^s(\cdot, y) \in \tilde{H}^1_{loc}(\Omega^R)$ the scattered field solution of $(\mathcal{P})$ with $v(\cdot, y) = \overline{\Phi(\cdot, y)}$. We introduce the near field operator $N : \tilde{L}^2(\Gamma^R) \to \tilde{L}^2(\Gamma^R)$ as

$$Ng(x) = \int_{\Gamma^R} u^s(x, y) g(y) ds(y).$$

Define $S : \tilde{L}^2(\Gamma^R) \longrightarrow \tilde{L}^2(D)$ as

$$Sg(x) = \int_{\Gamma^R} \overline{\Phi(x, y)} g(y) ds(y).$$

Then, the operator N can be decomposed as

$$N = GS,$$

where $G : \tilde{L}^2(D) \longrightarrow \tilde{L}^2(\Gamma^R)$ is the operator defined by

$$G(v) = w|_{\Gamma^R},$$

with w being the solution of $(\mathcal{P})$.

Link between N and ξ-quasi periodic solutions: For $\xi \in I$, we denote by $g_\xi := (\mathcal{J}g)(\xi, \cdot)$ and we observe that

$$Sg_\xi = S_\xi g_\xi(x), \tag{67}$$

with S_ξ being the operator given by (30), in fact

$$\begin{aligned} Sg_\xi(x) &= \int_{\mathbb{R}} \overline{\Phi(x,(y_1,R))} g_\xi(y_1,R)ds(y_1) = \sum_{l\in\mathbb{Z}} \int_{2\pi l}^{2\pi(l+1)} \overline{\Phi(x,(y_1,R))} g_\xi(y_1,R)ds(y_1), \\ &= \sum_{l\in\mathbb{Z}} \int_0^{2\pi} \overline{\Phi(x,(y_1+2\pi l,R))} g_\xi(y_1+2\pi l)ds(y_1) = \int_{\Gamma_0^R} \overline{\Phi_\xi(y,x)} g_\xi(y)ds(y), \\ &= S_\xi g_\xi(x), \end{aligned}$$

therefore

$$\int_I S_\xi g_\xi(x)d\xi = \int_{\Gamma^R} \overline{\Phi(x,y)} \int_I g_\xi(y)d\xi = Sg(x). \tag{68}$$

Let $\tilde{N}_\xi : L^2_\xi(\Gamma^R) \longrightarrow L^2_{loc}(\Gamma^R)$ be the operator defined by

$$\tilde{N}_\xi g_\xi(x) := \int_{\Gamma_0^R} g_\xi(y)u_\xi^{s,p}(x,y)ds(y) + \int_{\Gamma_0^R} g_\xi(y)\tilde{u}_\xi^s(x,y)ds(y),$$

with $u_\xi^{s,p}$ and $\tilde{u}_\xi^s$ are respectively the solutions of (23) and (24) with $v_\xi = \overline{\Phi_\xi(y,\cdot)}$. Then

$$\begin{aligned} \int_I (\tilde{N}_\xi g_\xi)(x)d\xi &= \int_I \int_{\Gamma_0^R} g_\xi(y)u_\xi^{s,p}(x,y)ds(y) + \int_I \int_{\Gamma_0^R} g_\xi(y)\tilde{u}_\xi^s(x,y)ds(y) \\ &= \int_I w_\xi^p(x)d\xi + \int_I \tilde{w}_\xi(x)d\xi, \end{aligned} \tag{69}$$

with w_ξ^p and $\tilde{w}_\xi$ are respectively solutions of (23) and (24) with $v_\xi = S_\xi g_\xi$. We denote by

$$w^p(x) := \int_I w_\xi^p(x)d\xi \quad \text{and} \quad \tilde{w}(x) := \int_I \tilde{w}_\xi(x)d\xi. \tag{70}$$

Using (68) we observe that w^p satisfies $(\mathcal{P})$ with $n = n_p$ and $v = Sg$. Moreover, $\tilde{w}$ satisfies

$$\begin{cases} \Delta \tilde{w} + k^2 n_p \tilde{w} = k^2(n_p - n)(w^p + v + \tilde{w}) & \text{in } \Omega^R, \\ \tilde{w} = 0 & \text{on } \Gamma^0, \\ \frac{\partial \tilde{w}}{\partial x_2}(\cdot, R) = T^R(\tilde{w}|_{\Gamma^R}) & \text{on } \Gamma^R, \end{cases} \tag{71}$$

Therefore, $w := w^p + \tilde{w}$ satisfies $(\mathcal{P})$ with $v = Sg$. Hence, we can equivalently define the operator $N : \tilde{L}^2(\Gamma^R) \longrightarrow \tilde{L}^2(\Gamma^R)$ as

$$Ng(x) = \int_I (\tilde{N}_\xi g_\xi)(x) d\xi. \tag{72}$$

We finally observe that $N_\xi g_\xi$ is not equal in general to $\mathcal{J}(Ng)(\xi)$ since the latter corresponds with the scattered field $w_\xi \in H^1_\xi(\Omega^R_0)$ satisfying

$$\Delta w_\xi + k^2 n_p w_\xi = k^2(1 - n_p)v_\xi + k^2(n_p - n)(w + v) \quad \text{in } \Omega^R_0,$$

where w satisfies $(\mathcal{P})$ with $v = Sg$ and $v_\xi = S_\xi g_\xi$. This equation is different from (53) that corresponds with the scattered field associated with $N_\xi g_\xi$. The main difference comes from the term $k^2(n_p - n)v$ in the right hand side.

3.2. Some properties of the operator S

The goal of this section is to prove that the operator $S : \tilde{L}^2(\Gamma^R) \longrightarrow \tilde{L}^2(D)$ is injective and characterize its range. The main difficulty here comes from the required continuity with respect to the Floquet-Bloch variable. This is why we first prove the uniform continuity of $\xi \longrightarrow S_\xi$ formalized in the technical Lemma 5.1 given in the Appendix.
Let $\chi \in C^0(I)$ and $\psi \in L^2_\sharp(\Gamma^R)$. We consider $g \in \tilde{L}^2(\Gamma^R)$ such that

$$g_\xi(x) := \mathcal{J}(g)(\xi, x) = e^{i\xi \cdot x_1}\psi(x)\chi(\xi), \text{ for } (\xi, x) \in I \times \Gamma^R_0. \tag{73}$$

Denoting

$$\widehat{\psi}_j = \frac{1}{2\pi}\int_0^{2\pi} e^{-ijy_1}\psi(y_1)dy_1,$$

we observe that the operator S_ξ given by (30) verifies for $x_2 < R$

$$\begin{aligned} S_\xi g_\xi(x) &= \int_0^{2\pi} \overline{\frac{i}{4\pi}\sum_{j\in\mathbb{Z}} e^{i\alpha_\xi(j)(y_1 - x_1)}\theta_\xi(j, R, x_2)} g_\xi(y_1)dy_1, \\ &= \overline{\frac{i}{4\pi}\sum_{j\in\mathbb{Z}} e^{-i\alpha_\xi(j)x_1}\theta_\xi(j, R, x_2)} \int_0^{2\pi} e^{-i\alpha_\xi(j)y_1} g_\xi(y_1)dy_1. \end{aligned}$$

Therefore,

$$S_\xi g_\xi(x) = -\frac{i}{2}\sum_{j\in\mathbb{Z}}\widehat{\psi}_j\chi(\xi)\overline{\theta_\xi(j,R,x_2)}e^{i\alpha_\xi(j)x_1}. \tag{74}$$

Let $0 < R_0 < R$, we define $\tilde{S}_\xi : L^2_\sharp(\Gamma^R) \longrightarrow L^2_\sharp(\Omega^{R_0})$ as

$$\tilde{S}_\xi\psi := e^{-i\xi\cdot x_1}S_\xi(e^{i\xi\cdot x_1}\psi). \tag{75}$$

Using (74) we have

$$\tilde{S}_\xi\psi(x) = -\frac{i}{2}\sum_{j\in\mathbb{Z}}\widehat{\psi}_j\overline{\theta_\xi(j,R,x_2)}e^{ijx_1}. \tag{76}$$

With help of Lemma 5.1 in the Appendix we prove the following.

