# Complex vs etale Abel Jacobi map for higher Chow groups and algebraicity of the zero locus of etale normal functions

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**Abstract**

We prove, using $p$-adic Hodge theory for open algebraic varieties, that for a smooth projective variety over a field $k \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ which is a finite extension of $\mathbb{Q}$, the complex abel jacobi map vanishes if the etale abel jacobi map vanishes. This implies that for a smooth projective morphism $f : X \to S$ of smooth complex algebraic varieties over a field $k \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ which is a finite extension of $\mathbb{Q}$, and $Z \in \mathcal{Z}^d(X, n)^{f,\partial=0}$ an algebraic cycle flat over $S$ whose cohomology class vanishes on fibers, the zero locus of the etale normal function associated to $Z$ is contained in the zero locus of the complex normal function associated to $Z$. From the work of Saito or Charles, we deduce that the zero locus of the complex normal function associated to $Z$ is defined over $\bar{\mathbb{Q}}$ if the zero locus of the etale normal function associated to $Z$ is not empty. We also prove an algebraicity result for the zero locus of an etale normal function associated to an algebraic cycle over a field of finite type over $\mathbb{Q}$. By the way, we get that the Hodge conjecture implies the Tate conjecture for smooth projective variety over a field which is a finite extension of $\mathbb{Q}$ (in particular we get Tate conjecture for divisors), and a for a smooth projective morphism $f : X \to S$ of smooth complex algebraic varieties over a field $k \subset \bar{\mathbb{Q}} \subset \mathbb{C}$, which is a finite extension of $\mathbb{Q}$, the locus of Hodge Tate classes as the image of a constructible algebraic subset of the de Rham vector bundle over $S_{\hat{k}}$, where $\hat{k}$ is a $p$ adic completion of $k$, inside the locus of Hodge classes of $f$.

## 1 Introduction

Let $X$ be a connected smooth projective variety over $\mathbb{C}$. The Abel-Jacobi map associates to a cycle $Z \in \mathcal{Z}^d(X)$ homologically equivalent to 0 of codimension $d$, a point $AJ(X)(Z) \in J^d(X)$ in the intermediate Jacobian of $X$. In family, if $f : X \to S$ is a smooth projective morphism of smooth connected complex varieties and $Z \in \mathcal{Z}^d(X)$ is a relative cycle on $S$ of codimension $d$ homologicaly trivial on fibers, the Abel-Jacobi map provides a holomorphic and horizontal section of the relative intermediate Jacobian : $\nu_Z : S \to J^d(X/S)$ called the associated normal function. By a Brosnan-Pearlstein theorem, the zero-locus $V(\nu_Z) \subset S$ of $\nu_Z$ is an algebraic subvariety ([3]). If $X/S$ and $Z$ are defined over a subfield $k \subset \mathbb{C}$, we conjecture in the spirit of the Bloch-Beilinson conjectures, that the zero-locus of the normal function is also defined over $k$ (see [7]).

On the other hand, if $k$ is of finite type over $\mathbb{Q}$, $X$ is a connected smooth projective variety over $k$, and $p$ is a prime number, we can define via the continuous étale cohomology with $\mathbb{Z}_p$ coefficients an etale Abel-Jacobi map which associate to a cycle $Z \in \mathcal{Z}^d(X, n)$ homologically equivalent to 0 of codimension $d$, an element $AJ^{et,p}(X)(Z) \in H^1(G, H^{2d-1-n}(X_{\bar{k}}, \mathbb{Z}_p))$ of the first degree Galois cohomology of the absolute galois group $G := Gal(\bar{k}/k)$ with value in the etale cohomology of $X_{\bar{k}}$ with $\mathbb{Z}_p$ coefficients. In family, we get, for $f : X \to S$ a smooth projective morphism of smooth connected varieties over $k$ and $Z \in \mathcal{Z}^d(X, n)$ a relative cycle on $S$ of codimension $d$ homologicaly trivial on fibers, a normal function $\nu_Z^{et,p}$ associated to $Z \in \mathcal{Z}^d(X, n)$ and thus a zero-locus $V(\nu_Z^{et,p}) \subset S$ and more generally a torsion locus $V(\nu_Z^{et,p}) \subset V_{tors}(\nu_Z^{et,p}) \subset S$ of $\nu_Z^{et}$ which are subsets of closed points of $S$. For $\sigma : k \hookrightarrow \mathbb{C}$ an embed-

ding, and $V \subset S$ a subset, we denote $V_{\mathbb{C}} := \pi_{k/\mathbb{C}}(S)^{-1}(V) \subset S_{\mathbb{C}}$ where $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \to S$ is the projection.

Let $k$ of finite type over $\mathbb{Q}$ and $f : X \to S$ a smooth projective morphism of smooth connected varieties over $k$. Let $Z \in \mathcal{Z}^d(X)$ a relative cycle. F. Charles then proves that for any embedding $\sigma : k \hookrightarrow \mathbb{C}$

- assuming that $R^{2d-1}f_*^{an}(\mathbb{C})$ has no global sections then $V(\nu_Z^{et,p})_{\mathbb{C}} = V(\nu_Z)$ ([7])
- if $V(\nu_Z)(\bar{k})$ is non-empty then it is defined over $k$ ([8]).

In this work, we show that, for a field $k \subset \bar{\mathbb{Q}}$ which is a finite extension of $\mathbb{Q}$, $X$ a connected smooth projective variety over $k$, $Z \in \mathcal{Z}^d(X, *)$ an (higher) algebraic cycle homologically equivalent to 0 of codimension $d$, and $p$ a prime number, if $AJ^{et,p}(X)(Z) = 0$ then $AJ_\sigma(X)(Z) := AJ(X_{\mathbb{C}})(Z_{\mathbb{C}}) = 0$ for any embedding $\sigma : k \hookrightarrow \mathbb{C}$ (c.f. theorem 6). This implies by definition that for $f : X \to S$ a smooth projective morphism of smooth connected varieties over a finite extension $k \subset \bar{\mathbb{Q}}$ of $\mathbb{Q}$, and $Z \in \mathcal{Z}^d(X, *)$ a relative cycle on $S$ of codimension $d$ homologicaly trivial on fibers,

$$V(\nu_Z^{et,p})_{\mathbb{C}} \subset V(\nu_Z),\ V_{tors}(\nu_Z^{et,p})_{\mathbb{C}} \subset V_{tors}(\nu_Z)$$

for any embedding $\sigma : k \hookrightarrow \mathbb{C}$ without any assumption (c.f. corollary 3(i)). We deduce that if $V(\nu_Z^{et,p})$ is not empty, $V(\nu_Z)$ is defined over $k$ (c.f. corollary 3(ii)).
The proof of theorem 6 uses $p$-adic Hodge theory for open varieties to relate de Rham cohomology and its Hodge filtration to $p$-adic étale coholomogy with its Galois action. Theorem 6 follows indeed from the fact that by proposition, 2 and proposition 1, we have for $k \subset \bar{\mathbb{Q}}$ a finite extension of $\mathbb{Q}$, $U \in \mathrm{SmVar}(k)$, and embeddings $\sigma : k \hookrightarrow \mathbb{C}$ and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ for all but finitely many prime $p \in \mathbb{N}$ (so that $k \cap \mathbb{Z}_p = \mathbb{Z}$), for each $j, l \in \mathbb{Z}$, a canonical injective map

$$H^j\iota_{p,ev}^{G,l}(U) : H^j_{et}(U_{\bar{k}}, \mathbb{Z}_p)(l)^G \hookrightarrow F^lH^j(U_{\mathbb{C}}^{an}, 2i\pi\mathbb{Z}),\ \alpha \mapsto H^j\iota_{p,ev}^{G,l}(U)(\alpha) := ev(U)(w(\alpha)),$$

which is by construction functorial in $U \in \mathrm{SmVar}(k)$ (see theorem 1). More precisely, if the étale Abel-Jacobi image of a cycle $Z$ is zero,

- there exist a Galois invariant class $\alpha$ in the étale cohomology of $((X \times \square^n)\backslash Z)_{\bar{k}}$ with non-zero boundary, where $\bar{k}$ is an algebraic closure of $k$.
- Then, by $p$ adic Hodge theory for $(X \times \square^n)\backslash Z$, $\alpha$ define a logarithmic de Rham class $w(\alpha)$ laying inside the right degree of the Hodge filtration of $((X \times \square^n)\backslash Z)_{\bar{k}}$ (c.f.proposition 2(ii)).
- Taking the image of this class by the complex period map with respect to the given embedding $\sigma : k \hookrightarrow \mathbb{C}$, we get a Betti cohomology class $ev((X \times \square^n)\backslash Z)(w(\alpha))$ of $((X \times \square^n)\backslash Z)^{an}_{\mathbb{C}}$ with $2i\pi\mathbb{Z}$ coefficients (c.f. proposition 1).
- This last class $(1/2i\pi)ev((X \times \square^n)\backslash Z)(w(\alpha))$ induce a splitting of the localization exact sequence of mixed Hodge structures :

  $$0 \to H^{2d-1-n}(X^{an}_{\mathbb{C}}, \mathbb{Z}) \xrightarrow{j^*} H^{2d-1}(((X \times \square^n)\backslash Z)^{an}_{\mathbb{C}}, \mathbb{Z})^{[Z]} \xrightarrow{\partial} H^{2d}_Z((X \times \square^n)^{an}_{\mathbb{C}}, \mathbb{Z})^{[Z]} = \mathbb{Z}^{Hdg}(d) \to 0,$$

  which means that the complex Abel-Jacobi image de $Z$ is zero.

By the way, since for $X \in \mathrm{PSmVar}(k)$, $\iota_{p,ev}^{G,d}(X)$ is compatible with cycle class maps, we get in particular that Hodge conjecture implies Tate conjecture. In particular, we get Tate conjecture for divisors over field of characteristic zero. We also show that for $U \in \mathrm{SmVar}(k)$,

$$H^j\gamma(U) \circ H^j\iota_{p,ev}^{G,d}(U) : H^j_{et}(U_{\bar{k}}, \mathbb{Z}_p)(d)^G \xrightarrow{H^j\iota_{ev}^{G,d}(U)} H^j(U_{\mathbb{C}}^{an}, 2i\pi\mathbb{Z}) \simeq H^j(U_{\mathbb{C}}^{an}, \mathbb{Z}) \xrightarrow{H^j\gamma(U)} H^j_{et}(U_{\bar{k}}, \mathbb{Z}_p)$$

is the canonical embedding for each $j \in \mathbb{Z}$, where $\gamma(U)$ is the canonical map given by reduction modulo $p^*$ and from the comparison theorem between etale and Betti cohomology with torsion coefficient (see theorem 1).
In section 6, we show that, for $f : X \to S$ a smooth projective morphism of connected smooth varieties over a field $k$ of finite type over $\mathbb{Q}$, $Z \in \mathcal{Z}^d(X, *)$ an (higher) algebraic cycle homologically equivalent to 0 of codimension $d$, and $p$ a prime number for all expect finitely many prime numbers $p \in \mathbb{N}$

$$V_{tors}(\nu_Z^{et,p}) = T \cap S_{(0)} \subset S,$$

where $T \subset S$ is the image of an constructible algebraic subset of $S_{\hat{k}_{\sigma_p}}$ by the projection $\pi_{k/\hat{k}_{\sigma_p}}(S) : S_{\hat{k}_{\sigma_p}} \to S$ where $\hat{k}_{\sigma_p}$ is the completion of $k$ with respect to $\sigma_p$ (c.f. definition 6 and theorem 7(i)). We also give a local version for $V_{tors}(\nu_{Z,\sigma_p}^{et,p}) \subset S$ (c.f. theorem 7(ii)). The proof use De Yong alterations to get a stratification $S = \sqcup_{\alpha\in\Lambda} S^\alpha$ by locally closed subsets and alterations $\pi^\alpha : (X_{S^\alpha} \times \square^n)^\alpha \to X_{S^\alpha} \times \square^n$ such that

$$f \circ \pi^\alpha : ((X \times \square^n)^\alpha_{\hat{k}_{\sigma_p}}, \pi^{\alpha,-1}(|Z|)) \to S^\alpha_{\hat{k}_{\sigma_p}}$$

is a semi-stable morphism. We then use the $p$ adic semi-stable comparison theorem for semi-stable morphisms $f' : (X', D') \to S'$, with $S', X', D' \in \mathrm{SmVar}(\hat{k}_{\sigma_p})$, that is satisfying

- $f' : X' \to S$ is smooth projective, $D' \subset X'$ is a normal crossing divisor,
- for all $s \in S'$, $D'_s \subset X'_s$ is a normal crossing divisor and $(X'_s, D'_s)$ has integral model with semi-stable reduction,

(or more generally for log smooth morphism of schemes over $(O_{\hat{k}_{\sigma_p}}, N_{\mathcal{O}})$), which gives, for each $j \in \mathbb{Z}$ a canonical filtered isomorphism

$$H^j f'_* \alpha(U') : R^j f'_* \mathbb{Z}_{p, U'^{et}_{\mathbb{C}_p}} \otimes_{\mathbb{Z}_p} O\mathbb{B}_{st,S'} \xrightarrow{\sim} R^j f'_{*Hdg}(O_{U'}, F_b) \otimes_{O_{S'}} O\mathbb{B}_{st,S'}.$$

which is for each $s \in S'$ compatible with the action of the Galois group $Gal(\mathbb{C}_p/k(s))$, the Frobenius and the monodromy: for all but finitely many primes $p$, $\hat{k}_{\sigma_p}(s)$ is unramified for all embeddings $\sigma_p : k \hookrightarrow \mathbb{C}_p$ and all $s \in S'$, so that we get a Frobenius action on $R^j f'_{*Hdg}(O_{U'}, F_b)_s = H^j_{DR}(U'_s)$ for all $s \in S'$.

This also give (see theorem 8) together with theorem 1, for $f : X \to S$ a smooth projective morphism, with $S, X$ smooth over a subfield $\sigma : k \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ which is a finite extension of $\mathbb{Q}$ and $p$ any but finitely many prime numbers, the locus of Hodge-Tate classes $HT^p_{j,d}(X/S) := (H^j Rf_{*\mathbb{Q}_{X^{et}_{\bar{k}}}}(d))^G$ where $G = Gal(\bar{k}/k)$ as the image under the projection $\pi_{k/\hat{k}_{\sigma_p}}(E_{DR}(X/S)) : E_{DR}(X_{\hat{k}_{\sigma_p}}/S_{\hat{k}_{\sigma_p}}) \to E_{DR}(X/S)$ of constructible algebraic subset

$$\begin{aligned}
\iota^{G,d}_{ev}(X/S) : HT^p_{j,d}(X/S)_{\mathbb{C}} \xrightarrow{\sim} \\
(F^d E_{DR}(X/S) \cap (\sqcup_{\alpha\in\Lambda}(\pi_{k/\hat{k}_{\sigma_p}}(X^\alpha)(E_{DR}(X^\alpha_{\hat{k}_{\sigma_p}}/S^\alpha_{\hat{k}_{\sigma_p}}) \otimes_{O_{X^\alpha_{\hat{k}_{\sigma_p}}}} O\mathbb{B}_{st,X}))^{\phi_p,N}))_{\mathbb{C}} \\
\hookrightarrow F^d E_{DR}(X_{\mathbb{C}}/S_{\mathbb{C}}) \cap R^j f_* \mathbb{Q}_{X^{an}_{\mathbb{C}}} =: HL_{j,d}(X_{\mathbb{C}}/S_{\mathbb{C}}), \\
\alpha_{s'} \mapsto ev(X^\alpha_{s'})(w(\alpha_{s'})),\ s' \in \pi_{k/\mathbb{C}}(S)^{-1}(s), s \in S
\end{aligned}$$

inside the locus of Hodge classes $HL_{j,d,\sigma}(X/S) := HL_{j,d}(X_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}(X_{\mathbb{C}}/S_{\mathbb{C}})$, (which is algebraic by the work of Deligne, Catani and Kaplan) where

- $E_{DR}(X/S) := H^j \int_f (O_X, F_b) = R^j f_* \Omega^\bullet_{X/S}$ and $E_{DR}(X_{\mathbb{C}}/S_{\mathbb{C}}) := H^j \int_f (O_{X_{\mathbb{C}}}, F_b)$ are the filtered algebraic vector bundles over $S$ and $S_{\mathbb{C}}$ respectively which have the Gauss-Manin integrable connexion,
- $S = \sqcup_{\alpha\in\Lambda} S^\alpha$ is a stratification by locally closed algebraic subsets, $\Lambda$ being a finite set,

- $\pi^\alpha : X^\alpha \to X_{S^\alpha}$ being alterations and $\pi_{k/\hat{k}_{\sigma_p}}(X^\alpha) : X^\alpha_{\hat{k}_{\sigma_p}} \to X_\alpha$ are the projections.

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# 2 Preliminaries and Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For $\mathcal{S} \in \mathrm{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \mathrm{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \to X$ is a morphism in $\mathcal{S}$, and whose morphisms $\mathrm{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \to Y$ in $\mathcal{S}$ such that $f \circ g = f'$.
- For $\mathcal{S} \in \mathrm{Cat}$ denote $\mathrm{Gr}\,\mathcal{S} := \mathrm{Fun}(\mathbb{Z}, \mathcal{S})$ is the category of graded objects.
- Denote by Ab the category of abelian groups. For $R$ a ring denote by $\mathrm{Mod}(R)$ the category of (left) $R$ modules. We have then the forgetful functor $o_R : \mathrm{Mod}(R) \to \mathrm{Ab}$.
- Denote by AbCat the category of small abelian categories.
- For $(\mathcal{S}, O_S) \in \mathrm{RCat}$ a ringed topos, we denote by
  - $\mathrm{PSh}(\mathcal{S})$ the category of presheaves of $O_S$ modules on $\mathcal{S}$ and $\mathrm{PSh}_{O_S}(\mathcal{S})$ the category of presheaves of $O_S$ modules on $\mathcal{S}$, whose objects are $\mathrm{PSh}_{O_S}(\mathcal{S})^0 := \{(M, m), M \in \mathrm{PSh}(\mathcal{S}), m : M \otimes O_S \to M\}$, together with the forgetful functor $o : \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}_{O_S}(\mathcal{S})$,
  - $C(\mathcal{S}) = C(\mathrm{PSh}(\mathcal{S}))$ and $C_{O_S}(\mathcal{S}) = C(\mathrm{PSh}_{O_S}(\mathcal{S}))$ the big abelian category of complexes of presheaves of $O_S$ modules on $\mathcal{S}$,
  - $C_{O_S(2)fil}(\mathcal{S}) := C_{(2)fil}(\mathrm{PSh}_{O_S}(\mathcal{S})) \subset C(\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of $O_S$ modules on $\mathcal{S}$ such that the filtration is biregular and $\mathrm{PSh}_{O_S(2)fil}(\mathcal{S}) = (\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$.
- Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$ a ringed topos with topology $\tau$. For $F \in C_{O_S}(\mathcal{S})$, we denote by $k : F \to E_\tau(F)$ the canonical flasque resolution in $C_{O_S}(\mathcal{S})$ (see [5]). In particular for $X \in \mathcal{S}$, $H^*(X, E_\tau(F)) \xrightarrow{\sim} \mathbb{H}^*_\tau(X, F)$.
- For $f : \mathcal{S}' \to \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \mathrm{RCat}$, endowed with topology $\tau$ and $\tau'$ respectively, we denote for $F \in C_{O_S}(\mathcal{S})$ and each $j \in \mathbb{Z}$,
  - $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^*F)$,
  - $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^{*mod}F)$,

  the canonical maps.
- For $\mathcal{X} \in \mathrm{Cat}$ a (pre)site and $p$ a prime number, we consider the full subcategory

$$\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X}) \subset \mathrm{PSh}(\mathbb{N} \times \mathcal{X}),\ \ F = (F_n)_{n\in\mathbb{N}},\ p^n F_n = 0,\ F_{n+1}/p^n \xrightarrow{\sim} F_n$$

  $C_{\mathbb{Z}_p}(\mathcal{X}) := C(\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X})) \subset C(\mathbb{N} \times \mathcal{X})$ and
  - $\mathbb{Z}_p := \mathbb{Z}_{p,\mathcal{X}} := ((\mathbb{Z}/p^*\mathbb{Z})_{\mathcal{X}}) \in \mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X})$ the diagram of constant presheaves on $\mathcal{X}$ indexed by $\mathbb{N}$,
  - $\underline{\mathbb{Z}_p} := \underline{\mathbb{Z}_p}_{\mathcal{X}} \in \mathrm{PSh}(\mathcal{X})$ the constant presheaf on $\mathcal{X}$,
  - $l_{p,\mathcal{X}} := (p(*)) : \underline{\mathbb{Z}_p}_{\mathcal{X}} \to \mathbb{Z}_{p,\mathcal{X}}$ the projection map in $\mathrm{PSh}(\mathbb{N} \times \mathcal{X})$

Recall that filtered colimits do NOT commute with infinite limits in general. In particular, for $f : \mathcal{X}' \to \mathcal{X}$ a morphism of (pre)site and $F = (F_n)_{n\in\mathbb{N}} \in \mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X})$, $\varprojlim_{n\in\mathbb{N}} f^* F_n$ is NOT isomorphic to $f^* \varprojlim_{n\in\mathbb{N}} F_n$ in $\mathrm{PSh}(\mathcal{X}')$ in general.

