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Complex vs etale Abel Jacobi map for higher Chow groups and algebraicity of the zero locus of etale normal functions

Johann Bouali

February 10, 2023

Abstract

We prove, using p -adic Hodge theory for open algebraic varieties, that for a smooth projective variety over a field $k \subset \mathbb{C}$ of finite type over \mathbb{Q} , the complex abel jacobian map vanishes if the etale abel jacobian map vanishes. This implies that for a smooth projective morphism $f : X \rightarrow S$ of smooth complex algebraic varieties over a field $k \subset \mathbb{C}$ of finite type over \mathbb{Q} and $Z \in \mathcal{Z}^d(X, n)^{f, \partial=0}$ an algebraic cycle flat over S whose cohomology class vanishes on fibers, the zero locus of the etale normal function associated to Z is contained in the zero locus of the complex normal function associated to Z . From the work of Saito or Charles, we deduce that the zero locus of the complex normal function associated to Z is defined over \bar{k} if the zero locus of the etale normal function associated to Z is not empty. We also prove an algebraicity result for the zero locus of an etale normal function associated to an algebraic cycle over a field k of finite type over \mathbb{Q} . By the way, we get that the Hodge conjecture implies the Tate conjecture for smooth projective varieties over fields of finite type over \mathbb{Q} and for smooth projective varieties over p -adic fields (in particular we get Tate conjecture for divisors), and a for a smooth projective morphism $f : X \rightarrow S$ of smooth complex algebraic varieties over a subfield $k \subset \mathbb{C}$ of finite type over \mathbb{Q} , the locus of Hodge Tate classes inside the locus of Hodge classes of f .

1 Introduction

Let X be a connected smooth projective variety over \mathbb{C} . The Abel-Jacobi map associates to a cycle $Z \in \mathcal{Z}^d(X)$ homologically equivalent to 0 of codimension d , a point $AJ(X)(Z) \in J^d(X)$ in the intermediate Jacobian of X . In family, if $f : X \rightarrow S$ is a smooth projective morphism of smooth connected complex varieties and $Z \in \mathcal{Z}^d(X)$ is a relative cycle on S of codimension d homologically trivial on fibers, the Abel-Jacobi map provides a holomorphic and horizontal section of the relative intermediate Jacobian $\nu_Z : S \rightarrow J^d(X/S)$ called the associated normal function. By a Brosnan-Pearlstein theorem, the zero-locus $V(\nu_Z) \subset S$ and more generally the torsion locus $V(\nu_Z) \subset V_{tors}(\nu_Z) \subset S$ of ν_Z is an algebraic subvariety ([3]). If X/S and Z are defined over a subfield $k \subset \mathbb{C}$, we conjecture in the spirit of the Bloch-Beilinson conjectures, that the zero-locus of the normal function is also defined over k (see [7]).

On the other hand, if k is of finite type over \mathbb{Q} , X is a connected smooth projective variety over k , and p is a prime number, we can define via the continuous étale cohomology with \mathbb{Z}_p coefficients an etale Abel-Jacobi map which associate to a cycle $Z \in \mathcal{Z}^d(X, n)$ homologically equivalent to 0 of codimension d , an element $AJ^{et,p}(X)(Z) \in H^1(G, H^{2d-1-n}(X_{\bar{k}}, \mathbb{Z}_p))$ of the first degree Galois cohomology of the absolute galois group $G := Gal(\bar{k}/k)$ with value in the etale cohomology of $X_{\bar{k}}$ with \mathbb{Z}_p coefficients. In family, we get, for $f : X \rightarrow S$ a smooth projective morphism of smooth connected varieties over k and $Z \in \mathcal{Z}^d(X, n)$ a relative cycle on S of codimension d homologically trivial on fibers, a normal function $\nu_Z^{et,p}$ associated to $Z \in \mathcal{Z}^d(X, n)$ and thus a zero-locus $V(\nu_Z^{et,p}) \subset S$ and more generally a torsion locus $V(\nu_Z^{et,p}) \subset V_{tors}(\nu_Z^{et,p}) \subset S$ of $\nu_Z^{et,p}$ which are subsets of closed points of S . For $\sigma : k \hookrightarrow \mathbb{C}$ an embedding, and $V \subset S$ a subset, we denote $V_{\mathbb{C}} := \pi_{k/\mathbb{C}}(S)^{-1}(V) \subset S_{\mathbb{C}}$ where $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \rightarrow S$ is the projection.

Let k of finite type over \mathbb{Q} and $f : X \rightarrow S$ a smooth projective morphism of smooth connected varieties over k . Let $Z \in \mathcal{Z}^d(X)$ a relative cycle. F. Charles then proves that for any embedding $\sigma : k \hookrightarrow \mathbb{C}$

- assuming that $R^{2d-1}f_*^{an}(\mathbb{C})$ has no global sections then $V(\nu_Z^{et,p})_{\mathbb{C}} = V(\nu_Z)$ ([7])
- if $V(\nu_Z)(\bar{k})$ is non-empty then it is defined over k ([8]).

In this work, we show that, for a field k of finite type over \mathbb{Q} , X a connected smooth projective variety over k , $Z \in \mathcal{Z}^d(X, *)$ an (higher) algebraic cycle homologically equivalent to 0 of codimension d , and $p \in \mathbb{N}$ a prime number, if $AJ^{et,p}(X)(Z) = 0$ then $AJ_{\sigma}(X)(Z) := AJ(X_{\mathbb{C}})(Z_{\mathbb{C}}) = 0$ for any embedding $\sigma : k \hookrightarrow \mathbb{C}$ (c.f. theorem 6). This implies by definition that for $f : X \rightarrow S$ a smooth projective morphism of smooth connected varieties over a field k of finite type over \mathbb{Q} , and $Z \in \mathcal{Z}^d(X, *)$ a relative cycle on S of codimension d homologically trivial on fibers,

$$V_{tors}(\nu_Z^{et,p})_{\mathbb{C}} \subset V_{tors}(\nu_Z)$$

for any embedding $\sigma : k \hookrightarrow \mathbb{C}$ without any assumption (c.f. corollary 3(i)). We deduce that if $V(\nu_Z^{et,p})$ is not empty, $V(\nu_Z)$ is defined over the algebraic closure \bar{k} of k (c.f. corollary 3(ii)).

The proof of theorem 6 uses p -adic Hodge theory for open varieties to relate de Rham cohomology and its Hodge filtration to p -adic étale cohomology with its Galois action. Theorem 6 follows indeed from the fact that by proposition 3(i), proposition 2 and proposition 1, we have for a field k of finite type over \mathbb{Q} , $U \in \text{SmVar}(k)$, and embeddings $\sigma : k \hookrightarrow \mathbb{C}$, $\sigma_p : k \hookrightarrow \mathbb{C}_p$ for a prime $p \in \mathbb{N}$, for each $j, l \in \mathbb{Z}$, a canonical injective map

$$H^j \iota_{p,ev}^{G,l}(U) : H_{et}^j(U_{\bar{k}}, \mathbb{Z}_p)(l)^G \hookrightarrow F^l H^j(U_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p, \alpha \mapsto H^j \iota_{p,ev}^{G,l}(U)(\alpha) := ev(U)(w(\alpha)),$$

which is by construction functorial in $U \in \text{SmVar}(k)$ (see theorem 1). More precisely, if the étale Abel-Jacobi image of a cycle Z is zero,

- there exist a Galois invariant class α in the étale cohomology of $((X \times \square^n) \setminus Z)_{\bar{k}}$ with non-zero boundary, where \bar{k} is an algebraic closure of k ,
- then, by p adic Hodge theory for $(X \times \square^n) \setminus Z$, α define a logarithmic de Rham class $w(\alpha)_L$ laying inside the right degree of the Hodge filtration of $((X \times \square^n) \setminus Z)_{\hat{k}_{\sigma_p}}$ where $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ is the p -adic completion of k with respect to an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$ (c.f. proposition 3(i) and proposition 2)
- taking the image of this class by the complex period map with respect to an embedding $\sigma' : k' \hookrightarrow \mathbb{C}$ extending the given embedding $\sigma : k \hookrightarrow \mathbb{C}$ where $k' \subset \hat{k}_{\sigma_p}$ is a subfield over which $w(\alpha)_L$ is defined, we get a Betti cohomology class $ev((X \times \square^n) \setminus Z)(w(\alpha)_L)$ of $((X \times \square^n) \setminus Z)_{\mathbb{C}}^{an}$ with $2i\pi\mathbb{Q}$ coefficients (c.f. proposition 1),
- this last class $(1/2i\pi)ev((X \times \square^n) \setminus Z)(w(\alpha)_L)$ induce a splitting of the localization exact sequence of mixed Hodge structures :

$$0 \rightarrow H^{2d-1-n}(X_{\mathbb{C}}^{an}, \mathbb{Q}) \xrightarrow{j^*} H^{2d-1}(((X \times \square^n) \setminus Z)_{\mathbb{C}}^{an}, \mathbb{Q})^{[Z]} \xrightarrow{\partial} H_Z^{2d}((X \times \square^n)_{\mathbb{C}}^{an}, \mathbb{Q})^{[Z]} = \mathbb{Q}^{Hdg}(d) \rightarrow 0,$$

which means that the complex Abel-Jacobi image de Z is zero.

By the way, since for $X \in \text{PSmVar}(k)$, $\iota_{p,ev}^{G,d}(X)$ is compatible with cycle class maps, we get in particular that Hodge conjecture implies Tate conjecture. In particular, we get Tate conjecture for divisors for smooth projective varieties over fields of characteristic zero.

In section 6, we show that, for $f : X \rightarrow S$ a smooth projective morphism of connected smooth varieties over a field k of finite type over \mathbb{Q} , $Z \in \mathcal{Z}^d(X, *)$ an (higher) algebraic cycle homologically equivalent to 0 of codimension d , and p a prime number for all expect finitely many prime numbers $p \in \mathbb{N}$

$$V_{tors}(\nu_Z^{et,p}) = T \cap S_{(0)} \subset S,$$

where $T \subset S$ is the image of closed points of a constructible algebraic subset of

$$E_{DR}^{2d-1}(((X \times \square^*) \setminus |Z|)_{\hat{k}_{\sigma_p}} / S_{\hat{k}_{\sigma_p}}) := H^{2d-1} \int_f (O_{((X \times \square^*) \setminus |Z|)_{\hat{k}_{\sigma_p}}}, F_b) \in \text{Vect}_{fil}(S_{\hat{k}_{\sigma_p}})$$

by the projection $p_S : E_{DR}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\hat{k}_{\sigma_p}} / S_{\hat{k}_{\sigma_p}}) \rightarrow S_{\hat{k}_{\sigma_p}}$, where $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ is the completion of k with respect to σ_p (c.f. definition 7 and theorem 7(i)). We also give a local version for $V_{tors}(U_{Z, \sigma_p}^{et, p}) \subset S$ (c.f. theorem 7(ii)). The proof use on the one hand proposition 3(ii), and on the other hand De Yong alterations to get a stratification $S = \sqcup_{\alpha \in \Lambda} S^\alpha$ by locally closed subsets and alterations $\pi^\alpha : (X_{S^\alpha} \times \square^n)^\alpha \rightarrow X_{S^\alpha} \times \square^n$ such that

$$f \circ \pi^\alpha : ((X \times \square^n)_{\hat{k}_{\sigma_p}}^\alpha, \pi^{\alpha, -1}(|Z|)) \rightarrow S_{\hat{k}_{\sigma_p}}^\alpha$$

is a semi-stable morphism. We then use the p adic semi-stable comparison theorem for semi-stable morphisms $f' : (X', D') \rightarrow S'$, with $S', X', D' \in \text{SmVar}(\hat{k}_{\sigma_p})$, that is satisfying

- $f' : X' \rightarrow S'$ is smooth projective, $D' \subset X'$ is a normal crossing divisor,
- for all $s \in S'$, $D'_s \subset X'_s$ is a normal crossing divisor and (X'_s, D'_s) has integral model with semi-stable reduction,

(or more generally for log smooth morphism of schemes over $(O_{\hat{k}_{\sigma_p}}, N_{\mathcal{O}})$), which gives, for each $j \in \mathbb{Z}$ a canonical filtered isomorphism

$$H^j f'_* \alpha(U') : R^j f'_* \mathbb{Z}_{p, U'^{et}} \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{st, S'} \xrightarrow{\sim} R^j f'_{*Hdg}(O_{U'}, F_b) \otimes_{O_{S'}} \mathcal{O}\mathbb{B}_{st, S'}.$$

which is for each $s \in S'$ compatible with the action of the Galois group $Gal(\mathbb{C}_p/k(s))$, the Frobenius and the monodromy: for all but finitely many primes p , $\hat{k}_{\sigma_p}(s)$ is unramified for all embeddings $\sigma_p : k \hookrightarrow \mathbb{C}_p$ and all $s \in S'$, so that we get a Frobenius action on $R^j f'_{*Hdg}(O_{U'}, F_b)_s = H_{DR}^j(U'_s)$ for all $s \in S'$.

This also give (see theorem 8) together with proposition 3(ii) and proposition 1, for $f : X \rightarrow S$ a smooth projective morphism, with S, X smooth over a subfield $\sigma : k \hookrightarrow \mathbb{C}$ of finite type over \mathbb{Q} , and p any but finitely many prime numbers and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding, the locus of Hodge-Tate classes $(H^j Rf_{* \mathbb{Q}_p, X_{\bar{k}}^{et}}(d))^G$, where $G = Gal(\bar{k}/k)$ and $\bar{k} \subset \mathbb{C}$ is the algebraic closure of k inside \mathbb{C} ,

$$\begin{aligned} & \iota_{ev}^{G, d}(X/S) : \pi_{k/\mathbb{C}}(S)^*(H^j Rf_{* \mathbb{Q}_p, X_{\bar{k}}^{et}}(d))^G \xrightarrow{\sim} \\ & < F^d E_{DR}^j(X/S) \cap (\sqcup_{\alpha \in \Lambda} (E_{DR}^j(X_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha) \otimes_{O_{S_{\hat{k}_{\sigma_p}}}} \mathcal{O}\mathbb{B}_{st, S_{\hat{k}_{\sigma_p}}^\alpha}^{\phi, N})) >_{\mathbb{Q}_p} \\ & \hookrightarrow (F^d E_{DR}(X_{\mathbb{C}}/S_{\mathbb{C}}) \cap R^j f_* \mathbb{Q}_{X_{\mathbb{C}}^{en}}) \otimes_{\mathbb{Q}} \mathbb{Q}_p =: HL_{j, d}(X_{\mathbb{C}}/S_{\mathbb{C}}) \otimes_{\mathbb{Q}} \mathbb{Q}_p, \\ & \alpha_s \mapsto ev(X_{k(s)})((1/2i\pi)w(\alpha_s)_{k(s)}), s' = \pi_{k/\mathbb{C}}^{-1}(s) : k(s) \hookrightarrow \mathbb{C}, s \in S \end{aligned}$$

inside the locus of Hodge classes $HL_{j, d, \sigma}(X/S) := HL_{j, d}(X_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^j(X_{\mathbb{C}}/S_{\mathbb{C}})$, which is a countable union of algebraic subvarieties by the work of Deligne, Catani and Kaplan, where

- $E_{DR}^j(X/S) := H^j \int_f (O_X, F_b) = R^j f_* \Omega_{X/S}^\bullet$ and $E_{DR}^j(X_{\mathbb{C}}/S_{\mathbb{C}}) := H^j \int_f (O_{X_{\mathbb{C}}}, F_b)$ are the filtered algebraic vector bundles over S and $S_{\mathbb{C}}$ respectively which have the Gauss-Manin integrable connexion,
- $S = \sqcup_{\alpha \in \Lambda} S^\alpha$ is a stratification by locally closed algebraic subsets, Λ being a finite set,
- $\pi^\alpha : X^\alpha \rightarrow X_{S^\alpha}$ being alterations, in particular we get sub vector bundles $\pi^{\alpha, *}: E_{DR}^j(X_{S^\alpha}/S^\alpha) := E_{DR}^j(X/S)|_{S^\alpha} \hookrightarrow E_{DR}^j(X^\alpha/S^\alpha)$,

- $\pi_{k/\hat{k}_{\sigma_p}}(S^\alpha)^* E_{DR}^j(X^\alpha/S^\alpha) \subset E_{DR}^j(X_{\hat{k}_{\sigma_p}}^\alpha/S_{\hat{k}_{\sigma_p}}^\alpha)$ is the canonical subset of closed points, $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ being the p adic completion of k with respect to σ_p ,
- $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \rightarrow S$ and $\pi_{k/\hat{k}_{\sigma_p}}(S) : S_{\hat{k}_{\sigma_p}} \rightarrow S$ are the projections.

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2 Preliminaries and Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topoi.
- For $\mathcal{S} \in \text{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \text{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \rightarrow X$ is a morphism in \mathcal{S} , and whose morphisms $\text{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \rightarrow Y$ in \mathcal{S} such that $f \circ g = f'$.
- For $\mathcal{S} \in \text{Cat}$ denote $\text{Gr } \mathcal{S} := \text{Fun}(\mathbb{Z}, \mathcal{S})$ is the category of graded objects.
- Denote by Ab the category of abelian groups. For R a ring denote by $\text{Mod}(R)$ the category of (left) R modules. We have then the forgetful functor $o_R : \text{Mod}(R) \rightarrow \text{Ab}$.
- Denote by AbCat the category of small abelian categories.
- For $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topoi, we denote by
 - $\text{PSh}(\mathcal{S})$ the category of presheaves of $O_{\mathcal{S}}$ modules on \mathcal{S} and $\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$ the category of presheaves of $O_{\mathcal{S}}$ modules on \mathcal{S} , whose objects are $\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})^0 := \{(M, m), M \in \text{PSh}(\mathcal{S}), m : M \otimes O_{\mathcal{S}} \rightarrow M\}$, together with the forgetful functor $o : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$,
 - $C(\mathcal{S}) = C(\text{PSh}(\mathcal{S}))$ and $C_{O_{\mathcal{S}}}(\mathcal{S}) = C(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}))$ the big abelian category of complexes of presheaves of $O_{\mathcal{S}}$ modules on \mathcal{S} ,
 - $C_{O_{\mathcal{S}}(2)\text{fil}}(\mathcal{S}) := C_{(2)\text{fil}}(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})) \subset C(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of $O_{\mathcal{S}}$ modules on \mathcal{S} such that the filtration is biregular and $\text{PSh}_{O_{\mathcal{S}}(2)\text{fil}}(\mathcal{S}) = (\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}), F, W)$.
- Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topoi with topology τ . For $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$, we denote by $k : F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{O_{\mathcal{S}}}(\mathcal{S})$ (see [5]). In particular for $X \in \mathcal{S}$, $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$.
- For $f : \mathcal{S}' \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \text{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ and each $j \in \mathbb{Z}$,
 - $f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F)$,
 - $f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*\text{mod}}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*\text{mod}}F)$,

the canonical maps.

- For $\mathcal{X} \in \text{Cat}$ a (pre)site and p a prime number, we consider the full subcategory

$$\text{PSh}_{\mathbb{Z}_p}(\mathcal{X}) \subset \text{PSh}(\mathbb{N} \times \mathcal{X}), \quad F = (F_n)_{n \in \mathbb{N}}, \quad p^n F_n = 0, \quad F_{n+1}/p^n \xrightarrow{\sim} F_n$$

$C_{\mathbb{Z}_p}(\mathcal{X}) := C(\text{PSh}_{\mathbb{Z}_p}(\mathcal{X})) \subset C(\mathbb{N} \times \mathcal{X})$ and

$$\mathbb{Z}_p := \mathbb{Z}_{p, \mathcal{X}} := ((\mathbb{Z}/p^* \mathbb{Z})_{\mathcal{X}}) \in \text{PSh}_{\mathbb{Z}_p}(\mathcal{X})$$

the diagram of constant presheaves on \mathcal{X} .

- Let $f : \mathcal{X}' \rightarrow \mathcal{X}$ a morphism of (pre)site with $\mathcal{X}, \mathcal{X}' \in \text{Cat}$. We will consider for $F = (F_n)_{n \in \mathbb{N}} \in C_{\mathbb{Z}_p}(\mathcal{X})$ the canonical map in $C(\mathcal{X}')$

$$T(f^*, \varprojlim)(F) : f^* \varprojlim_{n \in \mathbb{N}} F_n \rightarrow \varprojlim_{n \in \mathbb{N}} f^* F_n$$

Recall that filtered colimits do NOT commute with infinite limits in general. In particular, for $f : \mathcal{X}' \rightarrow \mathcal{X}$ a morphism of (pre)site and $F = (F_n)_{n \in \mathbb{N}} \in \text{PSh}_{\mathbb{Z}_p}(\mathcal{X})$, $\varprojlim_{n \in \mathbb{N}} f^* F_n$ is NOT isomorphic to $f^* \varprojlim_{n \in \mathbb{N}} F_n$ in $\text{PSh}(\mathcal{X}')$ in general.

- Denote by $\text{Sch} \subset \text{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). For $X \in \text{Sch}$, we denote by
 - $\text{Sch}^{ft}/X \subset \text{Sch}/X$ the full subcategory consisting of objects $X'/X = (X', f) \in \text{Sch}/X$ such that $f : X' \rightarrow X$ is an morphism of finite type
 - $X^{et} \subset \text{Sch}^{ft}/X$ the full subcategory consisting of objects $U/X = (X, h) \in \text{Sch}/X$ such that $h : U \rightarrow X$ is an etale morphism.
 - $X^{sm} \subset \text{Sch}^{ft}/X$ the full subcategory consisting of objects $U/X = (X, h) \in \text{Sch}/X$ such that $h : U \rightarrow X$ is a smooth morphism.

For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$. We then denote by

- $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
- $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
- $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

For a morphism of field $\sigma : k \hookrightarrow K$, we have the extension of scalar functor

$$\otimes_k K : \text{Sch}/k \rightarrow \text{Sch}/K, X \mapsto X_K := X_{K, \sigma} := X \otimes_k K, (f : X' \rightarrow X) \mapsto (f_K := f \otimes I : X'_K \rightarrow X_K).$$

which is left adjoint to the restriction of scalar

$$\text{Res}_{k/K} : \text{Sch}/K \rightarrow \text{Sch}/k, X = (X, a_X) \mapsto X = (X, \sigma \circ a_X), (f : X' \rightarrow X) \mapsto (f : X' \rightarrow X)$$

The adjunction maps are

- for $X \in \text{Sch}/k$, the projection $\pi_{k/K}(X) : X_K \rightarrow X$ in Sch/k , for $X = \cup_i X_i$ an affine open cover with $X_i = \text{Spec}(A_i)$ we have by definition $\pi_{k/K}(X_i) = \pi_{k/K}(A_i)$,
- for $X \in \text{Sch}/K$, $I \times \Delta_K : X \hookrightarrow X_K = X \times_K K \otimes_k K$ in Sch/K , where $\Delta_K : K \otimes_k K \rightarrow K$ is the diagonal which is given by for $x, y \in K$, $\Delta_K(x, y) = x - y$.

The extension of scalar functor restrict to a functor

$$\otimes_k K : \text{Var}(k) \rightarrow \text{Var}(K), X \mapsto X_K := X_{K, \sigma} := X \otimes_k K, (f : X' \rightarrow X) \mapsto (f_K := f \otimes I : X'_K \rightarrow X_K).$$

and for $X \in \text{Var}(k)$ we have $\pi_{k/K}(X) : X_K \rightarrow X$ the projection in Sch/k . An algebraic variety $X \in \text{Var}(K)$ is said to be defined over k if there exists $X_0 \in \text{Var}(k)$ such that $X \simeq X_0 \otimes_k K$ in $\text{Var}(K)$. For $X = (X, a_X) \in \text{Var}(k)$, we have $\text{Sch}^{ft}/X = \text{Var}(k)/X$ since for $f : X' \rightarrow X$ a morphism of schemes of finite type, $(X', a_X \circ f) \in \text{Var}(k)$ is the unique structure of variety over k of $X' \in \text{Sch}$ such that f becomes a morphism of algebraic varieties over k , in particular we have

- $X^{et} \subset \text{Sch}^{ft}/X = \text{Var}(k)/X$,
- $X^{sm} \subset \text{Sch}^{ft}/X = \text{Var}(k)/X$.