Lemma 3.1. *The operator $S : \tilde{L}^2(\Gamma^R) \longrightarrow \tilde{L}^2(D)$ is injective. The closure of its range is*

$$H^{inc}(D) := \left\{v \in L^2(D);\ v|_{D^p} \in \tilde{L}^2(D^p);\ \ \Delta v + k^2 v = 0 \text{ in } D\right\}.$$

Proof. Let $g \in \tilde{L}^2(\Gamma^R)$ such that $Sg = 0$ in D. Using the unique continuation principle we obtain $Sg = 0$ in Ω^R. Using the continuity and regularity of single layer potentials we have that $S \in H^2_{loc}(U^R)$ and verifies

$$\begin{cases} \Delta Sg + k^2 Sg = 0 & \text{in } U^R, \\ Sg = 0 & \text{on } \Gamma^R, \end{cases}$$

and the upper going radiation condition (2) with Γ^R replaced by $\Gamma^{R'}$ with $R' > R$. Therefore $Sg = 0$ in U^R. Using the jump relations for the normal derivative of S we obtain $g = 0$ which proves the injectivety of S.

We prove now the denseness of the range of S. Let $v \in \tilde{L}^2(D)$ and we denote by $v_\xi := (\mathcal{J}v)(\xi,\cdot)$. Fix $\epsilon > 0$ and consider a uniform partition of I into sub-domains $I^N_j := \overset{N}{\underset{j=1}{\cup}} I_j$ of size $\delta = \frac{1}{N}$. Using Lemma 2.2 we have, for all $\xi^N_j \in I^N_j$ there exists $\tilde{\psi^N_j} := e^{-i\xi^N_j\cdot x}\psi_{\xi^N_j} \in L^2_\sharp(\Gamma^R)$ such that

$$\|\tilde{S}_{\xi^N_j}\tilde{\psi}_{\xi^N_j} - \tilde{v}_{\xi^N_j}\|_{L^2(\Omega^{R_0}_0)} \leqslant \frac{\epsilon}{4}, \tag{77}$$

with $\tilde{v}_{\xi^N_j} = e^{-i\xi^N_j\cdot x}v_{\xi^N_j}$, where $\tilde{S}_\xi$ is defined by (75). We introduce the hat functions $\chi^N_j \in C^0(I)$ that are affine on each domain I_l and verifies $\chi^N_j(\xi_l) = \delta_{jl}$. We then define

$$\tilde{\psi}^N_\xi := \sum_{1\leqslant j\leqslant N}\tilde{\psi}_{\xi^N_j}\chi^N_j(\xi), \quad \hat{v}^N_\xi := \sum_{1\leqslant j\leqslant N}\tilde{v}_{\xi^N_j}\chi^N_j(\xi), \quad \hat{S}^N_\xi := \sum_{1\leqslant j\leqslant N}(\tilde{S}_{\xi^N_j}\tilde{\psi}_{\xi^N_j})\chi^N_j(\xi), \text{ for } 1 \leqslant j \leqslant N.$$

Then, we have

$$\|\tilde{S}_\xi\tilde{\psi}^N_\xi - \tilde{v}_\xi\|_{L^2(\Omega^{R_0}_0))} \leqslant \|\tilde{S}_\xi\tilde{\psi}^N_\xi - \hat{S}^N_\xi\|_{L^2(\Omega^{R_0}_0))} + \|\hat{S}^N_\xi - \hat{v}^N_\xi\|_{L^2(\Omega^{R_0}_0))} + \|\hat{v}^N_\xi - \tilde{v}_\xi\|_{L^2(\Omega^{R_0}_0))}. \tag{78}$$

Since $\tilde{S}_\xi(\tilde{\psi}_{\xi_j^N}\chi_j^N(\xi)) = (\tilde{S}_\xi\tilde{\psi}_{\xi_j^N})\chi_j^N(\xi)$, then the first term in the right hand side of (78) verifies

$$\tilde{S}_\xi\tilde{\psi}_{\xi_j^N} - \hat{S}_\xi^N = \sum_{1\leqslant j\leqslant N}(\tilde{S}_\xi\tilde{\psi}_{\xi_j^N} - \tilde{S}_{\xi_j^N}\tilde{\psi}_{\xi_j^N})\chi_j^N(\xi).$$

Therefore

$$\|\tilde{S}_\xi\tilde{\psi}_\xi^N - \hat{S}_\xi^N\|_{L^2(\Omega_0^{R_0}))} \leqslant \sup_{1\leqslant j\leqslant N}\ \sup_{\xi\in[\xi_{j-1}^N,\xi_{j+1}^N]}\|\tilde{S}_\xi\tilde{\psi}_{\xi_j^N} - \tilde{S}_{\xi_j^N}\tilde{\psi}_{\xi_j^N}\|_{L^2(\Omega_0^{R_0}))}.$$

Consider $\epsilon' := \frac{\epsilon}{4\sqrt{c}\left(\sup\limits_{\xi\in I}\|v_\xi\|_{L^2(\Omega_0^{R_0})}+1\right)}$, using Lemma 5.1 we chose $\delta > 0$ for which

$$\left\|\tilde{S}_\xi\tilde{\psi}_{\xi_j^N} - \tilde{S}_{\xi_j^N}\tilde{\psi}_{\xi_j^N}\right\|_{L^2(\Omega_0^{R_0})} \leqslant \sqrt{c}\epsilon'\|\tilde{S}_{\xi_j^N}\tilde{\psi}_{\xi_j^N}\|_{L^2(\Omega_0^{R_0})},$$

therefore

$$\sup_{1\leqslant j\leqslant N}\ \sup_{\xi\in[\xi_{j-1}^N,\xi_{j+1}^N]}\left\|\tilde{S}_\xi\tilde{\psi}_\xi^N - \tilde{S}_{\xi_j^N}\psi_{\xi_j^N}\right\|_{L^2_\sharp(\Omega_0^{R_0}))} \leqslant \sqrt{c}\epsilon'\left(\sup_{\xi\in I}\|v_\xi\|_{L^2(\Omega_0^{R_0})}+1\right) = \frac{\epsilon}{4}. \tag{79}$$

On the other hand, using (77) we deduce that the second term in the right hand side of (78) verifies

$$\|\hat{S}_\xi^N - \hat{v}_\xi^N\|_{L^2(\Omega_0^{R_0})} = \left\|\sum_{1\leqslant j\leqslant N}(\tilde{S}_{\xi_j^N}\tilde{\psi}_{\xi_j^N} - \tilde{v}_{\xi_j^N})\chi_j^N(\xi)\right\|_{L^2(\Omega_0^{R_0})} \leqslant \frac{\epsilon}{4}. \tag{80}$$

Moreover, since $v_\xi \in C^0_\sharp(I, L^2_\xi(D))$ then N could have been chosen from the beginning sufficiently large so that

$$\|\hat{v}_\xi^N - \tilde{v}_\xi\|_{L^2(\Omega_0^{R_0})} \leqslant \frac{\epsilon}{2}. \tag{81}$$

Finally, using (79), (80) and (81) we get

$$\sup_{\xi\in I}\|\tilde{S}_\xi\tilde{\psi}_\xi^N - \tilde{v}_\xi\|_{L^2(\Omega_0^{R_0})} = \sup_{1\leqslant j\leqslant N}\ \sup_{\xi\in[\xi_{j-1}^N,\xi_{j+1}^N]}\|\tilde{S}_\xi\tilde{\psi}_\xi^N - \tilde{v}_\xi\|_{L^2(\Omega_0^{R_0})} \leqslant \epsilon, \tag{82}$$

for sufficiently large N. This proves the denseness of the range of the operator S. □

For the analysis below we need to assume the well posedness of the following Interior Transmission Problem.