- Denote by $\mathrm{Sch} \subset \mathrm{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). For $X \in \mathrm{Sch}$, we denote by
  - $\mathrm{Sch}^{ft}/X \subset \mathrm{Sch}/X$ the full subcategory consisting of objects $X'/X = (X', f) \in \mathrm{Sch}/X$ such that $f : X' \to X$ is an morphism of finite type
  - $X^{et} \subset \mathrm{Sch}^{ft}/X$ the full subcategory consisting of objects $U/X = (X, h) \in \mathrm{Sch}/X$ such that $h : U \to X$ is an etale morphism.
  - $X^{sm} \subset \mathrm{Sch}^{ft}/X$ the full subcategory consisting of objects $U/X = (X, h) \in \mathrm{Sch}/X$ such that $h : U \to X$ is a smooth morphism.

  For a field $k$, we consider $\mathrm{Sch}/k := \mathrm{Sch}/\operatorname{Spec} k$ the category of schemes over $\operatorname{Spec} k$. We then denote by
  - $\mathrm{Var}(k) = \mathrm{Sch}^{ft}/k \subset \mathrm{Sch}/k$ the full subcategory consisting of algebraic varieties over $k$, i.e. schemes of finite type over $k$,
  - $\mathrm{PVar}(k) \subset \mathrm{QPVar}(k) \subset \mathrm{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
  - $\mathrm{PSmVar}(k) \subset \mathrm{SmVar}(k) \subset \mathrm{Var}(k)$ the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

  For a morphism of field $\sigma : k \hookrightarrow K$, we have the extention of scalar functor

  $$\otimes_k K : \mathrm{Sch}/k \to \mathrm{Sch}/K,\ X \mapsto X_K := X_{K,\sigma} := X \otimes_k K,\ (f : X' \to X) \mapsto (f_K := f \otimes I : X'_K \to X_K).$$

  which is left ajoint to the restriction of scalar

  $$Res_{k/K} : \mathrm{Sch}/K \to \mathrm{Sch}/k,\ X = (X, a_X) \mapsto X = (X, \sigma \circ a_X),\ (f : X' \to X) \mapsto (f : X' \to X)$$

  The adjonction maps are
  - for $X \in \mathrm{Sch}/k$, the projection $\pi_{k/K}(X) : X_K \to X$ in $\mathrm{Sch}/k$, for $X = \cup_i X_i$ an affine open cover with $X_i = \operatorname{Spec}(A_i)$ we have by definition $\pi_{k/K}(X_i) = n_{k/K}(A_i)$,
  - for $X \in \mathrm{Sch}/K$, $I \times \Delta_K : X \hookrightarrow X_K = X \times_K K \otimes_k K$ in $\mathrm{Sch}/K$, where $\Delta_K : K \otimes_k K \to K$ is the diagonal which is given by for $x, y \in K$, $\Delta_K(x, y) = x - y$.

  The extention of scalar functor restrict to a functor

  $$\otimes_k K : \mathrm{Var}(k) \to \mathrm{Var}(K),\ X \mapsto X_K := X_{K,\sigma} := X \otimes_k K,\ (f : X' \to X) \mapsto (f_K := f \otimes I : X'_K \to X_K).$$

  and for $X \in \mathrm{Var}(k)$ we have $\pi_{k/K}(X) : X_K \to X$ the projection in $\mathrm{Sch}/k$. An algebraic variety $X \in \mathrm{Var}(K)$ is said to be defined over $k$ if there exists $X_0 \in \mathrm{Var}(k)$ such that $X \simeq X_0 \otimes_k K$ in $\mathrm{Var}(K)$. For $X = (X, a_X) \in \mathrm{Var}(k)$, we have $\mathrm{Sch}^{ft}/X = \mathrm{Var}(k)/X$ since for $f : X' \to X$ a morphism of schemes of finite type, $(X', a_X \circ f) \in \mathrm{Var}(k)$ is the unique structure of variety over $k$ of $X' \in \mathrm{Sch}$ such that $f$ becomes a morphism of algebraic varieties over $k$, in particular we have
  - $X^{et} \subset \mathrm{Sch}^{ft}/X = \mathrm{Var}(k)/X$,
  - $X^{sm} \subset \mathrm{Sch}^{ft}/X = \mathrm{Var}(k)/X$.

- For $X \in \mathrm{Sch}$ and $s \in \mathbb{N}$, we denote by $X_{(s)} \subset X$ its points of dimension $s$, in particular $X_{(0)} \subset X$ are the closed points of $X$.

- For $X \in \mathrm{Sch}$ and $k$ a field we denote by $X(k) := \mathrm{Hom}_{Sch}(\mathrm{Spec}\, k, X)$ the $k$ points of $X$. We get $X(k)_{in} \subset X$ the image of the $k$-points of $X$. For $k \subset k'$ a subfield, $\mathbb{A}^N_{k'}(k)_{in} = k^N \subset k'^N \subset \mathbb{A}^N_{k'}$ and $\mathbb{A}^N_k(k')_{in} = \pi_{k/k'}(\mathbb{A}^N_k)(k'^N) \subset \mathbb{A}^N_k$.

- For $X \in \mathrm{Sch}$, we denote $X^{pet} \subset \mathrm{Sch}/X$ the pro etale site (see [2]) which is the full subcategory of $\mathrm{Sch}/X$ whose object consists of weakly etale maps $U \to X$ (that is flat maps $U \to X$ such that $\Delta_U : U \to U \times_X U$ is also flat) and whose topology is generated by fpqc covers. We then have the canonical morphism of site
$$\nu_X : X^{pet} \to X, (U \to X) \mapsto (U \to X)$$
For $F \in C(X^{et})$,
$$\mathrm{ad}(\nu_X^*, R\nu_{X*})(F) : F \to R\nu_{X*}\nu_X^* F$$
is an isomorphism in $D(X^{et})$, in particular, for each $n \in \mathbb{Z}$
$$H^n\Gamma(X, k) : \mathbb{H}^n_{et}(X, F) \xrightarrow{\sim} \mathbb{H}^n_{pet}(X, \nu_X^* F)$$
are isomorphisms, where
$$k := k \circ \mathrm{ad}(\nu_X^*, \nu_{X*})(E_{et}(F)) : E_{et}(F) \to \nu_{X*}E_{pet}(\nu_X^* F)$$
is the canonical map in $C(X^{et})$ which is a quasi-isomorphism.

- Let $k$ a field of characteristic zero and $k_0 \subset k$ a subfield. We say that $k$ is of finite type over $k_0$ if $k$ is generated as a field by $k_0$ and a finite set $\{\alpha_1, \ldots, \alpha_r\} \subset k$ of elements of $k$, that is $k = k_0(\alpha_1, \ldots, \alpha_r)$. If $k$ is of finite type over $k_0$ then it is of finite transcendence degree $d \in \mathbb{N}$ over $k_0$ and $k = k_0(\alpha_1, \ldots, \alpha_d)(\alpha_{d+1})$ with $\{\alpha_1, \ldots, \alpha_{d+1}\} \subset k$ such that $k_0(\alpha_1, \ldots, \alpha_d) = k_0(x_1, \ldots, x_d)$ and $\alpha_{d+1}$ is an algebraic element of $k$ over $k_0(\alpha_1, \ldots, \alpha_d)$. Note that if $k$ is of finite type over $k_0$ then it is NOT algebraicly closed. We denote $\bar{k}$ the algebraic closure of $k$. Then $\bar{k}$ is also transcendence degree $d$ over $k$.

- Let $C$ a field of characteristic zero. Let $X \in \mathrm{Var}(C)$. Then there exist a subfield $k \subset C$ of finite type over $\mathbb{Q}$ such that $X$ is defined over $k$ that is $X \simeq X_0 \otimes_k \mathbb{C}$ with $X_0 \in \mathrm{Var}(k)$.

- Let $X = (X, O_X) \in \mathrm{RCat}$ a ringed topos, we have in $C(X)$ the subcomplex of presheaves of abelian group
$$OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X, \text{ s.t. for } X^o \in X \text{ and } p \in \mathbb{N},$$
$$\Omega^p_{X,\log}(X^0) :=< df_{\alpha_1}/f_{\alpha_1} \wedge \cdots \wedge df_{\alpha_p}/f_{\alpha_p}, f_\alpha \in \Gamma(X^o, O_X)^* >\subset \Omega^p_X(X^0),$$
where $\Omega^\bullet_X := DR(X)(O_X) \in C(X)$ is the De Rham complex and $\Gamma(X^o, O_X)^* \subset \Gamma(X^o, O_X)$ is the subring consisting of inversible elements for the multiplication.

- Let $X \in \mathrm{Var}(k)$. Considering its De Rham complex $\Omega^\bullet_X := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X)$. If $X \in \mathrm{SmVar}(k)$, then $H^j_{DR}(X) = \mathbb{H}^j_{et}(X, \Omega^\bullet_X)$ since $\Omega^\bullet \in C(\mathrm{SmVar}(k))$ is $\mathbb{A}^1$ local and admits transfert (see [5]).

- Let $X \in \mathrm{Var}(k)$. Let $X = \cup_{i=1}^s X_i$ an open affine cover. For $I \subset [1, \ldots, s]$, we denote $X_I := \cap_{i \in I} X_i$. We get $X_\bullet \in \mathrm{Fun}(P([1, \ldots, s]), \mathrm{Var}(k))$. Since quasi-coherent sheaves on affine noetherian schemes are acyclic, we have for each $j \in \mathbb{Z}$, $H^j_{DR}(X) = \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet})$.

- Denote by $\mathrm{AnSp}(\mathbb{C}) \subset \mathrm{RTop}$ the full subcategory of analytic spaces over $\mathbb{C}$, and by $\mathrm{AnSm}(\mathbb{C}) \subset \mathrm{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). For $X \in \mathrm{AnSp}(\mathbb{C})$, we denote by $X^{et} \subset \mathrm{AnSp}(\mathbb{C})/X$ the full subcategory consisting of objects $U/X = (X, h) \in \mathrm{AnSp}(\mathbb{C})/X$ such that $h : U \to X$ is an etale morphism. By the Weirstrass preparation theorem (or the implicit function theorem if $U$ and $X$ are smooth), a morphism $r : U \to X$ with $U, X \in \mathrm{AnSp}(\mathbb{C})$ is etale if and only if it is an isomorphism local. Hence for $X \in \mathrm{AnSp}(\mathbb{C})$, the morphism of site $\pi_X : X^{et} \to X$ is an isomorphism of site.

- Denote by $\mathrm{CW} \subset \mathrm{Top}$ the full subcategory of $CW$ complexes. Denote by $\mathrm{Diff}(\mathbb{R}) \subset \mathrm{RTop}$ the full subcategory of differentiable (real) manifold.
- Let $k \subset \mathbb{C}$ a subfield. For $X \in \mathrm{Var}(k)$, we denote by

$$\mathrm{an}_X : X_{\mathbb{C}}^{an} \xrightarrow{\sim} X_{\mathbb{C}}^{an,et} \xrightarrow{\mathrm{an}_{X_{\mathbb{C}}^{et}}} X_{\mathbb{C}}^{et} \xrightarrow{\pi_{k/\mathbb{C}}(X^{et})} X^{et},$$

  the morphism of site given by the analytical functor.
- Let $k \subset \mathbb{C}$ a subfield. For $X \in \mathrm{Var}(k)$, we denote by

$$\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega^{\bullet}_{X_{\mathbb{C}}^{an}}$$

  the canonical embedding in $C(X_{\mathbb{C}}^{an})$. It induces the embedding in $C(X_{\mathbb{C}}^{an})$

$$\beta(X) : 2i\pi\mathbb{Z}_{X_{\mathbb{C}}^{an}} \xrightarrow{\iota_{2i\pi\mathbb{Z}/\mathbb{C}, X_{\mathbb{C}}^{an}}} \mathbb{C}_{X_{\mathbb{C}}^{an}} \xrightarrow{\alpha(X)} \Omega^{\bullet}_{X_{\mathbb{C}}^{an}}$$

  For $X \in \mathrm{SmVar}(k)$, $\alpha(X)$ is a quasi-isomorphism by the holomorphic Poincare lemma.
- Let $k \subset \mathbb{C}_p$ a subfield. For $X \in \mathrm{Var}(k)$, we denote by

$$\beta_{pet}(X_{\mathbb{C}_p}) : \underline{\mathbb{Z}_p}_{X_{\mathbb{C}_p}^{pet}} \xrightarrow{\iota_{X_{\mathbb{C}_p}^{pet}}} \Omega^{\bullet}_{X_{\mathbb{C}_p}^{pet},\log} \xrightarrow{OL_{X_{\mathbb{C}_p}^{et}}} \Omega^{\bullet}_{X_{\mathbb{C}_p}^{pet}}$$

  the canonical embedding in $C(X_{\mathbb{C}_p}^{pet})$.
- For $X \in \mathrm{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension $d$, which is the free abelian group generated by irreducible closed subsets of codimension $d$.
  - For $X \in \mathrm{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X, \bullet) \subset \mathcal{Z}^d(X \times \square^{\bullet})$ the Bloch cycle complex which is the subcomplex which consists of algebraic cycles which intersect $\square^*$ properly.
  - For $X \in \mathrm{Var}(k)$ irreducible and $d, n \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X, n)^{\partial=0}_{hom} \subset \mathcal{Z}^d(X, \bullet)^{\partial=0}$ the the subabelian group consisting of algebraic cycles which are homologicaly trivial.
  - For $f : X \to S$ a dominant morphism with $S, X \in \mathrm{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X, \bullet)^f \subset \mathcal{Z}^d(X, \bullet)$ the subcomplex consisting of algebraic cycles which are flat over $S$.
  - For $f : X \to S$ a dominant morphism with $S, X \in \mathrm{Var}(k)$ irreducible and $d, n \in \mathbb{N}$ we denote by $\mathcal{Z}^d(X, n)^{f,\partial=0}_{fhom} \subset \mathcal{Z}^d(X, n)^{f,\partial=0}$ the subabelian group consisting of algebraic cycles which are flat over $S$ and homological trivial on fibers.
- We denote $\mathbb{I}^n := [0, 1]^n \in \mathrm{Diff}(\mathbb{R})$ (with boundary). For $X \in \mathrm{Top}$ and $R$ a ring, we consider its singular cochain complex

$$C^*_{\mathrm{sing}}(X, R) := (\mathbb{Z}\,\mathrm{Hom}_{\mathrm{Top}}(\mathbb{I}^*, X)^{\vee}) \otimes R$$

  and for $l \in \mathbb{Z}$ its singular cohomology $H^l_{\mathrm{sing}}(X, R) := H^n C^*_{\mathrm{sing}}(X, R)$. In particular, we get by functoriality the complex

$$C^*_{X, R\,\mathrm{sing}} \in C_R(X),\ (U \subset X) \mapsto C^*_{\mathrm{sing}}(U, R)$$

  We will consider the canonical embedding

$$C^*\iota_{2i\pi\mathbb{Z}/\mathbb{C}}(X) : C^*_{\mathrm{sing}}(X, 2i\pi\mathbb{Z}) \hookrightarrow C^*_{\mathrm{sing}}(X, \mathbb{C}),\ \alpha \mapsto \alpha \otimes 1$$

whose image consists of cochains $\alpha \in C^j_{\text{sing}}(X, \mathbb{C})$ such that $\alpha(\gamma) \in 2i\pi\mathbb{Z}$ for all $\gamma \in \mathbb{Z}\operatorname{Hom}_{\text{Top}}(\mathbb{I}^*, X)$. We get by functoriality the embedding in $C(X)$

$$C^*\iota_{2i\pi\mathbb{Z}/\mathbb{C},X} : C^*_{X,2i\pi\mathbb{Z},\text{sing}} \hookrightarrow C^*_{X,\mathbb{C},\text{sing}},$$
$$(U \subset X) \mapsto (C^*\iota_{2i\pi\mathbb{Z}/\mathbb{C}}(U) : C^*_{\text{sing}}(U, 2i\pi\mathbb{Z}) \hookrightarrow C^*_{\text{sing}}(U, \mathbb{C}))$$

We recall we have

- For $X \in$ Top locally contractile, e.g. $X \in$ CW, and $R$ a ring, the inclusion in $C_R(X)$ $c_X : R_X \to C^*_{X,R\,\text{sing}}$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H^l_{\text{sing}}(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z}\operatorname{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \to C^*_{\text{sing}}(X, R),\ w \mapsto w : (\phi \mapsto w(\phi))$$

  is a quasi-isomorphism by Whitney approximation theorem.

- Let $X \in \text{AnSm}(\mathbb{C})$. Let $X = \cup_{i=1}^r \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0,1)^d$. Since a convex open subset of $\mathbb{C}^d$ is biholomorphic to an open ball we have $\mathbb{D}_I := \cap_{i\in I}\mathbb{D}_i \simeq D(0,1)^d$ (where $d$ is the dimension of a connected component of $X$). We get $\mathbb{D}_\bullet \in \text{Fun}(P([1,\dots,r]), \text{AnSm}(\mathbb{C}))$.

- For $k$ a field, we denote by $\text{Vect}(k)$ the category of vector spaces and $\text{Vect}_{fil}(k)$ the category of filtered vector spaces. Let $k \subset K$ a field extension of field of characteristic zero.
  - For $(V, F) \in \text{Vect}_{fil}(k)$, we get a filtered $K$ vector space $(V \otimes_k K, F) \in \text{Vect}_{fil}(K)$ by $F^j(V \otimes_k K) := (F^j V) \otimes_k K$. In this case, we say that the filtration $F$ on $V \otimes_k K$ is defined over $k$.
  - For $(V', F) \in \text{Vect}_{fil}(K)$ and $h : V \otimes_k K \xrightarrow{\sim} V'$ and isomorphism of $K$ vector space, we get $(V, F_h) \in \text{Vect}_{fil}(k)$ by $F^j_h V := h^{-1}(F^j V') \cap V$ (considering the canonical embedding $n : V \hookrightarrow V \otimes_k K$, $n(v) := v \otimes 1$).
  - For $(V, F) \in \text{Vect}_{fil}(k)$, we have $F^j(V \otimes_k K) \cap V = F^j V$.
  - For $(V', F) \in \text{Vect}_{fil}(K)$ and $h : V \otimes_k K \xrightarrow{\sim} V'$ and isomorphism of $K$ vector space, we have $h((F^j_h V) \otimes_k K) \subset F^j V'$. Of course this inclusion is NOT an equality in general. The filtration $F$ on $V'$ is NOT defined over $k$ in general.

- We also consider
  - $\text{Top}_2$ the category whose objects are couples $(X, Y)$ with $X \in$ Top and $Y \subset X$ a subset and whose set of morphisms $\text{Hom}((X', Y'), (X, Y))$ consists of $f : X' \to X$ continuous such that $Y' \subset f^{-1}(Y)$ (i.e. $f(Y') \subset Y$),
  - $\text{RTop}_2$ the category whose objects are couples $(X, Y)$ with $X = (X, O_X) \in$ RTop and $Y \subset X$ a subset and whose set of morphisms $\text{Hom}((X', Y'), (X, Y))$ consists of $f : X' \to X$ of ringed spaces such that $Y' \subset f^{-1}(Y)$,
  - $\text{Top}^2$ the category whose objects are couples $(X, Z)$ with $X \in$ Top and $Z \subset X$ a closed subset and whose set of morphisms $\text{Hom}((X', Z'), (X, Z))$ consists of $f : X' \to X$ continuous such that $f^{-1}(Z) \subset Z'$ (i.e. $f(X'\backslash Z') \subset X\backslash Z$), in particular we have the canonical functor $\text{Top}^2 \to \text{Top}_2$, $(X, Z) \mapsto (X, X\backslash Z)$,
  - $\text{RTop}^2$ the category whose objects are couples $(X, Z)$ with $X = (X, O_X) \in$ RTop and $Z \subset X$ a closed subset and whose set of morphisms $\text{Hom}((X', Z'), (X, Z))$ consists of $f : X' \to X$ of ringed spaces such that $f^{-1}(Z) \subset Z'$, in particular we have the canonical functor $\text{RTop}^2 \to \text{RTop}_2$, $(X, Z) \mapsto (X, X\backslash Z)$.

A (generalized) cohomology theory is in particular a functor $H^* : \mathrm{Top}_2 \to \mathrm{GrAb}$, e.g singular cohomology

$$H^*_{\mathrm{sing}} : \mathrm{Top}^2 \to \mathrm{Gr\,Ab}, (X,Y) \mapsto H^*_{\mathrm{sing}}(X,Y,R).$$

where $R$ is a commutative ring. It restrict to a functor $H^* : \mathrm{Top}^2 \to \mathrm{GrAb}, (X,Z) \mapsto H^*_Z(X) := H^*(X, X\backslash Z)$.

- Denote $\mathrm{Sch}^2 \subset \mathrm{RTop}^2$ the subcategory whose objects are couples $(X,Z)$ with $X = (X, O_X) \in \mathrm{Sch}$ and $Z \subset X$ a closed subset and whose set of morphisms $\mathrm{Hom}((X',Z'),(X,Z))$ consists of $f : X' \to X$ of locally ringed spaces such that $f^{-1}(Z) \subset Z'$.