A morphism $f : X' \rightarrow X$ with $X, X' \in \text{Var}(K)$ is said to be defined over k if $X \simeq X_0 \otimes_k K$ and $X' \simeq X'_0 \otimes_k K$ are defined over k and $\Gamma_f = \Gamma_{f_0} \otimes_k K \subset X' \times X$ is defined over k , so that $f_0 \otimes_k K = f$ with $f_0 : X'_0 \rightarrow X_0$.

- For $X \in \text{Sch}$ and $s \in \mathbb{N}$, we denote by $X_{(s)} \subset X$ its points of dimension s , in particular $X_{(0)} \subset X$ are the closed points of X .
- For $X \in \text{Sch}$ and k a field we denote by $X(k) := \text{Hom}_{\text{Sch}}(\text{Spec } k, X)$ the k points of X . We get $X(k)_{in} \subset X$ the image of the k -points of X . For $k \subset k'$ a subfield, $\mathbb{A}_{k'}^N(k)_{in} = k^N \subset k'^N \subset \mathbb{A}_{k'}^N$ and $\mathbb{A}_k^N(k')_{in} = \pi_{k/k'}(\mathbb{A}_k^N)(k'^N) \subset \mathbb{A}_k^N$.
- For $X \in \text{Sch}$, we denote $X^{pet} \subset \text{Sch}/X$ the pro etale site (see [2]) which is the full subcategory of Sch/X whose object consists of weakly etale maps $U \rightarrow X$ (that is flat maps $U \rightarrow X$ such that $\Delta_U : U \rightarrow U \times_X U$ is also flat) and whose topology is generated by fpqc covers. We then have the canonical morphism of site

$$\nu_X : X^{pet} \rightarrow X, (U \rightarrow X) \mapsto (U \rightarrow X)$$

For $F \in C(X^{et})$,

$$\text{ad}(\nu_X^*, R\nu_{X*})(F) : F \rightarrow R\nu_{X*}\nu_X^*F$$

is an isomorphism in $D(X^{et})$, in particular, for each $n \in \mathbb{Z}$

$$\nu_X^* := H^n \Gamma(X, k) : \mathbb{H}_{et}^n(X, F) \xrightarrow{\sim} \mathbb{H}_{pet}^n(X, \nu_X^*F)$$

are isomorphisms, where

$$k := k \circ \text{ad}(\nu_X^*, \nu_{X*})(E_{et}(F)) : E_{et}(F) \rightarrow \nu_{X*}E_{pet}(\nu_X^*F)$$

is the canonical map in $C(X^{et})$ which is a quasi-isomorphism. For $X \in \text{Sch}$, we denote

- $\underline{\mathbb{Z}}_{pX} := \varprojlim_{n \in \mathbb{N}} \nu_X^*(\mathbb{Z}/p^n\mathbb{Z})_{X^{et}} \in \text{PSh}(X^{pet})$ the constant presheaf on \mathcal{X} ,
- $l_{p, \mathcal{X}} := (p^*) : \underline{\mathbb{Z}}_{pX} \rightarrow \nu_X^*(\mathbb{Z}/p\mathbb{Z})_{X^{et}}$ the projection map in $\text{PSh}(\mathbb{N} \times \mathcal{X}^{pet})$.

- Let k a field of characteristic zero and $k_0 \subset k$ a subfield. We say that k is of finite type over k_0 if k is generated as a field by k_0 and a finite set $\{\alpha_1, \dots, \alpha_r\} \subset k$ of elements of k , that is $k = k_0(\alpha_1, \dots, \alpha_r)$. If k is of finite type over k_0 then it is of finite transcendence degree $d \in \mathbb{N}$ over k_0 and $k = k_0(\alpha_1, \dots, \alpha_d)(\alpha_{d+1})$ with $\{\alpha_1, \dots, \alpha_{d+1}\} \subset k$ such that $k_0(\alpha_1, \dots, \alpha_d) = k_0(x_1, \dots, x_d)$ and α_{d+1} is an algebraic element of k over $k_0(\alpha_1, \dots, \alpha_d)$. Note that if k is of finite type over k_0 then it is NOT algebraically closed. We denote \bar{k} the algebraic closure of k . Then \bar{k} is also transcendence degree d over k .
- Let C a field of characteristic zero. Let $X \in \text{Var}(C)$. Then there exist a subfield $k \subset C$ of finite type over \mathbb{Q} such that X is defined over k that is $X \simeq X_0 \otimes_k C$ with $X_0 \in \text{Var}(k)$.
- Let $k \subset \bar{\mathbb{Q}}$ be a number field, i.e. a finite extension of \mathbb{Q} . There exists a finite set of prime number $\delta(k)$ such that for all prime number $p \in \mathbb{N} \setminus \delta(k)$ we have for all each embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$ $\mathbb{Q}_p \cap k = \mathbb{Q}$.
- Let $X = (X, O_X) \in \text{RCat}$ a ringed topos, we have in $C(X)$ the subcomplex of presheaves of abelian group

$$\begin{aligned} OL_X : \Omega_{X, \log}^\bullet &\hookrightarrow \Omega_X^\bullet, \text{ s.t. for } X^o \in X \text{ and } p \in \mathbb{N}, \\ \Omega_{X, \log}^p(X^o) &:= \langle df_{\alpha_1}/f_{\alpha_1} \wedge \dots \wedge df_{\alpha_p}/f_{\alpha_p}, f_\alpha \in \Gamma(X^o, O_X)^* \rangle \subset \Omega_X^p(X^o), \end{aligned}$$

where $\Omega_X^\bullet := DR(X)(O_X) \in C(X)$ is the De Rham complex and $\Gamma(X^o, O_X)^* \subset \Gamma(X^o, O_X)$ is the subring consisting of invertible elements for the multiplication.

- Let $X \in \text{Var}(k)$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$. If $X \in \text{SmVar}(k)$, then $H_{DR}^j(X) = \mathbb{H}_{et}^j(X, \Omega_X^\bullet)$ since $\Omega^\bullet \in C(\text{SmVar}(k))$ is \mathbb{A}^1 local and admits transfert (see [5]).
- Let $X \in \text{Var}(k)$. Let $X = \cup_{i=1}^s X_i$ an open affine cover. For $I \subset [1, \dots, s]$, we denote $X_I := \cap_{i \in I} X_i$. We get $X_\bullet \in \text{Fun}(P([1, \dots, s]), \text{Var}(k))$. Since quasi-coherent sheaves on affine noetherian schemes are acyclic, we have for each $j \in \mathbb{Z}$, $H_{DR}^j(X) = \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet)$.
- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the full subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). For $X \in \text{AnSp}(\mathbb{C})$, we denote by $X^{et} \subset \text{AnSp}(\mathbb{C})/X$ the full subcategory consisting of objects $U/X = (X, h) \in \text{AnSp}(\mathbb{C})/X$ such that $h : U \rightarrow X$ is an etale morphism. By the Weirstrass preparation theorem (or the implicit function theorem if U and X are smooth), a morphism $r : U \rightarrow X$ with $U, X \in \text{AnSp}(\mathbb{C})$ is etale if and only if it is an isomorphism local. Hence for $X \in \text{AnSp}(\mathbb{C})$, the morphism of site $\pi_X : X^{et} \rightarrow X$ is an isomorphism of site.
- Denote by $\text{CW} \subset \text{Top}$ the full subcategory of CW complexes. Denote by $\text{Diff}(\mathbb{R}) \subset \text{RTop}$ the full subcategory of differentiable (real) manifold.
- Let $k \subset \mathbb{C}$ a subfield. For $X \in \text{Var}(k)$, we denote by

$$\text{an}_X : X_{\mathbb{C}}^{an} \xrightarrow{\sim} X_{\mathbb{C}}^{an, et} \xrightarrow{\text{an}_{X_{\mathbb{C}}^{et}}} X_{\mathbb{C}}^{et} \xrightarrow{\pi_{k/\mathbb{C}}(X^{et})} X^{et},$$

the morphism of site given by the analytical functor.

- Let $k \subset \mathbb{C}$ a subfield. For $X \in \text{Var}(k)$, we denote by

$$\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^\bullet$$

the canonical embedding in $C(X_{\mathbb{C}}^{an})$. It induces the embedding in $C(X_{\mathbb{C}}^{an})$

$$\beta(X) : 2i\pi\mathbb{Z}_{X_{\mathbb{C}}^{an}} \xrightarrow{\iota_{2i\pi\mathbb{Z}/\mathbb{C}, X_{\mathbb{C}}^{an}}} \mathbb{C}_{X_{\mathbb{C}}^{an}} \xrightarrow{\alpha(X)} \Omega_{X_{\mathbb{C}}^{an}}^\bullet$$

For $X \in \text{SmVar}(k)$, $\alpha(X)$ is a quasi-isomorphism by the holomorphic Poincare lemma.

- Let $k \subset \mathbb{C}_p$ a subfield. For $X \in \text{Var}(k)$, we denote by

$$\Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^\bullet := \varprojlim_{n \in \mathbb{N}} \nu_X^* \Omega_{X_{O_{\mathbb{C}_p}/p^n O_{\mathbb{C}_p}}, \log}^\bullet \in C(X_{\mathbb{C}_p}^{pet})$$

and

$$\beta_{pet}(X_{\mathbb{C}_p}) : \underline{\mathbb{Z}}_{p, X_{\mathbb{C}_p}^{pet}} := \varprojlim_{n \in \mathbb{N}} \nu_X^* (\mathbb{Z}/p^n \mathbb{Z})_{X^{pet}} \xrightarrow{\iota_{X_{\mathbb{C}_p}^{pet}}} \Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}^\bullet \xrightarrow{OL_{X_{\mathbb{C}_p}^{pet}, \mathcal{O}}} \varprojlim_{n \in \mathbb{N}} \nu_X^* \Omega_{X_{O_{\mathbb{C}_p}/p^n O_{\mathbb{C}_p}}^{pet}}^\bullet = \Omega_{X_{\mathbb{C}_p}^{pet}}^\bullet$$

the canonical embedding in $C(X_{\mathbb{C}_p}^{pet})$, where the last equality is standard. We get the following commutative diagram in $C(X_{\mathbb{C}_p}^{pet})$

$$\begin{array}{ccc} OL_{X^{pet}} : \Omega_{X_{\mathbb{C}_p}^{pet}, \log}^\bullet & \xrightarrow{T(\nu_X^*, \varprojlim_{n \in \mathbb{N}} (\Omega_{X_{O_{\mathbb{C}_p}/p^n O_{\mathbb{C}_p}}^{pet}))} & \Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}^\bullet & \xrightarrow{OL_{X_{\mathbb{C}_p}^{pet}, \mathcal{O}}} & \Omega_{X_{\mathbb{C}_p}^{pet}}^\bullet \\ & & \uparrow \iota_{X_{\mathbb{C}_p}^{pet}} & \nearrow \beta_{pet}(X_{\mathbb{C}_p}) & \\ \underline{\mathbb{Z}}_{p, X} := \varprojlim_{n \in \mathbb{N}} \nu_X^* (\mathbb{Z}/p^n \mathbb{Z})_{X^{pet}} & & & & \end{array}$$

We may write for short

$$OL_X := OL_{X_{\mathbb{C}_p}^{et}, \mathcal{O}} : \Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}^\bullet \hookrightarrow \Omega_{X_{\mathbb{C}_p}^{pet}}^\bullet$$

the embedding in $C(X_{\mathbb{C}_p}^{pet})$. For $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)$, we consider $w(\alpha) := H^j \beta_{pet}(X_{\mathbb{C}_p})(\alpha) \in H_{DR}^j(X_{\mathbb{C}_p})$.

- For $X \in \text{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension d , which is the free abelian group generated by irreducible closed subsets of codimension d .
 - For $X \in \text{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X, \bullet) \subset \mathcal{Z}^d(X \times \square^\bullet)$ the Bloch cycle complex which is the subcomplex which consists of algebraic cycles which intersect \square^\bullet properly.
 - For $X \in \text{Var}(k)$ irreducible and $d, n \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X, n)_{hom}^{\partial=0} \subset \mathcal{Z}^d(X, \bullet)^{\partial=0}$ the the subabelian group consisting of algebraic cycles which are homologically trivial.
 - For $f : X \rightarrow S$ a dominant morphism with $S, X \in \text{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X, \bullet)^f \subset \mathcal{Z}^d(X, \bullet)$ the subcomplex consisting of algebraic cycles which are flat over S .
 - For $f : X \rightarrow S$ a dominant morphism with $S, X \in \text{Var}(k)$ irreducible and $d, n \in \mathbb{N}$ we denote by $\mathcal{Z}^d(X, n)_{f, hom}^{f, \partial=0} \subset \mathcal{Z}^d(X, n)^{f, \partial=0}$ the subabelian group consisting of algebraic cycles which are flat over S and homological trivial on fibers.
- We denote $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$ (with boundary). For $X \in \text{Top}$ and R a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$. In particular, we get by functoriality the complex

$$C_{X, R \text{ sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We will consider the canonical embedding

$$C_{\text{sing}}^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(X) : C_{\text{sing}}^*(X, 2i\pi\mathbb{Z}) \hookrightarrow C_{\text{sing}}^*(X, \mathbb{C}), \alpha \mapsto \alpha \otimes 1$$

whose image consists of cochains $\alpha \in C_{\text{sing}}^j(X, \mathbb{C})$ such that $\alpha(\gamma) \in 2i\pi\mathbb{Z}$ for all $\gamma \in \mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)$. We get by functoriality the embedding in $C(X)$

$$\begin{aligned} C_{\text{sing}}^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}, X} : C_{X, 2i\pi\mathbb{Z}, \text{sing}}^* &\hookrightarrow C_{X, \mathbb{C}, \text{sing}}^* \\ (U \subset X) \mapsto (C_{\text{sing}}^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(U) : C_{\text{sing}}^*(U, 2i\pi\mathbb{Z}) &\hookrightarrow C_{\text{sing}}^*(U, \mathbb{C})) \end{aligned}$$

We recall we have

- For $X \in \text{Top}$ locally contractile, e.g. $X \in \text{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \rightarrow C_{X, R \text{ sing}}^*$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- Let $X \in \text{AnSm}(\mathbb{C})$. Let $X = \cup_{i=1}^r \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0, 1)^d$. Since a convex open subset of \mathbb{C}^d is biholomorphic to an open ball we have $\mathbb{D}_I := \cap_{i \in I} \mathbb{D}_i \simeq D(0, 1)^d$ (where d is the dimension of a connected component of X). We get $\mathbb{D}_\bullet \in \text{Fun}(P([1, \dots, r]), \text{AnSm}(\mathbb{C}))$.
- For k a field, we denote by $\text{Vect}(k)$ the category of vector spaces and $\text{Vect}_{\text{fil}}(k)$ the category of filtered vector spaces. Let $k \subset K$ a field extension of field of characteristic zero.
 - For $(V, F) \in \text{Vect}_{\text{fil}}(k)$, we get a filtered K vector space $(V \otimes_k K, F) \in \text{Vect}_{\text{fil}}(K)$ by $F^j(V \otimes_k K) := (F^j V) \otimes_k K$. In this case, we say that the filtration F on $V \otimes_k K$ is defined over k .
 - For $(V', F) \in \text{Vect}_{\text{fil}}(K)$ and $h : V \otimes_k K \xrightarrow{\sim} V'$ and isomorphism of K vector space, we get $(V, F_h) \in \text{Vect}_{\text{fil}}(k)$ by $F_h^j V := h^{-1}(F^j V') \cap V$ (considering the canonical embedding $n : V \hookrightarrow V \otimes_k K$, $n(v) := v \otimes 1$).
 - For $(V, F) \in \text{Vect}_{\text{fil}}(k)$, we have $F^j(V \otimes_k K) \cap V = F^j V$.
 - For $(V', F) \in \text{Vect}_{\text{fil}}(K)$ and $h : V \otimes_k K \xrightarrow{\sim} V'$ and isomorphism of K vector space, we have $h((F_h^j V) \otimes_k K) \subset F^j V'$. Of course this inclusion is NOT an equality in general. The filtration F on V' is NOT defined over k in general.
- We also consider
 - Top_2 the category whose objects are couples (X, Y) with $X \in \text{Top}$ and $Y \subset X$ a subset and whose set of morphisms $\text{Hom}((X', Y'), (X, Y))$ consists of $f : X' \rightarrow X$ continuous such that $Y' \subset f^{-1}(Y)$ (i.e. $f(Y') \subset Y$),
 - RTop_2 the category whose objects are couples (X, Y) with $X = (X, O_X) \in \text{RTop}$ and $Y \subset X$ a subset and whose set of morphisms $\text{Hom}((X', Y'), (X, Y))$ consists of $f : X' \rightarrow X$ of ringed spaces such that $Y' \subset f^{-1}(Y)$,
 - Top^2 the category whose objects are couples (X, Z) with $X \in \text{Top}$ and $Z \subset X$ a closed subset and whose set of morphisms $\text{Hom}((X', Z'), (X, Z))$ consists of $f : X' \rightarrow X$ continuous such that $f^{-1}(Z) \subset Z'$ (i.e. $f(X' \setminus Z') \subset X \setminus Z$), in particular we have the canonical functor $\text{Top}^2 \rightarrow \text{Top}_2$, $(X, Z) \mapsto (X, X \setminus Z)$,
 - RTop^2 the category whose objects are couples (X, Z) with $X = (X, O_X) \in \text{RTop}$ and $Z \subset X$ a closed subset and whose set of morphisms $\text{Hom}((X', Z'), (X, Z))$ consists of $f : X' \rightarrow X$ of ringed spaces such that $f^{-1}(Z) \subset Z'$, in particular we have the canonical functor $\text{RTop}^2 \rightarrow \text{RTop}_2$, $(X, Z) \mapsto (X, X \setminus Z)$.

A (generalized) cohomology theory is in particular a functor $H^* : \text{Top}_2 \rightarrow \text{GrAb}$, e.g singular cohomology

$$H_{\text{sing}}^* : \text{Top}^2 \rightarrow \text{Gr Ab}, (X, Y) \mapsto H_{\text{sing}}^*(X, Y, R).$$

where R is a commutative ring. It restrict to a functor $H^* : \text{Top}^2 \rightarrow \text{GrAb}$, $(X, Z) \mapsto H_Z^*(X) := H^*(X, X \setminus Z)$.

- Denote $\text{Sch}^2 \subset \text{RTop}^2$ the subcategory whose objects are couples (X, Z) with $X = (X, O_X) \in \text{Sch}$ and $Z \subset X$ a closed subset and whose set of morphisms $\text{Hom}((X', Z'), (X, Z))$ consists of $f : X' \rightarrow X$ of locally ringed spaces such that $f^{-1}(Z) \subset Z'$.
- Let k a field of characteristic zero. Denote $\text{SmVar}^2(k) \subset \text{Var}^2(k) \subset \text{Sch}^2/k$ the full subcategories whose objects are (X, Z) with $X \in \text{Var}(k)$, resp. $X \in \text{SmVar}(k)$, and $Z \subset X$ is a closed subset, and whose morphisms $\text{Hom}((X', Z') \rightarrow (X, Z))$ consists of $f : X' \rightarrow X$ of schemes over k such that $f^{-1}(Z) \subset Z'$.
- Let k a field of characteristic zero. Let

$$H^* : \text{SmVar}^2(k) \rightarrow \text{Gr AbCat}, (X, Z) \mapsto H_Z^*(X)$$

a mixed Weil cohomology theory in sense of [9] (e.g. (filtered) De Rham, etale or Betti cohomology, Hodge or p adic realization). For $X \in \text{SmVar}(k)$ and $Z \subset X$ a closed subset, we denote

$$H_Z^*(X)^0 := \ker(H_Z^*(X) \rightarrow H^*(X)).$$

For $X \in \text{SmVar}(k)$ and $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0}$, we consider the subobject $H^{2d-1}(U)^{[Z]} \subset H^{2d-1}(U)$ where $j : U := (X \times \square^n) \setminus |Z| \hookrightarrow X \times \square^n$ is the complementary open subset, given by the pullback by $H_Z^{2d}(X \times \square^n)^{[Z]} := [Z] \subset H_Z^{2d}(X \times \square^n)^0$

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{2d-1}(X \times \square^n) = H^{2d-n-1}(\dot{X}^*) & \longrightarrow & H^{2d-1}(U) & \xrightarrow{\partial} & H_Z^{2d}(X \times \square^n)^0 \longrightarrow 0 \\ & & \uparrow = & & \uparrow \subset & & \uparrow \subset \\ 0 & \longrightarrow & H^{2d-1}(X \times \square^n) = H^{2d-n-1}(\dot{X}^*) & \longrightarrow & H^{2d-1}(U)^{[Z]} & \xrightarrow{\partial} & H_Z^{2d}(X \times \square^n)^{[Z]} := [Z] \longrightarrow 0 \end{array}$$

of the first row exact sequence. In particular the second row is also an exact sequence.

- We denote by $\log \text{Sch}$ the category of log schemes whose objects are couples $(X, M) := (X, M, \alpha)$ where $X = (X, O_X) \in \text{Sch}$, $M \in \text{Shv}(X)$ is a sheaf of monoid and $\alpha : M \rightarrow O_X$ is a morphism of sheaves of monoid. In particular we have a canonical functor

$$\text{Sch}^2 \rightarrow \log \text{Sch}, (X, Z) \mapsto (X, M_Z), M_Z := (f \in O_X \text{ s.t. } f|_{X \setminus Z} \in O_{X \setminus Z}^* \subset O_X)$$

Let k a field of characteristic zero. We denote by $\log \text{Var}(k) \subset \log \text{Sch}/k$ the full subcategory of log varieties.

- Let p be a prime number. For K a p adic field (i.e. a finite extension of \mathbb{Q}_p), we consider the canonical functor of Huber (see [5])

$$\mathcal{R} : \text{Var}(K) \rightarrow \text{Sch}/O_K, X \mapsto \mathcal{R}(X) := (X, O_X^o) = X^o, O_X^o := (f \in O_X, \text{ s.t. } |f(x)|_p \leq 1 \forall x \in X(O_K))$$

where $O_K \subset K$ is the ring of integers of K .

- For k a field, we denote by $\text{Vect}(k) = \text{Mod}(k)$ the category of vector spaces over k and $C(k) := C(\text{Vect}(k))$ the category of complexes of vector spaces over k , by $\text{Vect}_{\text{fil}}(k)$ the category of filtered vector spaces over k and $C_{\text{fil}}(k) := C_{\text{fil}}(\text{Vect}(k))$ the category of filtered complexes of vector spaces over k .
- We assume all field have a transcendence basis at most countable, so that field of characteristic zero admit embeddings in \mathbb{C} and in \mathbb{C}_p for each prime number $p \in \mathbb{N}$.

We have the followings facts :

- Let k a field of characteristic zero. Denote $G := \text{Gal}(\bar{k}/k)$ its absolute Galois group. Then the functor

$$\Gamma(\bar{k})(-) : \text{PSh}(k^{et}) \rightarrow \text{Mod}(\bar{k}, G), F \mapsto \Gamma(\bar{k}, F)$$

is an equivalence of category whose inverse is

$$G(-) : \text{Mod}(\bar{k}, G) \rightarrow \text{PSh}(k^{et}), V \mapsto G(V) := V := ((k'/k) \mapsto V^{\text{Aut}(k'/k)}).$$

In particular, for each $V \in \text{Mod}(\bar{k}, G)$ and $j \in \mathbb{Z}$, we get an isomorphism

$$H^j G(V) : H^j(G, V) \xrightarrow{\sim} \text{Ext}_G^j(\bar{k}, V).$$

- For $X \in \text{Sch}$, we have

- $\nu_X^* \Omega_{X^{et}}^\bullet = \Omega_{X^{pet}}^\bullet$ and hence isomorphisms

$$H^* \Gamma(X, k) : \mathbb{H}_{et}^*(X, \Omega_{X^{et}}^\bullet) \xrightarrow{\sim} \mathbb{H}_{pet}^*(X, \Omega_{X^{pet}}^\bullet),$$

where

$$k := k \circ \text{ad}(\nu_X^*, \nu_{X^*})(-) : E_{et}(\Omega_{X^{et}}^\bullet) \rightarrow \nu_{X^*} E_{pet}(\Omega_{X^{pet}}^\bullet)$$

is the canonical map in $C(X^{et})$ (which is a quasi-isomorphism),

- $\nu_X^* \Omega_{X^{et}, \log}^\bullet = \Omega_{X^{pet}, \log}^\bullet$ and hence isomorphisms

$$H^* \Gamma(X, k) : \mathbb{H}_{et}^*(X, \Omega_{X^{et}, \log}^\bullet) \xrightarrow{\sim} \mathbb{H}_{pet}^*(X, \Omega_{X^{pet}, \log}^\bullet)$$

where

$$k := k \circ \text{ad}(\nu_X^*, \nu_{X^*})(-) : E_{et}(\Omega_{X^{et}, \log}^\bullet) \rightarrow \nu_{X^*} E_{pet}(\Omega_{X^{pet}, \log}^\bullet)$$

is the canonical map in $C(X^{et})$ (which is a quasi-isomorphism).