(**ITP3**): Seek $(u, v) \in \tilde{L}^2(D) \times \tilde{L}^2(D)$ such that $(u - v) \in \tilde{H}^2(D)$ satisfying

$$\begin{cases} \Delta u + k^2 n u = 0 \quad \text{in } D \\ \Delta v + k^2 v = 0 \quad \text{in } D \\ u - v = \varphi \quad \text{on } \partial D \\ \frac{\partial(u-v)}{\partial\nu} = \psi \quad \text{on } \partial D \end{cases} \tag{83}$$

for given $(\varphi, \psi) \in \tilde{H}^{3/2}(\partial D) \times \tilde{H}^{1/2}(\partial D)$. This problem has been extensively studied in the literature in the case of bounded domains D see for instance [5]. Indeed the results for bounded domain D extend easily to the case where D is unbounded but is the (infinite) union of disjoint bounded domains. This corresponds for instance to our case when D^p is the union of disjoint bounded domains. Consider the following assumption

Assumption 3.2. *Assume that k, n_p and n are such as $(\mathbf{ITP3})$ is well posed.*

Lemma 3.3. *Assume that Assumptions 1.1 and 3.2 hold. Then the operator G given by (67) is injective with dense range. Moreover,*

$$(z \in D) \quad \Longleftrightarrow \quad (\Phi(\cdot, z) \in \mathcal{R}ange(G))\,.$$

Proof. consider $v \in H^{inc}(D)$ such that $G(v) = 0$, i.e

$$w = 0 \quad \text{on } \Gamma^R,$$

with $w \in H^2_{loc}(\Omega^R)$ being the solution of $(\mathcal{P})$. Therefore, $w \in H^2_{loc}(U^R)$ and verifies

$$\begin{cases} \Delta w + k^2 w = 0 & \text{in } U^R, \\ w = 0 & \text{on } \Gamma^R, \end{cases}$$

and the upper going radiation condition (2) with Γ^R replaced by $\Gamma^{R'}$ with $R' > R$. Then $w = 0$ in U^R. Using the unique continuation principle we obtain that $w = 0$ in $\Omega^R \backslash D$. Setting $u := w + v$ we observe that the couple (u, v) verifies $(\mathbf{ITP3})$ with zero data. Therefore we deduce that $v = 0$ and then the injectivety of G.

Now we prove the denseness of the range of G. Let $g \in \overline{\mathcal{R}(G)}^{\perp}$, then

$$(G(v), g)_{L^2(D)} = 0 \quad \text{for all } v \in H^{inc}(D).$$

Let $f \in \tilde{L}^2(\Gamma^R)$ and consider $v = \overline{Sf}$. We then have

$$(G(\overline{Sf}), g)_{L^2(\Gamma^R)} = 0 \quad \text{for all } f \in \tilde{L}^2(\Gamma^R).$$

On the other hand, consider $w(f)$ and $w(g)$ solution of $(\mathcal{P})$ associated respectively to $v = \overline{Sf}$ and $v = \overline{Sg}$. Using similar arguments as in the proof of Lemma 2.5 one can prove that

$$\int_D (1 - n) w(f) \overline{Sg} dy = \int_D (1 - n) w(g) \overline{Sf} dx. \tag{84}$$

Therefore, from Theorem 1.4 and (84) we get

$$(G(\overline{Sf}), g)_{L^2(\Gamma^R)} = \int_D k^2 (1 - n)(w(f) + \overline{Sf}) \overline{Sg} dy = \int_D k^2 (1 - n)(w(g) + \overline{Sg}) \overline{Sf} dy$$

$$= (G(\overline{Sg}), f)_{L^2(\Gamma^R)}.$$

Consequently

$$(G(\overline{Sg}), f)_{L^2(\Gamma^R)} = 0 \quad \text{for all } f \in \tilde{L}^2(\Gamma^R)$$

which implies that $G(\overline{Sg}) = 0$. The injectivety of G gives that $Sg = 0$ and then $g = 0$ by Lemma 3.1.

Consider $z \in D$. We have that $\chi\Phi(\cdot, z) \in \tilde{L}^2(D)$, where χ is a regular cutoff function that vanishes in a neighborhood of z. Since $\Phi(\cdot, z)$ satisfies the Helmholtz equation outside z, elliptic regularity results applied to each component of D^p separately implies that $\chi\Phi \in \tilde{H}^2(D)$. Trace theorems then imply $(\Phi(\cdot, z), \frac{\partial\Phi(\cdot,z)}{\partial\nu}) \in \tilde{H}^{3/2}(\partial D) \times \tilde{H}^{1/2}(\partial D)$. We then consider $(u, v) \in \tilde{L}^2(D) \times \tilde{L}^2(D)$ to be the solution of **(ITP3)** with $(\varphi, \psi) = (\Phi(\cdot, z), \frac{\partial\Phi(\cdot,z)}{\partial\nu})$. We set

$$w := \begin{cases} u - v & \text{in } D \\ \Phi(\cdot, z) & \text{in } \Omega^R \backslash D. \end{cases}$$

We observe that $w \in H^2_{loc}(\Omega^R)$ and satisfies $(\mathcal{P})$. Hence $G(v) = \Phi(\cdot, z)$.

Consider now the case where $z \in \Omega^R \backslash D$. Assume that there exists $v \in H^{inc}(D)$ such that $G(v) = \Phi(\cdot, z)$. By unique continuation principle we obtain that $w = \Phi(\cdot, z)$ in $\Omega^R \backslash D$, which is a contradiction since $w \in H^2_{loc}(\Omega^R \backslash D)$ while $\Phi(\cdot, z) \notin H^2_{loc}(\Omega^R \backslash D)$. □

3.3. Application of the Generalized Linear Sampling Method (GLSM)

Let us consider the functional $J_\alpha(\phi, \cdot) : \tilde{L}^2(\Gamma^R) \longrightarrow \mathbb{R}$

$$J_\alpha(\phi; g) := \alpha \check{I}(g) + \|Ng - \phi\|^2, \quad \text{for all } g \in \tilde{L}^2(\Gamma^R)$$

where

$$\check{I}(g) := \sup_{\xi_0 \in I} I_{\xi_0}(\mathcal{J}g(\xi_0, \cdot)). \tag{85}$$

We denote by $j_\alpha(\phi) = \inf\limits_{g \in \tilde{L}^2(\Gamma^R)} J_\alpha(\phi; g)$. Let $c(\alpha) > 0$ verifying $\frac{c(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$

Theorem 3.4. *Assume that Assumptions 1.1 and 3.2 hold. Assume in addition that the hypothesis of Lemma 2.8 holds. Consider $z \in \Omega^R$, and let $g^\alpha \in \tilde{L}^2(\Gamma^R)$ such that*

$$J_\alpha(\Phi(\cdot, z), g^\alpha(z)) \leqslant j_\alpha(\Phi(\cdot, z)) + c(\alpha),$$

then

$$z \in D \iff \lim_{\alpha \to 0} \check{I}(g^\alpha(z)) < \infty. \tag{86}$$

Moreover, if $z \in D$ then $Sg|_D$ converges to some v in $L^2(D)$ where v is solution of ***(ITP3)***.