- Let $k$ a field of characteristic zero. Denote $\mathrm{SmVar}^2(k) \subset \mathrm{Var}^2(k) \subset \mathrm{Sch}^2/k$ the full subcategories whose objects are $(X,Z)$ with $X \in \mathrm{Var}(k)$, resp. $X \in \mathrm{SmVar}(k)$, and $Z \subset X$ is a closed subset, and whose morphisms $\mathrm{Hom}((X',Z') \to (X,Z))$ consists of $f : X' \to X$ of schemes over $k$ such that $f^{-1}(Z) \subset Z'$.

- Let $k$ a field of characteristic zero. Let

$$H^* : \mathrm{SmVar}^2(k) \to \mathrm{Gr\,AbCat}, (X,Z) \mapsto H^*_Z(X)$$

a mixed Weil cohomology theory in sense of [9] (e.g. (filtered) De Rham, etale or Betti cohomology, Hodge or $p$ adic realization). For $X \in \mathrm{SmVar}(k)$ and $Z \subset X$ a closed subset, we denote

$$H^*_Z(X)^0 := \ker(H^*_Z(X) \to H^*(X)).$$

For $X \in \mathrm{SmVar}(k)$ and $Z \in \mathcal{Z}^d(X,n)^{\partial=0}_{hom}$, we consider the subobject $H^{2d-1}(U)^{[Z]} \subset H^{2d-1}(U)$ where $j : U := (X \times \square^n)\backslash|Z| \hookrightarrow X \times \square^n$ is the complementary open subset, given by the pullback by $H^{2d}_Z(X \times \square^n)^{[Z]} := [Z] \subset H^{2d}_Z(X \times \square^n)^0$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{2d-1}(X \times \square^n) = H^{2d-n-1}(X) & \xrightarrow{j^*} & H^{2d-1}(U) & \xrightarrow{\partial} & H^{2d}_Z(X \times \square^n)^0 & \longrightarrow & 0 \\
 & & = \uparrow & & \subset \uparrow & & \subset \uparrow & & \\
0 & \longrightarrow & H^{2d-1}(X \times \square^n) = H^{2d-n-1}(X) & \xrightarrow{j^*} & H^{2d-1}(U)^{[Z]} & \xrightarrow{\partial} & H^{2d}_Z(X \times \square^n)^{[Z]} := [Z] & \longrightarrow & 0
\end{array}$$

of the first row exact sequence. In particular the second row is also an exact sequence.

- We denote by $\log \mathrm{Sch}$ the category of log schemes whose objects are couples $(X,M) := (X,M,\alpha)$ where $X = (X, O_X) \in \mathrm{Sch}$, $M \in \mathrm{Shv}(X)$ is a sheaf of monoid and $\alpha : M \to O_X$ is a morphism of sheaves of monoid. In particular we have a canonical functor

$$\mathrm{Sch}^2 \to \mathrm{logSch}, (X,Z) \mapsto (X, M_Z), M_Z := (f \in O_X s.t. f_{|X\backslash Z} \in O^*_{X\backslash Z}) \subset O_X$$

Let $k$ a field of characteristic zero. We denote by $\log \mathrm{Var}(k) \subset \log \mathrm{Sch}/k$ the full subcategory of log varieties.

- Let $p$ be a prime number. For $K$ a $p$ adic field (i.e. a finite extension of $\mathbb{Q}_p$), we consider the canonical functor of Huber (see [5])

$$\mathcal{R} : \mathrm{Var}(K) \to \mathrm{Sch}/O_K, X \mapsto \mathcal{R}(X) := (X, O^o_X) = X^{\mathcal{O}}, O^o_X := (f \in O_X, s.t. |f(x)|_p \leq 1 \forall x \in X(O_K))$$

where $O_K \subset K$ is the ring of integers of $K$.

We have the followings facts :

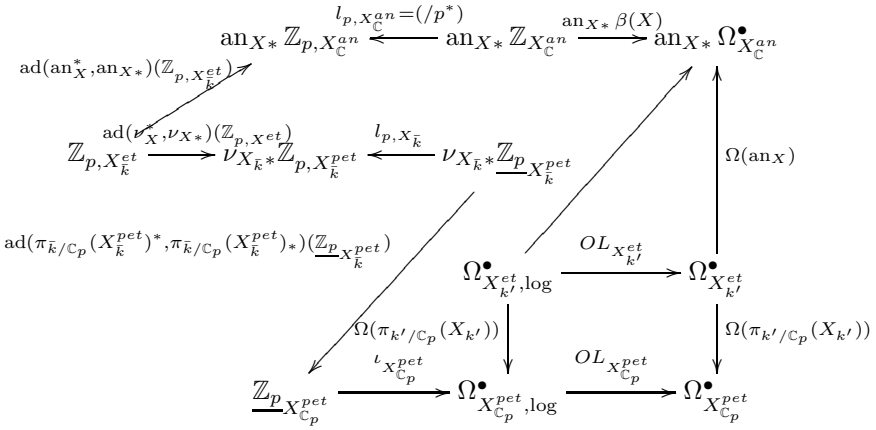
- Let $k$ a field of characteristic zero. Denote $G := Gal(\bar{k}/k)$ its absolute Galois group. Then the functor
$$\Gamma(\bar{k})(-) : \mathrm{PSh}(k^{et}) \to \mathrm{Mod}(\bar{k}, G),\ F \mapsto \Gamma(\bar{k}, F)$$
is an equivalence of category whose inverse is
$$G(-) : \mathrm{Mod}(\bar{k}, G) \to \mathrm{PSh}(k^{et}),\ V \mapsto G(V) := V := ((k'/k) \mapsto V^{Aut(k'/k)}).$$
In particular, for each $V \in \mathrm{Mod}(\bar{k}, G)$ and $j \in \mathbb{Z}$, we get an isomorphism
$$H^j G(V) : H^j(G, V) \xrightarrow{\sim} \mathrm{Ext}^j_G(\bar{k}, V).$$

- For $X \in \mathrm{Sch}$, we have
  - $\nu_X^* \Omega^\bullet_{X^{et}} = \Omega^\bullet_{X^{pet}}$ and hence isomorphisms
  $$H^*\Gamma(X, k) : \mathbb{H}^*_{et}(X, \Omega^\bullet_{X^{et}}) \xrightarrow{\sim} \mathbb{H}^*_{pet}(X, \Omega^\bullet_{X^{pet}}),$$
  where
  $$k := k \circ \mathrm{ad}(\nu_X^*, \nu_{X*})(-) : E_{et}(\Omega^\bullet_{X^{et}}) \to \nu_{X*} E_{pet}(\Omega^\bullet_{X^{pet}})$$
  is the canonical map in $C(X^{et})$ (which is a quasi-isomorphism),
  - $\nu_X^* \Omega^\bullet_{X^{et},\log} = \Omega^\bullet_{X^{pet},log}$ and hence isomorphisms
  $$H^*\Gamma(X, k) : \mathbb{H}^*_{et}(X, \Omega^\bullet_{X^{et},log}) \xrightarrow{\sim} \mathbb{H}^*_{pet}(X, \Omega^\bullet_{X^{pet},log})$$
  where
  $$k := k \circ \mathrm{ad}(\nu_X^*, \nu_{X*})(-) : E_{et}(\Omega^\bullet_{X^{et},\log}) \to \nu_{X*} E_{pet}(\Omega^\bullet_{X^{pet},\log})$$
  is the canonical map in $C(X^{et})$ (which is a quasi-isomorphism).

- Let $k$ a field of characteristic zero. The complex of presheaves
$$\Omega^\bullet_{log} \in C(\mathrm{SmVar}(k)), X \mapsto \Omega^\bullet_{X,log}(X)$$
is $\mathbb{A}^1$ local and admits transfers. Hence for $X \in \mathrm{SmVar}(k)$
$$H^*\Gamma(X, k) : \mathbb{H}^*(X, \Omega^\bullet_{X,log}) \xrightarrow{\sim} \mathbb{H}^*_{et}(X, \Omega^\bullet_{X^{et},log})$$
are isomorphisms.

- Let $k \subset K$ a field extension.
  - Let $X \in \mathrm{Var}(k)$. We have then the canonical isomorphism in $C_{Aut(K/k),fil}(X_K)$
  $$w(k/K) : (\Omega^\bullet_X \otimes_k K, F_b) \xrightarrow{\sim} (\Omega^\bullet_{X_K}, F_b)$$
  given by the universal property of derivation of a ring.
  - Let $X \in \mathrm{SmVar}(k)$. Let $\bar{X} \in \mathrm{PSmVar}(k)$ a smooth compactification of $X$ with $D := \bar{X} \backslash X$ a normal crossing divisor. We have then the canonical isomorphism in $C_{Aut(K/k),fil}(\bar{X}_K)$
  $$w(k/K) : (\Omega^\bullet_{\bar{X}}(\log D) \otimes_k K, F_b) \xrightarrow{\sim} (\Omega^\bullet_{\bar{X}_K}(\log D), F_b)$$
  given by the preceeding point. In particular, we get for all $j, l \in \mathbb{Z}$,
    - $F^l H^j w(k/K) : F^l H^j_{DR}(X) \otimes_k K \xrightarrow{\sim} F^l H^j_{DR}(X_K)$,
    - $H^j w(k/K) : H^j_{DR}(X) \xrightarrow{\sim} H^j_{DR}(X_K)^G$.

Let $\sigma : k \hookrightarrow \mathbb{C}$ a subfield of finite type over $\mathbb{Q}$. Consider $k \subset \bar{k} \subset \mathbb{C}$ the algebraic closure of $k$. Let $X \in \mathrm{Var}(k)$. Let $p$ a prime number. Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding. We have then for $k'/k$ a field extension and $\sigma' : k' \hookrightarrow \mathbb{C}$ and $\sigma'_p : k' \hookrightarrow \mathbb{C}_p$ field embedding such that $\sigma'_{|k} = \sigma$ and $\sigma'_{p|k} = \sigma_p$, the following diagram in $C(\mathbb{N} \times X^{et})$




with as above

$$\mathrm{an}_X : X^{an}_{\mathbb{C}} \simeq X^{an,et}_{\mathbb{C}} \xrightarrow{\mathrm{an}_X} X^{et}_{\mathbb{C}} \xrightarrow{\pi_{\bar{k}/\mathbb{C}}(X^{et}_{\bar{k}})} X^{et}_{\bar{k}},\ \ \mathrm{an}_X : X^{an}_{\mathbb{C}} \simeq X^{an,et}_{\mathbb{C}} \xrightarrow{\mathrm{an}_X} X^{et}_{\mathbb{C}} \xrightarrow{\pi_{k'/\mathbb{C}}(X^{et}_{k'})} X^{et}_{k'}$$

We also have for $X \in \mathrm{Var}(k)$, the canonical map

$$H^j\gamma(X) : H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}},\mathbb{Z}) \xrightarrow{H^j(/p^*)} H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}},\mathbb{Z}_p) \xrightarrow{\sim} H^j(X^{an}_{\mathbb{C}},\mathbb{Z}_p)$$
$$\xrightarrow{(\mathrm{an}^*_X)^{-1}} H^j_{et}(X_{\mathbb{C}},\mathbb{Z}_p) \xrightarrow{(\pi_{\bar{k}/\mathbb{C}}(X)^*)^{-1}} H^j_{et}(X_{\bar{k}},\mathbb{Z}_p)$$

for each $j \in \mathbb{Z}$, where $H^j(\mathrm{an}^*_X)$ is an isomorphism by the comparaison theorem between etale cohomology and Betti cohomology with torsion coefficients (see SGA4).

We have the following easy lemma

**Lemma 1.** *Let $l$ a prime number. Let $k \subset \mathbb{C}_l$ a subfield a finite type over $\mathbb{Q}$. Denote $G := Gal(\bar{k}/k)$ its absolute Galois group. Consider $\pi_{k/\mathbb{C}_l}(X_{\bar{k}}) : X_{\mathbb{C}_l} \to X_{\bar{k}}$ the projection. Let $X \in \mathrm{Var}(k)$. Let $p$ a prime number. Let $j \in \mathbb{Z}$ and $\alpha \in H^j_{et}(X_{\bar{k}},\mathbb{Z}_p)$. Then $\alpha \in H^j_{et}(X_{\bar{k}},\mathbb{Z}_p)^G$ if and only if*

$$\pi_{k/\mathbb{C}_l}(X_{\bar{k}})^*\alpha \in H^j_{et}(X_{\mathbb{C}_l},\mathbb{Z}_p)^{Aut(\mathbb{C}_l/k)},$$

*where $Aut(\mathbb{C}_l/k)$ denote the group of (algebraic) automorphism of $\mathbb{C}_l$ which fix $k$.*

*Proof.* Obvious. □

# 3 Integral complex and $p$-adic periods of a smooth algebraic variety over a field $k$ of finite type over $\mathbb{Q}$

## 3.1 Complex integral periods

Let $k$ a field of finite type over $\mathbb{Q}$.

Let $X \in \mathrm{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^s X_i$ an open affine cover. We have for $\sigma : k \hookrightarrow \mathbb{C}$ the evaluation period embedding map which is the morphism of bi-complexes

$$ev(X)^{\bullet}_{\bullet} : \Gamma(X_{\bullet},\Omega^{\bullet}_{X_{\bullet}}) \to \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^{\bullet},X^{an}_{\mathbb{C},\bullet})^{\vee} \otimes \mathbb{C},$$

$$w^l_I \in \Gamma(X_I,\Omega^l_{X_I}) \mapsto (ev(X)^l_I(w^l_I) : \phi^l_I \in \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^l,X^{an}_{\mathbb{C},I})^{\vee} \otimes \mathbb{C} \mapsto ev^l_I(w^l_I)(\phi^l_I) := \int_{\mathbb{I}^l} \phi^{l*}_I w^l_I)$$

given by integration. By taking all the affine open cover $(j_i : X_i \hookrightarrow X)$ of $X$, we get for $\sigma : k \hookrightarrow \mathbb{C}$, the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X)^\bullet_\bullet : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \to \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C}$$

It induces in cohomology, for $j \in \mathbb{Z}$, the evaluation period map

$$H^j ev(X) = H^j ev(X)^\bullet_\bullet : H^j_{DR}(X) = H^j \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \to H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C}) = H^j(\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

**Remark 1.** *We also have for $\sigma : k \hookrightarrow \mathbb{C}$ the composition*

$$\bar{ev}(X)^\bullet_\bullet : \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \xrightarrow{ev(X)^\bullet_\bullet} \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C}\bullet})^\vee \otimes \mathbb{C} \xrightarrow{\mathbb{Z}(X)(i) \circ \mathrm{an}^{-,-}} \mathrm{Hom}_{\mathrm{Fun}(\Delta^\bullet, \mathrm{Var}(k))}(\mathbb{D}^\bullet_{k,et}, X_\bullet)^\vee \otimes \mathbb{C}$$

*where $i : I^\bullet \mathbb{D}^\bullet_{k,et}$ is the embedding, which is given by integration : for $w^l_I \in \Gamma(X_I, \Omega^l_{X_I})$ and $\phi^l_I \in \mathrm{Hom}_{\mathrm{Fun}(\Delta^\bullet, \mathrm{Var}(k))}(\mathbb{D}^j_{k,et}, X_I)$,*

$$\bar{ev}^l_I(w^l_I)(\phi^l_I) = \int_{\mathbb{I}^l} \phi^{l,an*}_I w^l_I.$$

Let $X \in \mathrm{SmVar}(k)$. Note that

$$H^* ev(X_{\mathbb{C}}) : H^*_{DR}(X_{\mathbb{C}}) \xrightarrow{H^* R\Gamma(X^{an}_{\mathbb{C}}, E_{zar}(\Omega(\mathrm{an}_X)))} H^*_{DR}(X^{an}_{\mathbb{C}}) \xrightarrow{H^* R\Gamma(X^{an}_{\mathbb{C}}, \alpha(X))} H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and the quasi-isomorphism $\alpha(X) : \mathbb{C}_{X^{an}_{\mathbb{C}}} \hookrightarrow \Omega^\bullet_{X^{an}_{\mathbb{C}}}$ in $C(X^{an}_{\mathbb{C}})$. Hence,

$$H^* ev(X) = H^*_{DR}(X) \xrightarrow{\Omega(\pi_{k/\mathbb{C}}(X))} H^*_{DR}(X_{\mathbb{C}}) \xrightarrow{H^* ev(X_{\mathbb{C}})} H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is injective. The elements of the image $H^* ev(X)(H^*_{DR}(X)) \subset H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$ are the periods of $X$.

Let $X \in \mathrm{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^s X_i$ an open affine cover with $X_i := X \backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding and $X^{an}_{\mathbb{C}} = \cup_{i=1}^r \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0,1)^d$. Since a convex open subset of $\mathbb{C}^d$ is biholomorphic to an open ball we have $\mathbb{D}_I := \cap_{i \in I} \mathbb{D}_i \simeq D(0,1)^d$ (where $d$ is the dimension of a connected component of $X$). Denote by $j_\bullet : X^{an}_{\bullet,\mathbb{C}} \cap \mathbb{D}_\bullet \to X^{an}_{\bullet,\mathbb{C}}$ is the open embeddings. We then have the period morphism of tri-complexes

$$ev(X^{an}_{\mathbb{C}})^\bullet_{\bullet,\bullet} : \Gamma(X^{an}_{\bullet,\mathbb{C}} \cap \mathbb{D}_\bullet, \Omega^\bullet_{X^{an}_{\mathbb{C}}}) \to \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C},$$

$$w^l_{I,J} \in \Gamma(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \mathrm{an}^*_X \Omega^l_{X_I}) \mapsto$$

$$(ev^l_{I,J}(w^l_{I,J}) : \phi^l_{I,J} \in \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^l, X^{an}_{\mathbb{C},I} \cap \mathbb{D}_J)^\vee \otimes \mathbb{C} \mapsto ev^l_{I,J}(w^l_{I,J})(\phi^l_{I,J}) := \int_{\mathbb{I}^l} \phi^{l*}_{I,J} w^l_{I,J})$$

given by integration. We have then the factorization

$$H^j ev(X) : H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X) = \mathbb{H}^j_{et}(X, \Omega^\bullet_{X^{et}}) \xrightarrow{H^j \Omega(\pi_{k/\mathbb{C}}(X))}$$

$$H^j_{DR}(X_{\mathbb{C}}) := \mathbb{H}^j(X_{\mathbb{C}}, \Omega^\bullet_X) = \mathbb{H}^j_{et}(X_{\mathbb{C}}, \Omega^\bullet_{X^{et}}) \xrightarrow{j^*_\bullet \circ \mathrm{an}^*_{X_\bullet}} H^j \Gamma(X^{an}_{\bullet,\mathbb{C}} \cap \mathbb{D}_\bullet, \Omega^\bullet_{X^{an}_{\mathbb{C}}})$$

$$\xrightarrow{H^j ev(X^{an}_{\mathbb{C}})^\bullet_{\bullet,\bullet}} H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}} \cap \mathbb{D}_\bullet, \mathbb{C}) = H^j(\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet} \cap \mathbb{D}_\bullet)^\vee \otimes \mathbb{C}).$$

where for the left hand side, the first equality follows from the fact that $\Omega^\bullet \in C(\mathrm{SmVar}(k))$ is $\mathbb{A}^1$ local and admits transferts, and the equality of the right hand side follows from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces.