- Let k a field of characteristic zero. The complex of presheaves

$$\Omega_{\log}^\bullet \in C(\text{SmVar}(k)), X \mapsto \Omega_{X, \log}^\bullet(X)$$

is \mathbb{A}^1 local and admits transfers. Hence for $X \in \text{SmVar}(k)$

$$H^* \Gamma(X, k) : \mathbb{H}^*(X, \Omega_{X, \log}^\bullet) \xrightarrow{\sim} \mathbb{H}_{et}^*(X, \Omega_{X^{et}, \log}^\bullet)$$

are isomorphisms.

- Let $k \subset K$ a field extension.

- Let $X \in \text{Var}(k)$. We have then the canonical isomorphism in $C_{\text{Aut}(K/k), \text{fil}}(X_K)$

$$w(k/K) : (\Omega_X^\bullet \otimes_k K, F_b) \xrightarrow{\sim} (\Omega_{X_K}^\bullet, F_b)$$

given by the universal property of derivation of a ring.

- Let $X \in \text{SmVar}(k)$. Let $\bar{X} \in \text{PSmVar}(k)$ a smooth compactification of X with $D := \bar{X} \setminus X$ a normal crossing divisor. We have then the canonical isomorphism in $C_{\text{Aut}(K/k), \text{fil}}(\bar{X}_K)$

$$w(k/K) : (\Omega_{\bar{X}}^\bullet(\log D) \otimes_k K, F_b) \xrightarrow{\sim} (\Omega_{\bar{X}_K}^\bullet(\log D), F_b)$$

given by the preceding point. In particular, we get for all $j, l \in \mathbb{Z}$,

$$* F^l H^j w(k/K) : F^l H_{DR}^j(X) \otimes_k K \xrightarrow{\sim} F^l H_{DR}^j(X_K),$$

$$* H^j w(k/K) : H_{DR}^j(X) \xrightarrow{\sim} H_{DR}^j(X_K)^G.$$

- An affine scheme $U \in \text{Sch}$ is said to be w -contractible if any faithfully flat weakly etale map $V \rightarrow U$, $V \in \text{Sch}$, admits a section. We will use the facts that (see [2]):

- Any scheme $X \in \text{Sch}$ admits a pro-etale cover $(r_i : X_i \rightarrow X)_{i \in I}$ with for each $i \in I$, X_i a w -contractible affine scheme and $r_i : X_i \rightarrow X$ a weakly etale map. For $X \in \text{Var}(k)$ with k a field, we may assume I finite since the topological space X is then quasi-compact.
- If $U \in \text{Sch}$ is a w -contractible affine scheme, then for any sheaf $F \in \text{Shv}(U^{pet})$, $H_{pet}^i(U, F) = 0$ for $i \neq 0$ since $\Gamma(U, -)$ is an exact functor.

Let $\sigma : k \hookrightarrow \mathbb{C}$ a subfield of finite type over \mathbb{Q} . Consider $k \subset \bar{k} \subset \mathbb{C}$ the algebraic closure of k . Let $X \in \text{Var}(k)$. Let p a prime number. Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding. We have then for k'/k a field extension and $\sigma' : k' \hookrightarrow \mathbb{C}$ and $\sigma'_p : k' \hookrightarrow \mathbb{C}_p$ field embedding such that $\sigma'|_k = \sigma$ and $\sigma'_{p|k} = \sigma_p$, the following diagram in $C(\mathbb{N} \times X^{et})$

$$\begin{array}{ccccc}
& & \text{an}_{X^*} \mathbb{Z}_{p, X_{\mathbb{C}}^{an}} \xleftarrow{l_{p, X_{\mathbb{C}}^{an}} = (/p^*)} \text{an}_{X^*} \mathbb{Z}_{X_{\mathbb{C}}^{an}} \xrightarrow{\text{an}_{X^*} \beta(X)} \text{an}_{X^*} \Omega_{X_{\mathbb{C}}^{an}}^{\bullet} & & \\
& \nearrow \text{ad}(\text{an}_{X^*}, \text{an}_{X^*})(\mathbb{Z}_{p, X_{\mathbb{C}}^{et}}) & & & \nearrow \Omega(\text{an}_X) \\
& \mathbb{Z}_{p, X_{\bar{k}}^{et}} \xrightarrow{\text{ad}(\nu_X, \nu_{X^*})(\mathbb{Z}_{p, X_{\bar{k}}^{et}})} \nu_{X_{\bar{k}}^*} \mathbb{Z}_{p, X_{\bar{k}}^{pet}} \xleftarrow{l_{p, X_{\bar{k}}}} \nu_{X_{\bar{k}}^*} \mathbb{Z}_{p, X_{\bar{k}}^{pet}} & & & \\
& \nearrow \text{ad}(\pi_{\bar{k}/\mathbb{C}_p}(X_{\bar{k}}^{pet})^*, \pi_{\bar{k}/\mathbb{C}_p}(X_{\bar{k}}^{pet})^*)(\mathbb{Z}_{p, X_{\bar{k}}^{pet}}) & & & \nearrow \Omega(\pi_{k'/\mathbb{C}_p}(X_{k'})) \\
& & \Omega_{X_{k'}^{et}, \log}^{\bullet} \xrightarrow{OL_{X_{k'}^{et}}} \Omega_{X_{k'}^{et}}^{\bullet} & & \\
& \searrow T(\nu_X, \varprojlim)(-) \circ \Omega(\pi_{k'/\mathbb{C}_p}(X_{k'})) & \downarrow \Omega(\pi_{k'/\mathbb{C}_p}(X_{k'})) & & \downarrow \Omega(\pi_{k'/\mathbb{C}_p}(X_{k'})) \\
& & \mathbb{Z}_{p, X_{\mathbb{C}_p}^{pet}} \xrightarrow{l_{X_{\mathbb{C}_p}^{pet}}} \Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}^{\bullet} \xrightarrow{OL_{X_{\mathbb{C}_p}^{pet}}} \Omega_{X_{\mathbb{C}_p}^{pet}}^{\bullet} & & \\
& & \downarrow \Omega(\pi_{k'/\mathbb{C}_p}(X_{k'})) & & \downarrow \Omega(\pi_{k'/\mathbb{C}_p}(X_{k'}))
\end{array}$$

with as above

$$\text{an}_X : X_{\mathbb{C}}^{an} \simeq X_{\mathbb{C}}^{an, et} \xrightarrow{\text{an}_X} X_{\mathbb{C}}^{et} \xrightarrow{\pi_{\bar{k}/\mathbb{C}}(X_{\bar{k}}^{et})} X_{k'}^{et}, \quad \text{an}_X : X_{\mathbb{C}}^{an} \simeq X_{\mathbb{C}}^{an, et} \xrightarrow{\text{an}_X} X_{\mathbb{C}}^{et} \xrightarrow{\pi_{k'/\mathbb{C}}(X_{k'}^{et})} X_{k'}^{et}$$

We also have for $X \in \text{Var}(k)$, the canonical map

$$\begin{aligned}
H^j \gamma(X) : H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{Z}) &\xrightarrow{H^j(/p^*)} H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{Z}_p) \xrightarrow{\sim} H^j(X_{\mathbb{C}}^{an}, \mathbb{Z}_p) \\
&\xrightarrow{(\text{an}_X^*)^{-1}} H_{\text{et}}^j(X_{\mathbb{C}}, \mathbb{Z}_p) \xrightarrow{(\pi_{\bar{k}/\mathbb{C}}(X)^*)^{-1}} H_{\text{et}}^j(X_{\bar{k}}, \mathbb{Z}_p)
\end{aligned}$$

for each $j \in \mathbb{Z}$, where $H^j(\text{an}_X^*)$ is an isomorphism by the comparison theorem between etale cohomology and Betti cohomology with torsion coefficients (see SGA4).

We make the following key definition :

Definition 1. (i) Let $p \in \mathbb{N}$ a prime number. Let $k \subset \mathbb{C}_p$ a subfield. Let $X \in \text{SmVar}(k)$. Take a compactification $j : X \hookrightarrow \bar{X}$ with $\bar{X} \in \text{PSmVar}(k)$ such that $D := \bar{X} \setminus X \subset \bar{X}$ is a normal crossing divisor. Then for each $j \in \mathbb{Z}$,

$$H_{DR}^j(X_{\mathbb{C}_p}) = H^j \Gamma(\bar{X}_{\mathbb{C}_p}, E_{zar}(\Omega_{\bar{X}_{\mathbb{C}_p}}^{\bullet}(\log D_{\mathbb{C}_p}), F_b)) = H^j \Gamma(\bar{X}_{\mathbb{C}_p}, E_{pet}(\Omega_{\bar{X}_{\mathbb{C}_p}}^{\bullet}(\log D_{\mathbb{C}_p}), F_b)) \in \text{Vect}_{fil}(\mathbb{C}_p)$$

By the complex case which follows from the Hodge decomposition, we see after taking an isomorphism $\sigma : \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$ that the spectral sequence associated to $\Gamma(\bar{X}_{\mathbb{C}_p}, E_{zar}(\Omega_{\bar{X}_{\mathbb{C}_p}}^{\bullet}(\log D_{\mathbb{C}_p}), F_b)) \in C_{fil}(\mathbb{C}_p)$ is E_1 degenerate. However there is no canonical splitting of the filtration on $H_{DR}^j(X_{\mathbb{C}_p})$ (it depend on the choice of a basis of this \mathbb{C}_p vector space, or on the choice of such an isomorphism σ). We then consider, for each $j \in \mathbb{Z}$, a (non canonical) splitting

$$\theta_j(X) : H_{DR}^j(X_{\mathbb{C}_p}) \xrightarrow{\sim} \bigoplus_{0 \leq l \leq j} H_{zar}^l(X_{\mathbb{C}_p}, \Omega_{\bar{X}_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p})) = \bigoplus_{0 \leq l \leq j} H_{pet}^l(X_{\mathbb{C}_p}, \Omega_{\bar{X}_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p}))$$

in $\text{Vect}(\mathbb{C}_p)$. We then define the following map in $\text{Vect}(\mathbb{Z}_p)$

$$\begin{aligned}
OL_X^{\theta_j} &:= \theta_j(X)^{-1} \circ (\bigoplus_{0 \leq l \leq j} H^l OL_X^{l-j}) : \\
\mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{\bar{X}_{\mathbb{C}_p}, \log, \mathcal{O}}^{\bullet} \otimes \mathbb{Z}_p) &= \bigoplus_{0 \leq l \leq j} H_{pet}^l(X_{\mathbb{C}_p}, \Omega_{\bar{X}_{\mathbb{C}_p}, \log, \mathcal{O}}^{j-l} \otimes \mathbb{Z}_p) \\
\bigoplus_{0 \leq l \leq j} H^l OL_X^{l-j} &\rightarrow \bigoplus_{0 \leq l \leq j} H_{pet}^l(X_{\mathbb{C}_p}, \Omega_{\bar{X}_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p})) \xrightarrow{\theta_j(X)^{-1}} H_{DR}^j(X_{\mathbb{C}_p}),
\end{aligned}$$

where the equality follows from the fact that the differential of $\Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^\bullet$ vanishes (all the logarithmic forms are closed) so that we get a canonical splitting. We then have the commutative diagram in $\text{Vect}(\mathbb{Z}_p)$

$$\begin{array}{ccc} H_{\text{pet}}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p) & \xrightarrow{H^j \beta_{\text{pet}}(X_{\mathbb{C}_p})} & H_{\text{DR}}^j(X_{\mathbb{C}_p}) \xrightarrow{\theta_j(X)^{-1}} \oplus_{0 \leq l \leq j} H_{\text{pet}}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p})) . \\ & & \uparrow \text{OL}_X^{\theta_j} \nearrow \oplus_{0 \leq l \leq j} H^l \text{OL}_X^{l-j} \\ \mathbb{H}_{\text{pet}}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^\bullet \otimes \mathbb{Z}_p) & \xrightarrow{\cong} & \oplus_{0 \leq l \leq j} H_{\text{pet}}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^{j-l} \otimes \mathbb{Z}_p) \end{array}$$

Note that we also have the composition in $\text{Vect}(\mathbb{Z}_p)$

$$H^j \beta_{\text{pet}}(X_{\mathbb{C}_p}) : H_{\text{pet}}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p) \xrightarrow{H^j \iota_{X_{\mathbb{C}_p}}} \mathbb{H}_{\text{pet}}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^\bullet \otimes \mathbb{Z}_p) \xrightarrow{H^j \text{OL}_X} H_{\text{DR}}^j(X_{\mathbb{C}_p})$$

and that $H^j \text{OL}_X$ is NOT equal to $\text{OL}_X^{\theta_j}$. In fact we have for $w \in H_{\text{pet}}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^{j-l} \otimes \mathbb{Z}_p)$

$$\pi_l \circ \theta_j(X) \circ H^j \text{OL}_X(w) = H^l \text{OL}_X^{j-l}(w)$$

where $\pi_l : \oplus_{0 \leq l \leq j} H_{\text{pet}}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p})) \rightarrow H_{\text{pet}}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p}))$ is the projection.

- (ii) Let k a field of finite type over \mathbb{Q} . Let $X \in \text{SmVar}(k)$. Take a compactification $j : X \hookrightarrow \bar{X}$ with $\bar{X} \in \text{PSmVar}(k)$ such that $D := \bar{X} \setminus X \subset \bar{X}$ is a normal crossing divisor. Then for each $j \in \mathbb{Z}$, Let $p \in \mathbb{N} \setminus \delta(k, (X, D))$ a prime number (any but finitely many) and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding so that if $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ is the p adic completion of k with respect to σ_p then $\hat{k}_{\sigma_p} = \text{Frac}(W(O_{\hat{k}_{\sigma_p}}/p))$ and $(X_{O_{\hat{k}_{\sigma_p}}}, D_{O_{\hat{k}_{\sigma_p}}})$ has semi-stable reduction modulo p . Then the Frobenius endomorphism ϕ acts on $(X_{\hat{k}_{\sigma_p}}, D_{\hat{k}_{\sigma_p}})$. Then, for $j \in \mathbb{Z}$, a (non canonical) splitting (see (i))

$$\theta_j(X) : H_{\text{DR}}^j(X_{\mathbb{C}_p}) \xrightarrow{\sim} \oplus_{0 \leq l \leq j} H_{\text{zar}}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p})) = \oplus_{0 \leq l \leq j} H_{\text{pet}}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p}))$$

in $\text{Vect}(\mathbb{C}_p)$ is said to be invariant by Frobenius if $\theta_j(X)$ is defined over \hat{k}_{σ_p} , that is $\theta_j(X) = \theta_j(X)_0 \otimes_{\hat{k}_{\sigma_p}} \mathbb{C}_p$, and $\phi \circ \theta_j(X) = \theta_j(X) \circ \phi$. A splitting which is invariant by Frobenius is obtained by taking a basis of $H_{\text{DR}}^j(X_{\hat{k}_{\sigma_p}})$ which is invariant by Frobenius. We denote, for $j \in \mathbb{Z}$,

$$\theta_j(X) = \theta_j^\phi(X) : H_{\text{DR}}^j(X_{\mathbb{C}_p}) \xrightarrow{\sim} \oplus_{0 \leq l \leq j} H_{\text{zar}}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p})) = \oplus_{0 \leq l \leq j} H_{\text{pet}}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p}))$$

in $\text{Vect}(\mathbb{C}_p)$ a splitting which is invariant by Frobenius and

$$\begin{aligned} \text{OL}_X^{\theta_j^\phi} &:= \theta_j^\phi(X)^{-1} \circ (\oplus_{0 \leq l \leq j} H^l \text{OL}_X^{l-j}) : \\ \mathbb{H}_{\text{pet}}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^\bullet \otimes \mathbb{Z}_p) &= \oplus_{0 \leq l \leq j} H_{\text{pet}}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^\bullet \otimes \mathbb{Z}_p) \\ \xrightarrow{\oplus_{0 \leq l \leq j} H^l \text{OL}_X^{l-j}} &\oplus_{0 \leq l \leq j} H_{\text{pet}}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p})) \xrightarrow{\theta_j^\phi(X)^{-1}} H_{\text{DR}}^j(X_{\mathbb{C}_p}), \end{aligned}$$

the associated morphism.

3 Integral complex and p -adic periods of a smooth algebraic variety over a field k of finite type over \mathbb{Q}

3.1 Complex integral periods

Let k a field of characteristic zero.

Let $X \in \text{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^s X_i$ an open affine cover. We have for $\sigma : k \hookrightarrow \mathbb{C}$ an embedding, the evaluation period embedding map which is the morphism of bi-complexes

$$\begin{aligned} ev(X) \bullet \bullet : \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) &\rightarrow \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C}, \\ w_I^l \in \Gamma(X_I, \Omega_{X_I}^l) &\mapsto (ev(X)_I^l(w_I^l) : \phi_I^l \in \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^l, X_{\mathbb{C}, I}^{an})^\vee \otimes \mathbb{C} \mapsto ev_I^l(w_I^l)(\phi_I^l) := \int_{\mathbb{I}^l} \phi_I^{l*} w_I^l) \end{aligned}$$

given by integration. By taking all the affine open cover $(j_i : X_i \hookrightarrow X)$ of X , we get for $\sigma : k \hookrightarrow \mathbb{C}$, the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X) \bullet \bullet : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C}$$

It induces in cohomology, for $j \in \mathbb{Z}$, the evaluation period map

$$H^j ev(X) = H^j ev(X) \bullet \bullet : H_{DR}^j(X) = H^j \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{C}) = H^j(\text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

Remark 1. We also have for $\sigma : k \hookrightarrow \mathbb{C}$ the composition

$$\bar{ev}(X) \bullet \bullet : \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \xrightarrow{ev(X) \bullet \bullet} \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C} \xrightarrow{\mathbb{Z}(X)(i) \circ \text{an}^{-, -}} \text{Hom}_{\text{Fun}(\Delta^\bullet, \text{Var}(k))}(\mathbb{D}_{k, et}^\bullet, X_\bullet)^\vee \otimes \mathbb{C}$$

where $i : I^\bullet \mathbb{D}_{k, et}^\bullet$ is the embedding, which is given by integration : for $w_I^l \in \Gamma(X_I, \Omega_{X_I}^l)$ and $\phi_I^l \in \text{Hom}_{\text{Fun}(\Delta^\bullet, \text{Var}(k))}(\mathbb{D}_{k, et}^l, X_I)$,

$$\bar{ev}_I^l(w_I^l)(\phi_I^l) = \int_{\mathbb{I}^l} \phi_I^{l, an*} w_I^l.$$

Let $X \in \text{SmVar}(k)$. Note that

$$H^* ev(X_{\mathbb{C}}) : H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{H^* R\Gamma(X_{\mathbb{C}}^{an}, E_{\text{Zar}}(\Omega(\text{an} X)))} H_{DR}^*(X_{\mathbb{C}}^{an}) \xrightarrow{H^* R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X))} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and the quasi-isomorphism $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^\bullet$ in $C(X_{\mathbb{C}}^{an})$. Hence,

$$H^* ev(X) = H_{DR}^*(X) \xrightarrow{\Omega(\pi_{k/\mathbb{C}}(X))} H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{H^* ev(X_{\mathbb{C}})} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is injective. The elements of the image $H^* ev(X)(H_{DR}^*(X)) \subset H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$ are the periods of X .

Let $X \in \text{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^s X_i$ an open affine cover with $X_i := X \setminus D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding and $X_{\mathbb{C}}^{an} = \cup_{i=1}^r \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0, 1)^d$. Since a convex open subset of \mathbb{C}^d is biholomorphic to an open ball we have $\mathbb{D}_I := \cap_{i \in I} \mathbb{D}_i \simeq D(0, 1)^d$ (where d is the dimension of a connected component of X). Denote by $j_\bullet : X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet \rightarrow X_{\bullet, \mathbb{C}}^{an}$ is the open embeddings. We then have the period morphism of tri-complexes

$$\begin{aligned} ev(X_{\mathbb{C}}^{an}) \bullet \bullet \bullet : \Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet) &\rightarrow \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C}, \\ w_{I, J}^l \in \Gamma(X_{I, \mathbb{C}}^{an} \cap \mathbb{D}_J, \text{an}_X^* \Omega_{X_I}^l) &\mapsto \\ (ev_{I, J}^l(w_{I, J}^l) : \phi_{I, J}^l \in \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^l, X_{\mathbb{C}, I}^{an} \cap \mathbb{D}_J)^\vee \otimes \mathbb{C} &\mapsto ev_{I, J}^l(w_{I, J}^l)(\phi_{I, J}^l) := \int_{\mathbb{I}^l} \phi_{I, J}^{l*} w_{I, J}^l) \end{aligned}$$

given by integration. We have then the factorization

$$\begin{aligned} H^j \text{ev}(X) : H^j_{DR}(X) &:= \mathbb{H}^j(X, \Omega_X^\bullet) = \mathbb{H}^j_{\text{et}}(X, \Omega_{X^{\text{et}}}^\bullet) \xrightarrow{H^j \Omega(\pi_{k/\mathbb{C}}(X))} \\ H^j_{DR}(X_{\mathbb{C}}) &:= \mathbb{H}^j(X_{\mathbb{C}}, \Omega_X^\bullet) = \mathbb{H}^j_{\text{et}}(X_{\mathbb{C}}, \Omega_{X^{\text{et}}}^\bullet) \xrightarrow{j_{\bullet}^* \circ \text{an}_{X_{\bullet}}^*} H^j \Gamma(X_{\bullet, \mathbb{C}}^{\text{an}} \cap \mathbb{D}_{\bullet}, \Omega_{X_{\bullet, \mathbb{C}}^{\text{an}}}^\bullet) \\ &\xrightarrow{H^j \text{ev}(X_{\mathbb{C}}^{\text{an}})_{\bullet, \bullet}} H^j_{\text{sing}}(X_{\mathbb{C}}^{\text{an}} \cap \mathbb{D}_{\bullet}, \mathbb{C}) = H^j(\text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{\text{an}} \cap \mathbb{D}_{\bullet})^\vee \otimes \mathbb{C}). \end{aligned}$$

where for the left hand side, the first equality follows from the fact that $\Omega^\bullet \in C(\text{SmVar}(k))$ is \mathbb{A}^1 local and admits transferts, and the equality of the right hand side follows from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces.

Remark 2. Let $X \in \text{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^s X_i$ an open affine cover with $X_i := X \setminus D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding and $X_{\mathbb{C}}^{\text{an}} = \cup_{i=1}^r \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0, 1)^d$. Since a convex open subset of \mathbb{C}^d is biholomorphic to an open ball we have $\mathbb{D}_I := \cap_{i \in I} \mathbb{D}_i \simeq D(0, 1)^d$ (where d is the dimension of a connected component of X). Denote by $j_{\bullet} : X_{\bullet, \mathbb{C}}^{\text{an}} \cap \mathbb{D}_{\bullet} \rightarrow X_{\bullet, \mathbb{C}}^{\text{an}}$ the open embeddings. Then,

$$j_{\bullet}^* \circ \text{an}_{X_{\bullet}}^* := \Omega(j_{\bullet} \circ \text{an}_{X_{\bullet}}) : \Gamma(X_{\bullet, \mathbb{C}}, \Omega_{X^{\text{et}}}^\bullet) \rightarrow \Gamma(X_{\bullet, \mathbb{C}}^{\text{an}} \cap \mathbb{D}_{\bullet}, \Omega_{X_{\bullet, \mathbb{C}}^{\text{an}}}^\bullet)$$

is a quasi-isomorphism by the Grothendieck comparison theorem for De Rham cohomology and the acyclicity of quasi-coherent sheaves on noetherian affine schemes.