Proof. The proof of this theorem is an application of the abstract framework of GLSM given by Theorem 2.7 in [5] and Lemma 3.1, Lemma 3.3 and Lemma 2.8. Lemma 3.1 and Lemma 3.3 prove that the operator

$$N = GS,$$

has dense range. Moreover, we need to verify that the norm $\check{I}g$ is an equivalent norm to $\sup_{\xi \in I} \|S\mathcal{J}g(\xi,\cdot)\|_{L^2(D_0)}$ for all $g \in \tilde{L}^2(\Gamma^R)$. Lemma 2.8 and Theorem 2.9 prove this norm equivalence. The results of the theorem are then a straightforward application of Theorem 2.7 in [5] and Lemma 3.3. □

4. Application to differential imaging

The theoretical developments of the previous sections allow us to provide a theoretical justification of the algorithm proposed in [13] that provides an indicator function for the defect $\tilde{D}$ independently from D^p. This justification does not assume $\tilde{D}$ is also periodic (with a larger periodicity) which was the case in [13]. The principle idea behind this method is to consider the background as $2\pi M$ periodic with $M \in \mathbb{N}$ such that $M \geqslant 1$ and combine the application of the previous framework to different values of M. Indeed, the refractive index n_p is also $2\pi M$-periodic with respect to the first component x_1. Then, we can follow the same approach adopted in Section 2 by taking

$$\Omega_0^R = \Omega_0^{R,M} := [0, 2\pi M] \times [0, R] \quad \text{and} \quad D_0^p = D_0^{p,M} := \Omega_0^{R,M} \cap D^p,$$

in order to reconstruct $D_0 = D_0^M := D_0^{p,M} \cup \tilde{D}$ using the GLSM method.

In the following, for $m \geqslant 0$, the spaces $H^m_{\xi,M}(\Omega^R)$, $H^m_{\xi,M}(\Omega_0^{R,M})$ and $L^2_{\xi,M}(D)$ has respectively the same definition as $H^m_\xi(\Omega^R)$, $H^m_\xi(\Omega_0^R)$ and $L^2_\xi(D)$ with period 2π replaced by $2\pi M$. Moreover, we define for $\phi \in C_0^\infty(U^0)$ the one dimensional Floquet-Bloch transform with period $2\pi M$ as

$$\mathcal{J}_M\phi(\xi, x_1, x_2) = \sum_{j\in\mathbb{Z}} \phi(x_1 + 2\pi M j, x_2) e^{-i2\pi M\xi\cdot j}, \quad \xi \in I^M := [0, \frac{1}{M}],\ (x_1, x_2) \in U^0.$$

Fix $\xi \in I$ and we denote by $\xi_0 := \frac{\xi}{M} \in I^M$. We consider $\Phi_{\xi_0,M}(x,y) := (\mathcal{J}_M\Phi(\cdot,y))(\xi_0, x)$ the $\xi_0 M$-quasi-periodic Green function with period $2\pi M$. Similarly to (22), we define $u^{s,M}_{\xi_0} = w_{\xi_0}$ decomposed as

$$u^{s,M}_{\xi_0}(\cdot, y) = u^{s,p}_{\xi_0}(\cdot, y) + \tilde{u}^s_{\xi_0}(\cdot, y),$$

where $u^{s,p}_{\xi_0}(\cdot, y) \in H^1_{\xi_0,M}(\Omega_0^{R,M})$ solution of (23) with Γ_0^R is replaced by $\Gamma_0^{R,M} := [0, 2\pi M] \times \{R\}$ and $\tilde{u}^s_{\xi_0}(\cdot, y) \in \tilde{H}^1(\Omega^R)$ solution of (24). We introduce the $\xi_0 M$-quasi periodic near field operator $N^M_{\xi_0} : L^2_{\xi_0,M}(\Gamma^R) \longrightarrow L^2_{\xi_0,M}(\Gamma^R)$ given as

$$N^M_{\xi_0} = N^{p,M}_{\xi_0} + \tilde{N}^{p,M}_{\xi_0},$$

with $N^{p,M}_{\xi_0} : L^2_{\xi_0,M}(\Gamma^R) \longrightarrow L^2_{\xi_0,M}(\Gamma^R)$ and $\tilde{N}^{p,M}_{\xi_0} : L^2_{\xi_0,M}(\Gamma^R) \longrightarrow L^2_{\xi_0,M}(\Gamma^R)$ are given as (35) with $\Gamma^R_0 = \Gamma^{R,M}_0$ and $\mathcal{J}$ is replaced by $\mathcal{J}_M$. Define $S^M_{\xi_0} : L^2_{\xi_0,M}(\Gamma^R) \longrightarrow L^2_{\xi_0,M}(D)$ the operator given as (30) with $\Gamma^R_0 = \Gamma^{R,M}_0$ and $\overline{\Phi_{\xi_0}(y,x)} = \overline{\Phi_{\xi_0,M}(y,x)}$. Then, as in Section 2 the operator $N^M_{\xi_0}$ is decomposed as

$$N^M_{\xi_0} = G^M_{\xi_0}(S^M_{\xi_0}),$$

where $G^M_{\xi_0} : L^2_{\xi_0,M}(D) \longrightarrow L^2_{\xi_0,M}(\Gamma^R)$ is the operator defined by

$$G^M_{\xi_0}(v_{\xi_0}) = (w^p_{\xi_0} + \tilde{w}^p_{\xi_0})|_{\Gamma^{R,M}_0},$$

with $w^p_{\xi_0} \in H^1_{\xi_0,M}(\Omega^{R,M}_0)$ being the solution of (23) and $\tilde{w}^p_{\xi_0} = \mathcal{J}_M(\tilde{w}_{\xi_0})(\xi_0,\cdot)$ with $\tilde{w}_{\xi_0} \in \tilde{H}^1(\Omega^R)$ is the solution of (24). Moreover, we denote by $I^M_{\xi_0}$ the norm given as (57) with $N^p_{\xi_0} = N^{p,M}_{\xi_0}$, $\tilde{N}^p_{\xi_0} = \tilde{N}^{p,M}_{\xi_0}$ and $\Gamma^R_0 = \Gamma^{R,M}_0$.