**Remark 2.** *Let $X \in \mathrm{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^{s} X_i$ an open affine cover with $X_i := X \backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding and $X_{\mathbb{C}}^{an} = \cup_{i=1}^{r} \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0,1)^d$. Since a convex open subset of $\mathbb{C}^d$ is biholomorphic to an open ball we have $\mathbb{D}_I := \cap_{i \in I} \mathbb{D}_i \simeq D(0,1)^d$ (where $d$ is the dimension of a connected component of $X$). Denote by $j_\bullet : X_{\bullet,\mathbb{C}}^{an} \cap \mathbb{D}_\bullet \to X_{\bullet,\mathbb{C}}^{an}$ the open embeddings. Then,*

$$j_\bullet^* \circ \mathrm{an}_{X_\bullet}^* := \Omega(j_\bullet \circ \mathrm{an}_{X_\bullet}) : \Gamma(X_{\bullet,\mathbb{C}}, \Omega_{X^{et}}^\bullet) \to \Gamma(X_{\bullet,\mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet)$$

*is a quasi-isomorphism by the Grothendieck comparaison theorem for De Rham cohomology and the acyclicity of quasi-coherent sheaves on noetherian affine schemes.*

**Lemma 2.** *Let $X \in \mathrm{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^{s} X_i$ an open affine cover with $X_i := X \backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding and $X_{\mathbb{C}}^{an} = \cup_{i=1}^{r} \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0,1)^d$. Let*

$$w = [\sum_{I,J,l,cardI+cardJ+l=j} w_{I,J}^l] = \sum_{I,J,l,cardI+cardJ+l=j} [w_{I,J}^l] \in H^j\Gamma(X_{\bullet,\mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet).$$

*Then the following assertions are equivalent :*

*(i)* $H^j ev(X)(w) \in H_{\mathrm{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{Z}(2i\pi))$,

*(ii) for all $I, J, l$ such that $cardI + cardJ + l = j$ , there exist a lift*

$$[\tilde{w}_{IJ}^l] \in H^l\Gamma(X_{I,\mathbb{C}}^{an} \cap \mathbb{D}_J, \Omega_{X_{\mathbb{C}}^{an}}^\bullet)$$

*of $[w_{IJ}^l]$ with respect to the spectral sequence associated to the filtration on the total complex associated to the bi-complex structure such that*

$$H^l ev(X_{\mathbb{C}}^{an})_{I,J}^\bullet([\tilde{w}_{I,J}^l]) \in H_{\mathrm{sing}}^l(X_{I,\mathbb{C}}^{an} \cap \mathbb{D}_J, \mathbb{Z}(2i\pi)).$$

*Proof.* Follows immediately form the fact that $ev(X_{\mathbb{C}}^{an})_{\bullet,\bullet}^\bullet$ define by definition a morphism of spectral sequence for the filtration given by the bi-complex structure. □

The main proposition of this section is the following :

**Proposition 1.** *Let $X \in \mathrm{SmVar}(k)$. Let $X = \cup_{i=1}^{s} X_i$ an open affine cover with $X_i := X \backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. Let*

$$w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet) = \mathbb{H}_{et}^j(X, \Omega_X^\bullet).$$

*Then if $w \in H^j OL_{X^{et}}(\mathbb{H}_{et}^j(X, \Omega_{X^{et},\log}^\bullet))$ then $H^j ev(X)(w) \in H_{\mathrm{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{Z}(2i\pi))$, where we recall (see section 2) the inclusion in $C(X^{et})$*

$$OL_{X^{et}} : \Omega_{X^{et},\log}^\bullet \hookrightarrow \Omega_{X^{et}}^\bullet$$

*is the sub-complex of logarithmic forms.*

*Proof.* Let

$$w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet) = H^j\Gamma(X_\bullet, \Omega_X^\bullet).$$

where $X = \cup_{i=1}^{s} X_i$ is an open affine cover with $X_i := X \backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $X_{\mathbb{C}}^{an} = \cup_{i=1}^{r} \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0,1)^d$. Then by definition $H^j ev(X)(w) = H^j ev(X_{\mathbb{C}}^{an})(j_\bullet^* \circ \mathrm{an}_{X_\bullet}^* w)$ with

$$j_\bullet^* \circ \mathrm{an}_{X_\bullet}^* w \in H^j\Gamma(X_{\bullet,\mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet)$$

and we have the following commutative diagram

$$\begin{array}{ccc} \Gamma(X_\bullet, \Omega^\bullet_X) & \xrightarrow{j^*_\bullet \circ \mathrm{an}^*_{X_\bullet}} & \Gamma(X^{an}_{\bullet,\mathbb{C}} \cap \mathbb{D}_\bullet, \Omega^\bullet_{X^{an}_{\mathbb{C}}}) \ . \\ {\scriptstyle \Gamma(-, OL_{X^{et}})}\uparrow & \nearrow_{j^*_\bullet \circ \mathrm{an}^*_{X_\bullet}} & \\ \Gamma(X_\bullet, \Omega^\bullet_{X^{et},\log}), & & \end{array}$$

Now, if $w = H^j OL_{X^{et}}(w_L)$, since

$$(\Gamma(X_\bullet, k)) : \varinjlim_{r: X_\bullet \to X} H^j \Gamma(X_\bullet, \Omega^\bullet_{X^{et},\log}) \to \mathbb{H}^j_{et}(X, \Omega^\bullet_{X^{et},\log}) = \Gamma(X_\bullet, E_{et}(\Omega^\bullet_{X^{et},\log}))$$

where the limit is taken over the etale covers of $X$, and since $\Omega^\bullet_{X^{et},\log}$ admits transferts, there exist an open affine cover (depending on $w$) $X = \cup_{i=1}^s X_i$ with $X_i = X_i(w) := X \backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing such that

$$w_L = H^j \Gamma(X_\bullet, k)(w_L),\ w_L \in H^j \Gamma(X_\bullet, \Omega^\bullet_{X^{et},\log}).$$

We then have

$$j^*_\bullet \circ \mathrm{an}^*_{X_\bullet} w = j^*_\bullet \circ \mathrm{an}^*_{X_\bullet} w_L = [\sum_{I,J,l,cardI+cardJ+l=j} w^l_{L,I,J}] = \sum_{I,J,l,cardI+cardJ+l=j} [w^l_{L,I,J}]$$
$$\in H^j \Gamma(X^{an}_{\bullet,\mathbb{C}} \cap \mathbb{D}_\bullet, \Omega^\bullet_{X^{an}_{\mathbb{C}}}).$$

Let for each $(I, J, l)$ take a lift

$$[\tilde{w}^l_{L,I,J}] \in H^l \Gamma(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \Omega^\bullet_{X^{an}_{\mathbb{C}}})$$

of $[w^l_{L,I,J}]$ with respect to the spectral sequence associated to the filtration on the total complex associated to the bi-complex structure. We have by a standard computation, for each $(I, J)$

- $H^*_{\mathrm{sing}}(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \mathbb{Z}) =< \gamma_1, \cdots, \gamma_{cardI} >$.
- $w^l_{L,I,J} = \sum_{\mu' \in P([1,\cdots,s]), card\mu' = l} n^{\mu'}_{IJ} dz_{\mu'_1} \wedge \cdots \wedge dz_{\mu'_l} + d\alpha \in \Gamma(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \Omega^\bullet_{X^{an}_{\mathbb{C}}})$.

Then for $\mu \in P([1, \cdots, s])$ with $card\mu = l$, we get

$$H^l ev(X^{an}_{\mathbb{C}})_{I,J}([\tilde{w}^l_{L,I,J}])(\gamma_\mu) = \sum_k n^\mu_{IJ} \delta_{\mu',\mu} \in 2i\pi\mathbb{Z}.$$

where $\gamma_\mu := \gamma_{\mu_1} \cdots \gamma_{\mu_l}$. We conclude by lemma 2. □

Let $X \in \mathrm{SmVar}(k)$. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. By proposition 1, we have a commutative diagram of graded algebras

$$\begin{array}{ccc} H^*_{DR}(X) & \xrightarrow{H^* ev(X)} & H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C}) \\ {\scriptstyle H^* OL_{X^{et}}}\uparrow & & \uparrow{\scriptstyle H^* C^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(X^{an}_{\mathbb{C}})} \\ \mathbb{H}^*_{et}(X, \Omega_{X^{et},\log}) & \xrightarrow{H^* ev(X) \circ H^* OL_{X^{et}}} & H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Z}) \end{array}$$

where

$$C^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(X^{an}_{\mathbb{C}}) : C^\bullet_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Z}) \hookrightarrow C^\bullet_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is the subcomplex consiting of $\alpha \in C^j_{\text{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$ such that $\alpha(\gamma) \in 2i\pi\mathbb{Z}$ for all $\gamma \in C^{\text{sing}}_j(X^{an}_{\mathbb{C}}, \mathbb{Z})$. Recall that

$$H^*ev(X_{\mathbb{C}}) = H^*R\Gamma(X^{an}_{\mathbb{C}}, \alpha(X)) \circ \Gamma(X^{an}_{\mathbb{C}}, E_{zar}(\Omega(\text{an}_X))) : H^*_{DR}(X_{\mathbb{C}}) \xrightarrow{\sim} H^*_{\text{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and $\alpha(X) : \mathbb{C}_{X^{an}_{\mathbb{C}}} \hookrightarrow \Omega^{\bullet}_{X^{an}_{\mathbb{C}}}$, which gives the periods elements $H^*ev(X)(H^*_{DR}(X)) \subset H^*_{\text{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$. On the other side the induced map

$$H^*ev(X_{\mathbb{C}}) : H^*OL_{X^{et}}(\mathbb{H}^*_{et}(X_{\mathbb{C}}, \Omega_{X^{et},\log})) \hookrightarrow H^*\iota_{2i\pi\mathbb{Z}/\mathbb{C}}H^*_{\text{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Z})$$

is NOT surjective in general since the left hand side is invariant by the action of Galois group $Gal(\mathbb{C}/k)$ whereas the right hand side is not.

## 3.2 $p$ adic integral periods

Let $k$ a field of finite type over $\mathbb{Q}$. Denote $\bar{k}$ the algebraic closure of $k$ and $G = Gal(\bar{k}/k)$ the absolute Galois group of $k$. Let $X \in \text{SmVar}(k)$ a smooth variety. Take a compactification $\bar{X} \in \text{PSmVar}(k)$ of $X$ such that $D := \bar{X} \backslash X \subset X$ is a normal crossing divisor, and denote $j : X \hookrightarrow \bar{X}$ the open embedding. Let $p \in \mathbb{N}$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Then $k \subset \bar{k} \subset \mathbb{C}_p$ and $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$, where $\hat{k}_{\sigma_p}$ is the $p$-adic field which is the completion of $k$ with respect the $p$ adic norm given by $\sigma_p$. Denote $\hat{G}_{\sigma_p} = Gal(\mathbb{C}_p/\hat{k}_{\sigma_p}) = Gal(\bar{\mathbb{Q}}_p/\hat{k}_{\sigma_p})$ the Galois group of $\hat{k}_{\sigma_p}$. We have then the commutative diagram in $C_{\mathbb{B}_{dr}fil,\hat{G}_{\sigma_p}}(\bar{X}^{an,pet}_{\mathbb{C}_p})$

$$\begin{array}{ccc}
j_*E_{pet}(\mathbb{B}_{dr,X_{\mathbb{C}_p}}, F) & \xrightarrow{j_*E_{pet}(\alpha(X))} & j_*E_{pet}((\Omega^{\bullet}_{X_{\mathbb{C}_p}}, F_b) \otimes_{O_{X_{\mathbb{C}_p}}} (O\mathbb{B}_{dr,X_{\mathbb{C}_p}}, F)) \\
\uparrow \subset & & \uparrow \subset \\
E_{pet}(\mathbb{B}_{dr,\bar{X}_{\mathbb{C}_p},\log D_{\mathbb{C}_p}}, F) & \xrightarrow{\alpha(X)} & E_{pet}((\Omega^{\bullet}_{\bar{X}_{\mathbb{C}_p}}(\log D_{\mathbb{C}_p}), F_b) \otimes_{O_{\bar{X}_{\mathbb{C}_p}}} (O\mathbb{B}_{dr,\bar{X}_{\mathbb{C}_p},\log D_{\mathbb{C}_p}}, F)) \\
{\scriptstyle E_{pet}(j_*\iota'_{X^{pet}_{\mathbb{C}_p}})_j := E_{pet}(l\mapsto l.1)_j}\uparrow & & \uparrow{\scriptstyle E_{pet}(j_*(n\circ OL_{X^{pet}_{\mathbb{C}_p}}))_j := E_{pet}(w\mapsto (w\otimes 1))_j} \\
j_*E_{pet}(\underline{\mathbb{Z}_p}_{X_{\mathbb{C}_p}}) & \xrightarrow{j_*E_{pet}(\iota_{X^{pet}_{\mathbb{C}_p}}) := j_*E_{pet}(l\mapsto l.1)} & j_*E_{pet}(\Omega^{\bullet}_{X_{\mathbb{C}_p},\log}, F_b)
\end{array},$$

where for $j' : U' \hookrightarrow X'$ an open embedding with $X' \in \text{RTop}$ and $\tau$ a topology on RTop we denote for $m : j_*Q \to Q'$ with $Q \in \text{PSh}_O(U')$, $Q' \in \text{PSh}_O(X')$ the canonical map in $C_O(X')$

$$E^0_\tau(m)_j : j_*E^0_\tau(Q) \to E^0_\tau(j_*Q) \xrightarrow{E^0_\tau(m)} E^0_\tau(Q'),$$

giving by induction the canonical map $E_\tau(m)_j : j_*E_\tau(Q) \to E_\tau(Q')$ in $C_O(X')$. The main results of [15] state that

- the map in $C_{\mathbb{B}_{dr}fil}(\bar{X}^{an,pet}_{\hat{k}_{\sigma_p}})$

  $$\alpha(X) : (\mathbb{B}_{dr,\bar{X}_{\hat{k}_{\sigma_p}},\log D_{\hat{k}_{\sigma_p}}}, F) \hookrightarrow (\Omega^{\bullet}_{\bar{X}_{\hat{k}_{\sigma_p}}}(\log D_{\hat{k}_{\sigma_p}}), F_b) \otimes_{O_{\bar{X}_{\hat{k}_{\sigma_p}}}} (O\mathbb{B}_{dr,\bar{X}_{\hat{k}_{\sigma_p}},\log D_{\hat{k}_{\sigma_p}}}, F)$$

  is a filtered quasi-isomorphism, that is, the induced map in $C_{\mathbb{B}_{dr}fil,\hat{G}_{\sigma_p}}(\bar{X}^{an,pet}_{\mathbb{C}_p})$

  $$\alpha(X) := \alpha(X)_{\mathbb{C}_p} : (\mathbb{B}_{dr,\bar{X}_{\mathbb{C}_p},\log D_{\mathbb{C}_p}}, F) \hookrightarrow (\Omega^{\bullet}_{\bar{X}_{\mathbb{C}_p}}(\log D_{\mathbb{C}_p}), F_b) \otimes_{O_{\bar{X}_{\mathbb{C}_p}}} (O\mathbb{B}_{dr,\bar{X}_{\mathbb{C}_p},\log D_{\mathbb{C}_p}}, F)$$

  is thus a filtered quasi-isomorphism,

- the map in $D_{\mathbb{Z}_p fil}$

$$T(a_X, a_X, \otimes)(Rj_*\mathbb{Z}_{p,X^{et}}) : R\Gamma(X_{\mathbb{C}_p}, \mathbb{Z}_{p,X^{et}}) \otimes (\mathbb{B}_{dr,\mathbb{C}_p}, F) \to R\Gamma(\bar{X}_{\mathbb{C}_p}, (\mathbb{B}_{dr,\bar{X}_{\mathbb{C}_p},\log D_{\mathbb{C}_p}}, F))$$

  is an isomorphism.

Hence, we get the isomorphism in $D_{fil}(\mathbb{B}_{dr}, \hat{G}_{\sigma_p})$

$$\begin{aligned} R\alpha(X) := R\Gamma(\bar{X}_{\mathbb{C}_p}, \alpha(X)) \circ T(a_X, a_X, \otimes)(Rj_*\mathbb{Z}_{p,X^{et}}) : \\ R\Gamma(X_{\mathbb{C}_p}, \mathbb{Z}_{p,X^{et}}) \otimes (\mathbb{B}_{dr,\mathbb{C}_p}, F) \xrightarrow{\sim} R\Gamma(\bar{X}_{\mathbb{C}_p}, (\Omega^\bullet_{\bar{X}^{et}_{\mathbb{C}_p}}(\log D_{\mathbb{C}_p}), F_b) \otimes_{O_{\bar{X}_{\mathbb{C}_p}}} (O\mathbb{B}_{dr,\bar{X}_{\mathbb{C}_p},\log D_{\mathbb{C}_p}}, F)) \\ \xrightarrow{=} R\Gamma(\bar{X}_{\mathbb{C}_p}, (\Omega^\bullet_{\bar{X}^{et}_{\mathbb{C}_p}}, F_b) \otimes_{O_{\bar{X}_{\mathbb{C}_p}}} j_{*Hdg}(O_{X_{\mathbb{C}_p}}, F_b) \otimes_{O_{X_{\mathbb{C}_p}}} (O\mathbb{B}_{dr,X_{\mathbb{C}_p}}, F)) \end{aligned}$$

which gives for each $n \in \mathbb{Z}$ a filtered isomorphism of $\hat{G}_{\sigma_p}$-modules

$$H^n R\alpha(X) : H^n_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_{p,X^{et}}) \otimes \mathbb{B}_{dr,\mathbb{C}_p} \xrightarrow{\sim} H^n_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{dr,\mathbb{C}_p}$$

so that we can recover the Hodge filtration on $H^*_{DR}(X)$ by the action of $\hat{G}_{\sigma_p}$.

We have the following key proposition :

**Proposition 2.** *Let $k$ a field of finite type over $\mathbb{Q}$. Let $X \in \mathrm{SmVar}(k)$ a smooth variety. Take a compactification $\bar{X} \in \mathrm{PSmVar}(k)$ of $X$ such that $D := \bar{X} \backslash X \subset X$ is a normal crossing divisor, and denote $j : X \hookrightarrow \bar{X}$ the open embedding. Let $p \in \mathbb{N}$ a prime number. Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding. Consider its completion $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$ with respect to the $p$ adic norm induced by $\sigma_p$. Then $\hat{G}_{\sigma_p} := Gal(\mathbb{C}_p/\hat{k}_{\sigma_p}) \subset G := Gal(\bar{k}/k)$.*

*(i) Let $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)$. Consider then*

$$\nu^*_X \alpha := H^j\Gamma(X_{\mathbb{C}_p}, k)(\alpha) \in H^j_{pet}(X_{\mathbb{C}_p}, \mathbb{Z}_p) = H^j_{pet}(X_{\mathbb{C}_p}, \underline{\mathbb{Z}_p})$$

*where $k := k \circ \mathrm{ad}(\nu^*_X, \nu_{X*})(\mathbb{Z}_p) : E_{et}(\mathbb{Z}_p) \to \nu_{X*}E_{pet}(\mathbb{Z}_p)$ is the canonical map in $C(\mathbb{N}, X^{et})$ which is a quasi-isomorphism, and its associated De Rham log class*

$$w_L(\alpha) := H^j \iota_{X^{pet}_{\mathbb{C}_p}}(\nu^*_X \alpha) \in \mathbb{H}^j_{pet}(X_{\mathbb{C}_p}, \Omega^\bullet_{X^{pet}_{\mathbb{C}_p},\log}) = \mathbb{H}^j_{et}(X_{\mathbb{C}_p}, \Omega^\bullet_{X^{et}_{\mathbb{C}_p},\log})$$

*which gives*

$$\begin{aligned} w(\alpha) := H^j OL_{X^{et}_{\mathbb{C}_p}}(w_L(\alpha)) = H^j\beta(X^{pet}_{\mathbb{C}_p})(\alpha) \\ \in \mathbb{H}^j_{pet}(\bar{X}_{\mathbb{C}_p}, \Omega^\bullet_{\bar{X}^{pet}_{\mathbb{C}_p}}(\log D_{\mathbb{C}_p})) = \mathbb{H}^j_{et}(\bar{X}_{\mathbb{C}_p}, \Omega^\bullet_{\bar{X}^{et}_{\mathbb{C}_p}}(\log D_{\mathbb{C}_p})) = H^j_{DR}(X_{\mathbb{C}_p}) \end{aligned}$$

*If $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}}$ then*

$$H^j R\Gamma(X_{\mathbb{C}_p}, \alpha(X))(\alpha \otimes 1) = (w(\alpha) \otimes 1) \in F^l H^j_{DR}(X_{\hat{k}}) \otimes_{\hat{k}} \mathbb{B}_{dr,\mathbb{C}_p}$$

*so that $w(\alpha) \in F^l H^j_{DR}(X_{\mathbb{C}_p})$.*

*(ii) Let $\alpha \in H^j_{et}(X_{\bar{k}}, \mathbb{Z}_p)(l)$. Referring to (i), we consider*

- $w_L(\alpha) := w_L(\pi_{\bar{k}/\mathbb{C}_p}(X_{\bar{k}})^*\alpha) \in \mathbb{H}^j_{et}(X_{\mathbb{C}_p}, \Omega^\bullet_{X^{et}_{\mathbb{C}_p},\log})$,
- $w(\alpha) := H^j OL_{X^{et}_{\mathbb{C}_p}}(w_L(\alpha)) \in H^j_{DR}(X_{\mathbb{C}_p})$.