Lemma 1. Let $k \subset \mathbb{C}$ a subfield. Let $X \in \text{SmVar}(k)$ a smooth variety. Let $(r_i : X_i \rightarrow X)_{1 \leq i \leq s}$ an affine étale cover. Let $X_{\mathbb{C}}^{\text{an}} = \cup_{i=1}^r \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0, 1)^d$. Let

$$w = \left[\sum_{I, J, l, \text{card} I + \text{card} J + l = j} w_{I, J}^l \right] = \sum_{I, J, l, \text{card} I + \text{card} J + l = j} [w_{I, J}^l] \in H^j \Gamma(X_{\bullet, \mathbb{C}}^{\text{an}} \cap \mathbb{D}_{\bullet}, \Omega_{X_{\bullet, \mathbb{C}}^{\text{an}}}^\bullet).$$

Then the following assertions are equivalent :

- (i) $H^j \text{ev}(X)(w) \in H^j_{\text{sing}}(X_{\mathbb{C}}^{\text{an}}, 2i\pi\mathbb{Q})$,
- (ii) for all I, J, l such that $\text{card} I + \text{card} J + l = j$, there exist a lift

$$[\tilde{w}_{I, J}^l] \in H^l \Gamma(X_{I, \mathbb{C}}^{\text{an}} \cap \mathbb{D}_J, \Omega_{X_{I, \mathbb{C}}^{\text{an}}}^\bullet)$$

of $[w_{I, J}^l]$ with respect to the spectral sequence associated to the filtration on the total complex associated to the bi-complex structure such that

$$H^l \text{ev}(X_{\mathbb{C}}^{\text{an}})_{I, J}([\tilde{w}_{I, J}^l]) \in H^l_{\text{sing}}(X_{I, \mathbb{C}}^{\text{an}} \cap \mathbb{D}_J, 2i\pi\mathbb{Q}).$$

Proof. Follows immediately from the fact that $\text{ev}(X_{\mathbb{C}}^{\text{an}})_{\bullet, \bullet}$ define by definition a morphism of spectral sequence for the filtration given by the bi-complex structure. \square

Lemma 2. Let $k \subset \mathbb{C}$ a subfield. Let $\pi : X' \rightarrow X$ a proper generically finite (of degree d) morphism with $X, X' \in \text{SmVar}(k)$. Let $w \in H^j_{DR}(X)$. Then $\text{ev}(X)(w) \in H^j_{\text{sing}}(X_{\mathbb{C}}^{\text{an}}, 2i\pi\mathbb{Q})$ if and only if $\text{ev}(X')(\pi^* w) \in H^j_{\text{sing}}(X_{\mathbb{C}}^{\text{an}}, 2i\pi\mathbb{Q})$.

Proof. Since $\text{ev}(X')(\pi^* w) = \pi^* \text{ev}(X)(w) \in H^j_{\text{sing}}(X_{\mathbb{C}}^{\text{an}}, \mathbb{C})$, the only if part is obvious whereas the if part follows from the formula of $\pi_* \pi^* = dI$. \square

The main proposition of this section is the following :

Proposition 1. Let k a field of characteristic zero. Let $X \in \text{SmVar}(k)$. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. Let $p \in \mathbb{N}$ a prime number and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding.

(i) Take a compactification $X \hookrightarrow \bar{X}$, $\bar{X} \in \text{PSmVar}(k)$, with $D := \bar{X} \setminus X \subset \bar{X}$ a normal crossing divisor. Consider, for each $j \in \mathbb{Z}$, a (non canonical) splitting

$$\theta_j(X) : H_{DR}^j(X_{\mathbb{C}_p}) \xrightarrow{\sim} \bigoplus_{0 \leq l \leq j} H_{zar}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p})) = \bigoplus_{0 \leq l \leq j} H_{pet}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p}))$$

in $\text{Vect}(\mathbb{C}_p)$. Let $w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet) = \mathbb{H}_{pet}^j(X, \Omega_X^\bullet)$. If

$$w := \pi_{k/\mathbb{C}_p}(X)^* w \in OL_X^{\theta_j}(\mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}}^\bullet))$$

then $H^j ev(X)(w) \in H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q})$, where the map $OL_X^{\theta_j}$ is given in definition 1

(ii) Let $w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet) = \mathbb{H}_{pet}^j(X, \Omega_X^\bullet)$. If

$$w := \pi_{k/\mathbb{C}_p}(X)^* w \in H^j OL_X(\mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}}^{\bullet \geq 1}))$$

then $H^j ev(X)(w) \in H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q})$.

We recall (see section 2) the inclusion $OL_X : \Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}^\bullet \hookrightarrow \Omega_{X_{\mathbb{C}_p}^{pet}}^\bullet$ in $C(X_{\mathbb{C}_p}^{pet})$ is the sub-complex of logarithmic forms.

Proof. (i): Let

$$w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet) = H^j \Gamma(X_\bullet, \Omega_X^\bullet).$$

where $(r_i : X_i \rightarrow X)_{1 \leq i \leq s}$ is an affine etale cover. Let $X_{\mathbb{C}}^{an} = \cup_{i=1}^r \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0, 1)^d$. Denote $j_{IJ} : X_I \cap \mathbb{D}_J \hookrightarrow X_I$ the open embeddings. Then by definition $H^j ev(X)(w) = H^j ev(X_{\mathbb{C}}^{an})(j_\bullet^* \circ \text{an}_{X_\bullet}^* w)$ with

$$j_\bullet^* \circ \text{an}_{X_\bullet}^* w \in H^j \Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet).$$

Now, if $w = H^j OL_{X_{\mathbb{C}_p}^{pet}, \mathcal{O}}(\mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}}^\bullet))$, we have a canonical splitting

$$w = \sum_{l=0}^j w_L^{l, j-l} = \sum_{l=0}^j w^{l, j-l} \in H_{DR}^j(X_{\mathbb{C}_p}), \quad w_L^{l, j-l} \in H^{j-l}(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}^l), \quad w^{l, j-l} := H^j OL_{X_{\mathbb{C}_p}^{pet}}(w_L^{l, j-l}),$$

$\Gamma(X_{\mathbb{C}_p}, E_{pet}(\Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}^\bullet, F_b))$ being E_1 degenerated with a canonical splitting: the differentials at the E_1 level are trivial since all the logarithmic forms are closed. Let $0 \leq l \leq j$. Since $X \in \text{SmVar}(k)$ admits a pro-etale cover $(r_i : X_i \rightarrow X)_{1 \leq i \leq n}$ with X_i w -contractible (see section 2), and since $\Omega_{X_{\mathbb{C}_p}^{pet}}^l = \nu_X^* \Omega_{X_{\mathbb{C}_p}^{et}}^l$, there exists an affine etale cover $r = r(w^{l, j-l}) = (r_i : X_i \rightarrow X)_{1 \leq i \leq n}$ of X (depending on $w^{l, j-l}$) such that

$$w^{l, j-l} = [(w_{L, I}^{l, j-l})_I] \in H^j OL_{X_{\mathbb{C}_p}^{pet}}(H^{j-l} \Gamma(X_{\mathbb{C}_p, \bullet}, \Omega_{X_{\mathbb{C}_p, \log, \mathcal{O}}}^l)) \subset H^{j-l} \Gamma(X_{\mathbb{C}_p, \bullet}, \Omega_{\bar{X}_{\mathbb{C}_p}}^l(\log D_{\mathbb{C}_p})).$$

By lemma 2, we may assume, up to take a desingularization $\pi : X' \rightarrow X$ of $(X, \cup_i (r_i(X \setminus X_i)))$ and replace w with $\pi^* w$, that $r_i(X_i) = r_i(X_i(w)) = X \setminus D_i$ with $D_i \subset X$ smooth divisors with normal crossing For $1 \leq l \leq j$, we get

$$w_{L, I}^{l, j-l} = \sum_{\nu} df_{\nu_1} / f_{\nu_1} \wedge \cdots \wedge df_{\nu_l} / f_{\nu_l} \in \Gamma(X_{\mathbb{C}_p, I}, \Omega_{X_{\mathbb{C}_p}}^l).$$

For $l = 0$, we get

$$w^{0, j} = [(\lambda_I)] \in H^j \Gamma(X_{\mathbb{C}_p, I}, \mathcal{O}_{X_{\mathbb{C}_p, I}}), \quad \lambda_I \in \Gamma(X_{\mathbb{C}_p, I}, \mathbb{Z}_{p, X_{\mathbb{C}_p}^{pet}})$$

There exists $k' \subset \mathbb{C}_p$ containing k such that $w_{L,I}^{l,j-l} \in \Gamma(X_{k',I}, \Omega_{X_{k'}}^l)$ for all $0 \leq l \leq j$. Take an embedding $\sigma' : k' \hookrightarrow \mathbb{C}$ such that $\sigma'_k = \sigma$. We then have

$$j_\bullet^* \circ \text{an}_{X_\bullet}^* w = j_\bullet^* ((m_l \cdot w_{L,I}^{l,j-l})_{0 \leq l \leq j}) = (w_{L,I,J}^{l,j-l})_{l,I,J} \in H^j \Gamma(X_{\bullet,\mathbb{C}}^{\text{an}} \cap \mathbb{D}_\bullet, \Omega_{X_\bullet^{\text{an}}}^\bullet).$$

where for each (I, J, l) with $\text{card}I + \text{card}J + l = j$,

$$w_{L,I,J}^{l,j-l} := j_{IJ}^* w_{L,I}^{l,j-l} \in \Gamma(X_{I,\mathbb{C}}^{\text{an}} \cap \mathbb{D}_J, \Omega_{X_\mathbb{C}^{\text{an}}}^\bullet).$$

We have by a standard computation, for each (I, J, l) with $\text{card}I + \text{card}J + l = j$,

$$H_{\text{sing}}^*(X_{I,\mathbb{C}}^{\text{an}} \cap \mathbb{D}_J, \mathbb{Z}) = \langle \gamma_1, \dots, \gamma_{\text{card}I} \rangle.$$

On the other hand,

- $w_{L,I,J}^{l,j-l} = j_J^* (\sum_\nu df_{\nu_1}/f_{\nu_1} \wedge \dots \wedge df_{\nu_l}/f_{\nu_l}) \in \Gamma(X_{I,\mathbb{C}}^{\text{an}} \cap \mathbb{D}_J, \Omega_{X_\mathbb{C}^{\text{an}}}^\bullet)$ for $1 \leq l \leq j$,
- $w_{L,I,J}^{0,j} = \lambda_I$ is a constant by Baire theorem on $X_{I,\mathbb{C}}^{\text{an}} \cap \mathbb{D}_J$.

Hence, for $\mu \in P([1, \dots, s])$ with $\text{card}\mu = l$, we get, for $l = 0$ $H^l \text{ev}(X_\mathbb{C}^{\text{an}})_{I,J}(w_{L,I,J}^{0,j}) = 0$ and, for $1 \leq l \leq j$,

$$H^l \text{ev}(X_\mathbb{C}^{\text{an}})_{I,J}(w_{L,I,J}^{l,j-l})(\gamma_\mu) = \sum_k \delta_{\nu,\mu} \in 2i\pi\mathbb{Z}.$$

where $\gamma_\mu := \gamma_{\mu_1} \cdots \gamma_{\mu_l}$. We conclude by lemma 1.

(ii): Similar to the proof of (i). □

Let k a field of characteristic zero. Let $X \in \text{SmVar}(k)$. Let $X = \cup_{i=1}^s X_i$ an open affine cover with $X_i := X \setminus D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. Let $p \in \mathbb{N}$ a prime number and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding. A choice of an isomorphism $\sigma' : \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$ such that $\sigma'_k = \sigma$ gives a splitting

$$\theta_j(X)(\sigma') : H_{DR}^j(X_{\mathbb{C}_p}) \xrightarrow{\sim} \oplus_{0 \leq l \leq j} H_{zar}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p})) = \oplus_{0 \leq l \leq j} H_{pet}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p}))$$

in $\text{Vect}(\mathbb{C}_p)$. By proposition 1, we have a commutative diagram of graded algebras

$$\begin{array}{ccc} H_{DR}^*(X) & \xrightarrow{H^* \text{ev}(X)} & H_{\text{sing}}^*(X_\mathbb{C}^{\text{an}}, \mathbb{C}) \\ \uparrow \subset & & \uparrow H^* C^* \iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X_\mathbb{C}^{\text{an}}) \\ (OL_X^{\theta_j(\sigma')} \mathbb{H}_{pet}^*(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{\bullet, \log, \mathcal{O}})) \cap H_{DR}^*(X) & \xrightarrow{H^* \text{ev}(X)} & H_{\text{sing}}^*(X_\mathbb{C}^{\text{an}}, 2i\pi\mathbb{Q}) \end{array}$$

where

$$C^* \iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X_\mathbb{C}^{\text{an}}) : C_{\text{sing}}^\bullet(X_\mathbb{C}^{\text{an}}, 2i\pi\mathbb{Q}) \hookrightarrow C_{\text{sing}}^\bullet(X_\mathbb{C}^{\text{an}}, \mathbb{C})$$

is the subcomplex consisting of $\alpha \in C_{\text{sing}}^j(X_\mathbb{C}^{\text{an}}, \mathbb{C})$ such that $\alpha(\gamma) \in 2i\pi\mathbb{Z}$ for all $\gamma \in C_j^{\text{sing}}(X_\mathbb{C}^{\text{an}}, \mathbb{Q})$. Recall that

$$H^* \text{ev}(X_\mathbb{C}) = H^* R\Gamma(X_\mathbb{C}^{\text{an}}, \alpha(X)) \circ \Gamma(X_\mathbb{C}^{\text{an}}, E_{zar}(\Omega(\text{an}_X))) : H_{DR}^*(X_\mathbb{C}) \xrightarrow{\sim} H_{\text{sing}}^*(X_\mathbb{C}^{\text{an}}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and $\alpha(X) : \mathbb{C}_{X_\mathbb{C}^{\text{an}}} \hookrightarrow \Omega_{X_\mathbb{C}^{\text{an}}}^\bullet$, which gives the periods elements $H^* \text{ev}(X)(H_{DR}^*(X)) \subset H_{\text{sing}}^*(X_\mathbb{C}^{\text{an}}, \mathbb{C})$. On the other side the induced map

$$H^* \text{ev}(X_\mathbb{C}) : OL_X^{\theta_j(\sigma')}(\mathbb{H}_{pet}^*(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{\bullet, \log, \mathcal{O}})) \hookrightarrow H^* \iota_{2i\pi\mathbb{Q}/\mathbb{C}} H_{\text{sing}}^*(X_\mathbb{C}^{\text{an}}, 2i\pi\mathbb{Q})$$

is NOT surjective in general since the left hand side is invariant by the action of the group $\text{Aut}(\mathbb{C})$ (the group of field automorphism of \mathbb{C}) whereas the right hand side is not.

3.2 p adic integral periods

Let k a field of finite type over \mathbb{Q} . Denote \bar{k} the algebraic closure of k and $G = Gal(\bar{k}/k)$ the absolute Galois group of k . Let $X \in \text{SmVar}(k)$ a smooth variety. Take a compactification $\bar{X} \in \text{PSmVar}(k)$ of X such that $D := \bar{X} \setminus X \subset X$ is a normal crossing divisor, and denote $j : X \hookrightarrow \bar{X}$ the open embedding. Let $p \in \mathbb{N}$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Then $k \subset \bar{k} \subset \mathbb{C}_p$ and $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$, where \hat{k}_{σ_p} is the p -adic field which is the completion of k with respect the p adic norm given by σ_p . Denote $\hat{G}_{\sigma_p} = Gal(\mathbb{C}_p/\hat{k}_{\sigma_p}) = Gal(\bar{\mathbb{Q}}_p/\hat{k}_{\sigma_p})$ the Galois group of \hat{k}_{σ_p} . Recall (see section 2) that $\underline{\mathbb{Z}}_{p, X_{\mathbb{C}_p}} := \varprojlim_{n \in \mathbb{N}} \nu_X^*(\mathbb{Z}/p^n \mathbb{Z})_{X_{\mathbb{C}_p}^{et}} \in \text{Shv}(X_{\mathbb{C}_p}^{pet})$ and $\Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}} := \varprojlim_{n \in \mathbb{N}} \nu_X^* \Omega_{X_{\mathbb{C}_p}^{pet}/p^n \mathcal{O}_{\mathbb{C}_p}, \log} \in C(X_{\mathbb{C}_p}^{pet})$. We have then the commutative diagram in $C_{\mathbb{B}dr, \text{fil}, \hat{G}_{\sigma_p}}(\bar{X}_{\mathbb{C}_p}^{an, pet})$

$$\begin{array}{ccc}
j_* E_{pet}(\mathbb{B}_{dr, X_{\mathbb{C}_p}}, F) & \xrightarrow{j_* E_{pet}(\alpha(X))} & j_* E_{pet}((\Omega_{X_{\mathbb{C}_p}}^\bullet, F_b) \otimes_{\mathcal{O}_{X_{\mathbb{C}_p}}} (O_{\mathbb{B}dr, X_{\mathbb{C}_p}}, F)) \\
\uparrow \subset & & \uparrow \subset \\
E_{pet}(\mathbb{B}_{dr, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}, F) & \xrightarrow{\alpha(X)} & E_{pet}((\Omega_{\bar{X}_{\mathbb{C}_p}}^\bullet(\log D_{\mathbb{C}_p}), F_b) \otimes_{\mathcal{O}_{\bar{X}_{\mathbb{C}_p}}} (O_{\mathbb{B}dr, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}, F)) \\
E_{pet}(j_* \iota'_{X_{\mathbb{C}_p}^{pet}})_j := E_{pet}(l \rightarrow l.1)_j \uparrow & \xrightarrow{E_{pet}(j_* \beta_{pet}(X_{\mathbb{C}_p}))} & \uparrow E_{pet}(j_* (n \circ OL_{X_{\mathbb{C}_p}^{pet}}))_j := E_{pet}(w \rightarrow (w \otimes 1))_j \\
j_* E_{pet}(\underline{\mathbb{Z}}_{p, X_{\mathbb{C}_p}}) & \xrightarrow{j_* E_{pet}(\iota_{X_{\mathbb{C}_p}^{pet}}) := j_* E_{pet}(l \rightarrow l.1)} & j_* E_{pet}(\Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^\bullet, F_b)
\end{array}$$

where for $j' : U' \hookrightarrow X'$ an open embedding with $X' \in \text{RTop}$ and τ a topology on RTop we denote for $m : j_* Q \rightarrow Q'$ with $Q \in \text{PSh}_{\mathcal{O}}(U')$, $Q' \in \text{PSh}_{\mathcal{O}}(X')$ the canonical map in $C_{\mathcal{O}}(X')$

$$E_{\tau}^0(m)_j : j_* E_{\tau}^0(Q) \rightarrow E_{\tau}^0(j_* Q) \xrightarrow{E_{\tau}^0(m)} E_{\tau}^0(Q'),$$

giving by induction the canonical map $E_{\tau}(m)_j : j_* E_{\tau}(Q) \rightarrow E_{\tau}(Q')$ in $C_{\mathcal{O}}(X')$. The main results of [15] state that

- the map in $C_{\mathbb{B}dr, \text{fil}}(\bar{X}_{\hat{k}_{\sigma_p}}^{an, pet})$

$$\alpha(X) : (\mathbb{B}_{dr, \bar{X}_{\hat{k}_{\sigma_p}}, \log D_{\hat{k}_{\sigma_p}}}, F) \hookrightarrow (\Omega_{\bar{X}_{\hat{k}_{\sigma_p}}}^\bullet(\log D_{\hat{k}_{\sigma_p}}), F_b) \otimes_{\mathcal{O}_{\bar{X}_{\hat{k}_{\sigma_p}}}} (O_{\mathbb{B}dr, \bar{X}_{\hat{k}_{\sigma_p}}, \log D_{\hat{k}_{\sigma_p}}}, F)$$

is a filtered quasi-isomorphism, that is, the induced map in $C_{\mathbb{B}dr, \text{fil}, \hat{G}_{\sigma_p}}(\bar{X}_{\mathbb{C}_p}^{an, pet})$

$$\alpha(X) := \alpha(X)_{\mathbb{C}_p} : (\mathbb{B}_{dr, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}, F) \hookrightarrow (\Omega_{\bar{X}_{\mathbb{C}_p}}^\bullet(\log D_{\mathbb{C}_p}), F_b) \otimes_{\mathcal{O}_{\bar{X}_{\mathbb{C}_p}}} (O_{\mathbb{B}dr, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}, F)$$

is thus a filtered quasi-isomorphism,

- the map in $D_{\mathbb{Z}_p, \text{fil}}$

$$T(a_X, a_X, \otimes)(Rj_* \mathbb{Z}_{p, X^{et}}) : R\Gamma(X_{\mathbb{C}_p}, \mathbb{Z}_{p, X^{et}}) \otimes_{\mathbb{Z}_p} (\mathbb{B}_{dr, \mathbb{C}_p}, F) \rightarrow R\Gamma(\bar{X}_{\mathbb{C}_p}, (\mathbb{B}_{dr, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}, F))$$

is an isomorphism.

Hence, we get the isomorphism in $D_{\text{fil}}(\mathbb{B}_{dr}, \hat{G}_{\sigma_p})$

$$\begin{aligned}
R\alpha(X) &:= R\Gamma(\bar{X}_{\mathbb{C}_p}, \alpha(X)) \circ T(a_X, a_X, \otimes)(Rj_* \mathbb{Z}_{p, X^{et}}) : \\
R\Gamma(X_{\mathbb{C}_p}, \mathbb{Z}_{p, X^{et}}) \otimes_{\mathbb{Z}_p} (\mathbb{B}_{dr, \mathbb{C}_p}, F) &\xrightarrow{\sim} R\Gamma(\bar{X}_{\mathbb{C}_p}, (\Omega_{\bar{X}_{\mathbb{C}_p}}^{\bullet, et}(\log D_{\mathbb{C}_p}), F_b) \otimes_{\mathcal{O}_{\bar{X}_{\mathbb{C}_p}}} (O_{\mathbb{B}dr, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}, F)) \\
&\xrightarrow{=} R\Gamma(\bar{X}_{\mathbb{C}_p}, (\Omega_{\bar{X}_{\mathbb{C}_p}}^{\bullet, et}, F_b) \otimes_{\mathcal{O}_{\bar{X}_{\mathbb{C}_p}}} j_* \text{Hdg}(O_{X_{\mathbb{C}_p}}, F_b) \otimes_{\mathcal{O}_{X_{\mathbb{C}_p}}} (O_{\mathbb{B}dr, X_{\mathbb{C}_p}}, F))
\end{aligned}$$

which gives for each $n \in \mathbb{Z}$ a filtered isomorphism of \hat{G}_{σ_p} -modules

$$H^n R\alpha(X) : H_{et}^n(X_{\mathbb{C}_p}, \mathbb{Z}_p, X^{et}) \otimes \mathbb{B}_{dr, \mathbb{C}_p} \xrightarrow{\sim} H_{DR}^n(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{dr, \mathbb{C}_p}$$

so that we can recover the Hodge filtration on $H_{DR}^*(X)$ by the action of \hat{G}_{σ_p} .