4.1. Application of the GLSM for the reconstruction of D^M_0.

Following the same steps as in Section 2 we can present the free noise version of the GLSM. Introducing the functional $J^{\alpha,M}_{\xi_0} : L^2_{\xi_0,M}(\Gamma^R) \longrightarrow \mathbb{R}$ given as

$$J^{\alpha,M}_{\xi_0}(\phi; g_{\xi_0}) = \alpha I^M_{\xi_0}(g_{\xi_0}) + \|(N^{p,M}_{\xi_0} + \tilde{N}^{p,M}_{\xi_0})g_{\xi_0} - \phi\|^2.$$

We denote by $j^{\alpha,M}_{\xi_0}(\phi) = \inf_{g_{\xi_0} \in L^2_{\xi_0,M}(D)} J^{\alpha,M}_{\xi_0}(\phi; g_{\xi_0})$. Moreover, let $c(\alpha) > 0$ verifying $\frac{c(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$.

Theorem 4.1. *Assume that Assumptions 2.3 and 2.4 hold. Assume in addition that the hypothesis of Theorem 1.5 and Lemma 2.8 hold. Consider $z \in \Omega^R$, and let $g^\alpha_{\xi_0} \in L^2_{\xi_0,M}(\Gamma^R)$ such that*

$$J^{\alpha,M}_{\xi_0}(\Phi_{\xi_0,M}(\cdot,z), g^\alpha_{\xi_0}(z)) \leqslant j^{\alpha,M}_{\xi_0}(\Phi_{\xi_0,M}(\cdot,z)) + c(\alpha),$$

then

$$z \in D^M_0 \iff \lim_{\alpha\to 0} I^M_{\xi_0}(g^\alpha_{\xi_0}(z)) < \infty.$$

Proof. The prove is similar to the proof of Theorem 2.9. □

4.2. Application of the GLSM for the reconstruction of $D^{p,M}_0$

In this section we consider $M \geqslant 2$ and we explain how one can reconstruct only the domain $D^{p,M}_0$. Fix $\xi \in I$ and we denote by $\xi_0 := \frac{\xi}{M} \in I^M$. We observe that the Green function $\Phi_{\xi_0}(\cdot,z)$ is also $\xi_0 M$-quasi periodic with period $2\pi M$. Therefore, we can follow the same steps in the previous section by replacing $\Phi_{\xi_0,M}(x,y)$ by $\Phi_{\xi_0}(x,y)$ and we use that $\Phi_{\xi_0}(\cdot,z)$ admits singular points in $\Omega^{R,M}_0$ for $z \in \tilde{D}$ to reconstruct only the periodic domain $D^{p,M}_0$.

Lemma 4.2. *Assume that the assumptions of Theorem 1.5 hold and that Assumptions 2.3 holds. Then we have that*

$$(z \in D_0^{p,M}) \quad \Longleftrightarrow \quad \left(\Phi_{\xi_0}(\cdot, z) \in Range(G_{\xi_0}^M)\right).$$

Proof. Let $z \in D^p$. We consider $v_{\xi_0} \in H_{\xi_0}^{inc}(D)$ such that $v_{\xi_0}|_{\tilde{D}} = -\Phi_{\xi_0}(\cdot, z)$. Let $(u_{\xi_0}, v_{\xi_0}) \in L^2_{\xi_0}(D^p) \times L^2_{\xi_0}(D^p)$ be the solution of **(ITP1)** with $(\varphi, \psi) = \left(\Phi_{\xi_0}(\cdot, z), \frac{\partial \Phi_{\xi_0}(\cdot,z)}{\partial \nu}\right)$. We set

$$w_{\xi_0}^p := \begin{cases} u_{\xi_0} - v_{\xi_0} & \text{in } D^p, \\ \Phi_{\xi_0}(\cdot, z) & \text{in } \Omega^R \backslash D^p. \end{cases}$$

We observe that $v_{\xi_0} \in H_{\xi_0,M}^{inc}(D)$, where

$$H_{\xi_0,M}^{inc}(D) := \{v \in L^2_{\xi_0,M}(D),\ \Delta v + k^2 v = 0 \text{ in } D\},$$

and $w_{\xi_0}^p \in H^2_{\xi_0,M}(\Omega^R)$ satisfies (23). Moreover, let $\tilde{w}_{\xi_0}$ be the solution of (24). Since $(n_p - n)(w_{\xi_0}^p + v_{\xi_0}) = 0$, then $\tilde{w}_{\xi_0} = 0$ and consequently $G_{\xi_0}^M(v_{\xi_0}) = \Phi_{\xi_0}(\cdot, z)$.
Consider now the case where $z := (z_1, z_2) \notin D^p$. Assume that there exists $v_{\xi_0} \in H_{\xi_0,M}^{inc}(D)$ such that $G_{\xi_0}^M(v_{\xi_0}) = \Phi_{\xi_0}(\cdot, z)$. By the unique continuation principle we get $w_{\xi_0} := w_{\xi_0}^p + \tilde{w}_{\xi_0}^p = \Phi_{\xi_0}(\cdot, z)$ in $U^0 \backslash D_0^M$. Since $\tilde{D}$ is not distributed periodically, then for $z \in \tilde{D}$ there exists $j \in \mathbb{Z}$ such that $z_j := (z_1 + 2\pi j, z_2) \in \Omega_0^{R,M} \backslash D_0^M$. Therefore, $\Phi_{\xi_0}(\cdot, z) \notin H^2_{\xi_0,M}(\Omega^R \backslash D_0^M)$ for all $z \in \Omega^R \backslash D_0^{p,M}$ while $w_{\xi_0} \in H^2_{\xi_0,M}(\Omega^R \backslash D_0^M)$, which is a contradiction. □

Theorem 4.3. *Assume that Assumptions 2.3 and 2.4 hold. Assume in addition that the hypothesis of Theorem 1.5 and Lemma 2.8 hold. Consider $z \in \Omega^R$, and let $g_{\xi_0}^\alpha \in L^2_{\xi_0,M}(\Gamma^R)$ such that*

$$J_{\xi_0}^{\alpha,M}(\Phi_{\xi_0}(\cdot, z), g_{\xi_0}^\alpha(z)) \leqslant j_{\xi_0}^{\alpha,M}(\Phi_{\xi_0}(\cdot, z)) + c(\alpha),$$

then

$$z \in D_0^{p,M} \iff \lim_{\alpha \to 0} I_{\xi_0}^M(g_{\xi_0}^\alpha(z)) < \infty.$$

*Moreover, if $z \in D_0^{p,M}$ then $S_{\xi_0}^M g_{\xi_0}|_{D_0^{p,M}}$ converges to some v_{ξ_0} in $L^2(D_0^{p,M})$ where v_{ξ_0} is solution of **(ITP1)**.*

Proof. As in the proof of Theorem 2.9. By Lemma 2.1, Lemma 2.2 and Lemma 2.6 adopted to the $2\pi M$ periodic case we prove that the operator $N_{\xi_0}^M = N_{\xi_0}^{p,M} + \tilde{N}_{\xi_0}^{p,M}$ can be factorized as

$$N_{\xi_0}^M = G_{\xi_0}^M S_{\xi_0}^M = S_{\xi_0}^{*,M} T_{\xi_0}^M S_{\xi_0}^M, \tag{87}$$

and has dense range, with $S_{\xi_0}^{*,M} : L^2_{\xi_0,M}(D) \longrightarrow L^2_{\xi_0,M}(\Gamma^R)$ is the adjoint of the operator $S_{\xi_0}^M$ and $T_{\xi_0}^M$ is the operator defined as (39). Moreover, using Lemma 2.8 we prove that the norm $I_{\xi_0}^M g_{\xi_0}$ is an equivalent norm to $\|S_{\xi_0}^M g_{\xi_0}\|_{L^2(D_0^M)}$. Therefore, the results of the theorem are then a straightforward application of Theorem 2.7 in [5] and Lemma 4.2. □