*where $\pi_{\bar{k}/\mathbb{C}_p}(X_{\bar{k}}) : X_{\mathbb{C}_p} \to X_{\bar{k}}$ is the projection. If $\alpha \in H^j_{et}(X_{\bar{k}}, \mathbb{Z}_p)(l)^G$ then $w_L(\alpha) \in \mathbb{H}^j_{et}(X, \Omega^\bullet_{X^{et},\log})$ and $w(\alpha) = H^j OL_{X^{et}}(w_L(\alpha)) \in F^l H^j_{DR}(X)$.*

*Proof.* (i):It follows immediately from the fact that $H^jR\alpha(X)$ is a filtered quasi-isomorphism compatible with the Galois action of $\hat{G}_{\sigma_p}$ by [15] and from the commutative diagram

$$\begin{array}{ccccc} H^j_{pet}(X_{\mathbb{C}_p},\mathbb{Z}_p) & \xrightarrow{H^j\Gamma(X_{\mathbb{C}_p},\iota_X^{et})} & \mathbb{H}^j_{pet}(X_{\mathbb{C}_p},\Omega^\bullet_{X^{et}_{\mathbb{C}_p},\log}) & \xrightarrow{H^j\Gamma(X_{\mathbb{C}_p},(w\mapsto(w\otimes 1)))} & \mathbb{H}^j_{pet}(X_{\mathbb{C}_p},\Omega^\bullet_{X^{et}_{\mathbb{C}_p}}) \\ \downarrow{\scriptstyle \mathrm{an}^*_{X_{\mathbb{C}_p}}} & & \downarrow{\scriptstyle \mathrm{an}^*_{X_{\mathbb{C}_p}}} & & \downarrow{\scriptstyle \mathrm{an}^*_{X_{\mathbb{C}_p}}} \\ H^j_{pet}(X^{an}_{\mathbb{C}_p},\mathbb{Z}_p) & \xrightarrow{H^j\Gamma(X^{an}_{\mathbb{C}_p},\iota_X^{pet})} & \mathbb{H}^j_{pet}(X^{an}_{\mathbb{C}_p},\Omega^\bullet_{X^{pet}_{\mathbb{C}_p},\log}) & \xrightarrow{H^j\Gamma(X^{an}_{\mathbb{C}_p},(w\mapsto(w\otimes 1)))} & \mathbb{H}^j_{pet}(X^{an}_{\mathbb{C}_p},\Omega^\bullet_{X^{et}_{\mathbb{C}_p}}) \end{array}$$

where the left and right column are isomorphisms by GAGA (note that the middle column is NOT an isomorphism in general).
(ii): Assume by absurd that $w(\alpha)\notin H^j_{DR}(X)$. There exist a finite type extenion $k'/k$, $k'\subset\mathbb{C}_p$ (depending on $w(\alpha)$) such that $w(\alpha)\in H^j_{DR}(X_{k'})$. By assumption the orbit $Gal(k'/k)w(\alpha)\subset H_{DR}(X_{k'})$ of $w(\alpha)$ under $Gal(k'/k)$ is non trivial (i.e. contain more then one element). Then there exist a prime number $l$ and an embedding $\sigma'_l: k'\hookrightarrow\mathbb{C}_l$ such that the extension $\hat{k}'_{\sigma'_l}/\hat{k}_{\sigma_l}$ is non trivial (i.e. $\hat{k}'_{\sigma'_l}\neq\hat{k}_{\sigma_l}$) where $\sigma_l=\sigma'_{l|k}$. By injectivity of

$$\pi_{k'/\hat{k}'_{\sigma'_l}}(X_{k'})^*: H^j_{DR}(X_{k'})\to H^j_{DR}(X_{\hat{k}'_{\sigma'_l}})$$

where $\pi_{k'/\hat{k}'_{\sigma'_l}}(X_{k'}): X_{\hat{k}'_{\sigma'_l}}\to X_{k'}$ is the projection, the orbit $Gal(\hat{k}'_{\sigma'_l}/\hat{k}_{\sigma_l})w(\alpha)\subset H_{DR}(X_{\hat{k}'_{\sigma'_l}})$ of $w(\alpha):=\pi_{k'/\hat{k}'_{\sigma'_l}}(X_{k'})^*w(\alpha)$ under $Gal(\hat{k}'_{\sigma'_l}/\hat{k}_{\sigma_l})$ is non trivial (i.e. contain more then one element). By the main result of [15] (see above), we get

$$H^jR\alpha(X_{\mathbb{C}_l})^{-1}(w(\alpha))\notin(H^j_{et}(X_{\bar{k}'},\mathbb{Z}_l)\otimes\mathbb{B}_{dr,\mathbb{C}_l})^{\hat{G}_{\sigma'_l}}.$$

In particular $H^jR\alpha(X_{\mathbb{C}_l})^{-1}(w(\alpha))\notin H^j_{et}(X_{\bar{k}'},\mathbb{Z}_l)^{\hat{G}_{\sigma'_l}}$. Consider the pairing of $G$ modules

$$\delta(-,-): H^k_{pet}(X_{\mathbb{C}_p},\underline{\mathbb{Z}_p})\otimes_{\mathbb{Z}}H^l_{pet}(X_{\mathbb{C}_l},\underline{\mathbb{Z}_l})\to H^{k+l}_{pet}(X_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l},\underline{\mathbb{Z}_p}\otimes_{\mathbb{Z}}\underline{\mathbb{Z}_l})$$

and

$$\begin{aligned}\alpha(X_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l}): \pi_p^{*mod}\mathbb{B}_{dr,X_{\mathbb{C}_p}}\otimes_{O_{X_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l}}}\pi_l^{*mod}\mathbb{B}_{dr,X_{\mathbb{C}_l}}&\xrightarrow{\alpha(X_{\mathbb{C}_p})\otimes\alpha(X_{\mathbb{C}_l})}\\ \pi_p^{*mod}\Omega^\bullet_{X_{\mathbb{C}_p}}\otimes_{O_{X_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l}}}\pi_l^{*mod}\Omega^\bullet_{X_{\mathbb{C}_l}}&\xrightarrow{w(\Omega)}\Omega^\bullet_{X_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l}}\end{aligned}$$

is the canonical map in $C_{\hat{G}_{\sigma'_p}\times\hat{G}_{\sigma'_l}}(X^{pet}_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l})$ and

- $\pi_p:=\pi_{\mathbb{C}_p/\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l}(X_{\mathbb{C}_p}): X_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l}\to X_{\mathbb{C}_p}$
- $\pi_l:=\pi_{\mathbb{C}_l/\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l}(X_{\mathbb{C}_l}): X_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l}\to X_{\mathbb{C}_l}$

are the base change maps. By the commutative diagram of $\hat{G}_{\sigma'_p}\times\hat{G}_{\sigma'_l}$ modules

$$\begin{array}{ccccc} H^j_{pet}(X_{\bar{k}'},\underline{\mathbb{Z}_p})\otimes\mathbb{B}_{dr,\mathbb{C}_p} & \xrightarrow{i_{p/pl}} & H^j_{pet}(X_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l},\mathbb{B}_{dr,X_{\mathbb{C}_p}}\otimes_{O_{X_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l}}}\mathbb{B}_{dr,X_{\mathbb{C}_l}}) & \xleftarrow{i_{l/pl}} & H^j_{pet}(X_{\bar{k}'},\underline{\mathbb{Z}_l})\otimes_{\mathbb{B}_{dr,\mathbb{C}_l}} \\ \downarrow{\scriptstyle H^jR\alpha(X_{\mathbb{C}_l})} & & \downarrow{\scriptstyle H^j\alpha(X_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l})} & & \downarrow{\scriptstyle H^jR\alpha(X_{\mathbb{C}_l})} \\ H^j_{DR}(X_{\mathbb{C}_p})\otimes\mathbb{B}_{dr,\mathbb{C}_p} & \xleftarrow{\pi_{k'/\mathbb{C}_p}(X_{k'})^*} & H^j_{DR}(X_{k'})\otimes_{\mathbb{C}_p\otimes_{\mathbb{Q}}\mathbb{C}_l}\mathbb{B}_{dr,\mathbb{C}_p}\otimes_{\mathbb{Q}}\mathbb{B}_{dr,\mathbb{C}_l} & \xrightarrow{\pi_{k'/\mathbb{C}_l}(X_{k'})^*} & H^j_{DR}(X_{\mathbb{C}_l})\otimes\mathbb{B}_{dr,\mathbb{C}_l} \end{array}$$

where

$$i_{p/pl} : \underline{\mathbb{Z}_p} \hookrightarrow \underline{\mathbb{Z}_p} \otimes_{\mathbb{Z}} \underline{\mathbb{Z}_l},\ i_{l/pl} : \underline{\mathbb{Z}_l} \hookrightarrow \underline{\mathbb{Z}_p} \otimes_{\mathbb{Z}} \underline{\mathbb{Z}_l},\ i_{p/pl}(\Lambda) = \Lambda \otimes 1,\ i_{l/pl}(\Lambda) = \Lambda \otimes 1$$

are the canonical embeddings, we have for $\beta_p \in H^j_{pet}(X_{\bar{k}'}, \mathbb{Z}_p) \otimes \mathbb{B}_{dr,\mathbb{C}_p}$ $\beta_l \in H^j_{pet}(X_{\bar{k}'}, \mathbb{Z}_l) \otimes \mathbb{B}_{dr,\mathbb{C}_l}$,

$$\alpha(X_{\mathbb{C}_p \otimes_{\mathbb{Q}} \mathbb{C}_l})(\delta(\beta_p, \beta_l)) = \alpha(X_{\mathbb{C}_l}) \cdot \alpha(X_{\mathbb{C}_l}) \in H^j_{DR}(X_{\mathbb{C}_p \otimes_{\mathbb{Q}} \mathbb{C}_l}) \otimes_{\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_l} \mathbb{B}_{dr,\mathbb{C}_p} \otimes_{\mathbb{Q}} \mathbb{B}_{dr,\mathbb{C}_l}.$$

Hence

$$i_{p/pl}\alpha = \delta(\alpha, 1) = \delta(1, H^j R\alpha(X_{\mathbb{C}_l})^{-1}(w(\alpha))) \notin H^j_{pet}(X_{\mathbb{C}_p \otimes_{\mathbb{Q}} \mathbb{C}_l}, \underline{\mathbb{Z}_p} \otimes_{\mathbb{Z}} \underline{\mathbb{Z}_l})^{\hat{G}_{\sigma'_p} \times \hat{G}_{\sigma'_l}}.$$

Contradiction. We thus have $w(\alpha) \in H^j_{DR}(X)$. Now :

- By (i) $w(\alpha) \in F^l H^j_{DR}(X_{\mathbb{C}_p})$. Hence $w(\alpha) \in F^l H^j_{DR}(X) = F^l H^j_{DR}(X_{\mathbb{C}_p}) \cap H^j_{DR}(X)$. The equality $F^l H^j_{DR}(X) = F^l H^j_{DR}(X_{\mathbb{C}_p}) \cap H^j_{DR}(X)$ is given by the filtered isomorphism in $C_{fil}(\bar{X}_{\mathbb{C}_p})$

  $$w(k/\mathbb{C}_p) : (\Omega^\bullet_{\bar{X}}(\log D), F_b) \otimes_k \mathbb{C}_p \xrightarrow{\sim} (\Omega^\bullet_{\bar{X}_{\mathbb{C}_p}}(\log D_{\mathbb{C}_p}), F_b)$$

  which say that the Hodge filtration is defined over $k$: see section 2.

- We have $w(\alpha) \in H^j OL_{X^{et}}(\mathbb{H}^j_{et}(X, \Omega^\bullet_{X^{et},\log}))$ by the equality

  $$H^j OL_{X^{et}}(\mathbb{H}^j_{et}(X, \Omega^\bullet_{X^{et},\log})) = H^j OL_{X^{et}}(\mathbb{H}^j_{et}(X_{\mathbb{C}_p}, \Omega^\bullet_{X^{et}_{\mathbb{C}_p},\log})) \cap H^j_{DR}(X)$$

  given by the commutative diagram

$$\begin{array}{ccc} \Omega^\bullet_{X^{et},\log} \otimes_k \mathbb{C}_p & \xrightarrow{OL_{X^{et}} \otimes I} & \Omega^\bullet_{X^{et}} \otimes_k \mathbb{C}_p \\ \downarrow{\scriptstyle w(k/\mathbb{C}_p)} & & \downarrow{\scriptstyle w(k/\mathbb{C}_p)} \\ \Omega^\bullet_{X^{et}_{\mathbb{C}_p},\log} & \xrightarrow{OL_{X^{et}_{\mathbb{C}_p}}} & \Omega^\bullet_{X^{et}_{\mathbb{C}_p}} \end{array}$$

□

**Theorem 1.** *Let $k \subset \bar{\mathbb{Q}}$ a field which is a finite extension of $\mathbb{Q}$. Let $X \in \mathrm{SmVar}(k)$. For all but finitely many prime $p \in \mathbb{N}$, embeddings $\sigma_p : k \hookrightarrow \mathbb{C}_p$ which then satisfy $k \cap \mathbb{Z}_p = Z$ and $\sigma : k \hookrightarrow \mathbb{C}$, and each $j, l \in \mathbb{Z}$, we get from proposition 2(ii) and proposition 1 a canonical injective map*

$$H^j \iota^{G,d}_{p,ev}(X) : H^j_{et}(X_{\bar{k}}, \mathbb{Z}_p)(l)^G \hookrightarrow F^l H^j(X^{an}_{\mathbb{C}}, 2\pi i\mathbb{Z}),\ \alpha \mapsto H^j \iota^{G,d}_{p,ev}(X)(\alpha) := ev(X)(w(\alpha)).$$

*By construction, for $f : X' \to X$ a morphism with $X, X' \in \mathrm{SmVar}(k)$, we have the commutative diagram*

$$\begin{array}{ccc} H^j_{et}(X_{\bar{k}}, \mathbb{Z}_p)(l)^G & \xrightarrow{H^j \iota^{G,d}_{ev}(X)} & F^l H^j(X^{an}_{\mathbb{C}}, 2\pi i\mathbb{Z}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f*} \\ H^j_{et}(X'_{\bar{k}}, \mathbb{Z}_p)(l)^G & \xrightarrow{H^j \iota^{G,d}_{ev}(X')} & F^l H^j(X'^{an}_{\mathbb{C}}, 2\pi i\mathbb{Z}) \end{array}$$

*Moreover,*

$$H^j\gamma(X) \circ H^j \iota^{G,d}_{p,ev}(X) : H^j_{et}(X_{\bar{k}}, \mathbb{Z}_p)(d)^G \xrightarrow{H^j \iota^{G,d}_{p,ev}(X)} H^j(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Z}) \simeq H^j(X^{an}_{\mathbb{C}}, \mathbb{Z}) \xrightarrow{H^j\gamma(X)} H^j_{et}(X_{\bar{k}}, \mathbb{Z}_p)$$

*is the canonical embedding for each $j \in \mathbb{Z}$, where $\gamma(X)$ is the canonical map (see section 2) given by reduction modulo $p^*$ and from the comparison theorem between etale and Betti cohomology with torsion coefficient.*

*Proof.* Follows from proposition 2(ii) and proposition 1. The last assertion follows from the fact that for $\alpha \in H^j_{et}(X_{\bar{k}}, \mathbb{Z}_p)(d)^G$,

$$\alpha = [(\alpha^{l,I})] \in H^j\Gamma(X_{\bar{k},\bullet}, \mathbb{Z}_p),\ w(\alpha) = [w_L(\alpha)^{l,I}] \in H^j\Gamma(X_\bullet, \Omega^\bullet_{X^{et}})$$

for an open etale cover $r : X_\bullet \to X$, which gives

$$\gamma(X)((1/2i\pi)ev(X)(w(\alpha))) = \gamma(X)((ev(X)((1/2i\pi)w_L(\alpha)^{l,I}))) = (\alpha^{l,I}).$$

Note that for $w \in \mathbb{H}^j(X, \Omega^\bullet_{X^{et}})$ such that $ev(X)(w) \in H^j_{\text{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Z})$, $w$ is NOT logarithmic in general and $\gamma(X)(ev(X)(w)) \in H^j_{et}(X_{\bar{k}}, \mathbb{Z}_p)$ is NOT $G = Gal(\bar{k}/k)$ equivariant in general. □

**Corollary 1.** *Let $k$ a field which is a finite extension of $\mathbb{Q}$. Let $X \in \mathrm{PSmVar}(k)$. The the Hodge conjecture for $X$ implies the Tate conjecture for $X$. In particular we get Tate conjecture for divisors of $X$.*

*Proof.* Follows from theorem 1 : for $X \in \mathrm{PSmVar}(k)$ and $Z \in \mathcal{Z}^d(X, n)^{\partial=0}$, we get $H^{2d-n}\iota^{G,d}_{p,ev}(X)([Z]) = [Z]$, hence the Hodge conjecture for $X$ implies the Tate conjecture for $X$ since $H^{2d-n}\iota^{G,d}_{p,ev}(X)$ is injective. Since we have by the exponential sequence Hodge conjecture for divisors, we get Tate conjecture for divisors of $X$. □

# 4 The complex and etale Abel Jacobi maps and normal function

## 4.1 The complex Abel Jacobi map for higher Chow group and complex normal functions

Let $k$ a field of finite type over $\mathbb{Q}$. Consider an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then $k \subset \bar{k} \subset \mathbb{C}$, where $\bar{k}$ is the algebraic closure of $k$. We have then the quasi-isomorphism $\alpha(X) : \mathbb{C}_{X^{an}_{\mathbb{C}}} \hookrightarrow \Omega^\bullet_{X^{an}_{\mathbb{C}}}$ in $C(X^{an}_{\mathbb{C}})$.

- For $X \in \mathrm{SmVar}(k)$, we consider

  $$((H^j_{DR}(X), F), H^j(X^{an}_{\mathbb{C}}, \mathbb{Z}), H^j\alpha(X)) \in MHM_{k,gm}(k) \subset \mathrm{Vect}_{fil}(k) \times_I \mathrm{Ab}$$

  where $F$ is the Hodge filtration on $H^j_{DR}(X) \otimes_k \mathbb{C}$ and

  $$H^jR\Gamma(X^{an}_{\mathbb{C}}, \alpha(X)) : H^j(X^{an}_{\mathbb{C}}, \mathbb{C}) \xrightarrow{\sim} H^j_{DR}(X) \otimes_k \mathbb{C}$$

  Recall the geometric mixed Hodge structures (see [5]) are mixed Hodge structure by the Hodge decomposition theorem on smooth proper complex varieties.

- For $f : X \to S$ a morphism with $S, X \in \mathrm{SmVar}(k)$, we consider

  $$(H^jRf_{*Hdg}(O_X, F_b), H^jRf_*\mathbb{Z}_{X^{an}_{\mathbb{C}}}, H^jRf_*\alpha(X)) \in MHM_{k,gm}(S) \subset \mathrm{PSh}_{\mathcal{D}(1,0)fil}(S) \times_I P_{fil,k}(S^{an}_{\mathbb{C}})$$

  where

  $$H^jRf_*\alpha(X) : H^jRf_*\mathbb{C}_{X^{an}_{\mathbb{C}}} \xrightarrow{\sim} DR(S)(o_{fil}H^jRf_{*Hdg}(O_X, F_b) \otimes_k \mathbb{C} = H^j\int_f O_X \otimes_k \mathbb{C})$$

  Recall the geometric mixed Hodge modules (see [5]) are mixed Hodge modules by a theorem of Saito for proper morphisms of smooth complex varieties.