We then have :

Proposition 2. *Let k a field of finite type over \mathbb{Q} . Let $X \in \text{SmVar}(k)$. Take a compactification $\bar{X} \in \text{PSmVar}(k)$ of X such that $D := \bar{X} \setminus X \subset X$ is a normal crossing divisor. Let $p \in \mathbb{N} \in \delta(k, X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Consider, using definition 1, for $j \in \mathbb{Z}$,*

$$\theta_j^\phi(X) : H_{DR}^j(X_{\mathbb{C}_p}) \xrightarrow{\sim} \bigoplus_{0 \leq l \leq j} H_{zar}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p})) = \bigoplus_{0 \leq l \leq j} H_{pet}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p}))$$

in $\text{Vect}(\mathbb{C}_p)$ a splitting which is invariant by Frobenius. We have then

$$H^j \beta_{pet}(X_{\mathbb{C}_p})(H_{pet}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)) = OL_X^{\theta_j^\phi}(\mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^\bullet \otimes \mathbb{Z}_p)).$$

Proof. Follows from the fact that the logarithmic classes are the ones which are Frobenius invariant. \square

Proposition 3. *Let k a field of finite type over \mathbb{Q} . Denote \bar{k} the algebraic closure of k . Denote $G := \text{Gal}(\bar{k}/k)$ its absolute Galois group. Let $X \in \text{SmVar}(k)$ a smooth variety. Take a compactification $\bar{X} \in \text{PSmVar}(k)$ of X such that $D := \bar{X} \setminus X \subset X$ is a normal crossing divisor, and denote $j : X \hookrightarrow \bar{X}$ the open embedding. Let $p \in \mathbb{N}$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$ being the p -adic completion with respect to the p adic norm induced by σ_p . Then $\hat{G}_{\sigma_p} := \text{Gal}(\mathbb{C}_p/\hat{k}_{\sigma_p}) \subset G := \text{Gal}(\bar{k}/k)$.*

(i) Let $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)$. Consider then its associated De Rham class

$$\begin{aligned} w(\alpha) &:= H^j \beta(X_{\mathbb{C}_p}^{pet})(\alpha) = H^j R\alpha(X)(\alpha \otimes 1) \\ &\in \mathbb{H}_{pet}^j(\bar{X}_{\mathbb{C}_p}, \Omega_{\bar{X}_{\mathbb{C}_p}}^\bullet(\log D_{\mathbb{C}_p})) = \mathbb{H}_{et}^j(\bar{X}_{\mathbb{C}_p}, \Omega_{\bar{X}_{\mathbb{C}_p}}^\bullet(\log D_{\mathbb{C}_p})) = H_{DR}^j(X_{\mathbb{C}_p}) \end{aligned}$$

Then $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}}$ if and only if $w(\alpha) \in F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) = F^l H_{DR}^j(X_{\mathbb{C}_p}) \cap H_{DR}^j(X_{\hat{k}_{\sigma_p}})$. That is we have

$$H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \langle F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \cap H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p) \rangle_{\mathbb{Q}_p} \subset H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p, X^{et}) \otimes \mathbb{B}_{dr, \mathbb{C}_p},$$

where $\langle - \rangle_{\mathbb{Q}_p}$ denote the \mathbb{Q}_p vector space generated by $(-)$. Note that $H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ and $F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}})$ are canonically embedded as subabelian groups of $H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p, X^{et}) \otimes \mathbb{B}_{dr, \mathbb{C}_p}$ by $(-) \otimes 1$ and $\alpha(X) \circ ((-) \otimes 1)$ respectively.

(ii) Let $\alpha \in H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)(l)$. Consider, see (i), $w(\alpha) := w(\pi_{\bar{k}/\mathbb{C}_p}(X)^* \alpha) \in H_{DR}^j(X_{\mathbb{C}_p})$. Then $\alpha \in H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)(l)^G$ if and only if $w(\alpha) \in F^l H_{DR}^j(X) = F^l H_{DR}^j(X_{\mathbb{C}_p}) \cap H_{DR}^j(X)$. That is we have

$$H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)(l)^G \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \langle F^l H_{DR}^j(X) \cap H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p) \rangle_{\mathbb{Q}_p} \subset H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p, X^{et}) \otimes \mathbb{B}_{dr, \mathbb{C}_p},$$

where $\langle - \rangle_{\mathbb{Q}_p}$ denote the \mathbb{Q}_p vector space generated by $(-)$. Note that $H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)$ and $F^l H_{DR}^j(X)$ are canonically embedded as subabelian groups of $H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p, X^{et}) \otimes \mathbb{B}_{dr, \mathbb{C}_p}$ by $(-) \otimes 1$ and $\alpha(X) \circ ((-) \otimes 1)$ respectively.

Proof. (i): Follows immediately from the fact that $H^j R\alpha(X)$ is a filtered quasi-isomorphism compatible with the Galois action of \hat{G}_{σ_p} by [15].

(ii): Let $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)^G$. Take a basis $((\alpha_i)_{1 \leq i \leq t}, (\alpha_i)_{t+1 \leq i \leq s}) \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ such that for $1 \leq i \leq t$, $\alpha_i \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)^G$. We have

$$w(\alpha) = \sum_{1 \leq i \leq s} \lambda_{i, w(\alpha)} w(\alpha_i) \in H_{DR}^j(X_{\mathbb{C}_p})$$

Assume by absurd that $w(\alpha) \notin H_{DR}^j(X)$. There exist a finite type extension k'/k , $k' \subset \mathbb{C}_p$ (depending on $w(\alpha)$) such that $w(\alpha) \in H_{DR}^j(X_{k'})$. By assumption the orbit $Aut(k'/k) \cdot w(\alpha) \subset H_{DR}(X_{k'})$ of $w(\alpha)$ under $Aut(k'/k)$ is non trivial (i.e. contain more then one element). Then there exist a prime number l and an embedding $\sigma'_l : k' \hookrightarrow \mathbb{C}_l$ such that the extension $\hat{k}'_{\sigma'_l}/\hat{k}_{\sigma_l}$ is non trivial (i.e. $\hat{k}'_{\sigma'_l} \neq \hat{k}_{\sigma_l}$) where $\sigma_l = \sigma'_{l|k}$ and such that $w(\alpha) := \pi_{k'/\hat{k}'_{\sigma'_l}}(X_{k'})^* w(\alpha) \notin H_{DR}^j(X_{\hat{k}'_{\sigma'_l}})$, where $\pi_{k'/\hat{k}'_{\sigma'_l}}(X_{k'}) : X_{\hat{k}'_{\sigma'_l}} \rightarrow X_{k'}$ is the projection, recall that $H_{DR}^j(X_{k'}) = H_{DR}^j(X) \otimes_k k'$. By injectivity of

$$\pi_{k'/\hat{k}'_{\sigma'_l}}(X_{k'})^* : H_{DR}^j(X_{k'}) \rightarrow H_{DR}^j(X_{\hat{k}'_{\sigma'_l}}),$$

the orbit $Gal(\hat{k}'_{\sigma'_l}/\hat{k}_{\sigma_l}) \cdot \lambda w(\alpha) \subset H_{DR}(X_{\hat{k}'_{\sigma'_l}})$ of $w(\alpha) := \pi_{k'/\hat{k}'_{\sigma'_l}}(X_{k'})^* w(\alpha)$ under $Gal(\hat{k}'_{\sigma'_l}/\hat{k}_{\sigma_l}) \subset Aut(k'/k)$ is non trivial (i.e. contain more then one element). By the main result of [15] (see above), we get

$$H^j R\alpha(X_{\mathbb{C}_l})^{-1}(\lambda w(\alpha)) \notin (H_{et}^j(X_{\bar{k}'}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{B}_{dr, \mathbb{C}_l})^{\hat{G}_{\sigma'_l}}.$$

Consider the pairing of G modules

$$\delta(-, -) : H_{pet}^k(X_{\mathbb{C}_p}, \mathbb{B}_{dr, \mathbb{C}_p}) \otimes_{k'} H_{pet}^l(X_{\mathbb{C}_l}, \mathbb{B}_{dr, \mathbb{C}_l}) \rightarrow H_{pet}^{k+l}(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}, \mathbb{B}_{dr, \mathbb{C}_p} \otimes_{k'} \mathbb{B}_{dr, \mathbb{C}_l})$$

and

$$\begin{aligned} & \alpha(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}) : \pi_p^* \mathbb{B}_{dr, X_{\mathbb{C}_p}} \otimes_{k'} \pi_l^* \mathbb{B}_{dr, X_{\mathbb{C}_l}} \xrightarrow{\alpha(X_{\mathbb{C}_p}) \otimes \alpha(X_{\mathbb{C}_l})} \\ & \pi_p^*(\Omega_{X_{\mathbb{C}_p}}^\bullet \otimes_{O_{X_{\mathbb{C}_p}}} O_{\mathbb{B}_{dr, X_{\mathbb{C}_p}}}) \otimes_{k'} \pi_l^*(\Omega_{X_{\mathbb{C}_l}}^\bullet \otimes_{O_{X_{\mathbb{C}_l}}} O_{\mathbb{B}_{dr, X_{\mathbb{C}_l}}}) \xrightarrow{w(\Omega)} \\ & \Omega_{X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}}^\bullet \otimes_{\pi_p^* O_{X_{\mathbb{C}_p}} \otimes_{k'} \pi_l^* O_{X_{\mathbb{C}_l}}} O_{\mathbb{B}_{dr, X_{\mathbb{C}_p}} \otimes_{k'} O_{\mathbb{B}_{dr, X_{\mathbb{C}_l}}}} =: DR(X)(O_{\mathbb{B}_{dr, X_{\mathbb{C}_p}} \otimes_{k'} O_{\mathbb{B}_{dr, X_{\mathbb{C}_l}}}}) \end{aligned}$$

is the canonical map in $C_{\hat{G}_{\sigma'_p} \times \hat{G}_{\sigma'_l}}(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}^{pet})$ and

- $\pi_p := \pi_{\mathbb{C}_p/\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}(X_{\mathbb{C}_p}) : X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l} \rightarrow X_{\mathbb{C}_p}$
- $\pi_l := \pi_{\mathbb{C}_l/\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}(X_{\mathbb{C}_l}) : X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l} \rightarrow X_{\mathbb{C}_l}$

are the base change maps. By the commutative diagram of $\hat{G}_{\sigma'_p} \times \hat{G}_{\sigma'_l}$ modules

$$\begin{array}{ccccc} H_{pet}^j(X_{\bar{k}'}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{dr, \mathbb{C}_p} & \xrightarrow{((-) \otimes 1) \circ \pi_{\bar{k}'/\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}(X_{\bar{k}'})^*} & H_{pet}^j(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}, \mathbb{B}_{dr, X_{\mathbb{C}_p}} \otimes_{k'} \mathbb{B}_{dr, X_{\mathbb{C}_l}}) & \xleftarrow{((-) \otimes 1) \circ \pi_{\bar{k}'/\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}(X_{\bar{k}'})^*} & H_{pet}^j(X_{\bar{k}'}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{B}_{dr, \mathbb{C}_l} \\ \downarrow H^j R\alpha(X_{\mathbb{C}_l}) & & \downarrow H^j \alpha(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}) & & \downarrow H^j R\alpha(X_{\mathbb{C}_l}) \\ & \xrightarrow{((-) \otimes 1) \circ \pi_{\hat{k}'_{\sigma'_p}/\mathbb{C}_p}(X_{k'})^*} & \mathbb{H}_{pet}^j(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}, DR(X)(O_{\mathbb{B}_{dr, X_{\mathbb{C}_p}} \otimes_{k'} O_{\mathbb{B}_{dr, X_{\mathbb{C}_l}}})) & \xleftarrow{((-) \otimes 1) \circ \pi_{\hat{k}'_{\sigma'_l}/\mathbb{C}_l}(X_{k'})^*} & \\ & \xrightarrow{((-) \otimes 1) \circ \pi_{k'/(\mathbb{C}_p \otimes_{k'} \mathbb{C}_l)}(X_{k'})^*} & \uparrow & & \\ H_{DR}^j(X_{\mathbb{C}_p}) \otimes_{\mathbb{C}_p} \mathbb{B}_{dr, \mathbb{C}_p} & \xleftarrow{\pi_{k'/\mathbb{C}_p}(X_{k'})^*} & H_{DR}^j(X_{k'}) & \xrightarrow{((-) \otimes 1) \circ \pi_{k'/\mathbb{C}_l}(X_{k'})^*} & H_{DR}^j(X_{\mathbb{C}_l}) \otimes_{\mathbb{C}_l} \mathbb{B}_{dr, \mathbb{C}_l} \end{array}$$

we have for $\beta_p \in H_{pet}^j(X_{\bar{k}'}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{dr, \mathbb{C}_p}$, $\beta_l \in H_{pet}^j(X_{\bar{k}'}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{B}_{dr, \mathbb{C}_l}$,

$$\alpha(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l})(\delta(\beta_p, \beta_l)) = \alpha(X_{\mathbb{C}_l})(\beta_p) \cdot \alpha(X_{\mathbb{C}_l})(\beta_l) \in \mathbb{H}^j(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}, DR(X)(O_{\mathbb{B}_{dr, \mathbb{C}_p}} \otimes_{k'} O_{\mathbb{B}_{dr, \mathbb{C}_l}})).$$

Note that since $\pi_{\bar{k}'/(\mathbb{C}_p \otimes_{k'} \mathbb{C}_l)}(X_{\bar{k}'}) : X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l} \rightarrow X_{\bar{k}'}$ is flat $((-) \otimes 1) \circ \pi_{\bar{k}'/(\mathbb{C}_p \otimes_{k'} \mathbb{C}_l)}(X_{\bar{k}'})^*$ and $((-) \otimes 1) \circ \pi_{\bar{k}'/(\mathbb{C}_p \otimes_{k'} \mathbb{C}_l)}(X_{\bar{k}'})^*$ are injective, (the morphism involved in the base change are without torsion). Denote $d = \dim(X)$. Consider the canonical projection

$$\begin{aligned} \pi : X_{\mathbb{C}_p} \times X_{\mathbb{C}_l} &\rightarrow X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}, \text{ given by on } X^o \subset X \text{ open affine} \\ \pi((x_1, \dots, x_d), (x'_1, \dots, x'_d)) &:= (x_1 \otimes x'_1, \dots, x_d \otimes x'_d), \end{aligned}$$

where $X_{\mathbb{C}_p} \times X_{\mathbb{C}_l}$ is endowed with the product topology. Then the commutative diagram

$$\begin{array}{ccc} H_{pet}^j(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}, \mathbb{B}_{dr, \mathbb{C}_p} \otimes_{k'} \mathbb{B}_{dr, \mathbb{C}_l}) & \xrightarrow{\pi^*} & H_{pet}^j(X_{\mathbb{C}_p} \times X_{\mathbb{C}_l}, \pi^*(\mathbb{B}_{dr, \mathbb{C}_p} \otimes_{k'} \mathbb{B}_{dr, \mathbb{C}_l})) \\ \downarrow \alpha_{(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l})} & & \downarrow w(-) \circ (\pi^* \alpha_{(X_{\mathbb{C}_p})} \otimes \pi^* \alpha_{(X_{\mathbb{C}_l})}) \\ \mathbb{H}_{pet}^j(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}, DR(X)(O_{\mathbb{B}_{dr, X_{\mathbb{C}_p}}} \otimes_{k'} O_{\mathbb{B}_{dr, X_{\mathbb{C}_l}}})) & \xrightarrow{\pi^*} & \mathbb{H}_{pet}^j(X_{\mathbb{C}_p} \times X_{\mathbb{C}_l}, DR(X)(O_{\mathbb{B}_{dr, X_{\mathbb{C}_p}}} \otimes_{k'} O_{\mathbb{B}_{dr, X_{\mathbb{C}_l}}})) \end{array}$$

together with the p adic Poincare lemma on $X_{\mathbb{C}_p}$ and the l adic Poincare lemma on $X_{\mathbb{C}_l}$, the fact that π^* is injective (note that the product topology is less fine then the pro-etale topology on $X_{\mathbb{C}_p} \times_{k'} X_{\mathbb{C}_l}$ and that the map $O_{X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}} \rightarrow O_{X_{\mathbb{C}_p} \times_{k'} X_{\mathbb{C}_l}}$ is torsion free), show that $\alpha_{(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l})}$ is injective. Hence

$$\alpha \otimes 1 = \delta(\alpha, 1) = \delta(1, H^j R\alpha(X_{\mathbb{C}_l})^{-1}(w(\alpha))) \in H_{pet}^j(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}, \mathbb{B}_{dr, \mathbb{C}_p} \otimes_{k'} \mathbb{B}_{dr, \mathbb{C}_l}),$$

that is there exists $\lambda_\alpha \in k'$ such that

$$\alpha = \lambda_\alpha H^j R\alpha(X_{\mathbb{C}_l})^{-1}(w(\alpha)) = H^j R\alpha(X_{\mathbb{C}_l})^{-1}(\lambda_\alpha w(\alpha)) \in H_{pet}^j(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}, \mathbb{B}_{dr, \mathbb{C}_p} \otimes_{k'} \mathbb{B}_{dr, \mathbb{C}_l}).$$

Since α and $w(\alpha)$ are $\hat{G}_{\sigma'_l}$ invariant, $\lambda_\alpha \in \mathbb{Q}_p$. This gives

$$\alpha = H^j R\alpha(X_{\mathbb{C}_l})^{-1}(\lambda_\alpha w(\alpha)) \notin H_{pet}^j(X_{\mathbb{C}_p \otimes_{k'} \mathbb{C}_l}, \mathbb{Z}_p \otimes \mathbb{B}_{dr, \mathbb{C}_l})^{\hat{G}_{\sigma'_l}}.$$

Contradiction. We thus have $w(\alpha) \in H_{DR}^j(X)$. Conversely if $\alpha \notin H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)(l)^G$, we get similarly $w(\alpha) \notin H_{DR}^j(X)$. The result then follows from (i) and the equality $F^l H_{DR}^j(X) = F^l H_{DR}^j(X_{\mathbb{C}_p}) \cap H_{DR}^j(X)$ given by the filtered isomorphism in $C_{fil}(\bar{X}_{\mathbb{C}_p})$

$$w(k/\mathbb{C}_p) : (\Omega_{\bar{X}}^\bullet(\log D), F_b) \otimes_k \mathbb{C}_p \xrightarrow{\sim} (\Omega_{\bar{X}_{\mathbb{C}_p}}^\bullet(\log D_{\mathbb{C}_p}), F_b)$$

which say that the Hodge filtration is defined over k : see section 2. \square

Theorem 1. (i) Let k a field of finite type over \mathbb{Q} . Denote \bar{k} its algebraic closure and $G = Gal(\bar{k}/k)$ its absolute Galois group. Let $X \in \text{SmVar}(k)$. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. Let $p \in \mathbb{N} \setminus \delta(k, X)$ a prime number. Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding. For each $j, l \in \mathbb{Z}$, we get from proposition 3(i), proposition 2, and proposition 1, a canonical injective map

$$H^j \iota_{p, ev}^{G, l}(X) : H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)(l)^G \hookrightarrow F^l H^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p, \alpha \mapsto H^j \iota_{p, ev}^{G, d}(X)(\alpha) := ev(X)(w(\alpha)),$$

with $w(\alpha) := H^j \beta_{pet}(X_{\mathbb{C}_p})(\alpha) \in H_{DR}^j(X) \subset H_{DR}^j(X_{\mathbb{C}_p})$. By construction, for $f : X' \rightarrow X$ a morphism with $X, X' \in \text{SmVar}(k)$ and $p \in \mathbb{N} \setminus \delta(k, X, X')$ a prime number, we have the commutative diagram

$$\begin{array}{ccc} H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)(l)^G & \xrightarrow{H^j \iota_{p, ev}^{G, l}(X)} & F^l H^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \\ \downarrow f^* & & \downarrow f_* \\ H_{et}^j(X'_{\bar{k}}, \mathbb{Z}_p)(l)^G & \xrightarrow{H^j \iota_{p, ev}^{G, l}(X')} & F^l H^j(X_{\mathbb{C}}'^{an}, 2i\pi\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \end{array}$$

(ii) Let $K \subset \mathbb{C}_p$ a p -adic field such that $\text{Frac}(W(O_K)) = K$. Let $X \in \text{SmVar}(K)$. Denote $G_p = \text{Gal}(\bar{\mathbb{Q}}_p/K)$ its absolute Galois group. Let $\sigma : K \hookrightarrow \mathbb{C}$ an embedding. For each $j, l \in \mathbb{Z}$, we get from proposition 3(i), proposition 2, and proposition 1, a canonical injective map

$$H^j \iota_{ev}^{G_p, l}(X) : H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{G_p} \hookrightarrow F^l H^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p, \alpha \mapsto H^j \iota_{ev}^{G_p, l}(X)(\alpha) := ev(X)(w(\alpha)),$$

with $w(\alpha) := H^j \beta_{pet}(X_{\mathbb{C}_p})(\alpha) \in H_{DR}^j(X) \subset H_{DR}^j(X_{\mathbb{C}_p})$. By construction, for $f : X' \rightarrow X$ a morphism with $X, X' \in \text{SmVar}(K)$, we have the commutative diagram

$$\begin{array}{ccc} H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{G_p} & \xrightarrow{H^j \iota_{ev}^{G_p, l}(X)} & F^l H^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \\ \downarrow f^* & & \downarrow f^* \\ H_{et}^j(X'_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{G_p} & \xrightarrow{H^j \iota_{ev}^{G_p, l}(X')} & F^l H^j(X'_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \end{array}$$

Note that (ii) implies (i).

Proof. (i): Follows from (ii).

(ii): By proposition 3(i), $w(\alpha) \in F^l H_{DR}^j(X_{\bar{k}_{\sigma_p}}) \subset H_{DR}^j(X_{\mathbb{C}_p})$. Consider, using definition 1, for $j \in \mathbb{Z}$,

$$\theta_j^\phi(X) : H_{DR}^j(X_{\mathbb{C}_p}) \xrightarrow{\sim} \bigoplus_{0 \leq l \leq j} H_{zar}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p})) = \bigoplus_{0 \leq l \leq j} H_{pet}^l(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{j-l}(\log D_{\mathbb{C}_p}))$$

in $\text{Vect}(\mathbb{C}_p)$ a splitting which is invariant by Frobenius. By proposition 2,

$$w(\alpha) = \sum_{1 \leq i \leq r} \lambda_i w_{Li} \in H_{DR}^j(X_{\mathbb{C}_p}), (w_{Li})_{1 \leq i \leq r} \in \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^\bullet(\log, \mathcal{O})), (\lambda_i)_{1 \leq i \leq r} \in \mathbb{Z}_p.$$

Then, by proposition 1,

$$ev(X)(w(\alpha)) = \sum_{1 \leq i \leq r} \lambda_i ev(X)(w_{Li}) \in H_{sing}^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

□

Remark 3. Let k a field of finite type over \mathbb{Q} . Let $X \in \text{SmVar}(k)$. Let $p \in \mathbb{N}$ be a prime number. Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ and $\sigma : k \hookrightarrow \mathbb{C}$ be embeddings. Note that for $w \in \mathbb{H}^j(X, \Omega_{X_{et}}^\bullet)$ such that $ev(X)(w) \in H_{sing}^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q})$, w is NOT logarithmic in general and $\gamma(X)(ev(X)(w)) \in H_{et}^j(X_{\bar{k}}, \mathbb{Q}_p)$ is NOT $G = \text{Gal}(\bar{k}/k)$ equivariant in general, where

$$\gamma(X) : H_{sing}^j(X_{\mathbb{C}}^{an}, \mathbb{Q}) \xrightarrow{/p^*} H_{sing}^j(X_{\mathbb{C}}^{an}, \mathbb{Q}_p) \xrightarrow{\sim} H_{et}^j(X_{\mathbb{C}}^{an}, \mathbb{Q}_p) = H_{et}^j(X_{\bar{k}}, \mathbb{Q}_p).$$

is given in section 2.

Corollary 1. (i) Let k a field of finite type over \mathbb{Q} . Let $X \in \text{PSmVar}(k)$. The the Hodge conjecture for X implies the Tate conjecture for X . In particular we get Tate conjecture for divisors of X .

(ii) Let K a p -adic field. Let $X \in \text{PSmVar}(K)$. The the Hodge conjecture for X implies the Tate conjecture for X . In particular we get Tate conjecture for divisors of X .

(iii) Let k a finite field. Let $X \in \text{PSmVar}(k)$. The the Hodge conjecture for X implies the Tate conjecture for X . In particular we get Tate conjecture for divisors of X .

Proof. (i): Follows from theorem 1(i) : for $X \in \text{PSmVar}(k)$ and $Z \in \mathcal{Z}^d(X, n)^{\partial=0}$, we get $H^{2d-n} \iota_{p, ev}^{G, d}(X)([Z]) = [Z]$, hence the Hodge conjecture for X implies the Tate conjecture for X since $H^{2d-n} \iota_{p, ev}^{G, d}(X)$ is injective. Since we have by the exponential sequence Hodge conjecture for divisors, we get Tate conjecture for divisors of X .

(ii): As for (i) it follows from theorem 1(ii).