4.3. Application of the Differential Sampling Method for the reconstruction of $\tilde{D}$

As in [13], we explain in this section how one directly reconstruct $\tilde{D}$ using a differential indicator function. Consider $M \geqslant 2$, fix $\xi \in I$. We denote by $\xi_0 := \frac{\xi}{M} \in I^M$. Consider $g^\alpha \in \tilde{L}^2(\Gamma^R)$, $g^\alpha_{\xi_0} \in L^2_{\xi_0}(\Gamma^R)$ and $g^{\alpha,M}_{\xi_0} \in L^2_{\xi_0,M}(\Gamma^R)$ satisfying

$$
\begin{aligned}
&J_\alpha(\Phi(\cdot, z), g^\alpha(z)) \leqslant j_\alpha(\Phi(\cdot, z)) + c(\alpha),\\
&J^\alpha_{\xi_0}(\Phi_{\xi_0}(\cdot, z), g^\alpha_{\xi_0}(z)) \leqslant j^\alpha_{\xi_0}(\Phi_{\xi_0}(\cdot, z)) + c(\alpha),\\
&J^{\alpha,M}_{\xi_0}(\Phi_{\xi_0}(\cdot, z), g^{\alpha,M}_{\xi_0}(z)) \leqslant j^{\alpha,M}_{\xi_0}(\Phi_{\xi_0}(\cdot, z)) + c(\alpha),
\end{aligned}
$$

for $c(\alpha) > 0$ verifying $\frac{c(\alpha)}{\alpha} \to 0$ as $\alpha \to 0$. Let us define the indicator function to identify $\tilde{D}$ as

$$
\mathcal{I}^\alpha(z) := \left[\check{I}(g^\alpha)\left(1 + \frac{\check{I}(g^\alpha)}{I^M_{\xi_0}(g^{\alpha,M}_{\xi_0} - \frac{1}{M} g^\alpha_{\xi_0})}\right)\right]^{-1}, \tag{88}
$$

with $\check{I}$ is the norm defined by (85) and $I^M_{\xi_0}$ is the norm given as (57) with $N^p_{\xi_0} = N^{p,M}_{\xi_0}$, $\tilde{N}^p_{\xi_0} = \tilde{N}^{p,M}_{\xi_0}$ and $\Gamma^R_0 = \Gamma^{R,M}_0$.

Theorem 4.4. *Under the assumptions of Theorem 4.3 we have*

$$
\left(z \in \tilde{D}\right) \quad \Longleftrightarrow \quad \left(\lim_{\alpha\to 0} \mathcal{I}_\alpha > 0\right).
$$

Proof. Consider $z \in D^p$. By Theorem 2.9 and Theorem 4.3 we have that $S_{\xi_0} g^\alpha_{\xi_0}$ and $S^M_{\xi_0} g^{\alpha,M}_{\xi_0}$ converges respectively to $v_{\xi_0} \in H^{inc}_{\xi_0}(D)$ and $v^M_{\xi_0} \in H^{inc}_{\xi_0,M}(D)$ verifying

$$
G_{\xi_0}(v_{\xi_0}) = \Phi_{\xi_0}(\cdot, z) \quad \text{and} \quad G^M_{\xi_0}(v^M_{\xi_0}) = \Phi_{\xi_0}(\cdot, z). \tag{89}
$$

Moreover, from Lemma 2.7 and Lemma 4.2 we observe that v_{ξ_0} and $v^M_{\xi_0}$ are solutions of **(ITP1)** with $(\varphi, \psi) = \left(\Phi_{\xi_0}(\cdot, z), \frac{\partial \Phi_{\xi_0}(\cdot, z)}{\partial \nu}\right)$, then v_{ξ_0} coincides with $v^M_{\xi_0}$. On the other hand, from (58), Lemma 2.8 and Theorem 1.5 we have that $I^M_{\xi_0} g_{\xi_0}$ is equivalent (uniformly with respect to ξ_0) to $\|S^M_{\xi_0} g_{\xi_0}\|^2_{L^2(D^M_0)}$. In particular, there exists of constant $c_1 > 0$ independent from ξ_0 such that

$$
I^M_{\xi_0}(g^{\alpha,M}_{\xi_0} - \frac{1}{M} g^\alpha_{\xi_0}) \leqslant c_1 \left\| S^M_{\xi_0} g^{\alpha,M}_{\xi_0} - \frac{1}{M} S^M_{\xi_0} g^\alpha_{\xi_0} \right\|^2_{L^2(D^M_0)}. \tag{90}
$$

Moreover, we observe that $S^M_{\xi_0} g^\alpha_{\xi_0} = M S_{\xi_0} g^\alpha_{\xi_0}$. Therefore, the right hand side of (90) tends to zero as $\alpha \to 0$. On the other hand, from Theorem 3.4 we have that $\check{I}(g^\alpha) < \infty$ as $\alpha \to 0$. Hence

$$
\lim_{\alpha\to 0} \mathcal{I}_\alpha(z) = 0 \quad \text{for } z \in D^p.
$$

Consider now the case where $z \in \tilde{D}$. From Theorem 2.9 and Theorem 4.3 we have that $\|S^M_{\xi_0} g^{\alpha,M}_{\xi_0}\|_{L^2(D^M_0)} \to \infty$ and $\|S_{\xi_0} g_{\xi_0}\|_{L^2_{\xi_0}(D_0)}$ is bounded as $\alpha \to 0$. Moreover, we have that

$$I^M_{\xi_0}(g^{\alpha,M}_{\xi_0} - \frac{1}{M} g^{\alpha}_{\xi_0}) \geqslant c_2 \|S^M_{\xi_0} g^{\alpha,M}_{\xi_0} - S_{\xi_0} g^{\alpha}_{\xi_0}\|^2_{L^2(D^M_0)} \geqslant c_2 \|S^M_{\xi_0} g^{\alpha,M}_{\xi_0}\|^2_{L^2(D^M_0)} - c_2 \|S_{\xi_0} g^{\alpha}_{\xi_0}\|^2_{L^2(D_0)},$$

with $c_2 > 0$ is a constant independent from ξ_0. therefore

$$I^M_{\xi_0}(g^{\alpha,M}_{\xi_0} - \frac{1}{M} g^{\alpha}_{\xi_0}) \longrightarrow \infty \quad \text{as } \alpha \to 0,$$

which implies that

$$0 < \lim_{\alpha \to 0} \mathcal{I}_\alpha(z) < \infty \quad \text{for } z \in \tilde{D},$$

This ends the proof. □

Remark 4.5. *Consider $\hat{g}^{\alpha,M}_{\xi_0}(\Gamma^R)$ satisfying*

$$J^{\alpha,M}_{\xi_0}(\Phi_{\xi_0,M}(\cdot,z), \hat{g}^{\alpha,M}_{\xi_0}(z)) \leqslant j^{\alpha,M}_{\xi_0}(\Phi_{\xi_0,M}(\cdot,z)) + c(\alpha),$$

and define

$$\mathcal{I}^\alpha_M(z) := \left[I^M_{\xi_0}(\hat{g}^{\alpha,M}_{\xi_0}) \left(1 + \frac{I^M_{\xi_0}(\hat{g}^{\alpha,M})}{I^M_{\xi_0}(g^{\alpha,M}_{\xi_0} - \frac{1}{M} g^{\alpha}_{\xi_0})} \right) \right]^{-1}.$$

We observe that $\mathcal{I}^\alpha_M$ can be considered also as an indicator function for the identification of $\tilde{D}$, i.e Theorem 4.4 still holds if we change $\mathcal{I}^\alpha$ by $\mathcal{I}^\alpha_M$ and the proof follows the same arguments but with applying Theorem 4.1 instead of Theorem 3.4.