Let $X \in \mathrm{SmVar}(k)$. We have for $j, d \in \mathbb{N}$, the generalized Jacobian

$$J^{j,d}_\sigma(X) := H^j(X^{an}_{\mathbb{C}}, \mathbb{C})/(F^dH^j(X^{an}_{\mathbb{C}}, \mathbb{C}) \oplus H^j(X^{an}_{\mathbb{C}}, \mathbb{Z}))$$

where $F$ is given the Hodge filtration on $H^j_{DR}(X) \otimes_k \mathbb{C}$ and

$$H^j R\Gamma(X^{an}_{\mathbb{C}}, \alpha(X)) : H^j(X^{an}_{\mathbb{C}}, \mathbb{C}) \xrightarrow{\sim} H^j_{DR}(X) \otimes_k \mathbb{C}.$$

If $X \in \mathrm{PSmVar}(k)$ and $2d \geq n$, $J^{j,d}(X)$ is a complex torus since

$$((H^j_{DR}(X), F), H^j(X^{an}_{\mathbb{C}}, \mathbb{Z}), H^j R\Gamma(X^{an}_{\mathbb{C}}, \alpha(X))) \in HM_{k,gm}(k) \subset MHM_{k,gm}(k)$$

is a pure Hodge structure. For $X \in \mathrm{PSmVar}(k)$, we have a canonical isomorphism of abelian groups

$$I^{j,d}_{\sigma}(X) : J^{j,d}_{\sigma}(X) \xrightarrow{\sim} \mathrm{Ext}^1_{MHM_{k,gm}(k)}(\mathbb{Z}^{Hdg}_{\mathrm{pt}}(d), (H^j_{DR}(X), H^j(X^{an}_{\mathbb{C}}, \mathbb{Z}), H^j R\Gamma(X^{an}_{\mathbb{C}}, \alpha(X)))).$$

**Definition 1.** *Let $X \in \mathrm{SmVar}(k)$ irreducible. Let $\bar{X} \in \mathrm{PSmVar}(k)$ a compactification of $X$ with $D := \bar{X} \backslash X \subset \bar{X}$ a normal crossing divisor. The map of complexes of abelian groups (see [4])*

$$\mathcal{R}_X : \mathcal{Z}^d(X, \bullet) \to C^{\mathcal{D}}_{\bullet}(\bar{X}^{an}_{\mathbb{C}}, D^{an}_{\mathbb{C}}), Z \mapsto \mathcal{R}_X := (T_Z, \Omega_Z, R_Z)$$

*where $C^{\mathcal{D}}_{\bullet}(\bar{X}^{an}_{\mathbb{C}}, D^{an}_{\mathbb{C}})$ is the Deligne homology complex induces the complex Abel Jacobi map for higher Chow groups*

$$AJ_{\sigma}(X) : \mathcal{Z}^d(X, n)^{\partial=0}_{hom} \to J^{2d-1-n,d}_{\sigma}(X), Z \mapsto AJ_{\sigma}(X)(Z) := D^{-1}(R'_Z),$$
$$R'_Z = R_Z - \Omega'_Z + T'_Z, \text{ ; with } \partial T'_Z = T_Z, \partial \Omega'_Z = \Omega_Z$$

*where*

$$D : C^{\mathcal{D}}_{\bullet}(X^{an}_{\mathbb{C}}) \to C^{\mathcal{D}}_{\bullet}(\bar{X}^{an}_{\mathbb{C}}, D^{an}_{\mathbb{C}})$$

*is the Poincare dual for Deligne homology.*

**Theorem 2.** *Let $k \subset \mathbb{C}$ a subfield. Let $X \in \mathrm{PSmVar}(k)$.*

*(i) For $Z \in \mathcal{Z}^d(X, n)^{\partial=0}_{hom}$, we have*

$$AJ_{\sigma}(X)(Z) = I^{j,d}_{\sigma}(X)^{-1}(0 \to (H^{2d-1-n}_{DR}(X), H^{2d-1-n}_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{Z}), H^{2d-1-n} R\Gamma(X^{an}_{\mathbb{C}}, \alpha(X)))$$
$$\xrightarrow{(j^*, j^*, 0)}$$
$$(H^{2d-1}_{DR}((X \times \square^n) \backslash |Z|), H^{2d-1}_{\mathrm{sing}}(((X \times \square^n) \backslash |Z|)^{an}_{\mathbb{C}}, \mathbb{Z}), H^{2d-1-n} R\Gamma((X \times \square^n) \backslash |Z|)^{an}_{\mathbb{C}}, \alpha(X \times \square^n))^{[Z]}$$
$$\xrightarrow{(\partial, \partial, 0)}$$
$$(H^{2d}_{DR,|Z|}(X \times \square^n), H^{2d}_{\mathrm{sing},|Z|}((X \times \square^n)^{an}_{\mathbb{C}}, \mathbb{Z}), R\Gamma_{|Z|}((X \times \square^n)^{an}_{\mathbb{C}}, \alpha(X \times \square^n)))^{[Z]} = \mathbb{Z}^{Hdg}_{\mathrm{pt}}(n-d) \to 0)$$

*where $j : (X \times \square^n) \backslash |Z| \hookrightarrow X \times \square^n$ is the open embedding and*

$$H^{2d}_{Hdg,|Z|}(X \times \square^n)^{[Z]} \subset H^{2d}_{Hdg,|Z|}(X \times \square^n),\ H^{2d-1}_{Hdg}((X \times \square^n) \backslash |Z|)^{[Z]} \subset H^{2d-1}_{Hdg}((X \times \square^n) \backslash |Z|).$$

*are the subobjects given by the pullback of the class of $Z$ (see section 2).*

*(ii) Let $Z \in \mathcal{Z}^d(X, n)^{\partial=0}_{hom}$. Then $AJ_{\sigma}(X)(Z) = 0$ if and only if there exist $w \in H^{2d-1}_{DR}((X \times \square^n) \backslash |Z|)^{[Z]}$ such that*

- $w \in F^d H^{2d-1}_{DR}((X \times \square^n) \backslash |Z|)$,
- $ev((X \times \square^n) \backslash |Z|)(w) \in H^{2d-1}_{\mathrm{sing}}(((X \times \square^n) \backslash |Z|)^{an}_{\mathbb{C}}, \mathbb{Z}(2i\pi))$
- $\partial w \neq 0$.

*Proof.* (i):See [6].
(ii):Follows from (i) □

Let $f : X \to S$ a morphism with $S, X \in \mathrm{SmVar}(k)$. We have for $j, d \in \mathbb{N}$ such that $H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{an}}$ is a local system and $F^d H^j Rf_{*Hdg}(O_X, F_b) \subset H^j \int_f (O_X)$ are locally free sub $O_S$ modules, the generalized relative intermediate Jacobian

$$J_\sigma^{j,d}(X/S) := (H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{an}} \otimes O_{S_{\mathbb{C}}^{an}})/(F^d(H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{an}} \otimes O_{S_{\mathbb{C}}^{an}}) \oplus H^j Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{an}})$$

where $F$ is given by the Hodge filtration on $H^j \int_f (O_X)$ and

$$H^j Rf_* \alpha(X) : H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{an}} \xrightarrow{\sim} DR(S)(H^j \int_f (O_X)).$$

A generalized normal function is then a section $\nu \in \Gamma(S_{\mathbb{C}}^{an}, J_\sigma^{j,d}(X/S))$ which is horizontal (i.e. $\nabla \nu = 0$). For $s \in S$, we get immediately that $i_s^{*mod} J_\sigma^{j,d}(X/S) = J_\sigma^{j,d}(X_s)$. In particular we get for $f : X \to S$ a smooth projective morphism with $S, X \in \mathrm{SmVar}(k)$ and $j, d \in \mathbb{N}$, the relative intermediate Jacobian

$$J_\sigma^{j,d}(X/S) := (H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{an}} \otimes O_{S_{\mathbb{C}}^{an}})/(F^d(H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{an}} \otimes O_{S_{\mathbb{C}}^{an}}) \oplus H^j Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{an}})$$

where $F$ is given by the Hodge filtration on $H^j \int_f (O_X) = H^j Rf_* \Omega^\bullet_{X/S}$ and $H^j Rf_* \alpha(X)$. A normal function is then a section $\nu \in \Gamma(S_{\mathbb{C}}^{an}, J_\sigma^{j,d}(X/S))$ which is horizontal. For $f : X \to S$ a smooth projective morphism with $S, X \in \mathrm{SmVar}(k)$, we have a canonical isomorphism of abelian groups

$$I_\sigma^{j,d}(X/S) : J_\sigma^{j,d}(X/S) \xrightarrow{\sim} \mathrm{Ext}^1_{MHM(S)}(\mathbb{Z}_S^{Hdg}(d), (H^j Rf_{*Hdg}(O_X, F_b), H^j Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{an}}, H^j Rf_* \alpha(X))).$$

**Definition-Proposition 1.** *Let $f : X \to S$ a morphism with $S, X \in \mathrm{SmVar}(k)$. Let $j : S^o \hookrightarrow S$ an open subset such that for all $j, d \in \mathbb{Z}$, $j^* H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{an}}$ is a local system and $j^* F^d H^j Rf_{*Hdg}(O_X, F_b) \subset j^* H^j \int_f (O_X)$ is a locally free sub $O_S$ module. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. Let $d, n \in \mathbb{N}$. We have then, denoting $X^o := X \times_S S^o$ and using definition 1, the map*

$$AJ_\sigma(X^o/S^o) : \mathcal{Z}^d(X, n)^{f,\partial=0}_{fhom} \to \Gamma(S_{\mathbb{C}}^{o,an}, J_\sigma^{2d-n-1,d}(X^o/S^o)) \subset \Gamma(S_{\mathbb{C}}^{o,an}, \oplus_{s \in S_{\mathbb{C}}^{o,an}} i_{s*} J_{\sigma'}^{2d-n-1,d}(X_s)),$$
$$Z \mapsto AJ_\sigma(X^o/S^o)(Z) := \nu_Z := ((s \in S_{\mathbb{C}}^o) \mapsto (AJ_{\sigma'}(X_s)(Z_s) \in J_{\sigma'}^{2d-n-1,d}(X_s)))$$

*where $\mathcal{Z}^d(X, n)^{f,\partial=0}_{fhom} \subset \mathcal{Z}^d(X, n)^{f,\partial=0}$ denote the sub-abelian group consisting of algebraic cycles $Z$ with $Z_s := i_s^* Z \in \mathcal{Z}^d(X_s, n)^{\partial=0}_{hom}$, and $\sigma' : k(s) \hookrightarrow \mathbb{C}$ is the embedding given by $s$ extending $\sigma : k \hookrightarrow \mathbb{C}$, denoting again $s := \pi k/\mathbb{C}(S)(s) \in S$, $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \to S$ being the projection.*

*Proof.* Standard : to show that

$$\nu_Z := (s \in S_{\mathbb{C}}^o) \mapsto AJ_\sigma(X_s)(Z_s) \in \Gamma(S_{\mathbb{C}}^{o,an}, \oplus_{s \in S_{\mathbb{C}}^{o,an}} i_{s*} J_\sigma^{2d-n-1,d}(X_s))$$

is holomorphic and horizontal we consider a compactification $\bar{f} : \bar{X} \to S$ of $f$ with $\bar{X} \in \mathrm{SmVar}(k)$ and use trivializations of $f : (\bar{X}_{\mathbb{C}}^{o,an}, (\bar{X} \backslash X)_{\mathbb{C}}^{o,an}) \to S_{\mathbb{C}}^{o,an}$ which gives trivialization of the local system $j^* Rf_* \mathbb{Z}_{S_{\mathbb{C}}^{an}}$ (see [4] for example). □

**Corollary 2.** *Let $f : X \to S$ a smooth projective morphism with $S, X \in \mathrm{SmVar}(k)$. Let $d, n \in \mathbb{N}$. For $Z \in \mathcal{Z}^d(X, n)^{f,\partial=0}_{fhom}$, we have*

$$AJ_\sigma(X/S)(Z) = I_\sigma^{j,d}(X/S)^{-1}(0 \to (H^{2d-1-n} Rf_{*Hdg}(O_X, F_b), H^{2d-1-n} Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{an}}, H^{2d-1-n} Rf_* \alpha(X))$$
$$\xrightarrow{(j^*, j^*, 0)} (H^{2d-1} R(f \circ j)_{*Hdg}(O_{(X \times \square^n) \backslash |Z|}, F_b), H^{2d-1} R(f \circ j)_* \mathbb{Z}_{((X \times \square^n) \backslash |Z|)_{\mathbb{C}}^{an}},$$
$$H^{2d-1} R(f \circ j)_* \alpha((X \times \square^n) \backslash |Z|))^{[Z]} \xrightarrow{(\partial, \partial, 0)}$$
$$(H^{2d} Rf_{*Hdg} R\Gamma^{Hdg}_{|Z|}(O_{X \times \square^n}, F_b), (H^{2d} Rf_* R\Gamma_{|Z|} \mathbb{Z}_{(X \times \square^n)_{\mathbb{C}}^{an}}), Rf_* \Gamma_{|Z|} \alpha(X \times \square^n))^{[Z]} = \mathbb{Z}_S^{Hdg}(n-d) \to 0)$$

*where* $j : (X \times \square^n)\backslash|Z| \hookrightarrow X \times \square^n$ *is the open embedding and*

$$(H^{2d}Rf_{*Hdg}R\Gamma^{Hdg}_{|Z|}\mathbb{Z}^{Hdg}_{X\times\square^n})^{[Z]} \subset H^{2d}Rf_{*Hdg}R\Gamma^{Hdg}_{|Z|}\mathbb{Z}^{Hdg}_{X\times\square^n},$$
$$(H^{2d}R(f\circ j)_{*Hdg}\mathbb{Z}^{Hdg}_{(X\times\square^n)\backslash|Z|})^{[Z]} \subset H^{2d}R(f\circ j)_{*Hdg}\mathbb{Z}^{Hdg}_{(X\times\square^n)\backslash|Z|}.$$

*are the subobjects given by the pullback of the class of* $Z$*.*

*Proof.* Follows from theorem 2 by definition of the Abel Jacobi map and by the base change for mixed hodge modules. □

We have the following main result of [3] :

**Theorem 3.** *Let* $f : X \to S$ *a smooth projective morphism with* $S, X \in \mathrm{SmVar}(k)$*. Let* $d, n \in \mathbb{N}$*. Let* $\sigma : k \hookrightarrow \mathbb{C}$ *an embedding. For* $Z \in \mathcal{Z}^d(X,n)^{f,\partial=0}_{fhom}$*, the zero locus* $V(\nu_Z) \subset S_{\mathbb{C}}$ *of*

$$\mu_Z := AJ_\sigma(X/S)(Z) \in \Gamma(S^{an}_{\mathbb{C}}, J^{2d-1-n,d}_\sigma(X/S))$$

*is an algebraic subvariety.*

*Proof.* See [3]: if $\bar{S} \in \mathrm{PSmVar}(k)$ is a compactification of $S$ with $\bar{S}\backslash S = \cup_i D_i \subset \bar{S}$ a normal crossing divisor, there exist an analytic subset $\Sigma(\nu_Z) \subset \bar{S}_{\mathbb{C}}$ such that $V(\nu_Z) = \Sigma(\nu_Z) \cap S_{\mathbb{C}}$. By GAGA $\Sigma(\nu_Z) \subset \bar{S}_{\mathbb{C}}$ is algebraic subvariety. Hence $V(\nu_Z) \subset S_{\mathbb{C}}$ is an algebraic subvariety. □

## 4.2 The etale Abel Jacobi map for higher Chow group and etale normal functions

Let $k$ a field of finite type over $\mathbb{Q}$. Let $\bar{k}$ the algebraic closure of $k$ and denote by $G = Gal(\bar{k}/k)$ its galois group. Let $p \in \mathbb{N}$ a prime integer.

**Definition 2.** *Let* $X \in \mathrm{SmVar}(k)$ *irreducible. Let* $\bar{X} \in \mathrm{PSmVar}(k)$ *a compactification of* $X$ *with* $D := \bar{X}\backslash X \subset \bar{X}$ *a normal crossing divisor. Denote* $G = Gal(\bar{k}/k)$ *the absolute galois group. The cycle class map*

$$\mathcal{R}^{et,p}_X : \mathcal{Z}^d(X,n)^{\partial=0} \to H^{2d-n}_{p_X(|Z|),et}(X, D, \hat{\mathbb{Z}}_p) \to H^{2d-n}_{et}(X, D, \hat{\mathbb{Z}}_p),$$

*to continuous etale cohomology induces the etale Abel Jacobi map for higher Chow groups*

$$AJ_{et,p}(X) : \mathcal{Z}^d(X,n)^{\partial=0}_{hom} \to \mathrm{Ext}^1_G(\bar{k}, H^{2d-1-n}_{et}(X_{\bar{k}}, D_{\bar{k}}, \mathbb{Z}_p)),$$
$$Z \mapsto AJ_{et,p}(X)(Z) := L^1\mathcal{R}^{et,p}_X(Z)/L^2\mathcal{R}^{et,p}_X(Z),$$

*where* $L$ *is the filtration given by the Leray spectral sequence of the map of sites* $a_X :: X^{et} \to \mathrm{Spec}(k)^{et}$*.*

**Theorem 4.** *Let* $X \in \mathrm{PSmVar}(k)$*. Denote* $G = Gal(\bar{k}/k)$ *the absolute galois group.*

*(i) For* $Z \in \mathcal{Z}^d(X,n)^{\partial=0}_{hom}$*, we have*

$$AJ_{et,p}(X)(Z) = (0 \to H^{2d-1-n}_{et}(X_{\bar{k}}, \mathbb{Z}_p) \xrightarrow{j^*} H^{2d-1}_{et}(((X\times\square^n)\backslash|Z|)_{\bar{k}}, \mathbb{Z}_p)^{[Z]}$$
$$\xrightarrow{\partial} H^{2d}_{|Z|,et}((X\times\square^n)_{\bar{k}}, \mathbb{Z}_p)^{[Z]} = \bar{k}(n-d) \to 0)$$

*with* $j : (X \times \square^n)\backslash|Z| \hookrightarrow X \times \square^n$ *the open embedding, and*

$$H^{2d}_{|Z|,et}((X\times\square^n)_{\bar{k}}, \mathbb{Z}_p)^{[Z]} \subset H^{2d}_{|Z|,et}((X\times\square^n)_{\bar{k}}, \mathbb{Z}_p),$$
$$H^{2d-1}_{et}(((X\times\square^n)\backslash|Z|)_{\bar{k}}, \mathbb{Z}_p)^{[Z]} \subset H^{2d-1}_{et}(((X\times\square^n)\backslash|Z|)_{\bar{k}}, \mathbb{Z}_p)$$

*are the subobjects given by the pullback by the class of* $Z$ *(see section 2).*

*(ii) Let $Z \in \mathcal{Z}^d(X,n)^{\partial=0}_{hom}$. Then $AJ_{et,p}(X)(Z) = 0$ if and only if there exist*

$$\alpha \in H^{2d-1}_{et}(((X \times \square^n)\backslash|Z|)_{\bar{k}}, \mathbb{Z}_p)^{[Z]}$$

*such that $\alpha \in H^{2d-1}_{et}(((X \times \square^n)\backslash|Z|)_{\bar{k}}, \mathbb{Z}_p)^G$ and $\partial\alpha \neq 0$.*

*Proof.* (i):See [14].
(ii): Follows from (i). □

**Definition 3.** *Let $f : X \to S$ a morphism with $S, X \in \mathrm{SmVar}(k)$. Let $j : S^o \hookrightarrow S$ an open subset such that for all $j \in \mathbb{Z}$, $j^*H^jRf_*\mathbb{Z}_{p,X_{\bar{k}}}$ is a local system. Let $d, n \in \mathbb{N}$. We have then, denoting $X^o := X \times_S S^o$ and using definition 2, the map*

$$AJ_{et,p}(X^o/S^o) : \mathcal{Z}^d(X,n)^{f,\partial=0}_{fhom} \to \Gamma(S^o, \oplus_{s\in S^o_{(0)}} i_{s*}\,\mathrm{Ext}^1_{Gal(\bar{k}/k(s))}(\bar{k}, H^{2d-n-1}_{et}(X_{s,\bar{k}}, \mathbb{Z}_p))),$$
$$Z \mapsto AJ_{et,p}(X^o/S^o)(Z) := \nu^{et,p}_Z :=$$
$$((s \in S^o_{(0)}) \mapsto (AJ_{et,p}(X_s)(Z_s) \in \mathrm{Ext}^1_{Gal(\bar{k}/k(s))}(\bar{k}, H^{2d-n-1}_{et}(X_{s,\bar{k}}, \mathbb{Z}_p))))$$

*where $\mathcal{Z}^d(X,n)^{f,\partial=0}_{fhom} \subset \mathcal{Z}^d(X,n)^{f,\partial=0}$ denote the subabelian group consisting of algebraic cycles $Z$ with $Z_s := i_s^*Z \in \mathcal{Z}^d(X_s,n)_{hom}$. Recall that $i_s : \{s\} \hookrightarrow S_{(0)} \subset S$ is a closed Zariski point of $S$.*

We now localize, for each prime number $l$ and each embedding $\sigma_l : k \hookrightarrow \mathbb{C}_l$ the definition given above.