(iii): Follows from (ii) by base change of etale cohomology with respect to proper morphisms. □

4 The complex and etale Abel Jacobi maps and normal function

4.1 The complex Abel Jacobi map for higher Chow group and complex normal functions

Let k a field of finite type over \mathbb{Q} . Consider an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then $k \subset \bar{k} \subset \mathbb{C}$, where \bar{k} is the algebraic closure of k . We have then the quasi-isomorphism $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^{\bullet}$ in $C(X_{\mathbb{C}}^{an})$.

- For $X \in \text{SmVar}(k)$, we consider

$$((H_{DR}^j(X), F), H^j(X_{\mathbb{C}}^{an}, \mathbb{Z}), H^j \alpha(X)) \in MHM_{k, gm}(k) \subset \text{Vect}_{fil}(k) \times_I \text{Ab}$$

where F is the Hodge filtration on $H_{DR}^j(X) \otimes_k \mathbb{C}$ and

$$H^j R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X)) : H^j(X_{\mathbb{C}}^{an}, \mathbb{C}) \xrightarrow{\sim} H_{DR}^j(X) \otimes_k \mathbb{C}$$

Recall the geometric mixed Hodge structures (see [5]) are mixed Hodge structure by the Hodge decomposition theorem on smooth proper complex varieties.

- For $f : X \rightarrow S$ a morphism with $S, X \in \text{SmVar}(k)$, we consider

$$(H^j Rf_{*Hdg}(O_X, F_b), H^j Rf_{*}\mathbb{Z}_{X_{\mathbb{C}}^{an}}, H^j Rf_{*}\alpha(X)) \in MHM_{k, gm}(S) \subset \text{PSh}_{\mathcal{D}(1,0)fil}(S) \times_I P_{fil, k}(S_{\mathbb{C}}^{an})$$

where

$$H^j Rf_{*}\alpha(X) : H^j Rf_{*}\mathbb{C}_{X_{\mathbb{C}}^{an}} \xrightarrow{\sim} DR(S)(o_{fil} H^j Rf_{*Hdg}(O_X, F_b) \otimes_k \mathbb{C} = H^j \int_f O_X \otimes_k \mathbb{C})$$

Recall the geometric mixed Hodge modules (see [5]) are mixed Hodge modules by a theorem of Saito for proper morphisms of smooth complex varieties.

Let $X \in \text{SmVar}(k)$. We have for $j, d \in \mathbb{N}$, the generalized Jacobian

$$J_{\sigma}^{j,d}(X) := H^j(X_{\mathbb{C}}^{an}, \mathbb{C}) / (F^d H^j(X_{\mathbb{C}}^{an}, \mathbb{C}) \oplus H^j(X_{\mathbb{C}}^{an}, \mathbb{Z}))$$

where F is given the Hodge filtration on $H_{DR}^j(X) \otimes_k \mathbb{C}$ and

$$H^j R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X)) : H^j(X_{\mathbb{C}}^{an}, \mathbb{C}) \xrightarrow{\sim} H_{DR}^j(X) \otimes_k \mathbb{C}.$$

If $X \in \text{PSmVar}(k)$ and $2d \geq n$, $J^{j,d}(X)$ is a complex torus since

$$((H_{DR}^j(X), F), H^j(X_{\mathbb{C}}^{an}, \mathbb{Z}), H^j R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X))) \in HM_{k, gm}(k) \subset MHM_{k, gm}(k)$$

is a pure Hodge structure. For $X \in \text{PSmVar}(k)$, we have a canonical isomorphism of abelian groups

$$I_{\sigma}^{j,d}(X) : J_{\sigma}^{j,d}(X) \xrightarrow{\sim} \text{Ext}_{MHM_{k, gm}(k)}^1(\mathbb{Z}_{\text{pt}}^{Hdg}(d), (H_{DR}^j(X), H^j(X_{\mathbb{C}}^{an}, \mathbb{Z}), H^j R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X)))).$$

Definition 2. Let $X \in \text{SmVar}(k)$ irreducible. Let $\bar{X} \in \text{PSmVar}(k)$ a compactification of X with $D := \bar{X} \setminus X \subset \bar{X}$ a normal crossing divisor. The map of complexes of abelian groups (see [4])

$$\mathcal{R}_X : \mathcal{Z}^d(X, \bullet) \rightarrow C_{\bullet}^{\mathcal{D}}(\bar{X}_{\mathbb{C}}^{an}, D_{\mathbb{C}}^{an}), Z \mapsto \mathcal{R}_X := (T_Z, \Omega_Z, R_Z)$$

where $C_{\bullet}^{\mathcal{D}}(\bar{X}_{\mathbb{C}}^{an}, D_{\mathbb{C}}^{an})$ is the Deligne homology complex induces the complex Abel Jacobi map for higher Chow groups

$$AJ_{\sigma}(X) : \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0} \rightarrow J_{\sigma}^{2d-1-n, d}(X), Z \mapsto AJ_{\sigma}(X)(Z) := D^{-1}(R'_Z), \\ R'_Z = R_Z - \Omega'_Z + T'_Z, ; \text{ with } \partial T'_Z = T_Z, \partial \Omega'_Z = \Omega_Z$$

where

$$D : C_{\bullet}^{\mathcal{D}}(X_{\mathbb{C}}^{an}) \rightarrow C_{\bullet}^{\mathcal{D}}(\bar{X}_{\mathbb{C}}^{an}, D_{\mathbb{C}}^{an})$$

is the Poincare dual for Deligne homology.

Theorem 2. Let $k \subset \mathbb{C}$ a subfield. Let $X \in \text{PSmVar}(k)$.

(i) For $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0}$, we have

$$\begin{aligned} AJ_\sigma(X)(Z) &= I_\sigma^{j,d}(X)^{-1}(0 \rightarrow (H_{DR}^{2d-1-n}(X), H_{\text{sing}}^{2d-1-n}(X_{\mathbb{C}}^{\text{an}}, \mathbb{Z}), H^{2d-1-n}R\Gamma(X_{\mathbb{C}}^{\text{an}}, \alpha(X))) \\ &\quad \xrightarrow{(j^*, j^*, 0)} \\ & (H_{DR}^{2d-1}((X \times \square^n) \setminus |Z|), H_{\text{sing}}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}}^{\text{an}}, \mathbb{Z}), H^{2d-1-n}R\Gamma((X \times \square^n) \setminus |Z|)_{\mathbb{C}}^{\text{an}}, \alpha(X \times \square^n))^{[Z]} \\ &\quad \xrightarrow{(\partial, \partial, 0)} \\ & (H_{DR, |Z|}^{2d}(X \times \square^n), H_{\text{sing}, |Z|}^{2d}((X \times \square^n)_{\mathbb{C}}^{\text{an}}, \mathbb{Z}), R\Gamma_{|Z|}((X \times \square^n)_{\mathbb{C}}^{\text{an}}, \alpha(X \times \square^n))^{[Z]} = \mathbb{Z}_{\text{pt}}^{\text{Hdg}}(n-d) \rightarrow 0) \end{aligned}$$

where $j : (X \times \square^n) \setminus |Z| \hookrightarrow X \times \square^n$ is the open embedding and

$$H_{\text{Hdg}, |Z|}^{2d}(X \times \square^n)^{[Z]} \subset H_{\text{Hdg}, |Z|}^{2d}(X \times \square^n), \quad H_{\text{Hdg}}^{2d-1}((X \times \square^n) \setminus |Z|)^{[Z]} \subset H_{\text{Hdg}}^{2d-1}((X \times \square^n) \setminus |Z|).$$

are the subobjects given by the pullback of the class of Z (see section 2).

(ii) Let $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0}$. Then $AJ_\sigma(X)(Z) = 0$ if and only if there exist $w \in H_{DR}^{2d-1}((X \times \square^n) \setminus |Z|)^{[Z]}$ such that

- $w \in F^d H_{DR}^{2d-1}((X \times \square^n) \setminus |Z|)$,
- $ev((X \times \square^n) \setminus |Z|)(w) \in H_{\text{sing}}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}}^{\text{an}}, \mathbb{Z}(2i\pi))$
- $\partial w \neq 0$.

Proof. (i): See [6].

(ii): Follows from (i) □

Let $f : X \rightarrow S$ a morphism with $S, X \in \text{SmVar}(k)$. We have for $j, d \in \mathbb{N}$ such that $H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}}$ is a local system and $F^d H^j Rf_* \text{Hdg}(O_X, F_b) \subset H^j \int_f(O_X)$ are locally free sub O_S modules, the generalized relative intermediate Jacobian

$$J_\sigma^{j,d}(X/S) := (H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}} \otimes O_{S_{\mathbb{C}}^{\text{an}}}) / (F^d (H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}} \otimes O_{S_{\mathbb{C}}^{\text{an}}}) \oplus H^j Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{\text{an}}})$$

where F is given by the Hodge filtration on $H^j \int_f(O_X)$ and

$$H^j Rf_* \alpha(X) : H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}} \xrightarrow{\sim} DR(S)(H^j \int_f(O_X)).$$

A generalized normal function is then a section $\nu \in \Gamma(S_{\mathbb{C}}^{\text{an}}, J_\sigma^{j,d}(X/S))$ which is horizontal (i.e. $\nabla \nu = 0$). For $s \in S$, we get immediately that $i_s^{* \text{mod}} J_\sigma^{j,d}(X/S) = J_\sigma^{j,d}(X_s)$. In particular we get for $f : X \rightarrow S$ a smooth projective morphism with $S, X \in \text{SmVar}(k)$ and $j, d \in \mathbb{N}$, the relative intermediate Jacobian

$$J_\sigma^{j,d}(X/S) := (H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}} \otimes O_{S_{\mathbb{C}}^{\text{an}}}) / (F^d (H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}} \otimes O_{S_{\mathbb{C}}^{\text{an}}}) \oplus H^j Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{\text{an}}})$$

where F is given by the Hodge filtration on $H^j \int_f(O_X) = H^j Rf_* \Omega_{X/S}^\bullet$ and $H^j Rf_* \alpha(X)$. A normal function is then a section $\nu \in \Gamma(S_{\mathbb{C}}^{\text{an}}, J_\sigma^{j,d}(X/S))$ which is horizontal. For $f : X \rightarrow S$ a smooth projective morphism with $S, X \in \text{SmVar}(k)$, we have a canonical isomorphism of abelian groups

$$I_\sigma^{j,d}(X/S) : J_\sigma^{j,d}(X/S) \xrightarrow{\sim} \text{Ext}_{\text{MHM}(S)}^1(\mathbb{Z}_S^{\text{Hdg}}(d), (H^j Rf_* \text{Hdg}(O_X, F_b), H^j Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{\text{an}}}, H^j Rf_* \alpha(X))).$$

Definition-Proposition 1. Let $f : X \rightarrow S$ a morphism with $S, X \in \text{SmVar}(k)$. Let $j : S^o \hookrightarrow S$ an open subset such that for all $j, d \in \mathbb{Z}$, $j^* H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}}$ is a local system and $j^* F^d H^j Rf_* \text{Hdg}(O_X, F_b) \subset$

$j^*H^j \int_f(O_X)$ is a locally free sub O_S module. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. Let $d, n \in \mathbb{N}$. We have then, denoting $X^o := X \times_S S^o$ and using definition 2, the map

$$AJ_\sigma(X^o/S^o) : \mathcal{Z}^d(X, n)_{f_{hom}}^{f, \partial=0} \rightarrow \Gamma(S_{\mathbb{C}}^{o, an}, J_\sigma^{2d-n-1, d}(X^o/S^o)) \subset \Gamma(S_{\mathbb{C}}^{o, an}, \bigoplus_{s \in S_{\mathbb{C}}^{o, an}} i_{s*} J_{\sigma'}^{2d-n-1, d}(X_s)),$$

$$Z \mapsto AJ_\sigma(X^o/S^o)(Z) := \nu_Z := ((s \in S_{\mathbb{C}}^o) \mapsto (AJ_{\sigma'}(X_s)(Z_s) \in J_{\sigma'}^{2d-n-1, d}(X_s)))$$

where $\mathcal{Z}^d(X, n)_{f_{hom}}^{f, \partial=0} \subset \mathcal{Z}^d(X, n)^{f, \partial=0}$ denote the sub-abelian group consisting of algebraic cycles Z with $Z_s := i_s^* Z \in \mathcal{Z}^d(X_s, n)_{hom}^{\partial=0}$, and $\sigma' : k(s) \hookrightarrow \mathbb{C}$ is the embedding given by s extending $\sigma : k \hookrightarrow \mathbb{C}$, denoting again $s := \pi k/\mathbb{C}(S)(s) \in S$, $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \rightarrow S$ being the projection.

Proof. Standard : to show that

$$\nu_Z := (s \in S_{\mathbb{C}}^o) \mapsto AJ_\sigma(X_s)(Z_s) \in \Gamma(S_{\mathbb{C}}^{o, an}, \bigoplus_{s \in S_{\mathbb{C}}^{o, an}} i_{s*} J_\sigma^{2d-n-1, d}(X_s))$$

is holomorphic and horizontal we consider a compactification $\bar{f} : \bar{X} \rightarrow S$ of f with $\bar{X} \in \text{SmVar}(k)$ and use trivializations of $f : (\bar{X}_{\mathbb{C}}^{o, an}, (\bar{X} \setminus X)_{\mathbb{C}}^{o, an}) \rightarrow S_{\mathbb{C}}^{o, an}$ which gives trivialization of the local system $j^*Rf_*\mathbb{Z}_{S_{\mathbb{C}}^{o, an}}$ (see [4] for example). \square

Corollary 2. Let $f : X \rightarrow S$ a smooth projective morphism with $S, X \in \text{SmVar}(k)$. Let $d, n \in \mathbb{N}$. For $Z \in \mathcal{Z}^d(X, n)_{f_{hom}}^{f, \partial=0}$, we have

$$AJ_\sigma(X/S)(Z) = I_\sigma^{j, d}(X/S)^{-1}(0 \rightarrow (H^{2d-1-n}Rf_*Hdg(O_X, F_b), H^{2d-1-n}Rf_*\mathbb{Z}_{X_{\mathbb{C}}^{an}}, H^{2d-1-n}Rf_*\alpha(X))$$

$$\xrightarrow{(j^*, j^*, 0)} (H^{2d-1}R(f \circ j)_{*Hdg}(O_{(X \times \square^n) \setminus |Z|}, F_b), H^{2d-1}R(f \circ j)_{*\mathbb{Z}_{((X \times \square^n) \setminus |Z|)_{\mathbb{C}}^{an}}},$$

$$H^{2d-1}R(f \circ j)_{*\alpha}((X \times \square^n) \setminus |Z|)^{[Z]} \xrightarrow{(\partial, \partial, 0)}$$

$$(H^{2d}Rf_{*Hdg}R\Gamma_{|Z|}^{Hdg}(O_{X \times \square^n}, F_b), (H^{2d}Rf_{*}R\Gamma_{|Z|}\mathbb{Z}_{(X \times \square^n)_{\mathbb{C}}^{an}}), Rf_{*\Gamma_{|Z|}\alpha}(X \times \square^n)^{[Z]} = \mathbb{Z}_S^{Hdg}(n-d) \rightarrow 0)$$

where $j : (X \times \square^n) \setminus |Z| \hookrightarrow X \times \square^n$ is the open embedding and

$$(H^{2d}Rf_{*Hdg}R\Gamma_{|Z|}^{Hdg}\mathbb{Z}_{X \times \square^n}^{Hdg})^{[Z]} \subset H^{2d}Rf_{*Hdg}R\Gamma_{|Z|}^{Hdg}\mathbb{Z}_{X \times \square^n}^{Hdg},$$

$$(H^{2d}R(f \circ j)_{*Hdg}\mathbb{Z}_{(X \times \square^n) \setminus |Z|}^{Hdg})^{[Z]} \subset H^{2d}R(f \circ j)_{*Hdg}\mathbb{Z}_{(X \times \square^n) \setminus |Z|}^{Hdg}.$$

are the subobjects given by the pullback of the class of Z .

Proof. Follows from theorem 2 by definition of the Abel Jacobi map and by the base change for mixed hodge modules. \square

We have the following main result of [3] :

Theorem 3. Let $f : X \rightarrow S$ a smooth projective morphism with $S, X \in \text{SmVar}(k)$. Let $d, n \in \mathbb{N}$. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. For $Z \in \mathcal{Z}^d(X, n)_{f_{hom}}^{f, \partial=0}$, the zero locus $V(\nu_Z) \subset S_{\mathbb{C}}$ of

$$\mu_Z := AJ_\sigma(X/S)(Z) \in \Gamma(S_{\mathbb{C}}^{an}, J_\sigma^{2d-1-n, d}(X/S))$$

is an algebraic subvariety.

Proof. See [3]: if $\bar{S} \in \text{PSmVar}(k)$ is a compactification of S with $\bar{S} \setminus S = \cup_i D_i \subset \bar{S}$ a normal crossing divisor, there exist an analytic subset $\Sigma(\nu_Z) \subset \bar{S}_{\mathbb{C}}$ such that $V(\nu_Z) = \Sigma(\nu_Z) \cap S_{\mathbb{C}}$. By GAGA $\Sigma(\nu_Z) \subset \bar{S}_{\mathbb{C}}$ is algebraic subvariety. Hence $V(\nu_Z) \subset S_{\mathbb{C}}$ is an algebraic subvariety. \square

4.2 The etale Abel Jacobi map for higher Chow group and etale normal functions

Let k a field of finite type over \mathbb{Q} . Let \bar{k} the algebraic closure of k and denote by $G = Gal(\bar{k}/k)$ its galois group. Let $p \in \mathbb{N}$ a prime integer.

Definition 3. Let $X \in \text{SmVar}(k)$ irreducible. Let $\bar{X} \in \text{PSmVar}(k)$ a compactification of X with $D := \bar{X} \setminus X \subset \bar{X}$ a normal crossing divisor. Denote $G = Gal(\bar{k}/k)$ the absolute galois group. The cycle class map

$$\mathcal{R}_X^{et,p} : \mathcal{Z}^d(X, n)^{\partial=0} \rightarrow H_{p_X(|Z|), et}^{2d-n}(X, D, \hat{\mathbb{Z}}_p) \rightarrow H_{et}^{2d-n}(X, D, \hat{\mathbb{Z}}_p),$$

to continuous etale cohomology induces the etale Abel Jacobi map for higher Chow groups

$$\begin{aligned} AJ_{et,p}(X) : \mathcal{Z}^d(X, n)_{hom}^{\partial=0} &\rightarrow \text{Ext}_G^1(\bar{k}, H_{et}^{2d-1-n}(X_{\bar{k}}, D_{\bar{k}}, \mathbb{Z}_p)), \\ Z &\mapsto AJ_{et,p}(X)(Z) := L^1\mathcal{R}_X^{et,p}(Z)/L^2\mathcal{R}_X^{et,p}(Z), \end{aligned}$$

where L is the filtration given by the Leray spectral sequence of the map of sites $a_X :: X^{et} \rightarrow \text{Spec}(k)^{et}$.

Theorem 4. Let $X \in \text{PSmVar}(k)$. Denote $G = Gal(\bar{k}/k)$ the absolute galois group.

(i) For $Z \in \mathcal{Z}^d(X, n)_{hom}^{\partial=0}$, we have

$$\begin{aligned} AJ_{et,p}(X)(Z) &= (0 \rightarrow H_{et}^{2d-1-n}(X_{\bar{k}}, \mathbb{Z}_p) \xrightarrow{j^*} H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p)^{[Z]} \\ &\quad \xrightarrow{\partial} H_{|Z|, et}^{2d}((X \times \square^n)_{\bar{k}}, \mathbb{Z}_p)^{[Z]} = \bar{k}(n-d) \rightarrow 0) \end{aligned}$$

with $j : (X \times \square^n) \setminus |Z| \hookrightarrow X \times \square^n$ the open embedding, and

$$\begin{aligned} H_{|Z|, et}^{2d}(((X \times \square^n)_{\bar{k}}, \mathbb{Z}_p)^{[Z]} &\subset H_{|Z|, et}^{2d}((X \times \square^n)_{\bar{k}}, \mathbb{Z}_p), \\ H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p)^{[Z]} &\subset H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p) \end{aligned}$$

are the subobjects given by the pullback by the class of Z (see section 2).

(ii) Let $Z \in \mathcal{Z}^d(X, n)_{hom}^{\partial=0}$. Then $AJ_{et,p}(X)(Z) = 0$ if and only if there exist

$$\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p)^{[Z]}$$

such that $\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p)^G$ and $\partial\alpha \neq 0$.

Proof. (i): See [14].

(ii): Follows from (i). □

Definition 4. Let $f : X \rightarrow S$ a morphism with $S, X \in \text{SmVar}(k)$. Let $j : S^o \hookrightarrow S$ an open subset such that for all $j \in \mathbb{Z}$, $j^*H^j Rf_*\mathbb{Z}_{p, X_{\bar{k}}}$ is a local system. Let $d, n \in \mathbb{N}$. We have then, denoting $X^o := X \times_S S^o$ and using definition 3, the map

$$\begin{aligned} AJ_{et,p}(X^o/S^o) : \mathcal{Z}^d(X, n)_{f, hom}^{f, \partial=0} &\rightarrow \Gamma(S^o, \oplus_{s \in S^o_{(0)}} i_{s*} \text{Ext}_{Gal(\bar{k}/k(s))}^1(\bar{k}, H_{et}^{2d-n-1}(X_{s, \bar{k}}, \mathbb{Z}_p))), \\ Z &\mapsto AJ_{et,p}(X^o/S^o)(Z) := \nu_Z^{et,p} := \\ ((s \in S^o_{(0)}) &\mapsto (AJ_{et,p}(X_s)(Z_s) \in \text{Ext}_{Gal(\bar{k}/k(s))}^1(\bar{k}, H_{et}^{2d-n-1}(X_{s, \bar{k}}, \mathbb{Z}_p)))) \end{aligned}$$

where $\mathcal{Z}^d(X, n)_{f, hom}^{f, \partial=0} \subset \mathcal{Z}^d(X, n)^{f, \partial=0}$ denote the subabelian group consisting of algebraic cycles Z with $Z_s := i_s^*Z \in \mathcal{Z}^d(X_s, n)_{hom}$. Recall that $i_s : \{s\} \hookrightarrow S_{(0)} \subset S$ is a closed Zariski point of S .

We now localize, for each prime number l and each embedding $\sigma_l : k \hookrightarrow \mathbb{C}_l$ the definition given above.

Definition 5. Let $X \in \text{SmVar}(k)$ irreducible. Let $\bar{X} \in \text{PSmVar}(k)$ a compactification of X with $D := \bar{X} \setminus X \subset \bar{X}$ a normal crossing divisor. Let $\sigma_l : k \hookrightarrow \mathbb{C}_l$ an embedding. Then $k \subset \bar{k} \subset \mathbb{C}_l$, where \bar{k} is the algebraic closure of k and $k \subset \hat{k}_{\sigma_l} \subset \mathbb{C}_l$ where \hat{k}_{σ_l} is the completion of k with respect to σ_l . Denote $\hat{G}_{\sigma_l} := \text{Gal}(\mathbb{C}_l/\hat{k}_{\sigma_l})$. The cycle class map

$$\mathcal{R}_{X,\sigma_l}^{et,p} : \mathcal{Z}^d(X, n)^{\partial=0} \rightarrow H_{p_X(|Z|),et}^{2d-n}(X_{\hat{k}_{\sigma_l}}, D_{\hat{k}_{\sigma_l}}, \hat{\mathbb{Z}}_p) \rightarrow H_{et}^{2d-n}(X_{\hat{k}_{\sigma_l}}, D_{\hat{k}_{\sigma_l}}, \hat{\mathbb{Z}}_p),$$

to continuous etale cohomology induces the etale Abel Jacobi map for higher Chow groups

$$\begin{aligned} AJ_{et,p,\sigma_l}(X) : \mathcal{Z}^d(X, n)_{hom}^{\partial=0} &\rightarrow \text{Ext}_{\hat{G}_{\sigma_l}}^1(\mathbb{C}_l, H_{et}^{2d-1-n}(X_{\mathbb{C}_l}, D_{\mathbb{C}_l}, \mathbb{Z}_p)), \\ Z &\mapsto AJ_{et,p,\sigma_l}(X)(Z) := L^1\mathcal{R}_{X,\sigma_l}^{et,p}(Z)/L^2\mathcal{R}_{X,\sigma_l}^{et,p}(Z), \end{aligned}$$

where L is the filtration given by the Leray spectral sequence of the map of sites $a_X :: X_{\hat{k}_{\sigma_l}}^{et} \rightarrow \text{Spec}(\hat{k}_{\sigma_l})^{et}$.