5. Appendix

Lemma 5.1. *For all $\epsilon > 0$, there exists $\delta > 0$ such that for $|\xi' - \xi| < \delta$ we have*

$$\left\| \tilde{S}_\xi \psi - \tilde{S}_{\xi'} \psi \right\|^2_{L^2(\Omega^{R_0}_0)} \leqslant c\epsilon^2 \left\| \tilde{S}_\xi \psi \right\|^2_{L^2(\Omega^{R_0}_0)} \quad \forall\ \psi \in L^2_\sharp(\Gamma^R), \tag{91}$$

where $c > 0$ is a constant independent of ξ and $\xi' \in I$.

Proof. Let $\psi \in L^2_\sharp(\Gamma^R)$. From (74) we have

$$\left\| \tilde{S}_\xi \psi \right\|^2_{L^2(\Omega^{R_0}_0)} = \frac{\pi}{2} \sum_{j \in \mathbb{Z}} |\widehat{\psi}_j|^2 \int_0^{R_0} |\theta_\xi(j,R,x_2)|^2 dx_2,$$

$$\left\| (\tilde{S}_\xi - \tilde{S}_{\xi'}) \psi \right\|^2_{L^2(\Omega^{R_0}_0)} = \frac{\pi}{2} \sum_{j \in \mathbb{Z}} |\widehat{\psi}_j|^2 \int_0^{R_0} |\theta_\xi(j,R,x_2) - \theta_{\xi'}(j,R,x_2)|^2 dx_2.$$

Therefore, to prove (91) we prove for all $\epsilon > 0$ the existence of a constant $c > 0$ independent from ξ and ξ' such that

$$\int_0^{R_0} |\theta_\xi(j,R,x_2) - \theta_{\xi'}(j,R,x_2)|^2 dx_2 \leqslant c\epsilon^2 \int_0^{R_0} |\theta_\xi(j,R,x_2)|^2 dx_2.$$

Consider first $j \in \mathbb{Z}$ such that $k^2 \geqslant \alpha_\xi^2(j)$, i.e $\beta_\xi(j) = \sqrt{k^2 - |\xi + j|^2}$. There exists only a finite number of j for which this holds. Then

$$|\theta_\xi(j,R,x_2)|^2 = \frac{|e^{i\beta_\xi(j)R}|^2}{|\beta_\xi(j)|^2} |2\sin(\beta_\xi(j)x_2)|^2 = \frac{2}{|\beta_\xi(j)|^2} |1 - \cos(2\beta_\xi(j)x_2)|.$$

For $\beta_\xi(j) = 0$, we have

$$\lim_{\beta_\xi(j)\to 0} \int_0^{R_0} |\theta_\xi(j,R,x_2)|^2 dx_2 = \frac{4R_0^3}{3} =: c_1, \tag{92}$$

while if $\beta_\xi(j) > 0$, we have

$$\int_0^{R_0} |\theta_\xi(j,R,x_2)|^2 dx_2 = 8R_0^3 \left[\frac{y - \sin(y)}{y^3} \right], \tag{93}$$

with $y = 2R_0|\beta_\xi(j)|$. Since $y \longrightarrow \frac{y-\sin(y)}{y^3} > 0$ for $y \geqslant 0$ and since $\beta_\xi(j) > 0$ is bounded for $j \in \mathbb{Z}$ such that $k^2 \geqslant \alpha_\xi^2(j)$, then there exists a constant $c_1' > 0$ independent of ξ and j such that

$$\int_0^{R_0} |\theta_\xi(j,R,x_2)|^2 dx_2 \geqslant c_1'. \tag{94}$$

Consider $\epsilon > 0$, since $\theta_\xi(j,R,x_2)$ is continuous on the compact set $\bar{I}$. Then there exists $\delta > 0$ such that for $|\xi' - \xi| < \delta$ we have

$$|\theta_{\xi'}(j,R,x_2) - \theta_\xi(j,R,x_2)| \leqslant \epsilon. \tag{95}$$

Consequently, using (92)-(94) we get

$$\int_0^{R_0} |\theta_{\xi'}(j,R,x_2) - \theta_\xi(j,R,x_2)|^2 dy_2 \leqslant c_1'' R_0 \epsilon \int_0^{R_0} |\theta_\xi(j,R,x_2)|^2 dx_2, \tag{96}$$

with $c_1'' := 1/\min(c_1, c_1')$.

Consider the case where $k^2 < \alpha_\xi^2(j)$ for which $\beta_\xi(j) = i\sqrt{|\xi + j|^2 - k^2}$. Assume in addition that $|\beta_\xi(j)| \leqslant 1$. There exists only a finite number of j for which this holds. We have

$$|\theta_\xi(j,R,x_2)|^2 = \frac{e^{-2|\beta_\xi(j)|R}}{|\beta_\xi(j)|^2} \sinh(|\beta_\xi(j)|x_2)^2.$$

Therefore

$$\int_0^{R_0} |\theta_\xi(j,R,x_2)|^2 dx_2 = \frac{2R_0 e^{-2|\beta_\xi(j)|R}}{|\beta_\xi(j)|^2}\left[\frac{\sinh(2|\beta_\xi(j)|R_0)}{2R_0|\beta_\xi(j)|} - 1\right] \geqslant \frac{4R_0^3}{3}e^{-2|\beta_\xi(j)|R}, \tag{97}$$

where we used the inequality $\frac{\sinh(x)}{x} - 1 \geqslant \frac{x^2}{6}$ for $x \geqslant 0$. Since $|\beta_\xi(j)| \leqslant 1$, then we get

$$\int_0^{R_0} |\theta_\xi(j,R,x_2)|^2 dx_2 \geqslant \frac{4R_0^3}{3}e^{-2R} =: c_2'. \tag{98}$$

Consider $\epsilon > 0$, since $\theta_\xi(j,R,x_2)$ is continuous on the compact set $\bar{I}$, then there exists $\delta > 0$ such that for $|\xi' - \xi| < \delta$ we have

$$\int_0^{R_0} |\theta_{\xi'}(j,R,x_2) - \theta_\xi(j,R,x_2)|^2 dy_2 \leqslant c_2' R_0 \epsilon \int_0^{R_0} |\theta_\xi(j,R,x_2)|^2 dx_2. \tag{99}$$

Consider now $k^2 \leqslant \alpha_\xi^2(j)$ such that $|\beta_\xi(j)| > 1$. Let $\delta \geqslant 0$, $0 \leqslant \delta_0 < \delta$ such that $\xi' := \xi + \delta_0$, then we have

$$(\theta_{\xi+\delta_0} - \theta_\xi)(j,R,x_2) = \int_\xi^{\xi+\delta_0} \frac{\partial \theta_{\tilde{\xi}}}{\partial \tilde{\xi}} d\tilde{\xi}. \tag{100}$$

By the Cauchy-Schwartz inequality we get

$$\int_0^{R_0} |\theta_\xi(j,R,x_2) - \theta_{\xi+\delta_0}(j,R,x_2)|^2 dx_2 \leqslant \delta_0 \int_\xi^{\xi+\delta_0}\left(\int_0^{R_0} |\frac{\partial \theta_{\tilde{\xi}}}{\partial \tilde{\xi}}(j,R,x_2)|^2 dx_2\right) d\tilde{\xi}. \tag{101}$$