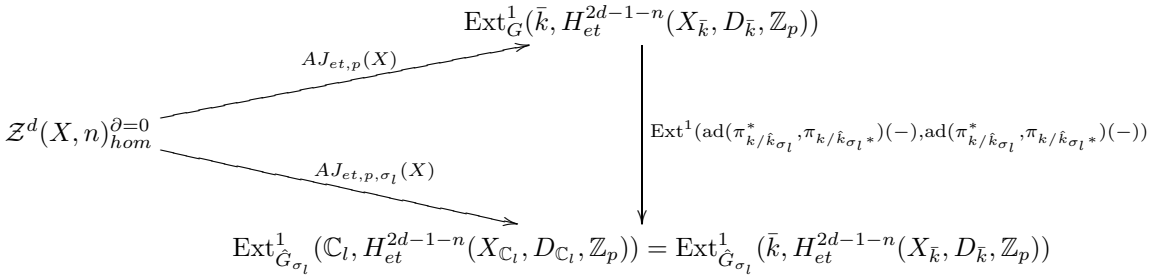
**Definition 4.** *Let $X \in \mathrm{SmVar}(k)$ irreducible. Let $\bar{X} \in \mathrm{PSmVar}(k)$ a compactification of $X$ with $D := \bar{X}\backslash X \subset \bar{X}$ a normal crossing divisor. Let $\sigma_l : k \hookrightarrow \mathbb{C}_l$ an embedding. Then $k \subset \bar{k} \subset \mathbb{C}_l$, where $\bar{k}$ is the algebraic closure of $k$ and $k \subset \hat{k}_{\sigma_l} \subset \mathbb{C}_l$ where $\hat{k}_{\sigma_l}$ is the completion of $k$ with respect to $\sigma_l$. Denote $\hat{G}_{\sigma_l} := Gal(\mathbb{C}_l\hat{k}_{\sigma_l})$. The cycle class map*

$$\mathcal{R}^{et,p}_{X,\sigma_l} : \mathcal{Z}^d(X,n)^{\partial=0} \to H^{2d-n}_{p_X(|Z|),et}(X_{\hat{k}_{\sigma_l}}, D_{\hat{k}_{\sigma_l}}, \hat{\mathbb{Z}}_p) \to H^{2d-n}_{et}(X_{\hat{k}_{\sigma_l}}, D_{\hat{k}_{\sigma_l}}, \hat{\mathbb{Z}}_p),$$

*to continuous etale cohomology induces the etale Abel Jacobi map for higher Chow groups*

$$AJ_{et,p,\sigma_l}(X) : \mathcal{Z}^d(X,n)^{\partial=0}_{hom} \to \mathrm{Ext}^1_{\hat{G}_{\sigma_l}}(\mathbb{C}_l, H^{2d-1-n}_{et}(X_{\mathbb{C}_l}, D_{\mathbb{C}_l}, \mathbb{Z}_p)),$$
$$Z \mapsto AJ_{et,p,\sigma_l}(X)(Z) := L^1\mathcal{R}^{et,p}_{X,\sigma_l}(Z)/L^2\mathcal{R}^{et,p}_{X,\sigma_l}(Z),$$

*where $L$ is the filtration given by the Leray spectral sequence of the map of sites $a_X :: X^{et}_{\hat{k}_{\sigma_l}} \to \mathrm{Spec}(\hat{k}_{\sigma_l})^{et}$. We have then the commutative diagram*




*where the right column arrow is given by the restriction $\pi_{k/\hat{k}_{\sigma_l}} : \hat{G}_{\sigma_l} \hookrightarrow G$.*

**Theorem 5.** *Let $X \in \mathrm{PSmVar}(k)$. Let $\sigma_l : k \hookrightarrow \mathbb{C}_l$ an embedding and $k \subset \hat{k}_{\sigma_l} \subset \mathbb{C}_l$ the completion of $k$ with respect to $\sigma_l$.*

*(i) For $Z \in \mathcal{Z}^d(X,n)^{\partial=0}_{hom}$, we have*

$$AJ_{et,p,\sigma_l}(X)(Z) = (0 \to H^{2d-1-n}_{et}(X_{\mathbb{C}_l},\mathbb{Z}_p) \xrightarrow{j^*} H^{2d-1}_{et}(((X\times\square^n)\backslash|Z|)_{\mathbb{C}_l},\mathbb{Z}_p)^{[Z]} \xrightarrow{\partial} H^{2d}_{|Z|,et}((X\times\square^n)_{\mathbb{C}_l},\mathbb{Z}_p)^{[Z]} = \mathbb{C}_l(n-d) \to 0)$$

*with $j : X\times\square^n\backslash|Z| \hookrightarrow X$ the open embedding and*

$$H^{2d}_{|Z|,et}((X\times\square^n)_{\mathbb{C}_l},\mathbb{Z}_p)^{[Z]} \subset H^{2d}_{|Z|,et}((X\times\square^n)_{\mathbb{C}_l},\mathbb{Z}_p),$$
$$H^{2d}_{|Z|,et}(((X\times\square^n)\backslash|Z|)_{\mathbb{C}_l},\mathbb{Z}_p)^{[Z]} \subset H^{2d}_{|Z|,et}(((X\times\square^n)\backslash|Z|)_{\mathbb{C}_l},\mathbb{Z}_p),$$

*are the subobjects given by the pullback by the class of $Z$ (see section 2).*

*(ii) Let $Z \in \mathcal{Z}^d(X,n)^{\partial=0}_{hom}$. Then $AJ_{et,p,\sigma_l}(X)(Z) = 0$ if and only if there exist*

$$\alpha \in H^{2d-1}_{et}(((X\times\square^n)\backslash|Z|)_{\mathbb{C}_l},\mathbb{Z}_p)^{[Z]}$$

*such that $\partial\alpha \neq 0$ and $\alpha \in H^{2d-1}_{et}(((X\times\square^n)\backslash|Z|)_{\mathbb{C}_l},\mathbb{Z}_p)^{\hat{G}_{\sigma_l}}$.*

*Proof.* Similar to the proof of theorem 4. □

**Definition 5.** *Let $f : X \to S$ a morphism with $S, X \in \mathrm{SmVar}(k)$. Let $j : S^o \hookrightarrow S$ an open subset such that for all $j \in \mathbb{Z}$, $j^*H^jRf_*\mathbb{Z}_{p,X_{\bar{k}}}$ is a local system. Let $d,n \in \mathbb{N}$. Let $\sigma_l : k \hookrightarrow \mathbb{C}_l$ an embedding and $k \subset \hat{k}_{\sigma_l} \subset \mathbb{C}_l$ the completion of $k$ with respect to $\sigma_l$. Denoting $X^o := X \times_S S^o$, we consider*

$$AJ_{et,p,\sigma_l}(X^o/S^o) : \mathcal{Z}^d(X,n)^{f,\partial=0}_{fhom} \to \Gamma(S^o, \oplus_{s\in S^o_{(0)}} i_{s*}\,\mathrm{Ext}^1(Gal(\mathbb{C}_l/\hat{k}_{\sigma_l}(s)), H^{2d-n-1}_{et}(X_{s,\bar{k}},\mathbb{Z}_p))),$$
$$Z \mapsto AJ_{et,p,\sigma_l}(X^o/S^o)(Z) := \nu^{et,p}_{Z,\sigma_l} :=$$
$$((s \in S^o_{(0)}) \mapsto (AJ_{et,p,\sigma_l}(X_s)(Z_s) \in \mathrm{Ext}^1_{Gal(\mathbb{C}_l/\hat{k}_{\sigma_l}(s))}(\bar{k}, H^{2d-n-1}_{et}(X_{s,\bar{k}},\mathbb{Z}_p))))$$

*where $\mathcal{Z}^d(X,n)^{f,\partial=0}_{fhom} \subset \mathcal{Z}^d(X,n)^{f,\partial=0}$ denote the subabelian group consisting of algebraic cycles $Z$ with $Z_s := i_s^*Z \in \mathcal{Z}^d(X_s,n)_{hom}$. Recall that $i_s : \{s\} \hookrightarrow S_{(0)} \subset S$ is a closed Zariski point of $S$.*

# 5 The vanishing of the etale Abel Jacobi map implies the vanishing of the complex Abel Jacobi map

The $p$ adic Hodge theory for open varieties implies the following main theorem :

**Theorem 6.** *Let $k \subset \bar{\mathbb{Q}}$ a field which is a finite extension of $\mathbb{Q}$. Denote $\bar{k}$ the algebraic closure of $k$ and $G = Gal(\bar{k}/k)$ its absolute Galois group. Let $X \in \mathrm{PSmVar}(k)$. Consider an embedding $\sigma : k \hookrightarrow \mathbb{C}$. For all but finitely many prime number $p \in \mathbb{N}$ a prime number, we have for all embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$, $k \cap \mathbb{Z}_p = \mathbb{Z}$. Let $Z \in \mathcal{Z}^d(X,n)^{\partial=0}_{hom}$. Consider the exact sequences*

- $0 \to H^{2d-1-n}_{et}(X_{\bar{k}},\mathbb{Z}_p) \xrightarrow{j^*} H^{2d-1}_{et}(((X\times\square^n)\backslash|Z|)_{\bar{k}},\mathbb{Z}_p) \xrightarrow{\partial} H^{2d}_{et,|Z|}((X\times\square^n)_{\bar{k}},\mathbb{Z}_p)^0 \to 0$
- $0 \to H^{2d-1-n}_{DR}(X) \xrightarrow{j^*} H^{2d-1}_{DR}((X\times\square^n)\backslash|Z|) \xrightarrow{\partial} H^{2d}_{DR,|Z|}(X\times\square^n)^0 \to 0,$

*where $j : (X\times\square^n)\backslash|Z| \hookrightarrow X\times\square^n$ is the open embedding. Consider the following assertions :*

*(i) $AJ_{et,p}(X)(Z) = 0 \in \mathrm{Ext}^G_1(\bar{k}, H^{2d-1-n}_{et}(X_{\bar{k}},\mathbb{Z}_p)(d-n))$,*

*(i)' there exist $\alpha \in H^{2d-1}_{et}(((X\times\square^n)\backslash|Z|)_{\bar{k}},\mathbb{Z}_p)(d)^{[Z]}$ such that $\alpha \in H^{2d-1}_{et}(((X\times\square^n)\backslash|Z|)_{\bar{k}},\mathbb{Z}_p)(d)^G$ and $\partial\alpha \neq 0$,*

*(ii) there exist* $w \in H^{2d-1}_{DR}((X \times \square^n)\backslash|Z|)^{[Z]}$ *such that* $w \in F^d H^{2d-1}_{DR}((X \times \square^n)\backslash|Z|)$,

$$w \in H^{2d-1}OL_{X^{et}}(\mathbb{H}^{2d-1}_{et}((X \times \square^n)\backslash|Z|, \Omega^\bullet_{X^{et},\log})),$$

*and* $\partial w \neq 0$,

*(iii) there exist* $w \in H^{2d-1}_{DR}((X \times \square^n)\backslash|Z|)^{[Z]}$ *such that* $w \in F^d H^{2d-1}_{DR}((X \times \square^n)\backslash|Z|)$,

$$H^{2d-1}ev((X \times \square^n)\backslash|Z|)(w) \in H^{2d-1}_{\mathrm{sing}}(((X \times \square^n)\backslash|Z|)^{an}_{\mathbb{C}}, 2i\pi\mathbb{Z}),$$

*and* $\partial w \neq 0$,

*(iii)'* $AJ_\sigma(X)(Z) = 0 \in J^{2d-1-n,d}_\sigma(X)$,

*where the inclusion* $OL_{X^{et}} : \Omega^\bullet_{X^{et},\log} \hookrightarrow \Omega^\bullet_{X^{et}}$ *of* $C(X^{et})$ *is the subcomplex of logarithmic forms and* $e_X : X^{et} \to X$ *is the canonical morphism of site (see section 2). Then (i) is equivalent to (i)', (i)' implies (ii), (ii) implies (iii), (iii) is equivalent to (iii)'. Hence (i) implies (iii)'.*

*Proof.* (i) is equivalent to (i)': see theorem 4(ii),
(i)' implies (ii):follows from proposition 2(ii): we take an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$ and $w = w(\alpha)$, $w_L = w_L(\alpha)$,
(ii) implies (iii):follows from proposition 1,
(iii) is equivalent to (iii)' : see theorem 2(ii). □

It implies the following :

**Corollary 3.** *(i) Let* $k \subset \bar{\mathbb{Q}}$ *a field which is a finite extension of* $\mathbb{Q}$. *Then* $\bar{k} = \bar{\mathbb{Q}}$ *is the algebraic closure of* $k$. *Let* $f : X \to S$ *a smooth projective morphism with* $S, X \in \mathrm{SmVar}(k)$. *Consider an embedding* $\sigma : k \hookrightarrow \mathbb{C}$. *Let* $p \in \mathbb{N}$ *a prime number. Then for* $Z \in \mathcal{Z}^d(X,n)^{f,\partial=0}_{fhom}$, *we have*

$$V(\nu^{et,p}_Z)_{\mathbb{C}} \subset V(\nu_Z) \subset S_{\mathbb{C}},\ V_{tors}(\nu^{et,p}_Z)_{\mathbb{C}} \subset V_{tors}(\nu_Z) \subset S_{\mathbb{C}}$$

*where*

- $V(\nu_Z) \subset V_{tors}(\nu_Z) \subset S_{\mathbb{C}}$ *is the zero locus, resp. torsion locus, of the complex normal function*

$$\nu_Z =: AJ_\sigma(X/S)(Z) \in \Gamma(S^{an}_{\mathbb{C}}, J^{2d-1-n,d}_\sigma(X/S))$$

*associated to* $Z$ *(see proposition-definition 1),*

- $V(\nu^{et,p}_Z) \subset V_{tors}(\nu^{et,p}_Z) \subset S$ *is the zero locus, resp. torsion locus of the etale normal function*

$$\nu^{et,p}_Z \in \Gamma(S, \oplus_{s \in S_{(0)}} i_{s*}\,\mathrm{Ext}^1_{Gal(\bar{k}/k(s))}(\bar{k}, H^{2d-n-1}_{et}(X_{s,\bar{k}}, \mathbb{Z}_p))(d-n))$$

*associated to* $Z$ *(see definition 3) and*

$$V(\nu^{et,p}_Z)_{\mathbb{C}} := \pi_{k/\mathbb{C}}(S)^{-1}(V(\nu^{et,p}_Z)),\ V_{tors}(\nu^{et,p}_Z)_{\mathbb{C}} := \pi_{k/\mathbb{C}}(S)^{-1}(V_{tors}(\nu^{et,p}_Z))$$

*where we recall* $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \to S$ *is the projection.*

*(ii) Let* $\sigma : k \hookrightarrow \mathbb{C}$ *a subfield which is a finite extension of* $\mathbb{Q}$. *Then* $\bar{k} = \bar{\mathbb{Q}}$ *is the algebraic closure of* $k$. *Let* $f : X \to S$ *a smooth projective morphism with* $S, X \in \mathrm{SmVar}(k)$. *Then for* $Z \in \mathcal{Z}^d(X,n)^{f,\partial=0}_{fhom}$, *the zero locus* $V(\nu_Z) \subset S_{\mathbb{C}}$ *of the complex normal function*

$$\nu_Z =: AJ_\sigma(X/S)(Z) \in \Gamma(S^{an}_{\mathbb{C}}, J^{2d-1-n,d}_\sigma(X/S))$$

*associated to* $Z$ *is defined over* $\bar{k}$ *if* $V(\nu^{et,p}_Z) \neq \emptyset$.

*Proof.* (i):Follows immediately from theorem 6 since for $s \in S_{(0)}$, $k(s)$ is of finite type over $\mathbb{Q}$ and for $s' \in \pi_{k/\mathbb{C}}(S)^{-1}(s)$ denoting $\sigma' : k(s) \hookrightarrow \mathbb{C}$ the embedding given by $s'$, we have by definition

- $\nu_Z(s') =: AJ_\sigma(X/S)(Z)(s') := AJ_{\sigma'}(X_s)(Z_s) \in J^{2d-1-n,d}_{\sigma'}(X_s)$.
- $\nu^{et,p}_Z(s) =: AJ_{et,p}(X/S)(Z)(s) := AJ_{et,p}(X_s)(Z_s) \in \mathrm{Ext}^1_{Gal(\bar{k}/k(s))}(\bar{k}, H^{2d-n-1}_{et}(X_{s,\bar{k}}, \mathbb{Z}_p)(d-n))$.

(ii): Since $V(\nu^{et,p}_Z) \subset S$ contain a $\bar{k}$ point, $V(\nu_Z) \subset S_{\mathbb{C}}$ contain a $\bar{k}$ point by (i). Hence by the work of [17] or [8], $V(\nu_Z) \subset S_{\mathbb{C}}$ is defined over $\bar{k}$. □

# 6 Algebraicity of the zero locus of etale normal functions

Let $k$ a field of finite type over $\mathbb{Q}$. Let $f : X \to S$ a smooth proper morphism with $S, X \in \mathrm{SmVar}(k)$ connected. Let $p$ a prime number. Let $Z \in \mathcal{Z}^d(X,n)^{\partial=0,f}_{hom}$. By the definition, we have

$$V(\nu^{et,p}_Z) \subset \cap_{l\in\mathbb{N},l\mathrm{prime}} \cap_{\sigma_l : k\hookrightarrow\mathbb{C}_l} .V(\nu^{et,p}_{Z,\sigma_l}) \subset S_{(0)}$$

and

$$V_{tors}(\nu^{et,p}_Z) \subset \cap_{l\in\mathbb{N},l\mathrm{prime}} \cap_{\sigma_l : k\hookrightarrow\mathbb{C}_l} .V_{tors}(\nu^{et,p}_{Z,\sigma_l}) \subset S_{(0)}.$$

In this section, we investigate the algebraicity of $V_{tors}(\nu^{et,p}_Z) \subset S$ and of $V_{tors}(\nu^{et,p}_{Z,\sigma_l}) \subset S$, $\sigma_l : k \hookrightarrow \mathbb{C}_l$.

**Remark 3.** *Let $k$ a field of finite type over $\mathbb{Q}$. Let $f : X \to S$ a smooth proper morphism with $S, X \in \mathrm{SmVar}(k)$ connected. Let $p$ a prime number. Let $Z \in \mathcal{Z}^d(X,n)^{\partial=0,f}_{hom}$. We can show, using [13], that we have in fact*

$$V_{tors}(\nu^{et,p}_Z) = \cap_{l\in\mathbb{N},l prime} \cap_{\sigma_l : k\hookrightarrow\mathbb{C}_l} .V_{tors}(\nu^{et,p}_{Z,\sigma_l}) \subset S_{(0)}.$$

*We don't need this result so we don't give the details.*

Let $k$ a field of finite type over $\mathbb{Q}$. Let $p$ be a prime number. Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ be an embedding. We have then $\hat{k}_{\sigma_p}$ the completion of $k$ with respect to $\sigma_p$ and we denote $O_{\hat{k}_{\sigma_p}} \subset \hat{k}_{\sigma_p}$ its ring of integers. We then consider the canonical functor of Huber (see section 2)

$$\mathcal{R} : \mathrm{Var}(\hat{k}_{\sigma_p}) \to \mathrm{HubSp}(\hat{k}_{\sigma_p}, O_{\hat{k}_{\sigma_p}}) \to \mathrm{Sch}\,/O_{\hat{k}_{\sigma_p}}, X \mapsto \mathcal{R}(X) = X^{\mathcal{O}}$$

which associated to a variety over a $p$ adic field its canonical integral model. Let $f : X \to S$ a smooth projective morphism with $S, X \in \mathrm{SmVar}(k)$. Let $Z \subset X$ a closed subset and $j : U = X\backslash Z \hookrightarrow X$ the open complementary subset. We have then $f := f_{\hat{k}_{\sigma_p}} : (X,Z)_{\hat{k}_{\sigma_p}} \to S_{\hat{k}_{\sigma_p}}$ the morphism in $\mathrm{SmVar}^2(\hat{k}_{\sigma_p})$ induced by the scalar extension functor and

$$f^{\mathcal{O}} := \mathcal{R}(f_{\hat{k}_{\sigma_p}}) : (X,Z)^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \to S^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$$

its canonical integral model in $\mathrm{Sch}^2\,/O_{\hat{k}_\sigma}$ to which we denote

$$f := f^{\mathcal{O}} : (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}}) := (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, (M_Z, N_{\mathcal{O}})) \to (S^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{\mathcal{O}})$$

the corresponding morphism in logSch, where for $K$ a $p$ adic field and $Y \in \mathrm{Sch}\,/O_K$, $(Y, N_{\mathcal{O}}) := (Y, M_{Y_k})$ with $k = O_K/(\pi)$ the residual field. We have then the morphisms of sites

$$v_{X,N} : (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})^{Falt} \to (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})^{ket},\ u_{X,N} : (X_{\mathbb{C}_p}, M_{Z_{\mathbb{C}_p}})^{ket} \to (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})^{Falt}$$

where $(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})^{Falt}$ denote the Falting site, and for $(Y,N) \in \mathrm{logSch}$, $(Y,N)^{ket} \subset \mathrm{logSch}\,/(Y,N)$ is the small Kummer etale site. If $(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})$ is log smooth, we consider an hypercover

$$a_\bullet : (X^{\mathcal{O},\bullet}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}}) \to (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})$$

in $\mathrm{Fun}(\Delta, \mathrm{logSch})$ by small log schemes in sense of [1]. The main result of [1] say that

- if $(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})$ is log smooth, the embedding in $C((X^{\mathcal{O}}_{\hat{k}_{\sigma_p}})^{Falt})$

$$\alpha(U) : (\mathbb{B}_{st,X_{\hat{k}_{\sigma_p}}}, N_{U,\mathcal{O}}) \hookrightarrow a_{\bullet *} DR(X^{\mathcal{O},\bullet}_{\hat{k}_{\sigma_p}}/S^{\mathcal{O}}_{\hat{k}_{\sigma_p}})(O\mathbb{B}_{st,X^{\bullet}_{\hat{k}_{\sigma_p}}}, N_{U,\mathcal{O}})$$

is a filtered quasi-isomorphism compatible with the action of $Gal(\mathbb{C}_p, \hat{k}_{\sigma_p})$, the Frobenius $\phi_p$ and the monodromy $N$,

- if $f : (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}}) \to (S^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{\mathcal{O}})$ is log smooth, the morphism in $D_{fil}((S^{\mathcal{O}}_{\hat{k}_{\sigma_p}})^{Falt})$

$$T(f, \mathbb{B}_{st}) : Rf_*(\mathbb{B}_{st,X_{\hat{k}_{\sigma_p}}}, N_{U,\mathcal{O}}) \xrightarrow{T(f,f,\otimes)(-)\circ \mathrm{ad}(u^*_{X,N}, Ru_{X,N,*})(-)} Rf_*\mathbb{Z}_{p,(X_{\mathbb{C}_p}, M_{Z_{\mathbb{C}_p}})^{ket}} \otimes_{\mathbb{Z}_p} \mathbb{B}_{st,S_{\hat{k}_{\sigma_p}}}$$
$$\xrightarrow{\mathrm{ad}(j^*, Rj_*)(\mathbb{Z}_{p,(X_{\mathbb{C}_p}, M_{Z_{\mathbb{C}_p}})^{et}})} R(f \circ j)_*\mathbb{Z}_{p,U^{et}_{\mathbb{C}_p}} \otimes_{\mathbb{Z}_p} \mathbb{B}_{st,S_{\hat{k}_{\sigma_p}}}$$

is an isomorphism, where the last map is an isomorphim by [12] theorem 7.4.