We have then the commutative diagram

$$\begin{array}{ccc} & & \text{Ext}_{\hat{G}}^1(\bar{k}, H_{et}^{2d-1-n}(X_{\bar{k}}, D_{\bar{k}}, \mathbb{Z}_p)) \\ & \nearrow^{AJ_{et,p}(X)} & \downarrow \text{Ext}^1(\text{ad}(\pi_{\bar{k}/\hat{k}_{\sigma_l}}^*, \pi_{\bar{k}/\hat{k}_{\sigma_l}^*})(-), \text{ad}(\pi_{\bar{k}/\hat{k}_{\sigma_l}}^*, \pi_{\bar{k}/\hat{k}_{\sigma_l}^*})(-)) \\ \mathcal{Z}^d(X, n)_{hom}^{\partial=0} & & \text{Ext}_{\hat{G}_{\sigma_l}}^1(\bar{k}, H_{et}^{2d-1-n}(X_{\bar{k}}, D_{\bar{k}}, \mathbb{Z}_p)) \\ & \searrow_{AJ_{et,p,\sigma_l}(X)} & \\ & & \text{Ext}_{\hat{G}_{\sigma_l}}^1(\mathbb{C}_l, H_{et}^{2d-1-n}(X_{\mathbb{C}_l}, D_{\mathbb{C}_l}, \mathbb{Z}_p)) = \text{Ext}_{\hat{G}_{\sigma_l}}^1(\bar{k}, H_{et}^{2d-1-n}(X_{\bar{k}}, D_{\bar{k}}, \mathbb{Z}_p)) \end{array}$$

where the right column arrow is given by the restriction $\pi_{\bar{k}/\hat{k}_{\sigma_l}} : \hat{G}_{\sigma_l} \hookrightarrow \hat{G}$.

Theorem 5. Let $X \in \text{PSmVar}(k)$. Let $\sigma_l : k \hookrightarrow \mathbb{C}_l$ an embedding and $k \subset \hat{k}_{\sigma_l} \subset \mathbb{C}_l$ the completion of k with respect to σ_l .

(i) For $Z \in \mathcal{Z}^d(X, n)_{hom}^{\partial=0}$, we have

$$\begin{aligned} AJ_{et,p,\sigma_l}(X)(Z) &= (0 \rightarrow H_{et}^{2d-1-n}(X_{\mathbb{C}_l}, \mathbb{Z}_p) \xrightarrow{j^*} H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_l}, \mathbb{Z}_p)^{[Z]} \\ &\quad \xrightarrow{\partial} H_{|Z|,et}^{2d}((X \times \square^n)_{\mathbb{C}_l}, \mathbb{Z}_p)^{[Z]} = \mathbb{C}_l(n-d) \rightarrow 0) \end{aligned}$$

with $j : X \times \square^n \setminus |Z| \hookrightarrow X$ the open embedding and

$$\begin{aligned} H_{|Z|,et}^{2d}((X \times \square^n)_{\mathbb{C}_l}, \mathbb{Z}_p)^{[Z]} &\subset H_{|Z|,et}^{2d}((X \times \square^n)_{\mathbb{C}_l}, \mathbb{Z}_p), \\ H_{|Z|,et}^{2d}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_l}, \mathbb{Z}_p)^{[Z]} &\subset H_{|Z|,et}^{2d}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_l}, \mathbb{Z}_p), \end{aligned}$$

are the subobjects given by the pullback by the class of Z (see section 2).

(ii) Let $Z \in \mathcal{Z}^d(X, n)_{hom}^{\partial=0}$. Then $AJ_{et,p,\sigma_l}(X)(Z) = 0$ if and only if there exist

$$\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_l}, \mathbb{Z}_p)^{[Z]}$$

such that $\partial\alpha \neq 0$ and $\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_l}, \mathbb{Z}_p)^{\hat{G}_{\sigma_l}}$.

Proof. Similar to the proof of theorem 4. □

Definition 6. Let $f : X \rightarrow S$ a morphism with $S, X \in \text{SmVar}(k)$. Let $j : S^\circ \hookrightarrow S$ an open subset such that for all $j \in \mathbb{Z}$, $j^* H^j Rf_* \mathbb{Z}_{p, X_{\bar{k}}}$ is a local system. Let $d, n \in \mathbb{N}$. Let $\sigma_l : k \hookrightarrow \mathbb{C}_l$ an embedding and $k \subset \hat{k}_{\sigma_l} \subset \mathbb{C}_l$ the completion of k with respect to σ_l . Denoting $X^\circ := X \times_S S^\circ$, we consider

$$AJ_{et,p,\sigma_l}(X^\circ/S^\circ) : \mathcal{Z}^d(X, n)_{f_{hom}}^{f, \partial=0} \rightarrow \Gamma(S^\circ, \bigoplus_{s \in S_{(0)}^\circ} i_{s*} \text{Ext}^1(\text{Gal}(\mathbb{C}_l/\hat{k}_{\sigma_l}(s)), H_{et}^{2d-n-1}(X_{s, \bar{k}}, \mathbb{Z}_p))),$$

$$Z \mapsto AJ_{et,p,\sigma_l}(X^\circ/S^\circ)(Z) := \nu_{Z, \sigma_l}^{et,p} :=$$

$$((s \in S_{(0)}^\circ) \mapsto (AJ_{et,p,\sigma_l}(X_s)(Z_s) \in \text{Ext}_{\text{Gal}(\mathbb{C}_l/\hat{k}_{\sigma_l}(s))}^1(\bar{k}, H_{et}^{2d-n-1}(X_{s, \bar{k}}, \mathbb{Z}_p))))$$

where $\mathcal{Z}^d(X, n)_{f_{hom}}^{f, \partial=0} \subset \mathcal{Z}^d(X, n)^{f, \partial=0}$ denote the subabelian group consisting of algebraic cycles Z with $Z_s := i_s^* Z \in \mathcal{Z}^d(X_s, n)_{hom}$. Recall that $i_s : \{s\} \hookrightarrow S_{(0)} \subset S$ is a closed Zariski point of S .

5 The vanishing of the etale Abel Jacobi map implies the vanishing of the complex Abel Jacobi map

The p adic Hodge theory for open varieties implies the following main theorem :

Theorem 6. Let k a field of finite type over \mathbb{Q} . Denote \bar{k} the algebraic closure of k and $G = \text{Gal}(\bar{k}/k)$ its absolute Galois group. Let $X \in \text{PSmVar}(k)$. Consider an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Let $p \in \mathbb{N} \setminus \delta(k, X)$ be a prime number (any by finitely many). Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ be an embedding. Denote $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the completion of k with respect to σ_p . Let $Z \in \mathcal{Z}^d(X, n)_{hom}^{\partial=0}$. Consider the exact sequences

- $0 \rightarrow H_{et}^{2d-1-n}(X_{\bar{k}}, \mathbb{Z}_p) \xrightarrow{j^*} H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p) \xrightarrow{\partial} H_{et, |Z|}^{2d}(((X \times \square^n)_{\bar{k}}, \mathbb{Z}_p)^0 \rightarrow 0$
- $0 \rightarrow H_{DR}^{2d-1-n}(X) \xrightarrow{j^*} H_{DR}^{2d-1}(((X \times \square^n) \setminus |Z|) \xrightarrow{\partial} H_{DR, |Z|}^{2d}(X \times \square^n)^0 \rightarrow 0,$

where $j : (X \times \square^n) \setminus |Z| \hookrightarrow X \times \square^n$ is the open embedding. Consider the following assertions :

- (i) $AJ_{et,p}(X)(Z) = 0 \in \text{Ext}_1^G(\bar{k}, H_{et}^{2d-1-n}(X_{\bar{k}}, \mathbb{Z}_p)(d-n)),$
- (i)' there exist $\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p)(d)^{[Z]}$ such that $\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p)(d)^G$ and $\partial\alpha \neq 0,$
- (ii) $AJ_{et,p,\sigma_p}(X)(Z) = 0 \in \text{Ext}_1^{\hat{G}_{\sigma_p}}(\mathbb{C}_p, H_{et}^{2d-1-n}(X_{\mathbb{C}_p}, \mathbb{Z}_p)(d-n)),$ obviously (i) implies (ii),
- (ii)' there exist $\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_p}, \mathbb{Z}_p)(d)^{[Z]}$ such that $\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_p}, \mathbb{Z}_p)(d)^{\hat{G}_{\sigma_p}}$ and $\partial\alpha \neq 0,$ obviously (i)' implies (ii)',
- (iii) there exist $w \in H_{DR}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\hat{k}_{\sigma_p}})^{[Z]}$ such that $w \in F^d H_{DR}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\hat{k}_{\sigma_p}}),$

$$w \in OL_{X \times \square^n}^{\theta_{2d-1}^\phi}(\mathbb{H}_{pet}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_p}, \Omega_{(X \times \square^n)_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}^\bullet)),$$

see definition 1, and $\partial w \neq 0,$

- (iv) there exist $w \in H_{DR}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}})^{[Z]}$ such that $w \in F^d H_{DR}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}}),$

$$H^{2d-1} ev(((X \times \square^n) \setminus |Z|)(w) \in H_{\text{sing}}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}),$$

and $\partial w \neq 0,$

- (iv)' there exist an integer $m \in \mathbb{N}$ such that $m \cdot AJ_\sigma(X)(Z) = 0 \in J_\sigma^{2d-1-n,d}(X),$

where the inclusion $OL_{X_{\mathbb{C}_p}^{pet}, \mathcal{O}} : \Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}^\bullet \hookrightarrow \Omega_{X_{\mathbb{C}_p}^{pet}}^\bullet$ of $C(X_{\mathbb{C}_p}^{pet})$ is the subcomplex of logarithmic forms. Then (i) is equivalent to (i)', (ii) is equivalent to (ii)', (ii)' implies (iii), (iii) implies (iv), (iv) is equivalent to (iv)'. Hence (i) implies (iv)'.

Proof. (i) is equivalent to (i)': see theorem 4(ii),

(ii) is equivalent to (ii)': see theorem 5(ii),

(ii)' implies (iii): follows from proposition 2 and proposition 3(i): We consider a basis

$$(\alpha_1, \dots, \alpha_s) \in H^{2d-1-n}(X_{\mathbb{C}_p}, \mathbb{Z}_p)(d-n)^{\hat{G}_{\sigma_p}}, \text{ with } w(\alpha_1), \dots, w(\alpha_s) \in OL_X^{\theta_\phi^{2d-1-n}}(\mathbb{H}_{pet}^{2d-1-n}(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}^\bullet)).$$

By assumption on α , it extend to a basis

$$(\alpha_1, \dots, \alpha_s, \alpha_{s+1}) \in H^{2d-1}((X \times \square^n \setminus |Z|)_{\mathbb{C}_p}, \mathbb{Z}_p)^{[Z]}(d)^{\hat{G}_{\sigma_p}}$$

with $w(\alpha_1), \dots, w(\alpha_{s+1}) \in OL_{X \times \square^n}^{\theta_\phi^{2d-1}}(\mathbb{H}_{pet}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_p}, \Omega_{(X \times \square^n)_{\mathbb{C}_p}^{pet}, \log, \mathcal{O}}^\bullet)).$

We then take $w = w(\alpha_{s+1})$.

(ii) implies (iii): follows from proposition 1 with $w := \pi_{k/\mathbb{C}}((X \times \square^n) \setminus |Z|)^* w$,

(iii) is equivalent to (iii)' : see theorem 2(ii). □

It implies the following :

Corollary 3. (i) Let k a field of finite type over \mathbb{Q} . Denote \bar{k} the algebraic closure of k . Let $f : X \rightarrow S$ a smooth projective morphism with $S, X \in \text{SmVar}(k)$. Consider an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then we have $\bar{k} \subset \mathbb{C}$ the canonical algebraic closure of k inside \mathbb{C} . Let $p \in \mathbb{N}$ a prime number. Then for $Z \in \mathcal{Z}^d(X, n)_{f_{\text{hom}}^{f, \partial=0}}$, we have

$$V_{\text{tors}}(\nu_Z^{et, p})_{\mathbb{C}} \subset V_{\text{tors}}(\nu_Z) \subset S_{\mathbb{C}}$$

where

– $V(\nu_Z) \subset V_{\text{tors}}(\nu_Z) \subset S_{\mathbb{C}}$ is the zero locus, resp. torsion locus, of the complex normal function

$$\nu_Z =: AJ_\sigma(X/S)(Z) \in \Gamma(S_{\mathbb{C}}^{an}, J_\sigma^{2d-1-n, d}(X/S))$$

associated to Z (see proposition-definition 1),

– $V(\nu_Z^{et, p}) \subset V_{\text{tors}}(\nu_Z^{et, p}) \subset S$ is the zero locus, resp. torsion locus of the etale normal function

$$\nu_Z^{et, p} \in \Gamma(S, \bigoplus_{s \in S_{(0)}} i_{s*} \text{Ext}_{Gal(\bar{k}/k(s))}^1(\bar{k}, H_{et}^{2d-n-1}(X_{s, \bar{k}}, \mathbb{Z}_p))(d-n))$$

associated to Z (see definition 4) and

$$V(\nu_Z^{et, p})_{\mathbb{C}} := \pi_{k/\mathbb{C}}(S)^{-1}(V(\nu_Z^{et, p})), \quad V_{\text{tors}}(\nu_Z^{et, p})_{\mathbb{C}} := \pi_{k/\mathbb{C}}(S)^{-1}(V_{\text{tors}}(\nu_Z^{et, p}))$$

where we recall $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \rightarrow S$ is the projection.

(ii) Let $\sigma : k \hookrightarrow \mathbb{C}$ a subfield which is of finite type over \mathbb{Q} . Denote $\bar{k} \subset \mathbb{C}$ the algebraic closure of k . Let $f : X \rightarrow S$ a smooth projective morphism with $S, X \in \text{SmVar}(k)$. Then for $Z \in \mathcal{Z}^d(X, n)_{f_{\text{hom}}^{f, \partial=0}}$, the zero locus $V(\nu_Z) \subset S_{\mathbb{C}}$ of the complex normal function

$$\nu_Z =: AJ_\sigma(X/S)(Z) \in \Gamma(S_{\mathbb{C}}^{an}, J_\sigma^{2d-1-n, d}(X/S))$$

associated to Z is defined over \bar{k} if $V(\nu_Z^{et, p}) \neq \emptyset$.

Proof. (i): Follows immediately from theorem 6 since for $s \in S_{(0)}$, $k(s)$ is of finite type over \mathbb{Q} and for $s' \in \pi_{k/\mathbb{C}}(S)^{-1}(s)$ denoting $\sigma' : k(s) \hookrightarrow \mathbb{C}$ the embedding given by s' , we have by definition

- $\nu_Z(s') =: AJ_\sigma(X/S)(Z)(s') := AJ_{\sigma'}(X_s)(Z_s) \in J_{\sigma'}^{2d-1-n, d}(X_s)$.

- $\nu_Z^{et, p}(s) =: AJ_{et, p}(X/S)(Z)(s) := AJ_{et, p}(X_s)(Z_s) \in \text{Ext}_{Gal(\bar{k}/k(s))}^1(\bar{k}, H_{et}^{2d-n-1}(X_{s, \bar{k}}, \mathbb{Z}_p)(d-n))$.

(ii): Since $V(\nu_Z^{et, p}) \subset S$ contain a \bar{k} point, $V(\nu_Z) \subset S_{\mathbb{C}}$ contain a \bar{k} point by (i). Hence by the work of [17] or [8], $V(\nu_Z) \subset S_{\mathbb{C}}$ is defined over \bar{k} . □

6 Algebraicity of the zero locus of etale normal functions and the locus of Hodge Tate classes

Let k a field of finite type over \mathbb{Q} . Let $f : X \rightarrow S$ a smooth proper morphism with $S, X \in \text{SmVar}(k)$ connected. Let p a prime number. Let $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0, f}$. By the definition, we have

$$V(\nu_Z^{\text{et}, p}) \subset \bigcap_{l \in \mathbb{N}, l \text{ prime}} \bigcap_{\sigma_l : k \hookrightarrow \mathbb{C}_l} V(\nu_{Z, \sigma_l}^{\text{et}, p}) \subset S_{(0)}$$

and

$$V_{\text{tors}}(\nu_Z^{\text{et}, p}) \subset \bigcap_{l \in \mathbb{N}, l \text{ prime}} \bigcap_{\sigma_l : k \hookrightarrow \mathbb{C}_l} V_{\text{tors}}(\nu_{Z, \sigma_l}^{\text{et}, p}) \subset S_{(0)}.$$

In this section, we investigate the algebraicity of $V_{\text{tors}}(\nu_Z^{\text{et}, p}) \subset S$ and of $V_{\text{tors}}(\nu_{Z, \sigma_l}^{\text{et}, p}) \subset S$, $\sigma_l : k \hookrightarrow \mathbb{C}_l$. Since Z is defined over k , we expect that the inclusion $V_{\text{tors}}(\nu_Z^{\text{et}, p}) \subset V(\nu_{Z, \sigma_l}^{\text{et}, p})$ is an equality for all primes $l \in \mathbb{N}$.

Remark 4. *Let k a field of finite type over \mathbb{Q} . Let $f : X \rightarrow S$ a smooth proper morphism with $S, X \in \text{SmVar}(k)$ connected. Let p a prime number. Let $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0, f}$. We can show, using [13], that we have in fact*

$$V_{\text{tors}}(\nu_Z^{\text{et}, p}) = \bigcap_{l \in \mathbb{N}, l \text{ prime}} \bigcap_{\sigma_l : k \hookrightarrow \mathbb{C}_l} V_{\text{tors}}(\nu_{Z, \sigma_l}^{\text{et}, p}) \subset S_{(0)}.$$

We don't need this result so we don't give the details.

Let k a field of finite type over \mathbb{Q} . Let p be a prime number. Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ be an embedding. We have then \hat{k}_{σ_p} the completion of k with respect to σ_p and we denote $O_{\hat{k}_{\sigma_p}} \subset \hat{k}_{\sigma_p}$ its ring of integers. We then consider the canonical functor of Huber (see section 2)

$$\mathcal{R} : \text{Var}(\hat{k}_{\sigma_p}) \rightarrow \text{HubSp}(\hat{k}_{\sigma_p}, O_{\hat{k}_{\sigma_p}}) \rightarrow \text{Sch}/O_{\hat{k}_{\sigma_p}}, X \mapsto \mathcal{R}(X) = X^{\mathcal{O}}$$

which associated to a variety over a p adic field its canonical integral model. Let $f : X \rightarrow S$ a smooth projective morphism with $S, X \in \text{SmVar}(k)$. Let $Z \subset X$ a closed subset and $j : U = X \setminus Z \hookrightarrow X$ the open complementary subset. We have then $f := f_{\hat{k}_{\sigma_p}} : (X, Z)_{\hat{k}_{\sigma_p}} \rightarrow S_{\hat{k}_{\sigma_p}}$ the morphism in $\text{SmVar}^2(\hat{k}_{\sigma_p})$ induced by the scalar extension functor and

$$f^{\mathcal{O}} := \mathcal{R}(f_{\hat{k}_{\sigma_p}}) : (X, Z)_{\hat{k}_{\sigma_p}}^{\mathcal{O}} \rightarrow S_{\hat{k}_{\sigma_p}}^{\mathcal{O}}$$

its canonical integral model in $\text{Sch}^2/O_{\hat{k}_{\sigma_p}}$ to which we denote

$$f := f^{\mathcal{O}} : (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}}) := (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, (M_Z, N_{\mathcal{O}})) \rightarrow (S_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{\mathcal{O}})$$

the corresponding morphism in logSch , where for K a p adic field and $Y \in \text{Sch}/O_K$, $(Y, N_{\mathcal{O}}) := (Y, M_{Y_k})$ with $k = O_K/(\pi)$ the residual field. We have then the morphisms of sites

$$v_{X, N} : (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})^{\text{Falt}} \rightarrow (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})^{\text{ket}}, u_{X, N} : (X_{\mathbb{C}_p}, M_{Z_{\mathbb{C}_p}})^{\text{ket}} \rightarrow (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})^{\text{Falt}}$$

where $(X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})^{\text{Falt}}$ denote the Falting site, and for $(Y, N) \in \text{logSch}$, $(Y, N)^{\text{ket}} \subset \text{logSch}/(Y, N)$ is the small Kummer etale site. If $(X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})$ is log smooth, we consider an hypercover

$$a_{\bullet} : (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}, \bullet}, N_{U, \mathcal{O}}) \rightarrow (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})$$

in $\text{Fun}(\Delta, \text{logSch})$ by small log schemes in sense of [1]. The main result of [1] say that

- if $(X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_U, \mathcal{O})$ is log smooth, the embedding in $C((X_{\hat{k}_{\sigma_p}}^{\mathcal{O}})^{Falt})$

$$\alpha(U) : (\mathbb{B}_{st, X_{\hat{k}_{\sigma_p}}}, N_U, \mathcal{O}) \hookrightarrow a_{\bullet*} DR(X_{\hat{k}_{\sigma_p}}^{\mathcal{O}, \bullet} / S_{\hat{k}_{\sigma_p}}^{\mathcal{O}})(\mathcal{O}_{\mathbb{B}_{st, X_{\hat{k}_{\sigma_p}}}}, N_U, \mathcal{O})$$

is a filtered quasi-isomorphism compatible with the action of $Gal(\mathbb{C}_p, \hat{k}_{\sigma_p})$, the Frobenius ϕ_p and the monodromy N , note that we have a commutative diagram in $C_{fil}(X_{\mathbb{C}_p}^{an, pet})$

$$\begin{array}{ccc} \mathbb{B}_{st, X_{\mathbb{C}_p}, \log} & \xrightarrow{\alpha(U)} & \mathcal{O}_{\mathbb{B}_{st, X_{\mathbb{C}_p}, \log}} \otimes_{\mathcal{O}_X} \Omega_{X_{\mathbb{C}_p}}^{\bullet}(\log D_{\mathbb{C}_p}) \\ \downarrow \subset & & \downarrow \subset \\ \mathbb{B}_{dr, X_{\mathbb{C}_p}} & \xrightarrow{\alpha(U)} & \mathcal{O}_{\mathbb{B}_{dr, X_{\mathbb{C}_p}, \log}} \otimes_{\mathcal{O}_X} \Omega_{X_{\mathbb{C}_p}}^{\bullet}(\log D_{\mathbb{C}_p}) \end{array}$$

see section 3,

- if $f : (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_U, \mathcal{O}) \rightarrow (S_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{\mathcal{O}})$ is log smooth, the morphism in $D_{fil}((S_{\hat{k}_{\sigma_p}}^{\mathcal{O}})^{Falt})$

$$\begin{aligned} T(f, \mathbb{B}_{st}) : Rf_*(\mathbb{B}_{st, X_{\hat{k}_{\sigma_p}}}, N_U, \mathcal{O}) &\xrightarrow{T(f, f, \otimes)(-) \circ \text{ad}(u_{X, N}^*, Ru_{X, N, *})(-)} Rf_* \mathbb{Z}_p, (X_{\mathbb{C}_p}, M_{\mathbb{Z}_{\mathbb{C}_p}})^{ket} \otimes_{\mathbb{Z}_p} \mathbb{B}_{st, S_{\hat{k}_{\sigma_p}}} \\ &\xrightarrow{\text{ad}(j^*, Rj_*)(\mathbb{Z}_p, (X_{\mathbb{C}_p}, M_{\mathbb{Z}_{\mathbb{C}_p}})^{et})} R(f \circ j)_* \mathbb{Z}_p, U_{\mathbb{C}_p}^{et} \otimes_{\mathbb{Z}_p} \mathbb{B}_{st, S_{\hat{k}_{\sigma_p}}} \end{aligned}$$

is an isomorphism, where the last map is an isomorphism by [12] theorem 7.4.