On the other hand, let us denote by $\gamma_0 := 2i(\xi + j)\frac{e^{-|\beta_\xi(j)|R}}{|\beta_\xi(j)|^2}$ and $\gamma_1 := R + \frac{1}{|\beta_\xi(j)|}$, we have that

$$\frac{\partial \theta_\xi}{\partial \xi}(j,R,x_2) = \gamma_0\left[\gamma_1 \sinh(|\beta_\xi(j)|x_2) - x_2\cosh(|\beta_\xi(j)|x_2)\right] \tag{102}$$

Using that $\sinh(2y) = 2\sinh(y)\cosh(y)$, $\cosh^2(y) - \sinh^2(y) = 1$ and $2\sinh^2(y) = \cosh(2y) - 1$ for all $y \in \mathbb{R}$, we get

$$\left|\frac{\partial \theta_\xi}{\partial \xi}(j,R,x_2)\right|^2 = |\gamma_0|^2\left[(\frac{\gamma_1^2 + x_2^2}{2})\cosh(2|\beta_\xi(j)|x_2) - \gamma_1 x_2 \sinh(2|\beta_\xi(j)|x_2) - \frac{\gamma_1^2}{2} + \frac{x_2^2}{2}\right]. \tag{103}$$

Therefore

$$\begin{aligned}\int_0^{R_0}\left|\frac{\partial \theta_\xi}{\partial \xi}(j,R,x_2)\right|^2 dx_2 = |\gamma_0|^2 \Bigg[& R_0^2\frac{\sinh(2|\beta_\xi(j)|R_0)}{4|\beta_\xi(j)|} - (\gamma_1 + \frac{1}{2|\beta_\xi(j)|})\frac{R_0\cosh(2|\beta_\xi(j)|R_0)}{2|\beta_\xi(j)|} \\ & + \frac{\gamma_1^2\sinh(2|\beta_\xi(j)|R_0)}{4|\beta_\xi(j)|} - \frac{\gamma_1^2 R_0}{2} + \frac{R_0^3}{6} + \frac{1}{4|\beta_\xi(j)|^2}(\gamma_1 + \frac{1}{2|\beta_\xi(j)|})\sinh(2|\beta_\xi(j)|R_0)\Bigg].\end{aligned}$$

In relation to (97) and the previous identity we consider the following functions defined for $x \in \mathbb{R}$ such that $x > 1$ as

$$f(x) := \frac{2e^{-2xR}}{x^2}\left[\frac{\sinh(2xR_0)}{2x} - R_0\right], \tag{104}$$

$$g(x) := 4\left|\frac{\sqrt{x^2+k^2}e^{-xR}}{x^2}\right|^2\left[\frac{R_0^2\sinh(2xR_0)}{4x} - (R+\frac{3}{2x})\frac{R_0\cosh(2xR_0)}{2x} + \frac{(R+\frac{1}{x})^2\sinh(2xR_0)}{4x}\right.$$
$$\left. - (R+\frac{1}{x})^2\frac{R_0}{2} + \frac{R_0^3}{6} + \frac{1}{4x^2}(R+\frac{3}{2x})\sinh(2xR_0)\right],$$

and we prove the existence of a constant $\alpha > 0$ such that $g(x) < \alpha f(x)$ for all $x > 1$. Let $M \in \mathbb{R}$ sufficiently large and consider first the case $1 < x \leqslant M$. Since f and g are continuous functions and $f(x) > 0$, then we have

$$g(x) < \alpha' f(x) \qquad \text{for } 1 < x \leqslant M, \tag{105}$$

with $\alpha' := \frac{\max\limits_{1\leqslant x\leqslant M} g(x)}{\min\limits_{1\leqslant x\leqslant M} f(x)} > 0$. For the case $x > M$, we compare f and g at infinity. We have

$$g(x) = \frac{(x^2+k^2)e^{-2xR}}{2x^4}\left(\frac{e^{2xR_0}}{x}\right)\left[(R_0-R)^2 + O\left(\frac{1}{x}\right)\right],$$
$$f(x) = \frac{e^{-2xR}}{2x^2}\left(\frac{e^{2xR_0}}{x}\right)\left[1 + O\left(\frac{1}{x}\right)\right].$$

Therefore $\frac{g(x)}{f(x)}$ is equivalent to $(R-R_0)^2$ at infinity. Using (105) we deduce the existence of a constant $\alpha > 0$ such that

$$g(x) \leqslant \alpha f(x), \quad \text{for all } x > 1,$$

which implies

$$g(|\beta_\xi(j)|) \leqslant \alpha f(|\beta_\xi(j)|),$$

for all $j \in \mathbb{Z}$ such that $|\beta_\xi(j)| > 1$. Then we deduce that

$$\int_0^{R_0} |\frac{\partial\theta_\xi}{\partial\xi}(j,R,x_2)|^2 dx_2 \leqslant \alpha\int_0^{R_0} |\theta_\xi(j,R,x_2)|^2 dx_2. \tag{106}$$

Therefore, using (101)-(106) we have

$$\int_0^{R_0} |\theta_\xi(j,R,x_2) - \theta_{\xi+\delta_0}(j,R,x_2)|^2 dx_2 \leqslant \delta_0\alpha\int_\xi^{\xi+\delta_0}\left(\int_0^{R_0} |\theta_{\tilde{\xi}}(j,R,x_2)|^2 dx_2\right) d\tilde{\xi}. \tag{107}$$

On the other hand, we show that $x \longrightarrow f(x)$ decreases in $\mathbb{R}_+$. Indeed , taking $y = 2xR_0$ and using that $\sinh(y) = \sum\limits_{k=0}^{\infty}\frac{y^{2k+1}}{(2k+1)!}$ we get

$$f'(y) = R_0^3 e^{-\frac{R}{R_0}y}h(y),$$

with

$$h(y) := \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k+3)!} \left(\frac{-R}{R_0} + \frac{2k+1}{(2k+4)(2k+5)} y \right).$$

Since $\cosh(y) := \sum\limits_{k=0}^{\infty} \frac{y^{2k}}{2k!}$ we observe that

$$y^3 h(y) \leqslant -\sum_{k=0}^{\infty} \frac{y^{2k+3}}{(2k+3)!} + \sum_{k=0}^{\infty} \frac{y^{2k+4}}{(2k+4)!} = -\sinh(y) + y + \cosh(y) - 1 - \frac{y^2}{2} \leqslant 0$$

for $y \geqslant 0$. Therefore $f'(y) \leqslant 0 \quad$ for all $\; y \geqslant 0$. Since $\xi \longrightarrow |\beta_\xi(j)|$ increases in $\bar{I}$ and $x \longrightarrow f(x)$ decreases in $\mathbb{R}_+$ we infer that $\xi \to \int_0^{R_0} |\theta_\xi(j, R, x_2)| dx_2$ decreases in $\bar{I}$. Therefore, from (107) we finally obtain that

$$\int_0^{R_0} |\theta_\xi(j, R, x_2) - \theta_{\xi+\delta_0}(j, R, x_2)|^2 dx_2 \leqslant \alpha \delta_0^2 \left(\int_0^{R_0} |\theta_\xi(j, R, x_2)|^2 dx_2 \right),$$

which ends the proof. $\square$

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