This gives if $(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})$ is log smooth, for each $j \in \mathbb{Z}$, a filtered isomorphism of filtered abelian groups

$$H^j R\alpha(U) : H^j_{et}(U_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{st,\hat{k}_{\sigma_p}} \xrightarrow{H^j T(a_X, \mathbb{B}_{st})^{-1}} H^j_{et}((X,N)^{Falt})(\mathbb{B}_{st,X_{\hat{k}_{\sigma_p}}}, N_{U,\mathcal{O}})$$
$$\xrightarrow{H^j R\Gamma((X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}}), \alpha(U))} H^j_{DR}(U_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{st,\hat{k}_{\sigma_p}}$$

compatible with the action of $Gal(\mathbb{C}_p/\hat{k}_{\sigma_p})$, of the Frobenius $\phi_p$ and the monodromy $N$. More generally, this gives if $\hat{k}_{\sigma_p}(s)$ is unramified for all $s \in S_{\hat{k}_{\sigma_p}}$ and if $f : (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}}) \to (S^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{\mathcal{O}})$ is log smooth, an isomorphism in $\mathrm{Shv}_{fil,G,\phi_p,N}(S_{\hat{k}_{\sigma_p}})$

$$H^j f_*\alpha(U) : R^j f_*\mathbb{Z}_{p,U^{et}_{\mathbb{C}_p}} \otimes_{\mathbb{Z}_p} \mathbb{B}_{st,S_{\hat{k}_{\sigma_p}}} \xrightarrow{H^j T(f, \mathbb{B}_{st})^{-1}} Rf_*(\mathbb{B}_{st,X_{\hat{k}_{\sigma_p}}}, N_{U,\mathcal{O}})$$
$$\xrightarrow{H^j Rf_*\alpha(U)} H^j \int_f j_{*Hdg}(O_{U_{\hat{k}_{\sigma_p}}}, F_b) \otimes_{O_{S_{\hat{k}_{\sigma_p}}}} O\mathbb{B}_{st,S_{\hat{k}_{\sigma_p}}}$$

that is a filtered isomorphism compatible with the action of $G = Gal(\mathbb{C}_p/\hat{k}_{\sigma_p})$, of the Frobenius $\phi_p$ and the monodromy $N$, writing for short again $f = f \circ j$.

**Definition 6.** *Let $k$ a field of finite type over $\mathbb{Q}$. Let $f : X \to S$ a smooth proper morphism with $S, X \in \mathrm{SmVar}(k)$. Let $p$ a prime number. Let $Z \in \mathcal{Z}(X,n)^{f,\partial=0}_{fhom}$. Denote $U := (X \times \square^n)\backslash|Z|$. We have then the following exact sequence in $\mathrm{Vect}_{\mathcal{D}fil}(S)$ (of filtered vector bundle with integrable connexion)*

$$0 \to E_{DR}(X/S) := H^{2d-1-n} \int_f (O_X, F_b) \xrightarrow{j^*} E_{DR}(U/S)^{[Z]} := (H^{2d-1} \int_f j_{*Hdg}(O_U, F_b))^{[Z]}$$
$$\xrightarrow{\partial} E_{DR,Z}(X/S)^{[Z]} := (H^{2d} \int_f \Gamma^{Hdg}_{|Z|}(O_{X\times\square^n}, F_b))^{[Z]} \to 0.$$

*Recall (see section 2) that $(X \times \square^n, M_{|Z|}) \in \mathrm{logVar}(k)$ denote log structure associated to $(X \times \square^n, |Z|) \in \mathrm{SmVar}^2(k)$. There exist a finite set of prime numbers $\delta(S)$ such that for all prime $p \in \mathbb{N}\backslash\delta(S)$, all embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$ and all $s \in S_{(0)}$, $\hat{k}_{\sigma_p}(s)$ is unramified over $\mathbb{Q}_p$. Let $p \in \mathbb{N}\backslash\delta(S)$ a prime number and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ be an embedding.*

- *Take, using [10] theorem 8.2 (after considering an integral model $(X,Z)^{\mathcal{R}} \in \mathrm{Sch}^2/\mathcal{R}$ over $\mathcal{R} \subset k$ of finite type over $\mathbb{Z}$ with function field $k$), for $p \in \mathbb{N}\backslash\delta^0(X/S)$, where $\delta^0(X/S)$ is a finite set, an alteration $\pi^0 : (X \times \square^n)^0 \to X \times \square^n$, that is a generically finite morphism such that ($(X \times$*

$\square^n)^0, \pi^{0,-1}(|Z|))^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ *is semi-stable pair, that is* $\pi^{0,-1}(|Z|) \subset ((X \times \square^n)^0$ *is a normal crossing divisor and* $(((X \times \square^n)^0, \pi^{0,-1}(|Z|))^{\mathcal{O}}_{\hat{k}_{\sigma_p}})_t$ *has semi-stable reduction where* $t \in \mathrm{Spec}(O_{\hat{k}_{\sigma_p}})$ *is the closed point. Then there exists a closed subset* $\Delta \subset S$ *such that for all* $s \in S^o := S \backslash \Delta$, $((X \times \square^n)^0, \pi^{0,-1}(|Z|))^{\mathcal{O}}_{\hat{k}_{\sigma_p}, s}$ *is a semi-stable pair.*

- *Take using [10] theorem 8.2, (after considering an integral model* $(X_\Delta, Z_\Delta)^{\mathcal{R}} \in \mathrm{Sch}^2/\mathcal{R}$ *over* $\mathcal{R} \subset k$ *of finite type over* $\mathbb{Z}$ *with function field* $k$*), for* $p \in \mathbb{N} \backslash \delta^1(X/S)$*, where* $\delta^1(X/S)$ *is a finite set, an alteration* $\pi^1 : (X_\Delta \times \square^n)^1 \to X_\Delta \times \square^n$ *such that* $((X_\Delta \times \square^n)^1, |Z_\Delta|)^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ *is a semi-stable pair. Then there exists a closed subset* $\Delta^2 \subset \Delta$ *such that for all* $s \in S^1 := \Delta \backslash \Delta^2$, $((X_\Delta \times \square^n)^1, |Z|_\Delta)^{\mathcal{O}}_{\hat{k}_{\sigma_p}, s}$ *is a semi-stable pair.*
- *Go on by induction.*

*We obtain by the above finite induction, for* $p \in \mathbb{N} \backslash \delta(S, X/S)$*, with* $\delta(S, X/S) := \delta(S) \cup (\cup_{\alpha \in \Lambda} \delta^\alpha(X/S))$*, a stratification* $S = \sqcup_{\alpha \in \Lambda} S^\alpha$*,* $\Lambda$ *being a finite set, by locally closed subset* $S^\alpha \subset S$*, and alterations (i.e. generically finite morphisms)* $\pi^\alpha : (X_{S^\alpha} \times \square^n)^\alpha \to X_{S^\alpha} \times \square^n$ *such that*

$$f \circ \pi^\alpha : (X_{S^\alpha} \times \square^n)^{\alpha, \mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U^\alpha, \mathcal{O}}) \to (S^\alpha_{\hat{k}_{\sigma_p}}, N_{\mathcal{O}})$$

*is log smooth, that is for all* $s \in S^\alpha$, $((X_{S^\alpha} \times \square^n)^\alpha, \pi^{\alpha,-1}(|Z_{S^\alpha}|))^{\mathcal{O}}_{\hat{k}_{\sigma_p}, s}$ *is a semi-stable pair. We then set*

$$T := p_S(((\sqcup_{\alpha \in \Lambda}(\pi_{k/\hat{k}_{\sigma_p}}(U^\alpha)((E_{DR}(U^\alpha_{\hat{k}_{\sigma_p}}/S^\alpha_{\hat{k}_{\sigma_p}}) \otimes_{O_{S^\alpha_{\hat{k}_{\sigma_p}}}} O\mathbb{B}_{st, S^\alpha_{\hat{k}_{\sigma_p}}})^{\phi_p, N})))$$
$$\cap F^d E_{DR}(U/S) \cap (E_{DR}(U/S)^{[Z]})) \backslash E_{DR}((X \times \square^n)/S)) \subset S$$

*and*

$$\hat{T}_{\sigma_p} := p_{S_{\hat{k}_\sigma}}(((\sqcup_{\alpha \in \Lambda}(E_{DR}(U^\alpha_{\hat{k}_{\sigma_p}}/S^\alpha_{\hat{k}_{\sigma_p}}) \otimes_{O_{S_{\hat{k}_{\sigma_p}}}} O\mathbb{B}_{st, S_{\hat{k}_{\sigma_p}}})^{\phi_p, N}) \cap$$
$$F^d E_{DR}(U_{\hat{k}_{\sigma_p}}/S_{\hat{k}_{\sigma_p}}) \cap \pi^{\alpha,-1}(E_{DR}(U_{\hat{k}_{\sigma_p}}/S_{\hat{k}_{\sigma_p}})^{[Z]})) \backslash E_{DR}((X \times \square^n)_{\hat{k}_{\sigma_p}}/S_{\hat{k}_{\sigma_p}})) \subset S_{\hat{k}_{\sigma_p}}$$

*where*

- $U^\alpha := (X_{S^\alpha} \times \square^n)^\alpha \backslash \pi^{\alpha,-1}(|Z_{S^\alpha}|)$,
- $E_{DR}(U^\alpha/S^\alpha) := H^{2d-1} R f_{*Hdg}(O_{U^\alpha}, F_b) \in \mathrm{Vect}_{fil}(S^\alpha)$,
- $\phi_p$ *is the Frobenius operator,* $N$ *is the monodromy operator, note that for* $K'/k'$ *a field extension,* $Y \in \mathrm{Var}(k')$ *and* $R \subset Y_{K'}$ *a closed subset, we have* $\overline{R}^{k'} = \pi_{k'/K'}(Y)^{-1}(\pi_{k'/K'}(Y)(R))$,
- $p_S : E_{DR}(U/S) \to S$ *and* $p_{S_{\hat{k}_\sigma}} : E_{DR}(U_{\hat{k}_{\sigma_p}}/S_{\hat{k}_{\sigma_p}}) \to S_{\hat{k}_{\sigma_p}}$ *are the projections.*

**Lemma 3.** *Let* $G$ *be a group. Consider a commutative diagram of* $G$ *modules*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & V & \xrightarrow{\partial} & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} & & \\ 0 & \longrightarrow & W' & \longrightarrow & V' & \xrightarrow{\partial'} & K' & \longrightarrow & 0 \end{array}$$

*whose rows are exact sequence and* $\pi^* : V \to V'$ *is injective. Let* $\alpha \in V$*. Then* $\alpha \in V^G$ *and* $\partial\alpha \neq 0$ *if and only if* $\pi^*\alpha \in V'^G$ *and* $\partial'\pi^*\alpha \neq 0$.

*Proof.* Follows from the fact that $< \alpha >$ define a splitting $W \oplus < \alpha > \subset V$ of $G$ modules. □

**Theorem 7.** *Let $k$ a field of finite type over $\mathbb{Q}$. Let $f : X \to S$ a smooth proper morphism with $S, X \in \mathrm{SmVar}(k)$ connected. Let $p \in \mathbb{N}\backslash\delta(S, X/S)$ a be prime number, where $\delta(S, X/S)$ is the finite set given in definition 6. Let $Z \in \mathcal{Z}^d(X, n)^{\partial=0,f}_{hom}$.*

*(i) We have*

$$V_{tors}(\nu_Z^{et,p}) = T \cap S_{(0)} \subset S,$$

*where $T \subset S$ is given in definition 6.*

*(ii) For each embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$, we have*

$$V_{tors}(\nu_{Z,\sigma_p}^{et,p})_{\hat{k}_{\sigma_p}} = \hat{T}_{\sigma_p} \cap S_{(0),\hat{k}_{\sigma_p}} \subset S_{\hat{k}_{\sigma_p}},$$

*where $\hat{T}_{\sigma_p} \subset S_{\hat{k}_{\sigma_p}}$ is given in definition 6, and for $V \subset S$ a subset, $V_{\hat{k}_{\sigma_p}} := \pi_{k/\hat{k}_{\sigma_p}}(S)^{-1}(V)$, where $\pi_{k/\hat{k}_{\sigma_p}}(S) : S_{\hat{k}_{\sigma_p}} \to S$ being the projection.*

*Proof.* Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ be an embedding. For each $\alpha \in \Lambda$, by the semi-stable comparaison theorem for

$$f^\alpha := f \circ \pi^\alpha : ((X_{S^\alpha} \times \square^n)^{\alpha,\mathcal{O}}_{\hat{k}_{\sigma_p}}, N^{\mathcal{O}}_{U^\alpha}) \to (S^{\alpha,\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{\mathcal{O}})$$

([1]) which is log smooth, we have the isomorphism in $\mathrm{Shv}_{fil,G,\phi_p,N}(S^\alpha_{\hat{k}_{\sigma_p}})$

$$H^j f^\alpha_* \alpha(U^\alpha) : R^j f^\alpha_* \mathbb{Z}_{p,U^{\alpha,et}_{\mathbb{C}_p}} \otimes_{\mathbb{Z}_p} \mathbb{B}_{st,S^\alpha_{\hat{k}_{\sigma_p}}} \xrightarrow{\sim} H^j \int_{f^\alpha} j_{*Hdg}(O_{U^\alpha_{\hat{k}_{\sigma_p}}}, F_b) \otimes_{O_{S_{\hat{k}_{\sigma_p}}}} O\mathbb{B}_{st,S^\alpha_{\hat{k}_{\sigma_p}}}, \tag{1}$$

recall that since $p \in \mathbb{N}\backslash\delta(S)$, $\hat{k}_{\sigma_p}(s)$ is unramified for all $s \in S_{\hat{k}_{\sigma_p}}$.

(i): Let $s \in S^\alpha_{(0)}$. Denote $G := Gal(\bar{k}/k(s))$.

- The map $\pi^\alpha_s : (X_s \times \square^n)^\alpha \to X_s \times \square^n$ is generically finite since $\pi^\alpha : (X \times \square^n)^\alpha \to X \times \square^n$ is generically finite and $f$ is flat. Thus by lemma 3 and theorem 4, $\nu_Z^{et,p}(s) := AJ^{et,p}(X_s)(Z_s) = 0$ if and only if there exists
$$\alpha \in H^{2d-1}_{et}(U_{S^\alpha,s}, \mathbb{Z}_p)^{[Z_s]}$$
such that $\pi^{\alpha,-1}(\alpha) \in H^{2d-1}_{et}(U^\alpha_s, \mathbb{Z}_p)(d)^G$ and $\partial\pi^{\alpha,-1}(\alpha) \neq 0$.
- On the other hand by (1) and lemma 1, $\alpha' \in H^{2d-1}_{et}(U^\alpha_s, \mathbb{Z}_p)(d)^G$ if and only if $w(\alpha') \in F^d H^{2d-1}_{DR}(U^\alpha_s)$ and
$$w(\alpha') \in (H^{2d-1}_{DR}(U^\alpha_{s,\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}(s)} \mathbb{B}_{st})^{\phi_p,N}.$$
Moreover $w(\pi^{\alpha,-1}(\alpha)) = \pi^{\alpha,-1}(w(\alpha))$.

(ii): Let $s \in S^\alpha_{(0)}$ and $s' \in \pi_{k/\hat{k}_{\sigma_p}}(S)^{-1}(s)$.

- Since $\pi^\alpha_s : (X_s \times \square^n)^\alpha \to X_s \times \square^n$ is generically finite (see the proof of (i)) we have by lemma 3 and theorem 5, $\nu^{et,p}_{Z,\sigma_p}(s) := AJ^{et,p}_{\sigma_p}(X_s)(Z_s) = 0$ if and only if there exists
$$\alpha \in H^{2d-1}_{et}(U_{S^\alpha,s,\hat{k}_{\sigma_p}}, \mathbb{Z}_p)^{[Z_s]}$$
such that $\pi^{\alpha,-1}(\alpha) \in H^{2d-1}_{et}(U^\alpha_{s,\hat{k}_{\sigma_p}}, \mathbb{Z}_p)(d)^{\hat{G}_{\sigma_p}}$ and $\partial\alpha \neq 0$.
- On the other hand by (1), $\alpha' \in H^{2d-1}_{et}(U^\alpha_{s,\hat{k}_{\sigma_p}}, \mathbb{Z}_p)(d)^{\hat{G}_{\sigma_p}}$ if and only if $w(\alpha') \in F^d H^{2d-1}_{DR}(U^\alpha_{\hat{k}_{\sigma_p},s})$ and
$$w(\alpha') \in (H^{2d-1}_{DR}(U^\alpha_{s,\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}(s)} \mathbb{B}_{st})^{\phi_p,N} = (H^{2d-1}_{DR}(U^\alpha_{s',\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}(s)} \mathbb{B}_{st})^{\phi_p,N}.$$
Moreover $w(\pi^{\alpha,-1}(\alpha)) = \pi^{\alpha,-1}(w(\alpha))$.

□

The main result of [1] also give together with theorem 1 the following

**Theorem 8.** *Let $f : X \to S$ be a smooth proper morphism, with $S, X$ smooth over $k \subset \mathbb{C}$ which is a finite extension of $\mathbb{Q}$. Consider for $j \in \mathbb{Z}$,*

$$E_{DR}(X/S) := H^j \int_f (O_X, F_b) = R^j f_* \Omega^\bullet_{X/S} \in \mathrm{Vect}_{\mathcal{D}fil}(S)$$

*Consider, using definition 6, a stratification $S = \sqcup_{\alpha \in \Lambda} S^\alpha$, $\Lambda$ being a finite set, by locally closed subset $S^\alpha \subset S$, and alterations (i.e. generically finite morphisms) $\pi^\alpha : X^\alpha \to X_{S^\alpha}$ such that*

$$f_\alpha := f \circ \pi^\alpha : (X^{\alpha,\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{\mathcal{O}}) \to (S^\alpha_{\hat{k}_{\sigma_p}}, N_{\mathcal{O}})$$

*is log smooth. Then, for each prime number $p \in \mathbb{N} \backslash \delta(S, X/S)$, we get by [1], for each $j, d \in \mathbb{Z}$ and each $\alpha \in \Lambda$, an isomorphism*

$$H^j f^\alpha_* \alpha(X^\alpha) : HT^p_{j,d}(X_{S^\alpha}/S^\alpha) \xrightarrow{\sim} F^d E_{DR}(X_{S^\alpha}/S^\alpha) \cap (\pi_{k/\hat{k}_{\sigma_p}}(X^\alpha)((E_{DR}(X^\alpha_{\hat{k}_{\sigma_p}}/S^\alpha_{\hat{k}_{\sigma_p}}) \otimes_{O_{X^\alpha_{\hat{k}_{\sigma_p}}}} O\mathbb{B}_{st,X})^{\phi_p, N}))$$

*where $HT^p_{j,d}(X_{S^\alpha}/S^\alpha) \subset R^j f_* \mathbb{Q}_{p, X^{et}_{S^\alpha, \bar{k}}}(d)$ is the locus of Hodge Tate classes. Using theorem 1, we get the locus of Tate classes $HT^p_{j,d}(X/S) := (R^j f_* \mathbb{Q}_{p, X^{et}_{\bar{k}}}(d))^G \subset R^j f_* \mathbb{Q}_{p, X^{et}_{\bar{k}}}(d)$ as the image of an algebraic constructible subset of $E_{DR}(X_{\hat{k}_{\sigma_p}}/S_{\hat{k}_{\sigma_p}})$*

$$\begin{aligned} \iota^{G,d}_{ev}(X/S) : (HT^p_{j,d}(X/S) = \sqcup_{\alpha \in \Lambda} HT^p_{j,d}(X_{S^\alpha}/S^\alpha))_{\mathbb{C}} &\xrightarrow{((H^j f^\alpha_* \alpha(X^\alpha))_{\alpha \in \Lambda})_{\mathbb{C}}, \sim} \\ (F^d E_{DR}(X/S) \cap (\sqcup_{\alpha \in \Lambda} \pi_{k/\hat{k}_{\sigma_p}}(X^\alpha)((E_{DR}(X^\alpha_{\hat{k}_{\sigma_p}}/S^\alpha_{\hat{k}_{\sigma_p}}) \otimes_{O_{X^\alpha_{\hat{k}_{\sigma_p}}}} O\mathbb{B}_{st, X^\alpha_{\hat{k}_{\sigma_p}}})^{\phi_p, N})))_{\mathbb{C}} \\ \hookrightarrow F^d E_{DR}(X_{\mathbb{C}}/S_{\mathbb{C}}) \cap R^j f_* \mathbb{Q}_{X^{an}_{\mathbb{C}}} =: HL_{j,d}(X_{\mathbb{C}}/S_{\mathbb{C}}), \\ \alpha_{s'} \mapsto \iota^{G,d}_{ev}(X_{s'})(\alpha_{s'}) := ev(X_{s'})(w(\alpha_{s'})),\ s' \in \pi_{k/\mathbb{C}}(S)^{-1}(s),\ s \in S, \end{aligned}$$

*inside the locus of Hodge classes $HL_{j,d}(X_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}(X_{\mathbb{C}}/S_{\mathbb{C}})$, where $\pi_{k/\hat{k}_{\sigma_p}}(X^\alpha) : X^\alpha_{\hat{k}_{\sigma_p}} \to X_\alpha$ and $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \to S$ are the projections. Note that the image consists of logarithmic classes. Note that for $Y \in \mathrm{Var}(k)$, the image of an algebraic constructible subset of $Y_{\hat{k}_{\sigma_p}}$ by $\pi_{k/\hat{k}_{\sigma_p}}(Y)$ is NOT algebraic constructible since $\pi_{k/\hat{k}_{\sigma_p}}(Y)$ is NOT a morphism of finite type.*

*Proof.* Follows from the result of [1] and theorem 1. □

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