This gives if $(X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_U, \mathcal{O})$ is log smooth, for each $j \in \mathbb{Z}$, a filtered isomorphism of filtered abelian groups

$$\begin{aligned} H^j R\alpha(U) : H_{et}^j(U_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{st, \hat{k}_{\sigma_p}} &\xrightarrow{H^j T(a_X, \mathbb{B}_{st})^{-1}} H_{et}^j((X, N)^{Falt})(\mathbb{B}_{st, X_{\hat{k}_{\sigma_p}}}, N_U, \mathcal{O}) \\ &\xrightarrow{H^j R\Gamma((X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_U, \mathcal{O}), \alpha(U))} H_{DR}^j(U_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{st, \hat{k}_{\sigma_p}} \end{aligned}$$

compatible with the action of $Gal(\mathbb{C}_p/\hat{k}_{\sigma_p})$, of the Frobenius ϕ_p and the monodromy N . More generally, this gives if $\hat{k}_{\sigma_p}(s)$ is unramified for all $s \in S_{\hat{k}_{\sigma_p}}$ and if $f : (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_U, \mathcal{O}) \rightarrow (S_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{\mathcal{O}})$ is log smooth, an isomorphism in $\text{Shv}_{fil, G, \phi_p, N}(S_{\hat{k}_{\sigma_p}})$

$$\begin{aligned} H^j f_* \alpha(U) : R^j f_* \mathbb{Z}_p, U_{\mathbb{C}_p}^{et} \otimes_{\mathbb{Z}_p} \mathbb{B}_{st, S_{\hat{k}_{\sigma_p}}} &\xrightarrow{H^j T(f, \mathbb{B}_{st})^{-1}} Rf_*(\mathbb{B}_{st, X_{\hat{k}_{\sigma_p}}}, N_U, \mathcal{O}) \\ &\xrightarrow{H^j Rf_* \alpha(U)} H^j \int_f j_* Hdg(\mathcal{O}_{U_{\hat{k}_{\sigma_p}}}, F_b) \otimes_{\mathcal{O}_{S_{\hat{k}_{\sigma_p}}}} \mathcal{O}_{\mathbb{B}_{st, S_{\hat{k}_{\sigma_p}}}} \end{aligned}$$

that is a filtered isomorphism compatible with the action of $G = Gal(\mathbb{C}_p/\hat{k}_{\sigma_p})$, of the Frobenius ϕ_p and the monodromy N , writing for short again $f = f \circ j$.

Definition 7. Let k a field of finite type over \mathbb{Q} . Let $f : X \rightarrow S$ a smooth proper morphism with $S, X \in \text{SmVar}(k)$. Let p a prime number. Let $Z \in \mathcal{Z}(X, n)_{f, hom}^{f, \partial=0}$. Denote $U := (X \times \square^n) \setminus |Z|$. We have then the following exact sequence in $\text{Vect}_{\mathcal{D}, fil}(S)$ (of filtered vector bundle with integrable connexion)

$$\begin{aligned} 0 \rightarrow E_{DR}^{2d-1-n}(X/S) := H^{2d-1-n} \int_f (\mathcal{O}_X, F_b) \xrightarrow{j^*} E_{DR}^{2d-1}(U/S)^{[Z]} := (H^{2d-1} \int_f j_* Hdg(\mathcal{O}_U, F_b))^{[Z]} \\ \xrightarrow{\partial} E_{DR, Z}^{2d}(X/S)^{[Z]} := (H^{2d} \int_f \Gamma_{|Z|}^{Hdg}(\mathcal{O}_{X \times \square^n}, F_b))^{[Z]} \rightarrow 0. \end{aligned}$$

Recall (see section 2) that $(X \times \square^n, M_{|Z|}) \in \text{logVar}(k)$ denote log structure associated to $(X \times \square^n, |Z|) \in \text{SmVar}^2(k)$. There exist a finite set of prime numbers $\delta(S)$ such that for all prime $p \in \mathbb{N} \setminus \delta(S)$, all embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$ and all $s \in S_{(0)}$, $\hat{k}_{\sigma_p}(s)$ is unramified over \mathbb{Q}_p , where $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p adic completion of k with respect to σ_p . Let $p \in \mathbb{N} \setminus \delta(S)$ a prime number and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ be an embedding and consider $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p adic completion of k with respect to σ_p .

- Take, using [10] theorem 8.2 (after considering an integral model $(X, Z)^{\mathcal{R}} \in \text{Sch}^2/\mathcal{R}$ over $\mathcal{R} \subset k$ of finite type over \mathbb{Z} with function field k), for $p \in \mathbb{N} \setminus \delta^0(X/S)$, where $\delta^0(X/S)$ is a finite set, an alteration $\pi^0 : (X \times \square^n)^0 \rightarrow X \times \square^n$, that is a generically finite morphism such that $((X \times \square^n)^0, \pi^{0,-1}(|Z|))_{\hat{k}_{\sigma_p}}^{\mathcal{O}}$ is semi-stable pair, that is $\pi^{0,-1}(|Z|) \subset ((X \times \square^n)^0)$ is a normal crossing divisor and $((X \times \square^n)^0, \pi^{0,-1}(|Z|))_{\hat{k}_{\sigma_p}}^{\mathcal{O}}|_t$ has semi-stable reduction where $t \in \text{Spec}(O_{\hat{k}_{\sigma_p}})$ is the closed point. Then there exists a closed subset $\Delta \subset S$ such that for all $s \in S^{\circ} := S \setminus \Delta$, $((X \times \square^n)^0, \pi^{0,-1}(|Z|))_{\hat{k}_{\sigma_p}, s}^{\mathcal{O}}$ is a semi-stable pair.
- Take using [10] theorem 8.2, (after considering an integral model $(X_{\Delta}, Z_{\Delta})^{\mathcal{R}} \in \text{Sch}^2/\mathcal{R}$ over $\mathcal{R} \subset k$ of finite type over \mathbb{Z} with function field k), for $p \in \mathbb{N} \setminus \delta^1(X/S)$, where $\delta^1(X/S)$ is a finite set, an alteration $\pi^1 : (X_{\Delta} \times \square^n)^1 \rightarrow X_{\Delta} \times \square^n$ such that $((X_{\Delta} \times \square^n)^1, |Z_{\Delta}|)_{\hat{k}_{\sigma_p}}^{\mathcal{O}}$ is a semi-stable pair. Then there exists a closed subset $\Delta^2 \subset \Delta$ such that for all $s \in S^1 := \Delta \setminus \Delta^2$, $((X_{\Delta} \times \square^n)^1, |Z_{\Delta}|)_{\hat{k}_{\sigma_p}, s}^{\mathcal{O}}$ is a semi-stable pair.
- Go on by induction.

We obtain by the above finite induction, for $p \in \mathbb{N} \setminus \delta(S, X/S)$, with $\delta(S, X/S) := \delta(S) \cup (\cup_{\alpha \in \Lambda} \delta^{\alpha}(X/S))$, a stratification $S = \sqcup_{\alpha \in \Lambda} S^{\alpha}$, Λ being a finite set, by locally closed subset $S^{\alpha} \subset S$, and alterations (i.e. generically finite morphisms) $\pi^{\alpha} : (X_{S^{\alpha}} \times \square^n)^{\alpha} \rightarrow X_{S^{\alpha}} \times \square^n$ such that

$$f \circ \pi^{\alpha} : (X_{S^{\alpha}} \times \square^n)_{\hat{k}_{\sigma_p}}^{\alpha, \mathcal{O}}, N_{U^{\alpha}, \mathcal{O}} \rightarrow (S_{\hat{k}_{\sigma_p}}^{\alpha}, N_{\mathcal{O}})$$

is log smooth, that is for all $s \in S^{\alpha}$, $((X_{S^{\alpha}} \times \square^n)^{\alpha}, \pi^{\alpha,-1}(|Z_{S^{\alpha}}|))_{\hat{k}_{\sigma_p}, s}^{\mathcal{O}}$ is a semi-stable pair. We then set

$$\begin{aligned} T &:= p_S(((\cup_{\alpha \in \Lambda} (E_{DR}^{2d-1}(U_{\hat{k}_{\sigma_p}}^{\alpha}/S_{\hat{k}_{\sigma_p}}^{\alpha})) \otimes_{O_{S_{\hat{k}_{\sigma_p}}}} O_{\mathbb{B}_{st, S_{\hat{k}_{\sigma_p}}}}))^{\phi_p, N}) \\ &\cap F^d E_{DR}^{2d-1}(U/S) \cap (E_{DR}^{2d-1}(U/S)^{[Z]}) \setminus E_{DR}^{2d-1}((X \times \square^n)/S) \subset S \end{aligned}$$

and

$$\begin{aligned} \hat{T}_{\sigma_p} &:= p_{S_{\hat{k}_{\sigma_p}}}(((\cup_{\alpha \in \Lambda} (E_{DR}^{2d-1}(U_{\hat{k}_{\sigma_p}}^{\alpha}/S_{\hat{k}_{\sigma_p}}^{\alpha})) \otimes_{O_{S_{\hat{k}_{\sigma_p}}}} O_{\mathbb{B}_{st, S_{\hat{k}_{\sigma_p}}}}))^{\phi_p, N}) \cap \\ &F^d E_{DR}^{2d-1}(U_{\hat{k}_{\sigma_p}}/S_{\hat{k}_{\sigma_p}}) \cap E_{DR}^{2d-1}(U_{\hat{k}_{\sigma_p}}/S_{\hat{k}_{\sigma_p}})^{[Z]} \setminus E_{DR}((X \times \square^n)_{\hat{k}_{\sigma_p}}/S_{\hat{k}_{\sigma_p}}) \subset S_{\hat{k}_{\sigma_p}} \end{aligned}$$

where

- $U^{\alpha} := \pi^{\alpha,-1}(U_{S^{\alpha}}) = (X_{S^{\alpha}} \times \square^n)^{\alpha} \setminus \pi^{\alpha,-1}(|Z_{S^{\alpha}}|)$,
- $E_{DR}^{2d-1}(U^{\alpha}/S^{\alpha}) := H^{2d-1} Rf_{*Hdg}(O_{U^{\alpha}}, F_b) \in \text{Vect}_{fil}(S^{\alpha})$,
- ϕ_p is the Frobenius operator, N is the monodromy operator,
- $E_{DR}^{2d-1}(U_{S_{\hat{k}_{\sigma_p}}}^{\alpha}/S_{\hat{k}_{\sigma_p}}^{\alpha}) \subset E_{DR}^{2d-1}(U_{\hat{k}_{\sigma_p}}^{\alpha}/S_{\hat{k}_{\sigma_p}}^{\alpha})$
- $\pi_{k/\hat{k}_{\sigma_p}}(S^{\alpha})^* E_{DR}^{2d-1}(U^{\alpha}/S^{\alpha}) \subset E_{DR}^{2d-1}(U_{\hat{k}_{\sigma_p}}^{\alpha}/S_{\hat{k}_{\sigma_p}}^{\alpha})$ is the canonical subset of closed points,
- $p_S : E_{DR}^{2d-1}(U/S) \rightarrow S$ and $p_{S_{\hat{k}_{\sigma_p}}} : E_{DR}(U_{\hat{k}_{\sigma_p}}/S_{\hat{k}_{\sigma_p}}) \rightarrow S_{\hat{k}_{\sigma_p}}$ are the projections.

Lemma 3. *Let G be a group. Consider a commutative diagram of G modules*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & V & \xrightarrow{\partial} & K & \longrightarrow & 0 \\ & & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* & & \\ 0 & \longrightarrow & W' & \longrightarrow & V' & \xrightarrow{\partial'} & K' & \longrightarrow & 0 \end{array}$$

whose rows are exact sequence and $\pi^* : V \rightarrow V'$ is injective. Let $\alpha \in V$. Then $\alpha \in V^G$ and $\partial\alpha \neq 0$ if and only if $\pi^*\alpha \in V'^G$ and $\partial'\pi^*\alpha \neq 0$.

Proof. Follows from the fact that $\langle \alpha \rangle$ define a splitting $W \oplus \langle \alpha \rangle \subset V$ of G modules. \square

Theorem 7. *Let k a field of finite type over \mathbb{Q} . Let $f : X \rightarrow S$ a smooth proper morphism with $S, X \in \text{SmVar}(k)$ connected. Let $p \in \mathbb{N} \setminus \delta(S, X/S)$ a be prime number, where $\delta(S, X/S)$ is the finite set given in definition 7. Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ be an embedding. Consider $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p adic completion of k with respect to σ_p . Let $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0, f}$.*

(i) *We have*

$$V_{\text{tors}}(\nu_Z^{\text{et}, p}) = T \cap S_{(0)} \subset S,$$

where $T \subset S$ is given in definition 7.

(ii) *For each embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$, we have*

$$V_{\text{tors}}(\nu_{Z, \sigma_p}^{\text{et}, p})_{\hat{k}_{\sigma_p}} = \hat{T}_{\sigma_p} \cap S_{(0), \hat{k}_{\sigma_p}} \subset S_{\hat{k}_{\sigma_p}},$$

where $\hat{T}_{\sigma_p} \subset S_{\hat{k}_{\sigma_p}}$ is given in definition 7, and for $V \subset S$ a subset, $V_{\hat{k}_{\sigma_p}} := \pi_{k/\hat{k}_{\sigma_p}}(S)^{-1}(V)$, where $\pi_{k/\hat{k}_{\sigma_p}}(S) : S_{\hat{k}_{\sigma_p}} \rightarrow S$ being the projection.

Proof. Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ be an embedding. For each $\alpha \in \Lambda$, by the semi-stable comparaison theorem for

$$f^\alpha := f \circ \pi^\alpha : ((X_{S^\alpha} \times \square^n)_{\hat{k}_{\sigma_p}}^{\alpha, \mathcal{O}}, N_{U^\alpha}^{\mathcal{O}}) \rightarrow (S_{\hat{k}_{\sigma_p}}^{\alpha, \mathcal{O}}, N_{\mathcal{O}})$$

([1]) which is log smooth, we have the isomorphism in $\text{Shv}_{\text{fil}, G, \phi_p, N}(S_{\hat{k}_{\sigma_p}}^\alpha)$

$$H^j f_*^\alpha \alpha(U^\alpha) : R^j f_*^\alpha \mathbb{Z}_{p, U_{\mathbb{C}_p}^{\alpha, \text{et}}} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{st}, S_{\hat{k}_{\sigma_p}}^\alpha} \xrightarrow{\sim} H^j \int_{f^\alpha} j_* \text{Hdg}(O_{U_{\hat{k}_{\sigma_p}}^\alpha}, F_b) \otimes_{O_{S_{\hat{k}_{\sigma_p}}^\alpha}} O_{\mathbb{B}_{\text{st}, S_{\hat{k}_{\sigma_p}}^\alpha}}, \quad (1)$$

recall that since $p \in \mathbb{N} \setminus \delta(S)$, $\hat{k}_{\sigma_p}(s)$ is unramified for all $s \in S_{\hat{k}_{\sigma_p}}$.

(i): Let $s \in S_{(0)}^\alpha$. Denote $G := \text{Gal}(\bar{k}/k(s))$.

- The map $\pi_s^\alpha : (X_s \times \square^n)^\alpha \rightarrow X_s \times \square^n$ is generically finite since $\pi^\alpha : (X \times \square^n)^\alpha \rightarrow X \times \square^n$ is generically finite and f is flat. Thus by lemma 3 and theorem 4, $\nu_Z^{\text{et}, p}(s) := AJ^{\text{et}, p}(X_s)(Z_s) = 0$ if and only if there exists

$$\alpha \in H_{\text{et}}^{2d-1}(U_{S^\alpha, s}, \mathbb{Z}_p)^{[Z_s]}$$

such that $\pi^{\alpha, -1}(\alpha) \in H_{\text{et}}^{2d-1}(U_s^\alpha, \mathbb{Z}_p)(d)^G$ and $\partial\pi^{\alpha, -1}(\alpha) \neq 0$.

- On the other hand by (1) and proposition 3(ii), $\alpha' \in H_{\text{et}}^{2d-1}(U_s^\alpha, \mathbb{Z}_p)(d)^G$ if and only if $w(\alpha')_k \in F^d H_{\text{DR}}^{2d-1}(U_s^\alpha)$ and

$$w(\alpha') \in (H_{\text{DR}}^{2d-1}(U_{s, \hat{k}_{\sigma_p}}^\alpha) \otimes_{\hat{k}_{\sigma_p}(s)} \mathbb{B}_{\text{st}})^{\phi_p, N}.$$

Moreover $w(\pi^{\alpha, -1}(\alpha)) = \pi^{\alpha, -1}(w(\alpha))$.

(ii): Let $s \in S_{(0)}^\alpha$ and $s' \in \pi_{k/\hat{k}_{\sigma_p}}(S)^{-1}(s)$.

- Since $\pi_s^\alpha : (X_s \times \square^n)^\alpha \rightarrow X_s \times \square^n$ is generically finite (see the proof of (i)) we have by lemma 3 and theorem 5, $\nu_{Z, \sigma_p}^{et, p}(s) := AJ_{\sigma_p}^{et, p}(X_s)(Z_s) = 0$ if and only if there exists

$$\alpha \in H_{et}^{2d-1}(U_{S^\alpha, s, \hat{k}_{\sigma_p}}, \mathbb{Z}_p)^{[Z_s]}$$

such that $\pi^{\alpha, -1}(\alpha) \in H_{et}^{2d-1}(U_{s, \hat{k}_{\sigma_p}}^\alpha, \mathbb{Z}_p)(d)^{\hat{G}_{\sigma_p}}$ and $\partial\alpha \neq 0$.

- On the other hand by (1), $\alpha' \in H_{et}^{2d-1}(U_{s, \hat{k}_{\sigma_p}}^\alpha, \mathbb{Z}_p)(d)^{\hat{G}_{\sigma_p}}$ if and only if $w(\alpha') \in F^d H_{DR}^{2d-1}(U_{\hat{k}_{\sigma_p}, s}^\alpha)$ and

$$w(\alpha') \in (H_{DR}^{2d-1}(U_{s, \hat{k}_{\sigma_p}}^\alpha) \otimes_{\hat{k}_{\sigma_p}(s)} \mathbb{B}_{st})^{\phi_p, N} = (H_{DR}^{2d-1}(U_{s', \hat{k}_{\sigma_p}}^\alpha) \otimes_{\hat{k}_{\sigma_p}(s)} \mathbb{B}_{st})^{\phi_p, N}.$$

Moreover $w(\pi^{\alpha, -1}(\alpha)) = \pi^{\alpha, -1}(w(\alpha))$.

□

Proposition 1 for (i), the main result of [1] together with proposition 3 for (ii), gives the following :

Theorem 8. *Let $f : X \rightarrow S$ be a smooth proper morphism, with $S, X \in \text{SmVar}(k)$ over a field $k \subset \mathbb{C}$ of finite type over \mathbb{Q} . Denote*

$$E_{DR}^j(X/S) := H^j \int_f (O_X, F_b) = R^j f_* \Omega_{X/S}^\bullet \in \text{Vect}_{\mathcal{D}fil}(S).$$

Let $p \in \mathbb{N} \setminus \delta(S, X/S)$ a prime number and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding. Denote $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p -adic completion of k with respect to σ_p . Consider, using definition 7, a stratification $S = \sqcup_{\alpha \in \Lambda} S^\alpha$, Λ being a finite set, by locally closed subset $S^\alpha \subset S$, and alterations (i.e. generically finite morphisms) $\pi^\alpha : X^\alpha \rightarrow X_{S^\alpha}$ such that

$$f_\alpha := f \circ \pi^\alpha : (X_{\hat{k}_{\sigma_p}}^{\alpha, \mathcal{O}}, N_{\mathcal{O}}) \rightarrow (S_{\hat{k}_{\sigma_p}}^\alpha, N_{\mathcal{O}})$$

is log smooth. Consider for $j, d \in \mathbb{Z}$,

- the locus of Hodge Tate classes

$$HT_{j,d}^p(X/S) := (\sqcup_{\alpha \in \Lambda} (E_{DR}^j(X_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha) \otimes_{O_{X_{\hat{k}_{\sigma_p}}^\alpha}} O_{\mathbb{B}_{st}, X}^{\phi_p, N}) \cap F^d E_{DR}^j(X_{\hat{k}_{\sigma_p}} / S_{\hat{k}_{\sigma_p}}) \subset E_{DR}^j(X_{\hat{k}_{\sigma_p}} / S_{\hat{k}_{\sigma_p}})$$

where

$$E_{DR}^j(X_{\hat{k}_{\sigma_p}} / S_{\hat{k}_{\sigma_p}}) := H^j \int_f (O_{X_{\hat{k}_{\sigma_p}}}, F_b) = E_{DR}^j(X/S) \otimes_k \hat{k}_{\sigma_p} \in \text{Vect}_{\mathcal{D}fil}(S_{\hat{k}_{\sigma_p}}),$$

- the locus of Hodge classes

$$HL_{j,d}(X/S) := HL_{j,d}(X_{\mathbb{C}}/S_{\mathbb{C}}) := F^d E_{DR}^j(X_{\mathbb{C}}/S_{\mathbb{C}}) \cap R^j f_* \mathbb{Q}_{X_{\mathbb{C}}^n} \subset E_{DR}^j(X_{\mathbb{C}}/S_{\mathbb{C}})$$

where

$$E_{DR}^j(X_{\mathbb{C}}/S_{\mathbb{C}}) := H^j \int_f (O_{X_{\mathbb{C}}}, F_b) = E_{DR}^j(X/S) \otimes_k \mathbb{C} \in \text{Vect}_{\mathcal{D}fil}(S_{\mathbb{C}}).$$

Then,

- (i) for each subfield $k' \subset \mathbb{C}$, we have a canonical embedding

$$\begin{aligned} ev(X/S) : HT_{j,d}^p(X/S)(k') &\hookrightarrow HL_{j,d}(X_{\mathbb{C}}/S_{\mathbb{C}}) \otimes_{\mathbb{Q}} \mathbb{Q}_p, \\ w_s &\mapsto (1/2i\pi) ev(X/S)(w_s) := ev(X_s)(w_s), \quad s \in S_{k'} \end{aligned}$$

note that the image consists of logarithmic classes, hence

$$p_{S_{\hat{k}_{\sigma_p}}} (HT_{j,d}^p(X/S)(k')) \subset p_{S_{\mathbb{C}}}(HL_{j,d}(X_{\mathbb{C}}/S_{\mathbb{C}})),$$

where $p_{S_{\mathbb{C}}} : E_{DR}^j(X_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow S_{\mathbb{C}}$ and $p_{S_{\hat{k}_{\sigma_p}}} : E_{DR}^j(X_{\hat{k}_{\sigma_p}} / S_{\hat{k}_{\sigma_p}}) \rightarrow S_{\hat{k}_{\sigma_p}}$ are the projections,

(ii) we have

$$(R^j f_* \mathbb{Q}_{p, X_k^{\text{ét}}}(d))^G = \langle HT_{j,d}^p(X/S) \cap F^d E_{DR}^j(X/S) \rangle_{\mathbb{Q}_p} \subset E_{DR}^j(X_{\hat{k}_{\sigma_p}}/S_{\hat{k}_{\sigma_p}}),$$

where $\langle - \rangle_{\mathbb{Q}_p}$ denote the \mathbb{Q}_p subvector bundle generated by $(-)$,

(iii) we have a canonical embedding in $\text{Shv}(S_{\mathbb{C}})$

$$\begin{aligned} \text{ev}(X/S) : \pi_{k/\mathbb{C}}(S)^*(R^j f_* \mathbb{Q}_{p, X_k^{\text{ét}}}(d))^G &\hookrightarrow HL_{j,d}(X_{\mathbb{C}}/S_{\mathbb{C}}) \otimes_{\mathbb{Q}} \mathbb{Q}_p, \\ \alpha_s &\mapsto (1/2i\pi)\text{ev}(X/S)(w(\alpha_s)) := \text{ev}(X_{k(s)}((1/2i\pi)w(\alpha_s))), \quad s' = \pi_{k/\mathbb{C}}^{-1}(s) : k(s) \hookrightarrow \mathbb{C}, \quad s \in S, \end{aligned}$$

where $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \rightarrow S$ is the projection.

Proof. (i): Follows from proposition 1 after taking an embedding $\sigma'_p : k' \hookrightarrow \mathbb{C}_p$.

(ii): Follows from the isomorphism $(H^j f_*^\alpha(X^\alpha))$ (c.f. theorem [1]) and proposition 3(ii).

(iii): Follows from (i) and (ii). □

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